Gambler's Ruin - Theory

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Introduction

The Gambler's Ruin problem frames a gambler who begins gambling with an initial fortune - in dollars say. At each successive gamble, the gambler either loses \$1 or gains \$1. The problem is to find the probability that the gambler goes bankrupt - loses the entirety of the fortune. This problem is a kind of random walk. Figures 1 and 2 below show simulation trajectories for this setup.

1 The Problem

A Gambler begins with \$k and repeatedly plays a game after which they may win \$1 with probability p or lose \$1 with probability $q = 1 - p$. The Gambler will stop playing if their fortune reaches \$0 or \$N. What is the probability that they go bankrupt?

2 The Solution

Let u_k be the probability that the Gambler bankrupts if the initial fortune is \$k. Then we can condition this probability on the first gamble as follows (utilising the law of total probability with the partitioning of lose/win):

$$u_k = P(wins) \times u_{k+1} + P(loses) \times u_{k-1}$$

This is a second order homogeneous difference equation. We look for solutions of the form $u_n = A \times \lambda^n$.

$$\begin{aligned} p \times u_{n+1} - u_n + qu_{n-1} &= 0 \\ \implies p \times A \times \lambda^{n+1} - A \times \lambda^n + q \times A \times \lambda^{n-1} &= 0 \\ \implies \lambda^2 - \frac{1}{p}\lambda + \frac{q}{p} &= 0 \end{aligned}$$

where $p, q \neq 0$. This has solution:

$$\lambda_{1,2} = \left\{ \frac{1-p}{p}, 1 \right\}$$

provided that $p \neq \frac{1}{2}$, this gives 2 different solutions. We have:

$$\begin{aligned} u_n &= A \left(\frac{1-p}{p} \right)^n + B(1)^n \\ &= A \left(\frac{1-p}{p} \right)^n + B \end{aligned}$$

We have that the Gambler stops gambling if either their fortune reaches \$0 or \$N. So we have the following boundary conditions:

$$u_0 = 1, u_N = 0$$

Figure 1: This is a plot of 10 simulations with $k = 12.5$, $p = 0.55$, $N = 25$. The theory results in a probability of 0.0753 of bankruptcy. The horizontal green line represents \$N and the hosrizontal red line represents \$0.

