Accelerating HE Operations Using Key Decomposition

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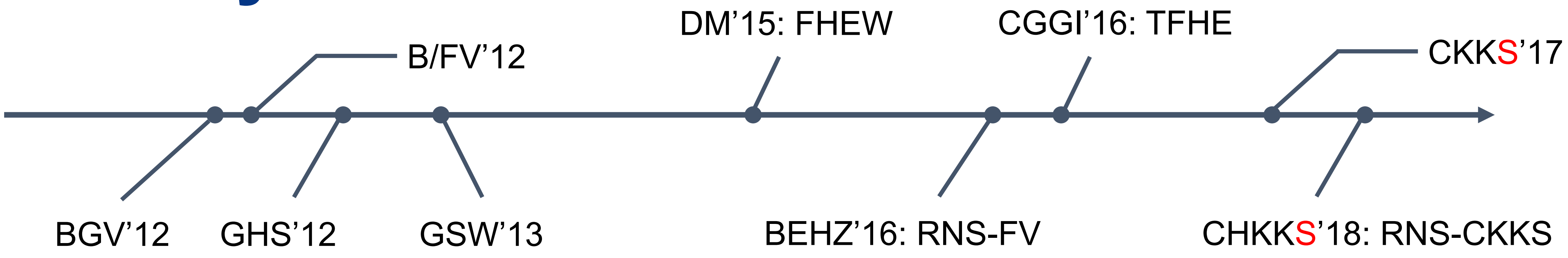
Table of Contents

01 Background

02 External Product & Key-switching

03 Our Contribution & Implementation Results

History



- Remarkable improvements of HE in 2012~2017

- Recent trends
  - Functionality (e.g., bootstrapping, non-arithmetic functions)
  - HE compilers
  - Hardware accelerator
  - Applications
  - Less interests in fundamental HE algorithms

Our Results

- There’s still room for algorithmic improvements!

- Accelerating **External Product**
  - A key building block of nonlinear HE operations
  - Key-switching (multiplication, automorphism) of BGV / BFV / CKKS
  - GSW operations

- Advantages & Implications
  - Better performance in terms of both asymptotic and concrete complexity
  - Compatible with the prior method and its implementation
  - May change future directions of hardware accelerator

Notation

Notation	Definition
N	Ring dimension (a power of two)
$R = \mathbb{Z}[X]/(X^N + 1)$	Ring of integers
$R_Q = \mathbb{Z}[X]/(X^N + 1)$	Residue ring
$Q = q_1 q_2 \ldots q_\ell$	Ciphertext modulus

RNS and NTT

- Residue Number System (RNS)
  - $Q = q_1 q_2 \dots q_d$, a product of distinct primes
  - Isomorphism: $R_Q \to \prod R_{q_i}$, $a \mapsto (a \bmod q_i)_{1 \le i \le \ell}$

- Number Theoretic Transform
  - ρ is a $(2N)$-th primitive root of unity ($q = 1 \pmod{2N}$).
  - NTT is a ring isomorphism $R_q \to \mathbb{Z}_q^N$ defined by $a(X) \mapsto \left(a(\rho), a(\rho^3), \dots, a(\rho^{2N-1})\right)$.

- Polynomial representations of $a(X) = a_0 + a_1 X + \cdots + a_{N-1} X^{N-1} \in R_q$:
  - Coefficient form: $(a_0, a_1, \dots, a_{N-1})$.
  - NTT form: $\left(a(\rho), a(\rho^3), \dots, a(\rho^{2N-1})\right)$.

Gadget Decomposition

- Gadget vector: $\boldsymbol{g} = (g_1, \dots, g_k) \in R_Q^k$.
  - Denote the inner product function as $g: R^k \to R_Q$, $g(x_1, \dots, x_k) = \sum_i x_i \cdot g_i \pmod{Q}$.
  - The product is well-defined since R_Q is an R-module.

- Gadget decomposition: $h: R_Q \to R^k$
  - $h(a) = (b_1, \dots, b_k)$ is a "short" vector in $g^{-1}(a)$
  - In other words, b_i are small and $\sum_i b_i \cdot g_i = a$ over R_Q.

An Example

- RNS Basis $Q = q_1 q_2 \ldots q_\ell$ where $q_1, \ldots, q_\ell$ are distinct primes

- **Prime decomposition:** $h(a) = \left([a]_{q_i}\right)_{1 \le i \le \ell} \in R^\ell$

- Gadget vector: $\boldsymbol{g} = (g_1, \ldots, g_\ell) \in R_Q^\ell$
  - $g_i \equiv 1 \pmod{q_i}$ and $g_i \equiv 0 \pmod{q_j}$ for $j \neq i$.

