Efficient & Correct Predictive Equivalence for Decision Trees

Joao Marques-Silva   JPMS@ICREA.CAT
ICREA & University of Lleida
Lleida, Catalunya, Spain

Alexey Ignatiev   ALEXEY.IGNATIEV@MONASH.EDU
Faculty of Information Technology
Monash University
Melbourne, Australia

Abstract

The Rashomon set of decision trees (DTs) finds several relevant uses. Recent work showed that DTs computing the same classification function, i.e. predictive equivalent DTs, can represent a significant fraction of the Rashomon set. Such redundancy can be undesirable. For example, feature importance based on the Rashomon set becomes inaccurate due the existence of predictive equivalent DTs, i.e. DTs that make the same prediction for every possible input. In recent work, McTavish et al. proposed solutions for several computational problems related with DTs, including that of deciding predictive equivalent DTs. This approach, which this paper refers to as MBDSR using the authors' last names initials, consists of applying the well-known method of Quine-McCluskey (QM) for obtaining minimum-size DNF (disjunctive normal form) representations of DTs, which are then used for comparing DTs for predictive equivalence. Furthermore, the minimum-size DNF representation was also applied to computing explanations for the predictions made by DTs, and to finding predictions in the presence of missing data. However, the problem of formula minimization is hard for the second level of the polynomial hierarchy, and there exist different reasons why the QM method may exhibit worst-case exponential running time and space. This paper first demonstrates that there exist decision trees that trigger the worst-case exponential running time and space of the QM method. Second, the paper shows that the MBDSR approach can produce incorrect results for the problem of deciding predictive equivalence. Third, the paper shows that any of the problems to which the minimum-size DNF representation has been applied to can in fact be solved in polynomial time, in the size of the DT. The experiments confirm that, for DTs for which the the worst-case of the QM method is triggered, the algorithms proposed in this paper are orders of magnitude faster than the ones proposed by McTavish et al. Furthermore, the algorithms proposed in this paper guarantee the correctness of computed results, in contrast with the the work of McTavish et al. Finally, the paper relates predictive equivalence with logic-based explanations and measures of importance.

Keywords: Decision Trees, Rashomon Sets, Predictive Equivalence

Contents

1 Introduction

In decision tree learning, the Rashomon set (Breiman, 2001b; Xin et al., 2022) captures the set of decision trees (DTs) for a given classification problem which are within some distance from some accepted definition of optimal DT (Xin et al., 2022). Several uses of the Rashomon set have been studied in recent years (Fisher et al., 2019; Semenova and Rudin, 2019; Rudin et al., 2022; Semenova et al., 2022; Xin et al., 2022; Donnelly et al., 2023; Paes et al., 2023; Biecek et al., 2023; Kobylinska et al., 2024; Hsu et al., 2024; Ciaperoni et al., 2024; Li et al., 2024b,a; Donnelly et al., 2025; Nguyen et al., 2025). For the same classification problem, different decision trees capture different classification functions, and it is often of interest to study a sample of the decision trees in the Rashomon set. Recent work (McTavish et al., 2025b,a) studies predictive equivalent DTs in the Rashomon set, i.e. DTs which represent the same classification function. This earlier work observes that the predictive equivalent DTs in the Rashomon set can represent a fairly significant percentage of the DTs, and so their removal enables keeping in the Rashomon set only DTs that capture different classification functions. For example, this yields better estimates in terms of feature importance (Donnelly et al., 2023).

To identify DTs that are predictive equivalent, McTavish et al. (McTavish et al., 2025b,a) propose a solution that revisits work on boolean formula minimization from the 1950s, namely the well-known Quine-McCluskey (QM) boolean formula minimization method (Quine, 1952, 1955; McCluskey, 1956).[1] Starting from some DT $\mathcal{T}$, the QM method is applied to computing a minimum-size boolean formula $\mathcal{T}_{\mathsf{DNF},c}$ for each class c, in disjunctive normal form (DNF). This formula is then used not only for deciding predictive equivalence, but also to answer a number of important queries regarding DT prediction in terms of non-availability of some features and their explanation (McTavish et al., 2025b,a). It should be underlined that the formula $\mathcal{T}_{\mathsf{DNF},c}$ is *not* any DNF representation of a DT for predicting class c. Since the language of DTs is less expressive that that of DNF formulas (Hyafil and Rivest, 1976), then any DT can be represented by a DNF of size no larger that the DT. For each class c, $\mathcal{T}_{\mathsf{DNF},c}$ represents a *minimum-size* DNF formula obtained from the set of prime implicants for predicting that class. Moreover, $\mathcal{T}_{\mathsf{DNF}}$ represents the set of DNFs, one for each class c. Throughout this paper, we will use the acronym MBDSR to refer to the approach proposed in the work of McTavish et al. (McTavish et al., 2025b,a).

In practice, the QM method starts by computing all the prime implicants of the boolean formula F, represented as a disjunction of terms. The resulting disjunction is often referred to as the Blake Canonical Form (BCF) (Blake, 1937; Brown, 1990) of F. For a DT $\mathcal{T}$ and predicted class c, the BCF will be referred to as $\mathsf{BCF}_c(\mathcal{T})$. Afterwards, the QM method enumerates all the minterms of the function and then it finds a minimum-cost set covering of the sets of terms in the formula using a (minimum-cost) subset of the prime implicants of the formula. The resulting minimum-size DNF formula is denoted as $\mathcal{T}_{\mathsf{DNF},c}$, which holds true for *any* partial assignment that is sufficient for predicting class c. Existing practical evidence indicates that the QM method scales poorly, with a few tens of variables representing what the QM method is known to be able to tackle (Brayton et al., 1984; Coudert, 1994). This is explained by the fact that the generation of prime implicants is worst-case exponential in time and space, and minimum-cost set covering corresponds to solving an NP-hard optimization problem, on an intermediate representation that is worst-case exponential on the

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1. There are different algorithms that implement variants of the QM method (McCluskey, 1956; Brayton et al., 1984; Coudert, 1994). The work of McTavish et al. (McTavish et al., 2025b,a) exploits the implementation of QM available in SymPy (Meurer et al., 2017).

size of the input. The fact that the QM method scales poorly motivated the development of a number of alternative methods (Brayton et al., 1984; Rudell, 1989; Coudert et al., 1993; Coudert, 1995; Ignatiev et al., 2015). More importantly, the problem of boolean formula minimization is known to be Σ_2^p-hard (Umans, 1998; Umans et al., 2006) and that the associated decision problem also lies in the second level of the polynomial hierarchy. In practical terms, this matches the complexity of deciding quantified boolean formulas (QBF) with two levels of quantification, which is known to scale much worse than solving (co)NP-complete decision problems on formulas (Biere et al., 2021), e.g. satisfiability or tautology.

While it is known that the number of prime implicants can grow exponentially with the number of variables for arbitrary boolean formulas (Chandra and Markowsky, 1978), this paper shows that this is also the case with DTs, i.e. there exist DTs for which the BCF is worst-case exponential in the size of the DT. The paper then improves on earlier work (McTavish et al., 2025b,a), by proving that the computation of $\mathsf{BCF}_c(\mathcal{T})$ (which is worst-case exponential in size) and $\mathcal{T}_{\mathsf{DNF},c}$ (which requires solving an NP-hard optimization problem on a worst-case exponential size representation) are both *unnecessary*, and can be replaced with polynomial-time algorithms. Concretely, this paper proves that the properties of: i) *completeness* (Theorem 3.2 in (McTavish et al., 2025a)); ii) *succinctness* (Proposition 3.3 in (McTavish et al., 2025a)); and iii) the *resolution of predictive equivalence* (Theorem 3.4 in (McTavish et al., 2025a)) can all be solved in polynomial time in the size of the given DT. In addition, we prove that all the algorithms proposed in this paper are correct.

Unfortunately, the issues with McTavish et al.'s approach go beyond the manifest inefficiency of the proposed algorithms (McTavish et al., 2025b,a). As this paper demonstrates, the algorithm for predictive equivalence proposed by McTavish et al. (McTavish et al., 2025b,a) is *unsound*, because it assumes a property of minimum-size DNF representations of boolean functions which does not necessarily hold. Concretely, the algorithm of McTavish et al. makes the explicit assumption of the canonicity of minimum-size DNF representations of boolean functions, and this is not the case (Brown, 1990). We briefly summarize here the main consequences from the unsoundness of McTavish et al.'s predictive equivalence algorithm. For two DTs that are not predictive equivalent, the algorithm of McTavish et al. will correctly report that there is no predictive equivalence. However, there can exist predictive equivalent DTs for which the algorithm will report non-equivalence. In light of the results in our paper, the conclusions of McTavish et al.'s work regarding the fraction of predictive equivalence DTs should be reassessed. As argued later in the paper, albeit formally incorrect, the MBDSR approach may or may not produce erroneous results, and this depends on the implementation of the QM method.

This paper is organized as follows. Section 2 introduces the notation and definitions used throughout. Section 3 develops an in-depth analysis of the work of McTavish et al. (McTavish et al., 2025b,a). Section 3.1 analyzes the MBDSR approach in terms of running time, proving that the MBDSR approach can require exponential time and space, and also demonstrating that the worst-case running time and space is exercised in the case of DTs. Section 3.2 goes a step further, and shows that MBDSR can produce incorrect results in the case of deciding predictive equivalence. Section 4 fixes both the issues of correctness and performance of the MBDSR approach, by proposing polynomial-time algorithms for the computational problems studied by McTavish et al. (McTavish et al., 2025b,a). Section 6 proves a number of additional results, relating predictive equivalence with logic-based explanations (Marques-Silva, 2022; Marques-Silva and Ignatiev, 2022), but also with measures of importance, including the Shapley values (Shapley, 1953; Chalkiadakis et al., 2012) and Banzhaf values (Banzhaf III, 1965; Chalkiadakis et al., 2012). These additional results

also allow developing an algorithm for predictive equivalence that asymptotically improves on the algorithm proposed in Section 4. Section 5 presents simple experiments demonstrating the massive difference in performance between the MBDSR approach, which can produce incorrect results, and the algorithms proposed in this paper. Section 7 concludes the paper.

Disclaimer. The authors of the MBDSR approach (McTavish et al., 2025b,a) have been informed of the efficiency and soundness issues in their work that the present paper reveals and corrects (Marques-Silva and Ignatiev, 2025).

