

The $N^3\text{LO}$ Twist-2 Matching of TMD Quark Transversity

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ABSTRACT: We present the first next-to-next-to-next-to-leading order ($N^3\text{LO}$) calculation of the twist-2 matching coefficients for transverse momentum dependent (TMD) quark transversity parton distribution and fragmentation functions in QCD. This matching relates the TMD quark transversity functions to their collinear counterparts in the large-transverse-momentum regime, and provides essential ingredients for precision TMD phenomenology involving transversely polarized beams. As part of our analysis, we also compute the complete set of next-to-next-to-leading order (NNLO) DGLAP splitting functions governing the QCD evolution of collinear transversity distributions. These results extend the perturbative toolkit for spin-dependent observables and establish the transversity sector on the same theoretical footing as unpolarized and helicity distributions. Our findings enable high-precision extractions of transversity PDFs and facilitate improved theoretical predictions for azimuthal asymmetries in semi-inclusive deep inelastic scattering (SIDIS), especially in light of forthcoming data from the Electron-Ion Collider (EIC).

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1 Introduction

Quark transversity distributions [1–6] represent one of the three leading-twist parton distribution functions (PDFs), alongside the unpolarized and helicity distributions. Unlike their chiral-even counterparts, transversity distributions are chiral-odd and therefore are inaccessible through inclusive deep inelastic scattering (DIS), but are only measurable in processes involving two chiral-odd objects that enter the cross section. The quark transversity PDFs carry essential information about the transverse spin structure of the nucleon, at the same time, they play a role in searches for physics Beyond-Standard-Model (BSM) through their contribution to tensor interactions [7]. In particular, precision measurements of low-energy β decays provide sensitive tests of possible BSM interactions encoded in the effective Lagrangians describing semi-leptonic transitions with dimension-six scalar and tensor operators. Their contributions interfere linearly with the Standard Model amplitude, making them directly proportional to the nucleon tensor charge g_T [8],

$$g_T = \delta^u(Q^2) - \delta^d(Q^2), \quad \delta^q(Q^2) = \int_0^1 dx [h_1^q(x, Q^2) - h_1^{\bar{q}}(x, Q^2)], \quad (1.1)$$

i.e., the isovector combination of first Mellin moments of up and down quark transversity. Tensor charges are also indispensable inputs to nucleon electric dipole moment (EDM), which provide a unique window to hadronic and semi-leptonic CP violation [9].

Over the past decades, significant progress has been made in unraveling the quark transversity, driven by increasingly refined theoretical frameworks and experimental observations [10–17]. Its extraction has been made possible either within TMD framework through Collins effect [18–25], or within collinear framework through the mechanism of di-hadron productions [26–28]. The first experimental evidence for transversity was obtained in semi-inclusive deep-inelastic scattering (SIDIS) with a transversely polarized proton target, $\ell + P^\uparrow \rightarrow \ell' + h + X$, by the HERMES Collaboration [29]. Later, both experimental analyses and theoretical studies advanced significantly, in particular through global analyses of SIDIS Collins modulation in conjunction with the chiral-odd Collins fragmentation function, as well as hadron-in-jet Collins effect and dihadron correlations [30–48]. In parallel, transversity distributions are also independently computed in Lattice simulations [49–54]. For a comprehensive review of recent progress, see Refs. [55, 56].

The large $p_T \gg \Lambda_{\text{QCD}}$ asymptotics has provided additional theoretical constraint relating TMD-based approach and the collinear framework. Significant progress has been made in recent years in developing the TMD formalism and computing perturbative ingredients at next-to-next-to-next-to-leading order (N³LO) for unpolarized and helicity TMDs [57–61]. However, the precision theory of transversely polarized TMDs has lagged behind, with matching coefficients and evolution kernels only known up to NLO (LO) [62]. The work of Gutierrez-Reyes, Scimemi, and Vladimirov [63] established the NNLO matching of the transversity TMDPDF and TMDFF. Their analysis also revealed interesting structures, such as the vanishing of the pretzelosity matching coefficients at two loops, suggesting that it does not match onto any twist-2 collinear distribution. This feature is consistent with quark-model studies indicating that the TMD pretzelosity distribution $h_{1T}^{\perp q}$ is directly

related to the higher-twist quark angular momentum [64–66]. The pioneering work of Gutierrez-Reyes, Scimemi, and Vladimirov has significantly advanced the theoretical understanding of quark transversity. Nevertheless, we find minor discrepancies compared to their results, as discussed in detail in the main body of the manuscript.

In this work, we extend the perturbative understanding of transversely polarized TMDs to the next level of precision. We present the first computation of the N³LO twist-2 matching coefficients for TMD quark transversity distributions and fragmentation functions, using an exponential regulator [67]. As an outcome, we derive the full set of NNLO DGLAP splitting functions in analytic form for quark transversity. Together, these developments provide a solid foundation for global analyses of spin-dependent data and for interpreting precision measurements at the forthcoming EIC.

2 The transversely polarized quark TMD PDFs and FFs

2.1 Operator definitions for TMD quark transversity PDFs and FFs

The transversely polarized TMD distribution can be defined in terms of SCET [68–72] collinear fields

$$\mathcal{B}_{q/N}^\mu(x, b_\perp, \vec{s}_T) = \int \frac{db^-}{2\pi} e^{-ixb^- P^+} \langle N(P), \vec{s}_T | \bar{\chi}_n(0, b^-, b_\perp) \frac{\not{n} \gamma_\perp^\mu}{2} \gamma_5 \chi_n(0) | N(P), \vec{s}_T \rangle, \quad (2.1)$$

where γ_\perp^μ is given by $\gamma_\perp^\mu \equiv g_\perp^{\mu\nu} \gamma_\nu$, and $N(P)$ is a hadron state with momentum $P^\mu = (\bar{n} \cdot P) n^\mu / 2 = P^+ n^\mu / 2$, with $n^\mu = (1, 0, 0, 1)$ and $\bar{n}^\mu = (1, 0, 0, -1)$, $\chi_n = W_n^\dagger \xi_n$ is the gauge invariant collinear quark field [73] in SCET, constructed from collinear quark field ξ_n and path-ordered collinear Wilson line $W_n(x) = \mathcal{P} \exp \left(ig \int_{-\infty}^0 ds \bar{n} \cdot A_n(x + \bar{n}s) \right)$. For a polarized hadron with transverse spin \vec{s}_T and mass M , the transversely polarized TMD distribution is parameterized as [74–77]

$$\mathcal{B}_{q/N}^\mu(x, b_\perp, \vec{s}_T) = \vec{s}_T^\mu h_1(x, b_\perp) + i \epsilon^{\mu\nu} b_{\perp\nu} M h_1^\perp(x, b_\perp) + \frac{M^2 b_T^2}{2} \left(\frac{g_\perp^{\mu\nu}}{2} + \frac{b_\perp^\mu b_\perp^\nu}{b_T^2} \right) \vec{s}_{T\nu} h_{1T}^\perp(x, b_\perp). \quad (2.2)$$

The physical meaning of the relevant TMDs are summarized as follows. The chiral-odd Boer-Mulders function [78] $h_1^\perp(x, b_\perp)$ describes the distribution of transversely polarized quarks in an unpolarized target, this function is studied at NLO in [79]. It is a T-odd function just like the Sivers function [80], both change sign when going from SIDIS to the Drell-Yan process [81–85]. Their existence is attributed to initial-state or final-state interactions between the active partons and the spectators. The pretzelosity distribution $h_{1T}^\perp(x, b_\perp)$ has vanishing twist-2 matching coefficients, as found in [63]. The quark may contribute a transverse angular momentum proportional to the transverse spin of the nucleon target, this contribution is described by the TMD quark transversity $h_1(x, b_\perp)$. In this work, we consider perturbative matching of $h_1(x, b_\perp)$ onto transversity PDFs

$$\vec{s}_T^\mu h_1(x) \equiv \int \frac{db^-}{2\pi} e^{-ixb^- P^+} \langle N(P), \vec{s}_T | \bar{\chi}_n(0, b^-, 0) \frac{\not{n} \gamma_\perp^\mu}{2} \gamma_5 \chi_n(0) | N(P), \vec{s}_T \rangle, \quad (2.3)$$

the definition above utilizes the consequence of rotation and parity symmetries of the operator matrix elements. The function $h_1(x)$ describes parton densities, in fact, it is asymmetries of quark transverse spin

$$\begin{aligned} h_1(x) &= \text{density of quark of spin parallel to target} \\ &\quad - \text{density of quark of spin antiparallel to target}, \end{aligned} \quad (2.4)$$

where the target is fully transversely polarized with respect to the beam axis. The OPE relation at small b_T is

$$h_1^q(x, b_\perp) = \int_x^1 \frac{dy}{y} \sum_{f=q, \bar{q}} \delta \mathcal{I}_{qf} \left(\frac{x}{y}, b_\perp \right) h_1^f(x) + \mathcal{O}(b_T^2 \Lambda_{\text{QCD}}^2), \quad (2.5)$$

where the sum over flavors is restricted to non-singlet combinations, as there is no gauge-invariant operator corresponding to a transversely polarized gluon. The matching coefficients $\delta \mathcal{I}_{qf}$ describes physics above the Λ_{QCD} scale, as a result, it can be computed from partonic TMD transversity PDFs

$$\delta h_{q/j}^{\text{bare}}(x, b_\perp) = \int \frac{db^-}{2\pi} e^{-ixb^- P^+} \langle P, j_\sigma | \bar{\chi}_n(0, b^-, b_\perp) \frac{\not{b} \gamma_\perp^\mu}{2} \gamma_5 \chi_n(0) | P, j_\rho \rangle \bar{\Gamma}_\mu^{\sigma\rho}, \quad (2.6)$$

where we introduce the dual spin projector $\bar{\Gamma}_\mu = \frac{\gamma_5 \gamma_\perp^\mu \not{P}}{2}$ to project out $\delta h_{q/j}(x, b_\perp)$, the partonic version of $h_1^q(x, b_\perp)$. By inserting a complete set of n -collinear state $\mathbb{1} = \int_{X_n} d\text{PS}_{X_n} |X_n\rangle \langle X_n|$ into the operator definition, the bare TMD PDFs can be computed from splitting amplitudes integrated over collinear phase space with a rapidity cutoff $\nu = 1/\tau$ through the exponential regulator $e^{-b_0 \tau \frac{P \cdot K}{P^+}}$ [67, 86]

$$\delta h_{qj}^{\text{bare}}(x, b_\perp) = \lim_{\tau \rightarrow 0} \int_{X_n} d\text{PS}_{X_n} e^{-iK_\perp \cdot b_\perp} e^{-b_0 \tau \frac{P \cdot K}{P^+}} \delta(K^+ - (1-x)P^+) \delta \mathbf{P}_{\mu i \leftarrow j}^{\sigma\rho} \bar{\Gamma}_\mu^{\sigma\rho}, \quad (2.7)$$

where K^μ is the total momentum of $|X_n\rangle$, and $d\text{PS}_{X_n}$ is the collinear phase space measure, and $\delta \mathbf{P}_{\mu i \leftarrow j}^{\sigma\rho}$ is the space-like spin correlator for the target

$$\delta \mathbf{P}_{\mu q \leftarrow j}^{\sigma\tau} \equiv \mathbf{S} \mathbf{p}_{X_n q_i^* \leftarrow j_\sigma}^* \Gamma_\mu^{ls} \mathbf{S} \mathbf{p}_{X_n q_s^* \leftarrow j_\rho} \equiv \langle j_\sigma | \bar{\chi}_n | X_n \rangle \frac{\not{b} \gamma_\perp^\mu}{2} \gamma_5 \langle X_n | \chi_n | j_\rho \rangle. \quad (2.8)$$