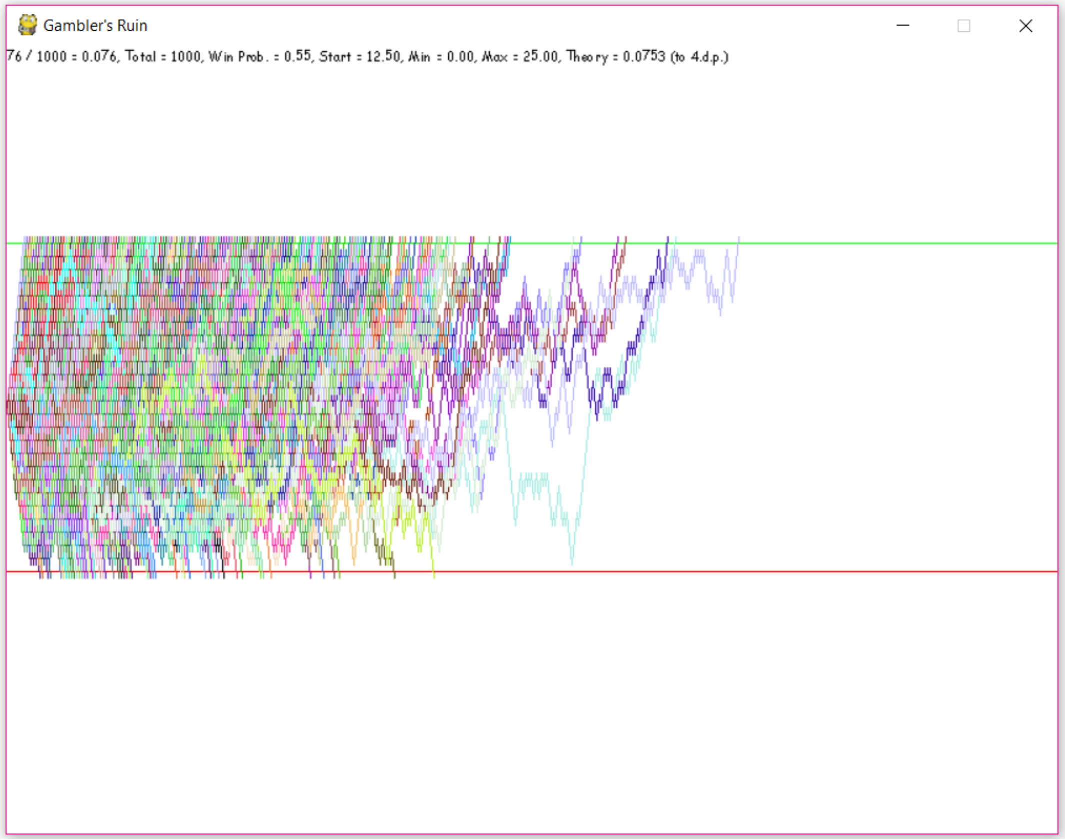

Figure 2: This is a plot of 1000 simulations with $k = 12.5$, $p = 0.55$, $N = 25$. The theory results in a probability of 0.0753 of bankruptcy. The horizontal green line represents \$N and the hosrizontal red line represents \$0.

Using these boundary conditions, we can solve for A and B:

$$\begin{aligned}u_0 = A\left(\frac{1-p}{p}\right)^0 + B = 1 \\ \implies A + B = 1 \\ \implies B = 1 - A\end{aligned}$$

and

$$\begin{aligned}u_N = A\left(\frac{1-p}{p}\right)^N + B = 0 \\ \implies B = -A\left(\frac{1-p}{p}\right)^N \\ \implies 1 - A = -A\left(\frac{1-p}{p}\right)^N \\ \implies A = \frac{1}{1-\left(\frac{1-p}{p}\right)^N} \\ \implies B = 1 - A = \frac{-\left(\frac{1-p}{p}\right)^N}{1-\left(\frac{1-p}{p}\right)^N}\end{aligned}$$

Giving the final solution:

$$u_n = \frac{\left(\frac{1-p}{p}\right)^n - \left(\frac{1-p}{p}\right)^N}{1-\left(\frac{1-p}{p}\right)^N} \tag{1}$$

For the case where $p = \frac{1}{2}$, we try the next most complex expression, let:

$$u_n = (An + B) \times \lambda^n$$

with $\lambda = 1$:

$$u_n = (An + B)$$

We can try this in the original equation with $p = q = 1/2$:

$$\frac{1}{2}u_{n+1} - u_n + \frac{1}{2}u_{n-1} = \frac{An}{2} + \frac{A}{2} + \frac{B}{2} - An - B + \frac{An}{2} - \frac{A}{2} + \frac{B}{2} = 0$$

Using the boundary conditions:

$$u_0 = B = 1$$

$$u_N = AN + B = 0 \implies A = \frac{-1}{N}$$

Giving the final equation as:

$$u_n = 1 - \frac{n}{N} \tag{2}$$

We have the final equations for the probability of bankruptcy as:

$$u_n = \begin{cases} 1 - \frac{n}{N} & \text{if } p = 0.5 \\ \frac{\left(\frac{1-p}{p}\right)^n - \left(\frac{1-p}{p}\right)^N}{1-\left(\frac{1-p}{p}\right)^N} & \text{if } p \neq 0.5 \text{ and } p \neq 0 \\ 1 & \text{if } p = 0 \end{cases} \tag{3}$$

Note: The case where $p = 0$ is obtained by multiplying the numerator and denominator of the $p \neq 0.5$ case by p^N and simplifying.

3 Expected Number of Steps

We can now ask the question "How many times is the Gambler expected to be able to gamble until he stops?". We can approach the solution in a similar manner as the probability calculation in the previous section. Namely, conditioning on the first gamble.

Let E_n be the expected number of steps until the Gambler's fortune reaches either \$0 or \$N if the fortune starts at \$n. We then condition on the first gamble as follows:

$$ExpectedStepsfromn = p \times (1 + ExpectedStepsfromn + 1) + q \times (1 + ExpectedStepsfromn - 1)$$

$$E_n = p \times (1 + E_{n+1}) + q \times (1 + E_{n-1})$$

Re-writing:

$$p \times E_{n+1} - E_n + q \times E_{n-1} = -1$$

Unlike in the previous section, this is a heterogeneous equation. A solution to this equation can be written in the form: $E_n = w_n + v_n$. We first find a solution to the homogeneous equation (w_n) and a solution to the heterogeneous equation (v_n).

