

Recursion in Python

- Concept of Recursion
- Recursion Practices
- Divide and Conquer

Definition of Recursion Functions

- A recursive function is one that calls itself.

```
def i_am_recursive(x) :  
    maybe do some work  
    if there is more work to do :  
        i_am_recursive(next(x))  
    return the desired result
```

- Infinite loop? Not necessarily, not if `next(x)` needs less work than `x`.

Recursive Definition [1/2]

- A description of something that refers to itself is called a *recursive definition*

$$n! = n(n-1)(n-2)\dots(1)$$

$$n! = n(n-1)!$$



base case

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)! & \text{otherwise} \end{cases}$$

- A recursive definitions should have two key characteristics:
 - There are one or more base cases for which no recursion is applied.
 - All chains of recursion eventually end up at one of the base cases.

Recursive Definition [1/2]

- Every recursive function definition includes two parts:
 - Base case(s) (non-recursive)
One or more simple cases that can be done right away
 - Recursive case(s)
One or more cases that require solving “simpler” version(s) of the original problem.
 - By “simpler”, we mean “smaller” or “shorter” or “closer to the base case”.

Recursive Computation Example: Factorial

- $n! = n \times (n-1) \times (n-2) \times \dots \times 1$

$$2! = 2 \times 1$$

$$3! = 3 \times 2 \times 1$$

$$4! = 4 \times 3 \times 2 \times 1$$

- alternatively:

