Proposition 1. *A PiePair is defined on the original data stream S, and symbolic regular expressions (SREs) are defined on the corresponding predicate-based stream $S_{\varphi,\psi}$, derived by applying the mapping function $\mathcal{F}_{\varphi,\psi}$. The relationship between PiePairs and SREs, based on precise temporal interval relations, is as follows:*

- ***Sufficiency:*** *If a sequence $sie \in \mathcal{L}(E(r(pie_\varphi, pie_\psi)))$ exists in $S_{\varphi,\psi}$, then the PiePair $r(pie^i_\varphi, pie^j_\psi)$ exists in S.*
- ***Necessity:*** *Conversely, if the PiePair $r(pie^i_\varphi, pie^j_\psi)$ exists in S, then a sequence $sie \in \mathcal{L}(E(r(pie_\varphi, pie_\psi)))$ must exist in $S_{\varphi,\psi}$.*

In other words, the existence of $r(pie^i_\varphi, pie^j_\psi)$ in S and the existence of $sie \in \mathcal{L}(E(r(pie_\varphi, pie_\psi)))$ in $S_{\varphi,\psi}$ are logically equivalent.

A Proof of Proposition 1

Proof. We prove the two directions of the equivalence.

Sufficiency: Below, we provide the sufficiency proofs for each case where $r \in \{$ followed-by, meets, overlaps, starts, during, finishes, equals$\}$. The proofs of the corresponding inverse relationship can be obtained by appropriately converting the predicates.

- followed-by: Let r be followed-by, and suppose that $sie \in \mathcal{L}(\varphi^+ \cdot (\neg\varphi \wedge \neg\psi)^+ \cdot \psi^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie'' \cdot ie'''$, where $ie' \in \mathcal{L}(\varphi^+)$, $ie'' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$, and $ie''' \in \mathcal{L}(\psi^+)$. Furthermore, the intervals ie', ie'', and ie''' correspond to the interval events ie^1, ie^2, and ie^3 in the original data stream S, where the point events within ie^1, ie^2, and ie^3 satisfy the predicates φ, $\neg\varphi \wedge \neg\psi$, and ψ, respectively.
  Since ie^1 satisfies φ and ie^2 satisfies $\neg\varphi$, there exists a PIE pie^1_φ such that $pie^1_\varphi.ts \leq ie^1.ts$ and $pie^1_\varphi.te = ie^1.te$. Similarly, since ie^3 satisfies ψ and ie^2 satisfies $\neg\psi$, there exists a PIE pie^2_ψ such that $pie^2_\psi.ts = ie^3.ts$ and $pie^2_\psi.te \geq ie^3.te$.
  Moreover, since ie^2 satisfies $\neg\varphi \wedge \neg\psi$ within the interval from $pie^1_\varphi.te$ to $pie^2_\psi.ts$, there is no point event pe such that $pe.p$ satisfies φ or ψ. Thus, pie^1_φ and pie^2_ψ satisfy the predicate followed-by, i.e, followed-by$(pie^1_\varphi, pie^2_\psi)$.
- meets: Let r be meets, and suppose that $sie \in \mathcal{L}((\varphi \wedge \neg\psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie''$, where $ie' \in \mathcal{L}((\varphi \wedge \neg\psi)^+)$ and $ie'' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$.
  Moreover, the intervals ie' and ie'' correspond to the interval events ie^1 and ie^2 in the original data stream S, where the point events within ie^1 satisfy $\varphi \wedge \neg\psi$, and the point events within ie^2 satisfy $\neg\varphi \wedge \psi$.
  Since $ie^1.te = ie^2.ts$, there exists a PIE pie^1_φ such that $pie^1_\varphi.ts \leq ie^1.ts$ and $pie^1_\varphi.te = ie^1.te$. Similarly, there exists a PIE pie^2_ψ such that $pie^2_\psi.ts = ie^2.ts$ and $pie^2_\psi.te \geq ie^2.te$.
  In conclusion , pie^1_φ and pie^2_ψ satisfy the meets relation, i.e., meets$(pie^1_\varphi, pie^2_\psi)$.

