# METASTABILITY FOR TELEGRAPH PROCESSES IN A DOUBLE-WELL POTENTIAL

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ABSTRACT. In this paper, we study equations with nonlinearity in the form of a double-well potential, randomised by a velocity-switching (telegraph) stochastic process. If the speed parameters of the randomisation are small, then this dynamics has one metastable uncertainty interval and two invariant attractors. The probabilities of leaving the metastable interval through the upper boundary are determined, as well as characteristics of the first crossing times. Invariant measures are also found.

When and if the direction of the telegraph process velocity coincides with the direction of the periodic change in potential, the system can go into a metastable state, having received a time window for the interwell transition.

The obtained results can be used as an alternative to stochastic resonance models. *Keywords:* piecewise deterministic process; first passage time; invariant measure; stochastic resonance

### 1. Introduction

Anomalous relativistic ([17], see also [9]) diffusion processes with finite propagation speed often seems more natural for modelling than the classical diffusion approach. There are several different terminological and ideological systems for dealing with such classes of anomalous diffusions. In physics, people prefer to say "continuous time random walks" [21], "flip-flop" [4], or "zig-zag processes" [9, 35]. In mathematics, the terms "piecewise-deterministic processes", [8], or "telegraph processes", [14, 19], are commonly used. The study of classical stochastic differential equations with regime switching (and their applications) is also widely represented, see for example [25, 40].

In this paper, we study a piecewise deterministic nonlinear system whose dynamics is specified by a double-well potential and a two-state Markov random process. It turns out that such a system can have metastable states as an important feature.

This problem setting continues the author's previous studies on stochastic differential equations based on piecewise deterministic processes. These studies have recently begun with Ornstein-Uhlenbeck-like processes, where the telegraph process replaces the Wiener process in the Langevin equation. For such stochastic dynamics with its various modifications, invariant measures and distributions of first passage times were obtained, see [26–30]. Here we develop similar ideas in a *nonlinear* setting.

Dynamic systems subject to random perturbations are actively studied in both physical and mathematical literature. In the simplest case, such systems are usually based on deterministic dynamics, which follows the ordinary differential equation  $d\gamma(t) = -U'(\gamma(t))dt$ , t > 0, equipped with a random perturbation, where the function U is the potential.

Let x be a stationary point, i.e., U'(x) = 0. If U''(x) > 0, then the position x is asymptotically stable, that is  $\gamma(t) \to x$  as  $t \to +\infty$  for any starting point near x. On the contrary, if U''(x) < 0, then x unstable and repulsive, i.e., any path that starts near x moves away from x. The interplay between different kinds of stationary points under different randomisations of this equation is of special interest. The first and most popular randomisation method is to add a stochastic term, such

<sup>&</sup>lt;sup>1</sup>Forthcoming in J.Appl.Probab.

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as white noise, to the deterministic differential equation, see [18]. Later, other randomisation options have emerged, for example, the Lévy process [10], "coloured" noise (e.g., the Ornstein-Uhlenbeck process) [38,41], the fractional Gaussian [15].

When the phases coexist, the phenomenon of hysteresis may occur, see the discussion in [7, 10, 37]. An introduction to metastability can be found in [3].

The simplest case of one-dimensional setting has recently been generalised in a number of works on metastability effects arising in the case of diffusion processes that are stopped or reflecting on the boundary of domain, see [11], and more generally, on metastability at and near the surfaces that remain invariant under diffusion processes, [12].

When using various randomisation methods, stochastic resonance can occur, where a small change in the input signal causes a large output signal in the system. Such dynamics, with a transition from one potential well to another, can be interpreted to qualitatively explain glacial cycles in the history of the Earth (from a warm state to an ice age and back), see [18]. Similar phenomena occur in various applied settings, from biology to metallurgy, [34]. For example, this approach is effective when modelling a single neurone, [5, 16, 24, 26] (see also a recent review of these models in [6]), and artificial neuronal sets [33], as well as dynamics of competing populations [39] and chemical reaction systems [36]. See also detailed overviews of other applications in [13, 20, 23].

The idea of stochastic resonance is successfully used for example by adding white noise  $\sigma(T)dW(t)$  to a deterministic differential equation, where W is standard Brownian motion and the coefficient  $\sigma(T)$  is chosen to maximise the coefficient of spectral power amplification. See the seminal works [1, 24]. The second approach involves the so-called 'effective dynamics', based on a two-state Markov chain with a time-periodic infinitesimal generator, which artificially ensures a spatial interwell transition, see e.g. [18].

Our randomisation methodology consists in replacing the diffusion term W(t) by a relativistic (finite velocity) process  $\mathbb{T}(t)$ . The idea of such replacement becomes more and more popular in physics, [17, 32]. With this randomisation, the model receives certain time windows for interwell transitions instead of stochastic resonance options. In this settings, we present an infinitesimal generator of the process (Theorem 2.1), compare the probabilities of which of the attractor the process will end up on when the velocities are small (Theorem 3.3), and derive formulae for the average first passage times when the process has already left the metastable position (Theorem 3.4). Invariant measures inside the attractors are also found (Theorem 4.1).

The paper is organised as follows. The next section is devoted to the problem setting and description of the structure of the underlying dynamics. In Section 3 we analyse the distributions of passage times. Section 4 presents invariant distributions.

### 2. Preliminaries and problem statement

Let  $\mathbb{T}(t)$  be a telegraph process with alternating velocities  $c_0, c_1, c_0 > c_1$ ,

$$\mathbb{T}(t) = \int_0^t c_{\xi(s)} \mathrm{d}s, \qquad t > 0, \tag{1}$$

where  $\xi = \xi(t) \in \{0,1\}$  is a two-state Markov process with alternating switching intensities  $\lambda_0$ ,  $\lambda_1$ , [31]. Consider a stochastic process  $X(t) = X^x(t)$  defined by the equation

$$dX(t) = -U'(X(t))dt + d\mathbb{T}(t), \qquad t > 0,$$
(2)

and the initial condition X(0) = x. Process X = X(t) can be considered as a telegraph process  $\mathbb{T}$  in the potential U.

Our previous studies [26–30] were based on the so-called Kac-Ornstein-Uhlenbeck processes, which are defined by the equation

$$d\bar{X}(t) = \left(c_{\mathcal{E}(t)} - \gamma_{\mathcal{E}(t)}\bar{X}(t)\right)dt, \qquad t > 0,$$

i.e. the case of alternating single-well potentials  $U_i = \gamma_i x^2/2 - c_i x$ .

For the quasilinear setting (2), suppose that the function U(x),  $U \in C^2(-\infty,\infty)$ , is a double-well potential, i.e. it has a local maximum at the point  $x_0$ , two local minima at the points  $x_-$ ,  $x_+$ ,  $x_$  $x_0 < x_+$ , and  $U''(x) \neq 0$ ,  $x \in \{x_-, x_0, x_+\}$ . As a typical example, one can use the symmetric function  $U(x) = \frac{x^4}{4} - \frac{x^2}{2}.$ 

The process X(t) is continuous, piecewise continuously differentiable and is formed by the sequential alternation of two deterministic patterns  $\gamma_0(t,x)$  and  $\gamma_1(t,x)$ , which follow the equations

$$\frac{\partial \gamma_0(t,x)}{\partial t} = c_0 - U'(\gamma_0(t,x)), \qquad \frac{\partial \gamma_1(t,x)}{\partial t} = c_1 - U'(\gamma_1(t,x)), \qquad t > 0,$$

$$\gamma_0(0,x) = \gamma_1(0,x) = x.$$
(3)

One pattern is replaced by another at random times  $\tau_n$ , when the underlying Markov process  $\xi(t)$ switches.

The initial value problems (3) are equivalent to the pair of integral equations

$$\int_{x}^{\gamma_{0}(t,x)} \frac{\mathrm{d}z}{c_{0} - U'(z)} = t, \quad \text{and} \quad \int_{x}^{\gamma_{1}(t,x)} \frac{\mathrm{d}z}{c_{1} - U'(z)} = t, \quad t \ge 0, \quad (4)$$

which can be rewritten in the equivalent form:

$$\Phi_0(\gamma_0(t,x)) = \Phi_0(x) + t$$
 and  $\Phi_1(\gamma_1(t,x)) = \Phi_1(x) + t$ . (5)

Here  $\Phi_0$  and  $\Phi_1$  are rectifying diffeomorphisms defined up to an arbitrary additive constant, by the equations

$$\Phi_0'(y) = \frac{1}{c_0 - U'(y)}, \qquad \Phi_1'(y) = \frac{1}{c_1 - U'(y)}. \tag{6}$$

Notice that the function  $\Phi_i = \Phi_i(y)$  is continuous and monotone between the critical points of the potential  $U_i = U(x) - c_i x$ .

