International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

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To cite this article: B. WITTENMARK (1975) Stochastic adaptive control methods: a survey, International Journal of Control, 21:5, 705-730, DOI: 10.1080/00207177508922026

To link to this article: https://doi.org/10.1080/00207177508922026

Published online: 12 Mar 2007.

INT. J. CONTROL, 1975, VOL. 21, NO. 5, 705–730

Stochastic adaptive control methods: a survey

B. WITTENMARK†

There is a vast literature about adaptive (self-organizing, self-optimizing) control systems. The aim of this paper is to point out some of the main ideas. Since there exists good survey papers about several of the classes of adaptive controllers the main part of this paper will concentrate on recent developments. The paper will cover stochastic adaptive controllers. The problem of dual control is given particular attention.

1. Adaptive control

For a long time control theorists and control engineers have dreamt of a controller that does not need to be tuned. This type of controller has been given many different names, for instance adaptive, self-organizing, self-optimizing and learning controller. The ultimate solution has not yet been found and it is questionable whether it exists. Many different solutions to the adaptive control problem have been suggested. Some solutions are designed from a very practical point of view, while others are based on highly technical theory.

Many adaptive regulators originated from the construction of high performance aeroplanes. Most autopilots are, however, based on gain scheduling rather than adaptive controllers. Apart from the aeroplane applications there have been very few reported applications of adaptive control until recently. One explanation might be that there have been difficulties in implementing the various ideas. The earlier adaptive controllers had to be implemented using the analogue technique which naturally limited the possibilities. It is only recently that computers have been extensively used for on-line control. The enormous development of computers has now made it possible to implement more complex regulators.

The necessity of adaptive control has also been discussed. There are many who mean that it is sufficient to use sophisticated, but constant, feedback controllers, while others are talking about a large demand for adaptive control. There are several reasons for using adaptive regulators in the process industry. The most obvious reason is that the dynamics of the process or the noise are changing due to changes in the production or wear of the equipment. So far the most common solution to this problem is to switch between different fix regulators. Another less obvious reason is that there can be many loops in a plant. The operator or system engineer simply does not have time to occasionally retune the regulators. It is also very difficult to tune more than two or three parameters manually, while an adaptive controller can easily tune many more parameters and thus make it possible to use a more complex structure of the control law. Adaptive control

Received 28 January 1975.

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or adaptive retuning of many loops can thus increase the total performance of the plant. These reasons can explain the growing interest in the process industry for adaptive control.

Many definitions of adaptive control have been suggested. A summary of definitions is given in Saridis *et al.* (1973). Most of the definitions of adaptive controllers are very vague and it is difficult to draw the boundary-lines between the different types of adaptive controllers. It is even difficult to determine if a controller is adaptive or not, since many adaptive controllers can be regarded as non-linear or time-varying controllers. The main idea is, however, that an adaptive controller has the ability to modify its behaviour depending on the performance of the closed-loop system. Most definitions of adaptive controllers point out some basic functions that are common for most adaptive regulators :

(*a*) identification of unknown parameters *or* measurement of a performance index ;

(*b*) decision of the control strategy ;

(*c*) on-line modification of the parameters of the controller.

Depending on how these functions are synthesized different types of regulators will be obtained.

The aim of this paper is to give a survey of *stochastic adaptive systems*. This class of adaptive controllers consists of systems where the variation of the process parameters have been described by stochastic models and have been taken into account in the derivation of the control algorithm.

In order to give a broader base for continued studies of adaptive controllers a few references will be given to some other survey papers on adaptive methods. The list is by no means complete but gives the reader starting points for further studies of the fascinating field of adaptive control.

Some earlier surveys are given by Aseltine *et al.* (1958), Stromer (1959), Truxal (1963), Donaldson and Kishi (1965), and Tsypkin (1966). These surveys cover the whole field of adaptive control. General surveys are also available in a number of books (see Mishkin and Braun 1961, Eveliegh 1967, Mendel and Fu 1970, and Tsypkin 1971).

There are also some good surveys of more specific fields of adaptive control. Extremum seaking methods are discussed for instance, by Li (1952), Morosanov (1957), Gibson (1960), and Hammond and Rees (1968). One popular adaptive scheme is the model-reference method which is dealt with by Maslov and Osovskii (1966), Landau (1972), and Hang and Parks (1973). Hyperstability concepts and Liapunov technique are techniques that can be used to investigate the stability properties of closed loop adaptive systems. This subject is reported on by Landau (1972), Hang and Parks (1973), Lindorff and Carroll (1973), and Lüders and Narendra (1972).

Finally the identification of unknown parameters in connection with adaptive control is discussed in Will (1963), Davies (1970), Tsypkin (1971) and Saridis (1972).

2. Stochastic adaptive control methods

The first adaptive controllers were developed using classical servomechanism theory and non-linear control theory. Very few controllers were constructed from a statistical point of view. If noise was considered it was mostly approximated by steps, ramps or other test signals. The systems were designed to be as insensitive as possible to these test signals. Today, however, many adaptive controllers are constructed which take into consideration the statistical nature of the fluctuations of the parameters or the disturbances acting on the system. The philosophy of stochastic adaptive control and ways to attack these problems are discussed in Bellman (1961). Many stochastic adaptive control problems are formulated as Bayesian control problems. Aoki (1967) has given a thorough penetration of this kind of problems.

2.1. *The two-armed bandit problem*

One of the first types of stochastic adaptive problems that were solved can be represented by the classical two armed bandit (TAB) problem. This problem is discussed for instance in Bellman (1961) and Yakowitz (1969). This type of problem comes from sequential design of statistical experiments and it arises for instance in connection with the use of drugs in critical medical cases. The TAB problem can be described in the following way. A player is faced with two slot machines, I and II. If he plays machine I the gain is one unit with probability p and the machine II gives a gain of one unit with probability q. In the simplest case p is known and q is unknown and chosen before each game of length N according to a given probability distribution. During the game the unknown quantity q has to be estimated and the player must at each step decide which machine to play in order to maximize his gain in each game of N plays.

Problems of the TAB type mainly consider static models where some parameter for the underlaying probabilities are unknown. This type of adaptive problems will not be further treated in this paper.

2.2. *Certainty equivalence and separation*

A block diagram which can represent many stochastic adaptive controllers is shown in Fig. 1. The design is based on the idea that the unknown process can be represented by a model, for example a transfer function. The estimator attempts to find the coefficients of the model. The vector $\mathscr{P}$ is the information

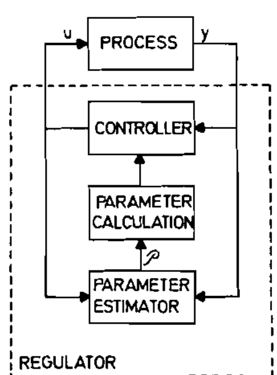

Figure 1. Schematic block diagram for an adaptive control system.

that is used to calculate the parameters of the controller. Finally the controller determines the input signal, u, to the process.