2 Preliminaries

2.1 Machine Learning (ML) Models

Classification models. A classification problem is defined on a set of features $\mathcal{F} = \{1, \ldots, m\}$. Each feature i has a domain $\mathbb{D}_i$. Domains can be categorical or ordinal. If ordinal, domains can be integer or real-valued. Feature space is defined as the cartesian product of the domains, $\mathbb{F} = \mathbb{D}_1 \times \cdots \times \mathbb{D}_m$. A classifier computes a classification function κ, that maps feature space to a set of classes $\mathbb{K} = \{c_1, \ldots, c_K\}$.

Given a feature i, x_i represents a variable that takes values from $\mathbb{D}_i$, whereas v_i represents a constant taken from $\mathbb{D}_i$. An instance (or sample) is a pair $(\mathbf{v}, c)$, with $\mathbf{v} \in \mathbb{F}$ and $c \in \mathbb{K}$, such that $\kappa(\mathbf{v}) = c$.

A classification model is defined as a tuple $C = (\mathcal{F}, \mathbb{F}, \mathbb{K}, \kappa)$. (The domains of features are left implicit for simplicity.)

Regression models. For regression models, a model similar to the classification case is assumed. Instead of classes we now assume a set of values $\mathbb{V}$. Moreover, the regression function ρ maps feature space to the set of values, $\rho : \mathbb{F} \to \mathbb{V}$. Although the paper focuses solely on classification models, we could easily adopt a unified ML model with a prediction function π, without explicitly distinguishing classification from regression (Marques-Silva, 2024).

In this general setting, an ML model is defined as a tuple $M = (\mathcal{F}, \mathbb{F}, \mathbb{T}, \pi)$, where $\mathbb{T}$ is a set of targets; these can represent ordinal or categorical values. (As before, the domains of features are left implicit for simplicity.)

2.2 Logic Foundations

Propositional logic. Standard notation and definitions are adopted (Biere et al., 2021). Boolean formulas are defined over propositional atoms (or variables) using the standard logical operators $\{\wedge, \vee, \neg\}$. A literal is an atom (x) or its negation $(\neg x)$. A term t is a conjunction of literals. A DNF (disjunctive normal form) formula ϕ is a disjunction of terms. Terms and formulas are also interpreted as functions, mapping feature space to $\{\top, \bot\}$.[2]

An *implicant* t for a function ϕ is a term that entails ϕ, i.e. $t \vDash \phi$, signifying that $\forall(\mathbf{x} \in \mathbb{F}).(t(\mathbf{x}) \to \phi(\mathbf{x}))$ (Brown, 1990; Crama and Hammer, 2011).[3] A *prime implicant* of ϕ is an implicant t such that any term t', obtained by dropping any literal from t, is no longer an implicant of ϕ. A prime implicant is *essential* if it covers at least one point of its domain that is not covered by any other prime implicant. A *minterm* is an implicant consistent with exactly one point in feature space. Prime implicants are often associated with *boolean* functions (Quine, 1952, 1955;

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2. For simplicity we do not explicitly define the semantics of propositional logic formulas, since it is quite standard.
3. In the boolean case, we will treat functions and formulas indistinctly.

McCluskey, 1956; Brown, 1990; Crama and Hammer, 2011). However, prime implicants have been studied in more general settings (Dillig et al., 2012). Similar to more recent works (Dillig et al., 2012), in this paper prime implicants are not restricted to boolean functions.

Partial assignments & consistency.   Let $\mathbb{OP}$ denote some universe of relational operators, e.g. we could consider $\mathbb{OP} = \mathbb{OP}_s \cup \mathbb{OP}_t$, with $\mathbb{OP}_t = \{\in\}$ and $\mathbb{OP}_s = \{=, <, \leq, \geq, >\}$. (Logic negation can be used to obtain additional operators.) Let $\mathbb{L} = \{(x_i, R_i, \mathsf{op}_i) \,|\, (i \in \mathcal{F}) \wedge (R_i \subseteq \mathbb{D}_i) \wedge (\mathsf{op}_i \in \mathbb{OP}_t)\} \cup \{(x_i, R_i, \mathsf{op}_i) \,|\, (i \in \mathcal{F}) \wedge (R_i \in \mathbb{D}_i) \wedge (\mathsf{op}_i \in \mathbb{OP}_s)\}$ denote all the literals that can be obtained from the variables associated with the features in $\mathcal{F}$ and subsets or values from their domains. (Throughout the paper, we will simplify the notation. For example, (x_i, v_i) will be used instead of $(x_i, v_i, =)$, and both represent the literal $x_i = v_i$. Moreover, $(x_i, v_i, =)$ is equivalent to writing $(x_i, \{v_i\}, \in)$.) The set of *partial assignments* is defined by the powerset of $\mathbb{L}$, i.e. $\mathbb{A} = 2^{\mathbb{L}}$. Thus, a partial assignment $\mathcal{A} \in \mathbb{A}$ is a subset of $\mathbb{L}$. In the case of complete assignments (or inputs to an ML model), we require that all features are assigned a single value from their domain, $\mathbb{I} = \{\mathcal{A} \in \mathbb{A} \,|\, \forall (i \in \mathcal{F}). |\{(x_i, R_i, \mathsf{op}_i) \in \mathcal{A}\}| = 1 \wedge (x_i, v_i, =) \in \mathcal{A}\}$. With each literal $(x_i, R_i, \mathsf{op}_i) \in \mathbb{L}$ we associate a domain $\mathsf{dom}((x_i, R_i, \mathsf{op}_i)) = \{\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_m) \in \mathbb{F} \,|\, x_i \,\mathsf{op}_i\, v_i\}$. Thus, the domain of a literal is the set of points in feature space for which the literal holds true. Similarly, we define the domain of a partial assignment $\mathcal{A}$, i.e. $\mathsf{dom}(\mathcal{A}) = \cap_{(x_i, v_i, \mathsf{op}_i)}\mathsf{dom}(x_i, v_i, \mathsf{op}_i)$. In addition, a partial assignment is said to be consistent, it if exhibits a non-empty domain, i.e. $\mathsf{Consistent}(\mathcal{A}) = [\mathsf{dom}(\mathcal{A}) \neq \emptyset]$. By default, we will require partial assignments to be consistent. Finally, we note that (prime) implicants can also be represented as partial assignments (Dillig et al., 2012; Amgoud, 2023). This provides a convenient framework for formalizing explanations in DTs.

2.3 Classification with Decision Trees

Decision trees.   A DT is a connected directed acyclic graph $\mathcal{T} = (V, E)$, with a single root node $r \in V$. $r = \mathsf{Root}(\mathcal{T})$ indicates that $r \in V$ is the root of DT $\mathcal{T}$. The tree nodes V are partitioned into non-terminal and terminal nodes. Each non-terminal node s (such that $\mathsf{NonTerminal}(s; \mathcal{T})$ holds true) is associated with a feature i, i.e. $i = \mathsf{Feature}(s; \mathcal{T})$. For a non-terminal node s, associated with feature i, the domain of i is partitioned among the children of node s. Each terminal node s (such that $\mathsf{Terminal}(s; \mathcal{T})$ holds true) is associated with a class from $\mathbb{K}$, denoted by $\mathsf{Class}(s; \mathcal{T})$. Whereas non-terminal nodes have two or more children, terminal nodes have no children. Moreover, for each node s in the DT, $\mathsf{Parent}(s; \mathcal{T})$ represents the parent of s in the tree. For the root node $r = \mathsf{Root}(\mathcal{T})$, $\mathsf{Parent}(r; \mathcal{T}) = \mathsf{none}$. As with any classifier, a DT computes a classification function $\kappa_{\mathcal{T}}$, mapping feature space to $\mathbb{K}$. For convenience, $\mathsf{NonTerminal}(\mathcal{T})$ denotes the nodes in V that are non-terminal. Similarly, $\mathsf{Terminal}(\mathcal{T})$ denotes the nodes in V that are terminal.

A path P in a DT is a sequence of nodes $P = \langle s_1, \ldots, s_r \rangle$, such that $s_1 = \mathsf{Root}(\mathcal{T})$, $s_j = \mathsf{Parent}(s_{j+1}; \mathcal{T})$, $j = 1, \ldots, r$, and $\mathsf{Terminal}(s_r; \mathcal{T})$ holds true. Moreover, each pair (s_j, s_{j+1}) represents both an edge of P and of $\mathcal{T}$, i.e. $(s_j, s_{j+1}) \in E$. We will use $\mathsf{Terminal}(P; \mathcal{T})$ to denote the terminal node of path P in DT $\mathcal{T}$. The paths of DT $\mathcal{T}$ are denoted by $\mathsf{Paths}(\mathcal{T})$. Moreover, if (s_p, s_q) is an edge of $\mathcal{T}$, then $\mathsf{Literal}(s_p, s_q; \mathcal{T})$ represents the literal associated with $\mathsf{Feature}(s_p; \mathcal{T})$ and edge (s_p, s_q) on the path of $\mathcal{T}$ containing that edge. For a path $P \in \mathsf{Paths}(\mathcal{T})$, $\mathsf{Literals}(P) = \{\mathsf{Literal}(s_p, s_q; \mathcal{T}) \,|\, (s_p, s_q) \in P\}$. We note that $\mathsf{Literals}(P)$ represents a generalized partial assignment and so the above definition of consistency also applies to the literals in a DT path.

