Gambler's Ruin - Theory

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Introduction

The Gambler's Ruin problem frames a gambler who begins gambling with an initial fortune - in dollars say. At each successive gamble, the gambler either loses \$1 or gains \$1. The problem is to find the probability that the gambler goes bankrupt - loses the entirety of the fortune. This problem is a kind of random walk. Figures 1 and 2 below show simulation trajectories for this setup.

1 The Problem

A Gambler begins with \$k and repeatedly plays a game after which they may win \$1 with probability p or lose \$1 with probability $q = 1 - p$. The Gambler will stop playing if their fortune reaches \$0 or \$N. What is the probability that they go bankrupt?

2 The Solution

Let u_k be the probability that the Gambler bankrupts if the initial fortune is \$k. Then we can condition this probability on the first gamble as follows (utilising the law of total probability with the partitioning of lose/win):

$$u_k = P(wins) \times u_{k+1} + P(loses) \times u_{k-1}$$

This is a second order homogeneous difference equation. We look for solutions of the form $u_n = A \times \lambda^n$.

$$\begin{aligned} p \times u_{n+1} - u_n + qu_{n-1} &= 0 \\ \implies p \times A \times \lambda^{n+1} - A \times \lambda^n + q \times A \times \lambda^{n-1} &= 0 \\ \implies \lambda^2 - \frac{1}{p}\lambda + \frac{q}{p} &= 0 \end{aligned}$$

where $p, q \neq 0$. This has solution:

$$\lambda_{1,2} = \left\{ \frac{1-p}{p}, 1 \right\}$$

provided that $p \neq \frac{1}{2}$, this gives 2 different solutions. We have:

$$\begin{aligned} u_n &= A\left(\frac{1-p}{p}\right)^n + B(1)^n \\ &= A\left(\frac{1-p}{p}\right)^n + B \end{aligned}$$

We have that the Gambler stops gambling if either their fortune reaches \$0 or \$N. So we have the following boundary conditions:

$$u_0 = 1, u_N = 0$$

Figure 1: This is a plot of 10 simulations with $k = 12.5$, $p = 0.55$, $N = 25$. The theory results in a probability of 0.0753 of bankruptcy. The horizontal green line represents \$N and the hosrizontal red line represents \$0.

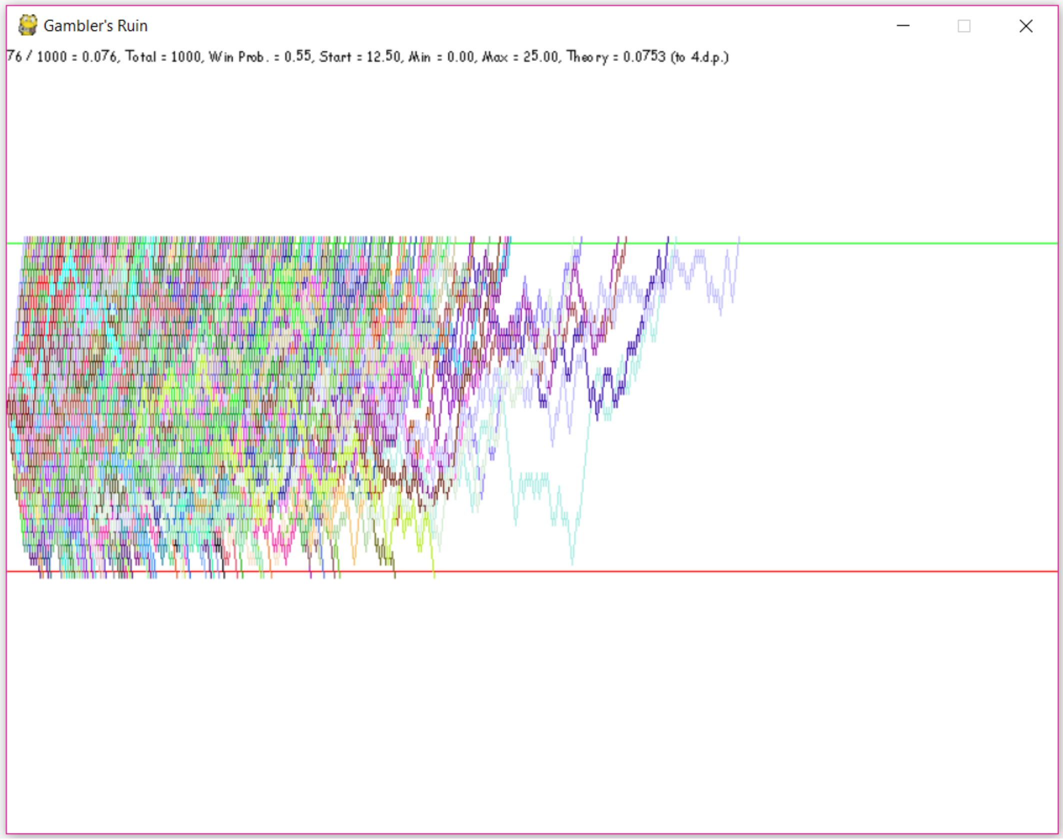

Figure 2: This is a plot of 1000 simulations with $k = 12.5$, $p = 0.55$, $N = 25$. The theory results in a probability of 0.0753 of bankruptcy. The horizontal green line represents $N and the hosrizontal red line represents $0.

Using these boundary conditions, we can solve for A and B:

$$\begin{aligned} u_0 = A\left(\frac{1-p}{p}\right)^0 + B = 1 \\ \implies A + B = 1 \\ \implies B = 1 - A \end{aligned}$$

and

$$\begin{aligned} u_N = A\left(\frac{1-p}{p}\right)^N + B = 0 \\ \implies B = -A\left(\frac{1-p}{p}\right)^N \\ \implies 1 - A = -A\left(\frac{1-p}{p}\right)^N \\ \implies A = \frac{1}{1-\left(\frac{1-p}{p}\right)^N} \\ \implies B = 1 - A = \frac{-\left(\frac{1-p}{p}\right)^N}{1-\left(\frac{1-p}{p}\right)^N} \end{aligned}$$

Giving the final solution:

$$u_n = \frac{\left(\frac{1-p}{p}\right)^n - \left(\frac{1-p}{p}\right)^N}{1-\left(\frac{1-p}{p}\right)^N} \tag{1}$$

For the case where $p = \frac{1}{2}$, we try the next most complex expression, let:

$$u_n = (An + B) \times \lambda^n$$

with $\lambda = 1$:

$$u_n = (An + B)$$

We can try this in the original equation with $p = q = 1/2$:

$$\frac{1}{2}u_{n+1} - u_n + \frac{1}{2}u_{n-1} = \frac{An}{2} + \frac{A}{2} + \frac{B}{2} - An - B + \frac{An}{2} - \frac{A}{2} + \frac{B}{2} = 0$$

Using the boundary conditions:

$$u_0 = B = 1$$

$$u_N = AN + B = 0 \implies A = \frac{-1}{N}$$

Giving the final equation as:

$$u_n = 1 - \frac{n}{N} \tag{2}$$