Besides, since the distribution is non-vanishing only in the non-singlet (ns) sectors, one may expect an equivalent definition of the integrand without the explicit appearance of γ_5 . Our strategy is therefore to eliminate γ_5 by deforming the operator definitions in four dimensions, as follows:

$$\frac{\not{b} \gamma_\perp^\mu}{2} \gamma_5 \dots \times \frac{\gamma_5 \gamma_\perp^\mu \not{P}}{2} \dots = -i \frac{1}{2} \varepsilon_\perp^{\mu\nu} \not{b} \gamma_\perp^\nu \dots \times i \frac{1}{2} \varepsilon_{\mu\sigma} \gamma_\perp^\sigma \not{P} \dots = \frac{\not{b} \gamma_\perp^\mu}{2} \dots \times \frac{\gamma_\perp^\mu \not{P}}{2} \dots, \quad (2.9)$$

note that it is not assumed here that various traces are topologically connected. With this method we have also verified that terms proportional to the cubic color structure d_{abc}^2 vanish identically. The TMD quark transversity FFs [87] can be obtained from crossing,

and frame-dependence must be carefully taken into account [1, 57, 86]. For a final state detected hadron N carrying momentum $P^\mu = (\bar{n} \cdot P)n^\mu/2 = P^+n^\mu/2$ and transverse spin \vec{s}_T , we define

$$\begin{aligned} \delta\mathcal{D}_{N/q}^\mu(z, b_\perp/z, \vec{s}_T) = & z^{1-2\epsilon} \not{\!\!\!\!\!\int\!\!\!\!\!\int}_{X_n} \int \frac{db^-}{2\pi} e^{iP^+b^-/z} \langle 0 | \bar{\chi}_n(0, b^-, b_\perp) | N(P), \vec{s}_T, X \rangle \\ & \times \frac{\not{\!\!\!\!\!\not{p}}_\perp \gamma_5^\mu}{2} \gamma_5 \langle N(P), \vec{s}_T, X | \chi_n(0) | 0 \rangle, \end{aligned} \quad (2.10)$$

where again we only consider contributions proportional to the detected hardron's transverse spin, i.e., the TMD transeversity FFs. Similar to Eq.(2.7), we can define the partonic TMD transversity FFs as

$$\delta f_{iq}^{\text{bare}}(z, b_\perp/z) = z^{1-2\epsilon} \lim_{\tau \rightarrow 0} \not{\!\!\!\!\!\int\!\!\!\!\!\int}_{X_n} d\text{PS}_{X_n} e^{-iK_\perp \cdot b_\perp} e^{-b_0 \tau \frac{P \cdot K}{P^+}} \delta \left(K^+ - \left(\frac{1}{z} - 1 \right) P^+ \right) \delta \mathbf{P}_{\mu i \leftarrow q}^{T, \sigma \rho} \bar{\Gamma}_{\sigma \rho}^\mu, \quad (2.11)$$

where $\delta \mathbf{P}_{\mu i \leftarrow q}^{T, \sigma \rho}$ is the square of the time-like splitting amplitude, which can be obtained from the space-like ones in Eq. (2.8) by analytical continuation. The dual spin projectors $\bar{\Gamma}$ in Eq. (2.11) are formally identical to those of the space-like ones.

2.2 Collinear mass factorization and renormalization group equations

The bare TMD helicity PDFs (or FFs) contain both ultraviolet (UV) and infrared (IR) divergences. The procedure of UV renormalization, zero-bin subtraction, and mass factorization is technically identical to that of the unpolarized case [58], and are summarized in the following collinear mass factorization formula

$$\begin{aligned} \frac{1}{Z^B} \frac{\delta h_{qj}^{\text{bare}}(x, b_\perp)}{\mathcal{S}_{0b}} &= \sum_k \delta \mathcal{I}_{qk}(x, b_\perp, \mu, \nu) \otimes \delta \phi_{kj}(x, \mu), \\ \frac{1}{Z^B} \frac{\delta f_{iq}^{\text{bare}}(z, b_\perp/z)}{\mathcal{S}_{0b}} &= \sum_k \delta d_{ik}(z, \mu) \otimes \delta \mathcal{C}_{kq}(z, b_\perp/z, \mu, \nu). \end{aligned} \quad (2.12)$$

where $\mathcal{S}_{0b}(\alpha_s)$ is the bare zero-bin soft function which is the same as the TMD soft function [88]. Z^B (see in Sec. C) are the multiplicative operator renormalization constants for TMD PDFs and FFs, and $\delta \mathcal{I}_{qj}$ ($\delta \mathcal{C}_{iq}$) are the finite coefficient functions. $\delta \phi_{ki}$ (and δd_{ik}) are the partonic transversity PDFs (and FFs), they evolve with the transversity splitting functions δP [89, 90], up to α_s^3 , they are given by

$$\begin{aligned} \delta \phi_{ij}(x, \alpha_s) = & \delta_{ij} \delta(1-x) - \frac{\alpha_s}{4\pi} \frac{\delta P_{ij}^{(0)}(x)}{\epsilon} \\ & + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[\frac{1}{2\epsilon^2} \left(\sum_k \delta P_{ik}^{(0)} \otimes \delta P_{kj}^{(0)}(x) + \beta_0 \delta P_{ij}^{(0)}(x) \right) - \frac{1}{2\epsilon} \delta P_{ij}^{(1)}(x) \right] \\ & + \left(\frac{\alpha_s}{4\pi} \right)^3 \left[\frac{-1}{6\epsilon^3} \left(\sum_{m,k} \delta P_{im}^{(0)} \otimes \delta P_{mk}^{(0)} \otimes \delta P_{kj}^{(0)}(x) + 3\beta_0 \sum_k \delta P_{ik}^{(0)} \otimes \delta P_{kj}^{(0)}(x) \right) \right. \end{aligned}$$

$$\begin{aligned}
& +2\beta_0^2\delta P_{ij}^{(0)}(x) \Big) + \frac{1}{6\epsilon^2} \left(\sum_k \delta P_{ik}^{(0)} \otimes \delta P_{kj}^{(1)}(x) + 2 \sum_k \delta P_{ik}^{(1)} \otimes \delta P_{kj}^{(0)}(x) \right. \\
& \left. + 2\beta_0\delta P_{ij}^{(1)}(x) + 2\beta_1\delta P_{ij}^{(0)}(x) \right) - \frac{1}{3\epsilon}\delta P_{ij}^{(2)}(x) \Big] + \mathcal{O}(\alpha_s^3), \tag{2.13}
\end{aligned}$$

and for the FFs

$$\begin{aligned}
\delta d_{ij}(z, \alpha_s) = & \delta_{ij}\delta(1-z) - \frac{\alpha_s}{4\pi} \frac{\delta P_{ij}^{T(0)}(z)}{\epsilon} \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{2\epsilon^2} \left(\sum_k \delta P_{ik}^{T(0)} \otimes \delta P_{kj}^{T(0)}(z) + \beta_0\delta P_{ij}^{T(0)}(z) \right) - \frac{1}{2\epsilon}\delta P_{ij}^{T(1)}(z) \right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[\frac{-1}{6\epsilon^3} \left(\sum_{m,k} \delta P_{im}^{T(0)} \otimes \delta P_{mk}^{T(0)} \otimes \delta P_{kj}^{T(0)}(z) + 3\beta_0 \sum_k \delta P_{ik}^{T(0)} \otimes \delta P_{kj}^{T(0)}(z) \right. \right. \\
& \left. + 2\beta_0^2\delta P_{ij}^{T(0)}(z) \right) + \frac{1}{6\epsilon^2} \left(2 \sum_k \delta P_{ik}^{T(0)} \otimes \delta P_{kj}^{T(1)}(z) + \sum_k \delta P_{ik}^{T(1)} \otimes \delta P_{kj}^{T(0)}(z) \right. \\
& \left. \left. + 2\beta_0\delta P_{ij}^{T(1)}(z) + 2\beta_1\delta P_{ij}^{T(0)}(z) \right) - \frac{1}{3\epsilon}\delta P_{ij}^{T(2)}(z) \right] + \mathcal{O}(\alpha_s^3), \tag{2.14}
\end{aligned}$$

where $\delta P_{ij}^{(n)}$ is the NⁿLO space-like transversity splitting function [91, 92], which is presently known to NLO [89, 90], and $\delta P_{ij}^{T(n)}$ is the NⁿLO time-like transversity splitting function.

The factorized finite coefficient functions obey the following μ -RG equations

$$\begin{aligned}
\frac{d}{d\ln\mu} \delta \mathcal{I}_{ji}(x, b_\perp, \mu, \nu) = & 2 \left[\Gamma_j^{\text{cusp}}(\alpha_s(\mu)) \ln \frac{\nu}{xP_+} + \gamma_j^B(\alpha_s(\mu)) \right] \delta \mathcal{I}_{ji}(x, b_\perp, \mu, \nu) \\
& - 2 \sum_k \delta \mathcal{I}_{jk}(x, b_\perp, \mu, \nu) \otimes \delta P_{ki}(x, \alpha_s(\mu)), \tag{2.15}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\ln\mu} \delta \mathcal{C}_{ij}(z, b_\perp/z, \mu, \nu) = & 2 \left[\Gamma_j^{\text{cusp}}(\alpha_s(\mu)) \ln \frac{z\nu}{P_+} + \gamma_j^B(\alpha_s(\mu)) \right] \delta \mathcal{C}_{ij}(z, b_\perp/z, \mu, \nu) \\
& - 2 \sum_k \delta P_{ik}^T(z, \alpha_s(\mu)) \otimes \delta \mathcal{C}_{kj}(z, b_\perp/z, \mu, \nu), \tag{2.16}
\end{aligned}$$

and the rapidity evolution equations [93, 94]

$$\begin{aligned}
\frac{d}{d\ln\nu} \delta \mathcal{I}_{ji}(x, b_\perp, \mu, \nu) = & -2 \left[\int_\mu^{b_0/b_T} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_j^{\text{cusp}}(\alpha_s(\bar{\mu})) + \gamma_j^R(\alpha_s(b_0/b_T)) \right] \delta \mathcal{I}_{ji}(x, b_\perp, \mu, \nu), \\
\frac{d}{d\ln\nu} \delta \mathcal{C}_{ij}(z, b_\perp/z, \mu, \nu) = & -2 \left[\int_\mu^{b_0/b_T} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_j^{\text{cusp}}(\alpha_s(\bar{\mu})) + \gamma_j^R(\alpha_s(b_0/b_T)) \right] \delta \mathcal{C}_{ij}(z, b_\perp/z, \mu, \nu). \tag{2.17}
\end{aligned}$$

We define the coefficient function order-by-order in terms of strong coupling $\alpha_s/(4\pi)$, up to $\mathcal{O}(\alpha_s^3)$, the solution to these evolution equations reads

$$\delta \mathcal{I}_{ji}^{(0)}(x, b_\perp, \mu, \nu) = \delta_{ji}\delta(1-x),$$

$$\begin{aligned}
\delta\mathcal{I}_{ji}^{(1)}(x, b_\perp, \mu, \nu) &= \left(-\frac{\Gamma_0^{\text{cusp}}}{2} L_\perp L_Q + \gamma_0^B L_\perp + \gamma_0^R L_Q \right) \delta_{ji} \delta(1-x) - \delta P_{ji}^{(0)}(x) L_\perp + \delta I_{ji}^{(1)}(x), \\
\delta\mathcal{I}_{ji}^{(2)}(x, b_\perp, \mu, \nu) &= \left[\frac{1}{8} \left(-\Gamma_0^{\text{cusp}} L_Q + 2\gamma_0^B \right) \left(-\Gamma_0^{\text{cusp}} L_Q + 2\gamma_0^B + 2\beta_0 \right) L_\perp^2 \right. \\
&\quad + \left(\left(-\Gamma_0^{\text{cusp}} L_Q + 2\gamma_0^B + 2\beta_0 \right) \frac{\gamma_0^R}{2} L_Q - \frac{\Gamma_1^{\text{cusp}}}{2} L_Q + \gamma_1^B \right) L_\perp \\
&\quad + \frac{(\gamma_0^R)^2}{2} L_Q^2 + \gamma_1^R L_Q \left. \right] \delta_{ji} \delta(1-x) + \left(\frac{1}{2} \sum_l \delta P_{jl}^{(0)} \otimes \delta P_{li}^{(0)}(x) \right. \\
&\quad + \frac{\delta P_{ji}^{(0)}(x)}{2} (\Gamma_0^{\text{cusp}} L_Q - 2\gamma_0^B - \beta_0) \left. \right) L_\perp^2 + \left[-\delta P_{ji}^{(1)}(x) - \delta P_{ji}^{(0)}(x) \gamma_0^R L_Q \right. \\
&\quad - \sum_l \delta I_{jl}^{(1)} \otimes \delta P_{li}^{(0)}(x) + \left(-\frac{\Gamma_0^{\text{cusp}}}{2} L_Q + \gamma_0^B + \beta_0 \right) \delta I_{ji}^{(1)}(x) \left. \right] L_\perp \\
&\quad + \gamma_0^R L_Q \delta I_{ji}^{(1)}(x) + \delta I_{ji}^{(2)}(x), \\
\delta\mathcal{I}_{ji}^{(3)}(x, b_\perp, \mu, \nu) &= L_\perp^3 \left[\left(\frac{1}{2} \beta_0 + \frac{1}{4} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) \sum_l \delta P_{jl}^{(0)} \otimes \delta P_{li}^{(0)}(x) \right. \\
&\quad - \frac{1}{6} \sum_{lk} \delta P_{jl}^{(0)} \otimes \delta P_{lk}^{(0)} \otimes \delta P_{ki}^{(0)}(x) + \delta_{ji} \delta(1-x) \left(\frac{1}{6} \beta_0^2 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right. \\
&\quad + \frac{1}{8} \beta_0 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^2 + \frac{1}{48} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^3 \left. \right) \\
&\quad + \delta P_{ji}^{(0)} \left(-\frac{1}{2} \beta_0 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) - \frac{1}{3} \beta_0^2 - \frac{1}{8} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^2 \right) \left. \right] \\
&\quad + L_\perp^2 \left[\left(-\frac{3}{2} \beta_0 - \frac{1}{2} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) \sum_l \delta I_{jl}^{(1)} \otimes \delta P_{li}^{(0)}(x) \right. \\
&\quad + \frac{1}{2} \sum_{lk} \delta I_{jl}^{(1)} \otimes \delta P_{lk}^{(0)} \otimes \delta P_{ki}^{(0)}(x) + \frac{1}{2} \sum_l \delta P_{jl}^{(0)} \otimes \delta P_{li}^{(1)}(x) \\
&\quad + \delta P_{ji}^{(0)}(x) \left(-\frac{1}{2} \beta_1 - \frac{1}{2} (2\gamma_1^B - \Gamma_1^{\text{cusp}} L_Q) \right) + \delta_{ji} \delta(1-x) \left(\frac{1}{4} \beta_1 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right. \\
&\quad + \frac{1}{2} \beta_0 (2\gamma_1^B - \Gamma_1^{\text{cusp}} L_Q) + \frac{1}{4} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) (2\gamma_1^B - \Gamma_1^{\text{cusp}} L_Q) \left. \right) \\
&\quad + \delta I_{ji}^{(1)}(x) \left(\frac{3}{4} \beta_0 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) + \beta_0^2 + \frac{1}{8} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^2 \right) \\
&\quad + \delta P_{ji}^{(1)}(x) \left(-\beta_0 - \frac{1}{2} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) + \frac{1}{2} \sum_l \delta P_{jl}^{(1)} \otimes \delta P_{li}^{(0)}(x) \left. \right] \\
&\quad + L_\perp \left[-\sum_l \delta I_{jl}^{(1)} \otimes \delta P_{li}^{(1)}(x) - \sum_l \delta I_{jl}^{(2)} \otimes \delta P_{li}^{(0)}(x) - \delta P_{ji}^{(0)}(x) \gamma_1^R L_Q - \delta P_{ji}^{(2)}(x) \right. \\
&\quad + \delta_{ji} \delta(1-x) \left(2\beta_0 \gamma_1^R L_Q + \frac{1}{2} \gamma_1^R (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) L_Q + \frac{1}{2} (2\gamma_2^B - \Gamma_2^{\text{cusp}} L_Q) \right)
\end{aligned}$$

