Image Reconstruction in Computed Tomography

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Overview

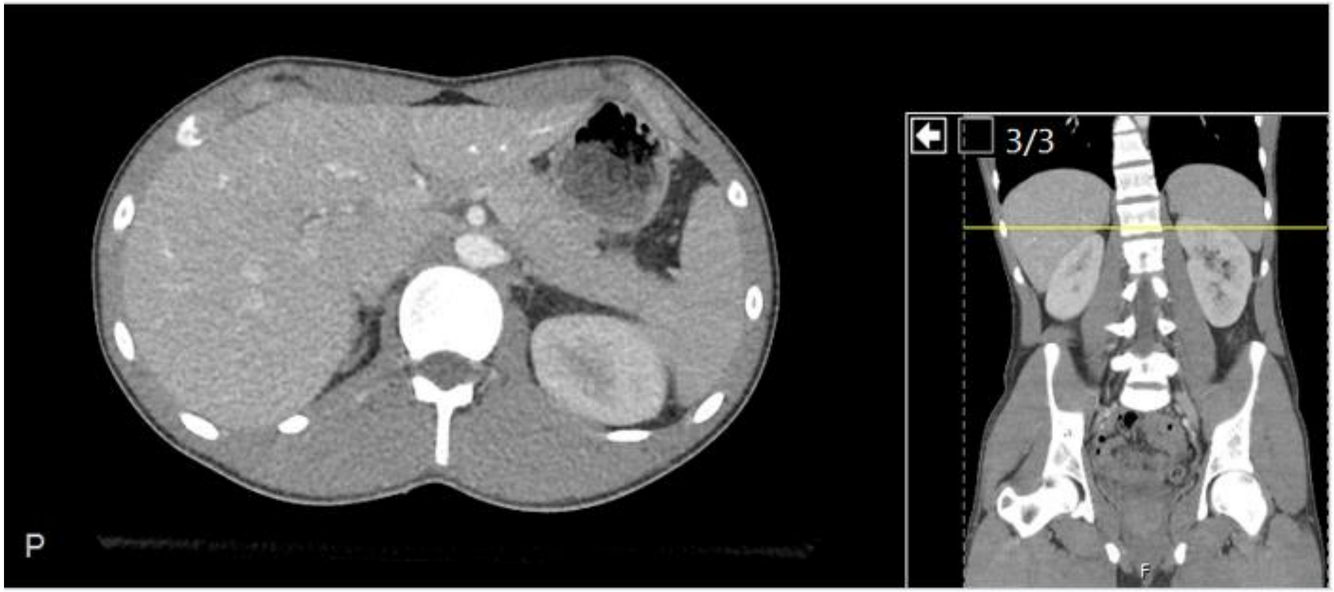
This week we introduce our first application of Fourier analysis to a vitally important real-world problem.

Computed Tomography (CT scanning) is indispensable in modern medicine. But what exactly is the nature of the “computation” here?

