# Infinitely many groups exhibiting intermediate growth in maximal sum-free sets

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#### Abstract

Given an Abelian groups G, denote  $\mu(G)$  the size of its largest sum-free subset and  $f_{\max}(G)$  the number of maximal sum-free sets in G. Confirming a prediction by Liu and Sharifzadeh, we prove that all even-order  $G \neq \mathbb{Z}_2^k$  have exponentially fewer maximal sum-free sets than  $\mathbb{Z}_2^k$ , i.e.  $f_{\max}(G) \leq 2^{(1/2-c)\mu(G)}$ , where  $c > 10^{-64}$ .

We construct an infinite family of Abelian groups G with intermediate growth in the number of maximal sum-free sets, i.e., with  $2^{(\frac{1}{2}+c)\mu(G)} \leq f_{\max}(G) \leq 3^{(\frac{1}{3}-c)\mu(G)}$ , where  $c=10^{-4}$ . This disproves a conjecture of Liu and Sharifzadeh and also answers a question of Hassler and Treglown in the negative.

Furthermore, we determine for every even-order group G, the number of maximal distinct sum-free sets (where a distinct sum is a+b=c with distinct a,b,c): it is  $2^{(1/2+o(1))\mu(G)}$  with the only exception being  $G=\mathbb{Z}_2^k\oplus\mathbb{Z}_3$ , when this function is  $3^{(1/3+o(1))\mu(G)}$ , refuting a conjecture of Hassler and Treglown.

Our proofs rely on a container theorem due to Green and Ruzsa. Other key ingredient is a sharp upper bound we establish on the number of maximal independent sets in graphs with given matching number, which interpolates between the classical results of Moon and Moser, and Hujter and Tuza. A special case of our bound implies that every n-vertex graph with a perfect matching has at most  $2^{n/2}$  maximal independent sets, resolving another conjecture of Hassler and Treglown.

## 1 Introduction

Sum-free subsets of the integers or Abelian groups have been widely studied in additive combinatorics. Let  $[n] := \{1, \ldots, n\}$ . We call a triple  $\{x, y, z\}$  a Schur triple if x + y = z. A set  $S \subseteq [n]$  is sum-free if S contains no Schur triple. Denote by f(n) the number of sum-free subsets of [n], and by  $f_{\max}(n)$  the number of maximal sum-free sets of [n]. Note that all subsets of the odd integers and  $\{\lfloor n/2 \rfloor + 1, \ldots, n\}$  are sum-free, hence  $f(n) \geq 2^{n/2}$ . In [4], Cameron and Erdős conjectured that  $f(n)/2^{n/2}$  tends to two different constants, depending on the parity of n, which was confirmed independently by Green [8] and Sapozhenko [20]. In a subsequent paper, Cameron and Erdős [5] posed the problem of determining  $f_{\max}(n)$ . This was resolved by the first and third authors, jointly with Sharifzadeh and Treglown [1, 2], who showed that  $f_{\max}(n)/2^{n/4}$  tends to four constants depending on n modulo 4.

The problem of counting sum-free sets in Abelian groups is more intricate. Let  $\mu(G)$  denote the maximum size of sum-free subsets of an Abelian group G. The study of  $\mu(G)$  dates back to the work of

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Diananda and Yap [6] in 1969, and it was finally determined for every group by Green and Ruzsa [10]. Let G be an Abelian group of order n. If n is divisible by a prime  $p \equiv 2 \pmod{3}$ , then we call G type I(p) for the smallest such p. If n is not divisible by any prime  $p \equiv 2 \pmod{3}$ , but 3|n, then G is type II. Otherwise, we say that G is type III. It was proved [6, 10] that if G is type I(p), then  $\mu(G) = (1/3 + 1/3p)n$ ; if G is type II, then  $\mu(G) = n/3$ ; and if G is type III, then  $\mu(G) = (1/3 - 1/3m)n$ , where m is the exponent of G.

Denote by f(G) and  $f_{\max}(G)$  the number of sum-free subsets and the number of maximal sum-free subsets of G, respectively. Green and Ruzsa [10] proved that  $f(G) = 2^{(1+o(1))\mu(G)}$ . Let us define the growth of  $f_{\max}(G)$  to be the smallest  $g \in \mathbb{R}$  such that

$$f_{\max}(G) \le g^{(1+o(1))\mu(G)}$$
.

Balogh, Liu, Sharifzadeh and Treglown [2] proved that, similarly to the integer setting,  $f_{\max}(G)$  is exponentially smaller than the total count f(G) for all finite Abelian groups G:  $f(G) \leq 3^{(1/3+o(1))\mu(G)}$ . In other words,  $f_{\max}(G)$  has growth at most  $3^{1/3}$  for every group G.

Recall that in the integer setting, the growth of  $f_{\text{max}}([n])$  is  $2^{1/2}$ . It was asked in [2] whether the growth of  $f_{\text{max}}(G)$  is also at most  $2^{1/2}$ ; they showed that it is the case for  $\mathbb{Z}_2^k$ . It was also conjectured [2] that there are infinitely many groups for which the growth of  $f_{\text{max}}$  is strictly less than  $2^{1/2}$ . For the former question, Liu and Sharifzadeh [16] gave a negative answer, showing that there are infinitely many groups for which  $f_{\text{max}}$  are of maximum growth  $3^{1/3}$ . They settled the latter conjecture affirmatively, showing that for almost all even-order groups, the growth of  $f_{\text{max}}$  are at most  $2^{1/2-c}$  for some absolute constant c > 0. In [16], it was mentioned that "Our result suggests that  $\mathbb{Z}_2^k$  might be the only exception among all even order groups achieving the bound  $2^{(1/2+o(1))\mu(G)}$ ."

Our first result confirms this belief.

**Theorem 1.1.** There exists a constant  $c > 10^{-64}$  and an integer  $n_0$  such that the following holds. Let G be an Abelian group with even order  $n > n_0$ . If  $G \neq \mathbb{Z}_2^k$ , then

$$f_{\max}(G) \le 2^{(1/2-c)\mu(G)}$$
.

In [16], Liu and Sharifzadeh made an even bolder conjecture that the same is true for all type I groups.

Conjecture 1.2 ([16]). All type I groups G except  $\mathbb{Z}_2^k$  have exponentially fewer maximal sum-free sets than  $2^{\mu(G)/2}$ .

To support this, they showed that this is the case for type I(5) groups. Recently, a related question was raised by Hassler and Treglown [13]. In light of no examples of intermediate growth between  $2^{1/2}$  and  $3^{1/3}$ , they asked the following question.

Question 1.3 ([13]). Given an Abelian group G of order n, is it true that either  $f_{\max}(G) = 3^{\mu(G)/3 + o(n)}$  or  $f_{\max}(G) \leq 2^{\mu(G)/2 + o(n)}$ ?

Our next result provides explicit constructions of infinitely many groups with intermediate growth in the number of maximal sum-free sets. This gives negative answers to both Theorem 1.2 and Theorem 1.3.

**Theorem 1.4.** Let  $c = 10^{-4}$ . There are infinitely many Abelian groups G of type I such that

$$2^{(\frac{1}{2}+c)\mu(G)} < f_{\max}(G) < 3^{(\frac{1}{3}-c)\mu(G)}$$
.

In [13], Hassler and Treglown initiated the study of distinct sum-free subsets of Abelian groups. A triple  $\{x,y,z\}$  is a distinct Schur triple if x+y=z and x,y,z are distinct. A set  $S\subseteq G$  is distinct sum-free if S contains no distinct Schur triple. Denote by  $\mu^*(G)$  the maximum size of distinct sum-free subsets of G, by  $f^*(G)$  the number of distinct sum-free sets of G, and by  $f^*_{\max}(G)$  the number of maximal distinct sum-free sets of G.

Hassler and Treglown [12] observed that using the removal lemma of Green [9], one has  $\mu(G) \leq$  $\mu^*(G) \le \mu(G) + o(n)$  and  $f(G) \le f^*(G) \le 2^{o(n)} \cdot f(G)$ . By a method similar to the proof of [2][Proposition 5.1], one also knows that  $f_{\max}^*(G) \le 3^{\mu^*(G)/3 + o(n)} = 3^{\mu(G)/3 + o(n)}$ . Hassler and Treglown [13] conjectured that  $f_{\text{max}}^*$  of all type I Abelian groups have growth  $2^{1/2}$ .

Conjecture 1.5 ([13]). For all type I groups G,  $f_{\max}^*(G) = 2^{(1/2+o(1))\mu(G)}$ .

They examined even-order groups in depth, giving a construction to prove the lower bound  $f_{\max}^*(G) \ge$  $2^{(n-2)/4}$ . For the upper bound, they showed the validity of their conjecture for almost all even-order groups.

We disprove Theorem 1.5 with counterexamples of type I(p) for p=2 and every  $p\geq 23$ . We show that a relaxed version of the conjecture is true, that is, it is true for all but one even-order group.

**Theorem 1.6.** For every even-order group G,

$$f_{\text{max}}^*(G) = 2^{(1/2 + o(1))\mu(G)},$$

with the only exception  $f_{\max}^*(\mathbb{Z}_2^k \oplus \mathbb{Z}_3) = 3^{(1+o(1))\mu(G)/3}$ . For every  $p \geq 23$  with  $p \equiv 2 \pmod 3$ , there are type I(p) groups with  $f_{\max}^*(G)$  exponentially larger than  $2^{\mu(G)/2}$ .

A key tool in our proofs is an optimal bound on the number of maximal independent sets in graphs with given matching number. Denote by  $MIS(\Gamma)$  the family of maximal independent sets in a graph  $\Gamma$ , and  $\operatorname{mis}(\Gamma) = |\operatorname{MIS}(\Gamma)|$ . Moon and Moser [17] proved that for every n-vertex graph  $\Gamma$ ,  $\operatorname{mis}(\Gamma) \leq 3^{n/3}$ , with equality if and only if  $\Gamma$  is the vertex disjoint union of n/3 triangles. Upper bounds on mis( $\Gamma$ ) have been provided for various classes of graphs, such as connected graphs [7, 11] and trees [19, 23]. An important extension is due to Hujter and Tuza [14]. They proved that for every n-vertex triangle-free graph  $\Gamma$ , mis( $\Gamma$ )  $\leq 2^{n/2}$ , with equality if and only if  $\Gamma$  is the vertex disjoint union of n/2 edges.