To avoid blow-up, assume that the integral  $\int_X \frac{dy}{U_i'(y)}$  diverges at any critical point of  $U_i$ :

$$\int_{x} \frac{\mathrm{d}y}{U_{i}'(y)} = \infty, \qquad x \in \{x_{-}, x_{0}, x_{+}\}. \tag{7}$$

From (4) it follows that the pattern  $\gamma_i(t,x)$  arising from the state (x,i),  $i \in \{0,1\}$ , also satisfies the initial value problem

$$\frac{\partial \gamma_{t}}{\partial t}(t,x) = L_{i}^{x}[\gamma_{t}(t,x)], \qquad t > 0,$$

$$\gamma_{t}(t,x)|_{t \downarrow 0} = x,$$
(8)

where 
$$L_i^x = -U_i'(x)\frac{\partial}{\partial x} = (c_i - U'(x))\frac{\partial}{\partial x}, \ i \in \{0,1\}.$$
 The properties of the process  $X$  depend significantly on the velocity amplitudes  $c_0$  and  $c_1$ .

## A: An unstable uncertainty interval and two attractors. Fig.1(a).

Let both parameters  $c_0$  and  $c_1$  be small enough so that  $V > c_0 > c_1 > v$ , where V and v are, respectively, a local maximum and a local minimum of U'. In this case, both  $U_0(x) = U(x) - c_0x$ and  $U_1(x) = U(x) - c_1x$  are still double-well potentials with alternation between them. Let  $a_0, a_{\pm}$  be the critical points of  $U_1$ ,  $a_- < a_0 < a_+$ , and  $b_0, b_\pm$  be the critical points of  $U_0$ ,  $b_- < b_0 < b_+$ , see Fig.1(a).

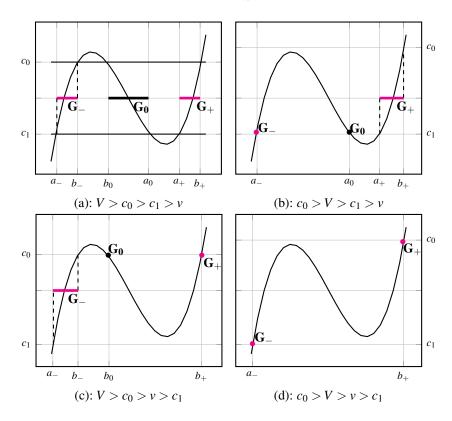


FIGURE 1. U'(x) and  $G_0$ ,  $G_{\pm}$ .

Note that  $X(t) \equiv a_-, X(t) \equiv a_+$  and  $X(t) \equiv a_0$  are stationary solutions of (2) under the initial state  $\xi(0)=1$ . Pattern  $\gamma_1=\gamma_1(t,x)$  is repelled by  $a_0$ , while being attracted to  $a_-$  or  $a_+$ . Respectively,  $X(t)\equiv b_-, X(t)\equiv b_+$  and  $X(t)\equiv b_0$  are stationary solutions under  $\xi(0)=0$ , and pattern  $\gamma_0$  is repelled by  $b_0$ , being attracted to points  $b_\pm$ .

As a result, the process X has two attractors:  $G_- = [a_-, b_-] \subset (-\infty, b_0)$ ,  $G_+ = [a_+, b_+] \subset (a_0, +\infty)$  and a metastable set  $G_0 = (b_0, a_0)$ . The process  $X^x$ , starting from the metastable set  $G_0$ ,  $x \in G_0$ , can temporarily oscillate inside, but once it leaves this interval through the upper or lower bound, it never returns, and then forever remains above  $a_0$  or below  $b_0$ , respectively, being attracted to  $G_+$  or  $G_-$ . See the sample of a path in Fig.2.

If the process  $X = X^x(t)$  starts from  $x = X^x(0) > a_0$ , then after a finite transition time it ends up in  $G_+$ , and when starting from  $x < b_0$  the process for a.s. finite time falls into  $G_-$ .

Note that both sets  $G_{-}$  and  $G_{+}$  are invariant under dynamics:

$$\mathbb{P}\{X(t) \in G_{-} \ \forall t > 0 \ | \ X(0) \in G_{-}\} = 1, \qquad \mathbb{P}\{X(t) \in G_{+} \ \forall t > 0 \ | \ X(0) \in G_{+}\} = 1. \tag{9}$$

Figure 3 shows plots of  $\Phi_0$  and  $\Phi_1$  in this case.

### B: Alternating single-well and double-well potentials, only one attractor. Fig. 1(b), (c).

When  $c_0$  becomes large, so that  $c_0 > V$ , the potential  $U_0(x)$  has only one critical point  $b_+$  becoming single-well, while  $U_1$  is still double-well, and has three critical points. In this case, the process X has only one attractor  $G_+$ , see Fig.1(b). After some transition period, with probability 1 X ends up in  $G_+$ . A symmetric situation occurs when  $c_1$  becomes less than v, see Fig.1(c).

## C: Alternating single-well potentials, one attractor. Fig.1(d).

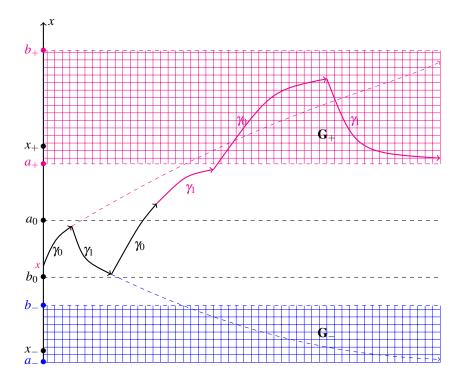


FIGURE 2. Sample path of X, case A.

In the case  $c_0 > V > v > c_1$ , both potentials  $U_0$  and  $U_1$  become single-well. The dynamics has only one attractor  $G = (a_-, b_+)$ . When the process X begins in  $x = X(0) \in G$ , it randomly fluctuates between the attractive levels  $a_-$  and  $b_+$ . If it starts outside, then it ends at G after some transition.

Two other less interesting single-well situations with a single attractor, are not shown in Fig.1. They correspond to  $c_0 > c_1 > V > v$  and  $c_1 < c_0 < v < V$ . See Fig.4 for a diagram of the different regimes.

The process  $\Xi = (X^x(t), \xi(t))$  is a Markov process with the state space  $\mathbb{R} \times \{0, 1\}$ .

Let  $\tau^0$  be the time of first switching from state 0 to state 1,  $\tau^1$  corresponds to the reverse switching. Due to the Markov property, the following equalities in distribution hold:

$$\begin{split} & \left[ X^{x}(t) \mid \xi(0) = 0 \right] \stackrel{D}{=} \gamma_{0}(t,x) \cdot \mathbb{1}_{\left\{\tau^{0} > t\right\}} + \mathbb{1}_{\left\{\tau^{0} < t\right\}} \cdot \left[ X^{\gamma_{0}(\tau^{0},x)}(t-\tau^{0}) \mid \xi(0) = 1 \right], \\ & \left[ X^{x}(t) \mid \xi(0) = 1 \right] \stackrel{D}{=} \gamma_{1}(t,x) \cdot \mathbb{1}_{\left\{\tau^{1} > t\right\}} + \mathbb{1}_{\left\{\tau^{1} < t\right\}} \cdot \left[ X^{\gamma_{1}(\tau^{1},x)}(t-\tau^{1}) \mid \xi(0) = 0 \right]. \end{split}$$

Based on (10) one can obtain an infinitesimal generator of  $\Xi$ .

**Theorem 2.1.** The infinitesimal generator of  $\Xi$  is given by  $\Lambda + \mathcal{L}$ , where

$$\mathcal{L} = \mathcal{L}^x = \begin{pmatrix} L_0^x & 0\\ 0 & L_1^x \end{pmatrix},\tag{11}$$

and  $\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}$  is the generator of the two-state Markov process  $\xi$ .