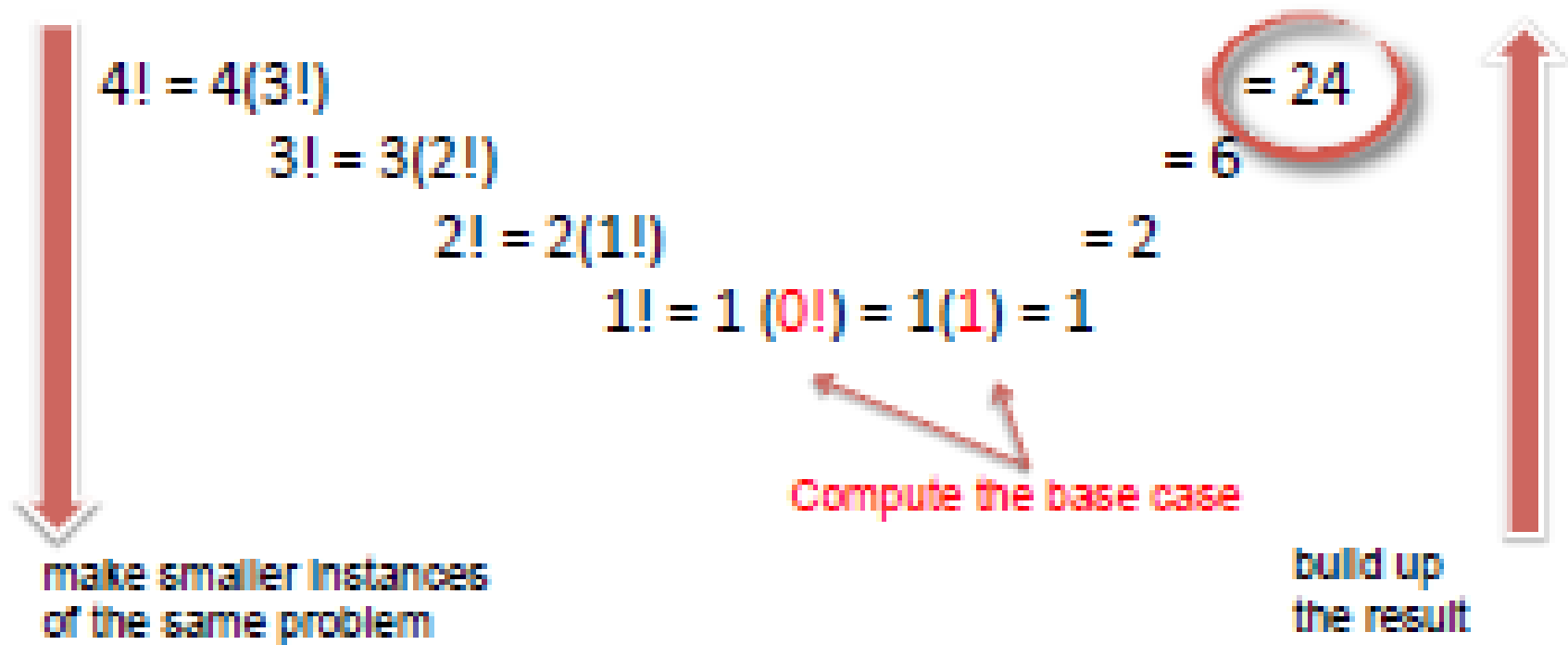
$$0! = 1 \text{ (Base case)}$$

$$n! = n \times (n-1)! \text{ (Recursive case)}$$

$$\text{So } 4! = 4 \times 3!$$

$$\text{And } 3! = 3 \times 2!, 2! = 2 \times 1!, 1! = 1 \times 0!$$

Conceptual Understanding of Recursion



Recursive Factorial Function in Python

```
# 0! = 1 (Base case)  
# n! = n * (n-1)! (Recursive case)  
def factorial(n):  
    if n == 0:      # base case  
        return 1  
    else:           # recursive case  
        return n * factorial(n-1)
```

Inside Python Recursion Processing

S

n-4 `factorial(4)? = 4 * factorial(3)`

T

n-3 `factorial(3)? = 3 * factorial(2)`

A

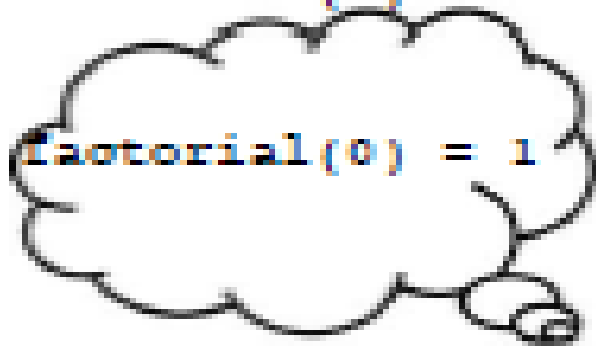
n-2 `factorial(2)? = 2 * factorial(1)`

C

n-1 `factorial(1)? = 1 * factorial(0)`

K

n-0 `factorial(0) = 1`



Recursive Solution vs. Iterative Solution

- For every recursive function, there is an equivalent iterative solution.
- For every iterative function, there is an equivalent recursive solution.
- But some problems are easier to solve one way than the other way.
- And be aware that most recursive programs need space for the stack, behind the scenes

Recursion in Python

- Concept of Recursion
- Recursion Practices
- Divide and Conquer

Factorial Function in Python (Iterative)

```
def factorial(n):  
    result = 1    # initialize accumulator var  
    for i in range(1, n+1):  
        result = result * i  
    return result
```

Versus (Recursive):

```
def factorial(n):  
    if n == 0:    # base case  
        return 1  
    else:         # recursive case  
        return n * factorial(n-1)
```

π Computation in Python [1/4]

Many Many Approximations

Bailey–Borwein–Plouffe formula

$$\pi = \sum_{i=0}^{\infty} \left[\frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \right].$$

Bellard's formula

$$\pi = \frac{1}{2^6} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \left(-\frac{2^5}{4n+1} - \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{2^2}{10n+7} + \frac{1}{10n+9} \right)$$

and

Chudnovsky algorithm

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}.$$

π Computation in Python [2/4]

The Algebraic Genius of Euler (1707, Switzerland)

- The Basel Problem

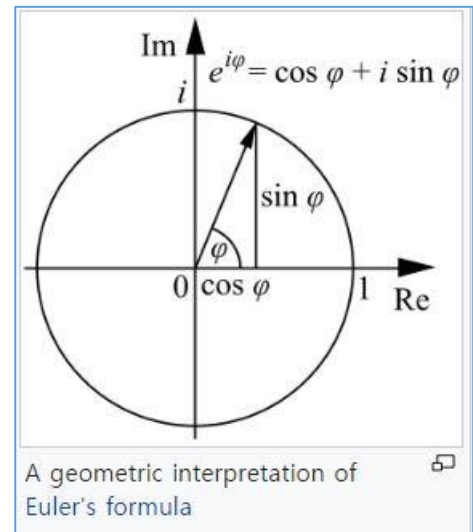
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right) = \frac{\pi^2}{6}$$



- Euler's formula

He also defined the exponential function for complex numbers, and discovered its relation to the trigonometric functions. For any real number φ (taken to be radians), Euler's formula states that the complex exponential function satisfies

$$e^{i\varphi} = \cos \varphi + i \sin \varphi.$$



π Computation in Python [3/4]

Iterative Version of π Computation

- Mathematicians have proved
$$\pi^2/6 = 1 + 1/4 + 1/9 + 1/16 + \dots$$
- We can use this to approximate π
- Compute the sum, multiply by 6, take the square root

```
def pi_series_iter(n) :  
    result = 0  
    for i in range(1, n+1) :  
        result = result + 1/(i**2)  
    return result  
  
def pi_approx_iter(n) :  
    x = pi_series_iter(n)  
    return (6*x)**(.5)
```

