A practical approach to perturbative corrections to few-body observables

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Abstract

We formulate two methods to facilitate the calculation of perturbative corrections to quantum few-body observables. Both techniques are designed for a numerical realization in combination with any tool that obtains either the entire spectrum or solely the eigenvalues of an operator corresponding to the observable of interest. We exemplify these methods in the context of the nuclear contact theory without pions (* EFT) and benchmark them in the deuteron channel with available analytical, field-theoretical calculations, as well as in the triton and 3-helium channels through earlier extractions within the dibaryon formalism, where in all three systems the point-proton root-mean-square charge radius (rms) was the perturbed observable of choice. Beyond these $A \le 3$ consistency and accuracy checks, we employ the numerical methods to predict the rms of the 4-helium nuclear ground state to assess three different ways of integrating the Coulomb interaction into * EFT. By comparing the respective results at leading and next-to-leading order for 3- and 4-helium, we find that the uncertainty due to the strong, short-range interaction is significantly larger compared with that due to the long-range Coulomb interaction for both bound states with their different binding momenta. Thereby, we provide strong support for simplifying extractions of bound-state observables by shutting off any Coulomb interaction if the strong part of the potential is considered only up to first order in the effective range expansion.

Keywords: perturbation, few-body nuclei, effective field theory, Coulomb interaction, root-mean-square radius

1. Introduction

Self-consistency of a physical theory sometimes demands the perturbative treatment of some of its parts. For instance, causality imposes constraints on the interaction range of a non-relativistic theory that is required to describe a system with correlations over a certain spatial extent, the so-called Wigner bound [1]. For effective field theories (EFTs) this often means that a resummation of range corrections leads to renormalization problems by violating the bound [2, 3]. To bypass this issue and thus maintain a clear hierarchy of EFT orders, the sub-leading EFT terms, including the range, are often treated perturbatively.

Counterintuitively, the practical extraction of perturbative corrections of a given order to few- and many-nucleon observables is often less straightforward than obtaining predictions through an infinite iteration of the perturbation in the Schrödinger equation. For our specific interest in A-body systems described by a Hamiltonian with non-perturbative kinetic (\hat{T}) and interaction terms $(\hat{V}^{(0)})$ plus a relatively weak part $(\alpha \hat{V}^{(1)})$ that acts as a perturbation,

$$\hat{H} = \hat{T} + \hat{V}^{(0)} + \alpha \hat{V}^{(1)} = \hat{H}^{(0)} + \alpha \hat{V}^{(1)}, \qquad (1)$$

the technical hurdle can be understood in terms of textbook quantum mechanics as follows: At 0^{th} order ($\alpha = 0$), the expectation value of an observable \hat{O} is given by

$$O_n^{(0)} = \left\langle \psi_n^{(0)} \middle| \hat{O} \middle| \psi_n^{(0)} \right\rangle, \tag{2}$$

where $|\psi_n^{(0)}\rangle$ is the n^{th} eigen solution corresponding to the zeroth order Hamiltonian $H^{(0)}$. The leading perturbative correction to the expectation value $O_n^{(0)}$ is linear in the small parameter α and is given by

$$\delta O_n^{(1)} = \left\langle \psi_n^{(1)} | \hat{O} | \psi_n^{(0)} \right\rangle + \text{h.c.}, \tag{3}$$

where it is the first order correction to the n^{th} eigenstate, expressed via the infinite sum

$$\left|\psi_{n}^{(1)}\right\rangle = \sum_{m \neq n} \frac{\left\langle\psi_{m}^{(0)} \middle| \hat{V}^{(1)} \middle| \psi_{n}^{(0)}\right\rangle}{E_{n} - E_{m}} \left|\psi_{m}^{(0)}\right\rangle,\tag{4}$$

which often precludes rigorous perturbative calculations, as it incorporates contributions from all possible eigenstates

 $|\psi_m^{(0)}\rangle$ together with their corresponding eigenenergies E_m .

For the specific case of the energy, however, the problem simplifies. While the first-order perturbative correction to the energy can be calculated as an expectation value $\langle \psi_n^{(0)} | \hat{V}^{(1)} | \psi_n^{(0)} \rangle$, the infinite sum, (4), appears at second order. Alternatively, these corrections can be extracted from a solution of the Schrödinger equation with Hamiltonian \hat{H} , (1), and a polynomial expansion of $E_n(\alpha)$ in the small parameter α . Such a procedure was successfully applied in Refs. [4, 5, 6], where perturbative corrections to nuclear ground states were extracted within χ EFT.

Being interested in a larger set of observables, we present two alternative methods to the canonical perturbative higher-order calculation to avoid accurate approximations of an infinite number of states. We formulate these methods for any observable \hat{O} and interaction $\hat{V}^{(0)} + \hat{V}^{(1)}$ for which approximate (or exact) solutions can be obtained for either the wave function and the energy or solely the energy E from

$$\hat{H}[\alpha,\beta]|\psi\rangle = (\hat{T} + \hat{V}^{(0)} + \alpha \cdot \hat{V}^{(1)} + \beta \cdot \hat{O})|\psi\rangle = E|\psi\rangle \tag{5}$$

as functions of the real multipliers α and β for the few-body system of interest.

