Relational event models for the analysis of social networks

an overview of network models

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On these slides we discuss

- what is special about **network** data;
- a range of statistical network models
  - for time-independent network data;
  - for longitudinal data given by snapshots;
  - for networks of relational events.

Outline.

Network data.

Models for time-independent networks and network snapshots.

Relational event models.

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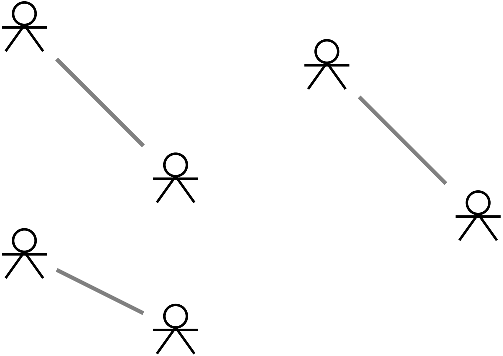
Relational event models.

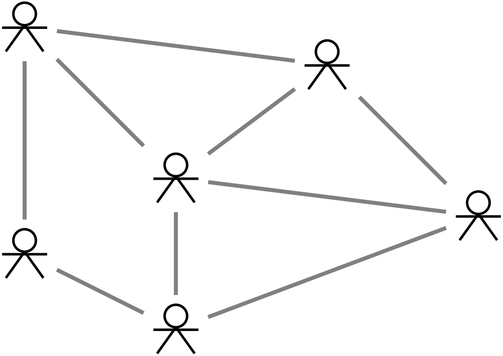
Network data.

Observations are associated with **dyads**



Network data.

Observations are associated with **overlapping dyads**



Network data.

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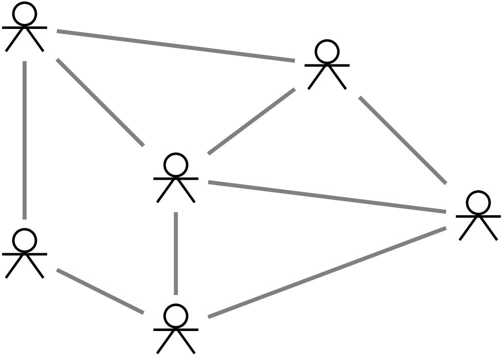
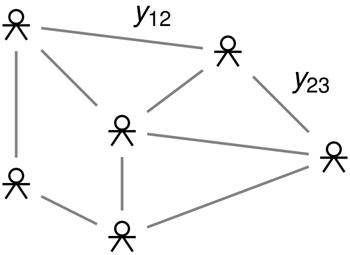


$\Rightarrow$ ***independence of observations is nearly unthinkable.***

Networks as realizations of random variables.

Observations are associated with **overlapping dyads.**



$$\begin{bmatrix} \cdot & 1 & 0 & 1 & 1 & 0 \\ 1 & \cdot & 1 & 0 & 1 & 0 \\ 0 & 1 & \cdot & 0 & 1 & 1 \\ 1 & 0 & 0 & \cdot & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdot & 1 \\ 0 & 0 & 1 & 1 & 1 & \cdot \end{bmatrix}$$

Observed network $y = (y_{ij})$ is realization of a matrix of random variables $Y = (Y_{ij})$.

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Models for time-independent networks and network snapshots.

Relational event models.

Cross-sectional data: time-independent networks.

A simple statistical model for networks.

Observed network $y = (y_{ij})$ is realization of a
matrix of random variables $Y = (Y_{ij})$.

Mutually independent model

$$P(Y = y) \;=\; \prod_{ij} P(Y_{ij} = y_{ij}) \; .$$

Dyadic tie probabilities are independent of each other.

- This model is completely unrealistic.

Exponential random graph models (ERGM).

very general framework for network models

Probability of a network y has the functional form

$$P_\theta(Y = y) = \frac{1}{z} \exp\left(\sum_k \theta_k \cdot s_k(y) \right) \quad , \text{ where}$$

- the θ_k are **parameters**
- the $s_k(y)$ are **statistics** of the network y, such as
  - number of edges (**density**)
  - number of edges connecting nodes with similar characteristics (**homophily**)
  - number of triangles (**triadic closure**)
  - number of stars (**activity** or **popularity**)
  - ...
- z is a normalizing constant.

Exponential random graph models (ERGM).

very general framework for network models

Probability of a network y has the functional form

$$P_\theta(Y = y) \quad = \quad \frac{1}{z} \exp\left(\sum_k \theta_k \cdot s_k(y)\right) \quad .$$

Given an observed network, parameters θ are estimated s. t.

- the expected values of all statistics are equal to
- the statistics of the observed network.

Estimated parameters reveal network effects (homophily, ...).

Some points about ERGMs.

- Can deal with complex dependence among dyadic observations;
- might lead to degenerate models;
- parameter estimation is quite involved (but software exists)

$\Rightarrow$ e. g., R package `ergm` (`https://statnet.org/`).

Is a model for cross-sectional networks without time information (does not apply to our setting).