π Computation in Python [4/4]

Recursive Version of π Computation

```
def pi_series_r(i) :  
    assert(i >= 0)  
    # base case  
    if i == 0:  
        return 0  
    # recursive case  
    else:  
        return pi_series_r(i-1) + 1 / i**2  
  
def pi_approx_r(n) :  
    x = pi_series_r(n)  
    return (6*x)**(.5)
```

```
def test_pi_approx() :  
    assert(pi_approx_iter(10) == 3.04936163598207)  
    assert(pi_approx_iter(100) == 3.1320765318091053)  
    assert(pi_approx_iter(1000) == 3.1406380562059946)  
    assert(pi_approx_iter(10000) == 3.1414971639472147)  
    # Python's default stack depth limit is 1000, so we can't compute pi_approx_r(1000)  
    for i in range(996) :  
        assert(pi_approx_r(i) == pi_approx_iter(i))  
    print("Done testing pi approximations")
```

Recursion on Lists: Sum of a List [1/2]

- First we need a way of getting a smaller input from a larger one:
 - Forming a sub-list of a list:

```
>>> a = [1, 11, 111, 1111, 11111, 111111]
>>> a[1:] ← the "tail" of list a
[11, 111, 1111, 11111, 111111]
>>> a[2:]
[111, 1111, 11111, 111111]
>>> a[3:]
[1111, 11111, 111111]
>>> a[3:5]
[1111, 11111]
>>>
```

Recursive Sum of a List

```
def sumlist(items):
    if items == []:
        return 0
    else:
        return items[0] + sumlist(items[1:])
```

What if we already know the sum of the list's tail? We can just add the list's first element!

Recursion on Lists: Sum of a List [2/2]

Tracing sumlist

```
def sumlist(items):  
    if items == []:  
        return 0  
    else:  
        return items[0] + sumlist(items[1:])
```

```
>>> sumlist([2,5,7])
```

```
sumlist([2,5,7]) = 2 + sumlist([5,7])
```

```
                    5 + sumlist([7])
```

```
                        7 + sumlist([])
```

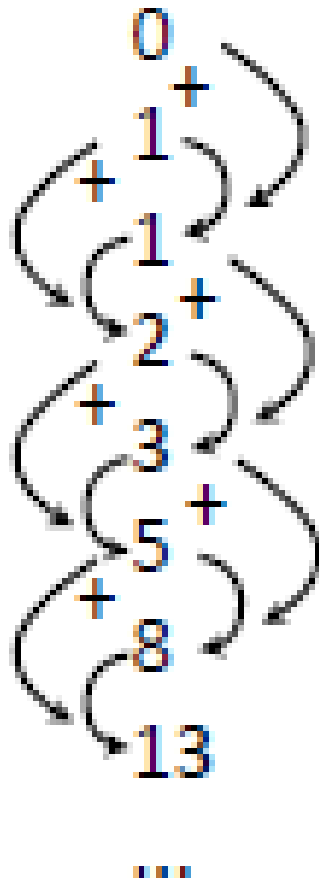
```
                            0
```

After reaching the base case, the final result is built up by the computer by adding 0+7+5+2.

Multiple Recursive Calls: Fibonacci Numbers

$$\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2), \quad n > 1$$

- A sequence of numbers:

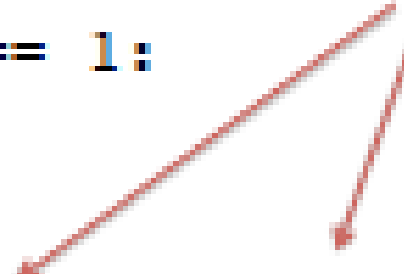


Recursive Definition of Fibonacci Numbers

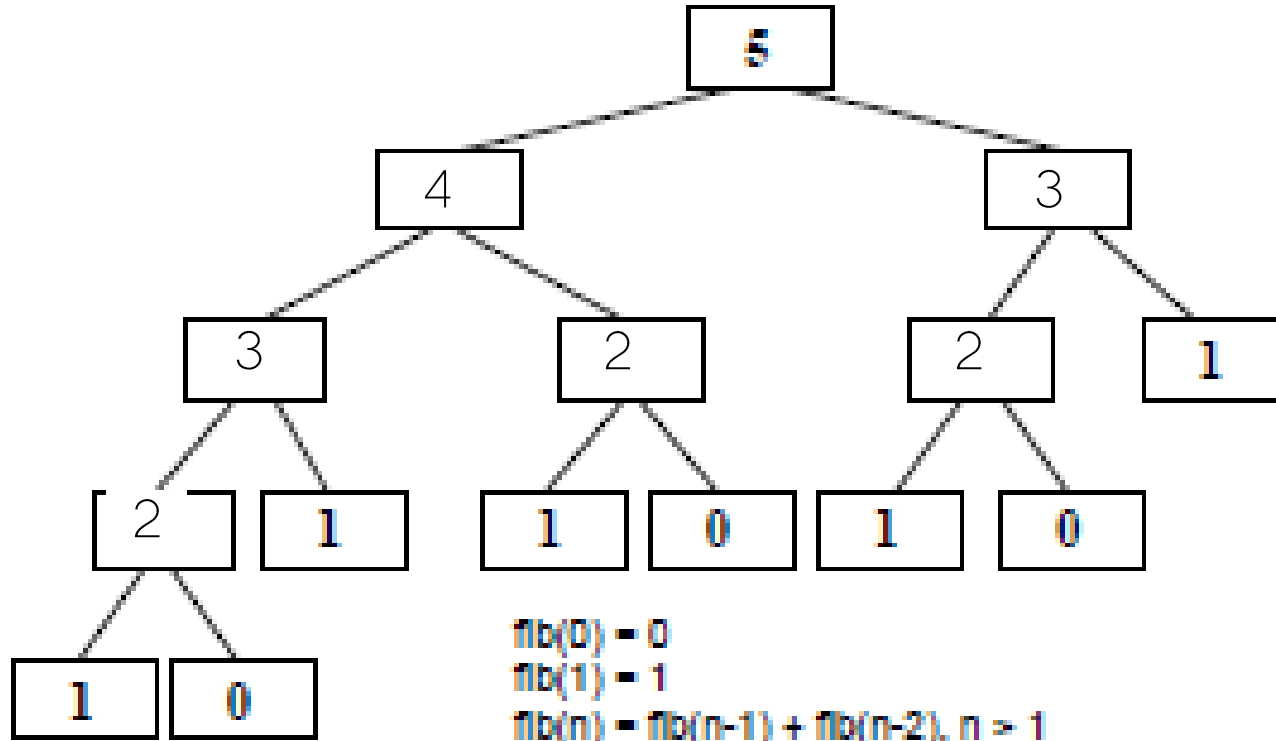
- Let $\text{fib}(n)$ = the n th Fibonacci number, $n \geq 0$
 - $\text{fib}(0) = 0$ (base case)
 - $\text{fib}(1) = 1$ (base case)
 - $\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2)$, $n > 1$

```
def fib(n):  
    if n == 0 or n == 1:  
        return n  
    else:  
        return fib(n-1) + fib(n-2)
```

Two recursive calls!



Recursive Call Tree of Fibonacci Number



Iterative Fibonacci Python Function

```
def fib(n):  
    x = 0  
    next_x = 1  
    for i in range(1, n+1):  
        x, next_x = next_x, x + next_x  
    return x
```

SIMULTANEOUS
ASSIGNMENT



Faster than the
recursive
version. Why?

Recursion on String: String Reversal [1]

- Write a function to reverse a given string
 - Divide it up into a first character and “all the rest”
 - Reverse the “rest” and append the first character to the end

```
>>> def reverse(s):  
    return reverse(s[1:]) + s[0]
```

```
>>> reverse("Hello")
```

```
Traceback (most recent call last):
```

```
File "<pyshell#6>", line 1, in -toplevel-  
    reverse("Hello")
```

```
File "C:/Program Files/Python 2.3.3/z.py", line 8, in reverse  
    return reverse(s[1:]) + s[0]
```

```
File "C:/Program Files/Python 2.3.3/z.py", line 8, in reverse  
    return reverse(s[1:]) + s[0]
```

```
...
```

```
File "C:/Program Files/Python 2.3.3/z.py", line 8, in reverse  
    return reverse(s[1:]) + s[0]
```

```
RuntimeError: maximum recursion depth exceeded
```

- What happened? There were 1000 lines of errors!

Recursion on String: String Reversal [2]

```
def reverse(s):  
    if s == "":  
        return s  
    else:  
        return reverse(s[1:]) + s[0]
```

```
>>> reverse("Hello")  
'olleH'
```

- Python stops it at 1000 calls, the default “maximum recursion depth.”
 - Each time a function is called it takes some memory.