$$\begin{aligned}
& +\delta I_{ji}^{(1)}(x) \left(\beta_1 + \frac{1}{2}(2\gamma_1^B - \Gamma_1^{\text{cusp}} L_Q) \right) + \delta I_{ji}^{(2)}(x) \left(2\beta_0 + \frac{1}{2}(2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) \Big] \\
& +\delta_{ji}\delta(1-x)\gamma_2^R L_Q + \delta I_{ji}^{(1)}(x)\gamma_1^R L_Q + \delta I_{ji}^{(3)}(x), \tag{2.18}
\end{aligned}$$

where $\delta I_{ji}^{(n)}(z)$ are the scale-independent coefficient functions. In $\delta \mathcal{I}_{ji}^{(3)}$ we have used $\gamma_0^R = 0$ to simplify the expression. The scale logarithms are defined by

$$L_\perp = \ln \frac{b_T^2 \mu^2}{b_0^2}, \quad L_Q = 2 \ln \frac{x P_+}{\nu}, \quad L_\nu = \ln \frac{\nu^2}{\mu^2}, \quad b_0 = 2e^{-\gamma_E}. \tag{2.19}$$

Similarly, the solution to the fragmentation coefficient functions are

$$\delta \mathcal{C}_{ji}^{(0)}(z, b_\perp/z, \mu, \nu) = \delta_{ji}\delta(1-z),$$

$$\delta \mathcal{C}_{ji}^{(1)}(z, b_\perp/z, \mu, \nu) = \left(-\frac{\Gamma_0^{\text{cusp}}}{2} L_\perp L_Q + \gamma_0^B L_\perp + \gamma_0^R L_Q \right) \delta_{ji}\delta(1-z) - \delta P_{ji}^{T(0)}(z) L_\perp + \delta C_{ji}^{(1)}(z),$$

$$\begin{aligned}
\delta \mathcal{C}_{ji}^{(2)}(z, b_\perp/z, \mu, \nu) = & \left[\frac{1}{8} \left(-\Gamma_0^{\text{cusp}} L_Q + 2\gamma_0^B \right) \left(-\Gamma_0^{\text{cusp}} L_Q + 2\gamma_0^B + 2\beta_0 \right) L_\perp^2 \right. \\
& + \left((-\Gamma_0^{\text{cusp}} L_Q + 2\gamma_0^B + 2\beta_0) \frac{\gamma_0^R}{2} L_Q - \frac{\Gamma_1^{\text{cusp}}}{2} L_Q + \gamma_1^B \right) L_\perp \\
& + \left. \frac{(\gamma_0^R)^2}{2} L_Q^2 + \gamma_1^R L_Q \right] \delta_{ji}\delta(1-z) + \left(\frac{1}{2} \sum_l \delta P_{jl}^{T(0)} \otimes \delta P_{li}^{T(0)}(z) \right. \\
& + \frac{\delta P_{ji}^{T(0)}(z)}{2} (\Gamma_0^{\text{cusp}} L_Q - 2\gamma_0^B - \beta_0) \Big) L_\perp^2 + \left[-\delta P_{ji}^{T(1)}(z) - \delta P_{ji}^{T(0)}(z) \gamma_0^R L_Q \right. \\
& - \sum_l \delta P_{jl}^{T(0)} \otimes \delta C_{li}^{(1)}(z) + \left(-\frac{\Gamma_0^{\text{cusp}}}{2} L_Q + \gamma_0^B + \beta_0 \right) \delta C_{ji}^{(1)}(z) \Big] L_\perp \\
& + \gamma_0^R L_Q \delta C_{ji}^{(1)}(z) + \delta C_{ji}^{(2)}(z),
\end{aligned}$$

$$\begin{aligned}
\delta \mathcal{C}_{ji}^{(3)}(z, b_\perp/z, \mu, \nu) = & L_\perp^3 \left[\left(\frac{1}{2} \beta_0 + \frac{1}{4} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) \sum_l \delta P_{jl}^{T(0)} \otimes \delta P_{li}^{T(0)}(z) \right. \\
& - \frac{1}{6} \sum_{lk} \delta P_{jl}^{T(0)} \otimes \delta P_{lk}^{T(0)} \otimes \delta P_{ki}^{T(0)}(z) + \delta_{ji}\delta(1-z) \left(\frac{1}{6} \beta_0^2 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right. \\
& + \frac{1}{8} \beta_0 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^2 + \frac{1}{48} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^3 \Big) \\
& + \left. \delta P_{ji}^{T(0)} \left(-\frac{1}{2} \beta_0 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) - \frac{1}{3} \beta_0^2 - \frac{1}{8} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q)^2 \right) \right] \\
& + L_\perp^2 \left[\left(-\frac{3}{2} \beta_0 - \frac{1}{2} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) \sum_l \delta P_{jl}^{T(0)} \otimes \delta C_{li}^{(1)}(z) \right. \\
& + \frac{1}{2} \sum_{lk} \delta P_{jl}^{T(0)} \otimes \delta P_{lk}^{T(0)} \otimes \delta C_{ki}^{(1)}(z) + \frac{1}{2} \sum_l \delta P_{jl}^{T(0)} \otimes \delta P_{li}^{T(1)}(z) \\
& + \left. \delta P_{ji}^{T(0)}(z) \left(-\frac{1}{2} \beta_1 - \frac{1}{2} (2\gamma_1^B - \Gamma_1^{\text{cusp}} L_Q) \right) + \delta_{ji}\delta(1-z) \left(\frac{1}{4} \beta_1 (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}\beta_0(2\gamma_1^B - \Gamma_1^{\text{cusp}}L_Q) + \frac{1}{4}(2\gamma_0^B - \Gamma_0^{\text{cusp}}L_Q)(2\gamma_1^B - \Gamma_1^{\text{cusp}}L_Q) \Big) \\
& + \delta C_{ji}^{(1)}(z) \left(\frac{3}{4}\beta_0(2\gamma_0^B - \Gamma_0^{\text{cusp}}L_Q) + \beta_0^2 + \frac{1}{8}(2\gamma_0^B - \Gamma_0^{\text{cusp}}L_Q)^2 \right) \\
& + \delta P_{ji}^{T(1)}(z) \left(-\beta_0 - \frac{1}{2}(2\gamma_0^B - \Gamma_0^{\text{cusp}}L_Q) \right) + \frac{1}{2} \sum_l \delta P_{jl}^{T(1)} \otimes \delta P_{li}^{T(0)}(z) \Big] \\
& + L_\perp \left[- \sum_l \delta P_{jl}^{T(1)} \otimes \delta C_{li}^{(1)}(z) - \sum_l \delta P_{jl}^{T(0)} \otimes \delta C_{li}^{(2)}(z) - \delta P_{ji}^{T(0)}(z) \gamma_1^R L_Q \right. \\
& - \delta P_{ji}^{T(2)}(z) + \delta_{ji} \delta(1-z) \left(2\beta_0 \gamma_1^R L_Q + \frac{1}{2} \gamma_1^R (2\gamma_0^B - \Gamma_0^{\text{cusp}}L_Q) L_Q + \frac{1}{2} (2\gamma_2^B - \Gamma_2^{\text{cusp}}L_Q) \right) \\
& + \delta C_{ji}^{(1)}(z) \left(\beta_1 + \frac{1}{2}(2\gamma_1^B - \Gamma_1^{\text{cusp}}L_Q) \right) + \delta C_{ji}^{(2)}(z) \left(2\beta_0 + \frac{1}{2}(2\gamma_0^B - \Gamma_0^{\text{cusp}}L_Q) \right) \Big] \\
& + \delta_{ji} \delta(1-z) \gamma_2^R L_Q + \delta C_{ji}^{(1)}(z) \gamma_1^R L_Q + \delta C_{ji}^{(3)}(z). \tag{2.20}
\end{aligned}$$

The anomalous dimensions appeared above are identical to those in the space-like case. The logarithms appeared in the fragmentation coefficient functions are defined as

$$L_\perp = \ln \frac{b_T^2 \mu^2}{b_0^2}, \quad L_Q = 2 \ln \frac{P_+}{z \nu}, \quad L_\nu = \ln \frac{\nu^2}{\mu^2}, \quad b_0 = 2e^{-\gamma_E}. \tag{2.21}$$

Both space-like and time-like coefficient functions depend on the rapidity regulator being used. Rapidity-regulator-independent TMD transversity PDFs and FFs can be obtained by multiplying the coefficient functions with the squared root of the TMD soft functions $\mathcal{S}(b_\perp, \mu, \nu)$ [57, 86]

$$\begin{aligned}
\delta h_{T,qj}(x, b_\perp, \mu) &= \delta \mathcal{I}_{qj}(x, b_\perp, \mu, \nu) \sqrt{\mathcal{S}(b_\perp, \mu, \nu)}, \\
\delta g_{T,iq}(z, b_\perp/z, \mu) &= \delta \mathcal{C}_{iq}(z, b_\perp/z, \mu, \nu) \sqrt{\mathcal{S}(b_\perp, \mu, \nu)}. \tag{2.22}
\end{aligned}$$

3 The NNLO transversity splitting functions

The next-to-leading order (NLO) space-like transversity splitting functions were first obtained in Refs. [89, 90]. In this work, we extend the precision to next-to-next-to-leading order (NNLO) for the first time. A detailed comparison with the known NLO results confirms complete agreement. In the following, we present our analytical results for both the space-like and time-like transversity splitting functions at NNLO accuracy.