The division can be justified either through the use of the certainty equivalence principle or the separation principle. These two concepts are discussed for instance in Patchell and Jacobs (1971) and Bar-Shalom and Tse (1974). The certainty equivalence principle holds if it is possible to first solve the deterministic problem with known parameters and then obtain the optimal controller for unknown parameters by substituting the true parameter values with the estimated values. One well-known class of problems for which the certainty equivalence principle holds is the linear-quadratic-gaussian control problems. In adaptive control there are very few cases where the certainty equivalence principle is applicable. One exception is when the unknown parameters are stochastic variables that are independent between different sampling intervals (see Tou 1963, Gunckel and Franklin 1963). However, certainty equivalence principle has been successfully used as an *ad hoc* design principle (for further details see § 3).

The separation principle (see for instance Witsenhausen 1971) is weaker than the certainty equivalence principle, and it is said to be valid if it is possible to make a separation between the identification of the parameters in the process and the determination of the parameters in the controller. The parameters of the controller are also allowed to be functions of, for instance, the uncertainties of the identified parameters.

Depending how the different blocks in Fig. 1 are synthesized different types of control systems will be obtained.

The *process* can be described through a state space model or through an input–output model. Further the models can be continuous time or sampled data models. Most of the stochastic adaptive controllers assume sampled data systems. The *estimator* can be using different types of identification methods, the least squares method, the generalized least squares method, instrumental variables, Kalman filter, extended Kalman filter, etc. Finally the *controller* can be determined using many types of performance indices. The purpose with the control can either be to make stationary control around a fixed reference value (the regulator problem) or to follow a time-varying reference value (the servo problem).

It is important to consider the information, $\mathscr{P}$, which is transferred from the estimator to the controller. The information, $\mathscr{P}$, might be the estimates of the unknown parameters, $\{\hat{\theta}\}$, or the estimates and the uncertainties of the estimates, $\{\hat{\theta}, P\}$, etc. The importance of the information pattern is discussed in Bohlin (1970). This information together with the performance index of the controller will in the sequel be used as a basis for the classification of different stochastic adaptive controllers.

The fundamental differences between the controllers obtained for different information pattern and different loss functions have been pointed out by Feldbaum (1960, 1961). Good definitions are also found in Bar-Shalom and Tse (1974).

The controllers obtained by enforcing the certainty equivalence principle, i.e. $\mathscr{P}=\{\hat{\theta}\}$, are called *certainty equivalence* controllers. These controllers do not take into consideration the fact that the estimated parameters are not equal to the true ones but are inaccurate. If the information pattern is changed to $\mathscr{P} = \{\hat{\theta}, P\}$, i.e. that the separation principle is applied, then the controller will be called *cautious*. In this case the controller is aware of the errors in the estimates and takes a more cautious control action.

Different types of controllers will also be obtained depending on the structure of the performance index. If the performance index only takes into account the previous measurements and does not assume that further information will be available then the resulting controller in Feldbaum's terminology will be called *non-dual*. On the other hand, the performance index can also be dependent on the future observations and this will result in a *dual* controller. The controller must, of course, be causal and the dependence of the future observations will be given as the probability distributions of the future observations given information up to the actual time.

According to the above discussion the minimization of a loss function one step ahead will give a non-dual controller, while a minimization several steps ahead will give a dual controller. In the first case it is useless for the controller to take a control action in order to increase the accuracies of the unknown parameters. In the second case it might be worth while for the controller to take some control actions in order to improve the estimates of the unknown parameters. The dual controller must thus ensure good control and good estimation. However, these two tasks are, in general, contradictory, since good estimation might require large control signals while good control might require that the control signals are small. A dual controller thus must compromise between these two tasks.

2.3. *Classification*

The stochastic adaptive controllers will be divided into non-dual and dual controllers. Within these two classes the controllers will be classified depending on the ways to synthesize the control system, see Fig. 2. Before going into the details of the different suggested types of stochastic adaptive controllers two examples will be given which illustrate the difference between the different classes of controllers.

The first example will illustrative the difference between a certainty equivalence and a cautious controller.

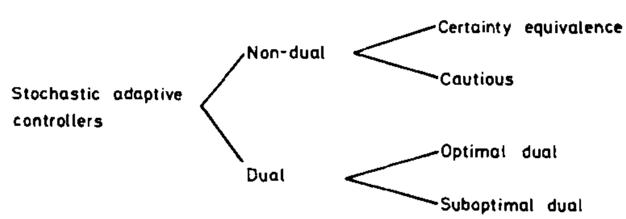

Figure 2. Schematic classification of stochastic adaptive controllers.

Example 2.1

Consider the stochastic system

$$y(t) - y(t-1) = bu(t-1) + e(t) \tag{2.1}$$

where b is an unknown constant and $\{e(t)\}$ is a sequence of independent equally distributed gaussian random variables with zero mean values and standard deviations σ. We want to select $u(t)$ in order to minimize the loss function

$$V = E\{y(t+1)^2 | y(t), y(t-1), \ldots, y(0), u(t-1), \ldots, u(0)\} \tag{2.2}$$

The unknown parameter b in (2.1) can be estimated using the least squares method (see Åström and Eykhoff 1971)

$$\begin{aligned} \hat{b}(t) &= E\{b | y(t), \ldots, u(t-1), \ldots\} \\ &= \frac{\sum_{s=1}^{t} \{y(s) - y(s-1)\} u(s-1)}{\sum_{s=1}^{t} u(s-1)^2} \\ p_b(t) &= \operatorname{Var}\{b | y(t), \ldots, u(t-1), \ldots\} \\ &= \frac{\sigma^2}{\sum_{s=1}^{t} u(s-1)^2} \end{aligned}$$

If b were known then the optimal loss is given as

$$\begin{aligned} \min_{u(t)} V &= \min_{u(t)} E\{(y(t) + bu(t) + e(t+1))^2 | y(t), \ldots, u(t-1), \ldots\} \\ &= \min \{(y(t) + bu(t))^2 + \sigma^2\} = \sigma^2 \end{aligned} \tag{2.3}$$

To obtain this we have used that $e(t+1)$ is independent of

$$b, y(t), \ldots, u(t-1), \ldots$$

The optimal control law is

$$u(t) = -\frac{1}{b} y(t) \tag{2.4}$$

If the estimated value, $\hat{b}(t)$, is used in (2.4) instead of the true value we get

$$u(t) = -\frac{1}{\hat{b}(t)} y(t) \tag{2.5}$$

i.e. we have assumed that the certainty equivalence principle can be used. The loss when using (2.5) will be

$$\begin{aligned} V &= E\left\{\left(y(t) - \frac{b}{\hat{b}(t)} y(t) + e(t+1)\right)^2 | y(t), \ldots, u(t-1), \ldots\right\} \\ &= \frac{p_b(t)}{\hat{b}(t)^2} y(t)^2 + \sigma^2 \end{aligned} \tag{2.6}$$