In this paper, we assume conditions that are often left implicit when studying DTs, and which can be formalized as follows:

$$\forall (P \in \mathsf{Paths}(\mathcal{T})). \, (\mathsf{Literals}(P; \mathcal{T}) \not\models \bot) \tag{1}$$

$$\forall (\mathbf{x} \in \mathbb{F}). \exists (P \in \mathsf{Paths}(\mathcal{T})). \, (\mathbf{x} \models \mathsf{Literals}(P; \mathcal{T})) \tag{2}$$

$$\forall (\mathbf{x} \in \mathbb{F}). \forall (P \in \mathsf{Paths}(\mathcal{T})). \, ((\mathbf{x} \models \mathsf{Literals}(P; \mathcal{T})) \rightarrow$$
$$(\forall (Q \in \mathsf{Paths}(\mathcal{T}) \setminus \{P\}). \, (\mathbf{x} \not\models \mathsf{Literals}(Q; \mathcal{T})))) \tag{3}$$
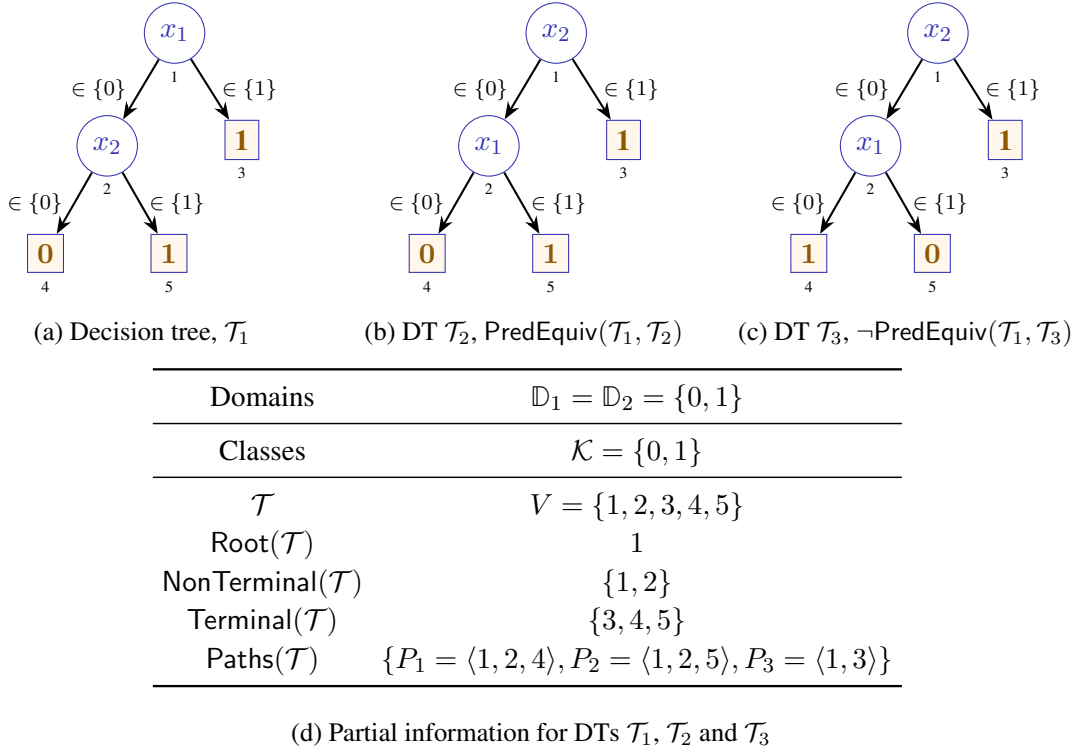
(For simplicity, Literals is interpreted as the conjunction of the literals.) Thus, DTs are assumed to respect: (i) any path in the DT is consistent; (ii) any point in feature space is consistent with at least one tree path; and (iii) any point in feature space that is consistent with some tree path must not be consistent with any other tree path.

Two DTs $\mathcal{T}$ and $\mathcal{T}'$ are *predictive equivalent* if it holds that: $\forall (\mathbf{x} \in \mathbb{F}). \, (\kappa_{\mathcal{T}}(\mathbf{x}) = \kappa_{\mathcal{T}'}(\mathbf{x}))$. Thus, predictive equivalence signifies that the two DTs cannot be distinguished over any point in feature space. For any pair of DTs $\mathcal{T}_1, \mathcal{T}_2$, the predicate $\mathsf{PredEquiv}(\mathcal{T}_1, \mathcal{T}_2)$ takes value $\top$ (i.e. it holds true) if the pair of DTs is predictive equivalent; otherwise it takes value $\bot$. Although the paper focuses mostly on DTs, it is plain that the definition of predictive equivalencd can be used to relate two arbitrary ML models. For example, we could have a DT that is predictive equivalent to random forest (RF) (Breiman, 2001a). In Section 6 we revisit predictive equivalence in this more general setting.

Although it is well-known that learning an optimal decision tree is NP-hard (Hyafil and Rivest, 1976), recent years witnessed a renewed interest in the learning of optimal decision trees (e.g. (Nijssen and Fromont, 2007; Bessiere et al., 2009; Nijssen and Fromont, 2010; Bertsimas and Dunn, 2017; Verwer and Zhang, 2017; Narodytska et al., 2018; Verwer and Zhang, 2019; Hu et al., 2019; Verhaeghe et al., 2020a; Avellaneda, 2020; Aglin et al., 2020a; Lin et al., 2020; Janota and Morgado, 2020; Hu et al., 2020; Aglin et al., 2020b; Verhaeghe et al., 2020b; Demirovic and Stuckey, 2021; Shati et al., 2021; Demirovic et al., 2022; Rudin et al., 2022; van der Linden et al., 2023), among many others), with the purpose of improving ML model interpretability. The rationale is that DTs are *interpretable* and so the model can be used for obtaining an explanation, e.g. the path consistent with the given instance (Molnar, 2020). Moreover, and although it is generally accepted that DTs are interpretable (Breiman, 2001b),[4] there is no rigourous definition of *interpretability* or *interpretable* ML model and this is unlikely to change (Lipton, 2018). Furthermore, a number of recent works (Izza et al., 2020, 2022; Audemard et al., 2022b; Marques-Silva and Ignatiev, 2023; McTavish et al., 2025a) observe that even optimal decision trees need to be explained, e.g. using a formal definition of explanation (Marques-Silva, 2022; Marques-Silva and Ignatiev, 2022). Concretely, recent work demonstrated that if DTs are explained by reporting the path consistent with some instance, then the explanation can be arbitrarily redundant on the number of features (Izza et al., 2020, 2022). Accordingly, one of the concerns of the MBDSR approach of McTavish et al. (McTavish et al., 2025b,a) is to also find rigorous explanations for DTs.

Running example. The DTs shown in Figure 1 will be used as the running example throughout the paper, with the reference DT being the first one. As an example, for $P_1 = \{1, 2, 4\}$, we have that $\mathsf{Literals}(P_1; \mathcal{T}) = \{(x_1, \{0\}, \in), (x_2, \{0\}, \in)\}$.

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4. The perception of DT interpretability can be traced back more than two decades. A quote from a 2001 paper by L. Breiman (Breiman, 2001b) illustrates this perception: *"On interpretability, trees rate an A+"*.

(a) Decision tree, $\mathcal{T}_1$ (b) DT $\mathcal{T}_2$, PredEquiv$(\mathcal{T}_1, \mathcal{T}_2)$ (c) DT $\mathcal{T}_3$, ¬PredEquiv$(\mathcal{T}_1, \mathcal{T}_3)$

Domains	$\mathbb{D}_1 = \mathbb{D}_2 = \{0, 1\}$
Classes	$\mathcal{K} = \{0, 1\}$
$\mathcal{T}$	$V = \{1, 2, 3, 4, 5\}$
Root$(\mathcal{T})$	1
NonTerminal$(\mathcal{T})$	$\{1, 2\}$
Terminal$(\mathcal{T})$	$\{3, 4, 5\}$
Paths$(\mathcal{T})$	$\{P_1 = \langle 1, 2, 4 \rangle, P_2 = \langle 1, 2, 5 \rangle, P_3 = \langle 1, 3 \rangle\}$

(d) Partial information for DTs $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$

Figure 1: Running example DT $\mathcal{T}_1$, predictive equivalent (PE) DT $\mathcal{T}_2$, and non-PE DT $\mathcal{T}_3$

2.4 Logic-Based Explanations

Recent years witnessed the emergence of rigorous, logic-based, explanability. This subsection provides a brief account of the progress in this field of research.

Abductive explanations (AXps).   AXps have been proposed in a number of works (Shih et al., 2018; Ignatiev et al., 2019a), and represent an example of formally defined explanations. An AXp answers a "Why (the prediction)?" question, and can be interpreted as a logic rule. In this paper, the notation used builds upon the one from (Amgoud, 2023), since it is preferred for describing the algorithms proposed later in the paper. A weak abductive explanation (WAXp) is a partial assignment $\mathcal{A} \in \mathbb{A}$ that entails the prediction:[5]

$$\forall(\mathbf{x} \in \mathbb{F}). \left(\bigwedge\nolimits_{(x_i, R_i, \mathsf{op}_i) \in \mathcal{A}} x_i \, \mathsf{op}_i \, R_i \right) \to (\kappa(\mathbf{x}) = c) \qquad (4)$$

Predicate WAXp$(\mathcal{A}; (\mathbf{v}, c), \mathcal{T})$ holds true when (4) holds.[6] A subset-minimal WAXp is referred to as an abductive explanation (AXp). WAXps (resp. AXps) are also referred to as sufficient (resp. minimal) reasons for a prediction (Darwiche and Hirth, 2023; Darwiche, 2023). Moreover, the WAXps for prediction $c \in \mathbb{K}$ represent the implicants of predicate $\pi_c : \mathbb{F} \to \{\bot, \top\}$, such that $\pi_c(\mathbf{x}) = [\kappa(\mathbf{x}) = c]$. We will use WAXps for κ and class c and the implicants of π_c interchangeably.

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5. As noted earlier, the notation used is adapted from (Amgoud, 2023), and it is more complicated than the one used in other works, e.g. (Marques-Silva, 2022). However, for the purposes of this paper, the more complicated notation will be helpful.

6. For predicates, parameterizations are shown after ';', but will be omitted for simplicity.

Example 1 *For the DT $\mathcal{T}_1$ and for class 1, $\{(x_1, 0), (x_2, 1)\}$ is a WAXp. Moreover, $\{(x_2, 1)\}$ is an AXp. By inspection, another AXp for class 1 is $\{(x_1, 1)\}$. Finally, for class 0, the only (W)AXp is $\{(x_1, 0), (x_2, 0)\}$.*

For a given classification model M, we define the set of AXps for an instance $(\mathbf{v}, c)$ by $\mathbb{A}((\mathbf{v}, c); M) = \{\mathcal{A} \subseteq \mathbb{L} \,|\, \mathsf{AXp}(\mathcal{A}; (\mathbf{v}, c), M)\}$. Furthermore, we observe that, given the definitions above, the predicate WAXp does not depend on $\mathbf{v}$, only on some partial assignment (that may be extracted from $\mathbf{v}$). As a result, when convenient, we will use the representation $\mathsf{WAXp}(\mathcal{A}; c, M)$ that holds true for partial assignments such that (4) holds, when M computes classification function κ and predicts $c \in \mathbb{K}$. This alternative representation will also be used to represent AXps. Thus, we can now define the set of all partial assignments that represent (W)AXps for a given prediction c:

$$\mathbb{WA}(M, c) = \{\mathcal{A} \subseteq \mathbb{L} \,|\, \mathsf{Consistent}(\mathcal{A}) \wedge \mathsf{WAXp}(\mathcal{A}; c, M)\} \tag{5}$$

$$\mathbb{A}(M, c) = \{\mathcal{A} \subseteq \mathbb{L} \,|\, \mathsf{Consistent}(\mathcal{A}) \wedge \mathsf{AXp}(\mathcal{A}; c, M)\} \tag{6}$$

(Since partial assignments are chosen from $\mathbb{L}$, we have to ensure that the partial assignment is consistent.)