Recursion on Greatest Common Denominator (GCD)

```
def gcd(a, b):  
    """Calculate the Greatest Common Divisor of a and b.  
  
    Unless b==0, the result will have the same sign as b (so that when  
    b is divided by it, the result comes out positive).  
    """  
    while b:  
        a, b = b, a%b  
    return a
```

```
def gcd(x,y):  
    while (y > 0):  
        oldX = x  
        x = y  
        y = oldX % y  
    return x  
  
print gcd(500, 420) # 20
```


Iterative Solution

Recursive Solution

Recursive Solution with Stack Trace

factorial	<pre>def factorial(n): factorial = 1 for i in range(2,n+1): factorial *= i return factorial print factorial(5)</pre>	<pre>def factorial(n): if (n < 2): return 1 else: return n*factorial(n-1) print factorial(5)</pre>	<pre>def factorial(n, depth=0): print " *depth, \"factorial(\"n, \"):" if (n < 2): result = 1 else: result = n*factorial(n-1,depth+1) print " *depth, \"-->\", result return result print factorial(5)</pre>
reverse	<pre>def reverse(s): reverse = "" for ch in s: reverse = ch + reverse return reverse print reverse("abcd")</pre>	<pre>def reverse(s): if (s == ""): return "" else: return reverse(s[1:]) + s[0] print reverse("abcd")</pre>	<pre>def reverse(s, depth=0): print " *depth, \"reverse(\"s, \"):" if (s == ""): result = "" else: result = reverse(s[1:], depth+1) + s[0] print " *depth, \"-->\", result return result print reverse("abcd")</pre>
gcd	<pre>def gcd(x,y): while (y > 0): oldX = x x = y y = oldX % y return x print gcd(500, 420) # 20</pre>	<pre>def gcd(x,y): if (y == 0): return x else: return gcd(y,x%y) print gcd(500, 420) # 20</pre>	<pre>def gcd(x,y,depth=0): print " *depth, \"gcd(\"x, \",\", y, \"):" if (y == 0): result = x else: result = gcd(y,x%y,depth+1) print " *depth, \"-->\", result return result print gcd(500, 420) # 20</pre>

Recursion in Python

- Concept of Recursion
- Recursion Practices
- Divide and Conquer

Family of Algorithms

- Greedy Methods
- **Divide and Conquer**
- Dynamic Programming
- Branch and Bound
- Back Tracking

전통적인 Computer Science Algorithms

- Machine Learning Algorithm
- Genetic Algorithm
- Randomized Algorithm

Approximation과 Prediction을 하는 Algorithms

- Mathematical Programming
 - Integer Programming
 - Linear Programming
 - Non-Linear Programming
 - Unconstrained Extrema
 - Constrained Extrema

Applied Mathematics or Industrial Engineering에서 하는 Algorithms

Recursion Example: Fast Exponentiation [1]

- One way to compute a^n : multiply a by itself n times.

```
def loopPower(a, n):  
    ans = 1  
    for i in range(n):  
        ans = ans * a  
    return ans
```

- Another way to compute a^n : divide and conquer!
 - $a^n = a^{n//2}(a^{n//2})$?

$$a^n = \begin{cases} a^{n//2}(a^{n//2}) & \text{if } n \text{ is even} \\ a^{n//2}(a^{n//2})(a) & \text{if } n \text{ is odd} \end{cases}$$

- $2^8 = 2^4(2^4)$
- $2^9 = 2^4(2^4)2$

Recursion Example: Fast Exponentiation [2]

```
def recursivePower(a, n) :  
    # raises a to the int power n  
    if n == 0:  
        return 1  
    else:  
        factor = recursivePower(a, n//2)  
        if n%2 == 0:                                # n is even  
            return factor * factor  
        else:                                       # n is odd  
            return factor * factor * a
```

- temporary variable *factor* is used so that we don't need to calculate $a^{n/2}$ more than once

Sorting Algorithms

- The sorting problem
 - take a list of n elements
 - and rearrange it so that the values are in increasing (or decreasing) order.
- *Selection sort*
 - For n elements, we find the smallest value and put it in the 0^{th} position.
 - Then we find the smallest remaining value from position 1 to $(n-1)$ and put it into position 1.
 - The smallest value from position 2 to $(n-1)$ goes in position 2.
 - ...

Naive Sorting: Selection Sort

```
def selSort(nums): # sort nums into ascending order

    n = len(nums)
    # For each position in the list (except the very last)
    for bottom in range(n-1):
        # find the smallest item in nums[bottom]...nums[n-1]

        mp = bottom                # bottom is smallest initially
        for i in range(bottom+1, n): # look at each position
            if nums[i] < nums[mp]:    # this one is smaller
                mp = i               # remember its index

        # swap smallest item to the bottom
        nums[bottom], nums[mp] = nums[mp], nums[bottom]
```