As before, let $w_n = A\lambda^n$, where A is a constant. We have:

$$pw_{n+1} - w_n + qw_{n-1} = 0$$
$$\implies \lambda^2 - \frac{1}{p}\lambda + \frac{q}{p} = 0$$

Giving solutions:

$$\lambda_{1,2} = (\frac{q}{p}, 1)$$

Our solution to the homogeneous equation is then:

$$w_n = A\lambda_1^n + B\lambda_2^n = A(\frac{q}{p})^n + B$$

For a particular solution (v_n) to the heterogeneous equation we try $v_n = Cn$:

$$pC(n+1) - Cn + qC(n-1) = -1$$
$$\implies pC - qC = -1$$

Giving:

$$C = \frac{-1}{p-q}$$

and our particular solution is

$$v_n = \frac{-n}{p-q}$$

Giving the full solution as

$$E_n = w_n + v_n = A(\frac{q}{p})^n + B - \frac{n}{p-q} \tag{4}$$

Using the boundary conditions $E_0 = 0, E_N = 0$

$$E_0 = A + B = 0 \implies B = -A$$

$$E_N = A(\frac{q}{p})^N + B - \frac{N}{p-q} = 0$$
$$\implies A((\frac{q}{p})^N - 1) = \frac{N}{p-q}$$

$$\implies A = \frac{N}{(p-q)((\frac{q}{p})^N - 1)}$$

Our final solution is then (for $p \neq \frac{1}{2}$)

$$E_n = \frac{N}{(p-q)((\frac{q}{p})^N - 1)}(\frac{q}{p})^n - \frac{N}{(p-q)((\frac{q}{p})^N - 1)} - \frac{n}{p-q} \tag{5}$$

Suppose $p = \frac{1}{2}$. We get a repeated solution $\lambda = 1$, meaning we try the next most complicated solution to the homogeneous equation

$$w_n = (An + B)\lambda^n = (An + B)$$

For the particular solution to the equation

$$E_{n+1} - 2E_n + E_{n-1} = -2$$

we try $v_n = Cn^2$ as the next most complicated particular solution. Substituting this into the heterogeneous equation above

$$C(n+1)^2 - 2Cn^2 + C(n-1)^2 = -1$$

giving

$$C = -1$$

The full equation is then

$$E_n = w_n + v_n = An + B - n^2$$

Applying the boundary conditions

$$E_0 = B = 0$$

$$E_N = AN - N^2 = 0 \implies A = N$$

Giving the final equation for E_n

$$E_n = w_n + v_n = Nn - n^2 \tag{6}$$

In summary

$$E_n = \begin{cases} Nn - n^2 & \text{if } p = 0.5 \\ \frac{N}{(p-q)((\frac{q}{p})^N - 1)}(\frac{q}{p})^n - \frac{N}{(p-q)((\frac{q}{p})^N - 1)} - \frac{n}{p-q} & \text{if } p \neq 0.5 \text{ and } p \neq 0 \\ n & \text{if } p = 0 \end{cases} \tag{7}$$

Note: The case where $p = 0$ is obtained by multiplying the numerator and denominator of the first to terms in the equation for the $p \neq 0.5$ case by p^N and simplifying.

4 Simulations

Simulations are carried out using the accompanying Python app utilising Pygame for visualisation.

It is expected that if the win probability is 1 ($p = 1$), the probability of the Gambler going bankrupt should be 0 and the expected number of steps should be the number of steps required to go from the current fortune to $N. This can be seen in figure 4 where each of the 1000 simulations carried out follow the exact same path. If the the win probability is 0 ($p = 0$), then we expect that the probability of the Gambler going bankrupt should be 1 and the expected number of steps should be the number of steps required to go from the current fortune to $0. This can be seen in figure 5 where each of the 1000 simulations carried out follow the exact same path. Two further simulations are carried out with identical parameters but different starting fortunes. Figures 3 and 6 show that a starting fortune closer to $0 increases the probability of bankruptcy.