- overlaps: Let r be overlaps, and suppose that $sie \in \mathcal{L}((\varphi \wedge \neg\psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie'' \cdot ie'''$, where $ie' \in \mathcal{L}((\varphi \wedge \neg\psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, and $ie''' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$. Furthermore, the intervals ie', ie'', and ie''' correspond to the interval events ie^1, ie^2, and ie^3 in the original data stream S, where the point events within ie^1, ie^2, and ie^3 satisfy the predicates $\varphi \wedge \neg\psi$, $\varphi \wedge \psi$, and $\neg\varphi \wedge \psi$, respectively.
  Since the point events in ie^1 and ie^2 satisfy φ, and $ie^1.te = ie^2.ts$, there exists a PIE pie^1_φ that $pie^1_\varphi.ts \leq ie^1.ts$ and $pie^1_\varphi.te = ie^2.te$.
  Similarly, since the point events in ie^2 and ie^3 satisfy ψ, and $ie^2.te = ie^3.ts$, there exists a PIE pie^2_ψ that $pie^2_\psi.ts = ie^2.ts$ and $pie^2_\psi.te \geq ie^3.te$.
  In conclusion, pie^1_φ and pie^2_ψ satisfy overlaps($pie^1_\varphi, pie^2_\psi$).
- starts: Let r be starts, and suppose that $sie \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie'' \cdot ie'''$, where $ie' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, and $ie''' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$. Furthermore, the intervals ie', ie'', and ie''' correspond to the interval events ie^1, ie^2, and ie^3 in the original data stream S, where the point events within ie^1, ie^2, and ie^3 satisfy the predicates $\neg\varphi \wedge \neg\psi$, $\varphi \wedge \psi$, and $\neg\varphi \wedge \psi$, respectively.
  Since the point events in ie^2 satisfy φ, and both ie^1 and ie^3 satisfy $\neg\varphi$, there exists a PIE pie^1_φ that corresponds to ie^2, with $pie^1_\varphi.ts = ie^2.ts$ and $pie^1_\varphi.te = ie^2.te$.
  Additionally, since all point events in ie^2 and ie^3 satisfy ψ, and the point events in ie^1 satisfy $\neg\psi$, there exists a PIE pie^2_ψ such that $pie^2_\psi.ts = ie^2.ts$ and $pie^2_\psi.te \geq ie^3.te$.
  Finally, since pie^1_φ and pie^2_ψ share the same starting time, and $pie^1_\varphi.te = ie^2.te < pie^2_\psi.te$, the pair $pie^1_\varphi, pie^2_\psi$ satisfies starts($pie^1_\varphi, pie^2_\psi$).
- during: Let r be during, and suppose that $sie \in \mathcal{L}((\neg\varphi \wedge \psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie'' \cdot ie'''$, where $ie' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, and $ie''' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$. Moreover, the intervals ie', ie'', and ie''' correspond to the interval events ie^1, ie^2, and ie^3 in the original data stream S, respectively. The point events in ie^1 and ie^3 satisfy $\neg\varphi \wedge \psi$, while those in ie^2 satisfy $\varphi \wedge \psi$. Since the point events in ie^2 satisfy φ, and both ie^1 and ie^3 satisfy $\neg\varphi$, there exists a PIE pie^1_φ that corresponds to ie^2, with $pie^1_\varphi.ts = ie^2.ts$ and $pie^1_\varphi.te = ie^2.te$. Additionally, since all point events in ie^1, ie^2, ie^3 satisfy ψ , there exists a PIE pie^2_ψ such that $pie^2_\psi.ts \leq ie^1.ts$ and $pie^2_\psi.te \geq ie^3.te$.
  Therefore, pie^1_φ and pie^2_ψ satisfy the during relation, i.e., during($pie^1_\varphi, pie^2_\psi$).
- finishes: Let r be finishes, and suppose that $sie \in \mathcal{L}((\neg\varphi \wedge \psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \neg\psi)^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie'' \cdot ie'''$, where $ie' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, and $ie''' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$. Furthermore, the intervals ie', ie'', and ie''' correspond to the interval events ie^1, ie^2, and ie^3 in the original data stream S, respectively. Within ie^1, the point events satisfy $\neg\varphi \wedge \psi$; within ie^2, the point events satisfy $\varphi \wedge \psi$; and within ie^3, the point events satisfy $\neg\varphi \wedge \neg\psi$.