In the following Sec. 2, we explain how to relate $|\psi_n(\alpha,\beta)\rangle$ and $E_n(\alpha,\beta)$ to

$$\langle \hat{O} \rangle_n = O_n^{(0)} + \delta O_n^{(1)} + \delta O_n^{(2)} + \dots,$$
 (6)

i.e., the perturbative corrections $\delta O_n^{(i)}$ proportional to the i^{th} powers of a small parameter α . Section 3 reports an application of the methods to the point-proton (root-)mean-squared charge radius, $\langle r_p^2 \rangle \stackrel{!}{=} \langle \psi | \hat{r}_p^2 | \psi \rangle$, of the deuteron, triton, helion and 4-helium ground states as predicted with the non-perturbative leading order (LO) *EFT interaction and its perturbative next-to-leading order (NLO) range correction *EFT (see reviews [7, 8]). For the helion and 4helium nuclei, we analyze in Sec. 3.2 *EFT variants that treat the static Coulomb interaction as non-perturbative, perturbative, and negligible. We conclude in Sec. 4 with an outlook on broader classes of problems to which the methodology can be applied.

2. Obtaining corrections perturbatively

We present practical methods to obtain the perturbative expansion, (6), without the need for an explicit infinite sum as in (4). The summation is replaced by the requirement of a numerically complete basis, in which the eigenstate or eigenvalue in (5), relevant to the expectation value $\langle \hat{O} \rangle$ of interest, can be expanded over a sufficient range of multipliers α and β . The practicality of the methods comes with the availability of several sophisticated techniques developed for the quantum-mechanical few-body problem, e.g., lattice methods [9, 10], Gaussian variational methods [11, 12], Monte Carlo techniques [13, 14], Hyperspherical bases [15, 16], Faddeev(-Yakubovsky) and AGS decompositions [17, 18, 19, 20], resonating-group methods [21, 22, 23], and no core shell model [24, 25]. Any of these methods may prove useful in the determination of the functional dependence of an expectation- or energy eigenvalue in step (I) of our methods. These methods entail nothing but a solution to Eq. (5) for the state of which an expectation value is of interest as a function of the single parameter α and an ensuing expansion of this dependence in a power series. We refer to this method as the *one-multiplier method* (OMM):

$$\langle \hat{O} \rangle_n \stackrel{(I)}{\equiv} O_n(\alpha; \beta = 0) \stackrel{(II)}{=} O_n^{(0)} + \sum_{i=1}^N \alpha^i \, \delta O_n^{(i)} \,. \tag{OMM}$$

While the coefficients of the power series furnish the perturbative expansion of the expectation value directly, a potential disadvantage of the approach is the demand of a sufficiently accurate solution for an eigenstate over a range of α 's, which must be chosen such that the convergence radius of the expansion in α times the interaction strength of $\hat{V}^{(1)}$ is nonzero.

For cases where this latter issue arises or if no solution to the eigenvectors of (5) is available, the *two-multiplier method* (TMM), in turn, takes an intermediate step by extracting solely eigenenergies of Eq. (5). However, they must be obtained on a two-dimensional (α, β) grid before adapting a two-dimensional power series to the resultant energy contour:

$$E_n \stackrel{(I)}{\equiv} E_n(\alpha, \beta) \stackrel{(II)}{=} E_n^{(0,0)} + \sum_{\substack{i,j=0\\i \neq i=0}}^N \alpha^i \beta^j \, \delta E_n^{(i,j)} \,. \tag{TMM}$$

As eigenenergies of the Schrödinger equation (5), the *Feynman-Hellmann theorem* [26, 27] applied with respect to the parameter β

$$\frac{\partial E_n(\alpha, \beta)}{\partial \beta} = \left\langle \frac{\partial \hat{H}[\alpha, \beta]}{\partial \beta} \right\rangle_n = \langle \hat{O} \rangle_n \tag{7}$$

relates the expansion coefficients linear in β , $\delta E_n^{(i,1)}$, to the i^{th} perturbation order of the observable:

$$O_n^{(0)} + \sum_{i=1}^N \alpha^i \, \delta O_n^{(i)} = \delta E_n^{(0,1)} + \sum_{i=1}^N \alpha^i \, \delta E_n^{(i,1)} \,. \tag{8}$$

The need to obtain eigenvalues $E_n(\alpha, \beta)$ on a two-dimensional grid significantly increases the computational cost for a sufficiently robust two-parameter fit and renders the (TMM) favorable only if the (OMM) turns out inadequate.

3. Perturbative corrections to nuclear radii

As a first application of (OMM) and (TMM) we investigate the point-proton root-mean-square charge radii (rms) of several small ($A \le 4$) nuclei, namely, 2 H, 3 He, and 4 He with a nuclear interaction theory that mandates the perturbative treatment of some of its parts. More specifically, we use the potential formulation of the renormalizable $^{\#}$ EFT up to NLO with local regulators [28, 29, 30, 31]. The LO interaction terms of this theory are iterated in the Schrödinger equation, while NLO corrections must be treated as a first-order perturbation. In general, the higher-order perturbative corrections are expected to improve the theoretical accuracy of the predictions as more physics at higher momenta is systematically accounted for by the EFT. To employ the two numerical methods above for this specific interaction allows for benchmarking with earlier, alternative $^{\#}$ EFT perturbative extractions of $\sqrt{\langle r_p^2 \rangle}$ in $A \le 3$ systems: a semi-analytical approach for the deuteron [32] and two three-body nuclei investigated with the *dibaryon* formalism [33, 34]. No previous $^{\#}$ EFT study of the point-proton root-mean-square charge radius beyond its non-perturbative LO is currently available for A > 3, and our results for 4 He are thus first-time predictions within $^{\#}$ EFT.