Image from Wikipedia

Body image

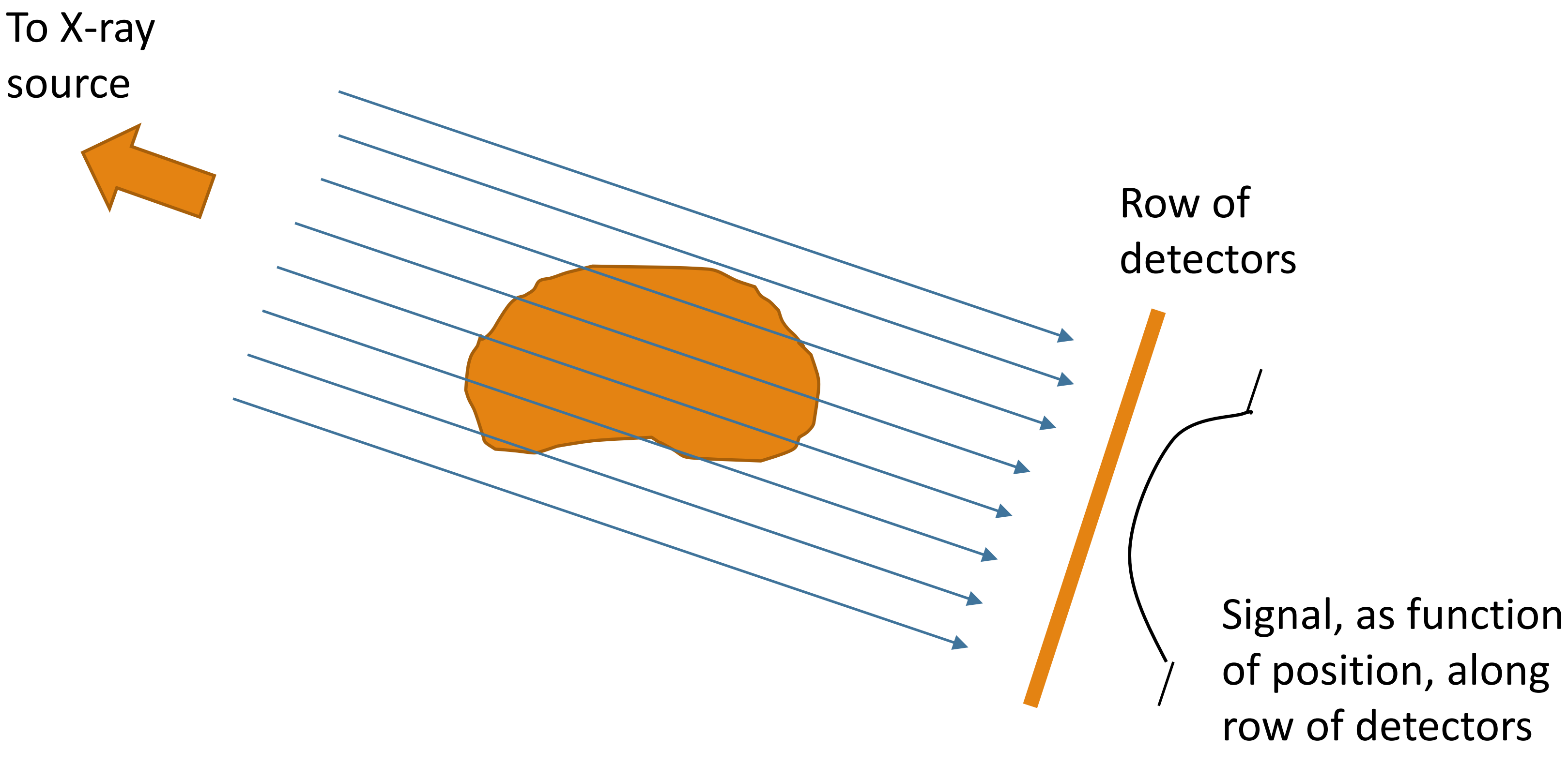
Primarily a CT scan yields an axial slice through the body, e.g. †



Modern scanners allow these "slices" to be stacked and visualized in 3D. We will focus how a single slice is reconstructed in the first place.