Furthermore, each AXp is uniquely defined by a partial assignment and a prediction. We can then talk about the sets of pairs, of partial assignments and predictions, such that the partial assignments are either sufficient for some prediction c,

$$\begin{aligned}
\mathbb{WA}(M) &= \{(\mathcal{A}, c) \,|\, c \in \mathbb{K} \wedge \mathcal{A} \subseteq \mathbb{L} \wedge \mathsf{Consistent}(\mathcal{A}) \wedge \mathsf{WAXp}(\mathcal{A}; c, M)\} \\
&= \{(\mathcal{A}, c) \,|\, c \in \mathbb{K} \wedge \mathcal{A} \in \mathbb{WA}(M, c)\} \\
\mathbb{A}(M) &= \{(\mathcal{A}, c) \,|\, c \in \mathbb{K} \wedge \mathcal{A} \subseteq \mathbb{L} \wedge \mathsf{Consistent}(\mathcal{A}) \wedge \mathsf{AXp}(\mathcal{A}; c, M)\} \\
&= \{(\mathcal{A}, c) \,|\, c \in \mathbb{K} \wedge \mathcal{A} \in \mathbb{A}(M, c)\}
\end{aligned}$$

Observe that the above definitions allow aggregating the (W)AXps for all of the classes in $\mathbb{K}$.

Contrastive explanations (CXps). CXps can be defined similarly to AXps (Ignatiev et al., 2020; Marques-Silva, 2022). However, given the option to work with partial assignments in this paper, we would have to adapt the definition of CXps to account for complete assignments. Since the paper's results do not dependent on CXps, we opt not to discuss CXps in this paper.

Progress in logic-based XAI. Besides the core references above, there has been rapid progress in logic-based XAI, by a growing number of researchers (Shih et al., 2018; Ignatiev et al., 2019a,b; Darwiche and Hirth, 2020; Marques-Silva et al., 2020; Barceló et al., 2020; Ignatiev et al., 2020; Wäldchen et al., 2021; Marques-Silva et al., 2021; Ignatiev and Marques-Silva, 2021; Izza and Marques-Silva, 2021; Malfa et al., 2021; Huang et al., 2021; Audemard et al., 2021; Cooper and Marques-Silva, 2021; Liu and Lorini, 2021; Boumazouza et al., 2021; Audemard et al., 2022b; Ignatiev et al., 2022; Huang et al., 2022; Audemard et al., 2022a; Pinchinat et al., 2022; Darwiche and Ji, 2022; Liu and Lorini, 2022; Audemard et al., 2023a, 2022c; Amgoud and Ben-Naim, 2022; Arenas et al., 2022; Cooper and Marques-Silva, 2023; Izza et al., 2023; Darwiche and Hirth, 2023; Liu and Lorini, 2023; Marques-Silva and Ignatiev, 2023; Yu et al., 2023; Huang et al., 2023b; Audemard et al., 2023c; Huang et al., 2023a; Bassan and Katz, 2023; Audemard et al., 2023b; Huang and Marques-Silva, 2023; Audemard et al., 2023d; Ji and Darwiche, 2023; Carbonnel et al., 2023; Wu et al., 2023; Izza et al., 2024b; Bassan et al., 2024; Izza et al., 2024c; Amir et al., 2024; Izza et al., 2024a; Létoffé et al., 2025; Marzouk et al., 2025; Bounia, 2025; Marques-Silva et al.,

2025; Izza et al., 2025; Barceló et al., 2025), among others. Surveys of some of this past work include (Marques-Silva and Ignatiev, 2022; Marques-Silva, 2022; Darwiche, 2023; Marques-Silva, 2023, 2024).

2.5 Additional Concepts

Complexity classes. The paper adopts complexity classes that are standard when studying the computational complexity of decision problems (Arora and Barak, 2009). Concretely, the paper considers the complexity classes P, NP, coNP and Σ_2^p. The class Σ_2^p corresponds to the class NP$^{\text{NP}}$ i.e. the class NP augmented with an oracle for class NP.

Non-determinism. Let $\mathbb{S}(P, I)$ represent the set of solutions for some computational problem P on input I. For example, in the case of the QM method, if P represents the QM method, and I denotes some boolean function, then any minimum-cost DNF would be included in $\mathbb{S}(P)$. An algorithm A for problem P on input I implements some form of non-determinism if multiple executions of A can produce different elements of $\mathbb{S}(P)$. If any execution of algorithm A for problem P on input I always computes the same solution $S \in \mathbb{S}(P, I)$, then A is deterministic. Non-determinism may represent an one or more of an algorithm's design decisions, e.g. the use of randomization, or it may manifest itself by the specificities of the programming language used for implementing the algorithm, e.g. the implementation of sorting not being stable, or the use of hashes returning elements in some arbitrary order. Most algorithms for solving combinatorial problems implement some sort of non-determinism (Gomes et al., 1998, 2000; Biere et al., 2021). In addition, the problem of non-determinism in sequential programs is the subject of continued research (Mudduluru et al., 2021; Miao et al., 2025).

3 Assessment of the MBDSR Approach

3.1 MBDSR's Worst-Case Exponential Time & Space

This section briefly reviews the recently proposed algorithm for deciding predictive equivalence (McTavish et al., 2025b,a), which builds on the well-known method of Quine-McCluskey (Quine, 1952, 1955; McCluskey, 1956). This algorithm solves the Σ_2^p-hard problem of boolean formula minimization (Umans, 1998), and exhibits well-known worst-case exponential running time and space. This section also proves that, even in the case of DTs, there exist cases of DTs for which the exponential worst-case running time and space is exercised.

The main goal of (McTavish et al., 2025b,a) is to develop a representation $\mathbb{M}$, which corresponds to $\mathcal{T}_{\text{DNF}}$ in (McTavish et al., 2025b,a), and which ensures the properties of faithfulness, completeness, succinctness and enables deciding predictive equivalence. Another goal is that ensuring these properties is computationally efficient in the size of the proposed representation. The property of faithfulness signifies that if a prediction is $c \in \mathbb{K}$ for the representation $\mathbb{M}$ then c is the correct prediction for the DT $\mathcal{T}$. Clearly, the original DT respects this property. Completeness means that in the presence of missing data, if all possible assignments to the features with missing values results in the same prediction $c \in \mathbb{K}$, then the prediction should be c. Succinctness captures the ability to report abductive explanations (Marques-Silva, 2022; Marques-Silva and Ignatiev, 2022) in the case of DTs (Izza et al., 2020, 2022; McTavish et al., 2025b,a). Polynomial-time algorithms for computing abductive explanations are described in earlier work (Izza et al., 2020, 2022). Finally,

deciding predictive equivalence signifies an algorithm for declaring that two trees represent the same classification function if and only if that is indeed the case.

BCF generation step. McTavish et al. (McTavish et al., 2025b,a) propose the enumeration of the DT terms for each class. For the DT in Figure 1, the terms for prediction 1 are x_1 and $\neg x_1 \wedge x_2$. Afterwards, each pair of terms is iteratively selected for a consensus operation (Brown, 1990), i.e. one term contains an atom, the other its negation, and the resulting term omits the atom. If a non-tautologous new term is obtained, then it is added to the set of terms. After all consensus operations are exhausted, i.e. no more non-tautologous consensus operations can be performed, the operation of *absorption* (Brown, 1990) is applied to remove terms that contain a superset of the literals of some other term.

Example 2 *For the running example, the consensus between the two terms yields x_2. Afterwards, no more new terms can be obtained by applying the consensus operation. Afterwards, absorption is applied, and so $\neg x_1 \wedge x_2$ is removed because of x_2. The resulting set of terms is:* $\mathsf{BCF}_1(\mathcal{T}_1) = \{x_1, x_2\}$. *Similarly, we obtain* $\mathsf{BCF}_0(\mathcal{T}_1) = \{\neg x_1 \wedge \neg x_2\}$.

In the general case, the number of prime implicants of a boolean formula is exponentially large in the size of the starting formula (Chandra and Markowsky, 1978; Brown, 1990).

Set covering step. Given the obtained set of terms (i.e. the prime implicants for the predicted class), the QM method then finds a cardinality minimum set of prime implicants that *covers* (i.e. entails) each of the original terms. For the running example, we keep the set $\{x_1, x_2\}$. Set covering is a well-known NP-hard problem (Karp, 1972). As a result, all known exact algorithms are worst-case exponential in the problem formulation size. A number of practical optimizations are usually implemented in practice (Coudert, 1994), and some are applied to the covering step of the implementation of the QM method proposed in the work of McTavish et al. (McTavish et al., 2025b,a).

For the running example $\mathcal{T}_1$, the BCFs cannot be further simplified, and so $\mathcal{T}_{1,\mathsf{DNF},1} = \mathsf{BCF}_1(\mathcal{T}_1) = x_1 \vee x_2$ and $\mathcal{T}_{1,\mathsf{DNF},0} = \mathsf{BCF}_0(\mathcal{T}_1) = \neg x_1 \wedge \neg x_2$.

Worst-case running time & space for DTs. Given some DT $\mathcal{T}$, one key question is the worst-case size of BCF_c for any $c \in \mathbb{K}$. We show that there exist DTs for which BCF_c is worst-case exponential, confirming worst-case exponential running time and space in the size of the original DT for the QM method.