6	4	1	2	10	11	12	7	4
0	1	2	3	4	5	6	7	8

1	4	6	2	10	11	12	7	4
0	1	2	3	4	5	6	7	8

bottom

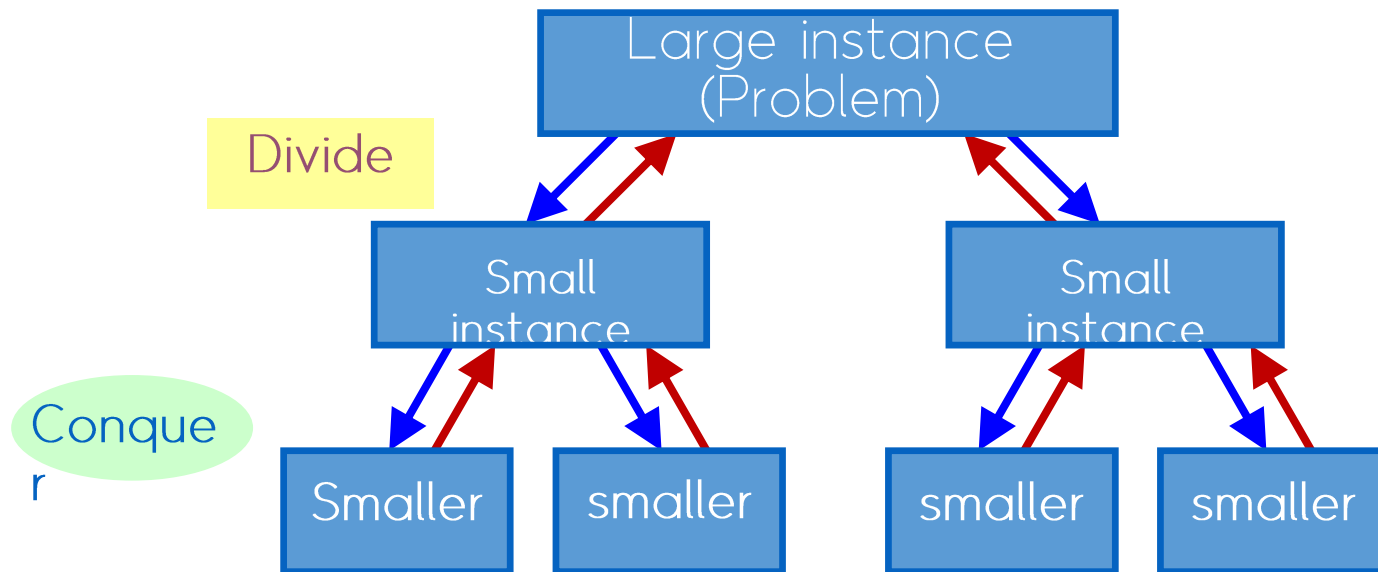
Divide and Conquer

- In computation:
 - **Divide** the problem into “simpler” versions of itself.
 - **Conquer** each problem using the same process (usually recursively).
 - **Combine** the results of the “simpler” versions to form your final solution.
- Examples: Towers of Hanoi, fractals, Binary Search, Merge Sort, Quicksort, and many, many more

Divide and Conquer style programming은 recursion이 자연스럽다!

Divide and Conquer Style Algorithm

- Distinguish between small and large instances
- Small instances solved differently from large ones
- All instances are **non-overlapping**