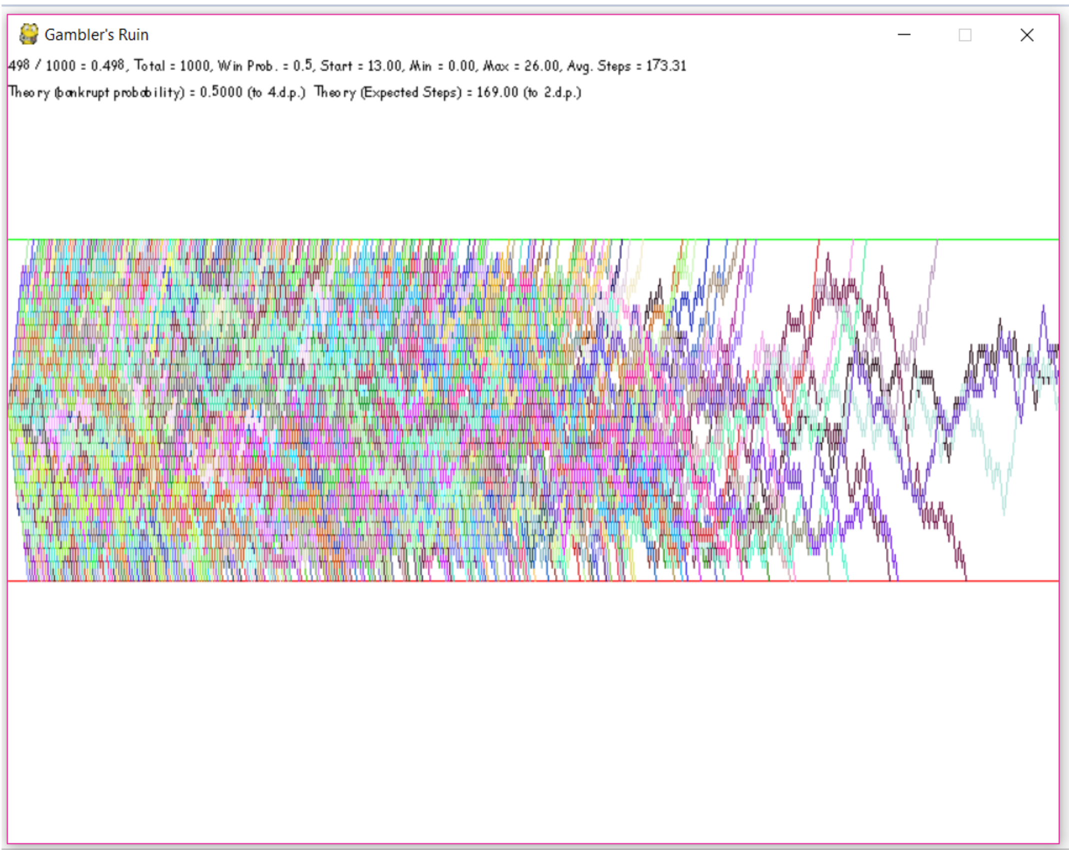

Figure 3: This is a plot of 1000 simulations with $k = 13$, $p = 0.5$, $N = 26$. The theory results in a probability 0.5 of bankruptcy and an expected number of steps of 169. The horizontal green line represents $N and the hosrizontal red line represents $0. We can see that the simulated probability of 0.498 and the simulated average number of steps of 173 is close to the theoretical values.