3.1 Space-like results in $\overline{\text{MS}}$

$$\delta P_{qq}^{(0)} = 4C_F \left[\frac{1}{1-x} \right]_+ + 3C_F \delta(1-x) - 4C_F, \quad (3.1)$$

$$\begin{aligned} \delta P_{qq}^{(1)} = & \left[\frac{1}{1-x} \right]_+ \left(\left(\frac{268}{9} - 8\zeta_2 \right) C_A C_F - \frac{40C_F N_f}{9} \right) \\ & + C_A C_F \left(\frac{x}{x-1} \left(\left(-\frac{44}{3} \right) H_0 - 8H_{0,0} \right) + 2x + 8\zeta_2 - \frac{286}{9} \right) \\ & + C_F^2 \left(\frac{x}{x-1} (-16H_{1,0} + 12H_0 - 16H_2) - 4x + 4 \right) \\ & + \delta(1-x) \left(\left(\frac{44}{3}\zeta_2 - 12\zeta_3 + \frac{17}{6} \right) C_A C_F + \left(\left(-\frac{8}{3} \right) \zeta_2 - \frac{1}{3} \right) C_F N_f + \left(-12\zeta_2 + 24\zeta_3 + \frac{3}{2} \right) C_F^2 \right) \\ & + C_F N_f T_F \left(\frac{16x}{3(x-1)} H_0 + \frac{80}{9} \right), \end{aligned} \quad (3.2)$$

$$\begin{aligned} \delta P_{qq}^{(2)} = & C_F^3 \left(\left(\left(-\frac{64}{3} \right) H_{-1,-1,0} + \left(-\frac{32}{3} \right) H_{-1,2} + \left(-\frac{16}{3} \right) H_{0,0,0} + \frac{8}{3} H_{-1,0} + \frac{16}{3} H_{-1,0,0} + \frac{32}{3} H_3 \right. \right. \\ & + \left. \frac{64}{3} H_{-2,0} \right) x^2 + \left(\left(-\frac{16}{3} \right) H_2 + \frac{40}{3} H_{-1,0} + 96H_3 + 192H_{-2,0} - 96H_{-1,2} - 16H_{1,0} \right. \\ & - \left. 192H_{-1,-1,0} + 48H_{-1,0,0} - 48H_{0,0,0} \right) x + \frac{40}{3} H_{-1,0} + \frac{80}{3} H_2 \\ & + \frac{1}{x-1} \left(\left(-\frac{8}{3} \right) H_0 + \frac{16}{3} H_{0,0} + \frac{200}{3} H_1 + \frac{104}{3} \right) \\ & + \frac{1}{x} \left(\left(-\frac{64}{3} \right) H_{-1,-1,0} + \left(-\frac{32}{3} \right) H_{-1,2} + \frac{8}{3} H_{-1,0} + \frac{16}{3} H_{-1,0,0} \right) + \left(-\frac{72x}{x-1} \right) \zeta_4 \\ & + \frac{x}{x-1} \left(\left(-\frac{400}{3} \right) H_1 + \left(-\frac{254}{3} \right) H_0 - 64H_4 + 64H_{-3,0} - 28H_{0,0} - 256H_{1,3} + 96H_{2,0} \right. \\ & - 128H_{2,2} - 128H_{3,0} - 128H_{3,1} - 384H_{-2,-1,0} + 192H_{-2,0,0} - 256H_{1,-2,0} + 192H_{1,0,0} \\ & - 128H_{1,2,0} - 64H_{2,0,0} - 128H_{2,1,0} + 64H_{0,0,0,0} + 128H_{1,0,0,0} - \left. \frac{208}{3} \right) \\ & + \frac{x^2}{x-1} \left(\left(-\frac{80}{3} \right) H_{0,0} + \frac{200}{3} H_1 + \frac{280}{3} H_0 + \frac{104}{3} \right) + \frac{128x^2}{3(x-1)} \zeta_3 + \left(-\frac{8x^3}{3(x-1)} \right) H_{0,0} \\ & + \frac{16x^3}{3(x-1)} \zeta_3 + \left(-\frac{16\zeta_2}{3(x-1)x} \right) H_1 + \frac{1}{x-1} \left(\frac{64}{3} - \frac{128H_1}{3} \right) \zeta_2 \\ & + \frac{x}{x-1} (-32H_0 - 384H_1 - 48) \zeta_3 + \frac{x}{x-1} \left(-192H_{-2} + 96H_0 + 96H_1 - \frac{104}{3} \right) \zeta_2 \\ & + \frac{x^2}{x-1} \left(-\frac{128H_0}{3} - \frac{128H_1}{3} + \frac{32}{3} \right) \zeta_2 + \frac{x^3}{x-1} \left(-\frac{16H_0}{3} - \frac{16H_1}{3} + \frac{8}{3} \right) \zeta_2 - 96H_{-1,2} \\ & + 16H_{1,0} - 192H_{-1,-1,0} + 48H_{-1,0,0} \Big) + N_f T_F C_F^2 \left(\frac{64}{x-1} \zeta_3 + \left(-\frac{320x}{3(x-1)} \right) \zeta_3 \right. \\ & + \frac{x}{x-1} \left(\frac{64}{3} H_{0,0,0} + \frac{128}{3} H_3 + \frac{128}{3} H_{2,0} + \frac{640}{9} H_2 + \frac{640}{9} H_{1,0} + \frac{256}{3} H_{1,0,0} - 24H_0 - 32H_{0,0} \right. \\ & + \left. \frac{436}{9} \right) + \frac{x^2}{x-1} \left(\frac{32}{3} H_0 + \frac{112}{9} \right) + \left(-\frac{64x\zeta_2}{3(x-1)} \right) H_0 - \frac{548}{9(x-1)} \Big) \end{aligned} \quad (3.3)$$

$$\begin{aligned}
& + N_f^2 T_F^2 C_F \left(-\frac{32x}{9} + \frac{x}{x-1} \left(\left(-\frac{320}{27} \right) H_0 + \left(-\frac{64}{9} \right) H_{0,0} \right) + \frac{160}{27} \right) \\
& + C_A^2 C_F \left(4H_{-1} \zeta_2 x^2 + \left(\left(-\frac{4}{3} \right) H_{-1,0} - 8H_{-1,2} - 8H_{-1,-1,0} \right) x^2 + 36H_{-1} \zeta_2 x \right. \\
& + (8H_2 + 4H_{-1,0} - 72H_{-1,2} - 72H_{-1,-1,0}) x + \frac{1}{x-1} \left(\left(-\frac{4}{3} \right) H_0 + \frac{92}{3} H_1 + \frac{1586}{9} \right) \\
& + \frac{88}{3(x-1)} \zeta_3 + \frac{88}{x-1} \zeta_4 + \frac{1}{x} \left(\left(-\frac{4}{3} \right) H_{-1,0} - 8H_{-1,2} - 8H_{-1,-1,0} \right) + \left(-\frac{192x}{x-1} \right) \zeta_4 \\
& + \frac{x}{x-1} \left(\left(-\frac{5396}{27} \right) H_0 + \left(-\frac{1592}{9} \right) H_{0,0} + \left(-\frac{304}{3} \right) H_3 + \left(-\frac{248}{3} \right) H_{0,0,0} + \left(-\frac{184}{3} \right) H_1 \right. \\
& - 32H_4 - 32H_{-3,0} - 24H_{-2,0} - 64H_{-2,2} - 128H_{1,3} - 128H_{-2,-1,0} + 32H_{-2,0,0} - 192H_{1,-2,0} \\
& + 88H_{1,0,0} + 64H_{2,0,0} + 32H_{0,0,0,0} + 96H_{1,0,0,0} + 128H_{1,1,0,0} - \frac{1702}{9} \left. \right) + \left(-\frac{32x^2}{x-1} \right) \zeta_3 \\
& + \frac{x^2}{x-1} \left(\frac{8}{3} H_{0,0} + \frac{92}{3} H_1 + \frac{140}{3} H_0 + 64H_3 + 64H_{-2,0} + \frac{116}{9} \right) + \left(-\frac{4x^3}{x-1} \right) \zeta_3 \\
& + \frac{x^3}{x-1} \left(\frac{4}{3} H_{0,0} + 8H_3 + 8H_{-2,0} \right) + \frac{4\zeta_2}{x} H_{-1} + \left(-\frac{4\zeta_2}{(x-1)x} \right) H_1 + \frac{1}{x-1} \left(-32H_1 - \frac{1000}{9} \right) \zeta_2 \\
& + \frac{x}{x-1} (-128H_0 - 288H_1 + 20) \zeta_3 + \frac{x}{x-1} \left(\frac{464H_0}{3} + 72H_1 + 32H_{1,0} + \frac{1036}{9} \right) \zeta_2 \\
& + \frac{x^2}{x-1} \left(-64H_0 - 32H_1 - \frac{8}{3} \right) \zeta_2 + \frac{x^3}{x-1} \left(-8H_0 - 4H_1 - \frac{4}{3} \right) \zeta_2 + 8H_2 + 36H_{-1} \zeta_2 + 4H_{-1,0} \\
& - 72H_{-1,2} - 72H_{-1,-1,0} \left. \right) + C_A C_F^2 \left(\left(\left(-\frac{8}{3} \right) H_{-1,0,0} + \frac{4}{3} H_{-1,0} + \frac{64}{3} H_{-1,2} + \frac{80}{3} H_{-1,-1,0} \right) x^2 \right. \\
& - 8H_{-1} \zeta_2 x^2 - 72H_{-1} \zeta_2 x + \left(\left(-\frac{44}{3} \right) H_{-1,0} + 192H_{-1,2} + 240H_{-1,-1,0} - 24H_{-1,0,0} \right) x \\
& + \left(-\frac{44}{3} \right) H_{-1,0} + \frac{1}{x-1} \left(\left(-\frac{328}{3} \right) H_1 + \left(-\frac{8}{3} \right) H_{0,0} + \frac{88}{3} H_2 + 4H_0 + 8H_{1,0} - \frac{872}{9} \right) \\
& + \frac{1}{x} \left(\left(-\frac{8}{3} \right) H_{-1,0,0} + \frac{4}{3} H_{-1,0} + \frac{64}{3} H_{-1,2} + \frac{80}{3} H_{-1,-1,0} \right) \\
& + \frac{x}{x-1} \left(\left(-\frac{992}{3} \right) H_{1,0,0} + \left(-\frac{2288}{9} \right) H_2 + \left(-\frac{2288}{9} \right) H_{1,0} + \left(-\frac{352}{3} \right) H_{2,0} + \frac{112}{3} H_{0,0,0} + \frac{512}{3} H_3 \right. \\
& + \frac{656}{3} H_1 + \frac{754}{3} H_0 - 32H_4 + 32H_{-3,0} + 144H_{-2,0} + 128H_{-2,2} + 100H_{0,0} + 256H_{1,3} - 32H_{3,0} \\
& + 448H_{-2,-1,0} - 160H_{-2,0,0} + 512H_{1,-2,0} - 160H_{2,0,0} - 160H_{0,0,0,0} - 352H_{1,0,0,0} - 256H_{1,1,0,0} + \frac{1744}{9} \left. \right) \\
& + \frac{276x}{x-1} \zeta_4 + \frac{x^2}{x-1} \left(\left(-\frac{640}{3} \right) H_{-2,0} + \left(-\frac{512}{3} \right) H_3 + \left(-\frac{464}{3} \right) H_0 + \left(-\frac{328}{3} \right) H_1 \right. \\
& + \left(-\frac{40}{3} \right) H_2 + \frac{64}{3} H_{0,0,0} + 8H_{0,0} + 8H_{1,0} - \frac{872}{9} \left. \right) + \frac{128x^2}{3(x-1)} \zeta_3 \\
& + \frac{x^3}{x-1} \left(\left(-\frac{80}{3} \right) H_{-2,0} + \left(-\frac{64}{3} \right) H_3 + \left(-\frac{4}{3} \right) H_{0,0} + \frac{8}{3} H_{0,0,0} \right) + \frac{16x^3}{3(x-1)} \zeta_3 + \left(-\frac{8\zeta_2}{x} \right) H_{-1} \\
& + \frac{32\zeta_2}{3(x-1)x} H_1 + \frac{1}{x-1} \left(\frac{256H_1}{3} - \frac{80}{3} \right) \zeta_2 + \frac{x}{x-1} \left(272H_0 + 768H_1 - \frac{224}{3} \right) \zeta_3 \\
& + \frac{x}{x-1} \left(96H_{-2} - \frac{832H_0}{3} - 192H_1 + 64H_2 + 128H_{0,0} + 64H_{1,0} + \frac{76}{3} \right) \zeta_2 + \frac{x^2}{x-1} \left(\frac{448H_0}{3} + \frac{256H_1}{3} \right) \zeta_2 \\
& + \frac{x^3}{x-1} \left(\frac{56H_0}{3} + \frac{32H_1}{3} + \frac{4}{3} \right) \zeta_2 - 72H_{-1} \zeta_2 + 192H_{-1,2} + 240H_{-1,-1,0} - 24H_{-1,0,0} \left. \right)
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
& +\delta(1-x) \left(\left(-32\zeta_3\zeta_2 + 18\zeta_2 + 68\zeta_3 + 144\zeta_4 - 240\zeta_5 + \frac{29}{2} \right) C_F^3 \right. \\
& + \left(\left(-\frac{136}{3} \right) \zeta_3 + \frac{20}{3} \zeta_2 + \frac{116}{3} \zeta_4 - 23 \right) N_f C_F^2 + \left(\left(-\frac{16}{9} \right) \zeta_3 + \frac{80}{27} \zeta_2 - \frac{17}{9} \right) N_f^2 C_F \\
& + C_A^2 \left(\left(-\frac{1552}{9} \right) \zeta_3 + \frac{4496}{27} \zeta_2 - 5\zeta_4 + 40\zeta_5 - \frac{1657}{36} \right) C_F \\
& + C_A \left(\left(\left(-\frac{494}{3} \right) \zeta_4 + \left(-\frac{410}{3} \right) \zeta_2 + \frac{844}{3} \zeta_3 + 16\zeta_2\zeta_3 + 120\zeta_5 + \frac{151}{4} \right) C_F^2 \right. \\
& + N_f \left(\left(-\frac{1336}{27} \right) \zeta_2 + \frac{200}{9} \zeta_3 + 2\zeta_4 + 20 \right) C_F \Bigg) \\
& + C_A N_f T_F C_F \left(\left(-\frac{8}{3} \right) H_1 + \left(-\frac{224}{3(x-1)} \right) \zeta_3 + \frac{320}{9(x-1)} \zeta_2 \right. \\
& + \frac{8x}{3} H_1 + \frac{x}{x-1} \left(\frac{32}{3} H_3 + \frac{64}{3} H_{0,0,0} + \frac{224}{3} H_{0,0} + \frac{2744}{27} H_0 - 32H_{1,0,0} + \frac{952}{27} \right) \\
& + \frac{96x}{x-1} \zeta_3 + \frac{x^2}{x-1} \left(\left(-\frac{8}{3} \right) H_0 + \frac{40}{3} \right) + \frac{x}{x-1} \left(-\frac{64H_0}{3} - \frac{320}{9} \right) \zeta_2 - \frac{1312}{27(x-1)} \Bigg) \\
& + \left[\frac{1}{1-x} \right]_+ \left(C_F \left(\left(-\frac{1072}{9} \right) \zeta_2 + 88\zeta_4 + \frac{88\zeta(3)}{3} + \frac{490}{3} \right) C_A^2 \right. \\
& + C_F N_f \left(\frac{160}{9} \zeta_2 - \frac{112\zeta(3)}{3} - \frac{836}{27} \right) C_A - \frac{16}{27} C_F N_f^2 + C_F^2 N_f \left(-\frac{110}{3} + 32\zeta(3) \right) \Bigg), \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
\delta P_{q\bar{q}}^{(1)} &= C_A C_F \left(\frac{x}{x+1} (8H_{0,0} - 16H_{-1,0}) - \frac{2x^2}{x+1} + \left(-\frac{8x}{x+1} \right) \zeta_2 + \frac{2}{x+1} \right) \\
&+ C_F^2 \left(\frac{x}{x+1} (32H_{-1,0} - 16H_{0,0}) + \frac{4x^2}{x+1} + \frac{16x}{x+1} \zeta_2 - \frac{4}{x+1} \right), \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
\delta P_{q\bar{q}}^{(2)} &= +N_f T_F C_F C_A \left(\frac{56x^2}{9(x+1)} + \frac{16}{3} H_1 + \left(-\frac{16x}{3} \right) H_1 + \left(-\frac{32x}{x+1} \right) \zeta_3 + \frac{x}{x+1} \left(\left(-\frac{128}{3} \right) H_{-1,2} \right. \right. \\
&+ \left(-\frac{320}{9} \right) H_{0,0} + \left(-\frac{64}{3} \right) H_{0,0,0} + \frac{64}{3} H_3 + \frac{64}{3} H_{-2,0} + \frac{64}{3} H_{-1,0,0} + \frac{640}{9} H_{-1,0} \Bigg) \\
&+ \frac{x}{x+1} \left(\frac{128H_{-1}}{3} - \frac{32H_0}{3} + \frac{320}{9} \right) \zeta_2 - \frac{56}{9(x+1)} \Bigg) \\
&+ N_f T_F C_F^2 \left(-\frac{112x^2}{9(x+1)} + \left(-\frac{32}{3} \right) H_1 + \frac{32x}{3} H_1 + \frac{x}{x+1} \left(\left(-\frac{1280}{9} \right) H_{-1,0} + \left(-\frac{128}{3} \right) H_3 \right. \right. \\
&+ \left(-\frac{128}{3} \right) H_{-2,0} + \left(-\frac{128}{3} \right) H_{-1,0,0} + \frac{128}{3} H_{0,0,0} + \frac{640}{9} H_{0,0} + \frac{256}{3} H_{-1,2} \Bigg) + \frac{64x}{x+1} \zeta_3 \\
&+ \frac{x}{x+1} \left(-\frac{256H_{-1}}{3} + \frac{64H_0}{3} - \frac{640}{9} \right) \zeta_2 + \frac{112}{9(x+1)} \Bigg) \\
&+ C_F^3 \left(\left(-\frac{184}{3} \right) H_1 + \left(-\frac{32}{3x} \right) H_{-1,-1,0} + \left(-\frac{32x^2}{3} \right) H_{-1,-1,0} \right. \\
&+ \frac{1}{x+1} \left(\left(-\frac{8}{3} \right) H_0 + \frac{16}{3} H_2 + \frac{16}{3} H_{0,0} + \frac{128}{3} H_{-1,0,0} + \frac{512}{3} H_{-1,2} + \frac{104}{3} \right) \\
&+ \frac{1}{x(x+1)} \left(\left(-\frac{64}{3} \right) H_{-1,2} + \left(-\frac{16}{3} \right) H_{-1,0,0} + \left(-\frac{8}{3} \right) H_{-1,0} \right) + \left(-\frac{280x}{x+1} \right) \zeta_4 \Bigg) \tag{3.7}
\end{aligned}$$