Divide and Conquer Example: Merge Sort

- *merge sort*

- Merging: combining two sorted lists into a single sorted list

split nums into two halves

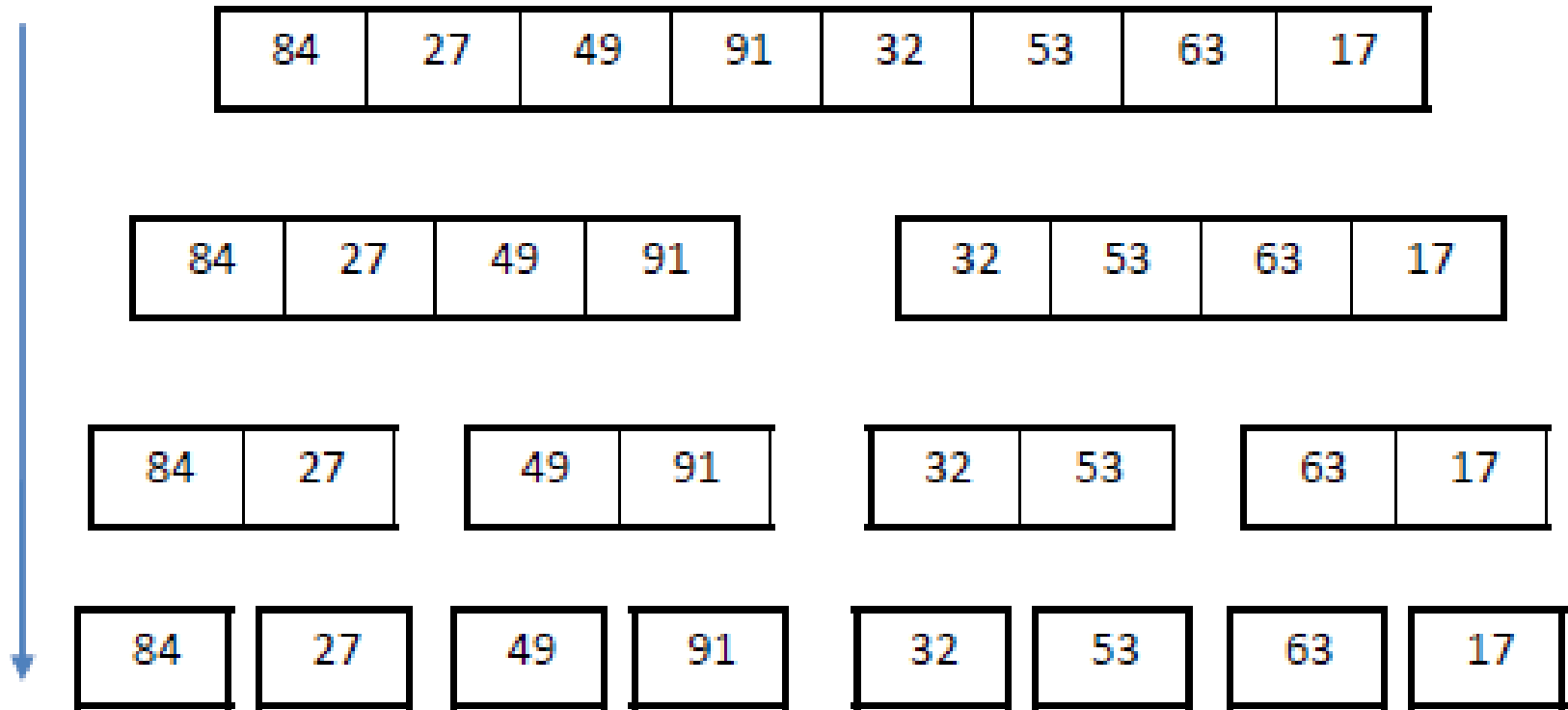
sort the first half

sort the second half

merge the two sorted halves back into nums

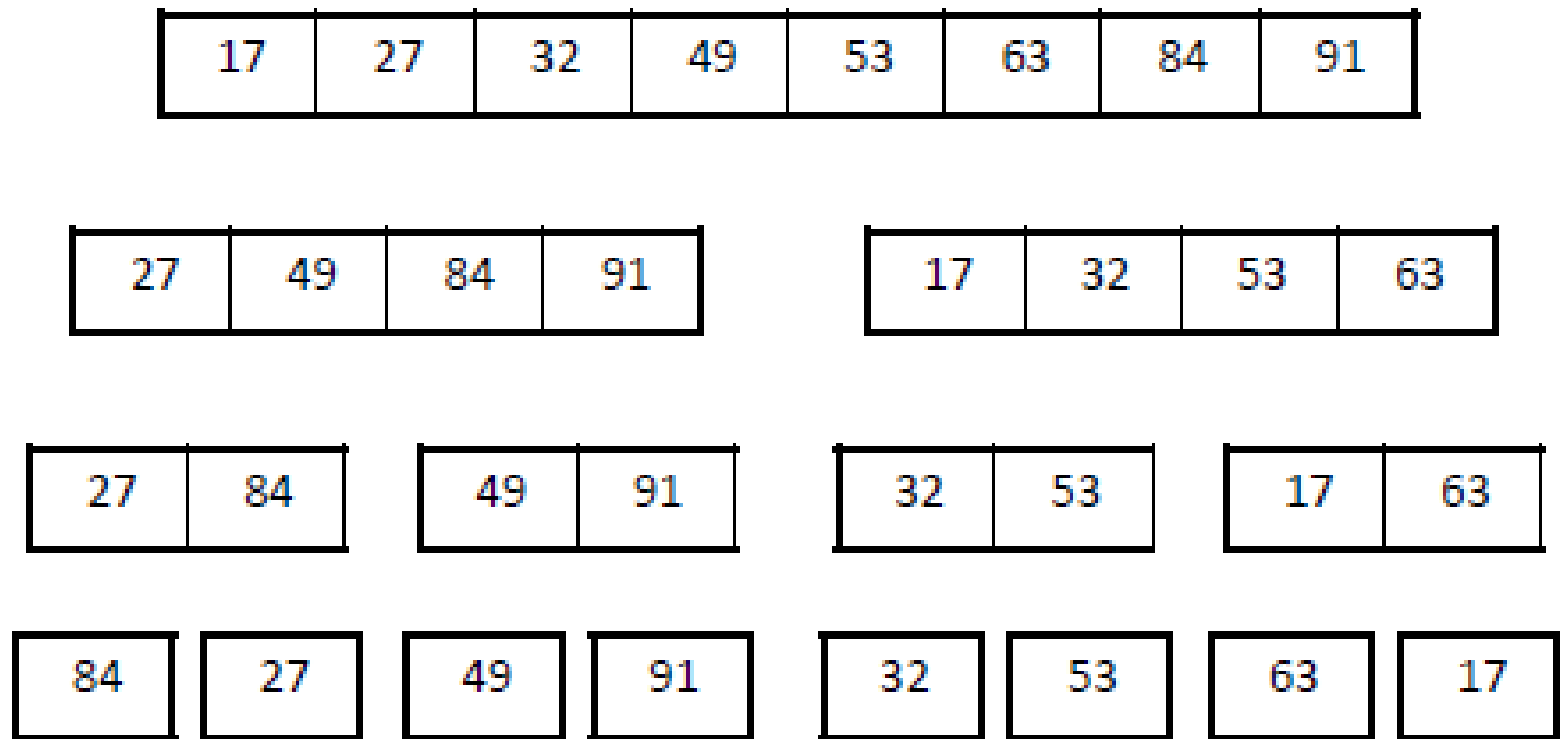
Divide (Split)