Longitudinal data given by network snapshots.

Longitudinal data given by network snapshots.

Networks, observed at given points in time $t_1, \ldots, t_h$

$$y = (y^{(1)}, \ldots, y^{(h)}) \ .$$

Each $y^{(\ell)}$ is an observed adjacency matrix.

Often results from repeated application of a questionnaire:
"*list all your friends within your school class*."

Stochastic actor-oriented models (SAOM).

Define probability distributions on

- **latent (unobserved)** sequences of micro-steps,
- where each micro-step changes the value of one dyad
  (deleting a tie or creating a tie),
- transforming a network $y^{(t)}$ into the next $y^{(t+1)}$.

Stochastic actor-oriented models (SAOM).

Define probability distributions on

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A micro-step is defined by two processes determining

1. the actor i who becomes active (gets the opportunity to change one out-going tie);
2. the target actor j (so that the tie y_{ij} will be flipped).

The active actor chooses micro-steps leading to desirable configurations (reciprocated ties, transitive closure, ...).

Distribution of micro-steps depends on current (latent) state.

Some points about SAOMs.

Define probability distributions on **latent (unobserved)** sequences of micro-steps,...

- Model network **dynamics** (rather than network state).
- Parameter estimation is quite involved (but software exists)

$\Rightarrow$ `RSiena` (`www.stats.ox.ac.uk/~snijders/siena`).

Is a model for network snapshots at given points in time (does not apply to our setting).

Outline.

Network data.

Models for time-independent networks and network snapshots.

Relational event models.

Networks of relational events.

Given by sequences of time-stamped dyadic events

$$E = (e_1, \ldots, e_N), \text{ where } e_i = (a_i, b_i, t_i, x_i)$$

- a_i **source** (sender) of the event;
- b_i **target** (receiver) of the event;
- t_i **time** of the event (potentially: ordinal);
- x_i **type** of the event (potentially: type and weight).

Who does when what to whom?

Relational event models (REM).

Given a sequence of time-stamped dyadic events

$$E = (e_1, \ldots, e_N) \ , \text{ where } e_i = (a_i, b_i, t_i, x_i) \ ,$$

define $G[E_{<t}] = G[\{e_i \in E\ ;\ t_i < t\}]$ **(network of past events).**

Relational event models (REM).

Given a sequence of time-stamped dyadic events

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define $G[E_{<t}] = G[\{e_i \in E \, ; \ t_i < t\}]$ **(network of past events).**

Relational event models specify probability distributions

$$P(E) = \prod_{i=1}^{N} P(e_i \,|\, G[E_{<t_i}]) \ .$$

- Events are assumed to be conditionally independent, given the network of past events.

$\Rightarrow$ With sufficient statistics of the network of past events, REMs are regression models(!).

Relational event models (REM).

Relational event models specify probability distributions

$$P(E) = \prod_{i=1}^{N} P(e_i \mid G[E_{<t_i}]) \ .$$

The conditional probabilities $P(e_i \mid G[E_{<t_i}])$ are specified by

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

- Response variables $y(e_i)$, encoding aspects of the event,
- are drawn from a probability distribution f,
- which is a function of the network of past events $G[E_{<t_i}]$.

Explanatory variables: statistics characterizing how the dyad (a_i, b_i) is embedded in the network of past events $G[E_{<t_i}]$.

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

Explanatory variables: event network statistics.

The distribution f in $y(e_i) \sim f(G[E_{<t_i}]; e_i)$ is typically a parametric function of dyad statistics $s = s_1, \ldots, s_k$

$$f(s(G[E_{<t_i}]; e_i); \theta), \text{ where } \theta = \theta_1, \ldots, \theta_k$$

Explanatory variables: event network statistics.

The distribution f in $y(e_i) \sim f(G[E_{<t_i}]; e_i)$ is typically a parametric function of dyad statistics $s = s_1, \ldots, s_k$

$$f(s(G[E_{<t_i}]; e_i); \theta), \text{ where } \theta = \theta_1, \ldots, \theta_k$$

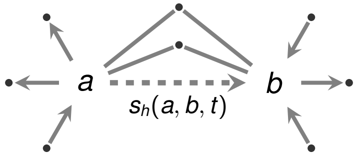
Statistics s_h characterize how dyads (A, B) are embedded into the network of past events.

- Distribution of events on the dyad (A, B) depend on past events
  - from A to B (**repetition**);
  - from B to A (**reciprocation**);
  - to/from A or B (**degree effects**);
  - to/from common third actors (**triadic effects**);
  - ...
- Parameters θ_h (to be estimated from empirical data) determine the effect of statistics on response variables.

Event network statistics $s_h(a, b, t)$.

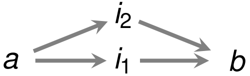
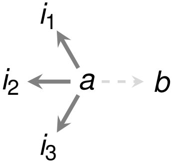
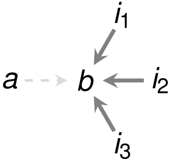
Statistics $s_h(a, b, t)$ assign time-varying values to dyads;

- are the variables which explain events on (a, b) at t;
- are functions of past events happening before t on the same or other dyads;
- introduce dependence among dyadic observations.