Furthermore, we utilize the results to obtain insight into the interplay of a short-range (*EFT) and long-range (static Coulomb) interactions. To this end, we adopt different potential incarnations of *EFT, each of which treating the static Coulomb interaction differently: as a LO and thereby a non-perturbative effect (cf. (9) and (10) below, and for more details, see Ref. [30]); as a NLO perturbation (cf. (11) and (12) below, and details in Ref. [31]); or entirely absent [28, 29]. For the first variant, namely, the *EFT potential up to NLO with non-perturbative Coulomb, the interaction potential reads:

$$V_{C}^{(0)} = \sum_{i < j} \left(C_{0}^{(0)}(\Lambda) \hat{\mathcal{P}}_{ij}^{(0,1;nn/np)} + C_{1}^{(0)}(\Lambda) \hat{\mathcal{P}}_{ij}^{(1,0;np)} \right) g_{\Lambda}(r_{ij}) + \sum_{i < j} \left(C_{2}^{(0)}(\Lambda) \hat{\mathcal{P}}_{ij}^{(0,1;pp)} g_{\Lambda}(r_{ij}) + \frac{e^{2}}{r_{ij}} \hat{\mathcal{P}}_{ij}^{(pp)} \right)$$

$$+ \sum_{i < j < k} D_{0}^{(0)}(\Lambda) \sum_{c \lor c} \hat{\mathcal{P}}_{ijk}^{(1/2,1/2;nnp/ppn)} g_{\Lambda}(r_{ij}) g_{\Lambda}(r_{ik}) , \qquad (9)$$

$$V_{C}^{(1)} = \sum_{i < j} \left(C_{0}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(0,1;nn/np)} + C_{2}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(1,0;np)} + C_{4}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(0,1;pp)} \right) g_{\Lambda}(r_{ij})$$

$$+ \sum_{i < j} \left(C_{1}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(0,1;nn/np)} + C_{3}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(1,0;np)} + C_{5}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(0,1;pp)} \right) \left(g_{\Lambda}(r_{ij}) \stackrel{\rightarrow}{\nabla}^{2} + \stackrel{\leftarrow}{\nabla}^{2} g_{\Lambda}(r_{ij}) \right)$$

$$+ \sum_{i < j < k} D_{0}^{(1)}(\Lambda) \sum_{\text{cyc}} \hat{\mathcal{P}}_{ijk}^{(1/2,1/2;nnp)} g_{\Lambda}(r_{ij}) g_{\Lambda}(r_{ik}) + \sum_{i < j < k} D_{1}^{(1)}(\Lambda) \sum_{\text{cyc}} \hat{\mathcal{P}}_{ijk}^{(1/2,1/2;ppn)} g_{\Lambda}(r_{ij}) g_{\Lambda}(r_{ik})$$

$$+ \sum_{i < j < k < l} E_{0}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ijkl}^{(0,0)} g_{\Lambda}(r_{ijkl}) \quad . \tag{10}$$

The $\hat{\mathcal{P}}_{...}^{(S,T;ch)}$ are generic projection operators into channels labeled by total spin (S), total isospin (T), and $T_z \longleftrightarrow (ch)$, $g_{\Lambda}(r) = \exp\left(-\frac{1}{4}\Lambda^2 r^2\right)$ are Gaussian regulators with momentum cutoffs Λ , $r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|$ is a relative distance between particles i and j, and $r_{ijkl}^2 \equiv \sum_{a < b \in \{i,j,k,l\}} r_{ab}^2$ is the squared four-body hyperradius. For further details on the operator

structure, the data used to calibrate the low-energy constants (LECs), and the numerical values of the latter, we refer the reader to Ref. [30]. The potential form of the second EFT variant, i.e., *EFT up to NLO with perturbative Coulomb interaction, reads:

$$V_{\mathcal{Q}}^{(0)} = \sum_{i < j} \left(C_0^{(0)}(\Lambda) \hat{\mathcal{P}}_{ij}^{(0,1)} + C_1^{(0)}(\Lambda) \hat{\mathcal{P}}_{ij}^{(1,0)} \right) g_{\Lambda}(r_{ij}) + \sum_{i < j < k} D_0^{(0)}(\Lambda) \sum_{\text{cyc}} \hat{\mathcal{P}}_{ijk}^{(1/2,1/2)} g_{\Lambda}(r_{ij}) g_{\Lambda}(r_{ik}) \quad , \tag{11}$$

$$V_{pC}^{(1)} = \sum_{i < j} \left(C_{0}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(0,1)} + C_{2}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(1,0)} \right) g_{\Lambda}(r_{ij}) + \sum_{i < j} \left(C_{1}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(0,1)} + C_{3}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ij}^{(1,0)} \right) \left(g_{\Lambda}(r_{ij}) \stackrel{\leftarrow}{\nabla}^{2} + \stackrel{\leftarrow}{\nabla}^{2} g_{\Lambda}(r_{ij}) \right)$$

$$+ \sum_{i < j} \left(C_{4;pC}^{(1)}(\Lambda) \hat{\mathcal{P}}_{ij}^{(0,1;pp)} g_{\Lambda}(r_{ij}) + \frac{e^{2}}{r_{ij}} \hat{\mathcal{P}}_{ij}^{(pp)} \right) + \sum_{i < j < k} D_{0}^{(1)}(\Lambda) \sum_{\text{cyc}} \hat{\mathcal{P}}_{ijk}^{(1/2,1/2)} g_{\Lambda}(r_{ij}) g_{\Lambda}(r_{ik})$$

$$+ \sum_{i < j < k < l} E_{0;pC}^{(1)}(\Lambda) \, \hat{\mathcal{P}}_{ijkl}^{(0,0)} g_{\Lambda}(r_{ijkl}) \quad . \tag{12}$$

Note the isospin symmetry of the LO potential, $V_{\mathcal{C}}^{(0)}$. Thence, it does not differentiate between projections into the third isospin axis (nn/np/pp). The Coulomb interaction is accounted for perturbatively as a part of the NLO range correction, $V_{pC}^{(1)}$.