See the proof in Appendix.

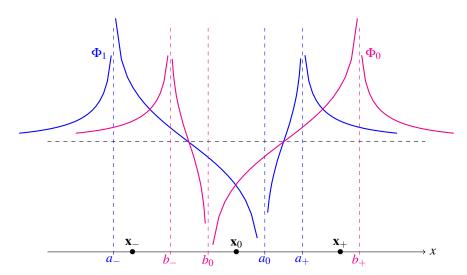


FIGURE 3. Functions  $\Phi_0(x)$  and  $\Phi_1(x)$ , case A.

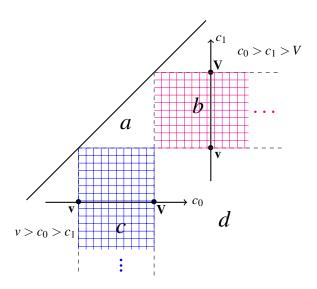


FIGURE 4. Scheme of different modes. Triangle **a**,  $V > c_0 > c_1 > v$  (case A); half-bands **b** and **c**, correspond to  $c_0 > V > c_1 > v$  and  $V > c_0 > v > c_1$  respectively (case B); quadrant **d**,  $c_0 > V > v > c_1$  (case C) (designations according to Fig.1).

## 3. Passage times

Let  $\mathcal{T}(x,y)$  be the time when the process X(t) first reaches a given threshold y, starting from point x,

$$\mathcal{T}(x,y) = \inf\{t > 0 \mid X(0) = x, X(t) = y\}.$$

The distribution of this random variable is determined by the moment generating function  $u = (u_0(q,x,y), u_1(q,x,y))^T$ ,

$$u_0(q,x,y) = \mathbb{E}[\exp(-q\mathcal{T}(x,y)) \mid \xi(0) = 0],$$
  

$$u_1(q,x,y) = \mathbb{E}[\exp(-q\mathcal{T}(x,y)) \mid \xi(0) = 1],$$
  $q > 0.$ 

By  $t_0^*(x,y)$  and  $t_1^*(x,y)$  we denote the time to reach level y without switching states, starting from (x,0) and (x,1), respectively. The value  $t_i^*(x,y)$  is finite if the equation  $\gamma_i(t,x) = y$  is satisfied for some positive t,  $t = t_i^*(x,y)$ . In this case we say that level y is reachable from state (x,i). Otherwise we set  $t_i^*(x,y) = \infty$ .

If y is reachable from the state (x,i), then both x and y belong to the domain of a common continuous piece of  $\Phi_i$ , and, moreover, since (5) holds,  $\Phi_i(y) > \Phi_i(x)$  and  $t_i^*(x,y) = \Phi_i(y) - \Phi_i(x)$ . Moreover,  $\lim_{x \to y} t_i^*(x,y) = 0$ .

By virtue of (6),

$$L_i^x t_i^*(x, y) = -L_i^x \Phi_i(x) = -1, \quad \text{when } t_i^*(x, y) \text{ is finite, } i \in \{0, 1\}.$$
 (12)

Let y be a level reachable from both (x,0) and (x,1). Conditioning on the first switching, like in (10), we obtain a system of integral equations:

$$\begin{cases}
 u_0(q, x, y) = e^{-(q + \lambda_0)t_0^*(x, y)} + \int_0^{t_0^*(x, y)} \lambda_0 e^{-(q + \lambda_0)\tau} u_1(q, \gamma_0(\tau, x), y) d\tau, \\
 u_1(q, x, y) = e^{-(q + \lambda_1)t_1^*(x, y)} + \int_0^{t_1^*(x, y)} \lambda_1 e^{-(q + \lambda_1)\tau} u_0(q, \gamma_1(\tau, x), y) d\tau.
\end{cases} (13)$$

On the contrary, in the case  $t_i^*(x,y) = +\infty$ , the corresponding integral equation of (13) can be interpreted as

$$u_{i}(q,x,y) = \int_{0}^{\infty} \lambda_{i} e^{-(q+\lambda_{i})\tau} u_{1-i}(q,\gamma_{i}(\tau,x),y) d\tau, \qquad i \in \{0,1\}.$$
(14)

**Theorem 3.1.** The moment generating function  $\mathbf{u} = (u_0(q,x,y), u_1(q,x,y))^T$  of the first passage time  $\mathcal{T}(x,y)$  satisfies the equation:

$$(\Lambda + \mathcal{L})\mathbf{u}(q, x, y) = q\mathbf{u}(q, x, y), \qquad q > 0,$$
(15)

where  $\Lambda + \mathcal{L}$  is the generator of Markov process  $\Xi$ .

Both entries  $u_0(q,x,y)$  and  $u_1(q,x,y)$  of u separately obey the telegraph ([2]) equations of the form,

$$L_1^x L_0^x u_0(q, x, y) - \left[ (q + \lambda_1) L_0^x + (q + \lambda_0) L_1^x \right] u_0(q, x, y) + q(q + \lambda_0 + \lambda_1) u_0(q, x, y) = 0,$$

$$L_0^x L_1^x u_1(q, x, y) - \left[ (q + \lambda_1) L_0^x + (q + \lambda_0) L_1^x \right] u_1(q, x, y) + q(q + \lambda_0 + \lambda_1) u_1(q, x, y) = 0.$$
(16)

Equations (15) and (16) are provided with boundary conditions that depend on whether the level y is reachable from (x,0) and (x,1).

*Proof.* Apply  $L_0^x$  and  $L_1^x$  to the first and the second equation of the system (13), respectively.

First, let the threshold y be reachable from (x,0). By virtue of (8) and (12), the first equation of (13) yields

$$\begin{split} L_0^x u_0(q,x,y) &= (q+\lambda_0) \mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} - \lambda_0 \mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} u_1(q,\gamma_0(t_0^*(x,y),x),y) \\ &+ \int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-(q+\lambda_0)\tau} \frac{\partial}{\partial \tau} \left[ u_1(q,\gamma_0(\tau,x),y) \right] \mathrm{d}\tau. \end{split}$$

Integrating by parts, one can see that

$$\begin{split} L_0^x u_0(q,x,y) &= (q+\lambda_0) \mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} - \lambda_0 u_1(q,x,y) \\ &+ (q+\lambda_0) \int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-(q+\lambda_0)\tau} u_1(q,\gamma_0(\tau,x),y) \mathrm{d}\tau. \end{split}$$

Proceeding similarly with the second equation of (13), both in the case of finite and infinite  $t_1^*(x, y)$ , as a result, we obtain the system

$$\begin{cases}
L_0^x u_0(q, x, y) = (q + \lambda_0) u_0(q, x, y) - \lambda_0 u_1(q, x, y), \\
L_1^x u_1(q, x, y) = -\lambda_1 u_0(q, x, y) + (q + \lambda_1) u_1(q, x, y),
\end{cases}$$
(17)

which in matrix form coincides with equation (15).

Equations (16) follow from (17) by eliminating variables. Indeed, applying  $L_1^x$  to the first equation of system (17), we get  $L_1^x L_0^x u_0(q,x,y) = (q+\lambda_0) L_1^x u_0(q,x,y) - \lambda_0 L_1^x u_1(q,x,y)$ . From the second equation of (17) one can obtain

$$L_1^x L_0^x u_0(q, x, y) = (q + \lambda_0) L_1^x u_0(q, x, y) + \lambda_0 \lambda_1 u_0(q, x, y) - \lambda_0 (q + \lambda_1) u_1(q, x, y).$$

The first equation of (17) can be rewritten as  $\lambda_0 u_1(q,x,y) = (q+\lambda_0)u_0(q,x,y) - L_0^x u_0(q,x,y)$ , which gives the first equation of (16); the second equation is obtained similarly.