It is known that the number of AXps for a concrete instance $(\mathbf{v}, c)$ for a decision tree is exponential in the worst-case (Audemard et al., 2022b, Proposition 1). This section proposes a different construction, adapted to the notation used in this paper. Figure 2 shows a motivating example for the generic construction proposed below. All features are binary, and the DT predicts either class 0 or class 1. The instance is $((1, 1, 1, 1, 1, 1, 1), 1)$. The prediction can be changed by changing the value of x_7, or the values of both x_1 and x_2, x_3 and x_4, or x_5 and x_6. Thus, to ensure that the prediction remains unchanged, we must fix x_7, and then fix one of x_1 and x_2, one of x_3 or x_4, and one of x_5 or x_6. Thus, the set of AXps for the given instance is shown in Figure 3. There are $8 = 2^3$ AXps, resulting from the two options in each of the three cases: x_1 and x_2, x_3 and x_4, and also
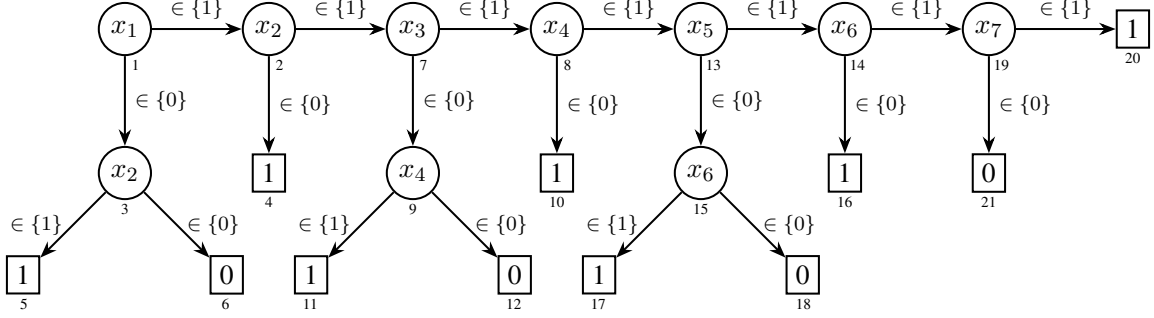
Figure 2: Motivating example. The instance is $((1, 1, 1, 1, 1, 1, 1), 1)$.

$$
\begin{aligned}
\{ \ & \{(x_1, 1), (x_3, 1), (x_5, 1), (x_7, 1)\}, \quad \{(x_1, 1), (x_3, 1), (x_6, 1), (x_7, 1)\}, \\
& \{(x_1, 1), (x_4, 1), (x_5, 1), (x_7, 1)\}, \quad \{(x_1, 1), (x_4, 1), (x_6, 1), (x_7, 1)\}, \\
& \{(x_2, 1), (x_3, 1), (x_5, 1), (x_7, 1)\}, \quad \{(x_2, 1), (x_3, 1), (x_6, 1), (x_7, 1)\}, \\
& \{(x_2, 1), (x_4, 1), (x_5, 1), (x_7, 1)\}, \quad \{(x_2, 1), (x_4, 1), (x_6, 1), (x_7, 1)\} \ \}
\end{aligned}
$$

Figure 3: AXps for Figure 2 and selected instance.

x_5 and x_6.[7] Finally, we note that the example of Figure 2 illustrates a more general construction, which is discussed below.

We now prove the following result.

Theorem 1 *There exist DTs $\mathcal{T}$ for which $\mathsf{BCF}_c(\mathcal{T})$ is exponential in the size of $\mathcal{T}$ for some class c.*

Proof (Sketch) We construct a DT with $2r+1$ features and $6r+3$ nodes, for some integer r. For this DT, a lower bound on the number of AXps is shown to be 2^r. Concretely, we select an instance, and prove that 2^r is a lower bound of the number of AXps, and so on the number of prime implicants in the BCF.

The DT is constructed with two gadgets, shown in Figure 4. The main gadget is replicated r times, as illustrated with the example in Figure 2. In addition, the last gadget is linked to the last of the r replicas of the main gadget. Clearly, the DT size is polynomial on the number of features. The chosen instance is $((1, \ldots, 1), 1)$. The rationale for the number of AXps is the same as the one used in the analysis of Figure 2. There are $r + 1$ ways to change the prediction of class 1. One way involves changing the value of feature x_{2r+1}. Each of the other ways involves a pair of variables x_{2k-1} and x_{2k}, $k = 1, \ldots, r$. Thus, in order to ensure that the prediction cannot be changed, one must choose from 2^r possible ways. Each such way is subset-minimal, and so it represents an AXp. Thus, $\mathsf{BCF}_1(\mathcal{T})$ has worst-case exponential size. ∎

---

7. For simplicity, we opt not to reason in terms of contrastive explanations (Miller, 2019; Ignatiev et al., 2020; Cooper and Marques-Silva, 2021, 2023), which would be well-defined in the case of a concrete instance, as is the case in this example. By using the minimal hitting-set duality between abductive and contrastive explanations (Ignatiev et al., 2020), it would also be possible to justify the existence of exponentially many AXps.
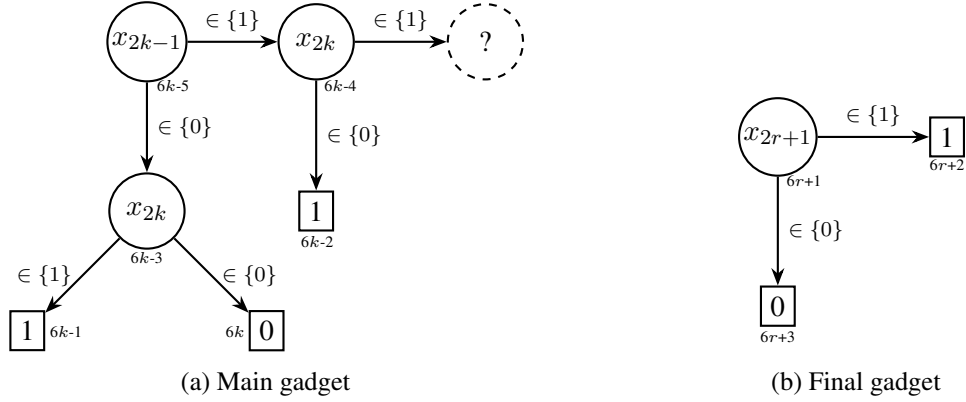
(a) Main gadget

(b) Final gadget

Figure 4: Gadgets used in the proof of Theorem 1. The main gadget is repeated r times, i.e. , $k = 1, \ldots, r$. The final gadget is connected at the end of the r main gadgets.

3.2 Unsoundness of MBDSR's Predictive Equivalence Algorithm

As stated in Section 1, the MBDSR algorithm for predictive equivalence proposed in (McTavish et al., 2025a,b) is incorrect. If two DTs are not predictive equivalent, then the MBDSR algorithm will correctly report non-equivalence. However, if two DTs are predictive equivalent, then the MBDSR algorithm is not guaranteed to report equivalence. The reason for why McTavish et al.'s (McTavish et al., 2025b,a) algorithm is incorrect is fairly simple.

The Blake canonical form is (as the name indicates) *canonical* (Blake, 1937; Brown, 1990). Canonicity signifies that there exists a single unique representation for some boolean function (Bryant, 1986; Wegener, 2000; Crama and Hammer, 2011). However, if a representation is not canonical, then there may exist *multiple distinct* representations for a given boolean function. The key flaw of McTavish et al.'s algorithm is that it computes a minimum-cost (e.g. number of literals or terms) representation of some boolean function, and this is *not* a canonical representation.[8] Indeed, it is well-known that a minimum-cost equivalent representation of the BCF is *not* canonical. Therefore, different minimum-cost representations of some BCF may be reported for the same function, hence disproving McTavish et al.'s claims about correctness of their algorithm. This section provides additional detail demonstrating the above arguments. It should be evident that a trivial fix to the algorithm of McTavish et al. would be to just use the BCF. However, that would defeat the purpose of attaining a *compact* representation for answering other queries of interest, e.g. prediction given missing data.

Example 3 analyzes an a boolean function with two distinct minimum cost DNFs. If implementations of QM use any form of non-determinism, i.e. they may not not always report the same minimum-cost DNF, then the use of QM for predictive equivalence, as proposed in the MBDSR approach, can declare predictive equivalent DTs not to be equivalent.

Example 3 *We consider an example boolean function $f : \{0, 1\}^4 \rightarrow \{0, 1\}$, with $f(x_4, x_3, x_2, x_1) = \sum_{m_i}, i = \{0, 2, 7, 8, 10, 14, 15\}$, where each m_i is a minterm. Figure 5 shows two Karnaugh maps*

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8. Evidently, the concept of canonicity is fundamental for ensuring uniqueness of representations (Bryant, 1986; Wegener, 2000; Crama and Hammer, 2011).

(a) Covering minterms with $\mathbb{I}_1$.

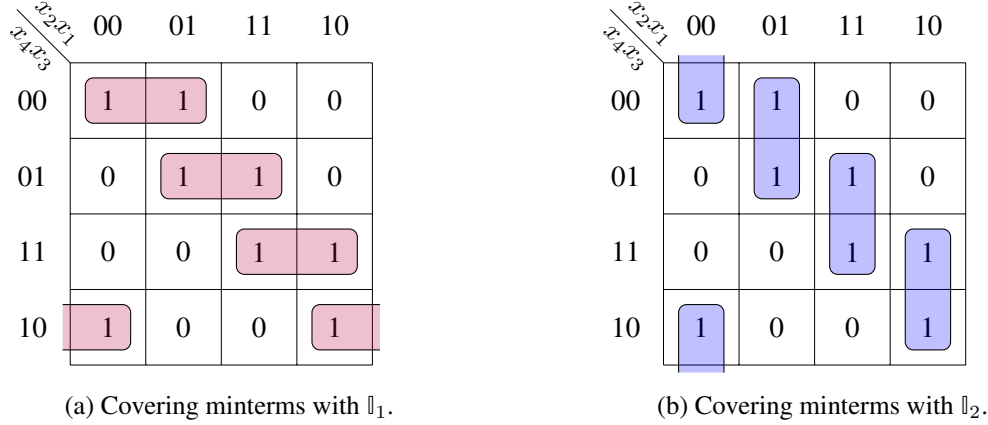(b) Covering minterms with $\mathbb{I}_2$.

Figure 5: Karnaugh maps for boolean function with two distinct minimum-cost DNFs.

for f.[9] By inspection of either map, it is plain that the set of prime implicants is defined by $\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2$, with

$$\mathbb{I}_1 = \{\bar{x}_4 \bar{x}_3 \bar{x}_2, \bar{x}_4 x_3 x_1, x_4 x_3 x_2, x_4 \bar{x}_3 \bar{x}_1\}$$
$$\mathbb{I}_2 = \{\bar{x}_3 \bar{x}_2 \bar{x}_1, \bar{x}_4 \bar{x}_2 x_1, x_3 x_2 x_1, x_4 x_2 \bar{x}_1\}$$

Furthermore, again by inspection of the Karnaugh maps, we can conclude that both $\mathbb{I}_1$ and $\mathbb{I}_2$ are minimum-cost DNF representations of f, as the two maps confirm. The set of prime implicants $\mathbb{I}_1$ covers all the minterms (in red), and the set of prime implicants $\mathbb{I}_2$ also covers all the minterms (in blue), i.e. both are minimum-cost DNFs (for some class). Either $\mathbb{I}_1$ or $\mathbb{I}_2$ could be reported by an implementation of the QM method. This holds true as long as the implementation of the QM method implements any form of non-determinism.

