Pre-University Mathematics

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Contents

Introduction

This paper consists of many different pre-university topics primarily from Cambridge A-Level H2 and H3 mathematics such as Fourier, Maclaurin Series, Conics, Equation of Planes etc. It covers the derivation and applications of the different topics. Some examples are also given.

1 Sequences and Series

Consider the following infinite series of elements:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^i}, \ldots$$

Can we assign this infinite sum to a numerical value? Indeed, we can. We must first understand what are **series** and **sequences**.

1.1 Sequence

A sequence is any number of elements arranged in a specific order.

Example 1. An infinite sequence of ascending odd numbers.

$$1, 3, 5, 7, \ldots$$

Sequences can be both infinite and finite. An example of a finite sequence is:

$$2, 4, 8, 16, \ldots, n$$

where n is the final element in the sequence.

More formally, the algebraic notation for a sequence is expressed as:

$$a_1, a_2, a_3, \ldots, a_n$$

where a_1 is the first term, a_2 is the second term, and a_n is the n-th term.

1.2 Series

A series is the total sum of all elements in a sequence. In the first section, we posed the question of obtaining a numerical value from an infinite sum. What we are really asking is: *How do we evaluate a series?*

With the initial sequence in **Section 1**, let's first change it to a finite sequence. We can rewrite it into this series:

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2^n}$$

We can also write the above in summation notation:

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k}$$

This describes the sum of elements of $\frac{1}{2^n}$ where $k = 1$ to $k = n$.

Suppose we want to find the sum of 10 elements in the series. We can write it as:

$$\begin{aligned} S_{10} &= \sum_{k=1}^{10} \frac{1}{2^k} \\ &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}} \\ &= 0.99902 \end{aligned}$$

1.3 Arithmetic Progression

The first type of sequence is known as an **arithmetic progression**.

Example 2. Consider the following sequence:

$$2, 5, 8, 11, \ldots$$

To obtain the above sequence, we start with the first term and add a fixed value to each term successively.

$$2, 2 + 3 = 5, 5 + 3 = 8, 8 + 3 = 11, \ldots$$

This fixed value is called the **common difference**. In the above example, the common difference would be 3.

More formally, an **arithmetic progression** or **AP** is a sequence whereby the difference between the preceding and succeeding terms is common. In algebraic notation, it can be written as:

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d$$

where a is the first term, d is the common difference, and n is the number of terms in the sequence.

From the above, we can observe that the n-th term is:

$$a_n = a + (n-1)d$$

1.3.1 Sum of AP Series

An AP series is simply the total sum of terms in a given arithmetic progression. For a general arithmetic progression:

$$a, (a+d), (a+2d), \ldots, (\ell-2d), (\ell-d), \ell$$

where a is the first term, d is the common difference and ℓ is the n-th term.

The AP series is written as:

$$S_n = a + (a+d) + (a+2d) + \cdots + (\ell-2d) + (\ell-d) + \ell$$

To derive a general formula for the AP series, let's first reverse the order of S_n:

$$S_n = \ell + (\ell-d) + (\ell-2d) + \cdots + (a+2d) + (a+d) + a$$

Then, let's add the original AP series to the reverse AP series by vertically summing each term.

$$2S_n = (a+\ell) + (a+\ell) + (a+\ell) + \cdots + (a+\ell) + (a+\ell) + (a+\ell)$$

Notice how all the terms have become $(a+\ell)$. There are now n number of $(a+\ell)$ terms. Now, we can make S_n the subject:

$$S_n = \frac{1}{2}n(a+\ell)$$

As seen in the previous section, the $n-th$ term, ℓ, can also be written as $a+(n-1)d$.
Derivation. Hence, the general formula for the sum of any AP series is:

$$S_n = \frac{1}{2}n(2a+(n-1)d)$$

All AP series with common difference, $d \neq 0$ are **divergent**. For any AP series,
If $d > 0$, then the sum of any AP series diverges to $+\infty$ as $n \to \infty$.
If $d < 0$, then the sum of any AP series diverges to $-\infty$ as $n \to \infty$.
If $d = 0$ and $a \neq 0$, then the sum of any AP series diverges to $+-\infty$ as $n \to$ depending on the sign of a.
If $d = 0$ and $a = 0$, then the sum of any AP series is 0.

1.4 Geometric Progression

The second type of sequence is known as a **geometric progression**.

Example 3. Consider the following sequence:

$$2, 6, 18, 54, \dots$$

To obtain the above sequence, we start with the first term and multiply it by a fixed value to each term successively.

$$2, 2 \times 3 = 6, 6 \times 3 = 18, 18 \times 3 = 54, \dots$$

This fixed value is called the **common ratio**. In the above example, the common ratio would be 3.