The third EFT version does not consider the static Coulomb interaction even at NLO $(V_{\mathbb{C}}^{(1)})$. Therefore, its form derives from (12) by setting the $C_{4;pC}^{(1)}$ and Coulomb terms in $V_{pC}^{(1)}$ to zero. The corresponding four-body-force LEC, $E_{0;\mathbb{C}}^{(1)}$ is always readjusted such that the ground-state binding energy of ⁴He is reproduced.

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In the course of renormalizing these three EFTs, i.e. in practice, fitting their LECs, we use a single set of data. For $C_{4;\mathcal{C}}^{(1)}$ of Eq. (10), in particular, we fit to the ground-state binding-energy difference $B(^3\mathrm{H}) - B(^3\mathrm{He})$ (see Ref. [31]).

Having thus split the interacting theory into a non-perturbative and a perturbative part - $(V_{\rm C}^{(0)} + V_{\rm C}^{(1)})$, $(V_{\it C}^{(0)} + V_{\rm pC}^{(1)})$, and $(V_{\it C}^{(0)} + V_{\it C}^{(1)})$, the resulting Eq. (5) must be solved for the various nuclear ground states on a one-dimensional α grid (OMM) and for the ground-state energies on a two-dimensional (α, β) grid (TMM). To do so, we use the Stochastic Variational Method with correlated Gaussian basis [11] for sets $\{\langle r_p^2 \rangle_0 (\alpha; \beta = 0), \alpha \in I \}$ and $\{E_0(\alpha, \beta), (\alpha \cdot \beta) \in I \otimes I \}$

on the logarithmic grid

$$I = \left\{ e^{\ln(x_1) + [\ln(x_M) - \ln(x_1)] \cdot n/M} , \ n \in [0, \dots, M] \right\} , \tag{13}$$

with M=25, $x_1=10^{-10}$, and $x_M=3\cdot 10^{-3}$. Here, each element (26 in case of the (OMM) and 26² for (TMM)) is the result of an independent variational solution of Eq. (5) for the approximation of the respective nuclear ground states. For each Λ , we diagonalize within one larger correlated-Gaussian basis, stochastically selected with the aid of auxiliary harmonic oscillator traps [35]. The subsequent least-square (OMM) and (TMM) fit in step (II) is employed with an increasing polynomial order up to the point where the numerical stabilization of extracted NLO perturbative corrections, $\delta O_0^{(1)} = \langle r_p^2 \rangle_0^{(1)}$, Eq. (6), is achieved.

3.1. Benchmarking in $A \le 4$ nuclei

For *EFT up to NLO with non-perturbative Coulomb interaction, Eq. (10), we calculate point-proton charge root-mean-square radii of light nuclei at different cutoff values $\Lambda \in \{1, 2, ..., 10\}$ fm⁻¹. Both (OMM) and (TMM) methods are applied to extract the non-perturbative LO values, $\langle r_p^2 \rangle_0^{(0)}$ —in the notation of Eq. (6), the subscript (0) marks the ground state and the superscript the perturbation order— and perturbative NLO corrections, $\delta \langle r_p^2 \rangle_0^{(1)}$, to them. The EFT prediction at lowest order for the square root of an observable is

$$\sqrt{\langle r_p^2 \rangle_0} \bigg|_{\text{LO}} \equiv \sqrt{\langle r_p^2 \rangle_0^{(0)}} \quad ,$$
(14)

while the square root must be expanded at NLO as

$$\sqrt{\langle r_p^2 \rangle_0} \bigg|_{\text{NLO}} = \sqrt{\langle r_p^2 \rangle_0} \bigg|_{\text{LO}} \left(1 + \frac{\delta \langle r_p^2 \rangle_0^{(1)}}{2 \langle r_p^2 \rangle_0^{(0)}} \right) \quad , \tag{15}$$

to have a polynomial order consistent with the EFT.

In Fig. 1, we display calculated 2 H, 3 H, 3 He, and 4 He ground state radii at LO and NLO as functions of an increasing momentum cutoff Λ . At NLO, we find that NLO corrections calculated with (OMM) and (TMM) differ by $\lesssim 0.1\%$ except for 4 He values at higher cutoff values extracted using (TMM). For the latter, we obtain relatively large uncertainties from the fits of the polynomials to the calculated energy grids, (13). We attribute this effect to a limited flexibility of the correlated Gaussian basis implying an increasing difficulty to attain numerical convergence at larger cutoff values. In contrast, (OMM) maintains its numerical stability, and fitting errors are smaller than the resolution of the figure for all results.