The previews arguments justify the following result.

Theorem 2 *Regarding the MBDSR algorithm for deciding predictive equivalence:*
  1. *MBDSR is incorrect for implementations of the QM method that exhibit non-determinism.*
  2. *MBDSR is correct for implementations of the QM method that do not exhibit non-determinism.*

As stated earlier, algorithms for solving combinatorial problems often implement some form of non-determinism, either deliberate as part of the algorithm itself or motivated by the programming language used (e.g. recent references on the topic of non-determinism in sequential programs include (Mudduluru et al., 2021; Miao et al., 2025)). Furthermore, it should also be underscored that the presentation in McTavish et al.'s work (McTavish et al., 2025b,a) does not address any of the issues resulting from the QM method producing a representation that is not canonical.

4 Polynomial-Time Algorithms

This section shows that the computational overhead of the MBDSR approach can be avoided. Concretely, this signifies that the computation of $\mathsf{BCF}\mathcal{T}$ and $\mathcal{T}_{\mathsf{DNF}}$ (and so the execution of the QM

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9. Karnaugh maps are a well-known method for the simplification of boolean functions with a small number of variables (Kohavi and Jha, 2009).

Algorithm 1 Decide consistency of tree path given some partial assignment $\mathcal{A}$

    Input: $\mathcal{A}$: partial assignment; P: path; $\mathcal{T}$: DT
    Output: $\top$ if path P consistent with $\mathcal{A}$; else $\bot$

1: **function** ConsistentPath($\mathcal{A}, P, \mathcal{T}$)
2:     $\forall(i \in \mathcal{F}). (\mathsf{Lits}[i] \leftarrow \top)$
3:     **for** $(x_i, R_i, \mathsf{op}_i) \in \mathcal{A}$ **do** $\triangleright$ Identify features to check for consistency
4:         $\mathsf{Lits}[i] \leftarrow \mathsf{Lits}[i] \wedge (x_i \; \mathsf{op}_i \; R_i)$
5:     $s_b \leftarrow \mathsf{Terminal}(P; \mathcal{T})$
6:     $s_a \leftarrow \mathsf{Parent}(s_b; \mathcal{T})$
7:     **while** $s_a \neq$ none **do** $\triangleright$ Pick literals from tree path, organize by feature
8:         $i \leftarrow \mathsf{Feature}(s_a)$
9:         $\mathsf{Lits}[i] \leftarrow \mathsf{Lits}[i] \wedge \mathsf{Literal}(s_a, s_b; \mathcal{T})$
10:       $(s_a, s_b) \leftarrow (\mathsf{Parent}(s_a; \mathcal{T}), s_a)$
11:     **for** $i \in \mathcal{F}$ **do** $\triangleright$ Check consistency of the literals for each feature
12:         **if** $\neg\mathsf{Consistent}(\mathsf{Lits}[i])$ **then**
13:            **return** $\bot$ $\triangleright$ Literals for some feature not consistent
14:     **return** $\top$ $\triangleright$ Literals of every feature in are consistent

---

method) is unnecessary for the computational problems studied in McTavish et al.'s work (Mc-Tavish et al., 2025b,a). To prove this claim, we show that each algorithm proposed in (McTavish et al., 2025b,a) that uses $\mathcal{T}_{\mathsf{DNF}}$ can be replaced by an algorithm with a running time polynomial in the size of the original DT.

Basic observations. In order to handle missing data, we propose to work with partial assignments instead of points in feature space. Thus, a feature whose value is missing is represented by omitting the feature from the partial assignment.

To decide whether a partial assignment $\mathcal{A}$ is consistent with a path P, we use Algorithm 1. (It should be noted that Algorithm 1 serves as a building block for all the algorithms described in this section.) For each feature $i \in \mathcal{F}$, we compute the conjunction of the literals in $\mathcal{A}$, if any, with the literals for feature i in the path P, if any. Afterwards, we assess the consistency of the conjunction of literals of each feature i. The algorithm could be made simpler in the case of binary features, since there can occur at most one literal on each feature in any path. However, we opt to describe the general case. Clearly, the running time of the algorithm is in $\mathcal{O}(|\mathcal{F}| + |P|)$, where we assume that the evaluation of consistency runs in constant time.

4.1 Completeness & Succinctness

WAXp invariance. The polynomial-time algorithms described in this section build on the following result, stating that WAXps are independent of the tree structure:

Theorem 3 *Let $\mathcal{T}$ be some DT. For any DT $\mathcal{T}'$ that is predictive equivalent to $\mathcal{T}$, $\mathcal{A} \in \mathbb{A}$ is a WAXp of $\mathcal{T}'$ iff $\mathcal{A}$ is a WAXp of $\mathcal{T}$.*

The proof idea is that WAXps of some class c are implicants of the predicate deciding whether the DT predicts c. Thus, if the two DTs compute the same function, then the set of implicants will

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Algorithm 2 Decide whether $\mathcal{A}$ is a WAXp for some class

    Input: $\mathcal{A}$: partial assignment; $\mathcal{T}$: DT
    Output: $\top$ if $\mathcal{A}$ is WAXp for some class c; else $\bot$

1: **function** IsSomeWAXp($\mathcal{A}, \mathcal{T}$)
2:     **for** $s \in$ Terminal($\mathcal{T}$) **do** $\triangleright$ Analyze all terminal nodes
3:         $c \leftarrow$ Class($s; \mathcal{T}$)
4:         **if** IsWAXp($\mathcal{A}, c, \mathcal{T}$) **then** $\triangleright$ Check if $\mathcal{A}$ is WAXp for c
5:             **return** c
6:     **return** none

---

be the same. (Observe that the set of implicants is not the BCF of a formula, but represents instead the Blake's *complete* canonical form (Blake, 1937; Brown, 1990). However, it is also a canonical representation.)

Proof (Sketch) The WAXps for each class c are implicants for the predicate π_c. Since implicants are a canonical representation (Blake, 1937; Brown, 1990; Darwiche and Marquis, 2002), then two predictive equivalent DTs will have the same sets of implicants for each class. Hence, a partial assignment is an implicant for π_c if and only if it is an implicant for π'_c and vice-versa. And also, implicants of π_c are WAXps of the classifier for class c. ∎

Given the above, a partial assignment $\mathcal{A}$ is a WAXp of DT $\mathcal{T}$ if and only if it is also a WAXp of any DT $\mathcal{T}'$ that is predictive equivalent to $\mathcal{T}$. Furthermore, prediction in the presence of missing data is tightly related with a partial assignment representing a WAXp.

Theorem 4 *A partial assignment $\mathcal{A}$ is a WAXp for class c for DT $\mathcal{T}$ iff for each $\mathbf{x} \in$ dom($\mathcal{A}$), $\kappa(\mathbf{x}) = c$.*

Proof This result follows from the definition of WAXp. ∎

Example 4 *For the running example DT $\mathcal{T}_1$, as argued earlier in the paper, one of the WAXps is $\{(x_1, 0), (x_2, 1)\}$. By inspection it is clear that the same partial assignment is also a WAXp for $\mathcal{T}_2$ for class 1, but not for $\mathcal{T}_3$.*

Given the above, an algorithm that decides whether a partial assignment is a WAXp suffices to ensure the property of completeness.
Deciding weak abductive explanations. Algorithms 2 and 3 summarize the main steps for deciding whether a partial assignment represents a WAXp for some class $c \in \mathbb{K}$.

Algorithm 2 checks whether a partial assignment is a WAXp for some class. Moreover, Algorithm 3 evaluates whether a partial assignment if a WAXp for a chosen class $c \in \mathbb{K}$. The rationale is that the paths with a prediction other than c are analyzed for consistency with the partial assignment $\mathcal{A}$. If there is consistency, then the partial assignment is not sufficient for the prediction of c; otherwise it is. Clearly, the algorithms run in polynomial time in the size of a DT.

Example 5 *For the running example DT $\mathcal{T}_2$, Algorithm 3 can be used to decide that $\{(x_1, 1)\}$ is a WAXp for class 1.*

Algorithm 3 Decide whether $\mathcal{A}$ is a WAXp for class c

---

    Input: $\mathcal{A}$: partial assignment; c: class; $\mathcal{T}$: DT
    Output: $\top$ if $\mathcal{A}$ is WAXp for class c; else $\bot$

1: **function** IsWAXp($\mathcal{A}, c, \mathcal{T}$)
2:     **for** $P \in$ Paths($\mathcal{T}$) **do**   ▷ Traverse paths of DT $\mathcal{T}$
3:          $s \leftarrow$ Terminal($P; \mathcal{T}$)
4:          **if** Class($s; \mathcal{T}$) $\neq c$ **then**   ▷ If path prediction differs from goal c
5:              **if** ConsistentPath($\mathcal{A}, P, \mathcal{T}$) **then**
6:                  **return** $\bot$ ▷ $\mathcal{A}$ consistent with path predicting class other than c
7:     **return** $\top$

Algorithm 4 Find one AXp

    Input: $\mathcal{A}$: partial assignment; c: class; $\mathcal{T}$: DT
    Output: AXp $\mathcal{X}$ for class c of DT $\mathcal{T}$

1: **function** FindAXp($\mathcal{A}, c, \mathcal{T}$)   ▷ Precondition: WAXp($\mathcal{A}, c; \mathcal{T}$)
2:      $\mathcal{X} \leftarrow \mathcal{A}$
3:      $\mathcal{W} \leftarrow$ Features($\mathcal{A}$)
4:     **for** $i \in \mathcal{W}$ **do**   ▷ Invariant: WAXp($\mathcal{X}, c; \mathcal{T}$)
5:          $(x_i, R_i, \mathsf{op}_i) \leftarrow$ Literal($i, \mathcal{X}$)
6:          $\mathcal{X} \leftarrow \mathcal{X} \setminus \{(x_i, R_i, \mathsf{op}_i)\}$
7:          **if** $\neg$IsWAXp($\mathcal{X}, c, \mathcal{T}$) **then**
8:              $\mathcal{X} \leftarrow \mathcal{X} \cup \{(x_i, R_i, \mathsf{op}_i)\}$
9:     **return** $\mathcal{X}$ ▷ Final result: AXp($\mathcal{X}, c; \mathcal{T}$)

Computing abductive explanations. As noted in recent work (Izza et al., 2022; Audemard et al., 2022b; McTavish et al., 2025b,a), decision trees can exhibit path literal redundancy, and so the explanation for a DT prediction can be smaller, possibly much smaller, than the path size in a DT. McTavish et al.'s propose to computing such explanations (and so deciding predictions given missing data) using $\mathcal{T}_{\mathsf{DNF}}$. However, as demonstrated in Section 3 this is worst-case exponential in time and space. In contrast, earlier work proposed a polynomial-time algorithm for explaining an instance $(\mathbf{v}, c)$ (e.g. (Izza et al., 2020, 2022; Audemard et al., 2022b)). Here, we opt instead for explaining a partial assignment that is known to be a WAXp for some class c. Algorithm 4 summarizes the computation of one AXp, which requires that the starting partial assignment is an WAXp for class c.

