A Scale-Invariant Ruler for Black Holes: From Stellar-Mass to Ultra-Massive with Unified Uncertainties

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Abstract

We present the *Black Hole Ruler*, a scale-invariant framework that maps astrophysical black holes onto a common set of gravitational units and orbital landmarks. The ruler uses mass M (and, where available, spin a_*) to compute a baseline triplet—gravitational time $t_g = GM/c^3$, Schwarzschild radius $r_s = 2GM/c^2$, and ISCO frequency $f_{\rm ISCO}$—and exploits the mass-invariant product $f_{\rm ISCO}\, t_g = 1/(6^{3/2}2\pi)$ as a universal sanity check. Deviations from the Schwarzschild baseline are modeled via Kerr spin corrections, and accretion physics is incorporated through a dual prescription that makes model dependence explicit: an efficiency bridge ($L = \eta_{\rm eff}\dot{M}c^2$) versus an ADAF/RIAF branch ($\lambda_{\rm Edd} = \kappa\dot{m}^2$). The ruler is extended with environmental context (sphere of influence $r_{\rm infl} = GM/\sigma^2$) and a tidal-disruption module with a logistic capture boundary. Case studies spanning eight orders of magnitude in mass—Cygnus X-1 (XRB), Sgr A* (quiescent SMBH), and M87* (LLAGN)—demonstrate predictive and diagnostic power: spin-aware ISCO shifts, horizon magnetic fields, and Blandford–Znajek jet powers that bracket observations without fine-tuning.

1 Introduction

Black holes span ~ 10 orders of magnitude in mass, yet observations are fragmented across wavelengths and techniques. We aim to unify these regimes with a minimal, Kerr-based template and explicit uncertainty propagation. Prior work includes horizon-scale imaging, gravitational-wave catalogs, maser dynamics, and reverberation mapping. Our contribution is a scale-invariant synthesis with a dual-branch accretion prescription and a transparent catalog schema for derived quantities.

2 The Black Hole Ruler

2.1 Gravitational units and baseline invariant

We adopt units $G = c = 1$ for derivations, restoring constants in reported values. The core definitions are

$$t_g = GM/c^3, \qquad r_g = GM/c^2, \qquad r_s = 2r_g, \tag{1}$$

$$r_{\rm ISCO}^{\rm Schw} = 6r_g, \qquad f_{\rm ISCO}^{\rm Schw} = \frac{c^3}{6^{3/2}2\pi GM}, \qquad f_{\rm ISCO}t_g = \frac{1}{6^{3/2}2\pi}\ (\approx 0.01083). \tag{2}$$

2.2 Kerr spin corrections

Using the Bardeen–Press–Teukolsky expressions, we compute $r_{\rm ISCO}(a_*)$ and

$$f_{\rm ISCO}(a_*) = \frac{c^3}{2\pi GM}\,\frac{1}{r_{\rm ISCO}^{3/2} + a_*} \tag{3}$$

for prograde/retrograde branches. Spin moves sources along a known one-parameter family relative to the Schwarzschild baseline.

3 Accretion Prescriptions and Jet Power

3.1 Efficiency bridge vs. ADAF/RIAF

Efficiency bridge: $\dot m = \lambda_{\rm Edd}/\eta_{\rm eff}$, $B_H \propto \dot m^{1/2} M^{-1/2}$, $P_{\rm BZ} \propto a_*^2(\lambda_{\rm Edd}/\eta_{\rm eff})M$.

ADAF/RIAF: $\lambda_{\rm Edd} = \kappa \dot m^2 \Rightarrow \dot m = (\lambda_{\rm Edd}/\kappa)^{1/2}$, hence $B_H \propto (\lambda_{\rm Edd}/\kappa)^{1/4} M^{-1/2}$, $P_{\rm BZ} \propto a_*^2(\lambda_{\rm Edd}/\kappa)^{1/2}M$. We provide both branches and propagate uncertainties in log-space.