$$\begin{aligned}
& + \frac{x}{x+1} \left(\frac{16}{3} H_{-1,0} + \frac{128}{3} H_2 + \frac{152}{3} H_{0,0} + 80 H_0 - 288 H_3 - 384 H_4 + 192 H_{-3,0} - 192 H_{-2,0} \right. \\
& + 896 H_{-2,2} + 576 H_{-1,2} + 1024 H_{-1,3} - 64 H_{3,0} - 256 H_{-2,-1,0} + 832 H_{-2,0,0} - 256 H_{-1,-2,0} \\
& \left. - 1536 H_{-1,-1,2} + 128 H_{-1,2,0} + 96 H_{0,0,0} - 1280 H_{-1,-1,0,0} + 704 H_{-1,0,0,0} - 192 H_{0,0,0,0} \right) \\
& + \frac{x^2}{x+1} \left(\left(-\frac{512}{3} \right) H_3 + \left(-\frac{256}{3} \right) H_{-2,0} + \left(-\frac{128}{3} \right) H_{0,0,0} + \frac{104}{3} H_0 + \frac{112}{3} H_2 \right. \\
& + \frac{128}{3} H_{-1,0,0} + \frac{512}{3} H_{-1,2} + 48 H_{0,0} - \frac{104}{3} \Big) + \frac{512x^2}{3(x+1)} \zeta_3 + \left(-\frac{64x^3}{3(x+1)} \right) \zeta_3 \\
& + \frac{x^3}{x+1} \left(\left(-\frac{64}{3} \right) H_{-1,2} + \left(-\frac{16}{3} \right) H_{-1,0,0} + \left(-\frac{8}{3} \right) H_{-1,0} + \frac{8}{3} H_{0,0} + \frac{16}{3} H_{0,0,0} + \frac{32}{3} H_{-2,0} \right. \\
& + \frac{64}{3} H_3 \Big) + \frac{32\zeta_2}{3x} H_1 + \left(-\frac{32}{3} x^2 \zeta_2 \right) H_1 + \frac{16\zeta_2}{x(x+1)} H_{-1} + \frac{1}{x+1} \left(-128 H_{-1} - \frac{32}{3} \right) \zeta_2 \\
& + \frac{x}{x+1} (-1152 H_{-1} + 416 H_0 + 336) \zeta_3 + \frac{x}{x+1} (-1024 H_{-2} - 480 H_{-1} + 288 H_0 - 192 H_2 \\
& + 1536 H_{-1,-1} - 1280 H_{-1,0} + 448 H_{0,0} - 40) \zeta_2 + \frac{x^2}{x+1} \left(-128 H_{-1} + \frac{640 H_0}{3} - 32 \right) \zeta_2 \\
& + \frac{x^3}{x+1} \left(16 H_{-1} - \frac{80 H_0}{3} - \frac{8}{3} \right) \zeta_2 + 96 x H_1 \zeta_2 - 96 H_1 \zeta_2 - 16 H_{1,0} + 96 H_{-1,-1,0} \\
& + x \left(\frac{184}{3} H_1 + 16 H_{1,0} + 96 H_{-1,-1,0} \right) \Big) + C_A C_F^2 \left(\frac{64}{3x} H_{-1,-1,0} + \frac{64x^2}{3} H_{-1,-1,0} \right. \\
& + \frac{1}{x+1} \left(\left(-\frac{640}{3} \right) H_{-1,2} + \left(-\frac{64}{3} \right) H_{-1,0,0} + \left(-\frac{56}{3} \right) H_2 + \left(-\frac{32}{3} \right) H_{-1,0} \right. \\
& + \left(-\frac{8}{3} \right) H_{0,0} + 4 H_0 - \frac{872}{9} \Big) + \frac{1}{x(x+1)} \left(\left(-\frac{4}{3} \right) H_{-1,0} + \frac{8}{3} H_{-1,0,0} + \frac{80}{3} H_{-1,2} \right) \\
& + \left(-\frac{20x}{x+1} \right) \zeta_4 + \frac{x}{x+1} \left(\left(-\frac{2432}{3} \right) H_{-1,2} + \left(-\frac{2444}{9} \right) H_{0,0} + \left(-\frac{640}{3} \right) H_{0,0,0} \right. \\
& + \left(-\frac{296}{3} \right) H_0 + \left(-\frac{160}{3} \right) H_2 + \frac{352}{3} H_{-1,0,0} + \frac{1072}{3} H_{-2,0} + \frac{1216}{3} H_3 + \frac{4120}{9} H_{-1,0} \\
& + 320 H_4 - 160 H_{-3,0} - 960 H_{-2,2} - 1024 H_{-1,3} + 32 H_{3,0} + 128 H_{-2,-1,0} - 672 H_{-2,0,0} \\
& + 128 H_{-1,-2,0} + 1792 H_{-1,-1,2} - 64 H_{-1,2,0} + 1152 H_{-1,-1,0,0} - 544 H_{-1,0,0,0} + 160 H_{0,0,0,0} \Big) \\
& + \left(-\frac{448x^2}{3(x+1)} \right) \zeta_3 + \frac{x^2}{x+1} \left(\left(-\frac{640}{3} \right) H_{-1,2} + \left(-\frac{104}{3} \right) H_2 + \left(-\frac{92}{3} \right) H_0 + \left(-\frac{88}{3} \right) H_{0,0} \right. \\
& + \left(-\frac{64}{3} \right) H_{-1,0,0} + \left(-\frac{32}{3} \right) H_{-1,0} + \frac{64}{3} H_{0,0,0} + \frac{512}{3} H_{-2,0} + \frac{640}{3} H_3 + \frac{872}{9} \Big) \\
& + \frac{x^3}{x+1} \left(\left(-\frac{80}{3} \right) H_3 + \left(-\frac{64}{3} \right) H_{-2,0} + \left(-\frac{8}{3} \right) H_{0,0,0} + \left(-\frac{4}{3} \right) H_{-1,0} + \frac{4}{3} H_{0,0} \right. \\
& + \frac{8}{3} H_{-1,0,0} + \frac{80}{3} H_{-1,2} \Big) + \frac{56x^3}{3(x+1)} \zeta_3 + \left(-\frac{40\zeta_2}{3x} \right) H_1 + \frac{40x^2\zeta_2}{3} H_1 + \left(-\frac{16\zeta_2}{x(x+1)} \right) H_{-1} \\
& + \frac{1}{x+1} \left(128 H_{-1} + \frac{64}{3} \right) \zeta_2 + \frac{x}{x+1} (1344 H_{-1} - 464 H_0 - 416) \zeta_3 + \frac{x}{x+1} \left(1024 H_{-2} \right. \\
& + \frac{1856 H_{-1}}{3} - \frac{896 H_0}{3} + 224 H_2 - 1792 H_{-1,-1} + 1344 H_{-1,0} - 416 H_{0,0} + \frac{2540}{9} \Big) \zeta_2 \\
& + \frac{x^2}{x+1} \left(128 H_{-1} - \frac{704 H_0}{3} + \frac{64}{3} \right) \zeta_2 + \frac{x^3}{x+1} \left(-16 H_{-1} + \frac{88 H_0}{3} - \frac{4}{3} \right) \zeta_2 + 76 H_1 \\
& \left. - 120 x H_1 \zeta_2 + 120 H_1 \zeta_2 + 8 H_{1,0} + x (-76 H_1 - 8 H_{1,0} - 192 H_{-1,-1,0}) - 192 H_{-1,-1,0} \right), \tag{3.8}
\end{aligned}$$