The algorithm's invariant is that the working partial assignment $\mathcal{X}$ is a WAXp, from which features are removed which the invariant is preserved. Due to monotonicity of the definition of WAXp, the partial assignment that is returned represents an AXp. Clearly, the algorithm ensures the property of succinctness of explanations, and runs in polynomial time in the size of the DT. Algorithm 4 mimics the algorithm proposed in earlier work (Izza et al., 2022), but adapted to working with partial assignments instead of a concrete instance.

Algorithm 5 Decide predictive equivalence

Input: $\mathcal{T}_1$: DT 1; $\mathcal{T}_2$: DT 2
Output: $\top$ if DTs are predictive equivalent; else $\bot$

1: **function** PredictivelyEquivalent($\mathcal{T}_1$, $\mathcal{T}_2$)
2:     **for** $P_1 \in$ Paths($\mathcal{T}_1$) **do**   ▷ Analyze the paths of $\mathcal{T}_1$
3:         $s_1 \leftarrow$ Terminal(P_1; $\mathcal{T}_1$)
4:         $c_1 \leftarrow$ Class(s_1; $\mathcal{T}_1$)
5:         $\mathcal{A}_1 \leftarrow$ Literals(P_1; $\mathcal{T}_1$)   ▷ Create partial assignment
6:         **for** $P_2 \in$ Paths($\mathcal{T}_2$) **do**   ▷ Analyzee the paths of $\mathcal{T}_2$
7:             $s_2 \leftarrow$ Terminal(P_2; $\mathcal{T}_2$)
8:             $c_2 \leftarrow$ Class(s_2; $\mathcal{T}_2$)
9:             **if** $c_1 \neq c_2$ **then**   ▷ Compare the paths if predictions differ
10:                **if** ConsistentPath($\mathcal{A}_1$, P_2, $\mathcal{T}$) **then**
11:                    **return** $\bot$ ▷ Not PE if partial assignment (of P_1) consistent with path P_2
12:     **return** $\top$ ▷ Unable to prove not PE; thus it is PE

Example 6 *Algorithm 4 can be used to prove that $\{(x_1, 1)\}$ is an AXp for either $\mathcal{T}_1$ or $\mathcal{T}_2$. This is not surprising, because the two DTs are predictive equivalent. Similarly, starting from $\{(x_1, 0), (x_2, 1)\}$, Algorithm 4 will yield $\{(x_2, 1)\}$ as the AXp, independently of which DT $\mathcal{T}_1$ or $\mathcal{T}_2$ is used.*

Although not studied in McTavish et al.'s work, one could also compute contrastive explanations for DTs in polynomial time (Huang et al., 2021; Izza et al., 2022). For example, the algorithms detailed in this section could be used with minor changes.

4.2 Predictive Equivalence

Deciding predictive equivalence. The main goal of the work of McTavish et al. (McTavish et al., 2025b,a) is to be able to deciding predictive equivalence efficiently. Accordingly, the algorithm proposed in earlier work (McTavish et al., 2025b,a) runs in polynomial-time in the size of the $\mathcal{T}_{\mathsf{DNF}}$ representation of the DT.

Theorem 5 *Two DTs, $\mathcal{T}_1$ and $\mathcal{T}_2$, are not predictively equivalent if there exists a point in feature space for which $\mathcal{T}_1$ and $\mathcal{T}_2$ predict different classes.*

Proof Consequence of predictive equivalence. ∎

From Theorem 5, it is immediate to conclude that deciding predictive equivalence is in coNP, and so can be decided with an NP oracle. (This also means that using the QM method for deciding predictive equivalence is unnecessary.) However, we show that deciding predictive equivalence is actually in P. The rationale of the algorithm is as follows. The algorithm visits the paths of one of the DTs; we will pick $\mathcal{T}_1$. For each path P_1 with prediction c in $\mathcal{T}_1$, we create a partial assignment $\mathcal{A}_1$ using the literals from the path P_1. Afterwards, we analyze the paths of $\mathcal{T}_2$ with a prediction *other* than c. If $\mathcal{A}_1$ is consistent with any such path, then Theorem 5 applies, and so the two DTs are not predictively equivalent. Algorithm 5 summarizes the main steps of the

		$\mathcal{T}_2$	
	P_1	P_2	P_3
P_1	—	$\bot$	$\bot$
$\mathcal{T}_1$　P_2	$\bot$	—	—
P_3	$\bot$	—	—

		$\mathcal{T}_3$	
	P_1	P_2	P_3
P_1	$\top$	—	$\bot$
$\mathcal{T}_1$　P_2	—	$\top$	—
P_3	—	$\bot$	—

(a) $\mathsf{dom}(\mathsf{Literals}(P_i)) \cap \mathsf{dom}(\mathsf{Literals}(P_j)) \neq \emptyset$? (b) $\mathsf{dom}(\mathsf{Literals}(P_i)) \cap \mathsf{dom}(\mathsf{Literals}(P_j)) \neq \emptyset$?

Figure 6: Execution of Algorithm 5 on running example. $\mathcal{T}_1$ and $\mathcal{T}_2$ are predictive equivalent, whereas $\mathcal{T}_1$ and $\mathcal{T}_3$ are not.

polynomial-time procedure to decide predictive equivalence. It is plain that the running time is polynomial on the product of the two DTs. By inspection, a simple upper bound on the running time is $\mathcal{O}(|\mathcal{T}_1| \times |\mathcal{T}_2| \times \min(|\mathcal{T}_1|, |\mathcal{T}_2|))$.[10]

Example 7 *Figure 6 summarizes the execution of Algorithm 5 on the running example DTs. As can be concluded, the algorithm reports that $\mathcal{T}_1$ and $\mathcal{T}_2$ are predictive equivalent, since no pair of paths (one from $\mathcal{T}_1$ and one from $\mathcal{T}_2$) with different predictions is consistent. In contrast, $\mathcal{T}_1$ and $\mathcal{T}_3$ are not predictive equivalent, since there exists at least one pair of paths (one from $\mathcal{T}_1$ and one from $\mathcal{T}_2$) with different predictions that is consistent. Figure 6(b) shows the two examples (P_1, P_1) and (P_2, P_2), with the first path taken from one of the trees (e.g. $\mathcal{T}_1$), and the second path taken from the other tree (e.g. resp. $\mathcal{T}_3$).*

Finally, it is important to observe that Algorithm 5 can be parallelized. Given enough processors, both loops can be parallelized, and so the total run time, if enough processors are available, can be reduced to $\mathcal{O}(\max(|\mathcal{T}_1|, |\mathcal{T}_2|))$. This complexity assumes that parallelization can be achieved in two phases, one for the first loop, and then another phase for the $\mathcal{O}(|\mathcal{T}_1|)$ executions of the second loop. For the second phase, parallelization of the second loop ensures that checking consistency between each pair of paths is done in parallel requiring at most $\mathcal{O}(\max(|\mathcal{T}_1|, |\mathcal{T}_2|))$ time.

Discussion. One might argue that the algorithms proposed in this section have complexity higher than those proposed in earlier work (McTavish et al., 2025b,a). However, the algorithms described in this section run in polynomial-time in the size of the DT, whereas the algorithms proposed in (McTavish et al., 2025b,a) require the prior computation of both $\mathsf{BCF}(\mathcal{T})$ and $\mathcal{T}_{\mathsf{DNF}}$, and $\mathsf{BCF}(\mathcal{T})$ was shown to be worst-case exponential in the size of the original DT. Besides larger running times, the algorithms proposed elsewhere (McTavish et al., 2025b,a) face a hard practical obstacle when $\mathsf{BCF}(\mathcal{T})$ is exponentially large, since the representation size can exceed the available computing resources.

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10. To attain this run-time complexity, the execution of Algorithm 5 should be changed to start by comparing the sizes of the two DTs, with the external loop being run on the largest, and the internal loop being run on the smallest.

Table 1: Results for different values of r, $r \leq 9$. sympy's QM stopped for $r = 7$ after 150000s. Running times in seconds. The number of DT nodes is: $6 \times r + 3$ and the number of features is $2 \times r + 1$.

r	sympy	$|\mathrm{BCF}_0(\mathcal{T})|$	$|\mathrm{BCF}_1(\mathcal{T})|$	BCF time
3	0.13	4	22	0.01
4	0.57	5	46	0.07
5	39.60	6	94	0.84
6	2789.45	7	190	11.28
7	>150000.00	8	382	161.25
8	– –	9	766	2264.62
9	– –	10	1534	64458.55

Table 2: Running times for DTs much larger than those in Table 1, with $r \in \{200, 500, 1000\}$.

r	# DT nodes	# features	$|\mathrm{BCF}_1(\mathcal{T})|$	One AXp	isWAXp?	PE?
200	1203	401	2^{200}	1.71s	0.005s	3.7s
500	3003	1001	2^{500}	26.98s	0.032s	57.1s
1000	6003	2001	2^{1000}	224.62s	0.126s	469.0s

5 Experimental Evidence

The work of McTavish et al. (McTavish et al., 2025b,a) offers comprehensive evidence regarding the importance of eliminating predictive equivalent DTs from the Rashomon set. Also, the algorithms proposed in the previous section exhibit negligible running times, even for DTs of fairly large size. (Our results concur with similar results were obtained in earlier work when computing instance-specific explanations (Izza et al., 2022).) As a result, the focus of our experiments is to demonstrate the practical effect of exercising the worst-case running time of the QM method. Since the proof Theorem 1 only considers a single instance, the experiments target the complete compilation of BCF for different values of r.