If 
$$\mathbb{P}\{\mathscr{T}(x,y) < \infty\} = 1$$
, then the mean value  $m(x,y) = (m_0(x,y), m_1(x,y))^T$ ,  $m_0(x,y) = \mathbb{E}[\mathscr{T}(x,y) \mid \xi(0) = 0], \qquad m_1(x,y) = \mathbb{E}[\mathscr{T}(x,y) \mid \xi(0) = 1],$ 

can be found by differentiation,

$$m_0(x,y) = -\frac{\partial u_0(q,x,y)}{\partial q}\bigg|_{q=0}, \qquad m_1(x,y) = -\frac{\partial u_1(q,x,y)}{\partial q}\bigg|_{q=0}.$$
 (18)

Theorem 3.1 implies the following result, general for Markov processes.

**Corollary 3.2.** Let the mean value of  $\mathcal{T}(x,y)$  be finite. Then the mean vector  $\mathbf{m}(x,y)$  satisfies the equation

$$(\Lambda + \mathcal{L}^x) m(x, y) = -1 = (1, 1)^{\mathrm{T}}.$$
 (19)

Let us first consider the case of small  $c_0$  and  $c_1$ , i.e.  $V > c_0 > c_1 > v$ , see Fig.1(a).

**3.1.** Small  $c_0$  and  $c_1$ : probabilities of leaving the uncertainty interval. In the case  $V > c_0 > c_1 > v$ , let the initial point x be inside the interval  $G_0$ , i.e.  $X(0) = x \in G_0 = (b_0, a_0)$ , case A, Fig.1(a).

Since both  $a_0$  and  $b_0$  are repulsive, the trajectory that has once crossed one of these thresholds never returns to the interval  $(b_0, a_0)$ . We are interested in the conditional probabilities of exiting the interval  $G_0 = (b_0, a_0)$  through the upper bound  $a_0$ ,

$$p_0(x) = \mathbb{P}\{\mathcal{T}(x, a_0) < \infty \mid \xi(0) = 0\}, \qquad p_1(x) = \mathbb{P}\{\mathcal{T}(x, a_0) < \infty \mid \xi(0) = 1\}.$$
 (20)

The moment generating functions  $u_0(q, x, a_0)$  and  $u_1(q, x, a_0)$ ,  $x \in G_0$ , are increasing. Further,  $0 < u_0(q, x, a_0) < 1$ ,  $0 < u_1(q, x, a_0) < 1$  for  $b_0 < x < a_0$ . Moreover,

$$\lim_{a \downarrow 0} [u_0(q, x, a_0)] = \mathbb{P} \{ \mathcal{T}(x, a_0) < \infty \mid \varepsilon(0) = 0 \} = p_0(x)$$

and

$$\lim_{q \downarrow 0} [u_1(q, x, a_0)] = \mathbb{P} \{ \mathcal{T}(x, a_0) < \infty \mid \varepsilon(0) = 1 \} = p_1(x)$$

are the probabilities to leave the interval  $(b_0, a_0)$  through the upper bound  $a_0$ , starting from the position X(0) = x with the initial state  $\xi(0) = 0$  and  $\xi(0) = 1$ , respectively.

The probabilities  $p_0(x)$  and  $p_1(x)$  depend on the following constants,

$$B_0 = \int_{b_0}^{a_0} \frac{\Psi(y)}{c_0 - U'(y)} dy, \qquad B_1 = \int_{b_0}^{a_0} \frac{\Psi(y)}{c_1 - U'(y)} dy.$$
 (21)

Here  $\Psi(y) = \exp(\lambda_0 \Phi_0(y) + \lambda_1 \Phi_1(y))$ .

The integrals in (21) converge, since

$$\left| \frac{\Psi(y)}{c_i - U'(y)} \right| \le const \cdot \exp\left(\lambda_i \Phi_i(y)\right) \cdot \left| \Phi_i'(y) \right|, \qquad y \in G_0,$$

and

$$\int_{b_0}^{a_0} e^{\lambda_i \Phi_i(y)} \left| \Phi_i'(y) \right| dy = \left| \exp \left( \lambda_i \Phi_i(y) \right) \right|_{y=b_0+}^{y=a_0-} < \infty.$$

See Fig.3.

**Theorem 3.3.** Let X(0) = x,  $x \in (b_0, a_0)$ . The probabilities  $p_0(x)$  and  $p_1(x)$  defined by (20) have the form:

$$p_0(x) = \mathbb{P}\{\mathscr{T}(x, a_0) < \infty \mid \xi(0) = 0\} = 1 - B_0^{-1} \int_x^{a_0} \frac{\exp(\lambda_0 \Phi_0(y) + \lambda_1 \Phi_1(y))}{c_0 - U'(y)} dy, \tag{22}$$

$$p_1(x) = \mathbb{P}\{\mathcal{T}(x, a_0) < \infty \mid \xi(0) = 1\} = B_1^{-1} \int_{b_0}^x \frac{\exp(\lambda_0 \Phi_0(y) + \lambda_1 \Phi_1(y))}{c_1 - U'(y)} dy, \tag{23}$$

where  $B_0, B_1$  are defined by (21).

Proof. By definition it follows that

$$p_0(x)|_{x \uparrow a_0} = 1, \qquad p_1(x)|_{x \downarrow b_0} = 0, \qquad q > 0.$$
 (24)

Conditioning on the first switching like in (10), similarly to (13), we obtain

$$u_0(0,x,a_0) = p_0(x) = e^{-\lambda_0 t_0^*(x,a_0)} + \int_0^{t_0^*(x,a_0)} \lambda_0 e^{-\lambda_0 \tau} p_1(\gamma_0(\tau,x)) d\tau.$$
 (25)

Since the process X leaving the interval  $G_0$  through the lower bound forever remains below  $b_0$ , then instead of (25) and (14) we obtain

$$u_1(0, x, a_0) = p_1(x) = \int_0^{t_1^*(x, b_0)} \lambda_1 e^{-\lambda_1 \tau} p_0(\gamma_1(\tau, x)) d\tau.$$
 (26)

Here  $t_0^*(x, a_0) = \Phi_0(a_0) - \Phi_0(x)$  and  $t_1^*(x, b_0) = \Phi_1(b_0) - \Phi_1(x)$ ; function  $\Phi_0$  is increasing, and function  $\Phi_1$  is decreasing on  $(b_0, a_0)$ .

From (25)-(26) in the usual way by applying  $L_0^x$  and  $L_1^x$  followed by integration by parts, differential equations similar to (16) and (17) are derived: first

$$(\Lambda + \mathcal{L})\mathbf{p}(x) = \mathbf{0}, \quad \mathbf{p}(x) = (p_0(x), p_1(x))^{\mathrm{T}},$$

and then

$$L_1^x L_0^x p_0(x) - (\lambda_1 L_0^x + \lambda_0 L_1^x) p_0(x) = 0,$$
  

$$L_0^x L_1^x p_1(x) - (\lambda_1 L_0^x + \lambda_0 L_1^x) p_1(x) = 0.$$
(27)

See Theorem 3.1.

Let

$$v_0(x) = L_0^x[p_0](x), v_1(x) = L_1^x[p_1](x).$$
 (28)

Note that the functions  $p_0(x)$  and  $p_1(x)$  increase,  $x \in G_0$ .

Since  $c_1 < U'(x) < c_0$ , then  $v_0(x) > 0$  and  $v_1(x) < 0, \forall x \in G_0$ . In these notations, equations (27) take a form

$$(c_1 - U'(x))\frac{\partial v_0}{\partial x}(x) - \lambda_1 v_0(x) - \lambda_0 \frac{c_1 - U'(x)}{c_0 - U'(x)} v_0(x) = 0,$$

$$(c_0 - U'(x))\frac{\partial v_1}{\partial x}(x) - \lambda_1 \frac{c_0 - U'(x)}{c_1 - U'(x)} v_1(x) - \lambda_0 v_1(x) = 0,$$

which can be rewritten as

$$\frac{dv_0}{dx}(x) = \psi(x)v_0(x), \qquad \frac{dv_1}{dx}(x) = \psi(x)v_1(x), b_0 < x < a_0,$$
(29)

where

$$\psi(x) = \frac{\lambda_0}{c_0 - U'(x)} + \frac{\lambda_1}{c_1 - U'(x)}.$$
(30)

From (29) and (30) by definition (6) we obtain

$$v_0(x) = A_0 \exp(\lambda_0 \Phi_0(x) + \lambda_1 \Phi_1(x)), \quad v_1(x) = A_1 \exp(\lambda_0 \Phi_0(x) + \lambda_1 \Phi_1(x))$$

with indefinite constants  $A_0$  and  $A_1$ .