Examples: repetition, reciprocation, (in-/out-/mixed-)degrees, triadic effects, four-cycle effects, covariate effects, ...

Examples: common statistics $s_h(a, b, t)$.

statistic	$s_h(a, b, t) =$	$a \dashrightarrow b$ depends on
`repetition`	$att(a, b, t)$	$a \longrightarrow b$
`reciproc.`	$att(b, a, t)$	$a \longleftarrow b$
`transitivity`	$\sum_i att(a, i, t) \cdot att(i, b, t)$	
`outDegSource`	$\sum_i att(a, i, t)$	
`inDegTarget`	$\sum_i att(i, b, t)$	

Families of probability distributions for the next event

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

Here: describe four variants

1. modeling relative event rates;
2. modeling the time to events;
3. modeling conditional event types;
4. modeling conditional event weights.

Families of probability distributions for the next event

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

(I) Modeling relative event rates.

$\Rightarrow$ Cox proportional-hazard models.

Modeling relative event rates.

Cox proportional-hazard models

On which dyad does the next event happen?

Specify time-varying dyadic event rates $\lambda(A, B; t)$
(expected number of events per time unit on (A, B) at t).

Decompose into time-varying baseline rate and a parametric relative part

$$\begin{aligned}\lambda(A, B; t) &= \lambda_0(t) \cdot \lambda_1(A, B; t; \theta) \\ \lambda_1(A, B; t; \theta) &= \exp\left(\sum_h \theta_h \cdot s_h(A, B; G[E_{<t}])\right)\end{aligned}$$

Baseline rate λ_0 may be estimated by non-parametric methods.

Modeling relative event rates.

Cox proportional-hazard models

Parametric relative event rate

$$\lambda_1(A, B; t; \theta) \quad = \quad \exp\left(\sum_h \theta_h \cdot s_h(A, B; G[E_{<t}])\right)$$

Leads to a partial likelihood for the i'th event $e_i = (a_i, b_i, t_i)$

$$P(e_i \mid G[E_{<t_i}]; \theta) = \frac{\lambda_1(a_i, b_i; t_i; \theta)}{\sum_{ab \in R_{t_i}} \lambda_1(a, b; t_i; \theta)}$$

Risk set R_{t_i}: all dyads that could experience an event at t_i;
potentially: sample from R_{t_i}.

Parameters: maximize $P(E; \theta) = \prod_i P(e_i \mid G[E_{<t_i}]; \theta)$.
`coxph` in the R-package **survival**.

Families of probability distributions for the next event

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

(II) Modeling the time to events.

Modeling the time to events.

assuming piecewise-constant event rates, changing only at event times

On which dyad – and when – does the next event happen?

Specify time-varying dyadic event rates $\lambda(A, B; t)$
(expected number of events per time unit on (A, B) at t).

$$\lambda(A, B; t; \theta) \quad = \quad \exp\left(\sum_h \theta_h \cdot s_h(A, B; G[E_{<t}])\right)$$

Leads to likelihood for the i'th event $e_i = (a_i, b_i, t_i)$

$$P(e_i \mid G[E_{<t_i}]; \theta) = \frac{\lambda(a_i, b_i; t_i; \theta)}{\exp\left((t_i - t_{i-1}) \cdot \sum_{ab \in R_{t_i}} \lambda(a, b; t_i; \theta)\right)}$$

Parameter estimation: `survreg` in R-package **survival**.

Families of probability distributions for the next event

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

(III) Modeling conditional event types.

Modeling conditional event types.

How do actors interact, given that they do interact?

Assume that there are two types of events:

cooperative and conflictive.

Specify probability that type x_i of given event $e_i = (a_i, b_i, t_i, x_i)$ is cooperative (e. g., by a logit model)

$$P(x_i = \text{coop} \mid a_i, b_i; G[E_{<t_i}]; \theta) = \text{logit}^{-1}\left(\sum_h \theta_h \cdot s_h(a_i, b_i; G[E_{<t}])\right)$$

Generalizes to, e. g., (ordered) multi-nominal event types.

Families of probability distributions for the next event

$$y(e_i) \sim f(G[E_{<t_i}]; e_i)$$

(VI) Modeling conditional event weights.

Modeling conditional event types.

How do actors interact, given that they do interact?

Assume that events have numeric weights $x \in \mathbb{R}$
e. g., x measuring the performance of interaction.

Specify distribution of weight x_i of given event $e_i = (a_i, b_i, t_i, x_i)$
e. g., by the normal distribution $\Rightarrow$ linear regression.

$$\begin{aligned} x_i \mid a_i, b_i; G[E_{<t_i}]; \theta &\sim \mathcal{N}(\mu, \sigma^2) \\ \mu &= \sum_h \theta_h \cdot s_h(a_i, b_i; G[E_{<t}]) \end{aligned}$$

Generalizes to other distributions of event weights.