Recently, Kahn and Park [15] proved stability versions of results of Moon-Moser and Hujter-Tuza. They improved the exponents by  $\Omega_{\delta}(n)$  in case  $\Gamma$  contains at most  $(1/3 - \delta)n$  vertex disjoint triangles with no edges between the triangles (i.e., the induced triangle matching number is at most  $(1/3 - \delta)n$ ), and in case when  $\Gamma$  is triangle-free and the induced matching number is at most  $(1/2-\delta)n$ , for some small constant  $\delta > 0$ . Kahn and Park's results were subsequently strengthened by Palmer and Park's [18]. In particular, they proved that if an n-vertex graph does not contain an induced triangle matching of size t+1, then it has at most  $3^t \cdot 2^{m/2}$  maximal independent sets, where  $0 \le t \le n/3$  and m=n-3t is even. In [13], Hassler and Treglown proposed the following conjecture.

Conjecture 1.7 ([13]). Let  $\Gamma$  be an n-vertex graph that contains a perfect matching. Then

$$\operatorname{mis}(\Gamma) \le 2^{n/2}$$
.

We resolve this conjecture in a stronger form by establishing a thorough relationship between the matching number and the number of maximal independent sets. We denote by  $D_6$  the graph which consists of two vertex disjoint triangles joined by an edge.

**Theorem 1.8.** Let  $\Gamma$  be an n-vertex graph with matching number  $\nu(\Gamma) = m$ . Then,

$$\operatorname{mis}(\Gamma) \le \begin{cases} 3^m & m \le \frac{n}{3}, \\ 2^{3m-n} \cdot 3^{n-2m} & m > \frac{n}{3}. \end{cases}$$

In the first case, equality holds if and only if each component of  $\Gamma$  is a  $K_3$  or an isolated vertex. In the second case, equality holds if and only if each component of  $\Gamma$  is a  $K_2$ ,  $K_3$ ,  $K_4$  or  $D_6$ .

The first part of Theorem 1.8 was proved by Shi, Tu and Wang [22], additionally, they also proved several results on connected graphs with given matching number. For completeness, we include a different proof.

The bulk of the proof is to handle the second part. Note that the special case of the second part when m=n/2 above verifies Theorem 1.7. To remember Theorem 1.8, a graph that is helpful to keep in mind is the vertex disjoint union of t triangles and s edges. In this case, n=2s+3t and m=s+t, and the number of maximal independent sets is exactly  $2^s3^t=2^{3m-n}\cdot 3^{n-2m}$ . Thus, Theorem 1.8 interpolates smoothly between the classical bounds given by Moon-Moser and Hujter-Tuza. Considering the wide applicability of the Moon-Moser and Hujter-Tuza Theorems, we anticipate that Theorem 1.8 will serve as a fundamental tool with applications beyond this work.

Comparing to the result of Palmer and Patkós [18], the advantage of Theorem 1.8 is that it is often difficult to estimate the *induced* triangle matching number of a graph, whereas getting bounds on the matching number is a more manageable task. For instance, this is how we achieve the upper bound in Theorem 1.4. Indeed, we use  $\operatorname{mis}(\Gamma)$  of certain auxiliary graph  $\Gamma$  to bound  $f_{\max}(G)$ , and then we identify a large matching in  $\Gamma$  to invoke Theorem 1.8. Also, Theorem 1.8 gives for some graphs stronger bounds. For instance, let  $\Gamma$  be vertex disjoint union of n/6 triangles and n/8  $K_4$ 's. Then  $\operatorname{mis}(\Gamma) = 3^{n/6} \cdot 2^{n/4}$ , which coincides with our upper bound in Theorem 1.8, while the bound by Palmer and Patkós gives only  $3^{7n/24} \cdot 2^{n/16} \gg 3^{n/6} \cdot 2^{n/4}$ , as  $\Gamma$  contains an induced triangle matching of size 7n/24.

Finally, we remark that the vanilla version of Theorem 1.8 does not suffice for all of our applications. For instance, for Theorems 1.1 and 1.6, we in fact need several stability versions of Theorem 1.8, which provides further (exponential) improvement when the graph is far away from the extremal examples (see Theorems 4.2, 4.3 and 5.1).

**Organization.** The rest of the paper is organized as follows. We first prove Theorem 1.8 in Section 2, then Theorem 1.4 in Section 3, Theorem 1.6 in Section 4, and Theorem 1.1 in Section 5, respectively.

# 2 Maximal independent sets given matching number

In this section we shall prove Theorem 1.8. We split the proof into two parts. The first part follows from the statement below.

**Theorem 2.1.** Suppose  $\Gamma$  is an n-vertex graph with matching number  $\nu(\Gamma) = m$ . If  $\Gamma$  is not a vertex-disjoint union of  $K_3$ 's and isolated vertices, then  $\min(\Gamma) < 3^m$ .

The second part of Theorem 1.8 follows from the following theorem.

**Theorem 2.2.** Suppose that  $\Gamma$  is an n-vertex graph with matching number  $\nu(\Gamma) = m$ . If  $\Gamma$  is not a vertex-disjoint union of  $K_2$ 's,  $K_3$ 's,  $K_4$ 's and  $D_6$ 's, then  $\min(\Gamma) < 2^{3m-n}3^{n-2m}$ .

Note that in Theorems 2.1 and 2.2 we do not have restriction on how large  $\nu(\Gamma)$  is.

To establish these bounds, we start with some notation and observations. For a set X and  $x \in X$ , we write  $X - x := X \setminus \{x\}$ . For a graph  $\Gamma$  and a vertex set X, we write  $\Gamma - X := \Gamma[V(\Gamma) \setminus X]$ . Denote by  $\mathrm{MIS}(\Gamma; v)$  the family of maximal independent sets containing v, and by  $\mathrm{MIS}(\Gamma; \overline{v}) = \mathrm{MIS}(\Gamma) \setminus \mathrm{MIS}(\Gamma; v)$  those not containing v, and let  $\mathrm{mis}(\Gamma; v)$  and  $\mathrm{mis}(\Gamma; \overline{v})$  denote the cardinality of these two sets, respectively. The lemma below collects some simple facts about how  $\mathrm{mis}(\Gamma)$  changes by removal of vertices.

**Lemma 2.3.** Let  $\Gamma$  be a graph and v be an arbitrary vertex of  $\Gamma$ . Then the followings hold.

- (i) For every induced subgraph  $\Gamma' \subseteq \Gamma$ , we have  $\min(\Gamma') \leq \min(\Gamma)$ .
- (ii)  $mis(\Gamma; v) = mis(\Gamma v N(v)).$
- (iii)  $\operatorname{mis}(\Gamma; \overline{v}) \leq \operatorname{mis}(\Gamma v)$ .
- (iv)  $\operatorname{mis}(\Gamma; \overline{v}) \leq \sum_{u \in N(v)} \operatorname{mis}(\Gamma; u)$ .

(v) Let uv be an edge. Then

$$\operatorname{mis}(\Gamma) \le \operatorname{mis}(\Gamma - v - N(v)) + \operatorname{mis}(\Gamma - u - N(u)) + \operatorname{mis}(\Gamma - v; \overline{u}).$$

*Proof.* (i) For each  $I' \in \mathrm{MIS}(\Gamma')$ , there is at least one maximal independent set  $I \in \mathrm{MIS}(\Gamma)$  such that  $I' \subseteq I$ . Furthermore, for  $I', J' \in \mathrm{MIS}(\Gamma')$  with  $I' \neq J'$ , by the maximality of I' and J' in  $\Gamma'$ , there is no  $I \in \mathrm{MIS}(\Gamma)$  such that  $I', J' \subseteq I$ . Hence  $\mathrm{mis}(\Gamma') \leq \mathrm{mis}(\Gamma)$ .

- (ii) Note that a set  $I \in MIS(\Gamma; v)$  if and only if  $I \cap N(v) = \emptyset$  and  $I v \in MIS(\Gamma v N(v))$ . Thus  $MIS(\Gamma; v) = MIS(\Gamma v N(v))$ .
  - (iii) If  $I \in MIS(\Gamma; \overline{v})$ , then  $I \in MIS(\Gamma v)$ , and thus  $MIS(\Gamma; \overline{v}) \subseteq MIS(\Gamma v)$ .
  - (iv) If  $I \in MIS(\Gamma; \overline{v})$ , then  $I \cap N(v) \neq \emptyset$ . We have  $MIS(\Gamma; \overline{v}) = \bigcup_{u \in N(v)} MIS(\Gamma; u)$ .
  - (v) Using (the proof of) part (iii), if uv is an edge in  $\Gamma$ , then we have

$$\begin{aligned} \operatorname{MIS}(\Gamma) &= \operatorname{MIS}(\Gamma; v) \cup \operatorname{MIS}(\Gamma; \overline{v}) \\ &\subseteq \operatorname{MIS}(\Gamma; v) \cup \operatorname{MIS}(\Gamma - v) \\ &= \operatorname{MIS}(\Gamma; v) \cup \operatorname{MIS}(\Gamma - v; u) \cup \operatorname{MIS}(\Gamma - v; \overline{u}) \\ &= \operatorname{MIS}(\Gamma; v) \cup \operatorname{MIS}(\Gamma; u) \cup \operatorname{MIS}(\Gamma - v; \overline{u}). \end{aligned}$$

The conclusion then follows from part (ii).

#### 2.1 Proof of Theorem 2.1

We say that  $\Gamma$  is type-A if each component of  $\Gamma$  is a  $K_3$  or an isolated vertex. It is easy to see that  $MIS(\Gamma) = 3^m$  for a type- $\Lambda$  graph  $\Gamma$ . Let  $\Gamma$  be a counterexample with minimum number of vertices. Then  $\Gamma$  is not type- $\Lambda$ , and  $\Gamma$  has at least one edge as the edgeless graph is type- $\Lambda$ , giving that  $m \geq 1$ . The only connected graphs with matching number 1 are  $K_3$  and stars. Hence, when m = 1, since  $\Gamma$  is not type- $\Lambda$ , it is a disjoint union of a star and isolated vertices, and  $mis(\Gamma) = 2 < 3^1$ . Hence, we may assume that  $m \geq 2$ .

Choose a maximum matching M and  $uv \in E(M)$ . Note that each of  $\Gamma - u - N(u)$ ,  $\Gamma - v - N(v)$  and  $\Gamma - v - u$  has matching number at most m - 1. Either those graphs are type-A, and we have the  $3^{m-1}$  upper bound, or by the minimality of  $\Gamma$  we have the same upper bound:

$$\operatorname{mis}(\Gamma - u - N(u)), \ \operatorname{mis}(\Gamma - v - N(v)), \ \operatorname{mis}(\Gamma - v; \overline{u}) \le 3^{m-1},$$
 (1)

where the last inequality follows from that  $\min(\Gamma - v; \overline{u}) \leq \min(\Gamma - v - u)$  by Theorem 2.3(iii). Now Theorem 2.3(v) gives that

$$\operatorname{mis}(\Gamma) \le \operatorname{mis}(\Gamma - u - N(u)) + \operatorname{mis}(\Gamma - v - N(v)) + \operatorname{mis}(\Gamma - v; \overline{u}) \le 3^m.$$

It remains to show that for a proper choice of uv, at least one of the three inequalities in (1) is strict.