Taking into account boundary conditions (24), we obtain

$$p_0(x) = -\int_x^{a_0} \frac{v_0(y)}{c_0 - U'(y)} dy + 1, \qquad p_1(x) = \int_{b_0}^x \frac{v_1(y)}{c_1 - U'(y)} dy.$$

By the no-blow-up condition (7), one can see that  $A_0 = B_0^{-1}$  and  $A_1 = B_1^{-1}$ , which gives the solution to equations (28) in the form (22)-(23).

## **3.2. Starting near attractor.** Let the initial point x and the threshold y be located near (or inside) the attractor.

We study only the case of the upper half-line,  $x, y > a_0$ , and the attractor  $G_+$ , see Fig.1(a) and (b). It is sufficient because the case of  $x, y < b_0$  and the attractor  $G_-$ , Fig.1(a) and (c), is symmetric.

Since both levels  $a_+$  and  $b_+$  are attractive, and  $a_0$ ,  $b_0$  are repulsive, the process X = X(t) increases a.s. as moving along the set  $(a_0, a_+)$  and decreases as moving along the set  $[b_+, +\infty)$ . Therefore,

$$\mathscr{T}(x,y)|_{x>y\geq a_0}=\infty, \qquad \mathscr{T}(x,y)|_{b_+\leq x\leq y}=\infty. \qquad a.s.$$

Inside the attractor  $G_+$ , the process X = X(t) randomly oscillates between the attractive levels  $a_+$  and  $b_+$ .

The mean value  $m(x,y) = (m_0(x,y), m_1(x,y))^T$ , can be expressed through the following integrals:

$$I_0(x,y) = \int_x^y \frac{\beta(z_0, y) dz_0}{c_0 - U'(z_0)}, \qquad I_1(x, y) = \int_x^y \frac{\beta(z_1, y) dz_1}{c_1 - U'(z_1)},$$
(31)

$$J_0(x,y) = \int_{\Delta_0(x,y)} \frac{\beta(z_0, z_1)}{(c_0 - U'(z_0))(c_1 - U'(z_1))} dz_0 dz_1,$$
(32)

$$J_1(x,y) = \int_{\Delta_1(x,y)} \frac{\beta(z_1, z_0)}{(c_0 - U'(z_0))(c_1 - U'(z_1))} dz_0 dz_1.$$
 (33)

Here  $\beta(x,y) = \frac{\Psi(x)}{\Psi(y)} = \exp\left(\lambda_0(\Phi_0(x) - \Phi_0(y)) + \lambda_1(\Phi_1(x) - \Phi_1(y))\right)$ , and  $\Delta_0(x,y), \Delta_1(x,y) \subset \mathbb{R}^2$  are two triangles defined by

$$\Delta_0(x,y) = \{ (z_0, z_1) \mid x < z_0 < y, z_0 < z_1 < y \}, \Delta_1(x,y) = \{ (z_0, z_1) \mid x < z_1 < y, z_1 < z_0 < y \}.$$

According to the above definitions, in both cases:  $a_0 < x < y \le a_+$  (with  $U'(x), U'(y) < c_1 < c_0$ ) and  $x > y \ge b_+$  (with  $U'(x), U'(y) > c_0 > c_1$ ), the integrals  $I_0(x,y), I_1(x,y)$  and  $J_0(x,y), J_1(x,y)$  converge.

## **Theorem 3.4.** • Let $x, y > a_0$ , see Fig. 1(a) and (b).

The mean values  $m_0(x, y)$  and  $m_1(x, y)$  are given explicitly as follows.

- In both cases,  $x < y < a_+$  and  $x > y > b_+$ , the following formulae are valid:

$$m_0(x,y) = I_0(x,y) + (\lambda_0 + \lambda_1)J_0(x,y),$$
 (34)

$$m_1(x, y) = I_1(x, y) + (\lambda_0 + \lambda_1)J_1(x, y).$$
 (35)

- *If* x < y *and*  $y ∈ [a_+, b_+)$ , *then* 

$$m_0(x,y) = (1 + \lambda_0/\lambda_1)I_0(x,y) + (\lambda_0 + \lambda_1)J_0(x,y),$$
 (36)

$$m_1(x,y) = \frac{1}{\lambda_1} + (\lambda_0 + \lambda_1)J_1(x,y).$$
 (37)

- If x > y and  $y \in (a_+, b_+]$ , then

$$m_0(x,y) = \frac{1}{\lambda_0} + (\lambda_0 + \lambda_1)J_0(x,b_+),$$
 (38)

$$m_1(x,y) = (1 + \lambda_0/\lambda_1)I_1(x,b_+) + (\lambda_0 + \lambda_1)J_1(x,b_+). \tag{39}$$

Here  $I_0$ ,  $I_1$  and  $J_0$ ,  $J_1$  are defined by (31)-(33).

• Let  $x, y < b_0$ , see Fig.1(a) and (c).

The mean values  $m_0(x,y)$  and  $m_1(x,y)$  are determined by the following formulae, symmetrical to those given above.

- In both cases,  $x > y > b_{-}$  and  $x < y < a_{-}$ , the following formulae are valid:

$$m_0(x, y) = I_0(x, y) + (\lambda_0 + \lambda_1)J_0(x, y),$$
 (40)

$$m_1(x,y) = I_1(x,y) + (\lambda_0 + \lambda_1)J_1(x,y).$$
 (41)

- If x > y and  $y \in [a_-, b_-)$ , then

$$m_0(x,y) = \frac{1}{\lambda_0} + (\lambda_0 + \lambda_1)J_0(x,y),$$
 (42)

$$m_1(x,y) = (1 + \lambda_1/\lambda_0)I_1(x,y) + (\lambda_0 + \lambda_1)J_1(x,y); \tag{43}$$

- If x < y and  $y ∈ (a_-, b_-]$ , then

$$m_0(x, y) = (1 + \lambda_1/\lambda_0)I_0(x, b_+) + (\lambda_0 + \lambda_1)J_0(x, b_+), \tag{44}$$

$$m_1(x,y) = \frac{1}{\lambda_1} + (\lambda_0 + \lambda_1)J_1(x,b_+).$$
 (45)

Notice that X is structured in such a way that if this process begins with  $x \in (a_0, a_+]$  or with  $x \in [b_+, +\infty)$ , (symmetrically,  $x \in [b_-, b_0)$  or  $x \in (-\infty, a_-]$ ) then with probability 1 it ends up inside the set  $G_+$   $(G_-)$  and remains there forever, randomly fluctuating between two attracting thresholds  $a_+$  and  $b_+$ .

In the rest of the paper, all proofs for the lower half-line ( $x < b_0$ ) are symmetric to the proofs for  $x > a_0$ , so we present only the latter.

To prove Theorem 3.4, we need some auxiliary statement, which is important in itself.

**Lemma 3.5.** Under the conditions of Theorem 3.4, the mean value of  $\mathcal{T}(x,y)$  is finite, and the entries  $m_0(x,y)$  and  $m_1(x,y)$  of the mean vector m(x,y) are positive solutions of equation (19). Furthermore,

- If  $a_0 < x < y < a_+$   $(b_0 > x > y > b_-)$  or  $x > y > b_+$   $(x < y < a_-)$ , then (19) is supplied with zero boundary condition  $m(x,y)|_{x\to y} = 0$ .
- *If* x < y *and*  $y \in [a_+, b_+)$ , *then*

$$m_0(x,y)|_{x\to y} = 0, \qquad m_1(x,y)|_{x\to y} = 1/\lambda_1;$$
 (46)

if x > y and  $y \in (a_-, b_-]$ , then

$$m_0(x,y)|_{x\to y} = 1/\lambda_0, \qquad m_1(x,y)|_{x\to y} = 0.$$
 (47)

• *If* x > y *and*  $y \in (a_+, b_+]$ , *then* 

$$m_0(x,y)|_{x\to y} = 1/\lambda_0, \qquad m_1(x,y)|_{x\to y} = 0;$$
 (48)

if x < y and  $y \in [a_-, b_-)$ , then

$$m_0(x,y)|_{x\to y} = 0, \qquad m_1(x,y)|_{x\to y} = 1/\lambda_1.$$
 (49)