3.2 Time-like results in $\overline{\text{MS}}$

$$\delta P_{qq}^{T(0)} = 4C_F \left[\frac{1}{1-z} \right]_+ + 3C_F \delta(1-z) - 4C_F, \quad (3.9)$$

$$\begin{aligned} \delta P_{qq}^{T(1)} = & \left[\frac{1}{1-z} \right]_+ \left(\left(\frac{268}{9} - 8\zeta_2 \right) C_A C_F - \frac{80}{9} C_F N_f T_F \right) \\ & + C_A C_F \left(\frac{z}{z-1} \left(\left(-\frac{44}{3} \right) H_0 - 8H_{0,0} \right) + 2z + 8\zeta_2 - \frac{286}{9} \right) \\ & + C_F^2 \left(\frac{z}{z-1} (32H_{0,0} + 16H_{1,0} - 12H_0 + 16H_2) - 4z + 4 \right) \\ & + \delta(1-z) \left(C_A C_F \left(\frac{44}{3} \zeta_2 - 12\zeta(3) + \frac{17}{6} \right) + \left(\left(-\frac{16}{3} \right) \zeta_2 - \frac{2}{3} \right) C_F N_f T_F \right. \\ & \left. + C_F^2 \left(-12\zeta_2 + 24\zeta(3) + \frac{3}{2} \right) \right) + C_F N_f T_F \left(\frac{16z}{3(z-1)} H_0 + \frac{80}{9} \right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} \delta P_{qq}^{T(2)} = & C_F^3 \left(\left(\left(-\frac{64}{3} \right) H_{-1,-1,0} + \left(-\frac{32}{3} \right) H_{-1,2} + \frac{8}{3} H_{-1,0} + \frac{16}{3} H_{-1,0,0} + \frac{32}{3} H_3 + \frac{64}{3} H_{-2,0} \right) z^2 \right. \\ & + \left(\frac{40}{3} H_{-1,0} + \frac{80}{3} H_2 + 96H_3 + 192H_{-2,0} - 96H_{-1,2} + 16H_{1,0} - 192H_{-1,-1,0} + 48H_{-1,0,0} \right) z \\ & + \left(-\frac{16}{3} \right) H_2 + \frac{40}{3} H_{-1,0} + \frac{1}{z-1} \left(\left(-\frac{80}{3} \right) H_0 + \frac{16}{3} H_{0,0} + \frac{200}{3} H_1 + \frac{104}{3} \right) \\ & + \frac{1}{z} \left(\left(-\frac{64}{3} \right) H_{-1,-1,0} + \left(-\frac{32}{3} \right) H_{-1,2} + \frac{8}{3} H_{-1,0} + \frac{16}{3} H_{-1,0,0} \right) + \left(-\frac{72z}{z-1} \right) \zeta_4 \\ & + \frac{z}{z-1} \left(\left(-\frac{400}{3} \right) H_1 + \left(-\frac{146}{3} \right) H_0 - 448H_4 + 64H_{-3,0} - 92H_{0,0} - 256H_{1,3} + 96H_{2,0} \right. \\ & - 128H_{2,2} - 256H_{3,0} - 128H_{3,1} - 384H_{-2,-1,0} + 192H_{-2,0,0} + 336H_{0,0,0} - 256H_{1,-2,0} \\ & + 192H_{1,0,0} - 128H_{1,2,0} - 192H_{2,0,0} - 128H_{2,1,0} - 448H_{0,0,0,0} - 256H_{1,0,0,0} - \frac{208}{3} \Big) \\ & + \frac{z^2}{z-1} \left(\left(-\frac{128}{3} \right) H_{0,0,0} + \frac{112}{3} H_{0,0} + \frac{200}{3} H_1 + \frac{208}{3} H_0 + \frac{104}{3} \right) + \frac{128z^2}{3(z-1)} \zeta_3 \\ & + \frac{z^3}{z-1} \left(\left(-\frac{16}{3} \right) H_{0,0,0} + \left(-\frac{8}{3} \right) H_{0,0} \right) + \frac{16z^3}{3(z-1)} \zeta_3 + \left(-\frac{16\zeta_2}{3(z-1)z} \right) H_1 \\ & + \frac{1}{z-1} \left(\frac{64}{3} - \frac{128H_1}{3} \right) \zeta_2 + \frac{z}{z-1} (32H_0 - 384H_1 - 48) \zeta_3 \\ & + \frac{z}{z-1} \left(-192H_{-2} + 192H_0 + 96H_1 + 256H_{0,0} - \frac{104}{3} \right) \zeta_2 \\ & + \frac{z^2}{z-1} \left(-\frac{128H_0}{3} - \frac{128H_1}{3} + \frac{32}{3} \right) \zeta_2 + \frac{z^3}{z-1} \left(-\frac{16H_0}{3} - \frac{16H_1}{3} + \frac{8}{3} \right) \zeta_2 - 96H_{-1,2} \\ & - 16H_{1,0} - 192H_{-1,-1,0} + 48H_{-1,0,0} \Big) + N_f T_F C_F^2 \left(\frac{64}{z-1} \zeta_3 + \left(-\frac{320z}{3(z-1)} \right) \zeta_3 \right. \\ & + \frac{z}{z-1} \left(\left(-\frac{992}{9} \right) H_{0,0} + \left(-\frac{320}{3} \right) H_{0,0,0} + \left(-\frac{640}{9} \right) H_2 + \left(-\frac{640}{9} \right) H_{1,0} + \left(-\frac{128}{3} \right) H_3 \right. \\ & + \left(-\frac{128}{3} \right) H_{2,0} + \frac{104}{3} H_0 + \frac{436}{9} \Big) + \frac{z^2}{z-1} \left(\frac{32}{3} H_0 + \frac{112}{9} \right) + \frac{64z\zeta_2}{3(z-1)} H_0 - \frac{548}{9(z-1)} \Big) \end{aligned} \quad (3.11)$$