Claim 2.4. We may assume that 
$$N(u) \cap V(M) = \{v\}$$
 and  $N(v) \cap V(M) = \{u\}$ .

Proof of claim. Suppose to the contrary that  $|N(u) \cap V(M)| \ge 2$  and  $x \in N(u) \cap V(M)$  with  $x \ne v$ . Let y be the neighbor of x in M. If  $\Gamma - u - v - x$  is a type-A graph, then as each of its components is either a  $K_3$  or an isolated vertex, removing any vertex from  $\Gamma - u - v - x$  would not decrease its matching number. Thus, by the maximality of M,

$$\nu(\Gamma - u - v - x) = \nu(\Gamma - u - v - x - y) \le m - 2,$$

giving that  $\operatorname{mis}(\Gamma - u - v - x) \leq 3^{m-2}$ . If  $\Gamma - u - v - x$  is not type-A, then by the minimality of  $\Gamma$  we have  $\operatorname{mis}(\Gamma - u - v - x) < 3^{m-1}$ . Hence using Theorem 2.3(i), we always have

$$\operatorname{mis}(\Gamma - u - N(u)) \le \operatorname{mis}(\Gamma - u - v - x) < 3^{m-1}$$

as desired. Similarly, if  $|N(v) \cap V(M)| \ge 2$ , then  $\min(\Gamma - v - N(v)) < 3^{m-1}$ .

Suppose  $\deg(u) = 1$ . Then every maximal independent set in  $\Gamma - v$  contains u, hence  $\operatorname{mis}(\Gamma - v; \overline{u}) = 0 < 3^{m-1}$  as desired. Thus, we may assume that  $\deg(u) \geq 2$  and similarly  $\deg(v) \geq 2$ .

Now by Theorem 2.4, each of u and v has at least one neighbor outside of M. By the maximality of M, it must be the case that  $N(u) = \{v, x\}$  with  $x \notin V(M)$  and  $N(v) = \{u, x\}$ .

If  $N(x) = \{u, v\}$ , then  $\Gamma$  is a disjoint union of a triangle  $\{u, v, x\}$  and  $\Gamma - u - v - x$ . As  $\Gamma$  is not type-A,  $\Gamma - u - v - x$  is not type-A either. Note that  $\nu(\Gamma - u - v - x) = m - 1$ . Hence by the minimality of  $\Gamma$ ,

$$mis(\Gamma) = 3mis(\Gamma - u - v - x) < 3 \cdot 3^{m-1} = 3^m,$$

as desired.

We may then assume that x has a neighbor  $y \notin \{u, v\}$ . By the maximality of M, y lies in M - u - v. Let y' be the neighbor of y in M. Consider now the maximum matching M' = M - yy' + xy. Notice that  $uv \in E(M')$  and  $N(u) \cap V(M') = \{v, x\}$ , contradicting Theorem 2.4. This completes the proof of Theorem 2.1.

#### 2.2 Proof of Theorem 2.2

We say that  $\Gamma$  is type-B if each component of  $\Gamma$  is a  $K_2$ ,  $K_3$ ,  $K_4$  or a  $D_6$ . It is easy to see that  $\operatorname{mis}(\Gamma) = 2^{3m-n}3^{n-2m}$  for a type-B graph  $\Gamma$ . Suppose for a contradiction that Theorem 2.2 is false, and let  $\Gamma$  be a counterexample with minimum number of vertices. Choose a maximum matching M in  $\Gamma$ , and an edge  $uv \in E(M)$ . Without loss of generality we may assume  $\deg(u) \leq \deg(v)$ . Define  $a(m,n) := 2^{3m-n}3^{n-2m}$ .

Considering the (n-2)-vertex graph  $\Gamma - u - v$  with  $\nu(\Gamma - u - v) = m-1$ , by Theorem 2.3(iii) and the minimality of  $\Gamma$ , we have

$$\operatorname{mis}(\Gamma - v; \overline{u}) \le \operatorname{mis}(\Gamma - u - v) \le a(m - 1, n - 2) = 2^{3m - n - 1}3^{n - 2m} = \frac{1}{2}a(m, n),$$
 (2)

and the second inequality is equality if and only if  $\Gamma - u - v$  is type-B.

Claim 2.5. For any  $x \in V(M)$  with deg(x) = d, we have

$$\min(\Gamma - x - N(x)) \le a(m - d, n - d - 1) = \frac{3^{d-1}}{2^{2d-1}} a(m, n),$$

and equality holds if and only if  $\Gamma - x - N(x)$  is type-B.

Proof of claim. Observe that  $\Gamma - x - N(x)$  is an (n - d - 1)-vertex graph, and since  $\{x\} \cup N(x)$  appears in at most d edges of the matching M, we have  $\nu(\Gamma - x - N(x)) \ge m - d$ . As a(m, n) is a decreasing function of m (when n is fixed), the minimality of  $\Gamma$  gives

$$\operatorname{mis}(\Gamma - x - N(x)) \le a(m - d, n - d - 1) = 2^{3m - n - 2d + 1}3^{n - 2m + d - 1} = \frac{3^{d - 1}}{2^{2d - 1}}a(m, n),$$

and equality holds if and only if  $\Gamma - x - N(x)$  is type-B.

We split the rest of the proof into cases depending on the degrees of u and v.

Case 1. deg(u) = deg(v) = 1.

Then  $\Gamma$  is a disjoint union of the edge uv and  $\Gamma - u - v$ . As  $\Gamma$  is not type-B,  $\Gamma - u - v$  is not type-B either. Hence we have the strict inequality in (2), and

$$\operatorname{mis}(\Gamma) = 2 \cdot \operatorname{mis}(\Gamma - u - v) < 2 \cdot \frac{1}{2}a(m, n) = a(m, n).$$

Case 2. deg(u) = 1 and  $deg(v) \ge 2$ .

In this case,  $MIS(\Gamma - v; \overline{u}) = \emptyset$ . By Theorem 2.5 we know that

$$\operatorname{mis}(\Gamma - v - N(v)) \le \frac{3}{8}a(m, n).$$

Note that in this case  $N(u) = \{v\}$ . Thus, by Theorem 2.3(v) and (2), we have

$$\mathrm{mis}(\Gamma) \leq \mathrm{mis}(\Gamma - u - v) + \mathrm{mis}(\Gamma - v - N(v)) + \mathrm{mis}(\Gamma - v; \overline{u}) \leq \frac{1}{2}a(m,n) + \frac{3}{8}a(m,n) + 0 = \frac{7}{8}a(m,n).$$

Since this case will be used later, we analyze what happens when the equalities hold. Note first that equality in (2) implies  $\Gamma - u - v$  is type-B. Next, equality in Theorem 2.5 implies that  $N(v) = \{u, x\}$  with some  $x \in V(M)$ , and  $\Gamma - u - v - x$  is type-B with  $\nu(\Gamma - u - v - x) = m - 2$ . As both  $\Gamma - u - v$  and  $\Gamma - u - v - x$  are type-B, x can only lie in a component of  $\Gamma - u - v$  that is either a  $K_3$ ,  $K_4$  or  $K_3$ .

If x is in a  $K_3$  component in  $\Gamma - u - v$ , then  $\nu(\Gamma - u - v - x) = \nu(\Gamma - u - v) = m - 1$ , a contradiction.

If x is in a  $K_4$  component in  $\Gamma - u - v$ , then uvx is an induced  $P_3$  and deg(x) = 4 in  $\Gamma$ .

If x is in a  $D_6$  component in  $\Gamma - u - v$ , then  $\deg(x) = 4$  in  $\Gamma$ .

Case 3.  $\deg(u) = \deg(v) = 2$ .

Let  $N(u) = \{v, x\}$  and  $N(v) = \{u, y\}$ , where we allow x = y.

Suppose first that  $x \in V(M)$  (the case when  $y \in V(M)$  is similar). Recall that  $\Gamma - u - v$  is an (n-2)-vertex graph with  $\nu(\Gamma - u - v) = m-1$ . Also, by Theorem 2.3(ii),  $\operatorname{mis}(\Gamma - u - v; x) \leq \operatorname{mis}(\Gamma - u - v - x - N(x))$ . Applying Theorem 2.5 to vertex x (which lies in a maximum matching M - uv) in  $\Gamma - u - v$ , we get that

$$\min(\Gamma - u - v; x) \le \min(\Gamma - u - v - x - N(x)) \le \frac{1}{2}a(m - 1, n - 2) = \frac{1}{4}a(m, n).$$

We also have  $\min(\Gamma - u - N(u)), \min(\Gamma - v - N(v)) \leq \frac{3}{8}a(m,n)$  from Theorem 2.5. Note that every  $I \in \min(\Gamma - v; \overline{u})$  contains x as  $N(u) = \{v, x\}$ . This, together with Theorem 2.3(iii), implies that

$$\operatorname{mis}(\Gamma - v; \overline{u}) \le \operatorname{mis}(\Gamma - u - v; x) \le \frac{1}{4}a(m, n).$$
 (3)

Then by Theorem 2.3(v), we have

$$\mathrm{mis}(\Gamma) \leq \frac{3}{8}a(m,n) + \frac{3}{8}a(m,n) + \frac{1}{4}a(m,n) = a(m,n).$$

When the equalities hold,  $\operatorname{mis}(\Gamma - u - v; x)$  attains a(m-1, n-2)/2 in Theorem 2.5, hence x has degree 1 (let  $x \sim z$ ) in  $\Gamma - u - v$  and  $\Gamma - u - v - x - z$  is type-B with  $\nu(\Gamma - u - v - x - z) = m - 2$ . Furthermore,  $\operatorname{mis}(\Gamma - u - N(u)) = \operatorname{mis}(\Gamma - u - v - x)$  attains a(m-2, n-3), hence  $\Gamma - u - v - x$  is also type-B with  $\nu(\Gamma - u - v - x) = m - 2$ . This implies that z is in a  $K_3$  component in  $\Gamma - u - v - x$ . Indeed, if z was part of a  $K_2$  then  $\Gamma - u - v - x - z$  would not be type-B, and if z was part of a  $K_4$  or a  $D_6$  then  $\nu(\Gamma - u - v - x) = m - 1$ , a contradiction.

If  $y \neq x$ , then  $\Gamma - u - v - y = \Gamma - v - N(v)$  is also type-B. Then  $y \neq z$ , otherwise x would be an isolated vertex in  $\Gamma - u - v - y$ . Now,  $x \sim z$  in  $\Gamma - u - v - y$ , where x has degree 1 and z has degree at least 2, which is impossible in a type-B graph. Therefore, x = y and  $\{u, v, x\}$  is a  $K_3$  in a  $D_6$  component in  $\Gamma$ , where the other 3 vertices form a  $K_3$  component in the type-B graph  $\Gamma - u - v - x$ . This contradicts to the fact that  $\Gamma$  is not type-B.