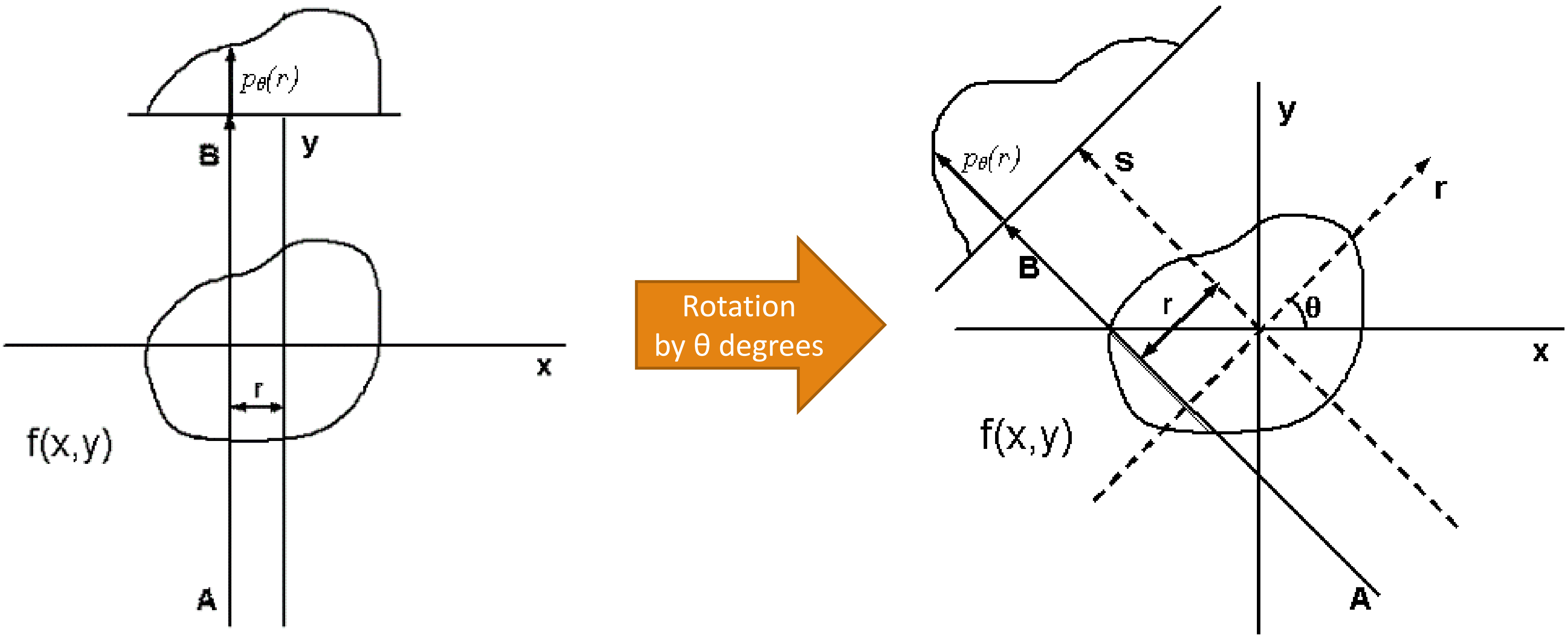
†From https://commons.wikimedia.org/wiki/Scrollable_computed_tomography_images_of_a_normal_abdomen_and_pelvis

Idealized CT with parallel rays



Beam Geometry

- Parameter r: the detector number (offset) receiving the signal
- Parameter θ: the angle of the scanner.

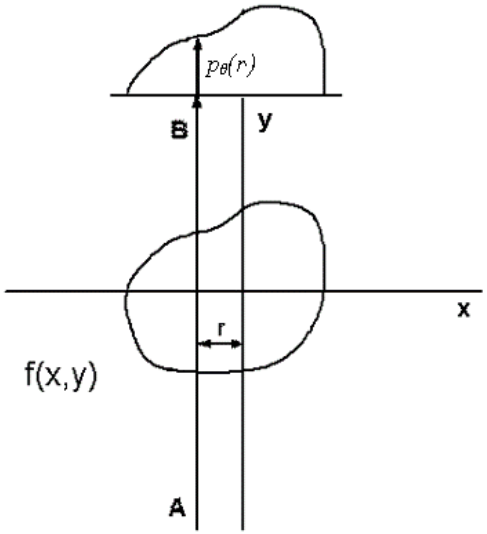


†From https://en.wikipedia.org/wiki/Tomographic_reconstruction

Attenuation of Rays

A single choice of angle for X-ray direction, and the perpendicular detector row, gives a single *one-dimensional* "X-ray slice" of the two-dimensional body slice of interest (which in turns forms part of a 3d body…)

The strength of X-rays arriving at any given detector depends on the sum of densities at all points along the path of the ray, which we can write mathematically as:



$$Signal\ Strength = \sum_{x=A}^{B} f(x, y)$$

where $f(x, y)$ is the density of the "object" as a two-dimensional function of position. In the integral, x, y are points on the ray (so along line A to B).

In the real world, function $f(x, y)$ is unknown. But the strength of signals can be measured by detectors.

The Radon Transform

Each X-ray beam is associated with the offset r and the angle θ.

