BHRuler Equations

Stephen L. Nagy
Independent Researcher

September 2025
Version 1.0.0

Introduction

This note lists the exact equations implemented by **BHRuler** (as in `bhruler.py`), and how they map to the output columns.

Notation and Constants

Symbols:

- M: black-hole mass
- $a_* \in [-1, 1]$: dimensionless spin (prograde > 0, retrograde < 0)
- $\lambda_{\mathrm{Edd}} = L_{\mathrm{bol}}/L_{\mathrm{Edd}}$: Eddington ratio (user input)
- η_{eff}: effective radiative efficiency (user input for η-bridge)
- κ: ADAF/RIAF proportionality ($\lambda_{\mathrm{Edd}} = \kappa \dot{m}^2$, user input)
- σ: stellar velocity dispersion (km s^{-1})
- R_e: galaxy effective radius (kpc)

Physical Constants (SI)

G	$6.67430 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$
c	$2.99792458 \times 10^{8}\ \mathrm{m\,s^{-1}}$
$M_\odot$	$1.98847 \times 10^{30}\ \mathrm{kg}$
$1\,\mathrm{pc}$	$3.09 \times 10^{16}\,\mathrm{m}$

1 Core Gravitational Scalings

Gravitational time/length:

$$t_g = \frac{GM}{c^3}, \qquad r_g = \frac{GM}{c^2}, \qquad r_s = 2r_g = \frac{2GM}{c^2}.$$

Schwarzschild ISCO baseline:

$$r_{\rm ISCO}^{\rm Schw} = 6r_g, \qquad f_{\rm ISCO}^{\rm Schw} = \frac{c^3}{6^{3/2} 2\pi GM}.$$

Mass-invariant sanity check (dimensionless):

$$f_{\rm ISCO}^{\rm Schw} t_g = \frac{1}{6^{3/2} 2\pi} \approx 0.01083.$$

2 Kerr Spin Corrections (Equatorial Orbits)

Bardeen–Press–Teukolsky ISCO radius (in r_g units):

$$\begin{aligned}
Z_1 &= 1 + (1 - a_*^2)^{1/3} \left[(1 + a_*)^{1/3} + (1 - a_*)^{1/3}\right], \\
Z_2 &= \sqrt{3a_*^2 + Z_1^2}, \\
r_{\rm ISCO}(a_*) &= \begin{cases} 3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} & \text{(prograde)} \\ 3 + Z_2 + \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} & \text{(retrograde)} \end{cases}
\end{aligned}$$

ISCO frequency (explicit branch sign):

$$f_{\rm ISCO}(a_*) = \frac{c^3}{2\pi GM} \times \begin{cases} \dfrac{1}{r_{\rm ISCO}^{3/2}(a_*) + a_*}, & \text{prograde} \\ \dfrac{1}{r_{\rm ISCO}^{3/2}(a_*) - a_*}, & \text{retrograde} \end{cases}$$

Diagnostic ratio:

$$\frac{f_{\rm ISCO}^{\rm Kerr}}{f_{\rm ISCO}^{\rm Schw}}.$$

3 Accretion Prescriptions (Toggle)

(a) η-bridge (efficient/thin-disk–like)

Dimensionless accretion rate and horizon magnetic field:

$$\dot{m} = \frac{\lambda_{\rm Edd}}{\eta_{\rm eff}}, \qquad B_H \simeq 4 \times 10^4 \text{ G } \dot{m}^{1/2} \left(\frac{10^9 M_\odot}{M}\right)^{1/2}.$$

Normalization is MAD-inspired (order-of-magnitude); the key physics is the scaling $B_H \propto \dot{m}^{1/2} M^{-1/2}$.

(b) ADAF/RIAF

$$\lambda_{\rm Edd} = \kappa \dot m^2 \;\Rightarrow\; \dot m = \sqrt{\frac{\lambda_{\rm Edd}}{\kappa}}, \qquad B_H \simeq 4\times10^4~{\rm G}\,\dot m^{1/2}\left(\frac{10^9 M_\odot}{M}\right)^{1/2}.$$

(c) Blandford–Znajek jet power (order-of-magnitude)

$$P_{\rm BZ} \approx 10^{45}~{\rm erg\,s^{-1}}\left(\frac{a_*}{0.9}\right)^2\left(\frac{B_H}{10^4\,{\rm G}}\right)^2\left(\frac{M}{10^9 M_\odot}\right)^2.$$

4 Environment and Host Coupling

Sphere of influence and host-normalized ratio:

$$r_{\rm infl} = \frac{GM}{\sigma^2}, \qquad \frac{r_{\rm infl}}{R_e}.$$

(If R_e is not provided, only $r_{\rm infl}$ is reported.)

5 Tidal-Disruption (TDE) Scalings

Fallback time (solar-type star):

$$t_{\rm fb} \approx 41~{\rm d}\left(\frac{M}{10^6 M_\odot}\right)^{1/2}\left(\frac{R_*}{R_\odot}\right)^{3/2}\left(\frac{M_*}{M_\odot}\right)^{-1}.$$

Spin-aware logistic capture boundary (for the TDE flag):

$$S(M; a_*) = \frac{1}{1+\exp\left(-\frac{\log_{10} M - \log_{10} M_{\rm crit}(a_*)}{w}\right)}, \qquad M_{\rm crit}(a_*) = M_{\rm crit,0}[1-0.6\,a_*],$$

with defaults $M_{\rm crit,0} = 3\times10^7 M_\odot$ and $w = 0.15$ dex. We label `tde_possible = true` when $S < 0.5$.

6 Output Column Mapping (from the scripts)

Core & spin: `t_g_s`, `r_s_km`, `f_ISCO_Schw_Hz`, `r_ISCO_rg`, `f_ISCO_Kerr_Hz`, `f_ratio`
Accretion / jets (per branch): `B_H_G_eta`, `P_BZ_erg_s_eta` and/or `B_H_G_adaf`, `P_BZ_erg_s_adaf`
Environment: `sigma_kms`, `Re_kpc`, `rinfl_pc`, `rinfl_over_Re`
TDE: `t_fb_days`, `tde_possible`

7 Built-in Validation Checks

- **Invariant:** $f_{\rm ISCO}^{\rm Schw} t_g \approx 0.01083$ (flat in M).
- **Spin effect:** $f_{\rm ISCO}^{\rm Kerr}/f_{\rm ISCO}^{\rm Schw}$ increases monotonically with prograde a_*.
- **Unit sanity:** $r_s = 2r_g$; $r_{\rm infl} \propto M/\sigma^2$.

References (Core Methods)

- Bardeen, Press & Teukolsky (1972) — Kerr geodesics/ISCO
- Blandford & Znajek (1977) — Jet power extraction
- GRAVITY Collaboration (2020) — S2 precession, Sgr A* dynamics
- EHT Collaboration (2019, 2021, 2022) — M87*, Sgr A* imaging and polarization