We may then assume that  $x, y \notin V(M)$ . Then by the maximality of M, we must have x = y. Note that  $\Gamma - u - N(u) = \Gamma - u - v - x$  is an (n-3)-vertex graph with  $\nu(\Gamma - u - N(u)) \ge m-1$ . Hence by Theorem 2.3(ii),

$$\mathrm{mis}(\Gamma; u) \leq \mathrm{mis}(\Gamma - u - v - x) \leq a(m - 1, n - 3) = 2^{3m - n} 3^{n - 2m - 1} = \frac{1}{3} a(m, n).$$

Similarly,  $\operatorname{mis}(\Gamma; v) \leq a(m, n)/3$ . On the other hand,

$$\min(\Gamma - v; \overline{u}) \le \min(\Gamma - v; x) \le \min(\Gamma - v - x - N(x)) \le \min(\Gamma - u - v - x) \le \frac{1}{3}a(m, n),$$

where the first inequality follows from  $N(u) = \{v, x\}$  and the second and third ones follow from Lemma 2.3(ii) and (i) respectively. Putting together, we also have  $\min(\Gamma) \leq a(m, n)$ .

When the equalities hold,  $\operatorname{mis}(\Gamma - v; x) = a(m, n)/3 = a(m, n - 1)/2$  and  $\nu(\Gamma - u - v - x) = m - 1$ . Now ux is an edge in a maximum matching M - uv + ux of  $\Gamma - v$ , and by (2) we know that ux is an isolated edge in  $\Gamma - v$  and  $\Gamma - u - v - x$  is type-B. Therefore,  $\{u, v, x\}$  is an isolated triangle in  $\Gamma$ , contradicting to the fact that  $\Gamma$  is not type-B.

Case 4. deg(u) = 2 and  $deg(v) \ge 3$ .

Let  $N(u) = \{v, x\}$ . If  $x \in V(M)$ , then by Theorem 2.5 and (3) we have

$$\min(\Gamma - u - N(u)) \le \frac{3}{8}a(m, n), \quad \min(\Gamma - v - N(v)) \le \frac{9}{32}a(m, n), \quad \min(\Gamma - v; \overline{u}) \le \frac{1}{4}a(m, n).$$

Putting these together we have  $mis(\Gamma) \leq 29a(m, n)/32$ .

If  $x \notin V(M)$ , then we still have  $\min(\Gamma - u - N(u)), \min(\Gamma - v; \overline{u}) \leq a(m, n)/3$  as in the previous case, and from Theorem 2.5 we have  $\min(\Gamma - v - N(v)) \leq 9a(m, n)/32$ . Putting these together we have  $\min(\Gamma) \leq 91a(m, n)/96$ .

Case 5.  $deg(u) \ge 3$  and  $deg(v) \ge 4$ .

By Theorem 2.3(iii) and (2) we have

$$\operatorname{mis}(\Gamma - v; \overline{u}) \le \operatorname{mis}(\Gamma - u - v) \le \frac{1}{2}a(m, n),$$

and Theorem 2.5 implies

$$\min(\Gamma - u - N(u)) \le \frac{9}{32}a(m, n), \quad \min(\Gamma - v - N(v)) \le \frac{27}{128}a(m, n).$$

Putting these together, by Theorem 2.3(v) we have  $mis(\Gamma) \leq 127a(m,n)/128$ .

The only remaining case is when  $\deg(u) = \deg(v) = 3$ . Since the choice of uv in M is arbitrary, from now on we may suppose that every vertex in M has degree 3. Denote  $N(u) = \{v, x, y\}$  and  $N(v) = \{u, z, w\}$ .

Case 6. Both  $\{x, y\}$  and  $\{z, w\}$  appear in at most one edge in M.

In this case  $\Gamma - u - N(u)$  is an (n-4)-vertex graph with  $\nu(\Gamma - u - N(u)) \ge m-2$ . Hence

$$\operatorname{mis}(\Gamma - u - N(u)) \le a(m - 2, n - 4) = 2^{3m - n - 2}3^{n - 2m} = \frac{1}{4}a(m, n).$$

Similarly,  $\min(\Gamma - v - N(v)) \le a(m, n)/4$ . Putting together with (2), we have from Theorem 2.3(v) that  $\min(\Gamma) \le a(m, n)$ .

When the equalities of (2) hold, we know that  $\Gamma - u - v$  is type-B, and each maximal independent set of  $\Gamma - u - v$  contains some neighbor of u. It implies that  $\{x, y\}$  is an isolated edge in  $\Gamma - u - v$ . Analogously,  $\{z, w\}$  is also an isolated edge. Depending on  $\{x, y\} = \{z, w\}$  or not,  $\{u, v, x, y, z, w\}$  forms an isolated copy of  $K_4$  or  $D_6$ , and the rest of the graph is type-B. This contradicts to that  $\Gamma$  is not type-B.

Case 7. One of  $\{x,y\}$  and  $\{z,w\}$  appears in at least two edges of M.

Without loss of generality we may suppose that  $xp, yq \in E(M)$  with  $p \neq y$  and  $q \neq x$ . We know from Theorem 2.5 that

 $\operatorname{mis}(\Gamma - u - N(u)), \ \operatorname{mis}(\Gamma - v - N(v)) \le \frac{9}{32}a(m, n).$ 

If  $N(x) = \{u, v, p\}$ , then we know that p cannot be adjacent to u (otherwise we have p = y). Apart from x and v, p is adjacent to another vertex. Now in  $\Gamma - u - v$  we have  $\deg(x) = 1$  and  $\deg(p) \ge 2$ , hence we may apply Case 2 to this case. Note that every vertex in M has degree 3, hence equality in Case 2 cannot be attained. Therefore,

$$\operatorname{mis}(\Gamma - v; \overline{u}) \le \operatorname{mis}(\Gamma - u - v) < \frac{7}{8}a(m - 1, n - 2) = \frac{7}{16}a(m, n).$$

The same argument works when  $N(y) = \{u, v, q\}$ .

If  $N(x) \neq \{u, v, p\}$  and  $N(y) \neq \{u, v, q\}$ , then applying Theorem 2.3(ii) and Theorem 2.5 to  $\Gamma - u - v$  gives

$$\min(\Gamma - u - v; x), \ \min(\Gamma - u - v; y) \le \frac{3}{8}a(m - 1, n - 2) = \frac{3}{16}a(m, n).$$

Together with Theorem 2.3(iv), we have

$$\operatorname{mis}(\Gamma - v; \overline{u}) \le \operatorname{mis}(\Gamma - u - v; x) + \operatorname{mis}(\Gamma - u - v; y) \le \frac{3}{8}a(m, n).$$

In both cases, by Theorem 2.3(v) we have

$$\operatorname{mis}(\Gamma) < \frac{9}{32}a(m,n) + \frac{9}{32}a(m,n) + \frac{7}{16}a(m,n) = a(m,n),$$

as desired. This completes the proof of Theorem 2.2.

## 3 Infinitely many groups with intermediate growth

In this section, we prove Theorem 1.4 with the following family of Abelian groups of type I(23).

**Theorem 3.1.** For each prime  $p \ge 29$ , let  $G = \mathbb{Z}_3^2 \oplus \mathbb{Z}_{23} \oplus \mathbb{Z}_p$  and n = |G| = 207p. Then one has

$$2^{\mu(G)/2} = 2^{4n/23} \ll 3^{n/9} \le f_{\max}(G) \le 3^{25n/216 + o(n)} \ll 3^{8n/69} = 3^{\mu(G)/3}.$$

The analysis of  $f_{\text{max}}(G)$  and  $f_{\text{max}}^*(G)$  involves the application of the hypergraph container method. The following lemma presented in [13] is an analogue of a result of Green and Ruzsa [10]. The hypergraph container method [3, 21] gives better quantitative bounds, but they are not needed for our proofs.

**Lemma 3.2** ([10],[13]). For every  $\delta > 0$  there exists an  $n_0 \in \mathbb{N}$  such that the following holds. If G is a finite Abelian group of order  $n \geq n_0$ , then there is a container family  $\mathcal{F}$  of subsets of G with the following properties.

- (i)  $|\mathcal{F}| \leq 2^{\delta n}$ .
- (ii) If  $I \subseteq G$  is distinct sum-free, then I is contained in some  $F \in \mathcal{F}$ .
- (iii) Every  $F \in \mathcal{F}$  is of the form  $F = A \cup B$ , where A is maximal sum-free and  $|B| \leq \delta n$ .

Note that every sum-free set is distinct sum-free, hence a sum-free set I is also contained in some  $F \in \mathcal{F}$ . We also need the following stability result for type I groups from Green and Ruzsa [10].

**Lemma 3.3** ([10]). Suppose that G is a type I(p) group of order n, with p = 3k + 2. Let  $A \subseteq G$  be maximal sum-free, and suppose that |A| > (p+2)n/3(p+1). Then there is a homomorphism  $\varphi \colon G \to \mathbb{Z}_p$  such that  $A = \varphi^{-1}(\{k+1,\ldots,2k+1\})$ .

We shall reduce the problem of bounding the number of maximal sum-free sets to a problem on maximal independent set of certain auxiliary graph.

**Definition 3.4.** For disjoint subsets  $A, S \subseteq G$ , let  $L_S[A]$  be the link graph of S on A defined as follows. Its vertex set is A and its edge set consists of the following edges:

- (i) two distinct vertices  $x, y \in A$  are adjacent if there exists  $s \in S$  such that  $\{x, y, s\}$  is a Schur triple;
- (ii) there is a loop at a vertex  $x \in A$  if there exist distinct  $s, t \in S$  such that  $\{x, s, t\}$  is a Schur triple.
- (iii) there is a loop at a vertex  $x \in A$  if there exist  $s \in S$  such that  $\{x, x, s\}$  or  $\{x, s, s\}$  is a Schurtriple.

Note that although Theorem 1.8 is stated for simple graphs, it also works on graphs with loops, by simply deleting the vertices having loops.

Now we prove that there are groups whose growth of  $f_{\text{max}}$  is between  $2^{1/2}$  and  $3^{1/3}$ . For subsets  $S \subseteq F$  of a group, we use the symbol MSF(F) ( $\text{MSF}^*(F)$ ) for the set of the maximal (distinct) sum-free sets contained in F, and MSF(F;S) ( $\text{MSF}^*(F;S)$ ) for the family of such sets containing S. Denote by  $\text{msf}(\cdot)$  the cardinality of  $\text{MSF}(\cdot)$ .

Proof of Theorem 3.1. Let  $\delta > 0$ . We shall prove that  $3^{n/9} \leq f_{\text{max}}(G) \leq 3^{25n/216+2\delta n}$ .