To get the last equality the standard formula

$$E(b^2) = (Eb)^2 + \operatorname{Var} b$$

has been used. The loss has increased with the term

$$\frac{p_b(t)}{\hat{b}(t)^2} y(t)^2$$

compared with the optimal loss when b was known. The control law (2.5) does not minimize (2.2) because

$$\begin{aligned}\min_{u(t)} V &= \min_{u(t)} E\{(y(t)+bu(t)+e(t+1))^2|y(t), \ldots, u(t-1), \ldots\}\\ &= \min_{u(t)} \{(y(t)+\hat{b}(t)u(t))^2+p_b(t)u(t)^2+\sigma^2\}\\ &= \frac{p_b(t)}{\hat{b}(t)^2+p_b(t)} y(t)^2+\sigma^2 \end{aligned} \tag{2.7}$$

and the minimum is assumed for the control law

$$u(t) = -\frac{\hat{b}(t)}{\hat{b}(t)^2+p_b(t)} y(t) \tag{2.8}$$

The loss in eqn. (2.7) is less than in (2.6) since $p_b(t) \geqslant 0$. The first term in (2.7) is the loss due to the uncertainty of the parameter and the second term is due to the process noise $e(t)$.

The optimal controller (2.8) is cautious since it considers the inaccuracy of the estimate of b. If $p_b(t) \to 0$ then (2.5) and (2.8) will be the same and the loss approaches the optimal loss for known b, (2.3).

Example 2.2

In Sternby (1974) a Markov chain with four states, x_1–x_4, is described. The transition probabilities depend on the control signal u $(0 \leqslant u \leqslant 1)$. The transition probabilities are piecewise linear in the control signal. The loss function is

$$V_N = E \sum_{t_0+1}^{t_0+N} v(x(t))$$

where

$$v(x_1) = v(x_4) = 1$$
$$v(x_2) = v(x_3) = 0$$

The example is constructed in such a way that the future trajectories from the states x_1 and x_2 are the same. Also the trajectories from x_3 and x_4 are the same. It is then convenient to introduce q as the probability that the initial state is x_1 or x_2. The corresponding probability for x_3 or x_4 will be $1-q$.

The following control strategies are derived analytically :

(*a*) Open loop control.

(*b*) Open loop optimal feedback control.

(*c*) One-step regulator.

(*d*) Two-step regulator.

(*e*) Optimal dual regulator.

In the open loop case the control strategy is determined without making any measurements.

The notation open loop optimal feedback (OLOF) control (see Dreyfus 1964), is confusing. The controller is derived by at each time interval minimizing the multistep loss function under the assumption that no future measurements will be available, i.e. determine the open loop control sequence. The first step in the control sequence is then used and the performance of the system is measured. Based on the new information (feedback) a new minimization is done, etc. The first step in the OLOF control is thus the same as the first step in the open loop control.

The one-step and the two-step regulators minimize V_N for $N=1$ and 2, respectively.

Figure 3 shows the expected loss, V_∞, for the different controllers as a function of the initial probability q. The open loop control gives the largest loss while the optimal dual controller gives the smallest loss for any value of q. In this simple example the one-step regulator is better than the OLOF control.

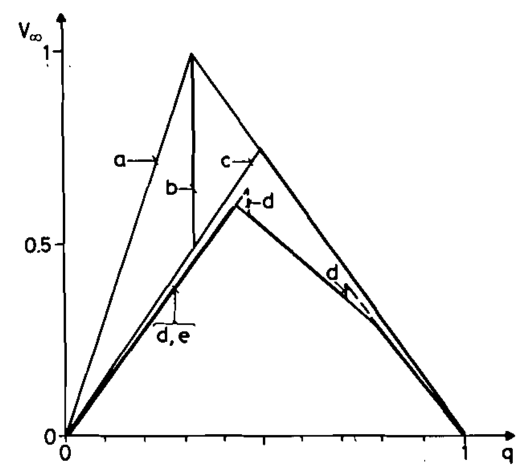

Figure 3. The expected loss V_∞ as a function of the initial probability q for different control strategies: (*a*) open loop control; (*b*) open loop optimal feedback control; (*c*) one-step regulator; (*d*) two-step regulator; (*e*) optimal dual regulator.

The two-step regulator and the optimal dual regulator have dual properties which make them superior over the other regulators. The largest gain with the dual controllers is obtained when the initial state is very uncertain, $q \approx 0{\cdot}5$.

The figure also shows the difference between the one-step and the two-step regulators. When the loss function is minimized two steps ahead it might be advantageous to make a control action in order to get a better knowledge of the state. This can be used in the next step to decrease the expected loss. The one-step regulator just tries to make the best short term control.

In the following two sections non-dual and dual adaptive controllers will be described. Within each class different ways to solve the adaptive control problem will be presented. The intention has been to select some methods within each subclass and describe them in some detail. At the end of each subsection references will be given to other variants. Although some methods will fall between two classes and others do not fit anywhere the main ideas of stochastic adaptive control are hopefully covered.

2.4. *Notations*

In order to describe the different controllers it is necessary to specify the type of models and the performance indices that are commonly used. Most controllers are designed for single-input single-output sampled data systems. (For convenience the sampling period is normalized to one unit of time.) Input–output models can be described by the difference equation

$$\begin{aligned} y(t)+a_1y(t-1)+\dots+a_ny(t-n) \\ =b_0u(t-k)+b_1u(t-k-1)+\dots+b_nu(t-k-n) \\ +\sigma(e(t)+c_1e(t-1)+\dots+c_ne(t-n)) \end{aligned} \tag{2.9}$$

where $y(t)$ is the output, $u(t)$ is the input and $\{e(t)\}$ is a sequence of independent equally distributed gaussian variables with zero mean value and unit variance. Introduce the forward shift operator, q, and the polynomials

$$A^*(q^{-1})=1+a_1q^{-1}+\dots+a_nq^{-n}$$
$$B^*(q^{-1})=b_0+b_1q^{-1}+\dots+b_nq^{-n}$$
$$C^*(q^{-1})=1+c_1q^{-1}+\dots+c_nq^{-n}$$

The model (2.9) can now be written in the compact form

$$A^*(q^{-1})y(t)=B^*(q^{-1})u(t-k)+\sigma C^*(q^{-1})e(t) \tag{2.10}$$

Some or all of the parameters in eqn. (2.10) can be unknown. For convenience the vector of unknown parameters is denoted by θ.