- Correctness: $\sum_i [a]_{q_i} \cdot g_i = a \ \ (mod\ Q)$
  - Chinese Remainder Theorem

Table of Contents

01 Background

02 External Product & Key-switching

03 Our Contribution & Implementation Results

Definition

- External Product $\boxdot: R_Q \times R_Q^{\ell} \rightarrow R_Q$
  - Input: $a \in R_Q$ & $\boldsymbol{u} = (u_1, \ldots, u_\ell) \in R_Q^d$
  - Output: $c \in R_Q$
  - Step 1 (decomposition): $(b_1, \ldots, b_\ell) \leftarrow h(a) \in R^{\ell}$
  - Step 2 (linear combination): $c = b_1 u_1 + \cdots + b_\ell u_\ell \in R_Q$

- Note : $a \boxdot \boldsymbol{g} = a$

- We write $a \boxdot \boldsymbol{U} = (a \boxdot \boldsymbol{u}_0, a \boxdot \boldsymbol{u}_1) \in R_Q^2$ for $\boldsymbol{U} = [\boldsymbol{u}_0 \mid \boldsymbol{u}_1] \in R_Q^{\ell \times 2}$

Key-switching

- Key-switching key: $\boldsymbol{U} = [\boldsymbol{u}_0 \mid \boldsymbol{u}_1] = [s \cdot \boldsymbol{u}_1 + \boldsymbol{e} \mid \boldsymbol{u}_1] + [s' \cdot \boldsymbol{g} \mid 0]$

- Input: a ciphertext component a (under a secret s')

- Output: $(c_0, c_1) \leftarrow a \boxdot \boldsymbol{U} \in R_Q^2$ is an RLWE ciphertext:

$$c_0 + c_1 s \approx a s' \pmod{Q}.$$

- Examples:
  - Relinearization: $s' = s^2$.
  - Automorphism: $s' = \varphi(s)$.

Implementing External Product

- Input: $a \in R_Q$ & $\boldsymbol{u} = (u_1, \dots, u_\ell) \in R_Q^d$
  - Step 1 (decomposition): $(b_1, \dots, b_\ell) \leftarrow h(a) \in R^\ell$
  - Step 2 (linear combination): $c = b_1 u_1 + \cdots + b_\ell u_\ell \in R_Q$

- **Key question:** How to compute the product of $b_i \in R$ and $u_i \in R_Q$?

- **Previously:**
  - Compute the RNS representation of $b_i \in R$ modulo Q : ℓ^2 NTTs
  - Then perform the inner product over R_Q: ℓ^2 mults

Table of Contents

01 Background

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Our Method

- **Goal:** Compute the linear combination $c = b_1 u_1 + \cdots + b_\ell u_\ell \in R_Q$ more efficiently.

- **Main Idea:** perform the computation over R as much as possible
  - Suppose that the gadget decompositions of u_i are given, say $h(u_i) = \left(v_{i,1}, \dots, v_{i,\ell}\right) \in R^\ell$.
  - From $u_i = \sum_j v_{i,j} \cdot g_j \pmod{Q}$,

$$c = \sum_i b_i u_i = \sum_i b_i \cdot \left(\sum_j v_{i,j} \cdot g_j\right) = \sum_j \left(\sum_i b_i v_{i,j}\right) \cdot g_j \pmod{Q}.$$

Important Facts

$$c = \sum_j \left(\sum_i b_i v_{i,j} \right) \cdot g_j \pmod{Q}.$$

- $v_{i,j}$ can be precomputed
- $c_j = \sum_i b_i v_{i,j}$ are **small** elements of R.
  - Its upper bound is determined by h, not Q.
- $c = c_j \ (mod\ q_j)$, so $(c_1, \ldots, c_\ell)$ is the RNS representation of c modulo Q.

New External Product

- Input: $a \in R_Q$ and $h(u_i) = (v_{i,1}, \dots, v_{i,\ell}) \in R^\ell$ for $1 \le i \le \ell$.
  - $v_{i,j}$ are precomputed and given in the DFT form over R

- Step 1: Compute $b_1 = [a]_{q_1}, \dots, b_\ell = [a]_{q_\ell}$ over R: ℓ DFTs
- Step 2: Compute $c_j = \sum_i b_i v_{i,j}$ for each j: ℓ^2 mults in total
- Step 3: Convert c_j back into the coefficient form: ℓ inverse DFTs

- Output: $(c_1, c_2, \dots, c_\ell)$, the RNS rep. of c modulo Q.

This is purely an algorithmic optimization, computing the same ciphertext!

In the full version..