So if we rotate the whole assembly of X-ray source and detectors around the (human?) subject in the middle, a slice of whom has unknown density $f(x, y)$, we build up a certain function:

$$\boldsymbol{p_\theta}(\boldsymbol{r}) = Strength\ of\ a\ signal\ with\ offset\ \boldsymbol{r}\ and\ angle\ \boldsymbol{\theta}$$

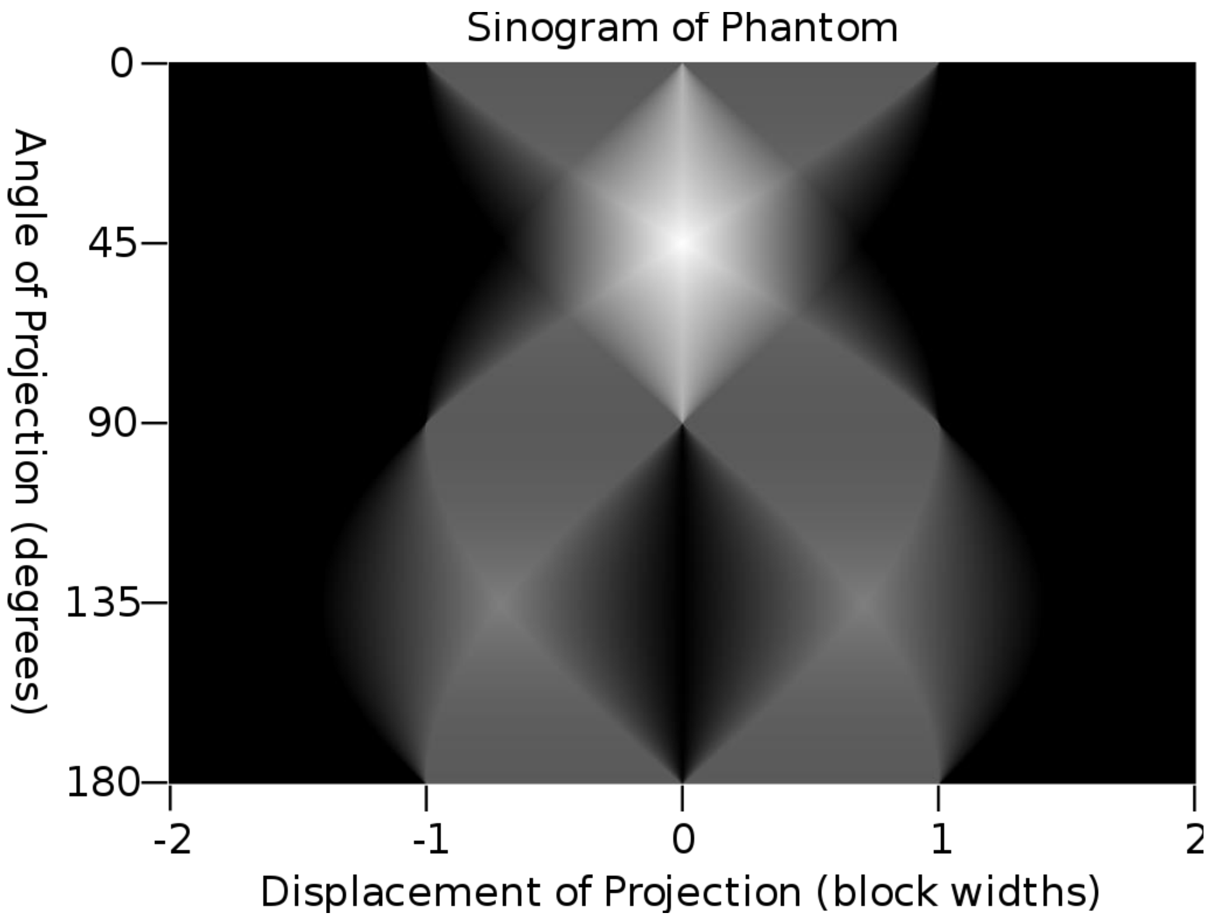
where θ is the angle of the assembly and r labels a particular detector, based on it's offset from the centre of the row of detectors.

The value of $p_\theta(r)$ can be extracted directly from the signal value recorded by the detector, and can also be mathematically related to the function $f(x, y)$ by the line integral on the previous slide.

The Sinogram

The *Sinogram* is a way of presenting the raw output of a CT scan.

Essentially it just plots the value of the Radon transform as a function of r and θ, using a grey-scale to represent the values of $p_\theta(r)$.



Can we get $f(x, y)$?

The problem of image reconstruction is to find the density of the original body as a function of position x, y. This 2d function is essentially the slice image we desire.

The measurable Radon transform is mathematically some integral or sum over the desired function $f(x, y)$, and is itself a function of two variables r and θ.

The question is whether given $p_\theta(r)$ can we reconstruct the unknown density $f(x, y)$?