Unsorted Data



Conquer (Merge)

Final Sorted Data



Outline of Merging 2 Lists

- **Input:** Two lists a and b , already sorted
- **Output:** A new list containing the elements of a and b merged together in sorted order.
- **Algorithm:**
 1. Create an empty list c , set $index_a$ and $index_b$ to 0
 2. While $index_a < \text{length of } a$ and $index_b < \text{length of } b$
 - a. Add the smaller of $a[index_a]$ and $b[index_b]$ to the end of c , and increment the index of the list with the smaller element
 3. If any elements are left over in a or b , add them to the end of c , in order
 4. Return c

Divide and Conquer Example: Merge Sort [1]

```
def merge(lst1, lst2, lst3):
    # merge sorted lists lst1 and lst2 into lst3
    # these indexes keep track of current position in each list
    i1, i2, i3 = 0, 0, 0 # all start at the front
    n1, n2 = len(lst1), len(lst2)
    # Loop while both lst1 and lst2 have more items
    while i1 < n1 and i2 < n2:
        if lst1[i1] < lst2[i2]: # top of lst1 is smaller
            lst3[i3] = lst1[i1] # copy it into current spot in lst3
            i1 = i1 + 1
        else: # top of lst2 is smaller
            lst3[i3] = lst2[i2] # copy it into current spot in lst3
            i2 = i2 + 1
        i3 = i3 + 1 # item added to lst3, update position

    # Here either lst1 or lst2 is done. One of the following loops
    # will execute to finish up the merge.
    while i1 < n1: # Copy remaining items (if any) from lst1
        lst3[i3] = lst1[i1]
        i1 = i1 + 1
        i3 = i3 + 1
    while i2 < n2: # Copy remaining items (if any) from lst2
        lst3[i3] = lst2[i2]
        i2 = i2 + 1
        i3 = i3 + 1
```

Divide and Conquer Example: Merge Sort [2]

```
def mergeSort(nums):  
    # Put items of nums into ascending order  
    n = len(nums)  
  
    if n > 1:      # Do nothing if nums contains 0 or 1 items  
  
        m = n/2    # split the two sublists  
        nums1, nums2 = nums[:m], nums[m:]  
                    # recursively sort each piece  
        mergeSort(nums1)  
        mergeSort(nums2)  
                    # merge the sorted pieces back  
        merge(nums1, nums2, nums)
```

Comparing Sorts

- Selection Sort (n^2 algorithm)
- For a list of size n .
 - To find the smallest element, the algorithm inspects all n items.
 - The next time through the loop, it inspects the remaining $n-1$ items.
- The total number of iterations is:

$$n + (n-1) + (n-2) + (n-3) + \dots + 1 = \frac{n(n+1)}{2}$$

- contains an n^2 term: the number of steps in the algorithm is proportional to the square of the size of the list

- Merge Sort ($n \cdot \log n$ algorithm)

- $n, n/2, n/4, \dots, 1$

$\Rightarrow \log_2 n$ levels

\Rightarrow total work required to sort n items: $n \cdot \log_2 n$

Comparing Algorithms

© The McGraw-Hill Companies, Inc. all rights reserved.