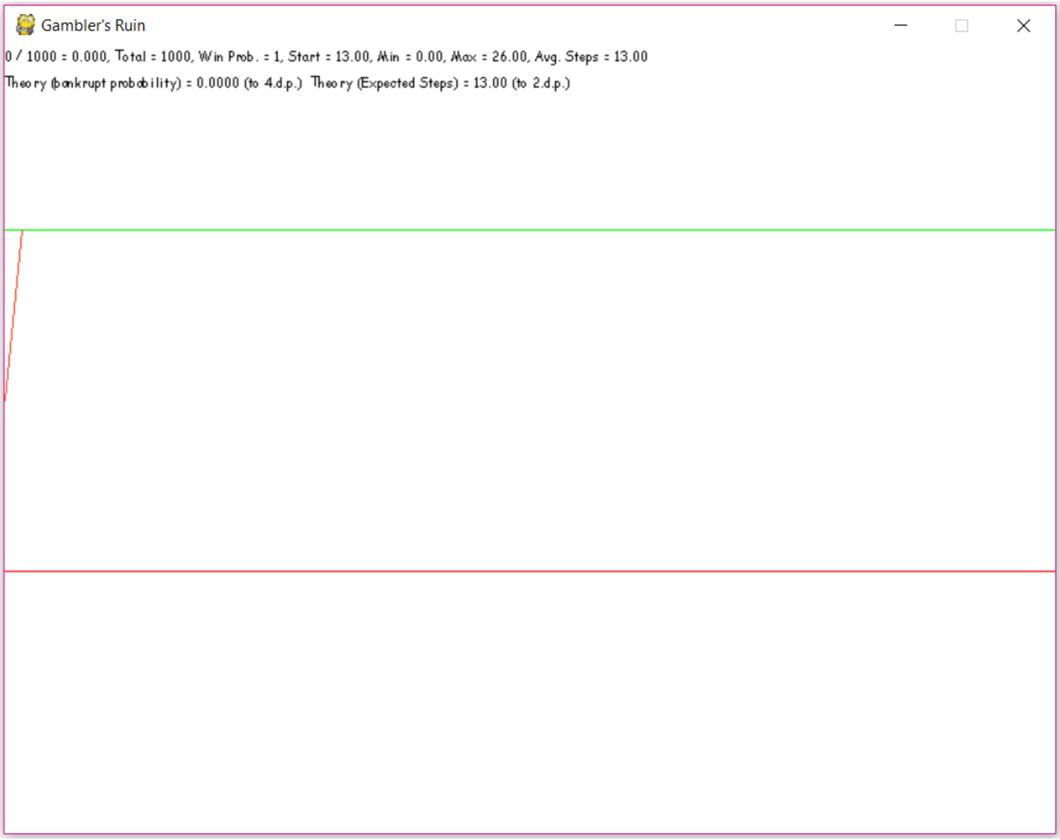

Figure 4: This is a plot of 1000 simulations with $k = 13$, $p = 1$, $N = 26$. The theory results in a probability 0 of bankruptcy and an expected number of steps of 13. The horizontal green line represents $N and the hosrizontal red line represents $0. We can see that the simulated probability of 0 and the simulated average number of steps of 13 exactly match the theoretical values.