Since the point events in ie^2 satisfy φ, and both ie^1 and ie^3 satisfy $\neg\varphi$, there exists a PIE pie^1_φ that corresponds to ie^2, with $pie^1_\varphi.ts = ie^2.ts$ and $pie^1_\varphi.te = ie^2.te$.

Additionally, since all point events in ie^1 and ie^2 satisfy ψ, and the point events in ie^3 satisfy $\neg\psi$, there exists a PIE pie^2_ψ such that $pie^2_\psi.te = ie^2.te$ and $pie^2_\psi.ts \leq ie^1.ts$.

Finally, since pie^1_φ and pie^2_ψ share the same ending time, and $pie^2_\psi.ts < pie^1_\varphi.ts = ie^2.ts$, the pair $pie^1_\varphi, pie^2_\psi$ satisfies $\mathsf{finishes}(pie^1_\varphi, pie^2_\psi)$.

- equals: Let r be equals, and suppose that $sie \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \neg\psi)^+)$. By the concatenation rule of regular expressions, there exists a sequence $sie = ie' \cdot ie'' \cdot ie'''$, where $ie' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, and $ie''' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$. On the original data stream S, the intervals ie', ie'', and ie''' correspond to the interval events ie^1, ie^2, and ie^3, respectively. Within ie^2, all point events satisfy $\varphi \wedge \psi$, while within ie^1 and ie^3, the point events satisfy $\neg\varphi \wedge \neg\psi$.

  Since every point event in ie^2 satisfies both φ and ψ, there must exist PIE pie^1_φ and a PIE pie^2_ψ that $pie^1_\varphi.ts = pie^2_\psi.ts = ie^2.ts$ and $pie^1_\varphi.te = pie^2_\psi.te = ie^2.te$.

  Therefore, pie^1_φ and pie^2_ψ satisfy the equals relation, i.e., $\mathsf{equals}(pie^1_\varphi, pie^2_\psi)$.

Necessity:

Below, we present the necessity proofs for each $r \in \{\mathsf{followed\text{-}by}, \mathsf{meets}, \mathsf{overlaps}, \mathsf{starts}, \mathsf{during}, \mathsf{finishes}, \mathsf{equals}\}$. In particular, we show that if the PiePair $r(pie^i_\varphi, pie^j_\psi)$ exists in S, then there must exist a sequence $sie \in \mathcal{L}(E(r(pie_\varphi, pie_\psi)))$ in $S_{\varphi,\psi}$. The inverse relations follow by appropriately converting the predicates.

- $\mathsf{followed\text{-}by}$: When r is $\mathsf{followed\text{-}by}$, according to the definition of $\mathsf{followed\text{-}by}$, we have $pie^i_\varphi.te < pie^j_\psi.ts$.

  First, choose an interval event ie^1 contained in pie^i_φ such that $ie^1.ts \geq pie^i_\varphi.ts$ and $ie^1.te = pie^i_\varphi.te$. All point events within ie^1 satisfy φ.

  Next, choose an interval event ie^3 contained in pie^j_ψ such that $ie^3.ts = pie^j_\psi.ts$ and $ie^3.te \leq pie^j_\psi.te$. All point events within ie^3 satisfy ψ.

  Finally, Choose an interval event ie^2 such that $ie^2.ts = pie^i_\varphi.te$ and $ie^2.te = pie^j_\psi.ts$. All point events within ie^2 satisfy $\neg\varphi \wedge \neg\psi$ since $\mathsf{followed\text{-}by}(pie^i_\varphi, pie^j_\psi)$. In the predicate-based data stream S' corresponding to φ, ψ, the interval events ie^1, ie^2, ie^3 correspond to sequences ie', ie'', ie''', where $ie' \in \mathcal{L}(\varphi^+)$, $ie'' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$, and $ie''' \in \mathcal{L}(\psi^+)$. Construct $sie = ie' \cdot ie'' \cdot ie'''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}(\varphi^+ \cdot (\neg\varphi \wedge \neg\psi)^+ \cdot \psi^+)$.
- meets: When r is meets, according to the definition of meets, we have $pie^i_\varphi.te = pie^j_\psi.ts$.