Since we benchmark our results against $^{\sharp}$ EFT, we detail the $\Lambda \to \infty$ extrapolation of our finite-cutoff results. We assume

$$\left\langle r_p^2 \right\rangle (\mathbf{N}^m \mathbf{LO}) = \sum_{n=0}^{m+1} \frac{b_n^{(m)}}{\Lambda^n} \quad , \tag{16}$$

for the residual dependence of all radii on the Gaussian regulator parameter Λ at order m of the EFT. Thus, we fit two parameters $(b_0^{(0)}$ and $b_1^{(0)})$ at LO and three $(b_0^{(1)}, b_1^{(1)}, \text{ and } b_2^{(1)})$ at NLO.

In the upper-left panel of Fig. 1, we compare our results for $\sqrt{\langle r_p^2 \rangle}$ as predicted by the LO and NLO of *EFT for the deuteron with the corresponding results calculated in Ref. [32]. The observable, interaction structure, and the data used to calibrate its strengths are identical. However, the earlier work obtains the leading perturbative correction within the semi-analytical framework of perturbative EFT. This prediction starts from the same *EFT Lagrangian density of which the potential operators used in Eq. (5) are derived. A noteworthy difference, however, is their use of a power-divergence subtraction scheme [32] in contrast to the cutoff regularization employed in this work. It is therefore a non-trivial observation and consistency check that, in the zero-range limit $\Lambda \to \infty$, our results for the non-perturbative and perturbative orders yield the same numerical result.

Given this agreement for the deuteron, our NLO $\sqrt{\langle r_p^2 \rangle}$ results for ³H (lower-left panel of Fig. 1) extrapolated to the contact limit differ from those of Ref. [33]: 0.9% at LO and 9% at NLO. This hints towards an increase of the NLO over the LO contribution in the three-body case compared with the two-body system. A plausible explanation for this difference¹, and especially its increase, when comparing LO to NLO between the two works, is the employed

¹Note that a naive (10)30% estimate of the theoretical (N)LO uncertainty in the [#]EFT prediction does cover the observed difference.

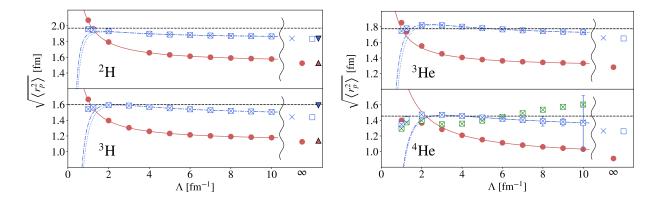


Figure 1: Cutoff regulator (Λ) dependence of $\sqrt{\langle r_p^2 \rangle}$ predictions at LO \longrightarrow and NLO [(OMM) \cdots [\cdots , (TMM) $-\cdot$ \times \cdot] of *EFT for $A \le 4$ nuclei. $\Lambda \to \infty$ limits are split by \ge and compared for 2 H with [32] (LO \triangle , NLO \blacktriangledown) and for 3 H with [33] (LO \triangle , NLO \blacktriangledown). For 4 He, we also show results without the four-body counter-term at NLO [(OMM) \Box , (TMM) \times]. The Coulomb interaction is considered non-perturbatively. All results are shown next to experimental data (---), which are evaluated as $\sqrt{\langle r_p^2 \rangle} = \sqrt{\langle r_p^2 \rangle_{ch} - \langle R_p^2 \rangle - \frac{N}{Z} \langle R_n^2 \rangle_{ch} - \frac{3}{4M_p^2}}$. We take experimental charge mean-squared radii $\langle r_p^2 \rangle_{ch}$, from Ref. [36], proton/neutron mean-squared charge radii $\langle R_p^2 \rangle / \langle R_n^2 \rangle$ from Ref. [37], and a value of Darwin-Foldy term $3/(4M_p^2) = 0.033$ fm².

renormalization condition. The agreement between the LO and NLO deuteron radii arises when both the diagrammatic and our numerical approaches renormalize the interaction strength to an amplitude that is expanded in the effective range of the spin-triplet neutron-proton system (the so-called ρ -parametrization). In contrast, the 3 H calculation in Ref. [33] chose to renormalize the two-nucleon propagators at NLO by fixing the residues of the bound-state and virtual poles in the two (iso)spin channels. This approach (so-called the Z-parametrization) has been developed in Ref. [38, 39] to improve the convergence of ‡ EFT, especially for observables which are sensitive to external probes that induce (virtual) transitions from a bound to a scattering state. The operator \hat{r}_p^2 falls into this category, and our results are therefore expected to converge more slowly to the experimental 3 H datum. We intend to revisit this explanation in future work once the Z-parametrized ‡ EFT potential is available. Furthermore, we expect a less pronounced effect when switching from ρ - to Z-parametrization in 4 He as its ground state is far below the deuteron-deuteron threshold.

Furthermore, the earlier EFT studies of the ³He point-proton root-mean-square charge radius [34] represent a qualitatively different theory as they treat the proton-proton system without any electromagnetic contribution. Our results (upper-right panel of Fig. 1), in turn, employ a non-perturbative iteration of the static Coulomb potential. An intuitive argument suggests, however, a perturbative character of the Coulomb force for sufficiently large momenta. Whether this justifies its demotion or complete nullification in the study of bound systems relates to the question of the characteristic momenta at which charges approach each other within those compounds. Naively, the deeper the binding, the larger the typical relative binding momentum, and the more perturbative the Coulomb interaction becomes. We revisit this issue of 'perturbativeness' of the Coulomb interaction in the following subsection.