It is somewhat akin to Fourier transforms where $\tilde{X}(k, l)$ is some sum or integral of an original function $X(m, n)$. Can we invert the Radon transform in the same way we invert $\tilde{X}(k, l)$ to recover the original function?

The Projection-Slice Theorem

The answer of course is yes – pretty well.

The solution hinges on a little theorem about Fourier Transforms called the Projection-Slice Theorem.

Suppose, for any fixed value of θ, we take the 1d Fourier Transform of the Radon transform function $p_\theta(r)$, treated as a function of the spatial variable *r*. We may call the transformed function $\tilde{p}_\theta(K)$, where *K* is a wave number representing frequency of change with respect to *r*.

The Projection-Slice Theorem says that $\tilde{p}_\theta(K)$ is exactly the same as the value of the two-dimensional Fourier Transform of $f(x, y)$, which we may call $\tilde{f}(k, l)$, evaluated at a point *k*, *l* in wave number space, at distance *K* from the origin along a line at angle θ from *k* axis.

- $\tilde{p}_\theta(K) = \tilde{f}(k, l)$

†From https://upload.wikimedia.org/wikipedia/commons/9/97/Radon_transform_via_Fourier_transform.png

Projection Slice Theorem

Creating $p_\theta(r)$ using detectors

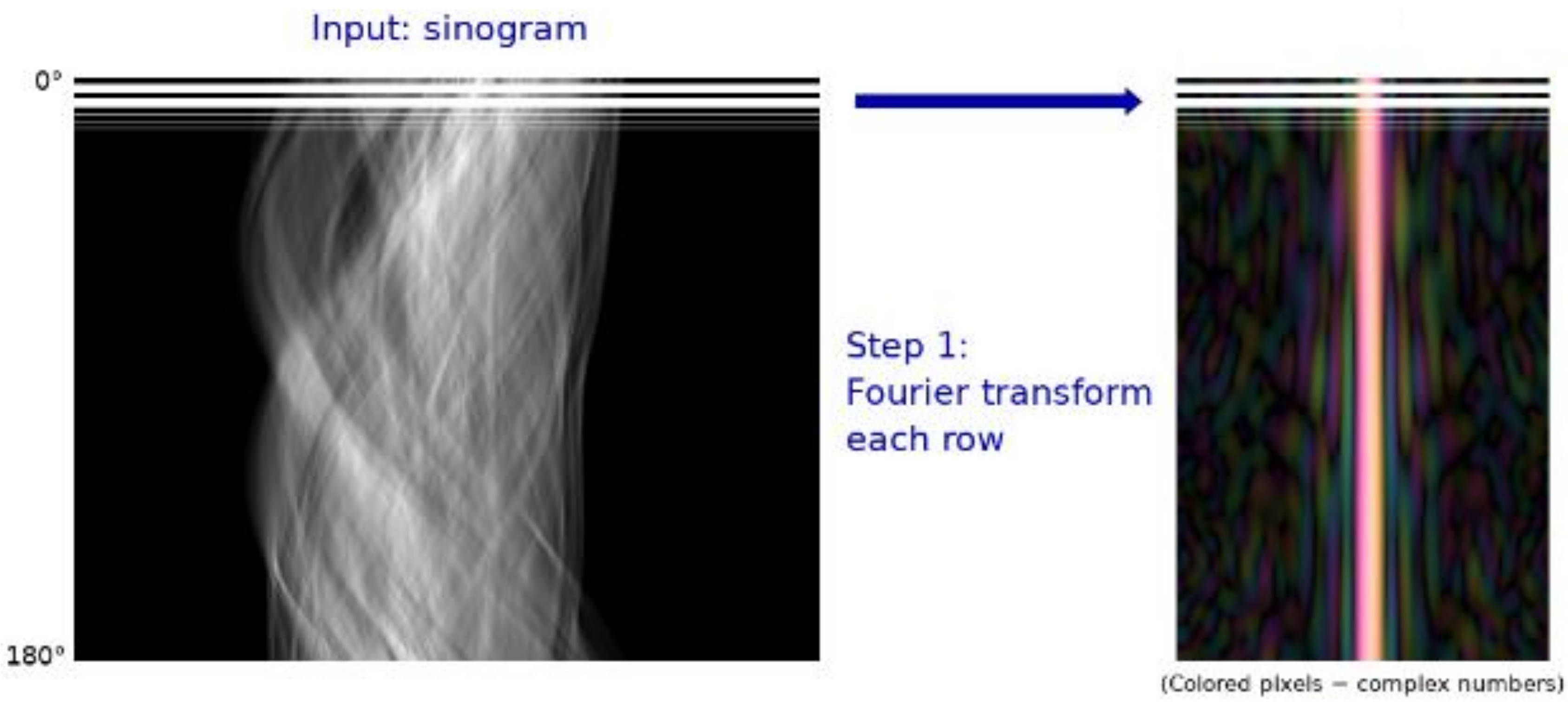
$\Big\downarrow$ 1D FFT

$$\tilde{p}_\theta(K) = \tilde{f}(k, l)$$

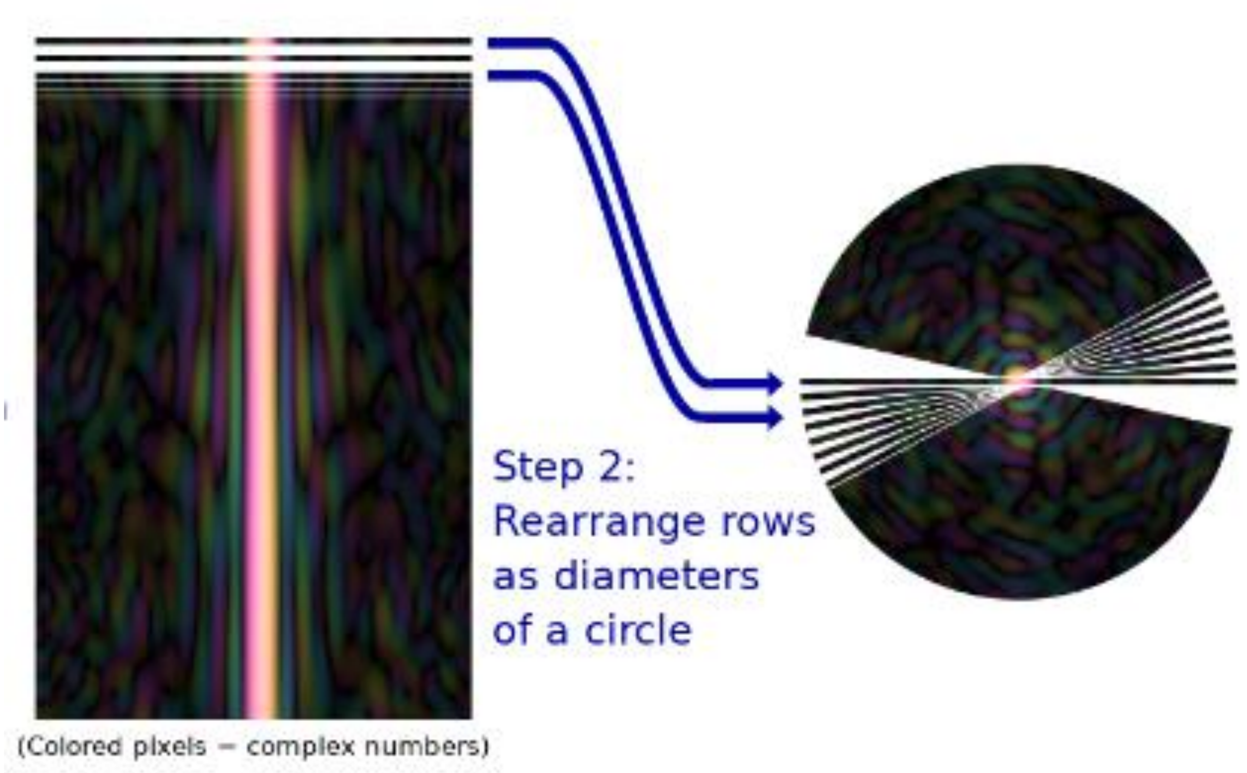
$\Big\downarrow$ 2D IFFT

Density function $f(x, y)$

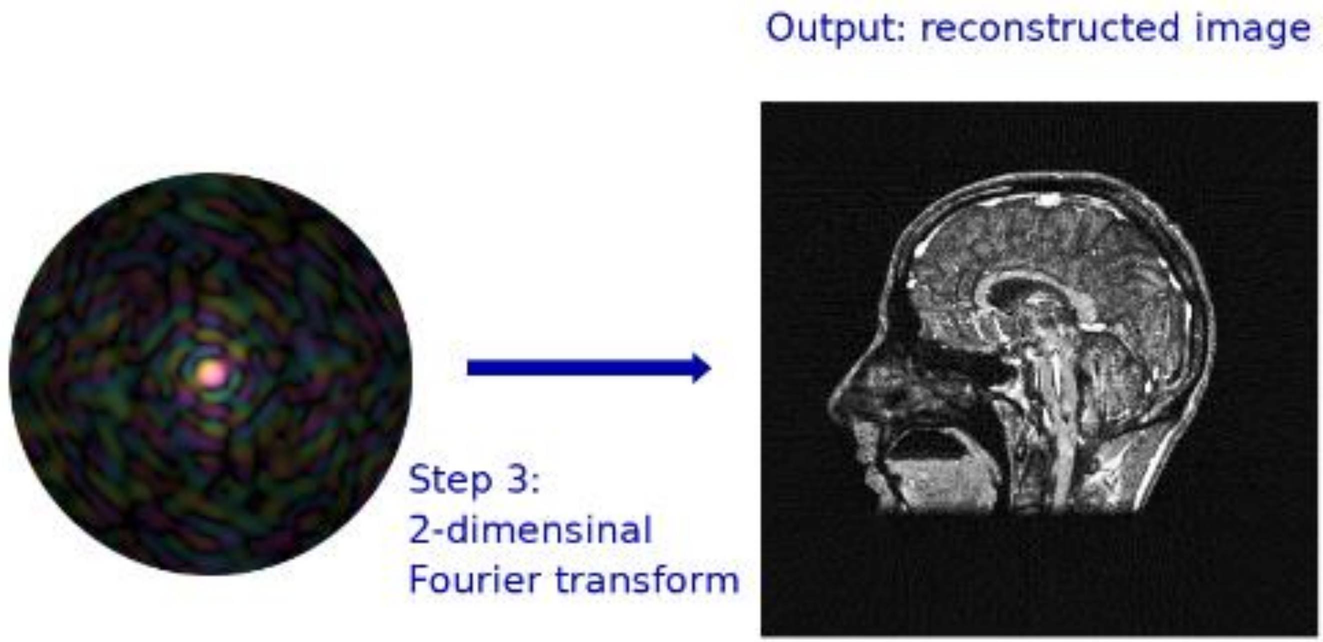
Theoretical Reconstruction I



Theoretical Reconstruction II



Theoretical Reconstruction II



Reconstruction in Practice

Sequence of pictures on previous 3 slides is correct mathematically. BTW I have butchered the images from:
https://upload.wikimedia.org/wikipedia/commons/9/97/Radon_transform_via_Fourier_transform.png

Unfortunately the last step is not as easy to implement numerically as one might hope.

The obvious idea would be to use an inverse 2D FFT to reconstruct the image. But in the real discretized data the "known" FT is on some "polar" grid of K, θ, points, and interpolating that to a rectangular *k*, *l* grid for FFT is problematic.

We need to use some more mathematics to simplify the inverse FT.

Back Projection

Let's calculate the 2D inverse FT of $\tilde{f}(k,l)$ to obtain the density function $f(x,y)$:

$$f(x,y) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{f}(k,l) e^{i(kx+ly)}$$

Replace/interpolate $\tilde{f}(k,l)$ with $\tilde{p}_\theta(K)$ and apply 1D FT on each line:

$$f(x,y) = \sum_{j=0}^{N-1} \frac{\pi}{N} p_{\theta_j}(\cos(\theta_j)\, x + \sin(\theta_j)\, y)$$

where $\theta_j = \pi j/N$.