  First, Choose an interval event ie^1 such that $ie^1.te = pie^j_\psi.ts$ and $|ie^1| = 1$. Due to the longest subsequence property of pie^j_ψ, ie^1 satisfies $\neg\psi$. Furthermore, since $|pie^i_\varphi| \geq 1$ and $pie^i_\varphi.te = ie^1.te$, ie^1 satisfies $\varphi \wedge \neg\psi$.

Next, choose an interval event ie^2 such that $ie^2.ts = pie^i_\varphi.te$ and $|ie^2| = 1$. Due to the longest subsequence property of pie^i_φ, ie^2 satisfies $\neg\varphi$. Moreover, since $|pie^j_\psi| \geq 1$ and $pie^j_\psi.ts = ie^2.ts$, ie^2 satisfies $\neg\varphi \wedge \psi$.

In the predicate-based data stream S' corresponding to φ, ψ, the interval events ie^1 and ie^2 correspond to sequences ie' and ie'', respectively, so $ie' \in \mathcal{L}((\varphi\wedge\neg\psi)^+)$ and $ie'' \in \mathcal{L}((\neg\varphi\wedge\psi)^+)$. Since $ie^1.te = ie^2.ts$, we can construct $sie = ie' \cdot ie''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}((\varphi \wedge \neg\psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$.

– overlaps: When r is overlaps, by definition we have $pie^i_\varphi.ts < pie^j_\psi.ts < pie^i_\varphi.te < pie^j_\psi.te$.

  First, choose an interval event ie^1 with $ie^1.te = pie^j_\psi.ts$ and $|ie^1| = 1$. By the longest subsequence property of pie^j_ψ, ie^1 satisfies $\neg\psi$. Since $pie^i_\varphi.ts < pie^j_\psi.ts$, it follows that $pie^i_\varphi.ts \leq ie^1.ts$, so ie^1 also satisfies φ. Thus, ie^1 satisfies $\varphi \wedge \neg\psi$.

  Next, choose an interval event ie^2 such that $ie^2.ts = pie^j_\psi.ts$, $ie^2.te = pie^i_\varphi.te$, and $|ie^2| \geq 1$. Since ie^2 is contained in both pie^i_φ and pie^j_ψ, ie^2 satisfies $\varphi\wedge\psi$.

  Finally, choose an interval event ie^3 with $ie^3.ts = pie^i_\varphi.te$ and $|ie^3| = 1$. By the longest subsequence property of pie^i_φ, ie^3 satisfies $\neg\varphi$. Since $pie^i_\varphi.te < pie^j_\psi.te$, ie^3 also lies within the time span of pie^j_ψ, hence ie^3 satisfies ψ. Thus, ie^3 satisfies $\neg\varphi \wedge \psi$.

  In the predicate-based data stream S', ie^1, ie^2, ie^3 correspond to ie', ie'', ie''', where $ie' \in \mathcal{L}((\varphi \wedge \neg\psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, and $ie''' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$.

  Construct $sie = ie' \cdot ie'' \cdot ie'''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}((\varphi \wedge \neg\psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$.

– starts: When r is starts, by definition: $pie^i_\varphi.ts = pie^j_\psi.ts < pie^i_\varphi.te < pie^j_\psi.te$.

  First, choose an interval event ie^1 ending at $pie^i_\varphi.ts = pie^j_\psi.ts$ with $|ie^1| = 1$. By the longest subsequence property, all point events in ie^1 satisfy $\neg\varphi \wedge \neg\psi$.