To date, no other assessment exists of the effect of a perturbative NLO range correction on $\sqrt{\langle r_p^2 \rangle}$ of ⁴He. At LO, our results agree (lower-right panel in Fig. 1) with an earlier *EFT Monte-Carlo calculation [40] which entails a remarkable consistency in regulator-parameter convergencies, $\Lambda \to \infty$, of equal accuracy as those observed in the LO three-body calculations. In contrast to the three-nucleon system, ⁴He is characterized by a new four-body scale which is probed with NLO operators [41]. Leaving this scale unconstrained can enhance the regulator dependence to a degree that may render the theory impractical (green markers - (OMM) × and (TMM) \square). This demonstrates, for the first time, the effect of the NLO deviation from naive power counting in a four-body observable other than the binding energy. (cf. Ref. [41]).

A_X	Coulomb	Λ^*	# _{LO}	$\pi_{ m NLO}$	$\pi_{ m NLO}$	Exp.
	versions		7°LO	(OMM)	(TMM)	$\sqrt{\langle r_p^2 \rangle}$
² H	-	-	1.52	1.84	1.84	1.990
³ H	-	$B(^3H)$	1.13	1.45	1.44	1.608
³ He	¢	$B(^3\text{He})$	1.33	1.69	1.68	
	\mathcal{C}	$B(^3H)$	1.25	1.60	1.60	1.784
	pC	$B(^3H)$	1.25	1.63	1.63	
	C	$B(^3H)$	1.28	1.66	1.65	
⁴ He	¢	$B(^3\text{He})$	0.961	1.24	-	
	\mathcal{C}	$B(^3H)$	0.905	1.30	-	1.477
	pC	$B(^3H)$	0.905	1.24	-	
	C	$B(^3H)$	0.912	1.27	1.26	

Table 1: Predictions for various nuclear ($A \le 4$) point-proton root-mean-square charge radii $\sqrt{\langle r_p^2 \rangle}$ (in fm) of *EFT versions differing in their Coulomb treatment [absent (\mathcal{C})/perturbative (pC)/non-perturbative (C)] and three-body scale input Λ^* . All numbers are considered in the scaling limit, $\Lambda \to \infty$, and the associated extrapolation uncertainty is $< 10^{-3}$ fm. For references to experimental data, see the caption of Fig. 1.

3.2. Accounting for the Coulomb Interaction

For another non-trivial application of the introduced perturbative tools, we chose to investigate differences in $\sqrt{\langle r_p^2 \rangle}$ predictions at LO and NLO of the earlier introduced static-Coulomb versions of *EFT. This provides a quantitative guide to the circumstances, namely, the observables, for which one formulation converges order-by-order more rapidly than the other. In technical terms, we compare the effects of the absent $(V_{\mathbb{C}}^{(0)} + V_{\mathbb{C}}^{(1)})$, perturbative $(V_{\mathbb{C}}^{(0)} + V_{\mathbb{C}}^{(1)})$ static-Coulomb interaction on $\sqrt{\langle r_p^2 \rangle}$ of ³He and ⁴He. The corresponding LO and NLO predictions in their scaling limits $(\Lambda \to \infty)$ are compiled ² in Table 1.

First, we observe that isospin-symmetric \mathcal{C} predicts a smaller ³He radius if the binding energy of the triton sets the three-body scale, while it gets larger using the helion binding energy. The results are thereby consistent with the naively expected negative-slope correlation between binding energy and radius. For ⁴He, the \mathcal{C} LO root-mean-square radius prediction is found larger, too, if the shallower $B(^3\text{He})$ scale is employed instead of the deeper $B(^3\text{H})$ as renormalization condition. However, at NLO, the radius is predicted to be larger with the latter constraint – an effect which has to be related to the new four-body NLO scale and the chosen calibration of the same $B(^4\text{He})$ datum for both three-body scenarios.

An important indicator of the convergence rate of the respective EFTs is the difference between LO and NLO predictions. In this regard, the three EFT variants do not differ significantly; all implying a rate consistent with the naïve 1/3 [#]EFT estimate. Quantitatively, the Coulomb treatment affects the NLO contribution to vary by $\lesssim 0.03$ fm for 3 He, and $\lesssim 0.07$ fm for 4 He, while the [#]EFT NLO uncertainty estimates based on cutoff variation including the $\Lambda \to \infty$ limit are, respectively, ≈ 0.16 fm and 0.2 fm.

²The adequacy of the OMM was assessed in ⁴He for the non-perturbative Coulomb EFT and we subsequently abstained from obtaining the computationally more costly alternative TMM value for the other cases.

In our study, the * EFT's theoretical uncertainty is caused by a truncation of a double-expansion in two small parameters– (p_{ty}/m_{π}) associated with a contact interaction whose perturbation series breaks down at large typical momenta, p_{ty} , but converges for lower momenta, and $(\alpha m_N/p_{ty})$ characterizing a long-range Coulomb interaction where the power series diverges at small momenta. The observation of the * EFT being dominated for a specific observable by one of those expansions is the key result. From our predictions, we conclude that the ground-state root-mean-square radius is insensitive to long-range Coulomb effects up to NLO in * EFT. The momenta contributing significantly to the ground state radii appear sufficiently large to suppress static Coulomb exchange relative to the short-distance interaction.