Equation (2.10) is a canonical form for describing single-input single-output models with noise. In the sequel (2.10) will be referred to as the maximum likelihood model (ML model) since it is possible to get unbiased estimates of all the parameters in the model using for instance the maximum likelihood model (see Åström *et al.* 1965). A simpler model is

$$A^*(q^{-1})y(t)=B^*(q^{-1})u(t-k)+\sigma e(t) \tag{2.11}$$

which will be called the least squares model (LS model) since it is possible to use the least squares method (Åström–Eykhoff 1971), to get unbiased estimates of the parameters in the A^*- and B^*-polynomials.

State space models will be described through the standard model

$$\left.\begin{aligned} x(t+1)&=A(\theta)x(t)+B(\theta)u(t)+v(t) \\ y(t)&=C(\theta)x(t)+e(t) \end{aligned}\right\} \tag{2.12}$$

where the matrices A, B and C are functions of the vector of the unknown parameters. $\{v(t)\}$ and $\{e(t)\}$ are sequences of independent equally distributed random variables. In some cases non-linear models are considered and those will be described through the equations

$$\left.\begin{aligned} x(t+1) &= f(x(t), u(t), \theta, v(t)) \\ y(t) &= g(x(t), u(t), \theta, e(t)) \end{aligned}\right\} \tag{2.13}$$

The unknown parameters, θ, use to be modelled as a stochastic process

$$\theta(t+1) = \phi\theta(t) + w(t) \tag{2.14}$$

where $\{w(t)\}$ is a sequence of independent equally distributed gaussian random variables with zero mean value and the covariance matrix R_w. The important special case where the parameters are constant is obtained by letting $R_w = 0$ and $\phi = I$ (the unit matrix).

For instance if all the parameters in the LS model or the ML model are unknown then

$$\theta = \{a_1 \ldots a_n \quad b_0 \ldots b_n\}$$

and

$$\theta = \{a_1 \ldots a_n \quad b_0 \ldots b_n \quad c_1 \ldots c_n\}$$

respectively.

For input–output models the most commonly used performance indices (loss functions) are

$$V_1 = E(y(t+k) - y_r)^2 \tag{2.15}$$

and

$$V_2 = \frac{1}{N} E \sum_{s=t+k}^{t+N+k-1} (y(s) - y_r)^2 \tag{2.16}$$

where y_r is a desired reference value. For the state space models the loss function uses to be

$$V_3 = E\left\{x(N)^T Q_0 x(N) + \sum_{t=0}^{N-1} (x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t))\right\} \tag{2.17}$$

It is also convenient to introduce the notation

$$\mathscr{Y}_t = [y(t), y(t-1), \ldots, y(0), u(t-1), u(t-2), \ldots, u(0)]$$

i.e. a vector containing all inputs and outputs respectively that are available at time t. This notation is used to specify the information available when minimizing the loss functions given above.

3. Non-dual adaptive controllers

The non-dual adaptive controllers will be divided into two classes, certainty equivalence and cautious controllers. The first class includes methods where enforced certainty equivalence has been used as an *ad hoc* design method. The second class contains methods obtained by using the separation principle.

3.1. *Certainty equivalence controllers*

Three different types of methods will be described. In the first class the estimation is done in order to find the parameters describing the process and the disturbances. Second is described a special class of stochastic adaptive controllers called self-tuning regulators. These are designed to control processes with constant but unknown parameters. Finally the third class contains controllers where it is assumed that the unknown parameters of the process belong to a known finite set of values.

3.1.1. *Methods based on process parameter estimation*

An obvious idea for designing an adaptive controller is to use some real time identification method to find the parameters in the process and then use the estimated parameters to determine the control law by minimizing some loss function assuming that the estimated parameters are true. This idea has been used for input–output models as well as state space models.

The basic idea is described by Kalman (1958) where he applies the least squares to determine the unknown parameters in the model (2.11). Based on the estimated parameters a dead-beat controller is determined.

Kalman also discusses a modification of the least squares method which makes it possible to follow slowly time-varying parameters. This can be done by using an exponential decreasing weighting of old data, i.e. the estimator has a fading memory.

In the paper by Kalman noise is not considered, but the same idea has been used later for noisy systems. For instance Schwartz and Steiglitz (1971) use this idea to determine a minimum variance controller, i.e. they use the performance index V_1 (2.15).

This type of methods work very well for constant or slowly time-varying parameters when the estimation is done using the method of least squares and when the process actually is described by the LS model. The maximum likelihood method is not suitable for real time calculations which has made it difficult to use the basic separation for models of ML type. For models of ML type it is necessary to get information about the system dynamics as well as the characteristics of the noise. Different approximative identification methods have been suggested (see for instance Hasting-James and Sage 1969, Panuska 1968, Young 1968).

Sandoz and Swanick (1972) propose an adaptive controller where the ML model is approximated by a higher-order LS model. This is done by first dividing eqn. (2.10) by $C^*(q^{-1})$ which gives

$$C^*(q^{-1})^{-1}A^*(q^{-1})y(t) = C^*(q^{-1})^{-1}B^*(q^{-1})u(t-k) + \sigma e(t)$$

The rational functions $C^*(q^{-1})^{-1}A^*(q^{-1})$ and $C^*(q^{-1})^{-1}B^*(q^{-1})$ are then approximated by higher-order polynomials which are identified using the least squares method. A Riccati equation is then used to derive the control law.

Another method to estimate the process and the noise is suggested by Turtle and Phillipson (1971). Their method essentially separates the input and the noise dependences by making a model for the dynamic and a predictor for the noise. The model and the predictor is updated from an autoregression model obtained every Nth step.

Kotnour *et al.* (1966) have proposed an adaptive regulator which contains a predictor. This method can be regarded as an extension of the extremum seaking method presented by Draper and Li (1951). The method of Kotnour *et al.* consists of two parts, one sinusoidal perturbation adaptive part and a discrete time predictor. The predictor, where the parameters are not determined on line, is used to predict the future influence of the disturbances. This information is then used to make a correcting control action. Experiments on a natural gas combustion system show that the insertion of the predictor in the adaptive loop provides significant improvement over the conventional perturbation adaptive system.

Method using state space models will usually give a control law which is a function of the unknown parameters as well as of the state of the system. The estimation of the state and the parameters leads to a non-linear estimation problem even if the model is linear. If it is assumed that the whole state is measurable then the problem will be reduced to a linear estimation problem. The non-linear estimation problem is usually attacked by using extended Kalman filters. The controller is mostly obtained by minimizing the loss function V_3 (2.17). This will in general imply that a new optimization problem has to be solved each time the estimated parameters have changed. Methods based on this principle are given by Jenkins and Roy (1966) and Luxat and Lees (1973). The last method allows non-linear systems.