$$\begin{aligned}
& + C_A^2 C_F \left(4H_{-1}\zeta_2 z^2 + \left(\left(-\frac{4}{3} \right) H_{-1,0} - 8H_{-1,2} - 8H_{-1,-1,0} \right) z^2 + 36H_{-1}\zeta_2 z \right. \\
& + (8H_2 + 4H_{-1,0} - 72H_{-1,2} - 72H_{-1,-1,0})z + \frac{1}{z-1} \left(\left(-\frac{4}{3} \right) H_0 + \frac{92}{3}H_1 + \frac{1586}{9} \right) \\
& + \frac{88}{3(z-1)}\zeta_3 + \frac{88}{z-1}\zeta_4 + \frac{1}{z} \left(\left(-\frac{4}{3} \right) H_{-1,0} - 8H_{-1,2} - 8H_{-1,-1,0} \right) + \left(-\frac{192z}{z-1} \right) \zeta_4 \\
& + \frac{z}{z-1} \left(\left(-\frac{5396}{27} \right) H_0 + \left(-\frac{1592}{9} \right) H_{0,0} + \left(-\frac{304}{3} \right) H_3 + \left(-\frac{248}{3} \right) H_{0,0,0} + \left(-\frac{184}{3} \right) H_1 \right. \\
& - 32H_4 - 32H_{-3,0} - 24H_{-2,0} - 64H_{-2,2} - 128H_{1,3} - 128H_{-2,-1,0} + 32H_{-2,0,0} - 192H_{1,-2,0} \\
& + 88H_{1,0,0} + 64H_{2,0,0} + 32H_{0,0,0,0} + 96H_{1,0,0,0} + 128H_{1,1,0,0} - \frac{1702}{9} \left. \right) + \left(-\frac{32z^2}{z-1} \right) \zeta_3 \\
& + \frac{z^2}{z-1} \left(\frac{8}{3}H_{0,0} + \frac{92}{3}H_1 + \frac{140}{3}H_0 + 64H_3 + 64H_{-2,0} + \frac{116}{9} \right) + \left(-\frac{4z^3}{z-1} \right) \zeta_3 \\
& + \frac{z^3}{z-1} \left(\frac{4}{3}H_{0,0} + 8H_3 + 8H_{-2,0} \right) + \frac{4\zeta_2}{z} H_{-1} + \left(-\frac{4\zeta_2}{(z-1)z} \right) H_1 + \frac{1}{z-1} \left(-32H_1 - \frac{1000}{9} \right) \zeta_2 \\
& + \frac{z}{z-1} (-128H_0 - 288H_1 + 20) \zeta_3 + \frac{z}{z-1} \left(\frac{464H_0}{3} + 72H_1 + 32H_{1,0} + \frac{1036}{9} \right) \zeta_2 \\
& + \frac{z^2}{z-1} \left(-64H_0 - 32H_1 - \frac{8}{3} \right) \zeta_2 + \frac{z^3}{z-1} \left(-8H_0 - 4H_1 - \frac{4}{3} \right) \zeta_2 + 8H_2 + 36H_{-1}\zeta_2 + 4H_{-1,0} \\
& - 72H_{-1,2} - 72H_{-1,-1,0} \left. \right) + C_A C_F^2 \left(\left(\left(-\frac{8}{3} \right) H_{-1,0,0} + \frac{4}{3}H_{-1,0} + \frac{64}{3}H_{-1,2} + \frac{80}{3}H_{-1,-1,0} \right) z^2 \right. \\
& - 8H_{-1}\zeta_2 z^2 - 72H_{-1}\zeta_2 z + \left(\left(-\frac{44}{3} \right) H_{-1,0} + 192H_{-1,2} + 240H_{-1,-1,0} - 24H_{-1,0,0} \right) z \\
& + \left(-\frac{44}{3} \right) H_{-1,0} + \frac{1}{z-1} \left(\left(-\frac{328}{3} \right) H_1 + \left(-\frac{8}{3} \right) H_{0,0} + \frac{40}{3}H_2 + 16H_0 - 8H_{1,0} - \frac{872}{9} \right) \\
& + \frac{1}{z} \left(\left(-\frac{8}{3} \right) H_{-1,0,0} + \frac{4}{3}H_{-1,0} + \frac{64}{3}H_{-1,2} + \frac{80}{3}H_{-1,-1,0} \right) \\
& + \frac{z}{z-1} \left(\frac{352}{3}H_{2,0} + \frac{656}{3}H_1 + \frac{736}{3}H_{0,0,0} + \frac{2288}{9}H_2 + \frac{2288}{9}H_{1,0} + \frac{1216}{3}H_3 \right. \\
& + \frac{3892}{9}H_{0,0} + 26H_0 + 160H_4 + 32H_{-3,0} + 144H_{-2,0} + 128H_{-2,2} + 256H_{1,3} + 32H_{3,0} + 448H_{-2,-1,0} \\
& - 160H_{-2,0,0} + 512H_{1,-2,0} - 96H_{1,0,0} - 96H_{2,0,0} + 96H_{0,0,0,0} - 160H_{1,0,0,0} - 256H_{1,1,0,0} + \frac{1744}{9} \left. \right) \\
& + \frac{276z}{z-1}\zeta_4 + \frac{z^2}{z-1} \left(\left(-\frac{640}{3} \right) H_{-2,0} + \left(-\frac{512}{3} \right) H_3 + \left(-\frac{428}{3} \right) H_0 \right. \\
& + \left(-\frac{328}{3} \right) H_1 + \left(-\frac{88}{3} \right) H_2 + \frac{64}{3}H_{0,0,0} - 24H_{0,0} - 8H_{1,0} - \frac{872}{9} \left. \right) + \frac{128z^2}{3(z-1)}\zeta_3 \\
& + \frac{z^3}{z-1} \left(\left(-\frac{80}{3} \right) H_{-2,0} + \left(-\frac{64}{3} \right) H_3 + \left(-\frac{4}{3} \right) H_{0,0} + \frac{8}{3}H_{0,0,0} \right) + \frac{16z^3}{3(z-1)}\zeta_3 + \left(-\frac{8\zeta_2}{z} \right) H_{-1} \\
& + \frac{32\zeta_2}{3(z-1)z} H_1 + \frac{1}{z-1} \left(\frac{256H_1}{3} - \frac{80}{3} \right) \zeta_2 + \frac{z}{z-1} \left(240H_0 + 768H_1 - \frac{224}{3} \right) \zeta_3 \\
& + \frac{z}{z-1} \left(96H_{-2} - \frac{1040H_0}{3} - 192H_1 - 64H_2 - 128H_{0,0} - 64H_{1,0} + \frac{76}{3} \right) \zeta_2 \\
& + \frac{z^2}{z-1} \left(\frac{448H_0}{3} + \frac{256H_1}{3} \right) \zeta_2 + \frac{z^3}{z-1} \left(\frac{56H_0}{3} + \frac{32H_1}{3} + \frac{4}{3} \right) \zeta_2 - 72H_{-1}\zeta_2 + 192H_{-1,2} \\
& + 240H_{-1,-1,0} - 24H_{-1,0,0} \left. \right) + N_f T_F C_A C_F \left(\left(-\frac{8}{3} \right) H_1 + \left(-\frac{224}{3(z-1)} \right) \zeta_3 + \frac{320}{9(z-1)}\zeta_2 \right. \\
& + \frac{8z}{3}H_1 + \frac{z}{z-1} \left(\frac{32}{3}H_3 + \frac{64}{3}H_{0,0,0} + \frac{224}{3}H_{0,0} + \frac{2744}{27}H_0 - 32H_{1,0,0} + \frac{952}{27} \right) \\
& + \frac{96z}{z-1}\zeta_3 + \frac{z^2}{z-1} \left(\left(-\frac{8}{3} \right) H_0 + \frac{40}{3} \right) + \frac{z}{z-1} \left(-\frac{64H_0}{3} - \frac{320}{9} \right) \zeta_2 - \frac{1312}{27(z-1)} \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + N_f^2 T_F^2 C_F \left(-\frac{32z}{9} + \frac{z}{z-1} \left(\left(-\frac{320}{27} \right) H_0 + \left(-\frac{64}{9} \right) H_{0,0} \right) + \frac{160}{27} \right) \\
& + \left[\frac{1}{1-z} \right]_+ \left(C_F \left(\left(-\frac{1072}{9} \right) \zeta_2 + 88\zeta_4 + \frac{88\zeta(3)}{3} + \frac{490}{3} \right) C_A^2 \right. \\
& + C_F N_f T_F \left(\frac{320}{9} \zeta_2 - \frac{224\zeta(3)}{3} - \frac{1672}{27} \right) C_A - \frac{64}{27} C_F N_f^2 T_F^2 + C_F^2 N_f T_F \left(-\frac{220}{3} + 64\zeta(3) \right) \Big) \\
& + \delta(1-z) \left(\left((18 - 32\zeta(3))\zeta_2 + 144\zeta_4 - 240\zeta(5) + 68\zeta(3) + \frac{29}{2} \right) C_F^3 \right. \\
& + N_f T_F \left(\frac{40}{3} \zeta_2 + \frac{232}{3} \zeta_4 - \frac{272\zeta(3)}{3} - 46 \right) C_F^2 + N_f^2 T_F^2 \left(\frac{320}{27} \zeta_2 - \frac{64\zeta(3)}{9} - \frac{68}{9} \right) C_F \\
& + C_A^2 \left(\frac{4496}{27} \zeta_2 - 5\zeta_4 + 40\zeta(5) - \frac{1552\zeta(3)}{9} - \frac{1657}{36} \right) C_F \\
& + C_A \left(\left(\left(-\frac{494}{3} \right) \zeta_4 + \left(-\frac{410}{3} + 16\zeta(3) \right) \zeta_2 + 120\zeta(5) + \frac{844\zeta(3)}{3} + \frac{151}{4} \right) C_F^2 \right. \\
& \left. \left. + N_f T_F \left(\left(-\frac{2672}{27} \right) \zeta_2 + 4\zeta_4 + \frac{400\zeta(3)}{9} + 40 \right) C_F \right) \right), \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
\delta P_{\bar{q}q}^{T(1)} &= C_A C_F \left(\frac{z}{z+1} (8H_{0,0} - 16H_{-1,0}) - \frac{2z^2}{z+1} + \left(-\frac{8z}{z+1} \right) \zeta_2 + \frac{2}{z+1} \right) \\
&+ C_F^2 \left(\frac{z}{z+1} (32H_{-1,0} - 16H_{0,0}) + \frac{4z^2}{z+1} + \frac{16z}{z+1} \zeta_2 - \frac{4}{z+1} \right), \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
\delta P_{\bar{q}q}^{T(2)} &= +C_F C_A^2 \left(-4H_1 \zeta_2 z^2 - 8H_{-1,-1,0} z^2 + 36H_1 \zeta_2 z + \left(\frac{68}{3} H_1 + 72H_{-1,-1,0} \right) z + \left(-\frac{68}{3} \right) H_1 \right. \\
&+ \left(-\frac{8}{z} \right) H_{-1,-1,0} + \frac{1}{z+1} \left(\left(-\frac{4}{3} \right) H_0 + \frac{16}{3} H_{-1,0} + 8H_2 + 64H_{-1,2} + \frac{358}{9} \right) \\
&\quad \left. + \frac{1}{z(z+1)} \left(\frac{4}{3} H_{-1,0} - 8H_{-1,2} \right) \right) \\
&+ \frac{z}{z+1} \left(\left(-\frac{2072}{9} \right) H_{-1,0} + \left(-\frac{392}{3} \right) H_3 + \left(-\frac{392}{3} \right) H_{-2,0} + \left(-\frac{176}{3} \right) H_{-1,0,0} + \frac{88}{3} H_0 \right. \\
&+ \frac{248}{3} H_{0,0,0} + \frac{1108}{9} H_{0,0} + \frac{784}{3} H_{-1,2} + 16H_2 - 64H_4 + 32H_{-3,0} + 256H_{-2,2} + 256H_{-1,3} \\
&\quad \left. + 128H_{-2,0,0} - 512H_{-1,-1,2} - 256H_{-1,-1,0,0} + 96H_{-1,0,0,0} - 32H_{0,0,0,0} \right) + \frac{80z}{z+1} \zeta_4 \\
&\quad + \frac{z^2}{z+1} \left(\frac{8}{3} H_{0,0} + \frac{16}{3} H_{-1,0} + \frac{20}{3} H_0 + 8H_2 - 64H_3 - 64H_{-2,0} + 64H_{-1,2} - \frac{358}{9} \right) \\
&\quad + \frac{32z^2}{z+1} \zeta_3 + \left(-\frac{4z^3}{z+1} \right) \zeta_3 + \frac{z^3}{z+1} \left(\left(-\frac{4}{3} \right) H_{0,0} + \frac{4}{3} H_{-1,0} + 8H_3 + 8H_{-2,0} - 8H_{-1,2} \right) \\
&+ \frac{4\zeta_2}{z} H_1 + \frac{4\zeta_2}{z(z+1)} H_{-1} + \frac{1}{z+1} (-32H_{-1} - 8) \zeta_2 + \frac{z}{z+1} (-384H_{-1} + 128H_0 + 124) \zeta_3 \\
&\quad + \frac{z}{z+1} \left(-256H_{-2} - \frac{568H_{-1}}{3} + \frac{232H_0}{3} - 64H_2 + 512H_{-1,-1} - 352H_{-1,0} + 96H_{0,0} \right. \\
&\quad \left. - \frac{1180}{9} \right) \zeta_2 + \frac{z^2}{z+1} \left(-32H_{-1} + 64H_0 - \frac{8}{3} \right) \zeta_2 + \frac{z^3}{z+1} \left(4H_{-1} - 8H_0 + \frac{4}{3} \right) \zeta_2 - 36H_1 \zeta_2 \\
&\quad \left. + 72H_{-1,-1,0} \right) \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
& C_F^3 \left(\left(-\frac{184}{3} \right) H_1 + \left(-\frac{32}{3z} \right) H_{-1,-1,0} + \left(-\frac{32z^2}{3} \right) H_{-1,-1,0} \right. \\
& + \frac{1}{z+1} \left(\left(-\frac{80}{3} \right) H_0 + \frac{16}{3} H_{0,0} + \frac{112}{3} H_2 + \frac{128}{3} H_{-1,0,0} + \frac{512}{3} H_{-1,2} + \frac{104}{3} \right) \\
& + \frac{1}{z(z+1)} \left(\left(-\frac{64}{3} \right) H_{-1,2} + \left(-\frac{16}{3} \right) H_{-1,0,0} + \left(-\frac{8}{3} \right) H_{-1,0} \right) + \left(-\frac{280z}{z+1} \right) \zeta_4 \\
& + \frac{z}{z+1} \left(\left(-\frac{40}{3} \right) H_{0,0} + \frac{16}{3} H_{-1,0} + \frac{128}{3} H_2 + 80H_0 - 288H_3 - 320H_{-3,0} + 640H_{-2,2} \right. \\
& + 576H_{-1,2} + 512H_{-1,3} + 64H_{3,0} + 256H_{-2,-1,0} - 64H_{-2,0,0} + 256H_{-1,-2,0} - 1536H_{-1,-1,2} \\
& + 384H_{-1,0,0} - 128H_{-1,2,0} - 192H_{0,0,0} - 256H_{-1,-1,0,0} - 448H_{-1,0,0,0} + 320H_{0,0,0,0} \left. \right) \\
& + \frac{z^2}{z+1} \left(\left(-\frac{512}{3} \right) H_3 + \left(-\frac{256}{3} \right) H_{-2,0} + \left(-\frac{128}{3} \right) H_{0,0,0} + \frac{16}{3} H_2 \right. \\
& + \frac{128}{3} H_{-1,0,0} + \frac{176}{3} H_0 + \frac{512}{3} H_{-1,2} - 16H_{0,0} - \frac{104}{3} \left. \right) + \frac{512z^2}{3(z+1)} \zeta_3 \\
& + \left(-\frac{64z^3}{3(z+1)} \right) \zeta_3 + \frac{z^3}{z+1} \left(\left(-\frac{64}{3} \right) H_{-1,2} + \left(-\frac{16}{3} \right) H_{-1,0,0} + \left(-\frac{8}{3} \right) H_{-1,0} \right. \\
& + \frac{8}{3} H_{0,0} + \frac{16}{3} H_{0,0,0} + \frac{32}{3} H_{-2,0} + \frac{64}{3} H_3 \left. \right) + \frac{32\zeta_2}{3z} H_1 + \left(-\frac{32}{3} z^2 \zeta_2 \right) H_1 \\
& + \frac{16\zeta_2}{z(z+1)} H_{-1} + \frac{1}{z+1} \left(-128H_{-1} - \frac{32}{3} \right) \zeta_2 + \frac{z}{z+1} (-1152H_{-1} - 32H_0 + 336) \zeta_3 \\
& + \frac{z}{z+1} (-512H_{-2} - 480H_{-1} + 384H_0 - 192H_2 + 1536H_{-1,-1} - 768H_{-1,0} - 64H_{0,0} - 40) \zeta_2 \\
& + \frac{z^2}{z+1} \left(-128H_{-1} + \frac{640H_0}{3} - 32 \right) \zeta_2 + \frac{z^3}{z+1} \left(16H_{-1} - \frac{80H_0}{3} - \frac{8}{3} \right) \zeta_2 \\
& + 96zH_1\zeta_2 - 96H_1\zeta_2 + 16H_{1,0} + 96H_{-1,-1,0} + z \left(\frac{184}{3} H_1 - 16H_{1,0} + 96H_{-1,-1,0} \right) \left. \right) \\
& + N_f T_F C_F^2 \left(-\frac{112z^2}{9(z+1)} + \left(-\frac{32}{3} \right) H_1 + \frac{32z}{3} H_1 + \frac{z}{z+1} \left(\left(-\frac{1280}{9} \right) H_{-1,0} + \left(-\frac{128}{3} \right) H_3 \right. \right. \\
& + \left(-\frac{128}{3} \right) H_{-2,0} + \left(-\frac{128}{3} \right) H_{-1,0,0} + \frac{128}{3} H_{0,0,0} + \frac{640}{9} H_{0,0} + \frac{256}{3} H_{-1,2} \left. \right) \\
& + \frac{64z}{z+1} \zeta_3 + \frac{z}{z+1} \left(-\frac{256H_{-1}}{3} + \frac{64H_0}{3} - \frac{640}{9} \right) \zeta_2 + \frac{112}{9(z+1)} \left. \right) \\
& + N_f T_F C_F C_A \left(\frac{56z^2}{9(z+1)} + \frac{16}{3} H_1 + \left(-\frac{16z}{3} \right) H_1 + \left(-\frac{32z}{z+1} \right) \zeta_3 \right. \\
& + \frac{z}{z+1} \left(\left(-\frac{128}{3} \right) H_{-1,2} + \left(-\frac{320}{9} \right) H_{0,0} + \left(-\frac{64}{3} \right) H_{0,0,0} + \frac{64}{3} H_3 + \frac{64}{3} H_{-2,0} \right. \\
& + \frac{64}{3} H_{-1,0,0} + \frac{640}{9} H_{-1,0} \left. \right) + \frac{z}{z+1} \left(\frac{128H_{-1}}{3} - \frac{32H_0}{3} + \frac{320}{9} \right) \zeta_2 - \frac{56}{9(z+1)} \left. \right), \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
& +C_A C_F^2 \left(\frac{64}{3z} H_{-1,-1,0} + \frac{64z^2}{3} H_{-1,-1,0} + \frac{1}{z+1} \left(\left(-\frac{640}{3} \right) H_{-1,2} + \left(-\frac{104}{3} \right) H_2 \right. \right. \\
& + \left(-\frac{64}{3} \right) H_{-1,0,0} + \left(-\frac{32}{3} \right) H_{-1,0} + \left(-\frac{8}{3} \right) H_{0,0} + 16H_0 - \frac{872}{9} \Big) \\
& + \frac{1}{z(z+1)} \left(\left(-\frac{4}{3} \right) H_{-1,0} + \frac{8}{3} H_{-1,0,0} + \frac{80}{3} H_{-1,2} \right) + \left(-\frac{20z}{z+1} \right) \zeta_4 \\
& + \frac{z}{z+1} \left(\left(-\frac{2432}{3} \right) H_{-1,2} + \left(-\frac{2156}{9} \right) H_{0,0} + \left(-\frac{296}{3} \right) H_0 + \left(-\frac{224}{3} \right) H_{-1,0,0} \right. \\
& + \left(-\frac{208}{3} \right) H_{0,0,0} + \left(-\frac{160}{3} \right) H_2 + \frac{784}{3} H_{-2,0} + \frac{1216}{3} H_3 + \frac{4120}{9} H_{-1,0} + 128H_4 \\
& + 96H_{-3,0} - 832H_{-2,2} - 768H_{-1,3} - 32H_{3,0} - 128H_{-2,-1,0} - 224H_{-2,0,0} \\
& \left. - 128H_{-1,-2,0} + 1792H_{-1,-1,2} + 64H_{-1,2,0} + 640H_{-1,-1,0,0} + 32H_{-1,0,0,0} - 96H_{0,0,0,0} \right) \\
& + \left(-\frac{448z^2}{3(z+1)} \right) \zeta_3 + \frac{z^2}{z+1} \left(\left(-\frac{640}{3} \right) H_{-1,2} + \left(-\frac{128}{3} \right) H_0 + \left(-\frac{64}{3} \right) H_{-1,0,0} \right. \\
& + \left(-\frac{56}{3} \right) H_2 + \left(-\frac{32}{3} \right) H_{-1,0} + \frac{8}{3} H_{0,0} + \frac{64}{3} H_{0,0,0} + \frac{512}{3} H_{-2,0} + \frac{640}{3} H_3 + \frac{872}{9} \Big) \\
& + \frac{z^3}{z+1} \left(\left(-\frac{80}{3} \right) H_3 + \left(-\frac{64}{3} \right) H_{-2,0} + \left(-\frac{8}{3} \right) H_{0,0,0} + \left(-\frac{4}{3} \right) H_{-1,0} + \frac{4}{3} H_{0,0} \right. \\
& + \frac{8}{3} H_{-1,0,0} + \frac{80}{3} H_{-1,2} \Big) + \frac{56z^3}{3(z+1)} \zeta_3 + \left(-\frac{40\zeta_2}{3z} \right) H_1 + \frac{40z^2\zeta_2}{3} H_1 + \left(-\frac{16\zeta_2}{z(z+1)} \right) H_{-1} \\
& + \frac{1}{z+1} \left(128H_{-1} + \frac{64}{3} \right) \zeta_2 + \frac{z}{z+1} (1344H_{-1} - 240H_0 - 416) \zeta_3 + \frac{z}{z+1} \left(768H_{-2} \right. \\
& + \frac{1856H_{-1}}{3} - \frac{1040H_0}{3} + 224H_2 - 1792H_{-1,-1} + 1088H_{-1,0} - 160H_{0,0} + \frac{2540}{9} \Big) \zeta_2 \\
& + \frac{z^2}{z+1} \left(128H_{-1} - \frac{704H_0}{3} + \frac{64}{3} \right) \zeta_2 + \frac{z^3}{z+1} \left(-16H_{-1} + \frac{88H_0}{3} - \frac{4}{3} \right) \zeta_2 + 76H_1 \\
& \left. - 120zH_1\zeta_2 + 120H_1\zeta_2 - 8H_{1,0} + z(-76H_1 + 8H_{1,0} - 192H_{-1,-1,0}) - 192H_{-1,-1,0} \right) \tag{3.17}
\end{aligned}$$