Experimental setup. The algorithm described in (McTavish et al., 2025a) uses sympy's implementation of QM (Meurer et al., 2017).[11] We also implemented a prototype in Python, that mimics a standard implementation of the prime implicant generation step of the QM method, i.e. BCF generation. The experiments were run on a Macbook with an M3 Pro processor. For the DT used in the proof of Theorem 1, different values of r were considered. The parallelization of Algorithm 5, summarized in Section 4.2, was not considered in a large-scale cluster; this would be simple to do, for example with Apache Spark.[12]

Results. Table 1 summarizes the results for values of r ranging from 3 to 9, i.e. DTs with up to $6 \times r + 3 = 57$ nodes (for $r = 9$). As can be observed, for class 1 (resp. class 0), the size of $|\mathrm{BCF}_1(\mathcal{T})|$ (resp. $|\mathrm{BCF}_0(\mathcal{T})|$) grows exponentially (resp. linearly) with the value of r. sympy's QM algorithm shows large running times, and it was stopped for $r = 7$, i.e. DT with 45 nodes. For the independent generation of BCF, the running times also grow exponentially (as expected).

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11. https://www.sympy.org/.
12. https://spark.apache.org/.

For $r = 9$, the running times for computing the BCF's are already massive. The inefficiency observed is attributed to the fact that Python was used and, more importantly and as noted earlier in the paper, the computational overhead of generating the BCF. This overhead is significant, and worsens for trees with more AXps. For the algorithms proposed in this paper, we consider the cases $r = 200, 500, 1000$, as shown in 2. For the case $r = 200$, this represents a DT with $6 \times r + 3 = 1203$ nodes, as shown, and so a DT that is almost 20 times larger than the ones considered for computing the BCF and running QM). In this case, we have $2 \times r + 1 = 401$ features, which makes the use of the QM method impractical. For this DT and prediction 1, and from the proof of Theorem 1, we have a lower bound of 2^{200} prime implicants for π_1. The computation of one AXp for a partial assignment consistent with the longest tree path takes 1.71s. Deciding if the same partial assignment is a WAXp takes 5ms. Moreover, deciding predictive equivalence of a tree with the same size, but with one terminal node with a changed prediction, takes 3.7s. These running times are negligible when compared with the first (or both) phase(s) of the QM method on a DT with 57 nodes (for $r = 9$), i.e. a DT that is less than a tenth of the case $r = 200$. The cases $r = 500$ and $r = 1000$ represent much more complex DTs, with far more nodes and features. Concretely, for $r = 500$ (resp. $r = 1000$), a lower bound on the number of prime implicants is 2^{500} (resp. 2^{1000}). The running times for computing one AXp and for deciding predictive equivalence increase noticeably. However, as noted in Section 4, Algorithm 5 can be parallelized, and so, on a cluster, the observed run times can be significantly reduced. Furthermore, the use of a programming language other than Python, e.g. C/C++, would enable significant reductions in the running times.

Finally, we underscore that the experimental results consider a special DT construction aiming at highlighting the performance limitations of using an implementation of the QM method. However, DTs of large size can occur in some high-risk (Rudin, 2019) application domains, where high accuracy is paramount. For example, (Ghiasi et al., 2020) reports a DT with 161 nodes and 20 real-valued features. In those cases, the use of the QM method will also not scale, even if high-performance solutions are exploited (Coudert, 1994). (We note that sympy is known to be inefficient for formulas with more than 8 boolean variables.[13])

6 Additional Results

This section relates predictive equivalence with several concepts studied in logic-based XAI.

6.1 Predictive Equivalence and AXps

Theorem 6 *Let $C_1 = (\mathcal{F}, \mathbb{F}, \mathbb{K}, \kappa_1)$ and $C_2 = (\mathcal{F}, \mathbb{F}, \mathbb{K}, \kappa_2)$ be two classifiers. Then, C_1 is predictive equivalent to C_2 if and only if for every instance $(\mathbf{v}, c)$, with $\kappa(\mathbf{v}) = c$, $\mathbb{A}(C_1, c)) = \mathbb{A}(C_2, c))$.*

Proof From Section 2.4, it is the case that $\mathbb{A}(M, c)$ represents all the prime implicants of π_c. Thus, $\mathbb{A}(M, c)$ represents the BCF for π_c. The BCF is a canonical (i.e. unique) representation for a boolean function (Blake, 1937; Quine, 1952; Brown, 1990). Hence, for predictive equivalent classifiers, and for each $c \in \mathbb{K}$, only one BCF exists. The same applies to every $c \in \mathbb{K}$. Thus, the result follows. ∎

Remark 7 *Theorem 6 also holds if AXps are replaced by weak AXps.*

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13. This is discussed for example in https://docs.sympy.org/latest/modules/logic.html.

Remark 8 *Theorem 6 can be generalized to arbitrary ML models, e.g. regression models, that compute some prediction function.*

Although revealing interesting connections between predictive equivalence and the sets of AXps/CXps, it may also seem unclear how Theorem 6 could be useful in practice.

We illustrate one example use of the connection between predictive equivalence and (W)AXps.

Theorem 9 *Suppose we have two ML models M_1 and M_2, and two partial assignments $\mathcal{A}_1$ and $\mathcal{A}_2$, representing (W)AXps for classes $c_1 \neq c_2$, respectively for M_1 and M_2. It is the case that, if* $\mathsf{Consistent}(\mathcal{A}_1, \mathcal{A}_2)$, *then M_1 and M_2 are not predictive equivalent.*

6.2 Predictive Equivalence and Corrected Measures of Importance

It is well-known that SHAP scores approximate a theoretical definition of Shapley values for XAI (Strumbelj and Kononenko, 2010, 2014). Thus, the quality of the approximation provided by SHAP scores depends on a number of factors. However, recent work proved that the proposed definition of Shapley values for XAI (Strumbelj and Kononenko, 2010, 2014; Lundberg and Lee, 2017) can mislead human decision makers (Marques-Silva and Huang, 2024; Huang and Marques-Silva, 2024), by assigning no importance to features that are critical for a given prediction, and by assigning some importance to features that bear no influence in that prediction. Since it is unclear when Shapley values for XAI (and so SHAP scores) can mislead, the use of the existing definition of Shapley values for XAI should be discouraged.

Recent works proposed different solutions to this flaw (Biradar et al., 2024; Yu et al., 2024; Létoffé et al., 2025). Of these, we study the one proposed by Létoffé et al. (Létoffé et al., 2025), since it can also be based on Shapley values.

The key issue with past definitions of Shapley values for XAI is the characteristic function used, which is based on the expected value of the classifier. The proposed solution for this flaw is to adopt a different characteristic function that is defined as follows, where $\mathcal{S} \subseteq \mathcal{F}$:

$$v(\mathcal{S}; (\mathbf{v}, c), M) = \begin{cases} 1 & \text{if } \mathsf{WAXp}(\mathcal{S}; (\mathbf{v}, c), M) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(As shown, the definition of v is parameterized on the given instance $(\mathbf{v}, c)$ and the ML model M.) Given this modified characteristic function, the issues reported with Shapley values for XAI are eliminated (Létoffé et al., 2025). We refer to the definition of Shapley values using the characteristic function above as *corrected SHAP scores*.

We can now relate predictive equivalence with corrected SHAP scores.

Theorem 10 *Let C_1, C_2 be two classifiers defined on the same features, with the same feature space, and mapping to the same set of classes. Then, if C_1 and C_2 are predictive equivalent then, for any instance, the corrected SHAP scores for C_1 equal the corrected SHAP scores for C_2.*

Proof If C_1, C_2 are predictive equivalence then, by Theorem 6, C_1, C_2 have equal sets of (W)AXps. Given the definition of characteristic function Equation (7), then the corrected SHAP scores will be the same. ∎

As argued earlier in this section, in practice appears harder to compute the Shapley values for *every* instance than simply deciding predictive equivalence. However, Theorem 10 shows the connections that logic-based XAI reveals. Nevertheless, Theorem 10 is one example of several measures of importance that can be considered:

Remark 11 *Following the framework proposed in (Létoffé et al., 2024), Theorem 10 can be restated for several power indices studied in the context of a priori voting power (Felsenthal and Machover, 1998), and generalized to the case of feature attribution in XAI by Létoffé et al. (Létoffé et al., 2024).*

7 Conclusions

Recent work by McTavish et al. (McTavish et al., 2025b,a) demonstrated the importance of eliminating predictive equivalent decision trees from the Rashomon set. Indeed, this work showed that a significant percentage of decision trees in the Rashomon sets for different classification problems are predictively equivalent. Specifically, the work of McTavish et al. proposed an approach (MBDSR) for solving a number of computational problems of interest for working with Rashomon sets of DTs, including deciding predictive equivalence of DTs. At its core, the MBDSR approach consists of using the well-known Quine-McCluskey (QM) method for computing minimimum-size DNF representations of DTs. The additional algorithms proposed by McTavish et al. run in linear time on the size of the computed minimimum-size DNF.

In this paper, we show two critical issues with the MBDSR approach proposed by McTavish et al. (McTavish et al., 2025b,a). As a first issue, we show that the MBDSR approach is unnecessarily complex in that it requires running the QM method for solving a problem that is hard for the second level of the polynomial hierarchy. In addition, we also prove that the worst-case exponential running time and space of the QM method can be exercised in the case of decision trees. In contrast, in this paper, we propose novel polynomial-time algorithms, in the size of the original DT, for *all* of the computational problems studied in the work of McTavish et al. (McTavish et al., 2025b,a). The experiments confirm negligible running times in practice for all the algorithms proposed in this paper. As a second issue, we prove that the MBDSR approach is flawed. Concretely, we demonstrate that the use of the QM method for deciding predictive equivalence can produce incorrect results. Therefore, the MBDSR is not only inefficient, but it can produce incorrect results, depending on the implementation of the QM method that is used. Namely, for implementations of the QM method that exhibit non-determinism, then incorrect results can be obtained with the MBDSR approach.

Future work will investigate improvements to the asymptotic (but polynomial) running times of the algorithms proposed in this paper, but also assessing the alternatives revealed by the results in Section 6. In addition, another topic of research would be to devise asymptotically optimal algorithms for solving the computational problems studied in Section 4. Given the massive improvements that our algorithms yield, this line of research is at present interesting mostly from a theoretical perspective. Finally, Section 6 reveals several lines of research that are also interesting from a theoretical standpoint. Future research will address these lines of research.

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