- The **special-modulus** technique is applied
  - A key-switching key has a larger modulus $PQ > Q$.
- General RNS-based gadget decompositions
  - $h(a) = \left([a]_{D_1}, \ldots, [a]_{D_k}\right)$ for some $D_1 \ldots D_k = Q$
  - Another gadget decomposition over R_{PQ} for u_i.
- Need multi-precision integral polynomial arithmetic
  - Precision depends on the decomposition bounds

Implementation

- Used the Lattigo library

- Base Ring of Integers
  - Instantiate R by R_B for sufficiently large $B > 0.$
  - B is a product of r' prime numbers p_i
  - A DFT over R corresponds to r' NTTs

- Parameter setup
  - The usual HE parameters
  - Choose the best-performing decomposition over R_{PQ}

Experimental Results (CPU)

N	ℓ	r	Prev	Ours	Speedup
32768	24	1	0.317	0.140	2.3
		2	0.204	0.110	1.9
		3	0.151	0.101	1.5
65536	48	1	2.688	0.818	3.3
		2	1.980	0.655	3.0
		3	1.438	0.585	2.5
		4	1.191	0.550	2.2

N: RLWE dimension, r: decomposition size (no. of primes in each digit)

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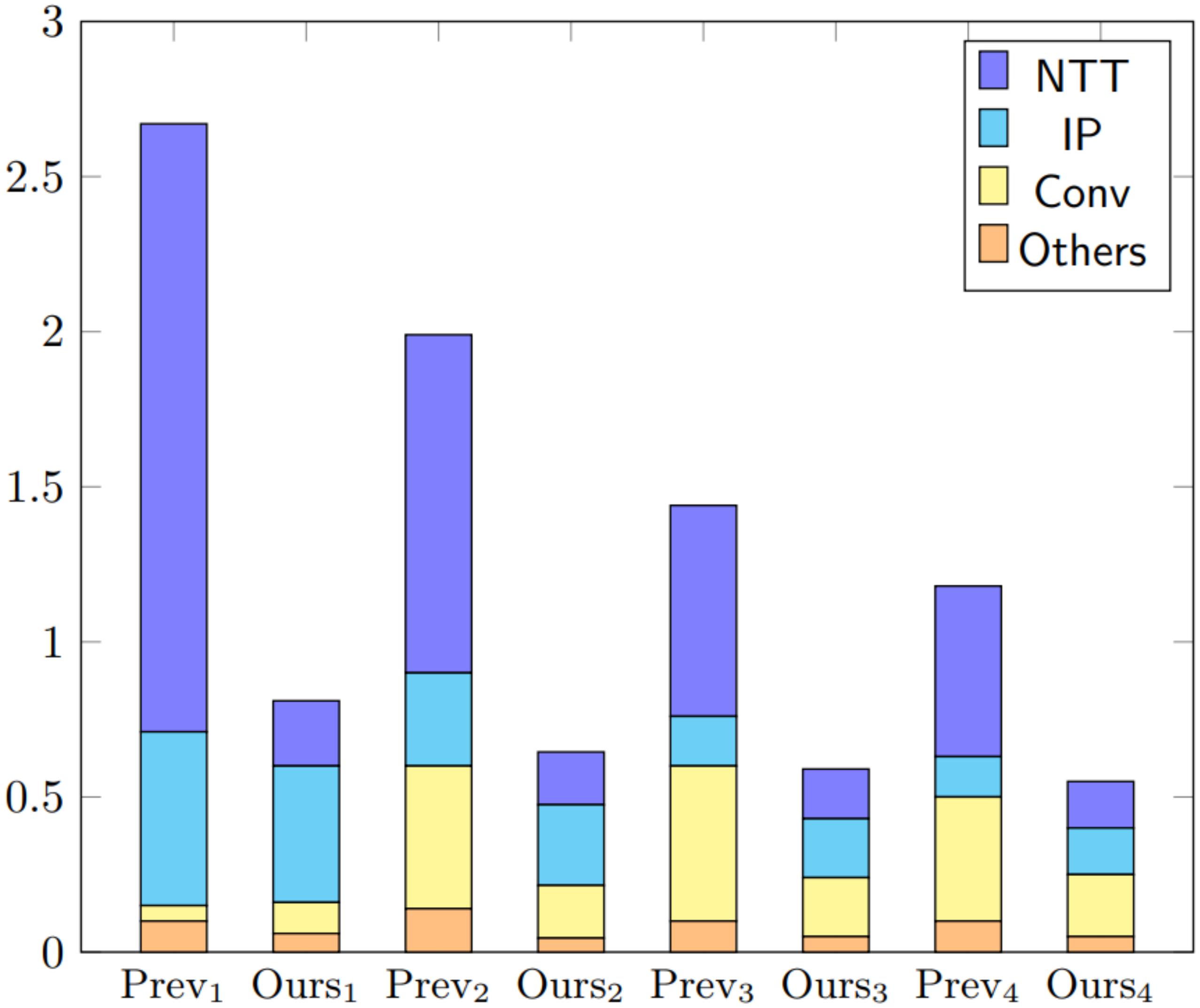

$r = 1$ $r = 2$ $r = 3$ $r = 4$