Let us first establish the lower bound. Let s=(0,1,0,0). We shall make use of subsets of the set  $A=\{1\}\oplus\mathbb{Z}_3\oplus\mathbb{Z}_2\oplus\mathbb{Z}_p$ . For each function  $f\colon\mathbb{Z}_{23}\oplus\mathbb{Z}_p\to\mathbb{Z}_3$ , denote  $A_f=\{(1,f(x),x):x\in\mathbb{Z}_{23}\oplus\mathbb{Z}_p\}\cup\{s\}$ . It is not hard to see that each  $A_f$  is sum-free, and can be extended to a maximal sum-free set. Moreover, for functions  $f\neq g, A_f\cup A_g$  is not sum-free. Indeed, without loss of generality we may assume that g(x)=f(x)+1 for some  $x\in\mathbb{Z}_{23}\oplus\mathbb{Z}_p$ , then  $(1,f(x),x),(1,f(x)+1,x),s\in A_f\cup A_g$ , with (1,f(x),x)+s=(1,f(x)+1,x). Hence  $A_f$  and  $A_g$  cannot extend to the same maximal sum-free set. Consequently,  $f_{\max}(G)$  is at least the number of such functions f, which is  $3^{n/9}$ .

To obtain the upper bound, apply Theorem 3.2 to G with  $\delta$  to obtain a family  $\mathcal{F}$ . Then, by Theorem 3.2(ii) and (iii), we have

$$\mathrm{MSF}(G) = \bigcup_{F \in \mathcal{F}} \mathrm{MSF}(F) \subseteq \bigcup_{F \in \mathcal{F}} \bigcup_{\substack{S \subseteq B \\ S \text{ sum-free}}} \mathrm{MSF}(A \cup S; S).$$

Indeed, every maximal sum-free set I contained in  $F = A \cup B$  can be constructed by first choosing a sum-free  $S \subseteq B$  and then extending S in A to a maximal one.

Now, fix  $F = A \cup B \in \mathcal{F}$  and a sum-free  $S \subseteq B$ . By Theorem 3.2(i) and (iii), there are at most  $2^{2\delta n}$  such choices. It then suffices to bound  $\operatorname{msf}(A \cup S; S)$  by  $3^{25n/216}$ .

To upper bound  $\operatorname{msf}(A \cup S; S)$ , construct the link graph  $\Gamma = L_S[A]$ . Notice that each maximal sum-free set in  $\operatorname{MSF}(A \cup S; S)$  corresponds to a maximal independent set in  $\Gamma$ . Thus,

$$\operatorname{msf}(A \cup S; S) \leq \operatorname{mis}(\Gamma)$$
.

If  $|A| \leq 25n/72$ , then Moon-Moser Theorem tells us that  $mis(\Gamma) \leq 3^{25n/216}$ , as desired.

We may then assume that |A| > 25n/72. Apply Theorem 3.3 to find a homomorphism  $\varphi \colon G \to \mathbb{Z}_{23}$  with  $A = \varphi^{-1}(M)$ , where  $M = \{8, 9, 10, 11, 12, 13, 14, 15\} \subseteq \mathbb{Z}_{23}$ . Thus, |A| = 8n/23. If S is an empty set, then  $\Gamma$  is an edgeless graph and  $\min(\Gamma) = 1$ , as desired. Otherwise, choose some  $s \in S$ .

If  $\varphi(s) = 0$ , then as 0 - M = M in  $\mathbb{Z}_{23}$ , we see that for each  $x \in A$ , s - x is also in A. Moreover, since  $0 \notin 2M$ , there is no solution to 2x = s in A. Hence  $\Gamma$  contains a perfect matching that matches x and s - x, then by Theorem 1.8, we have

$$\operatorname{mis}(\Gamma) \le 2^{|A|/2} = 2^{4n/23} < 3^{25n/216}.$$

If  $\varphi(s) \in \pm \{1, 2, 3, 4\}$ , then for each  $x \in A$ , at least one of  $\{x + s, x - s\}$  is in A. Denote

$$P_s(x) = A \cap \{x, x + s, \dots, x + 22s\}.$$

Notice that as 23 is a prime, A has the decomposition

$$A = \bigsqcup_{\varphi(x)=8} P_s(x).$$

We construct a large matching as follows. For each x with  $\varphi(x) = 8$ , the set  $P_s(x)$  contains a number of vertex disjoint paths of the form  $\{x + a \cdot s, \dots, x + b \cdot s\}$  with  $a, b \in \mathbb{Z}_{23}$  in  $L_S[A]$ , and since at least one of  $\{x - s, x + s\}$  is in A, we see that  $a \neq b$ , i.e., each path has length at least 1. As  $P_s(x)$  has 8 vertices and contains a linear forest with no isolated vertex, it has a matching of size 3. Therefore,  $\nu(\Gamma) \geq 3|A|/8$ , and by Theorem 1.8,

$$\operatorname{mis}(\Gamma) < 2^{|A|/8} 3^{|A|/4} = 2^{n/23} 3^{2n/23} < 3^{25n/216}$$

We may now assume that for all  $s \in S$ ,  $\varphi(s) \in \pm \{5, 6, 7\}$ . For each  $x \in A$  with  $\varphi(x) \in \{11, 12\}$ , one has

$$\varphi(x+s), \varphi(x-s), \varphi(s-x) \in \{4, 5, 6, 7, 16, 17, 18, 19\}, \forall s \in S,$$

none of them is in A. Hence x is isolated in  $\Gamma$ , and by the Moon-Moser Theorem,

$$\operatorname{mis}(\Gamma) \le 3^{(3|A|/4)/3} = 3^{2n/23} < 3^{25n/216}.$$

This completes the proof.

## 4 Maximal distinct sum-free sets

In this section we establish an upper bound for  $f_{\text{max}}(G)$  and  $f_{\text{max}}^*(G)$  for even-order groups. Theorem 1.6 follows from the following theorem and Theorems 4.5 and 4.6.

**Theorem 4.1.** Let G be an even-order n-element group, which is not  $\mathbb{Z}_2^k \oplus \mathbb{Z}_3$  for any integer k. Then

$$f_{\max}(G), f_{\max}^*(G) \le 2^{n/4 + o(n)}.$$

To prove Theorem 4.1, we need the following two strengthenings of Theorem 1.8. We postpone the proof of these two lemmas to the end of this section. Here,  $T_{\ell}$  denotes the graph consisting of two vertex disjoint  $C_{\ell}$ 's,  $u_1 \cdots u_{\ell}$  and  $v_1 \cdots v_{\ell}$ , with a perfect matching  $u_i v_i$ ,  $1 \leq i \leq \ell$  between them.

**Lemma 4.2.** Let  $\Gamma$  be an n-vertex graph, and M be a matching of m edges of  $\Gamma$ . Suppose that  $V_1, \ldots, V_k$  are pairwise disjoint vertex sets, each containing at least one edge from M, and for each i, every vertex in  $V(M) \cap V_i$  has degree at least 4 within  $V_i$ . Then,

$$\operatorname{mis}(\Gamma) \le a(k, m, n) := \left(\frac{59}{64}\right)^k 2^{3m-n} 3^{n-2m}.$$

**Lemma 4.3.** Let  $\ell \geq 3$  and  $\Gamma$  be an n-vertex graph. Suppose  $\Gamma$  has k copies of  $T_{\ell}$ 's and m edges, all of which are vertex disjoint. Then,

$$\operatorname{mis}(\Gamma) \le b(k, m, n) := \left(\frac{31}{32}\right)^k 2^{3(m+\ell k)-n} 3^{n-2(m+\ell k)}.$$

The following simple observation is also needed for this and later in the next section.

**Observation 4.4.** Let  $G = \mathbb{Z}_{2^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus K$  be an even order group such that  $r, \alpha_1, \ldots, \alpha_r \in \mathbb{N}$  and K is odd. Let  $\varphi \colon G \to \mathbb{Z}_2$  be a homomorphism. Then the equation 2x = 0 has exactly  $2^r$  solutions, with at most half of them satisfying  $\varphi(x) = 1$ . Furthermore, if 2x' = s for some  $s \in G$  and  $\varphi(x') = 1$ , then 2x = s has at least  $2^{r-1}$  solutions with  $\varphi(x) = 1$ .

*Proof.* Let  $x_i = (0, ..., 0, 2^{\alpha_i - 1}, 0, ..., 0)$ , where the only non-zero coordinate is the *i*-th coordinate, for  $1 \le i \le r$ . Then,  $S_0 := \operatorname{span}\{x_1, ..., x_r\}$  is the set of solutions to 2x = 0, which has size  $2^r$ .

If  $\varphi(S_0) = \{0\}$ , then all the solutions satisfy  $\varphi(x) = 0$ . Otherwise, there exists  $x \in S_0$  with  $\varphi(x) = 1$ . For each  $y \in S_0$  with  $\varphi(y) = 1$ , we have 2(x + y) = 2x + 2y = 0 and hence  $x + y \in S_0$ . Moreover,  $\varphi(x + y) = \varphi(x) + \varphi(y) = 0$ . Hence at most half of the elements in  $S_0$  satisfy  $\varphi(x) = 1$ .

Now suppose 2x' = s with  $\varphi(x') = 1$ . Then,  $S' := x' + S_0$  is the set of solutions to 2x' = s, which also has size  $2^r$ . Moreover, for each  $y \in S_0$  with  $\varphi(y) = 0$ , we have  $x + y \in S'$  and  $\varphi(x + y) = 1$ . Hence at least half of the elements in S' satisfy  $\varphi(x) = 0$ .

Proof of Theorem 4.1. We first present the proof for  $f_{\max}^*(G)$ . We consider the distinct link graph  $\Gamma^* := L_S^*[A]$  defined as in Theorem 3.4 but using only edges from (i) and (ii). Since A is sum-free, we have  $|A| \leq \mu(G) = n/2$ . Following the analysis in the proof of Theorem 3.1, it suffices to prove that for each pair of A and S such that A is maximal distinct sum-free, S is distinct sum-free and  $A \cap S = \emptyset$ , one has

$$\operatorname{mis}(\Gamma^*) \le 2^{n/4}$$
.

If  $|A| \leq 4n/9$ , then by the Moon-Moser Theorem, mis  $(\Gamma^*) \leq 3^{4n/27} < 2^{n/4}$ , as desired.

We may then assume that |A| > 4n/9. Apply Theorem 3.3 to find a homomorphism  $\varphi \colon G \to \mathbb{Z}_2$  with  $A = \varphi^{-1}(1)$ , and thus |A| = n/2 and  $S \subseteq \varphi^{-1}(0)$ . If S is empty, then  $\Gamma^*$  is an edgeless graph and thus  $\operatorname{mis}(\Gamma^*) = 1$ , as desired. We now assume that S is non-empty.