Saridis and Lobbia's (1972) estimations of the parameters and the state are separated. The parameters are estimated by using the observable canonical form and by introducing a perturbation signal in the system. A Kalman filter is then designed based on the estimated parameters and the state feedback controller is obtained by minimizing a quadratic loss function.

Continuous time systems are reported by Balakrishnan (1973), he also discusses the convergence of the method. The identification is made using the maximum likelihood method. If the same control law is used over long periods of time it is possible to show that the method converges to the optimal control law that could be obtained if the parameters were known. The infrequent change in the control law must be assumed in order to get stationarity in the process.

3.1.2. *Self-tuning regulators*

One way to solve the minimum variance control problem, i.e. when using the performance index V_1, for models of ML type will now be discussed. The problem is attacked by making a prediction model. The output of (2.10) can be predicted k steps ahead by using a predictor of the form

$$\hat{y}(t+k) = -\sum_{i=1}^{m} \alpha_i y(t-i+1) + \sum_{i=0}^{l} \beta_i u(t-i)$$

where $m=n$ and $l=n+k-1$. For known systems the parameters of the predictor are easily determined from the parameters in the system (see Åström 1970). The minimum square error predictor has the property that the prediction error

$$\epsilon(t+k) = y(t+k) - \hat{y}(t+k)$$

is a moving average of order $k-1$. This means that the optimal predictor has the property that $\epsilon(t+k)$ is independent of $y(t), y(t-1), \ldots$ and $u(t)$, $u(t-1), \ldots$. The parameters of the predictor α_i and β_i are now estimated from the equation

$$y(t+k)-\hat{y}(t+k)=y(t+k)+\sum_{i=1}^{m}\alpha_i y(t-i+1)-\sum_{i=0}^{l}\beta_i u(t-i)=\epsilon(t+k)$$

by minimizing $\epsilon(t+k)$ with respect to α_i and β_i, i.e. by using the method of least squares. When the parameters of the predictor are given the loss function (2.15) is minimized by the control signal

$$u(t)=\frac{1}{\beta_0}\left[\sum_{i=1}^{m}\alpha_i y(t-i+1)-\sum_{i=1}^{l}\beta_i u(t-i)\right]$$

This particular algorithm, which is similar to Kalman's method described above, was first described by Peterka (1970). The properties of the algorithm were first analysed by Åström and Wittenmark (1973). The latter showed that if the predictor contained sufficient parameters and if the estimations converge then the resulting controller is the optimal minimum variance controller that could be obtained if the parameters of the system were known. The regulator thus has the desired asymptotic properties. The method allows tuning of parameters in feedback as well as feedforward loops. This algorithm is also one of the few algorithms for which the convergence properties have been analysed (see Ljung and Wittenmark 1974).

The good transient as well as the good asymptotic properties make the method attractive for industrial applications. The regulator has successfully been used in the paper industry (see Cegrell and Hedqvist 1973, Cegrell and Hedqvist 1974, Borisson and Wittenmark 1974) and in the mining industry (see Borisson and Syding 1974).

The self-tuning regulator discussed above can only be used for minimum phase systems. Variants that can be used for non-minimum phase systems are discussed by Peterka and Åström (1973) and Åström and Wittenmark (1974). A survey of the theoretical and practical results of the self-tuning regulators is given by Åström *et al.* (1975). An adaptive feedforward control scheme is discussed by Wade and Schoeffler (1970). A technique is proposed for updating the feedforward controller parameters based upon observations of the load disturbance and process output.

3.1.3. *Methods for finite parameter sets*

The estimation of the states and the parameters in the state space model (2.12) is a complicated non-linear estimation problem. Many approximations to this problem have been proposed in the literature. One possible approximation is based on a discretization of the parameters pace, Θ, into a finite set, i.e.

$$\Theta=\{\theta_1, \theta_2, \ldots, \theta_r\}$$

A Kalman filter is then designed for each of the r parameter vectors, θ_i, and a loss function of type V_3 (2.17) is minimized for each parameter vector in order to get r state feedback controllers

$$u_i(t)=-L_i\hat{x}_i(t)$$

where L_i is the optimal feedback provided that the true parameter vector was θ_i, and $\hat{x}_i(t)$ is the estimate of the state vector from the ith Kalman filter. The control signal to the system is then constructed as a weighted sum

$$u(t)=\sum_{1}^{r} p_i(t)u_i(t)$$

where $0 \leqslant p_i(t) \leqslant 1$ and

$$\sum_{i=1}^{r} p_i = 1$$

The problem is now reduced to make an on-line determination of the weights $p_i(t)$. In Saridis and Dao (1972) and Deshpande *et al.* (1973) this is done for sampled data systems by making *a posteriori* estimates of the weights using the residuals from the Kalman filters. Athans and Willner (1973) consider time continuous systems where the weights are determined through solving a system of r non-linear differential equations driven by the residual signals from the Kalman filters.

A variant where the states are weighted together instead of the control signals is discussed by Stein and Saridis (1969).

The method with several Kalman filters which are weighted together gives a smoother control than if the adaptive system switches between the different filters, i.e. $p_i(t)$ is either 0 or 1. The method is also attractive since the optimal gains in the Kalman filters and the feedback controllers can be pre-computed off-line. It is only the updating of the weighting factors that has to be done on-line. So far very little has, however, been reported about the properties of the method when the true parameters of the system do not coincide with a member of the set Θ.

3.2. *Cautious controllers*

The control law for this type of adaptive methods is a function of the estimated parameters as well as of the accuracies of the estimates. The control action will for the same values of the estimates be different depending on if the estimates are good or poor. It is appropriate to distinguish between methods based on input–output models and on state space models.

3.2.1. *Methods based on input–output models*

This section will contain methods which are generalizations of Example 2.1.

Consider the LS model with $k=1$ and where the unknown parameters are modelled as a stochastic process (2.14), further assume that the purpose with the control is to minimize the loss function V_1. This problem has been discussed in Aoki (1967), Bohlin (1969) and Åström and Wittenmark (1971). A similar problem is also treated in Drenick and Shaw (1964).