Moreover,  $m_0(x, y)$  and  $m_1(x, y)$  satisfy separate second-order equations

$$\begin{cases}
L_1^x L_0^x m_0(x, y) - (\lambda_1 L_0^x + \lambda_0 L_1^x) m_0(x, y) = \lambda_0 + \lambda_1, \\
L_0^x L_1^x m_1(x, y) - (\lambda_1 L_0^x + \lambda_0 L_1^x) m_1(x, y) = \lambda_0 + \lambda_1.
\end{cases}$$
(50)

*Proof.* First note that if  $a_0 < x < y < a_+$  or  $x > y > b_+$ , then  $t_0^*(x,y)$  and  $t_1^*(x,y)$  are both finite,  $t_0^*(x,y)|_{x\to y} = t_1^*(x,y)|_{x\to y} = 0$ , functions  $u_0$ ,  $u_1$  follow the coupled integral equations equations (13),  $\mathcal{T}(x,y) < \infty$  a.s. and  $u_0(0,x,y) = u_1(0,x,y) = 1$ . Differentiating in (13) with respect to q and then setting q = 0, we obtain the integral equations

$$\begin{cases}
m_{0}(x,y) = t_{0}^{*}(x,y)e^{-\lambda_{0}t_{0}^{*}(x,y)} \\
+ \int_{0}^{t_{0}^{*}(x,y)} \lambda_{0}\tau e^{-\lambda_{0}\tau} d\tau + \int_{0}^{t_{0}^{*}(x,y)} \lambda_{0}e^{-\lambda_{0}\tau} m_{1}(\gamma_{0}(\tau,x),y) d\tau, \\
m_{1}(x,y) = t_{1}^{*}(x,y)e^{-\lambda_{1}t_{1}^{*}(x,y)} \\
+ \int_{0}^{t_{1}^{*}(x,y)} \lambda_{1}\tau e^{-\lambda_{1}\tau} d\tau + \int_{0}^{t_{1}^{*}(x,y)} \lambda_{1}e^{-\lambda_{1}\tau} m_{0}(\gamma_{1}(\tau,x),y) d\tau.
\end{cases} (51)$$

After simplification, system (51) becomes

$$\begin{cases}
m_0(x,y) = \frac{1 - e^{-\lambda_0 t_0^*(x,y)}}{\lambda_0} + \int_0^{t_0^*(x,y)} \lambda_0 e^{-\lambda_0 \tau} m_1(\gamma_0(\tau,x), y) d\tau, \\
m_1(x,y) = \frac{1 - e^{-\lambda_1 t_1^*(x,y)}}{\lambda_1} + \int_0^{t_1^*(x,y)} \lambda_1 e^{-\lambda_1 \tau} m_0(\gamma_1(\tau,x), y) d\tau.
\end{cases} (52)$$

Note that in the cases A and B, the interval  $G_+$  attracts both patterns,  $\gamma_0(t,x)$  and  $\gamma_1(t,x)$ . Therefore,  $t_i^*(x,y)|_{x\to y}=0$  if  $x< y< a_+$  or  $x>y>b_+$ . By (52), this yields the zero boundary condition for (19):  $m_i(x,y)|_{x\to y}=0$ ,  $i\in\{0,1\}$ .

If  $x < y = a_+$ , then  $t_0^*(x,y)$  is still finite,  $t_0^*(x,y)\big|_{x\uparrow y} = 0$  and  $t_1^*(x,y) = \infty$ . In this case, the first equations in (51) and (52) remain the same. Hence,  $m_0(x,y)\big|_{x\uparrow y} = 0$ . The second equation in (52) takes the form

$$m_1(x,y) = \frac{1}{\lambda_1} + \int_0^\infty \lambda_1 e^{-\lambda_1 \tau} m_0(\gamma_1(\tau,x), y) d\tau,$$

which gives the same differential equation as before and the boundary condition (46).

If  $x > y = b_+$ , then similar reasoning again leads to the boundary condition (48).

By eliminating variables (as in Theorem 3.1) one can see that  $m_0(x, y)$  and  $m_1(x, y)$  satisfy two separate second-order equations (50).

The proof of Theorem 3.4 is similar to the proof of Theorem 3.3. It consists of two steps. First, we transform the system (50) of the second-order equations into a system of first-order equations for the functions

$$v_0(x,y) := L_0^x m_0(x,y), \qquad v_1(x,y) := L_1^x m_1(x,y).$$
 (53)

Note that (53) can be solved in the form:

$$m_0(x,y) = m_0(x,y)|_{x=y} - \int_x^y \frac{v_0(z_0,y)}{c_0 - U'(z_0)} dz_0,$$

$$m_1(x,y) = m_1(x,y)|_{x=y} - \int_x^y \frac{v_1(z_1,y)}{c_1 - U'(z_1)} dz_1,$$
(54)

Second, formulae (54) will give the answer if we find  $v_0$  and  $v_1$ . *1st step*. By definition of  $L_0^x$  and  $L_1^x$ ,

$$L_0^x m_1(x,y) = \frac{c_0 - U'(x)}{c_1 - U'(x)} \cdot v_1(x,y), \qquad L_1^x m_0(x,y) = \frac{c_1 - U'(x)}{c_0 - U'(x)} \cdot v_0(x,y).$$

Applying the change of variables (53) to (50), we get

$$\begin{split} &(c_1 - U'(x)) \frac{\partial v_0(x,y)}{\partial x} - \lambda_1 v_0(x,y) - \lambda_0 \frac{c_1 - U'(x)}{c_0 - U'(x)} v_0(x,y) = \lambda_0 + \lambda_1, \\ &(c_0 - U'(x)) \frac{\partial v_1(x,y)}{\partial x} - \lambda_1 \frac{c_0 - U'(x)}{c_1 - U'(x)} v_1(x,y) - \lambda_0 v_1(x,y) = \lambda_0 + \lambda_1. \end{split}$$

Thus, system (50) is equivalent to

$$\begin{cases}
\frac{\partial v_0}{\partial x}(x,y) = \psi(x)v_0(x,y) + \frac{\lambda_0 + \lambda_1}{c_1 - U'(x)}, \\
\frac{\partial v_1}{\partial x}(x,y) = \psi(x)v_1(x,y) + \frac{\lambda_0 + \lambda_1}{c_0 - U'(x)},
\end{cases} (55)$$

where  $\psi(x)$  is defined by (30),  $\psi(x) = \frac{\lambda_0}{c_0 - U'(x)} + \frac{\lambda_1}{c_1 - U'(x)} = \Psi'(x)/\Psi(x)$ . The solution of (55) is determined by the boundary conditions that follow from Lemma 3.5.

2d step. Consider the case  $a_0 < x < y < a_+$ , with  $U'(x), U'(y) < c_1 < c_0$ . The case  $x > y > b_+$  is symmetric with  $U'(x), U'(y) > c_0 > c_1$ .

Since in this case,  $m(x,y)|_{x\to y} = 0$  (Lemma 3.5), then by virtue of (19), system (55) is supplied with the boundary conditions

$$|v_0(x,y)|_{x\uparrow y} = |v_1(x,y)|_{x\uparrow y} = -1.$$
 (56)

Therefore.

$$v_{0}(z_{0}, y) = \Psi(z_{0})\Psi(y)^{-1} \left[ -1 - (\lambda_{0} + \lambda_{1}) \int_{z_{0}}^{y} \frac{\Psi(y)\Psi(z_{1})^{-1}}{c_{1} - U'(z_{1})} dz_{1} \right]$$

$$= -\beta(z_{0}, y) - (\lambda_{0} + \lambda_{1}) \int_{z_{0}}^{y} \frac{\beta(z_{0}, z_{1})}{c_{1} - U'(z_{1})} dz_{1},$$

$$v_{1}(z_{1}, y) = \Psi(z_{1})\Psi(y)^{-1} \left[ -1 - (\lambda_{0} + \lambda_{1}) \int_{z_{1}}^{y} \frac{\Psi(y)\Psi(z_{0})^{-1}}{c_{0} - U'(z_{0})} dz_{0} \right]$$

$$= -\beta(z_{1}, y) - (\lambda_{0} + \lambda_{1}) \int_{z_{1}}^{y} \frac{\beta(z_{1}, z_{0})}{c_{0} - U'(z_{0})} dz_{0}, \quad z_{0}, z_{1} < y.$$
(58)

Formulae (34)-(35) follow from (54) and (57)-(58).