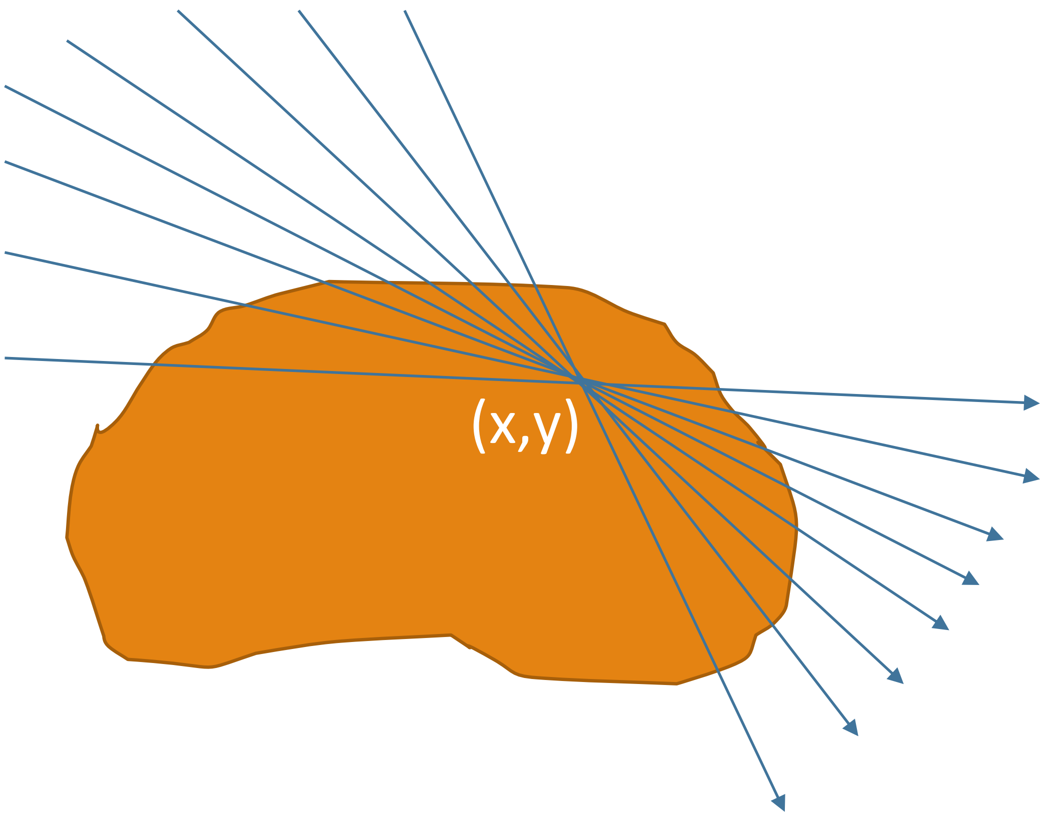
That last sum is called Back Projection that converts the original Radon transform $p_\theta(r)$ to the density function $f(x,y)$.

Physical Interpretation

With some geometry and trigonometry, referring to slide 5, can show that the rays with $r = x\cos(\theta) + y\sin(\theta)$ are the rays that pass through the point (x, y).

So "pure" back projection corresponds to simply adding together signals for all rays that pass through this point.

Intuitively plausible way of estimating the density at (x,y).



Filtered Sinogram

In practice pure back projection, without any filtering of the Radon transform, will often yield the broadly recognizable form of the image – albeit very blurred.

To get a much better quality result in practice, we have to first apply a filter, like the $|K|$ filter discussed in Lab 1, to the Radon transform $p_\theta(r)$. The filtered Radon Transform is called $p_\theta^{filtered}(r)$.

Steps to obtain $p_\theta^{filtered}(r)$ from $p_\theta(r)$:

1. Do an 1D FFT to obtain $\tilde{p}_\theta(K)$ which is a function of integer K
2. Multiply $\tilde{p}_\theta(K)$ by $|K|$, like what we did it lab 1
3. Apply inverse 1D FFT to the each row of $\tilde{p}_\theta(K)$ to obtain $p_\theta^{filtered}(r)$

Filtered Back Projection

In slide 16, we used Back Projection that converts the pure Radon transform $p_\theta(r)$ to the density function $f(x, y)$.

Let's replace the $p_\theta(r)$ with $p_\theta^{filtered}(r)$:

$$f(x, y) = \sum_{j=0}^{N-1} \frac{\pi}{N} p_{\theta_j}^{filtered}(\cos(\theta_j)\, x + \sin(\theta_j)\, y)$$

where $\theta_j = \pi j / N$.

The resulting sum is called *Filtered Back Projection*.

Practical Details

The pure $|K|$ filter is a kind of *high-pass filter*, and can lead to noisy images. In practice it is common to apply some kind of cut-off for high wave numbers to reduce the noise (again similar to what we have done in earlier labs for simple image processing).