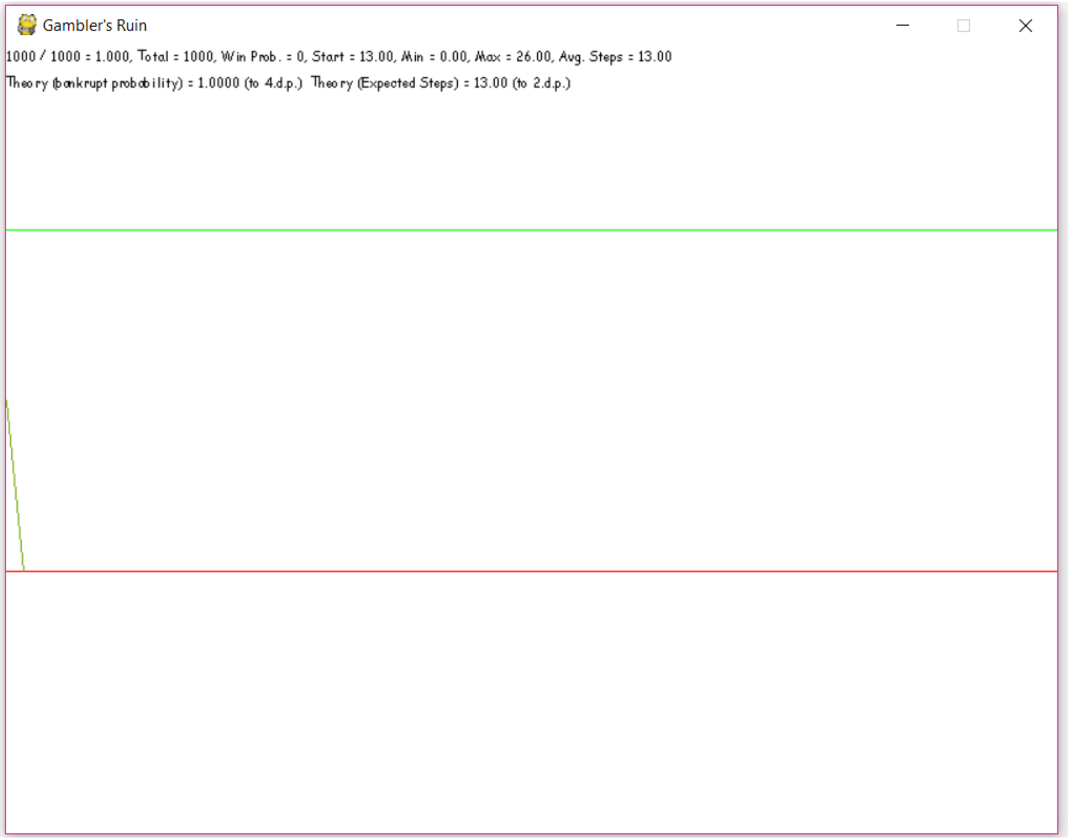

Figure 5: This is a plot of 1000 simulations with $k = 13$, $p = 0$, $N = 26$. The theory results in a probability 1 of bankruptcy and an expected number of steps of 13. The horizontal green line represents \$N and the hosrizontal red line represents \$0. We can see that the simulated probability of 1 and the simulated average number of steps of 13 exactly match the theoretical values.

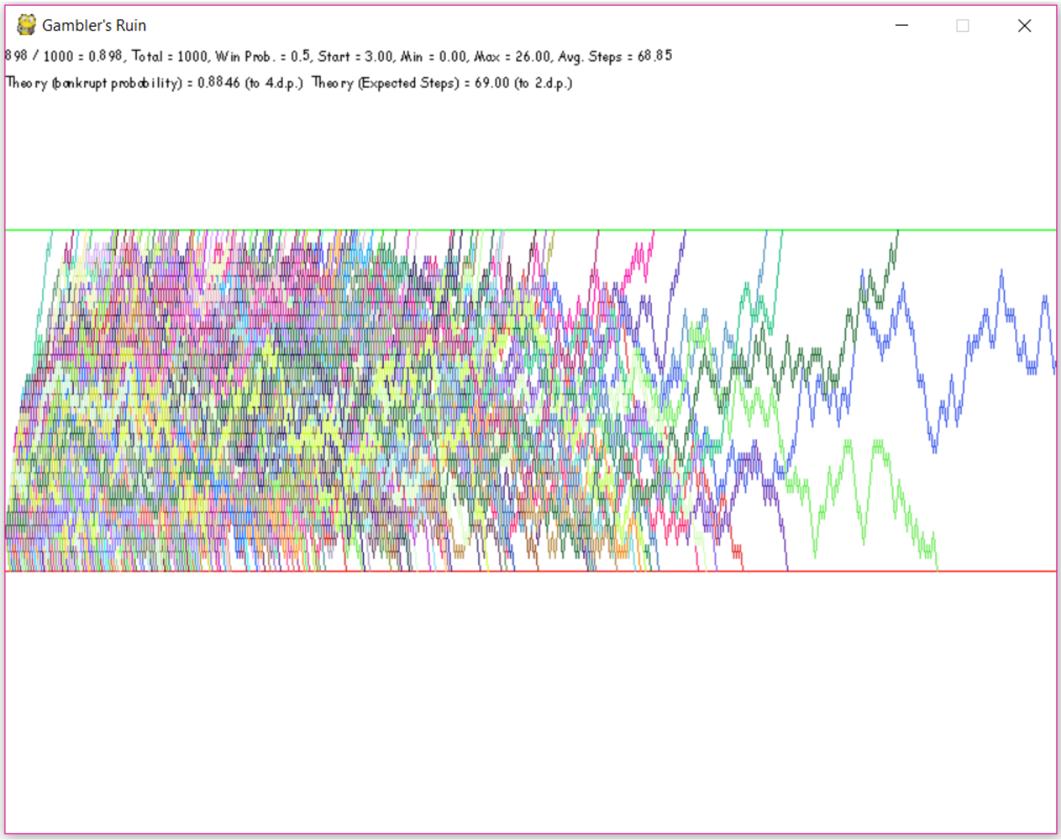

Figure 6: This is a plot of 1000 simulations with $k = 3$, $p = 0.5$, $N = 26$. The theory results in a probability 0.8846 of bankruptcy and an expected number of steps of 69. The horizontal green line represents \$N and the hosrizontal red line represents \$0. We can see that the simulated probability of 0.898 and the simulated average number of steps of 68.85 match the theoretical values closely.