In the case  $a_0 < x < y = a_+$ , by (46) and (19), we obtain the boundary conditions for (55):

$$v_0(x,y)|_{x\uparrow y} = -1 - \lambda_0/\lambda_1, \qquad v_1(x,y)|_{x\uparrow y} = 0.$$

Similar to (57), we obtain

$$v_0(z_0, y) = -\beta(z_0, y) (1 + \lambda_0/\lambda_1) - (\lambda_0 + \lambda_1) \int_{z_0}^{y} \frac{\beta(z_0, z_1)}{c_1 - U'(z_1)} dz_1,$$
 (59)

and

$$v_1(z_1, y) = -(\lambda_0 + \lambda_1) \int_{z_1}^{y} \frac{\beta(z_1, z_0)}{c_0 - U'(z_0)} dz_0.$$
 (60)

Formulae (36) and (37) follow from (54) and (59)-(60).

The rest of the proof follows by symmetry.

**3.3. Large parameters.** If both parameters are large, i.e.  $c_0 > V$  and  $c_1 < v$ , see Fig.1(d), then both potentials,  $U(x) - c_0 x$  and  $U(x) - c_1 x$  are single-well, repulsion points disappear, and the attractors  $G_-$  and  $G_+$  merge into one  $G = [a_-, b_+]$ . In this case, process  $X^x(t), x \in G$ , randomly oscillates between two attracting stationary states  $x = a = a_-$  and  $x = b = b_+$ .

If both parameters are large, the mean first passage times is derived in closed form using the technique of Theorem 3.4.

**Theorem 3.6.** Let the parameters  $c_0, c_1$  be large, as described above.

The average values  $m_0(x,y) = \mathbb{E}[\mathscr{T}(x,y) \mid \xi(0) = 0]$  and  $m_1(x,y) = \mathbb{E}[\mathscr{T}(x,y) \mid \xi(0) = 1]$  are determined as follows:

- if x < y < b or x > y > a, then  $m_0$  and  $m_1$  are determined by formulae (34)-(35);
- if a < x < y < b, then formulae (36)-(37) are valid;
- if a < y < x < b, then formulae (38)-(39) are valid.

## 4. Stationary measure

Let the interval G = (a, b) be invariant under the dynamics X, see (9), so that  $c_1 \le U'(x) \le c_0$ ,  $x \in G$ , and  $U'(b) = c_0$ ,  $U'(a) = c_1$ , and thus  $G = G_-$  or  $G = G_+$ , see Fig. 1.

The explicit form of the operator  $\mathcal{L}^*$  adjoint to  $\mathcal{L}$ , (11), on the interval G is obtained by integrating by parts in the integral

$$\langle \mathscr{L}[\mathbf{f}](x), \mathbf{g}(x) \rangle_{L_2(G)} = \int_G \left[ (c_0 - U'(x)) f_0'(x) g_0(x) + (c_1 - U'(x)) f_1'(x) g_1(x) \right] dx$$

for any test-functions  $f = (f_0, f_1)$  and  $g = (g_0, g_1)$ . We have

$$\langle \mathscr{L}[\mathbf{f}](x), \, \mathbf{g}(x) \rangle_{L_2(G)} = \left[ (c_0 - U'(x)) f_0(x) g_0(x) \right] |_{\partial G} + \left[ (c_1 - U'(x)) f_1(x) g_1(x) \right] |_{\partial G}$$

$$- \int_G \left\{ f_0(x) \frac{\mathrm{d}}{\mathrm{d}x} \left[ (c_0 - U'(x)) g_0(x) \right] + f_1(x) \frac{\mathrm{d}}{\mathrm{d}x} \left[ (c_1 - U'(x)) g_1(x) \right] \right\} \mathrm{d}x.$$

Therefore, the formal adjoint to the infinitesimal generator  $\mathcal{L}$  is given by

$$\mathcal{L}^* \mathbf{g}(x) = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} [(c_0 - U'(x))g_0(x)] & 0\\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} [(c_1 - U'(x))g_1(x)] \end{pmatrix}$$
(61)

with a singular part corresponding to the boundary condition

$$g_0(a) = 0, g_1(b) = 0.$$
 (62)

As we show below, there exist invariant measures  $\mu^-$  and  $\mu^+$  supported on the attractors  $G_- = (a_-, b_-)$  and  $G_+ = (a_+, b_+)$ , respectively.

**Theorem 4.1.** Let G = (a,b) be an invariant attractor  $(G_- \text{ or } G_+)$ . The invariant measure with support G exists and is determined by the probability density functions  $\pi_0(x)$  and  $\pi_1(x)$ :

$$\pi_0(x) = C_0 \frac{\Psi(x)^{-1}}{c_0 - U'(x)}, \qquad \pi_1(x) = C_1 \frac{\Psi(x)^{-1}}{U'(x) - c_1}, \quad x \in G.$$

Here  $\Psi(x)^{-1} = \exp(-\lambda_0 \Phi_0(x) - \lambda_1 \Phi_1(x))$ , and  $C_0$ ,  $C_1$  are normalising constants,

$$C_0 = \frac{\lambda_1}{\lambda_0 + \lambda_1} \left( \int_a^b \frac{\Psi(x)^{-1} dx}{c_0 - U'(x)} \right)^{-1}, \qquad C_1 = \frac{\lambda_0}{\lambda_0 + \lambda_1} \left( \int_a^b \frac{\Psi(x)^{-1} dx}{U'(x) - c_1} \right)^{-1}.$$
 (63)

*Proof.* The invariant density  $\pi = (\pi_0(x), \pi_1(x)), x \in G$ , which is defined by

$$\mathbb{P}\{X(t) \in dx, \, \xi(t) = 0\} \equiv \pi_0(x)dx, \qquad \mathbb{P}\{X(t) \in dx, \, \xi(t) = 1\} \equiv \pi_1(x)dx,$$

follows the Fokker-Planck equation

$$(\Lambda^* + \mathcal{L}^*)\pi = \mathbf{0}, \quad x \in G$$

 $\Lambda^* = \Lambda^T$ , with boundary conditions  $\pi_0(a) = 0$  and  $\pi_1(b) = 0$ , see (61)-(62). That is, the entries  $\pi_0$  and  $\pi_1$  are positive solutions to the coupled equations

$$-\frac{\mathrm{d}}{\mathrm{d}x} \left[ (c_0 - U'(x))\pi_0(x) \right] - \lambda_0 \pi_0(x) + \lambda_1 \pi_1(x) = 0, \tag{64}$$

$$-\frac{d}{dx}[(c_1 - U'(x))\pi_1(x)] + \lambda_0\pi_0(x) - \lambda_1\pi_1(x) = 0, \tag{65}$$

a < x < b.

Summing up (64) and (65) we obtain

$$(c_0 - U'(x))\pi_0(x) + (c_1 - U'(x))\pi_1(x) \equiv const.$$

The constant is zero. It follows, for example, from  $U'(a) = c_1$ , and due to the boundary conditions, (62),  $\pi_0(a) = 0$ .