  Next, choose an interval event ie^2 with $ie^2.ts = pie^i_\varphi.ts = pie^j_\psi.ts$ and $ie^2.te = pie^i_\varphi.te$, where $|ie^2| \geq 1$. Since ie^2 lies within both pie^i_φ and pie^j_ψ from their common start until pie^i_φ ends, all point events in ie^2 satisfy $\varphi \wedge \psi$.

  Finally, choose an interval event ie^3 such that $ie^3.ts = pie^i_\varphi.te$ and $|ie^3| = 1$. Since $pie^i_\varphi.te < pie^j_\psi.te$, we have $ie^3.te \leq pie^j_\psi.te$, and all point events in ie^3 satisfy $\neg\varphi \wedge \psi$.

  In the predicate-based data stream S', the interval events ie^1, ie^2, ie^3 correspond to sequences ie', ie'', ie''' where $ie' \in \mathcal{L}((\neg\varphi\wedge\neg\psi)^+)$, $ie'' \in \mathcal{L}((\varphi\wedge\psi)^+)$, $ie''' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$. Construct $sie = ie' \cdot ie'' \cdot ie'''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$.

– during: When r is during, by definition: $pie^j_\psi.ts < pie^i_\varphi.ts < pie^i_\varphi.te < pie^j_\psi.te$.

  First, choose an interval event ie^1 with $ie^1.te = pie^i_\varphi.ts$ and $|ie^1| = 1$. By the longest subsequence property of pie^i_φ, all point events in ie^1 satisfy $\neg\varphi$. Moreover, since $pie^j_\psi.ts \leq ie^1.ts$, ie^1 also satisfies ψ. Thus, ie^1 satisfies $\neg\varphi\wedge\psi$.

Next, choose an interval event ie^2 such that $ie^2.ts = pie^i_\varphi.ts$ and $ie^2.te = pie^i_\varphi.te$, with $|ie^2| \geq 1$. Because ie^2 is contained in both pie^i_φ and pie^j_ψ throughout pie^i_φ's duration, all point events in ie^2 satisfy $\varphi \wedge \psi$.
Finally, choose an interval event ie^3 with $ie^3.ts = pie^i_\varphi.te$ and $|ie^3| = 1$. By the longest subsequence property of pie^i_φ, all point events in ie^3 satisfy $\neg\varphi$. Since $ie^3.te \leq pie^j_\psi.te$, ie^3 also satisfies ψ. Thus, ie^3 satisfies $\neg\varphi \wedge \psi$.
In the predicate-based data stream S', ie^1, ie^2, ie^3 correspond to ie', ie'', ie''' where $ie' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, $ie''' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$.
Construct $sie = ie' \cdot ie'' \cdot ie'''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}((\neg\varphi \wedge \psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \psi)^+)$.