The problem is solved by first observing that the unknown parameters can be estimated using a Kalman filter, which gives the estimates, $\hat{\theta}(t+1)$, and the covariance matrix, $P(t+1)$, based on data obtained up to and including time t, i.e. $\mathscr{Y}_t$. Using a fundamental lemma in stochastic optimal control, see e.g. Åström (1970), it is possible to show that it is sufficient to minimize

$$\begin{aligned} E[(y(t+1)-y_r)^2|\mathscr{Y}_t] &= E[(\varphi(t+1)\theta(t+1)+e(t+1)-y_r)^2|\mathscr{Y}_t] \\ &= (\varphi(t+1)\hat{\theta}(t+1)-y_r)^2 \\ &\quad + \varphi(t+1)P(t+1)\varphi(t+1)^{\mathrm{T}}+\sigma^2 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \varphi(t+1) &= [\varphi_1(t+1), \ldots, \varphi_{2n+1}(t+1)] \\ &= [-y(t), -y(t-1), \ldots, -y(t-n+1), u(t-1), \ldots, u(t-n-1)] \end{aligned}$$

The control signal, $u(t)$, minimizing (3.1) is given by

$$u(t) = \frac{\hat{\theta}_{n+1}(t+1)y_r - \sum_{i=1}^{2n+1}{}' [\hat{\theta}_{n+1}(t+1)\hat{\theta}_i(t+1) + p_{n+1i}(t+1)]\varphi_i(t+1)}{\hat{\theta}_{n+1}{}^2(t+1) + p_{n+1n+1}(t+1)} \tag{3.2}$$

where $\sum'$ means that the term corresponding to $i=n+1$ is excluded and p_{ij} is the i, jth element in P.

The control law (3.2) clearly shows the influence of the uncertainties of the parameter estimates. The covariance matrix will help the controller to make a more cautious control action when the estimates are poor. If the estimates are very poor then $u(t)$ might become very small. This implies that the controller will not excite the system and the estimates in the future steps can be still worse which will give still smaller control signals and so on. The control can thus unintentionally be turned off for some period of time until the noise excites the system in such a way that better estimates are achieved. Examples with turn-off are given for instance by Åström and Wittenmark (1971) and Hughes and Jacobs (1974). The turn-off phenomenon is due to the fact that the controller minimizes the loss function only one step ahead and that the controller is not rewarded if it makes a control action in order to get better estimates which can be used to improve the control in future steps. Several ways to avoid the turn-off phenomenon will be discussed in § 4.

The minimization of V_1 when $k=2$ is discussed by Wieslander and Wittenmark (1971). The equations will be more and more complex as k increases. Nahorski and Vidal (1974) handle the case with general k by using the model

$$y(t+k)+\alpha_1 y(t)+\ldots+\alpha_m y(t-m-1)=\beta_0 u(t)+\ldots+\beta_l u(t-l)+e(t+k)$$

where $e(t+k)$ is a stochastic variable independent of $\mathscr{Y}_t$. The unknown parameters are

$$\theta = [\alpha_1 \ldots \alpha_m \quad \beta_0 \ldots \beta_l]$$

and these parameters are now modelled according to (2.14).

Regulators for multivariable input–output models are described by Mendes (1971) and Peterka and Åström (1973). The methods described above can be extended to the ML model by also calculating the regression with respect to past prediction errors, i.e. by using some approximate maximum likelihood identification method.

3.2.2. *Methods based on state space models*

The basic problem discussed in this section is defined by the state space model (2.12) with unknown, possibly stochastic, parameters and by the loss function (2.17). The input–output model corresponding to (2.12) is in general of ML type. As pointed out above this will give rise to difficulties in the simultaneous estimation of the parameters and the states. Another difficulty is the multistep minimization.

If the parameters in (2.12) are known we get the standard linear quadratic control problem. In that case it is possible to divide the derivation of the control signal into two parts, state estimation and determination of a state feedback law for the deterministic case, i.e. the certainty equivalence principle holds. When the parameters of the system are unknown it is in general not possible to apply the certainty equivalence principle. One exception is the case with 'white parameters', i.e. when the stochastic process describing the parameters is independent between the sampling times. This case is studied by Tou (1963) and Gunckel and Franklin (1963). The general problem can, however, be attacked by applying the separation principle. The estimation of the state as well as the unknown parameters leads in general to a non-linear problem and only approximate solutions can be obtained.

Variants of the stated problem have been treated by many authors under different assumptions. We will first discuss how to approximate the multi-step minimization problem. (The optimal solution is discussed in the next section.) One common way is to use open loop optimal feedback (OLOF) control (see Dreyfus 1964), which can be described in the following way: The estimates of the parameters and the states are assumed to be available at the current time t, $0 \leqslant t \leqslant N-1$, and we want to determine the control signal for the remaining control period, $t+1-N-1$. The control sequence is determined under the additional assumption that no future measurements will be available. The first control in the determined control sequence is then applied to the system. When a new measurement is obtained the same procedure is repeated. An optimization problem from time t to time N thus has to be solved at each sampling time. The resulting controller will be a cautious controller since the variance of the estimates will influence the control signal. The OLOF controller might, however, be overly cautious because of the assumption that no further measurements will be available to correct for erroneous control actions. The properties of the OLOF controller are further discussed by Bar-Shalom and Sivan (1969) and Tse and Athans (1972). Some illustrative examples which show the difference between optimal open loop, OLOF and optimal controllers are given by Dreyfus (1964).

The estimation problem has been attacked by making different assumptions about the process. In Farison, Graham and Shelton (1967) it is assumed that the whole state vector is available, which is equivalent to assuming that the LS model (2.11) is used. In this case the estimation problem will be linear and the parameters are obtained by using a Kalman filter. The estimation of the state and the parameters will also be reduced to a linear problem if it is assumed that the A and C matrices in (2.12) are known and that only the B matrix contains unknown elements. This is equivalent to assuming that the poles of the system are known while the zeroes are unknown. This problem is treated by Tse and Athans (1972), and is generalized to both unknown poles and zeroes by Ku and Athans (1973), where the estimation is done using an extended Kalman filter. In Frost (1970) one variant is discussed where the parameters are estimated using a Bayesian approach.

4. Dual adaptive controllers

As shown in the previous sections most adaptive controllers are based on a separation between the estimation of unknown parameters and state variables from the determination of the control signal. The methods discussed so far have not considered the interaction between the identification and the control. This means that the control laws have not been designed to facilitate the identification. The control strategies have been designed in order to minimize some control error sometimes under the assumption that some parameters are uncertain. The learning procedure has not been active but 'accidental'. In a *dual controller* there is, however, an interaction between the identification and the control in the sense that the controller must compromise between a control action and a probing action (see Feldbaum 1960, 1961). The interaction is obtained by considering that the future uncertainties of the parameters are functions of the control signals applied to the system. The loss function, which has to be minimized with respect to the control signal, thus contains some information of the future observations through the statistics of the observations given the present information (Bar-Shalom and Tse 1974). The necessity of dual control has been illustrated in the previous section. For instance the one-stage cautious controller could give 'turn-off' of the control signal, and the OLOF (Open Loop Optimal Feedback) controller might be too cautious since it does not expect that more information will be obtained.

The formal solution of the dual control problem has been known for a long time (see for instance Feldbaum 1960, 1961, and Aoki 1967). The solution leads, however, to a functional equation which in most cases is difficult to solve. In order to understand the difficulties involved we will, for a moment, return to the problem discussed in Example 2.1. The optimal one-step controller was easy to determine by just minimizing the loss function (2.15). The control strategy that minimizes the loss function (2.16) is much more difficult to determine.