We are thus led to conclude that observables on A-body nuclear states with a typical momentum scale, $p_{ty}^{(A)}$, – which may be associated with energies well below some breakup threshold, e.g., in our case, we have

$$\frac{p_{\text{ty}}^{(3)}}{p_{\text{ty}}^{(4)}} := \frac{\sqrt{\frac{2}{3}m_N B(^3\text{He})}}{\sqrt{\frac{3}{4}m_N B(^4\text{He})}} \approx \frac{1}{3}$$
(17)

– of the order considered here can be accurately predicted without having to deal with the long-range part of the interaction, at all. To see significant differences in the predictions of the three introduced EFTs, it is now obvious to turn to observables on states which involve much lower typical momenta, e.g., radii of halo nuclei, or transitions between the ground and shallow nuclear states, such as the first excited state of ⁴He.

4. Perspectives

The methods described in this article extract from numerical solutions of the Schrödinger equation corrections to observables expressed as polynomials of arbitrary order in a general perturbation operator. The efficacy of the methods was demonstrated by assessing the perturbative effect of the long- and short-range components of a nuclear interaction theory on the second spatial moment of the proton distribution within the nuclear bound states comprising less than five nucleons.

The renormalizable NLO *EFT prediction of the ⁴He point-proton root-mean-square charge radius, together with our quantification of the impact of excluding the Coulomb interaction, treating it in first-order perturbation theory, and iterating it to all orders on $\sqrt{\langle r_p^2 \rangle}$ of ³He and ⁴He nuclei, are results original to our work.

The application of the methods to calibrate the coupling strengths of * EFT beyond NLO renders them stepping stones to assess with these higher orders usefulness and convergence rates of * EFT for different observables. Furthermore, we deem the methods practical to analyze the "perturbativeness" of certain parts of other nuclear interaction potentials. For instance, the perturbative character of an exchange pion or Δ -excitation as part of high-precision interactions (e.g., [42, 43, 44]) can readily be studied in analogy to our above assessment of the Coulomb interaction. Finally, any prediction of scattering observables that extracts amplitudes from expectation values, e.g., via the finite-volume or trap-size dependence of energy spectra, can be combined with our methods. The associated perturbation-induced transformation of the character of S-matrix poles—a controversial issue in the development of contact theories (see [45, 46])—from, e.g., bound \rightarrow virtual/resonance, or vice-versa, is thus transparent: the pole moves as a consequence of an infinite iteration, and it is from this infinite series one isolates the various orders.

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References

- [1] Eugene P. Wigner. Lower limit for the energy derivative of the scattering phase shift. Phys. Rev., 98:145, 1955.
- [2] Daniel R. Phillips and Thomas D. Cohen. How short is too short? constraining zero-range interactions in nucleon-nucleon scattering. *Phys. Lett. B*, 390:7, 1997.
- [3] H. W. Hammer and Dean Lee. Causality and the effective range expansion. Annals Phys., 325:2212, 2010.
- [4] C.-J. Yang. Chiral potential renormalized in harmonic-oscillator space. Phys. Rev. C, 94:064004, 2016.
- [5] C. J. Yang, A. Ekström, C. Forssén, and G. Hagen. Power counting in chiral effective field theory and nuclear binding. *Phys. Rev. C*, 103:054304, 2021.
- [6] C. J. Yang, A. Ekström, C. Forssén, G. Hagen, G. Rupak, and U. van Kolck. The importance of few-nucleon forces in chiral effective field theory. *Eur. Phys. J. A*, 59:233, 2023.
- [7] U. Van Kolck. Effective field theory of nuclear forces. Prog. Part. Nucl. Phys., 43:337, 1999.
- [8] H. W. Hammer, S. König, and U. van Kolck. Nuclear effective field theory: status and perspectives. *Rev. Mod. Phys.*, 92:025004, 2020.
- [9] Dean Lee. Lattice simulations for few- and many-body systems. Prog. Part. Nucl. Phys., 63:117, 2009.
- [10] Evgeny Epelbaum, Hermann Krebs, Dean Lee, and Ulf-G. Meißner. Ab initio calculation of the hoyle state. *Phys. Rev. Lett.*, 106:192501, 2011.
- [11] Y. Suzuki and K. Varga. *Stochastic variational approach to quantum-mechanical few body problems*, volume 54. Springer, 1998.
- [12] Bernard Silvestre-Brac and Vincent Mathieu. The Few-body problem in terms of correlated gaussians. *Phys. Rev. E*, 76:046702, 2007.
- [13] J. Carlson, S. Gandolfi, F. Pederiva, Steven C. Pieper, R. Schiavilla, K. E. Schmidt, and R. B. Wiringa. Quantum monte carlo methods for nuclear physics. *Rev. Mod. Phys.*, 87:1067, 2015.
- [14] J.E. Lynn, I. Tews, S. Gandolfi, and A. Lovato. Quantum monte carlo methods in nuclear physics: Recent advances. *Annu. Rev. Nucl. Sci.*, 69:279, 2019.
- [15] Esben Nielsen, Dmitri Vladimir Fedorov, Aksel S Jensen, and Eduardo Garrido. The three-body problem with short-range interactions. *Phys. Rep.*, 347(5):373, 2001.
- [16] Seth T. Rittenhouse, N. P. Mehta, and Chris H. Greene. Green's functions and the adiabatic hyperspherical method. *Phys. Rev. A*, 82:022706, 2010.
- [17] LD119854 Faddeev. Scattering theory for a three-particle system. Zhur. Eksptl'. i Teoret. Fiz., 39, 1960.
- [18] LD Faddeev. Scattering theory for a three-particle system. In *Fifty Years of Mathematical Physics: Selected Works of Ludwig Faddeev*, pages 37–42. World Scientific, 2016.
- [19] OA Yakubovskii. On the integral equations in the theory of n particle scattering. Technical report, Leningrad State Univ., 1967.
- [20] E. O. Alt, P. Grassberger, and W. Sandhas. Reduction of the three particle collision problem to multichannel two particle lippmann-schwinger equations. *Nucl. Phys. B*, 2:167, 1967.