More formally, a **geometric progression** or **GP** is a sequence whereby each term is produced by multiplying each preceding term by a constant value. In algebraic notation, it can be written as:

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}$$

where a is the first term, r is the common ratio, and n is the number of terms in the sequence.

From the above, we can observe that the n-th term is:

$$a_n = ar^{n-1}$$

1.4.1 Sum of GP Series

Similar to a series, a GP Series is simply the total sum of terms in a given geometric progression. For a general geometric progression, its GP series can be written as:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

We can evaluate the series by multiplying both sides by r and then subtracting:

$$\begin{aligned} rS_n &= ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\ S_n - rS_n &= a - ar^n \\ S_n(1-r) &= a - ar^n \end{aligned}$$

Now, we can make S_n the subject.
Derivation. Hence, the general formula for the sum of any GP series is:

$$S_n = \frac{a - ar^n}{1-r} \quad \text{for } r \neq 1$$

A GP series can either be **convergent** or **divergent**.

1.5 p-Series

The **p-Series** is a type of series in the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \ldots$$

where p is an exponent and $p \geq 0$.

Interestingly, when $p = 1$, it forms a special type of series called a **Harmonic Series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots$$

In general, a p-series converges if $p > 1$ as the partial sum grows smaller fast enough. Conversely, it diverges if $p \leq 1$.

1.6 Convergence and Divergence of Series

A **convergent** series is any series where the sum converges to a numerical value. For a given series

$$\sum_{n=1}^{\infty} a_n$$

we say it is convergent if the series converges to the sum, $S \in \mathbb{R}$, if

$$\lim_{n\to\infty} S_n = S$$

where the partial sum

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Otherwise, the series is **divergent**.

Distributive and associative arithmetic laws still apply for any series.

1.6.1 Absolute Convergence

The series

$$\sum_{n=1}^{\infty} a_n$$

converges **absolutely** if

$$\sum_{n=1}^{\infty} |a_n| \text{ converges}$$

Otherwise, it converges **conditionally** if

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty} |a_n| \text{ diverges}$$

In the next few sections, we will explore how we can test if a given series is convergent or divergent.

1.6.2 Divergence Test

The divergence test states that a series is divergent if the limit of the term $a_n \neq 0$ or does not exist.

Definition. The series diverges if

$$\lim_{n\to\infty} a_n \neq 0 \text{ or}$$
$$\lim_{n\to\infty} a_n \text{ does not exist}$$

If the limit is 0, more tests would need to be carried out to determine if the series is truly convergent.

Example 4.1 Determine if the following series $\sum_{n=1}^{\infty} \frac{n-1}{n}$ is convergent or divergent.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n-1}{n} \\
=& \lim_{n\to\infty} (1 - \frac{1}{n}) \\
=& \lim_{n\to\infty} (1 - 0) \\
=& 1
\end{aligned}$$

$\therefore$ Since the limit $\neq 0$, the series is divergent.

1.6.3 Geometric Series Test

The geometric series test applies to any geometric series in the form $\sum_{n=1}^{\infty} ar^{n-1}$.

Definition. The series converges or diverges based on the following

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{for } |r| < 1 \\ \text{diverges} & \text{for } |r| \geq 1 \end{cases}$$

Example 4.2 Determine if the following series $\sum_{n=1}^{\infty} \frac{5^{n-1}}{3^{n+1}}$ is convergent or divergent. If it converges, find the value it converges to.

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^{n+1}}$$
$$= \sum_{n=1}^{\infty} \frac{1}{25} \left(\frac{3}{5}\right)^{n-1}$$
$$= \frac{\frac{1}{25}}{1 - \frac{3}{5}}$$
$$= \frac{1}{10}$$

1.6.4 Direct Comparison Test

The **direct comparison test** or **comparison test** involves the comparison of 2 series.

Definition. For a given series $\sum_{n=1}^{\infty} a_n$ where its convergence is unknown, and a series $\sum_{n=1}^{\infty} b_n$ where its convergence is known

$$0 \leq a_n \leq b_n \text{ for all } n$$
$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$
$$\sum_{n=1}^{\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

The rules above are not associative.

Example 4.3. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$.

$$a_n = \frac{n^2-1}{3n^4+1}$$
$$\text{Let } b_n = \frac{n^2}{n^4} = \frac{1}{n^2}, \text{ such that } a_n < b_n \text{ for all } n$$
$$\text{Since, } p = 2 > 1, b_n \text{ converges}$$
$$\therefore \text{By the comparison test, the series converges.}$$

The direct comparison test is mainly used for series similar to geometric or p-Series. It is more challenging to find an appropriate series for this test.