3.2 Blandford–Znajek scaling

We estimate

$$P_{\rm BZ} \approx 10^{45}\ \mathrm{erg\ s^{-1}}\ \left(\frac{a_*}{0.9}\right)^2 \left(\frac{B_H}{10^4\ \mathrm{G}}\right)^2 \left(\frac{M}{10^9 M_\odot}\right)^2 \tag{4}$$

and report ranges based on the branch selection, guided by polarimetry and variability near $\sim 10\,t_g$.

4 Environmental Context and TDE Module

4.1 Sphere of influence and morphology

We compute $r_{\rm infl} = GM/\sigma^2$ and $r_{\rm infl}/R_e$ with a three-tier σ acquisition strategy (IFU/long-slit; dynamical fallback with k-factor provenance; scaling fallback with morphology warnings). Errors are propagated via

$$\frac{\delta r_{\rm infl}}{r_{\rm infl}} = \sqrt{\left(\frac{\delta M}{M}\right)^2 + \left(2\frac{\delta\sigma}{\sigma}\right)^2}, \tag{5}$$

$$\frac{\delta(r_{\rm infl}/R_e)}{(r_{\rm infl}/R_e)} = \sqrt{\left(\frac{\delta r_{\rm infl}}{r_{\rm infl}}\right)^2 + \left(\frac{\delta R_e}{R_e}\right)^2}. \tag{6}$$

4.2 TDE critical mass and rates

We set a spin-aware disruption boundary with a logistic capture factor $S(M; a_*)$ and model per-galaxy rates $\Gamma_{\rm gal} = \Gamma_0(M/10^6 M_\odot)^\alpha(\rho_\star/\rho_0)^\beta(\sigma/\sigma_0)^\gamma S$. A hierarchical fit to current TDE samples can calibrate $(\Gamma_0, \alpha, \beta, \gamma)$; volumetric rates follow by convolving with host demographics.

5 Case Studies

5.1 Cygnus X-1 (stellar XRB)

High spin ($a_* \gtrsim 0.9$) compresses $r_{\rm ISCO}$ and approximately doubles the ISCO tone relative to Schwarzschild; inferred B_H and $P_{\rm BZ} \sim 10^{36\text{–}37}$ erg s^{-1} align with microquasar jets.

5.2 Sgr A* (quiescent SMBH)

Extremely low $\lambda_{\rm Edd}$ with RIAF-like efficiency boosts B_H moderately while keeping $P_{\rm BZ}$ small, consistent with weak radio emission. Sphere of influence $\sim$ 1–2 pc; TDEs possible.

5.3 M87* (LLAGN with jet)

MAD/RIAF branch yields $P_{\rm BZ} \sim 10^{44\pm1}$ erg s^{-1}, bracketing jet energetics; TDEs suppressed by direct capture. $r_{\rm infl} \sim 0.2$–0.27 kpc (3–4% of R_e).

6 Results: Cross-Scale Atlas and Invariants

We verify the baseline invariant $f_{\rm ISCO}t_g$ across a 10-object atlas and a 2024–2025 cross-scale set (Gaia BH3, GW231123 remnant, ω Cen IMBH candidate, CEERS-1019, J0529-4351). Spin-aware frequencies are reported where credible; ADAF/bridge jet-power ranges accompany SMBH entries.

7 Uncertainties and Reproducibility

Uncertainties are propagated analytically in log form. Automated QC flags include M–σ outliers, aperture sanity, and morphology warnings. The accompanying tables follow a consistent schema with per-entry provenance and quality grades (Gold/Silver/Bronze).

8 Discussion and Outlook

The ruler enables instrument matching, variability rescaling, imaging forecasts, and growth tests from XRBs to UMBHs. Future work: full spin posteriors via a hierarchical combiner; expanded environment metrics (gas content, nuclear profiles); calibrated TDE rates; and a $\sim$ 50-object catalog advancing toward a community standard.

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