To generalize the example we introduce a loss function

$$V = E \sum_{s=0}^{N-1} L_s(x(s), u(s)) \tag{4.1}$$

where $x(s)$ might be the state vector or the output, compare the loss functions (2.16) and (2.17). Further introduce

$$J^*(\mathscr{P}_t, N-t) = \min_{u(t), \ldots, u(N-1)} E\left\{ \sum_{s=t}^{N-1} L_s(x(s), u(s)) \,|\, \mathscr{P}_t \right\} \tag{4.2}$$

The function $J^*(\mathscr{P}_t, N-t)$ can be interpreted as the optimal cost-to-go ($N-t$ steps) given the information state $\mathscr{P}_t$ at time t. The information state can either be the vector $\mathscr{Y}_t$ or more usually the conditional distribution of the state and the parameters. In the first case the information state is growing with time while the dimension is fixed in the second case.

The minimum of V with respect to $u(0), \ldots, u(N-1)$ is given through

$$\min_{u(0), \ldots, u(N-1)} V = J^*(\mathscr{P}_t, N)$$
$$= \min_{u(0)} E\{L_0 + \min_{u(1)} E\{L_1 + \ldots + \min_{u(N-1)} E\{L_{N-1} | \mathscr{P}_{N-1}\} | \mathscr{P}_{N-2} \cdots \} | \mathscr{P}_0\} \tag{4.3}$$

It is in essence these nested expectations and minimizations that create the large difficulties in the determination of the control sequence, $u(0), \ldots, u(N-1)$. The OLOF controller is obtained from (4.3) by formally assuming that the expectations and the minimization operations commute. This approximation will, however, not preserve the dual property.

Using the principle of optimality and dynamic programming it is possible to show that the optimal cost-to-go satisfies the functional equation

$$J^*(\mathscr{P}_t, N-t) = \min_{u(t)} E\{L_t + J^*(\mathscr{P}_{t+1}(\mathscr{P}_t, u(t)), N-t-1) | \mathscr{P}_t\} \tag{4.4}$$

In principle the optimal control problem now is solved, but the practical problems how to solve the functional equation are large. This is due to the fact that the conditional distribution of the estimated state and the estimated parameters is infinitely dimensional. Further, the control law generally is non-linear in the information state, which makes it necessary to determine the control signal from a control table. The number of variables in the control table increases with the square of the number of unknown variables (states and parameters: see Åström and Wittenmark 1971). This will soon make the problem unsolvable even on large computers. Numerical solutions of the functional equation have, however, been made for some simple examples. These solutions have given valuable insight into the properties of the optimal solution even if it is not a method that can be used for any practical problems.

There are many ways to approximate the functional equation (4.4). The approximations must, however, be done with some care in order to preserve the dual property of the optimal solution.

4.1. *Optimal dual controllers*

In this subsection we will give examples of numerical solutions of the functional equation. We will also discuss some special cases where it is possible to solve the functional equation.

4.1.1. *Numerical solutions*

There are rather few reports about numerical solutions. The explanation might be that only very simple examples can be solved due to the ' curse of dimensionality '. The numerical solutions have, however, given insight into the properties of the optimal controllers which can be useful in the construction of suboptimal dual controllers.

Florentine (1962) uses the first-order system

$$x(t+1) = x(t) + bu(t) + v(t)$$

where the gain, b, is fixed but unknown with a given *a priori* distribution. The purpose with the control is to minimize

$$E \sum_{t=0}^{N-1} (x(t+1)^2 + u(t)^2)$$

The problem is solved by discretizing $x(t)$ and $u(t)$. The functional equation is iterated backwards from the final time N. After three steps it is reported that a stationary control law (control table) is obtained as a function of the known state variable, $x(t)$, the estimate of b and the variance of the estimation error. The fast convergence to a constant control table can be explained by the fact the b is constant. The control policy could then be approximated by an analytic expression. It turned out that the approximated control law was less cautious than the controller obtained by minimizing the loss function just one step ($N=1$).

Another numerically solved problem is given by Jacobs (1967) and Jacobs and Langdon (1970). The system is given by

$$\begin{cases} x(t+1) - x(t) = u(t+1) - u(t) + v(t) \\ \qquad\qquad y(t) = x(t)^2 \end{cases}$$

The absolute value of the state variable can be measured through $y(t)$ while the sign is unknown. Introducing the probability for $x(t)$ to be positive it is possible to derive the corresponding functional equation. The solution is given as a control table which has two entries, the probability for positive x, and the absolute value of x.

Bohlin (1969) and Åström and Wittenmark (1971) have discussed in slightly different ways, a zero-order system with an unknown gain,

$$y(t) = b(t)u(t-1) + e(t)$$

The unknown gain, $b(t)$, is assumed to be described by the known stochastic process

$$b(t+1) - b_0 = a(b(t) - b_0) + v(t)$$

where a and b_0 are known constants. The purpose with the control is to minimize a loss function of the type

$$V = E \sum_{t=0}^{N-1} (y(t) - y_r)^2$$

where y_r is a desired reference value.

The solution leads again to a control table which is a function of the estimate of the gain and the variance of the estimation error. In Åström and Wittenmark (1971) 20 iterations ($N=20$) had to be done before a stationary control table was obtained.

The usefulness of the examples above is two-fold. First, they give examples of optimal control which can be used to evaluate different sub-optimal controllers. Second, at least the three last examples clearly show the dual property of the optimal control law. This can be seen by studying the control variable as a function of the variance of the unknown parameter.

If the estimate of the unknown parameter is accurate then the optimal control strategy will be about the same as the one-stage control strategy. When the variance increases there is a discontinuity in the control variable, which now makes it different from the one-stage controller. The discontinuity occurs when the controller decides that some control action must be done in order to increase the accuracy of the estimate of the unknown parameter.

4.1.2. *Special cases of dual control*

There are very few cases where it is possible to solve the functional eqn. (4.4). The assumptions that have to be done are in general very unrealistic.

In Gorman and Zaborszky (1968) and Grammaticos and Horowitz (1970) continuous and discrete time systems, respectively, are considered under the assumption that the whole state is measurable. Further it is assumed that the poles of the system are known while the zeroes are unknown. In both cases it is possible to find the solution of the functional eqn. (4.4) by solving a set of differential or difference equations corresponding to the Riccati equation in the standard linear quadratic case.

Sternby (1974), discusses a Markov chain with four states (compare Example 2.2). The transition probabilities are functions of the control signal. In that particular example it is possible to find the analytical expression for the optimal dual controller as well as for other control strategies.

4.2. *Suboptimal dual controllers*

Since it is difficult to find the optimal dual controllers much effort has been given to the problem of finding suboptimal solutions with dual properties. There have been two ways to approach this problem. First it is possible to start with a cautious controller and modify the controller in order to assure good estimation and control. The second way is to start from the opposite direction with the functional equation and try to find useful approximations.