- [21] John Archibald Wheeler. Molecular viewpoints in nuclear structure. Phys. Rev., 52(11):1083, 1937.
- [22] Y. C. Tang, M. Lemere, and D. R. Thompson. Resonating-group method for nuclear many-body problems. *Phys. Rept.*, 47:167, 1978.
- [23] Karl Wildermuth. A unified theory of the nucleus. Springer-Verlag, 2013.
- [24] P. Navrátil, V. G. Gueorguiev, J. P. Vary, W. E. Ormand, and A. Nogga. Structure of a = 10 -13 nuclei with two-plus three-nucleon interactions from chiral effective field theory. *Phys. Rev. Lett.*, 99:042501, 2007.
- [25] Bruce R. Barrett, Petr Navrátil, and James P. Vary. Ab initio no core shell model. Prog. Part. Nucl. Phys., 69:131, 2013.
- [26] R. P. Feynman. Forces in molecules. *Phys. Rev.*, 56:340, 1939.
- [27] Hans Hellmann. Einführung in die Quantenchemie. Franz Deuticke, Leipzig and Wien, 1937.
- [28] Martin Schäfer and Betzalel Bazak. Few-nucleon scattering in pionless effective field theory. *Phys. Rev. C*, 107:064001, 2023.
- [29] Mirko Bagnarol, Martin Schäfer, Betzalel Bazak, and Nir Barnea. Five-body calculation of s-wave n-4he scattering at next-to-leading order pionless effective field theory.
- [30] Matúš Rojik, Martin Schäfer, Mirko Bagnarol, and Nir Barnea. Charged particle scattering in renormalizable pionless effective field theory at next-to-leading order: The *pd*, *dd*, and *p*³He case, 2025.
- [31] L. Contessi, M. Schäfer, A. Gnech, A. Lovato, and U. van Kolck. A renormalizable theory for not-so-light nuclei, 2025.
- [32] Jiunn-Wei Chen, Gautam Rupak, and Martin J. Savage. Nucleon-nucleon effective field theory without pions. *Nucl. Phys. A*, 653:386, 1999.
- [33] Jared Vanasse. Triton charge radius to next-to-next-to-leading order in pionless effective field theory. *Phys. Rev. C*, 95:024002, 2017.
- [34] Jared Vanasse. Charge and magnetic properties of three-nucleon systems in pionless effective field theory. *Phys. Rev. C*, 98:034003, 2018.
- [35] M. Schäfer, B. Bazak, N. Barnea, and J. Mareš. Nature of the $\Lambda nn(J^{\pi} = 1/2^+, i = 1)$ and ${}^{3}_{\Lambda}H^*(J^{\pi} = 3/2^+, i = 0)$ states. *Phys. Rev. C*, 103:025204, 2021.
- [36] I. Angeli and K.P. Marinova. Table of experimental nuclear ground state charge radii: An update. *Atomic Data and Nuclear Data Tables*, 99:69, 2013.
- [37] S. Navas and et al. Review of particle physics. Phys. Rev. D, 110:030001, 2024.
- [38] Daniel R. Phillips, Gautam Rupak, and Martin J. Savage. Improving the convergence of nn effective field theory. *Phys. Lett. B*, 473:209, 2000.
- [39] Harald W. Griesshammer. Improved convergence in the three-nucleon system at very low energies. *Nucl. Phys. A*, 744:192, 2004.
- [40] L. Contessi, A. Lovato, F. Pederiva, A. Roggero, J. Kirscher, and U. van Kolck. Ground-state properties of ⁴he and ¹⁶o extrapolated from lattice qcd with pionless eft. *Phys. Lett. B*, 772:839, 2017.
- [41] B. Bazak, J. Kirscher, S. König, M. Pavón Valderrama, N. Barnea, and U. van Kolck. Four-body scale in universal few-boson systems. *Phys. Rev. Lett.*, 122:143001, 2019.
- [42] E. Epelbaum, H.-W. Hammer, and Ulf-G. Meißner. Modern theory of nuclear forces. Rev. Mod. Phys., 81:1773.

- [43] M. Piarulli, L. Girlanda, R. Schiavilla, A. Kievsky, A. Lovato, L. E. Marcucci, Steven C. Pieper, M. Viviani, and R. B. Wiringa. Local chiral potentials with δ -intermediate states and the structure of light nuclei. *Phys. Rev. C*, 94:054007, 2016.
- [44] R. Somasundaram, J. E. Lynn, L. Huth, A. Schwenk, and I. Tews. Maximally local two-nucleon interactions at nlo in pion-less chiral effective field theory. *Phys. Rev. C*, 109:034005, 2024.
- [45] M. Schäfer, L. Contessi, J. Kirscher, and J. Mareš. Multi-fermion systems with contact theories. *Phys. Lett. B*, 816:136194, 2021.
- [46] C.-J. Yang. Feasibility of perturbative generation of bound states from resonances or virtual states. *Phys. Rev. C*, 109:054003, 2024.