If  $S = \{0\}$ , then  $\Gamma^*$  consists of a matching and some isolated vertices, which also gives mis  $(\Gamma^*) \le 2^{|A|/2} = 2^{n/4}$ . Otherwise, pick  $0 \ne s \in S$  with minimal order.

If s has order |s| = 2, then  $\Gamma^*$  admits a perfect matching  $\{x, x + s\}_{x \in A}$ . Directly applying Theorem 1.8 we obtain mis  $(\Gamma^*) \leq 2^{|A|/2}$ .

Now we suppose that all elements in S have order at least 3, and we first discuss the cases when  $S \subseteq \{s, -s\}$ .

Recall that  $T_{\ell}$  is the graph consisting of two vertex disjoint  $C_{\ell}$ 's and a perfect matching between them. We denote by  $T_{\ell}^+$  the graph consisting of two vertex disjoint  $C_{\ell}$ 's,  $u_1 \cdots u_{\ell}$  and  $v_1 \cdots v_{\ell}$ , with two perfect matchings  $u_i v_{\ell-i+1}, u_i v_{\ell-i+3}, 1 \leq i \leq \ell$  between them. Here, i+1=1 for  $i=\ell$ . If  $-x \in \{x, x+s, \ldots, x+(|s|-1)s\}$ , then the component containing x in  $\Gamma^*$  is  $\{x, x+s, \ldots, x+(|s|-1)s\}$ , constituting a  $C_{|s|}$ . If not, then the component is a  $T_{|s|}$  or  $T_{|s|}^+$ , depending on  $S = \{s\}$  or  $S = \{s, -s\}$ .

When |s|=3, we count the  $C_3$  components in  $\Gamma^*$ . Each  $C_3$  component is of the form  $\{x, x+s, x+2s\}$  with  $-x \in \{x, x+s, x+2s\}$ , implying that there is a  $k \in \mathbb{Z}_3$  with -x = x+ks. It follows that 2(x+(k/2)s)=0. Hence each  $C_3$  component contains a solution to 2y=0. On the other hand, we know that 3 divides  $n/2^r$  as G has an order 3 element. Since the group is not  $\mathbb{Z}_2^k \oplus \mathbb{Z}_3$ , we know that  $n/2^r \ge 6$  in the setting of Theorem 4.4. Hence 2y=0 has at most n/12 solutions in A, and the number of  $C_3$  components in  $\Gamma^*$  is at most n/12. We know that  $\min(T_3)=6$ ,  $\min(T_3^+)=3$ , and  $\min(C_3)=3$ . Therefore,

$$\operatorname{mis}(\Gamma^*) \le 3^{n/12} 6^{(|A| - n/4)/6} = 3^{n/12} 6^{n/24} < 2^{n/4}.$$
 (4)

For  $|s| \ge 4$ , it is known from [7] that  $\min(C_{|s|}) < 1.4^{|s|} < 2^{|s|/2}$  and from Theorem 4.3 that both  $\min(T_{|s|})$  and  $\min(T_{|s|}^+)$  are at most  $31 \cdot 2^{|s|-5} < 2^{|s|}$ . Hence  $\min(\Gamma^*) < 2^{|A|/2}$ , as desired.

Now we proceed to handle the case when  $S \nsubseteq \{s, -s\}$ .

Let  $t \in S$  with  $t \notin \{s, -s\}$ . For each x, let  $V_x = \{x + is + jt : 0 \le i \le |s| - 1, 0 \le j \le |t| - 1\}$ . Then for each  $x \in A$ , x + s, x + t, x - s, x - t are four distinct neighbors of x in  $V_x$ . Furthermore, we know that

$$A = \bigcup_{x \in A} V_x,$$

where for each pair  $x, y \in A$ , either  $V_x = V_y$  or  $V_x \cap V_y = \emptyset$ .

Let  $\ell = |V_x|$  (we know that all  $V_x$  have the same size). Then  $A = V(\Gamma)$  is partitioned into  $n/2\ell$  disjoint vertex subsets  $V_x$ . We find in each  $V_x$  an almost perfect matching (i.e., a matching of size  $\lfloor \ell/2 \rfloor$ ).

Indeed, if |s| or |t| is even (without loss of generality suppose |s| is even), then for each  $0 \le j \le |t| - 1$  and odd  $0 \le i \le |s| - 1$ , we match x + is + jt with x + (i - 1)s + jt, which results in a perfect matching. If both |s| and |t| are odd, then for each  $0 \le j \le |t| - 1$  and odd  $0 \le i \le |s| - 1$ , we match x + is + jt with x + (i - 1)s + jt; for each odd  $0 \le j \le |t| - 1$  and i = |s| - 1, we match x + is + jt with x + is + (j - 1)t. Then all vertices are matched except x + (|s| - 1)s + (|t| - 1)t. Hence the matching has size  $(\ell - 1)/2$ , which is almost perfect.

Let M be the matching of  $\Gamma$  consisting of such matchings of  $V_x$ , then M has size at least  $n(\ell-1)/4\ell$ . Theorem 4.2 gives that

$$\operatorname{mis}(\Gamma^*) \le 59^{n/2\ell} 2^{n(\ell-15)/4\ell} 3^{n/2\ell} < 2^{n/4}$$
.

This completes the proof for  $f_{\max}^*(G)$ .

For  $f_{\text{max}}(G)$ , we analogously consider the link graph  $L_S[A]$ , and it suffices to show that

$$\operatorname{mis}(\Gamma) \le \operatorname{mis}(\Gamma^*)$$

for all disjoint sum-free sets A and S. Note that if (x, y, s) is a Schur triple with  $x \neq y \in A$  and  $s \in S$ , then it is a distinct triple since  $s \notin A$ . Hence, the only difference between  $\Gamma$  and  $\Gamma^*$  is that  $\Gamma$  might have some loops of type (iii) in Theorem 3.4.

Let X be the set of vertices with loops in  $\Gamma$ . Then by Theorem 2.3(i), we have

$$\operatorname{mis}(\Gamma) = \operatorname{mis}(\Gamma - X) = \operatorname{mis}(\Gamma^* - X) \le \operatorname{mis}(\Gamma^*),$$

completing the proof.

Remark. For the distinct sum-free sets, the lower bound was known in [13] that  $f_{\text{max}}^*(G) \geq 2^{(n-2)/4}$ .

### 4.1 Counterexamples

**Proposition 4.5.** Let  $k \in \mathbb{N}$  and  $n = 3 \cdot 2^k$ . Then  $f_{\max}^* (\mathbb{Z}_2^k \oplus \mathbb{Z}_3) = 3^{n/6 + o(n)}$ .

*Proof.* The upper bound follows from the known facts that  $\mu^*(\mathbb{Z}_2^k \oplus \mathbb{Z}_3) = n/2$ , and that for every group G we have  $f_{\max}^*(G) \leq 3^{\mu^*(G)/3 + o(n)}$ .

For the lower bound, let  $s=(0,\ldots,0,1)$ . We define subsets of  $A=\{1\}\oplus\mathbb{Z}_2^{k-1}\oplus\mathbb{Z}_3$  as follows. For each function  $f\colon\mathbb{Z}_2^{k-1}\to\mathbb{Z}_3$ , denote  $A_f=\left\{(1,x,f(x)):x\in\mathbb{Z}_2^{k-1}\right\}\cup\{s\}$ . Then each  $A_f$  is distinct sum-free and extends to a maximal distinct sum-free set. Moreover, for any functions  $f\neq g, A_f\cup A_g$  is not distinct sum-free. Indeed, without loss of generality suppose g(x)=f(x)+1 for some  $x\in\mathbb{Z}_2^{k-1}$ , then  $(1,x,f(x)),(1,x,f(x)+1),s\in A_f\cup A_g$ , with (1,x,f(x))+s=(1,x,f(x)+1). Thus  $A_f$  and  $A_g$  cannot extend to the same maximal distinct sum-free set, and  $f_{\max}^*(G)$  is at least the number of such functions f, which is  $3^{n/6}$ .

For odd groups, there are several ways to construct counterexamples. We present one of them.

**Proposition 4.6.** For each prime  $p \ge 23$  with  $p \equiv 2 \pmod{3}$ , one has

$$f_{\max}^* \left( \mathbb{Z}_3^2 \oplus \mathbb{Z}_p \right) \ge 3^p \ge 2^{\mu(G)/2} = 2^{3(p+1)/2}.$$

*Proof.* Let s = (0, 1, 0) and define  $A_f = \{(1, f(x), x) : x \in \mathbb{Z}_p\} \cup \{s\}$ , with  $f : \mathbb{Z}_p \to \mathbb{Z}_3$ . Similarly to the proof of Theorem 4.5, each  $A_f$  is distinct sum-free and two sets  $A_f$  and  $A_g$  cannot extend to the same maximal distinct sum-free set. Thus we have  $f_{\max}^*(G) \geq 3^p$ .

#### 4.2 Proofs of Theorems 4.2 and 4.3

Proof of Theorem 4.2. Suppose  $\Gamma$  is a counterexample with minimum number of vertices. As Theorem 1.8 implies the case k=0, we may assume  $k\geq 1$ . Choose  $u\sim v$  in  $M\cap V_1$ . Recall that v has  $d\geq 4$  neighbors in  $V_1$ .

Similarly to Theorem 2.5, after deleting v and the neighborhood of v in  $V_1$ ,  $\Gamma - v - (N(v) \cap V_1)$  is an (n-1-d)-vertex graph, and  $\{v\} \cup (N(v) \cap V_1)$  appears in at most d edges in M. Furthermore, this deletion does not affect  $V(M) \cap V_i$  for any  $2 \le i \le k$ . Hence by Theorem 2.3(i) and the minimality of  $\Gamma$  we obtain

$$\min(\Gamma - v - N(v)) \le \min(\Gamma - v - (N(v) \cap V_1)) \le a(k - 1, m - d, n - d - 1)$$
$$= \frac{64 \cdot 3^{d-1}}{59 \cdot 2^{2d-1}} a(k, m, n) \le \frac{27}{118} a(k, m, n).$$

Similarly,  $\min(\Gamma - u - N(u)) \le 27a(k, m, n)/118$ . By Theorem 2.3(iii), we have

$$\operatorname{mis}(\Gamma - v; \overline{u}) \le \operatorname{mis}(\Gamma - u - v) \le a(k - 1, m - 1, n - 2) = \frac{32}{59}a(k, m, n).$$

Finally, we get from Theorem 2.3(v) that

$$\min(\Gamma) \le \min(\Gamma - u - N(u)) + \min(\Gamma - v - N(v)) + \min(\Gamma - v; \overline{u})$$

$$\le \frac{27}{118}a(k, m, n) + \frac{27}{118}a(k, m, n) + \frac{32}{59}a(k, m, n) = a(k, m, n),$$

as desired.  $\Box$ 

Proof of Theorem 4.3. Suppose to the contrary that there is a counterexample  $(\Gamma, \mathcal{T}, M)$  with minimum  $k = |\mathcal{T}|$ , where  $\mathcal{T}$  is the collection of  $T_{\ell}$  and M is the collection of vertex disjoint edges, which are also vertex disjoint from  $\mathcal{T}$ .