1.6.5 Limit Comparison Test

The **limit comparison test** states that if the limit of the ratio of 2 sequences is 0 or ∞, then the denominator or numerator grew much faster respectively. If the limit is a finite number, then both sequences and by extension, both series behave similarly as $n \to \infty$.

Definition. For the positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \text{ and}$$

$$L > 0, L \neq \infty$$

Then, both series are either convergent, or divergent.

Example 4.4. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+n}}{n^4-n^2}$.

$$a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+n}}{n^4-n^2}$$

$$\text{Let } b_n = \frac{\sqrt{n^3}}{n^4} = \frac{1}{n^{\frac{5}{2}}}$$

$$\text{Since, } p = \frac{5}{2} > 1, b_n \text{ converges}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+n}}{n^4-n^2}}{\frac{\sqrt{n^3}}{n^4}} &= \lim_{n\to\infty} \frac{\sqrt{n^3+n}}{n^4-n^2} \cdot \frac{n^4}{\sqrt{n^3}} \\
&= \lim_{n\to\infty} \sqrt{\frac{n^3+n}{n^3}} \cdot \frac{n^2}{n^2-1} \\
&= \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} \cdot \frac{n^2}{n^2-1} \\
&= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}}{1-\frac{1}{n^2}} \\
&= \frac{\sqrt{1}}{1} \\
&= 1
\end{aligned}$$

$\therefore$ By the limit comparison test, the series converges.

The limit comparison test applies to a wider range of series. For geometric and p-Series, it is often easier to apply this test than the direct comparison test.

1.6.6 Integral Test

The **integral test** states that

Definition. For a series $\sum_{n=1}^{\infty} a_n$, $f(x)$ is a continuous, positive, and decreasing function for the domain $[1, \infty)$, and $f(n) = a_n$ then,

$$\int_1^{\infty} f(x)\, dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$
$$\int_1^{\infty} f(x)\, dx \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Example 4.5. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$.

$$\begin{aligned}
\text{Let } f(x) &= \frac{e^x}{1+e^{2x}} \\
\int \frac{e^x}{1+e^{2x}}\, dx & \\
\text{Let } u &= e^x, \\
du &= e^x\, dx \\
\int \frac{1}{1+u^2}\, du &= \arctan(u) + c \\
&= \arctan(e^x) + c \\
\int_1^{\infty} \frac{e^x}{1+e^{2x}}\, dx &= [\arctan(e^x)]_1^{\infty} \\
&= \arctan(\infty) - \arctan(e) \\
&= \frac{\pi}{2} - \arctan(e)
\end{aligned}$$

$\therefore$ Since $f(x)$ is positive, continuous, and decreasing, by the integral test, the series converges.

1.6.7 Alternating Series Test

The **alternating series test** only applies to alternating series.

Definition. For a series with the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, it converges if

$$b_n \text{ is positive, decreasing, and}$$
$$\lim_{n\to\infty} b_n = 0$$

Otherwise, it diverges.

Example 4.6. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

$$\begin{aligned} b_n &= \frac{1}{\sqrt{n}} \\ \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \\ &= 0 \end{aligned}$$

$\therefore$ Since b_n is positive and decreasing, by the alternating series test, the series converges.

1.6.8 Ratio Test

The **ratio test** determines the convergence of a series if the limit of the ratio of consecutive terms in the series shrinks fast enough.

Definition. For a given series $\sum_{n=1}^{\infty} a_n$,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$$
$$L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges absolutely}$$
$$L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$
$$L = 1 \Rightarrow \text{inconclusive}$$

Example 4.7. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

$$a_n = \frac{3^n}{n!}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \\
&= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} 3 \cdot \frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1}{(n+1) \cdot n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1} \\
&= \lim_{n\to\infty} \frac{3}{n+1} \\
&= 0
\end{aligned}$$

$\therefore$ By the ratio test, the series is absolutely convergent.

Generally, the ratio test is most useful when dealing with series with powers or factorial.

1.6.9 Root Test

The **root test** determines the convergence of a series if the root of the sequence which forms a ratio, shrinks fast enough.

Definition. For a given series $\sum_{n=1}^{\infty} a_n$,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L$$

$$\begin{aligned}
L < 1 &\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges absolutely} \\
L > 1 &\Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges} \\
L = 1 &\Rightarrow \text{inconclusive}
\end{aligned}$$

Example 4.8. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{2n}\right)^n$.

$$
\begin{aligned}
a_n &= \frac{3n+1}{2n}^n \\
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+1}{2n}\right)^n} \\
&= \lim_{n\to\infty} \frac{3n+1}{2n} \\
&= \frac{3}{2}
\end{aligned}
$$

$\therefore$ By the root test, the series is divergent.

Generally, the root test is most useful when dealing with series with powers of n.