- finishes: When r is finishes, by definition: $pie^j_\psi.ts < pie^i_\varphi.ts < pie^i_\varphi.te = pie^j_\psi.te$.
  First, choose an interval event ie^1 with $ie^1.te = pie^i_\varphi.ts$ and $|ie^1| = 1$. By the longest subsequence property of pie^i_φ, all point events in ie^1 satisfy $\neg\varphi$. Since $pie^j_\psi.ts \leq ie^1.ts$, ie^1 also satisfies ψ, and thus $\neg\varphi \wedge \psi$.
  Next, choose an interval event ie^2 with $ie^2.ts = pie^i_\varphi.ts$ and $ie^2.te = pie^i_\varphi.te$, where $|ie^2| \geq 1$. Because ie^2 is contained within both pie^i_φ and pie^j_ψ for the entire duration of pie^i_φ, all point events in ie^2 satisfy $\varphi \wedge \psi$.
  Finally, choose an interval event ie^3 such that $ie^3.ts = pie^i_\varphi.te = pie^j_\psi.te$ and $|ie^3| = 1$. By the longest subsequence property of pie^i_φ and pie^j_ψ, all point events in ie^3 satisfy $\neg\varphi \wedge \neg\psi$.
  In the predicate-based data stream S', ie^1, ie^2, ie^3 correspond to ie', ie'', ie''' where $ie' \in \mathcal{L}((\neg\varphi \wedge \psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, $ie''' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$.
  Construct $sie = ie' \cdot ie'' \cdot ie'''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}((\neg\varphi \wedge \psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \neg\psi)^+)$.
- equals: When r is equals, by definition: $pie^j_\psi.ts = pie^i_\varphi.ts < pie^i_\varphi.te = pie^j_\psi.te$.
  First, choose an interval event ie^1 with $ie^1.te = pie^i_\varphi.ts = pie^j_\psi.ts$ and $|ie^1| = 1$. By the longest subsequence property of both pie^i_φ and pie^j_ψ, all point events in ie^1 satisfy $\neg\varphi \wedge \neg\psi$.
  Next, choose an interval event ie^2 such that $ie^2.ts = pie^i_\varphi.ts = pie^j_\psi.ts$ and $ie^2.te = pie^i_\varphi.te = pie^j_\psi.te$, with $|ie^2| \geq 1$. Since ie^2 lies entirely within both pie^i_φ and pie^j_ψ, all point events in ie^2 satisfy $\varphi \wedge \psi$.
  Finally, choose an interval event ie^3 such that $ie^3.ts = pie^i_\varphi.te = pie^j_\psi.te$ and $|ie^3| = 1$. By the longest subsequence property, all point events in ie^3 also satisfy $\neg\varphi \wedge \neg\psi$.
  In the predicate-based data stream S', ie^1, ie^2, ie^3 correspond to ie', ie'', ie''' where $ie' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$, $ie'' \in \mathcal{L}((\varphi \wedge \psi)^+)$, $ie''' \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+)$.
  Construct $sie = ie' \cdot ie'' \cdot ie'''$. By the concatenation rules of regular expressions, $sie \in \mathcal{L}((\neg\varphi \wedge \neg\psi)^+ \cdot (\varphi \wedge \psi)^+ \cdot (\neg\varphi \wedge \neg\psi)^+)$. ■

Theorem 1. *The detection result of a Pattern query is equivalent to the result obtained by detecting mPiePairs and then performing a natural join on the results. More formally:*

The Pattern query consists of n mPiePairs, represented by the set $\mathcal{MPP} = \{C_i(pie_{\varphi_i}, pie_{\psi_i}) \mid 0 < i < n\}$, connected by the logical operator AND. The query also includes all the m PIEs in the set $\mathcal{X} = \{pie_i \mid 0 < i < m\}$, where m is the total number of PIEs involved in the query. Let $\mathcal{Y} = \{ie_i \mid 0 < i < m\}$ represent the a corresponding detection result. Then:

- ***Sufficiency:*** *If the result $\mathcal{Y}$ satisfies the Pattern query, then it must also satisfy the detection result of the mPiePairs followed by natural joins.*
- ***Necessity:*** *Conversely, if the result $\mathcal{Y}$ satisfies the mPiePair detection followed by the natural join, it must also satisfy the Pattern query.*

B Proof of Theorem 1

Proof. We prove the two directions of the equivalence.

Sufficiency: Assume that a result $\mathcal{Y}$ satisfies the Pattern query. For each mPiePair $C_i(pie_{\varphi_i}, pie_{\psi_i})$ in the Pattern query, there must exist a pair (ie_j, ie_k) such that $C(ie_j, ie_k)$ holds. Since the interval events in $\mathcal{Y}$ correspond to unique PIEs, the result set $\mathcal{Y}$ consists of distinct event pairs. Therefore, after performing the natural join on these mPiePairs, the resulting set will be unique and consistent, and the final result will still satisfy the detection result of the mPiePairs followed by natural joins.

Necessity: Now assume that a result $\mathcal{Y}$ is obtained by detecting mPiePairs and performing the natural join on these results. For each mPiePair $C_i(pie_{\varphi_i}, pie_{\psi_i})$ in the Pattern query, there must exist a pair (ie_j, ie_k) such that $C(ie_j, ie_k)$ holds. The natural join operation ensures that only compatible event pairs are combined, so the final result $\mathcal{Y}$ will satisfy the conditions of the Pattern query.

Thus, we have shown that the detection result of a Pattern query is equivalent to the result of detecting mPiePairs followed by natural joins. ■