By Theorem 1.8, we may assume that  $k \geq 1$ . Choose a  $T \in \mathcal{T}$  to be a copy of  $T_{\ell}$ , and let  $(u_1, u_2, \ldots, u_{\ell})$  and  $(v_1, v_2, \ldots, v_{\ell})$  be the two  $C_{\ell}$ 's of T, with  $u_i \sim v_i$  for all  $1 \leq i \leq \ell$ . By Theorem 2.3(v), we have

$$\min(\Gamma) \le \min(\Gamma - v_2 - N(v_2)) + \min(\Gamma - u_2 - N(u_2)) + \min(\Gamma - v_2; \overline{u_2}).$$

We proceed to bound each of the terms above. After deleting  $u_1, u_2, u_3, v_2$ , we see that  $(\mathcal{T} - T, M + u_4v_4 + \ldots + u_\ell v_\ell)$  is a disjoint union of  $T_\ell$ 's and edges in  $\Gamma - u_1 - u_2 - u_3 - v_2$ , which satisfies the inequality in Theorem 4.3 due to the minimality of k. Hence by Theorem 2.3(i), we have

$$\min(\Gamma - u_2 - N(u_2)) \le \min(\Gamma - u_1 - u_2 - u_3 - v_2) \le b(k - 1, m + \ell - 3, n - 4) = \frac{9}{31}b(k, m, n).$$

Similarly, we have  $\operatorname{mis}(\Gamma - v_2 - N(v_2)) \leq 9b(k, m, n)/31$ .

For  $mis(\Gamma - v_2; \overline{u_2})$ , if  $deg(u_2) = 3$ , from Theorem 2.3(iv), (ii) and (i) and the minimality of k, we have

$$\begin{aligned} \min(\Gamma - v_2; \overline{u_2}) &\leq \min(\Gamma - v_2 - u_2; u_1) + \min(\Gamma - v_2 - u_2; u_3) \\ &\leq \min(\Gamma - v_2 - u_2 - u_1 - v_1 - u_\ell) + \min(\Gamma - v_2 - u_2 - u_3 - v_3 - u_4) \\ &\leq b(k - 1, m + \ell - 3, n - 5) + b(k - 1, m + \ell - 3, n - 5) \leq \frac{12}{31}b(k, m, n). \end{aligned}$$

Putting together, we have  $mis(\Gamma) \leq 30b(k, m, n)/31$ . The same method works if  $deg(v_2) = 3$ .

The remaining case is when  $deg(u_2), deg(v_2) \geq 4$ . Let x be a neighbor of  $u_2$  other than  $u_1, u_3, v_2$ . Classifying by  $x \in V(T) \cup V(M), x \in V(T) \setminus V(T)$  or otherwise, we can bound  $mis(\Gamma - u_1 - u_2 - u_3 - v_2 - x)$ , using the minimality of k, in each of these cases by  $b(k-1, m+\ell-4, n-5), b(k-2, m+2\ell-4, n-5)$  and

 $b(k-1, m+\ell-3, n-5)$  respectively; among these three terms, the maximum is  $b(k-2, m+2\ell-4, n-5)$ . Hence, we have

$$\min(\Gamma - u_2 - N(u_2)) \le \min(\Gamma - u_1 - u_2 - u_3 - v_2 - x)$$
  
$$\le b(k - 2, m + 2\ell - 4, n - 5) = \frac{216}{961}b(k, m, n).$$

Similarly,  $\min(\Gamma - v_2 - N(v_2)) \le 216b(k, m, n)/961$ , and

$$\min(\Gamma - v_2; \overline{u_2}) \le \min(\Gamma - u_2 - v_2) \le a(k - 1, m + \ell - 1, n - 2) = \frac{16}{31}b(k, m, n).$$

Altogether we have  $mis(\Gamma) \leq 928b(k, m, n)/961$ . This completes the proof.

## 5 All but one even-order group has few maximal sum-free sets

In this section, we prove Theorem 1.1, i.e., that  $f_{\max}(G)$  is exponentially smaller than  $2^{n/4}$  for every even group G other than  $\mathbb{Z}_2^k$ . We need the following strengthening of Theorem 1.8. We postpone its proof to the end of this section.

**Lemma 5.1.** Let  $\Gamma$  be an n-vertex graph. Suppose  $\Gamma$  has k induced copies of  $C_4$  and m edges, all of which are vertex disjoint. Then,

$$\operatorname{mis}(\Gamma) \le c(k, m, n) := \left(\frac{4}{7}\right)^k 2^{6k+3m-n} 3^{n-2m-4k}.$$

We also need the following statement.

**Theorem 5.2** ([16]). There exists a constant  $c > 10^{-4}$  such that the following holds. For every  $C \ge 10^{30}$ , there is an integer  $n_0 = n_0(C)$  such that for every Abelian group  $G = \mathbb{Z}_{2^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus K$  with even order  $n > n_0$ , |K| odd and  $n/2^r \ge C$ , we have

$$f_{\max}(G) \le 2^{(1/4-c)n}$$
.

We shall prove the following counterpart of Theorem 5.2, from which Theorem 1.1 follows.

**Theorem 5.3.** For each  $C \geq 2$  there exists an integer  $n_0$  such that the following holds. For every Abelian group  $G = \mathbb{Z}_{2^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus K$  with even order  $n > n_0$ , |K| odd and  $2 \leq n/2^r \leq C$ , we have

$$f_{\max}(G) \le 2^{(1/4-c)n},$$

where  $c = 1/300C^2$ .

*Proof.* For a maximal sum-free set A and a sum-free set S with  $A \cap S = \emptyset$ , we construct the link graph  $\Gamma = L_S[A]$ . It suffices to prove that for all such pairs (A, S), one has

$$\operatorname{mis}(\Gamma) \le 2^{n/4-cn}$$
.

We follow the case analysis as in the proof of Theorem 4.1. Note that as S is sum-free,  $0 \notin S$ .

If  $|A| \leq 4n/9$ , then mis $(\Gamma) \leq 3^{4n/27} < 2^{n/4-cn}$  by the choice of c.

We may assume that |A| > 4n/9. Apply Theorem 3.3 to find a homomorphism  $\varphi \colon G \to \mathbb{Z}_2$  with  $A = \varphi^{-1}(1)$ . Pick  $s \in S$  with minimal order. We know that if  $G = \mathbb{Z}_{2^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus K$  with  $\alpha_1 \leq \ldots \leq \alpha_r$ , then s has order

$$|s| \le 2^{\alpha_r} \cdot |K| = \frac{n}{2^{\alpha_1} 2^{\alpha_2} \cdot \ldots \cdot 2^{\alpha_{r-1}}} \le \frac{n}{2^{r-1}} \le 2C.$$

We first discuss the case when |s|=2. For each  $x\in A$ ,  $\{x,x+s\}$  is an edge in  $\Gamma$ .

If 2x = s' has a solution in A for some  $s' \in S$ , then by Theorem 4.4, it has at least  $2^{r-1}$  solutions in A. Fix such an s' and let  $x_1, \ldots, x_{2^{r-1}} \in A$  be solutions to 2x = s'. Then they cannot appear in any independent set of  $\Gamma$ , since there are loops at them. Since  $\Gamma - x_1 - \ldots - x_{2^{r-1}}$  has matching number at least  $n/4 - 2^{r-1}$ , Theorem 1.8 gives the desired bound

$$\operatorname{mis}(\Gamma) \le 2^{n/4 - 2^r} 3^{2^r} = \left(\frac{2}{3}\right)^{2^r} 2^{n/4} \le \left(\frac{2}{3}\right)^{n/C} 2^{n/4} < 2^{n/4 - cn}.$$

We can now assume that  $2A \cap S = \emptyset$ . If the two edges  $\{x, x+s\}$  and  $\{s-x, -x\}$  are vertex disjoint, then there is a matching between them (edges  $\{x, s-x\}, \{-x, x+s\}$ ), and they together form a  $C_4$  in  $\Gamma$  (we argue later that it is actually an induced  $C_4$ ). If they are not vertex disjoint, then they coincide, and it must be that 2x = 0, since x = s - x or x + s = -x would imply  $2A \cap S \neq \emptyset$ .

Denote m the number of coincided edges. Recall that each such edge is of the form  $\{x_i, x_i + s\}$  with both endpoints being a solution to the equation 2x = 0. As there are at most  $2^{r-1}$  solutions to 2x = 0 in A due to Theorem 4.4 and  $A = \varphi^{-1}(1)$ , we have  $m \leq 2^{r-2} \leq \frac{n}{8}$ , and hence the number of  $C_4$ 's is

$$k := \frac{\frac{n}{2} - 2 \cdot \# \text{ coincided edges}}{4} = \frac{\frac{n}{2} - 2m}{4} = \frac{n}{8} - \frac{m}{2} \ge \frac{n}{16}.$$

We claim that each such 4-cycle  $\{x, x+s, -x, s-x\}$  is an induced  $C_4$ . If not, then  $\{x, -x\}$  or  $\{s+x, s-x\}$  is an edge in  $\Gamma$ . Either  $x+(-x)=0 \in S$ , or  $\pm(x-(-x))=\pm 2x \in S$ , contradicting to the fact that S is sum-free and  $2A \cap S = \emptyset$ . Thus by Theorem 5.1 with  $(k, m, n)_{5.1} = (\frac{n}{8} - \frac{m}{2}, m, \frac{n}{2})$ , we have

$$\operatorname{mis}(\Gamma) \le \left(\frac{4}{7}\right)^k 2^{n/4} \le \left(\frac{4}{7}\right)^{n/16} 2^{n/4} < 2^{n/4 - cn}.$$