The first approach has mainly been used for input–output models of LS type (2.11) where the control objective have been to minimize the loss function (2.15). Using only a cautious controller can give rise to 'turn-off' of the control if the unknown parameters are strongly time-varying. Several ways have been suggested to avoid the turn-off phenomenon. One way discussed by Wieslander and Wittenmark (1971), Nakamura and Nakamura (1973) and Jacobs and Patchell (1972) is to add a perturbation signal to the cautious controller. The perturbation signal, which can be a square-wave or pseudo random signal, etc., is used to excite the system in order to get good estimation. Another way to avoid turn-off is to prevent the control signal from being too small. This can be done with the following control law

$$u(t) = \begin{cases} u_c(t) & \text{if } |u_c(t)| \geqslant M \\ M & \text{if } |u_c(t)| < M \end{cases}$$

where $u_c(t)$ is the control signal given by the cautious controller (Hughes and Jacobs 1974). These methods can be regarded as passive suboptimal dual methods since there is no coupling between the accuracy of the estimates and the perturbation signal.

By changing the one-step loss function (2.15) it is possible to get active suboptimal dual controllers. In Alster and Bélanger (1974) the loss function

$$E\{(y(t+1)-y_r)^2|\mathscr{Y}_t\}$$

is minimized with respect to $u(t)$ under the additional constraint that

$$\text{tr } P^{-1}(t+1) \geqslant M$$

This gives the one-stage cautious controller when the estimates are good. When the accuracies are poor the controller makes a control action in order to get better estimates. This method does not prevent turn-off, but it will rapidly turn on the control again if a turn-off has occurred.

In Wittenmark (1975) the loss function is changed to

$$V = E\{(y(t+1)-y_r)^2 + \lambda f(P(t+2))|\mathscr{Y}_t\}$$

This modification forces the control signal to compromise between good control and good estimation. The loss function must, however, be minimized numerically. In Wittenmark (1975) there is also a comparison between dual control and different suboptimal dual controllers in a couple of simple examples.

A second way to get suboptimal dual controllers is to find approximations of the functional equation. Some care must, however, be taken in the approximations in order to maintain the dual property, compare the discussion of eqn. (4.3) above.

One way to approximate the functional equation is proposed in Alspach (1972) where the *a posteriori* distributions needed in the evaluation of the functional equation are approximated by a sum of normal distributions. The minimization of the loss function is, however, still difficult to perform.

Murphy (1968) gives an approximate dual solution in the special case when the poles of the system are known but the zeroes are unknown. In this case the estimation problem reduces to a linear problem as discussed in the previous section. The loss function is minimized by linearization around predicted states and control trajectories, under the assumption that the control law is a linear feedback from the predicted value of an augmented state vector. The minimization of the loss function is reduced to a non-linear boundary value problem. This procedure is repeated each time a new measurement is obtained.

The idea of linearization is used for more general systems by Tse *et al.* (1973), Tse and Bar-Shalom (1973) and Bar-Shalom *et al.* (1974). They consider non-linear time-varying systems with non-linear measurements (2.13). It is assumed that the estimation problem is solved by using for instance an extened Kalman filter. This will of course only give an approximation of the desired information state, which is the conditional distributions for the state and the parameters given the present information $\mathscr{Y}_t$. In order to minimize the cost-to-go over the remaining steps the loss function is linearized around a nominal trajectory. The initial conditions for the nominal trajectory are a function of the control signal that has to be chosen. The nominal trajectory can be chosen as the one given by the OLOF controller. The linearized loss function is then minimized using a search method. Since the nominal trajectory is a function of the control signal, through the initial value, a new nominal trajectory must be found and a new linearization must be done for each iteration in the search procedure. This implies that the computation times might be long, especially for large N.

In Tse *et al.* (1973) and Tse and Bar-Shalom (1973) a couple of examples are given which show the improvement of using a suboptimal dual controller compared with a controller based on enforced use of the certainty equivalence principle. The average loss as well as the standard deviation of the loss could be considerably reduced.

5. Summary

The list of references indicates that there has been a substantial research and development of adaptive controllers during the last years. This paper does not cover all the work on stochastic adaptive controllers, but hopefully the main ideas have been covered.

There are a few trends in the field of stochastic adaptive control that will be stressed in this summary :

(i) Available tools for analysis.
(ii) Better understanding of dual controllers.
(iii) Applications.

Simulations have been the main tool in the investigation of adaptive controllers and will probably remain so. By just making simulations it is, however, not possible to determine if the controller really converges to the optimal controller or if the closed loop system is stable, etc. Simulations should thus be used together with analysis. On the other hand simulations can be very useful in order to verify the theory. During the last years new tools have been available for analysing different adaptive regulators. This makes it possible to more systematically develop regulators with the desired properties. Among the tools can be mentioned the use of hyperstability concepts and Liapunov theory which have been used to develop regulators which will guarantee stability of the closed loop system (see Landau 1972, Lindorff and Carroll 1973). These methods have, however, mainly been used for deterministic systems.

Balakrishnan (1973) has proved convergence for one stochastic controller under the assumption that the parameters of the controller are fixed during long periods of time.

In Ljung and Wittenmark (1974) tools are given for the analysis of stochastic difference equations with time-varying parameters. These tools have been used to show convergence and stability for rather complex stochastic adaptive controllers. For instance it can be shown that the self-tuning regulator discussed in § 3 under weak conditions can stabilize any system.

The research in the area of dual control has given deeper understanding of the properties of dual controllers. This has been very valuable for the construction of different suboptimal dual controllers.

There are two cases where it can be very important that the regulator has the dual property. The first case is when the parameters of the system are rapidly time-varying. In that case a non-dual controller can give rise to the turn-off phenomenon. Most non-dual controllers designed for steady state control have, however, good properties when the process has constant or slowly time-varying parameters. When the time horizon is very short it is necessary that the controller quickly can get good estimates of the unknown parameters. In this case it might thus be necessary to have a dual controller. An illustrative example is given in Tse and Bar-Shalom (1973) where the system has constant parameters but the control period is only 20 steps. A controller with dual properties has for this system a performance that is superior to that of a non-dual controller.

Finally, during the last years there has been an increasing interest, for instance, in the process industry, to use adaptive controllers. The results reported for instance in Cegrell and Hedqvist (1973) and Borisson and Syding (1974) show that a more effective control can be obtained with an adaptive controller.

Acknowledgments

The author would like to express his gratitude to Professor K. J. Åström for many stimulating discussions.

The work has partly been done during a 6 months visit to Canada and the United States. The visit was made possible through economic support from The Sweden–America Foundation, Svenska Träforskningsinstitutet and the National Research Council of Canada.

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