Therefore,

$$\frac{\pi_0(x)}{\pi_1(x)} = \frac{U'(x) - c_1}{c_0 - U'(x)}.$$
(66)

Equation (64) can be rewritten as

$$-\frac{[(c_0-U'(x))\pi_0(x)]'}{(c_0-U'(x))\pi_0(x)}-\frac{\lambda_0}{c_0-U'(x)}+\frac{\lambda_1}{c_0-U'(x)}\cdot\frac{\pi_1(x)}{\pi_0(x)}=0.$$

By (66), we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\log[(c_0-U'(x))\pi_0(x)] = -\psi(x), \qquad x \in G = (a,b),$$

where  $\psi(x)$  is defined before, (30), and  $\pi_0(a) = 0$ . Integrating, we obtain

$$\pi_0(x) = C_0 \frac{\exp(-\lambda_0 \Phi_0(x) - \lambda_1 \Phi_1(x))}{c_0 - U'(x)}, \qquad a < x < b,$$

where  $\Phi_0$  and  $\Phi_1$  are defined in (6). The boundary condition holds, since

$$\Phi_0|_{x\to a} < \infty, \qquad \Phi_1(x)|_{x\to a} = +\infty.$$

Similarly,

$$\pi_1(x) = C_1 \frac{\exp(-\lambda_0 \Phi_0(x) - \lambda_1 \Phi_1(x))}{U'(x) - c_1}, \qquad a < x < b.$$

Therefore,  $C_0$  and  $C_1$  are given by (63), since the invariant distribution of  $\xi(t)$  is determined by

$$\int_{a}^{b} \pi_{0}(x) dx = \mathbb{P}_{0} \{ \xi(t) = 0 \} = \frac{\lambda_{1}}{\lambda_{0} + \lambda_{1}}, \qquad \int_{a}^{b} \pi_{1}(x) dx = \mathbb{P}_{1} \{ \xi(t) = 0 \} = \frac{\lambda_{0}}{\lambda_{0} + \lambda_{1}}.$$

### Discussion

The mathematical interpretation of physical phenomena often uses very sophisticated and sometimes very artificial tools. For example, a very popular climate change model, widely studied in recent decades, is based on stochastic differential equation with periodic potential.

$$dX_{\varepsilon}(t) = -U'(X_{\varepsilon}(t), t/T_{\varepsilon})dt + \sigma(T_{\varepsilon})dW(t), \qquad t > 0.$$
(67)

Here the period  $T_{\varepsilon}$  is usually assumed to be exponentially large in  $\varepsilon$  and the periodic fluctuations of the potential are due to periodic changes in the parameters of the Earth's orbit (the so called Milankovitch astronomical cycles, [22]).

The key property postulated for this model is stochastic resonance. This methodology involves choosing the diffusion coefficient  $\sigma(T_{\varepsilon})$  taking into account the spectral properties of the mean solution of (67). In my opinion, this method looks artificial, since the using of white noise for such modelling (especially for climate models) is not confirmed. Moreover, it is worth noting that the global and long-term behaviour of the atmosphere does not undergo abrupt and rapid changes, and the climate usually changes very slowly. This makes the classical diffusion process with infinite variation and infinitely fast propagation to be problematic for describing climate change.

We propose to add to the (deterministic) continuous periodic changes of potential randomly atternating trends corresponding to internal processes in the atmosphere and at the surface, see (2). In other words, one can the modify model (67) replacing the white-noise term  $\sigma(T_{\varepsilon})dW(t)$  by a telegraph process  $\mathbb{T}(t)$ .

When and if this trend coincides with the direction of the periodic change in potential, the system can go into a metastable state receiving a time window for the interwell transition. Otherwise, this internal trend has only a retarding effect on the global changes that occur from the laws of celestial mechanics.

## Appendix: the proof of Theorem 2.1

Let  $\tau^0 \sim \text{Exp}(\lambda_0)$  be the time of the first switching of the process  $\xi(t)$  from state 0 to state 1, and  $\tau^1 \sim \text{Exp}(\lambda_1)$  be the time of the first switching from state 1 to state 0.

Let  $\mathcal{P}(t, dy \mid x) = (p_{ij}(t, dy \mid x))_{i,j \in \{0,1\}}$  be the transition probability matrix, where

$$p_{ij}(t, dy \mid x) = \mathbb{P}\{X(t) \in dy, \xi(t) = j \mid X(0) = x, \xi(0) = i\}.$$

The corresponding Markov semigroup  $P_t$  is defined on the test-function  $f = (f_0, f_1)$  as follows:

$$(P_{t}\mathbf{f})_{i}(x) = \mathbb{E}\left[\mathbf{f}(\Xi(t)) \mid \Xi(0) = (x,i)\right] = \int_{-\infty}^{\infty} f_{0}(y)p_{i0}(t,dy \mid x) + \int_{-\infty}^{\infty} f_{1}(y)p_{i1}(t,dy \mid x),$$

$$i \in \{0,1\}.$$

Conditioning on the first switching, see (10), we obtain the following system of integral equations:

$$\begin{cases} p_{0j}(t, dy \mid x) = e^{-\lambda_0 t} \delta_{\gamma_0(t, x)}(dy) + \int_0^t \lambda_0 e^{-\lambda_0 \tau} p_{1j}(t - \tau, dy \mid \gamma_0(\tau, x)) d\tau, \\ p_{1j}(t, dy \mid x) = e^{-\lambda_1 t} \delta_{\gamma_1(t, x)}(dy) + \int_0^t \lambda_0 e^{-\lambda_0 \tau} p_{0j}(t - \tau, dy \mid \gamma_1(\tau, x)) d\tau, \\ j \in \{0, 1\}. \end{cases}$$
(68)

Here  $\delta_a(dy)$  is Dirac's  $\delta$ -measure at point a.

By applying the operator  $\frac{\partial}{\partial t} - L_0^x$  to the first equation of (68), and the operator  $\frac{\partial}{\partial t} - L_1^x$  to the second, one can find that system of integral equations (68) is equivalent to some initial value problem for partial differential equations. Indeed, by virtue of (8),

$$\begin{cases} \frac{\partial p_{0j}(t,\mathrm{d}y\mid x)}{\partial t} - L_0^x[p_{0j}(t,\mathrm{d}y\mid x)] = -\lambda_0 \mathrm{e}^{-\lambda_0 t} \delta_{\gamma_0(t,x)}(\mathrm{d}y) \\ + \lambda_0 \mathrm{e}^{-\lambda_0 t} p_{1j}(0,\mathrm{d}y\mid \gamma_0(t,x)) - \int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 \tau} \frac{\partial}{\partial \tau} \left[ p_{1j}(t-\tau,\mathrm{d}y\mid \gamma_0(\tau,x)) \right] \mathrm{d}\tau, \\ \frac{\partial p_{1j}(t,\mathrm{d}y\mid x)}{\partial t} - L_1^x[p_{0j}(t,\mathrm{d}y\mid x)] = -\lambda_1 \mathrm{e}^{-\lambda_1 t} \delta_{\gamma_1(t,x)}(\mathrm{d}y) \\ + \lambda_1 \mathrm{e}^{-\lambda_1 t} p_{0j}(0,\mathrm{d}y\mid \gamma_1(t,x)) - \int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 \tau} \frac{\partial}{\partial \tau} \left[ p_{0j}(t-\tau,\mathrm{d}y\mid \gamma_1(\tau,x)) \right] \mathrm{d}\tau. \end{cases}$$

Integrating by parts, we obtain

$$\begin{cases} \frac{\partial p_{0j}(t, \operatorname{d}y \mid x)}{\partial t} - L_0^x[p_{0j}(t, \operatorname{d}y \mid x)] = -\lambda_0 p_{0j}(t, \operatorname{d}y \mid x) + \lambda_0 p_{1j}(t, \operatorname{d}y \mid x), \\ \frac{\partial p_{1j}(t, \operatorname{d}y \mid x)}{\partial t} - L_1^x[p_{1j}(t, \operatorname{d}y \mid x)] = \lambda_1 p_{0j}(t, \operatorname{d}y \mid x) - \lambda_1 p_{1j}(t, \operatorname{d}y \mid x), \\ j \in \{0, 1\}, \end{cases}$$

which can be rewritten in matrix form

$$\frac{\partial \mathscr{D}(t, \mathrm{d}y \mid x)}{\partial t} = (\Lambda + \mathscr{L}^x) \left[ \mathscr{D}(t, \mathrm{d}y \mid x) \right], \qquad t > 0,$$

with initial condition

$$\mathscr{P}(t, \mathrm{d}y \mid x)|_{t\downarrow 0} = \begin{pmatrix} \delta_x(\mathrm{d}y) & 0\\ 0 & \delta_x(\mathrm{d}y) \end{pmatrix}.$$

The research was supported by the Russian Science Foundation (RSF), project number 24-21-00245, https://rscf.ru/project/24-21-00245

## Acknowlegements

This research was inspired by a report given at the United Seminar of the Department of Probability Theory of Moscow State University, by Professor A.N.Shiryaev, https://www.youtube.com/watch?v=KWzp5ruOSP4

I am very grateful for opportunity to participate in this seminar.

I am also grateful to the reviewers and the Associate Editor for their careful reading of the manuscript and very helpful suggestions that made the text much better.

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