We next consider the case when |s| > 2, and  $S \subseteq \{s, -s\}$ . If |s| = 3 and  $G \neq \mathbb{Z}_2^k \oplus \mathbb{Z}_3$ , then we follow the proof of Theorem 4.1 (see (4)) to obtain

$$\operatorname{mis}(\Gamma) \le 3^{n/12} 6^{n/24} < 2^{n/4 - cn}$$

If  $G = \mathbb{Z}_2^k \oplus \mathbb{Z}_3$ , then  $s = \pm (0, \dots, 0, 1)$ , and  $\Gamma$  is the graph consisting of  $2^{k-1}$  isolated triangles  $\{x, x + s, x + 2s\}_{x \in A}$ , each with a loop (at a vertex that corresponds to a solution to 2x = s). We have

$$\operatorname{mis}(\Gamma) = 2^{2^{k-1}} = 2^{n/6} < 2^{n/4 - cn}$$

If  $4 \le |s| \le 2C$ , we have from the proof of Theorem 4.1 that the distinct link graph  $\Gamma^* = L_S^*[A]$  is a disjoint union of  $C_{|s|}$  and  $T_{|s|}$  or  $T_{|s|}^+$ . Let the number of  $C_{|s|}$  and  $T_{|s|}$  (or  $T_{|s|}^+$ ) be  $\ell$  and k, respectively. Then we have

$$\ell|s| + 2k|s| = |A| = \frac{n}{2}.$$

Hence, since  $|s| \leq 2C$  and recall that  $\Gamma$  is a supergraph of  $\Gamma^*$  with some additional loops, we see that

$$mis(\Gamma) \le mis(\Gamma^*) < 1.4^{|s|\ell} 31^k 2^{|s|k-5k}$$

$$= \left(\frac{1.4^2}{2}\right)^{|s|\ell/2} \left(\frac{31}{32}\right)^k 2^{|s|\ell/2 + |s|k} < \left(\frac{31}{32}\right)^{n/4|s|} 2^{n/4} \le \left(\frac{31}{32}\right)^{n/8C} 2^{n/4} < 2^{n/4 - cn}.$$

It remains to handle the case that  $S \nsubseteq \{s, -s\}$ . We use the notation  $t, V_x, \ell$  as in the proof of Theorem 4.1. Then

$$\ell \le |s| \cdot |t| < 2C \cdot 2C = 4C^2.$$

Hence

$$\operatorname{mis}(\Gamma) \le \operatorname{mis}(\Gamma^*) \le 59^{n/2\ell} 2^{n(\ell-15)/4\ell} 3^{n/2\ell} = \left(\frac{3^2 \cdot 59^2}{2^{15}}\right)^{n/4\ell} 2^{n/4} < 0.96^{n/16C^2} 2^{n/4} < 2^{n/4-cn}.$$

This completes the proof.

#### 5.1 Proof of Theorem 5.1

For  $U, V \subset \Gamma$ , we denote by  $\operatorname{mis}(\Gamma; U; \overline{V})$  the number of maximal independent sets I satisfying  $I \cap U \neq \emptyset$  and  $I \cap V = \emptyset$ . We also write  $\operatorname{mis}(\Gamma; U) = \operatorname{mis}(\Gamma; U; \emptyset)$ ,  $\operatorname{mis}(\Gamma; \overline{V}) = \operatorname{mis}(\Gamma; V(\Gamma); \overline{V})$ , and  $\operatorname{mis}(\Gamma; U, W; \overline{V})$  for the number of maximal independent sets I satisfying  $I \cap U \neq \emptyset$ ,  $I \cap W \neq \emptyset$  and  $I \cap V = \emptyset$ .

Suppose for a contradiction that there is a counterexample  $(\Gamma, \mathcal{C}, M)$  with minimum  $k = |\mathcal{C}|$ , where  $\mathcal{C}$  is a collection of pairwise vertex disjoint induced  $C_4$ 's and M is a matching, which is also vertex disjoint from  $\mathcal{C}$ . By Theorem 1.8, we may assume that  $k \geq 1$ . Choose  $C = u_1 u_2 v_2 v_1 \in \mathcal{C}$ , an induced  $C_4$ , where  $u_1 u_2, u_2 v_2, v_2 v_1, v_1 u_1$  are the edges.

For each pair of  $i, j \in \{1, 2\}$ , we construct a graph  $\Gamma_{i,j}$  as follows: starting from  $\Gamma$ , we change the neighborhood of  $u_1, u_2, v_2, v_1$  in  $V(\Gamma - C)$  by letting  $N_{\Gamma_{i,j}}(u_1) = N_{\Gamma_{i,j}}(u_2) = N_{\Gamma}(u_i)$  and  $N_{\Gamma_{i,j}}(v_1) = N_{\Gamma_{i,j}}(v_2) = N_{\Gamma}(v_j)$ . We shall prove that

$$\operatorname{mis}(\Gamma) \le \frac{2}{7} \sum_{i,j \in \{1,2\}} \operatorname{mis}(\Gamma_{i,j}). \tag{5}$$

We show that each  $I \in MIS(\Gamma)$  corresponds to at least 3.5 maximal independent sets on the right hand side, by classifying the maximal independent sets by their intersections with  $N(u_1), N(u_2), N(v_1), N(v_2)$ . Denote  $I' = I - u_1 - u_2 - v_1 - v_2$ .

(i) When none of  $I' \cap N(u_1)$ ,  $I' \cap N(u_2)$ ,  $I' \cap N(v_1)$ ,  $I' \cap N(v_2)$  is an empty set, then I' = I, and it is also a maximal independent set in each of  $\Gamma_{i,j}$  for  $i,j \in \{1,2\}$ . The ratio is 1:4.

In the rest of the proof, we discuss only one case from each class of similar cases.

- (ii) When  $I' \cap N(u_1) = \emptyset$  and all other intersections are non-empty, we have that  $I = I' + u_1$ ; while  $I' + u_1, I' + u_2$  are two maximal independent sets in each of  $\Gamma_{1,1}$  and  $\Gamma_{1,2}$ . The ratio is 1:4.
- (iii) When  $I' \cap N(u_1) = \emptyset$  and  $I' \cap N(u_2) = \emptyset$ , and all other intersections are non-empty, we have that  $I' + u_1, I' + u_2$  are two maximal independent sets in  $\Gamma$ ; while

$$I' + u_1, I' + u_2 \in MIS(\Gamma_{1,1}), MIS(\Gamma_{1,2}), MIS(\Gamma_{2,1}), MIS(\Gamma_{2,2}),$$

altogether the ratio is 2:8.

(iv) When  $I' \cap N(u_1) = I' \cap N(v_1) = \emptyset$  and all other intersections are non-empty, we have that  $I' + u_1$ ,  $I' + v_1$  are two maximal independent sets in  $\Gamma$ ; while

$$I' + u_1 + v_2, \ I' + u_2 + v_1 \in MIS(\Gamma_{1,1}), \quad I' + u_1, \ I' + u_2 \in MIS(\Gamma_{1,2}),$$
  
 $I' + v_1, \ I' + v_2 \in MIS(\Gamma_{2,1}), \quad I' \in MIS(\Gamma_{2,2}),$ 

altogether the ratio is 2:7.

(v) When  $I' \cap N(u_1) = I' \cap N(v_2) = \emptyset$  and all other intersections are non-empty, we have that  $I = I' + u_1 + v_2$ ; while

$$I' + u_1 + v_2, \ I' + u_2 + v_1 \in MIS(\Gamma_{1,2}), \quad I' + u_1, I' + u_2 \in MIS(\Gamma_{1,1}),$$
  
 $I' + v_1, \ I' + v_2 \in MIS(\Gamma_{2,2}), \quad I' \in MIS(\Gamma_{2,1}),$ 

altogether the ratio is 1:7.

(vi) When  $I' \cap N(u_1) \neq \emptyset$  and all other intersections are empty, we have that  $I' + v_2$ ,  $I' + u_2 + v_1$  are two maximal independent sets in  $\Gamma$ ; while

$$I' + v_1, I' + v_2 \in MIS(\Gamma_{1,1}), MIS(\Gamma_{1,2}), \quad I' + u_1 + v_2, I' + u_2 + v_1 \in MIS(\Gamma_{2,1}), MIS(\Gamma_{2,2}),$$

altogether the ratio is 2:8.

(vii) When each of the four intersections is  $\emptyset$ , then I is also a maximal independent set in each of  $\Gamma_{i,j}$  for  $i,j \in \{1,2\}$ . Altogether the ratio is 1:4.

Combining all the above cases, (5) is proved, and by the pigeonhole principle there exists  $i, j \in \{1, 2\}$  with

$$\operatorname{mis}(\Gamma_{i,j}) \ge \frac{7}{8} \operatorname{mis}(\Gamma).$$
 (6)

We let  $\Gamma' = \Gamma - C$ . Then, (C - C, M) is a collection of induced  $C_4$ 's and edges of  $\Gamma'$ , which satisfies, following from the minimality of k that

$$\min(\Gamma') \le c(k-1, m, n-4) = \frac{7}{16}c(k, m, n). \tag{7}$$

We may partition  $MIS(\Gamma')$  into four classes, according to the intersection pattern of a maximal independent set with  $N(u_i)$  and  $N(v_j)$  (where again, these neighborhoods are in  $V(\Gamma) - C$  spanned by the graph  $\Gamma$ ):

$$\operatorname{mis}(\Gamma') = \operatorname{mis}(\Gamma'; N(u_i), N(v_j)) + \operatorname{mis}(\Gamma'; N(u_i); \overline{N(v_j)}) + \operatorname{mis}(\Gamma'; N(v_j); \overline{N(u_i)}) + \operatorname{mis}(\Gamma'; \overline{N(u_i) \cup N(v_j)}).$$

For MIS( $\Gamma_{i,j}$ ) with i,j as in (6), and the neighborhoods considered in  $V(\Gamma) - C$ , each independent set  $I' \in \text{MIS}(\Gamma'; N(u_i), N(v_j))$  corresponds to one independent set  $I' \in \text{MIS}(\Gamma_{i,j}; N(u_i), N(v_j))$ . Each independent set  $I' \in \text{MIS}(\Gamma'; N(u_i); \overline{N(v_j)})$  corresponds to two independent sets  $I' + v_1, I' + v_2 \in \text{MIS}(\Gamma_{i,j}; N(u_i); \overline{N_{\Gamma_{i,j}}(v_j)})$  and each independent set  $I' \in \text{MIS}(\Gamma'; \overline{N(u_i) \cup N(v_j)})$  corresponds to two independent sets  $I' + u_1 + v_2, I' + u_2 + v_1 \in \text{MIS}(\Gamma_{i,j}; \overline{N(u_i) \cup N(v_j)})$ . Therefore, we have

$$\min(\Gamma_{i,j}) = \min(\Gamma'; N(u_i), N(v_j)) + 2\min(\Gamma'; N(u_i); \overline{N(v_j)}) + 2\min(\Gamma'; N(v_j); \overline{N(u_i)}) + 2\min(\Gamma'; \overline{N(u_i)}) + 2\min(\Gamma'; \overline{N(u_i)}) + 2\min(\Gamma'; \overline{N(u_i)})$$

Therefore,  $\operatorname{mis}(\Gamma_{i,j}) \leq 2\operatorname{mis}(\Gamma')$  and together with (6) and (7) it follows

$$\operatorname{mis}(\Gamma) \le \frac{8}{7} \operatorname{mis}(\Gamma_{i,j}) \le \frac{16}{7} \operatorname{mis}(\Gamma') \le c(k, m, n),$$

as desired.

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