4 N³LO coefficients for transversity TMDs in $\overline{\text{MS}}$

The analytic expressions for the coefficient functions will be provided in the ancillary files along with the arXiv submission. In this section we study the asymptotic behaviors and present their numerical fits.

4.1 Small- x and threshold limit

The coefficient functions develop end-point divergences both in the threshold and high energy limit. We first present here the results for leading threshold limit. The results for high energy limit will be discussed in next section. In the $z \rightarrow 1$ limit, we have exactly the same results as the unpolarized case [58, 95]

$$\lim_{z \rightarrow 1} \delta \mathcal{I}_{ij}^{(2)}(z) = \lim_{z \rightarrow 1} \delta \mathcal{C}_{ji}^{(2)}(z) = \frac{2\gamma_{1,i}^R}{(1-z)_+} \delta_{ij}, \quad \lim_{z \rightarrow 1} \delta \mathcal{I}_{ij}^{(3)}(z) = \lim_{z \rightarrow 1} \delta \mathcal{C}_{ji}^{(3)}(z) = \frac{2\gamma_{2,i}^R}{(1-z)_+} \delta_{ij}, \tag{4.1}$$

where $\gamma_{1(2)}^R$ are the two(three)-loop rapidity anomalous dimensions [88, 96]. The relation between threshold limit and rapidity anomalous dimension has been anticipated in [97–99]. The explicit expressions up to three-loop read [95]

$$\begin{aligned}
\delta\mathcal{I}_{qq}^{(1)}(z) &= \delta\mathcal{C}_{qq}^{(1)}(z) = 0, \\
\delta\mathcal{I}_{qq}^{(2)}(z) &= \delta\mathcal{C}_{qq}^{(2)}(z) = \frac{1}{(1-z)_+} \left[\left(28\zeta_3 - \frac{808}{27} \right) C_A C_F + \frac{224}{27} C_F N_f T_F \right], \\
\delta\mathcal{I}_{qq}^{(3)}(z) &= \delta\mathcal{C}_{qq}^{(3)}(z) = \frac{1}{(1-z)_+} \left[\left(-\frac{1648\zeta_2}{81} - \frac{1808\zeta_3}{27} + \frac{40\zeta_4}{3} + \frac{125252}{729} \right) C_A C_F N_f T_F \right. \\
&\quad + \left(-\frac{176}{3}\zeta_3\zeta_2 + \frac{6392\zeta_2}{81} + \frac{12328\zeta_3}{27} + \frac{154\zeta_4}{3} - 192\zeta_5 - \frac{297029}{729} \right) C_A^2 C_F \\
&\quad \left. + \left(-\frac{608\zeta_3}{9} - 32\zeta_4 + \frac{3422}{27} \right) C_F^2 N_f T_F + \left(-\frac{128}{9}\zeta_3 - \frac{7424}{729} \right) C_F N_f^2 T_F^2 \right]. \quad (4.2)
\end{aligned}$$

It is also instructive to investigate the small- x behavior of the transversity coefficient functions and to compare them with the corresponding unpolarized case. Unlike the unpolarized coefficient functions, which exhibit a power-like divergence $\sim 1/x$ in the small- x limit, the transversity coefficient functions diverge only logarithmically. For clarity, we summarize below the small- x expansions of the coefficient functions. In particular, we focus on the scale-independent functions that appear in the RG solutions, cf. Eqs.(2.18) and (2.20).

4.1.1 Small- x expansion of TMD transversity PDFs

$$\delta I_{qq}^{(1)}(x) = 0, \quad (4.3)$$

$$\delta I_{qq}^{(2)}(x) = \frac{14C_A C_F}{3} - \frac{4}{3} C_F N_f T_F - 2C_F^2, \quad (4.4)$$

$$\begin{aligned}
\delta I_{qq}^{(3)}(x) &= C_A C_F N_f T_F \left(\left(-\frac{8}{9} \right) \ln^2 x + \left(-\frac{200}{27} \right) \ln x + \frac{16}{9} \zeta_2 - \frac{2672}{81} \right) \\
&\quad + C_A C_F^2 \left(\left(-\frac{16}{3} \right) \ln^2 x + \left(-\frac{148}{9} \right) \ln x - 16\zeta_2 + 24\zeta_3 - \frac{12616}{81} \right) \\
&\quad + C_A^2 C_F \left(\frac{22}{9} \ln^2 x + \frac{296}{27} \ln x + \left(-\frac{44}{3} \right) \zeta_3 + \frac{4}{9} \zeta_2 + \frac{2384}{27} \right) \\
&\quad + C_F^2 N_f T_F \left(\frac{16}{9} \ln^2 x + \frac{64}{9} \ln x + \frac{2600}{81} \right) + C_F N_f^2 T_F^2 \left(\frac{32}{27} \ln x + \frac{160}{81} \right) \\
&\quad + C_F^3 \left(\frac{8}{9} \ln^2 x + \left(-\frac{40}{9} \right) \ln x + \frac{32}{3} \zeta_2 + \frac{32}{3} \zeta_3 + \frac{464}{9} \right), \quad (4.5)
\end{aligned}$$

$$\delta I_{q\bar{q}}^{(1)}(x) = 0, \quad (4.6)$$

$$\delta I_{q\bar{q}}^{(2)}(x) = 2C_F^2 - C_A C_F, \quad (4.7)$$

$$\begin{aligned} \delta I_{q\bar{q}}^{(3)}(x) = & C_A C_F N_f T_F \left(\frac{8}{9} \ln^2 x + \frac{8}{9} \ln x + \left(-\frac{16}{9} \right) \zeta_2 + \frac{1192}{81} \right) \\ & + C_A C_F^2 \left(\frac{16}{3} \ln^2 x + \frac{16}{9} \ln x + \left(-\frac{232}{9} \right) \zeta_2 - 24\zeta_3 + \frac{11428}{81} \right) \\ & + C_A^2 C_F \left(\left(-\frac{22}{9} \right) \ln^2 x - 2 \ln x + \frac{92}{9} \zeta_2 + \frac{44}{3} \zeta_3 - \frac{4670}{81} \right) \\ & + C_F^2 N_f T_F \left(\left(-\frac{16}{9} \right) \ln^2 x + \left(-\frac{16}{9} \right) \ln x + \frac{32}{9} \zeta_2 - \frac{2384}{81} \right) \\ & + C_F^3 \left(\left(-\frac{8}{9} \right) \ln^2 x + \frac{40}{9} \ln x + \left(-\frac{32}{3} \right) \zeta_3 + \frac{32}{3} \zeta_2 - \frac{464}{9} \right), \end{aligned} \quad (4.8)$$

4.1.2 Small- z expansion of TMD transversity FFs

$$\delta C_{qq}^{(1)}(z) = 0, \quad (4.9)$$

$$\delta C_{qq}^{(2)}(z) = C_A C_F \left(\frac{14}{3} - 2 \ln z \right) - \frac{4}{3} C_F N_f T_F + C_F^2 (4 \ln z - 2), \quad (4.10)$$

$$\begin{aligned} \delta C_{qq}^{(3)}(z) = & C_A C_F N_f T_F \left(\left(-\frac{32}{9} \right) \ln^2 z + \frac{40}{27} \ln z + \frac{16}{9} \zeta_2 - \frac{2672}{81} \right) \\ & + C_A C_F^2 \left(\left(\left(-\frac{56}{3} \right) \zeta_2 + \frac{310}{3} \right) \ln z + \left(-\frac{260}{9} \right) \ln^2 z - 116\zeta_2 - \frac{11932}{81} \right) \\ & + C_A^2 C_F \left(\left(8\zeta_2 - \frac{838}{27} \right) \ln z + \frac{88}{9} \ln^2 z + \frac{172}{9} \zeta_2 + \frac{2348}{27} \right) \\ & + C_F^2 N_f T_F \left(\frac{64}{9} \ln^2 z - 16 \ln z + \frac{32}{3} \zeta_2 + \frac{2600}{81} \right) + C_F N_f^2 T_F^2 \left(\frac{32}{27} \ln z + \frac{160}{81} \right) \\ & + C_F^3 \left(\left(\frac{16}{3} \zeta_2 - \frac{140}{3} \right) \ln z + \frac{56}{3} \ln^2 z + \frac{232}{3} \zeta_2 + 40 \right), \end{aligned} \quad (4.11)$$

$$\delta C_{\bar{q}q}^{(1)}(z) = 0, \quad (4.12)$$

$$\delta C_{\bar{q}q}^{(2)}(z) = C_A C_F (2 \ln z - 1) + C_F^2 (2 - 4 \ln z), \quad (4.13)$$

$$\delta C_{\bar{q}q}^{(3)}(z) = C_A C_F N_f T_F \left(\frac{32}{9} \ln^2 z - 8 \ln z + \left(-\frac{64}{9} \right) \zeta_2 + \frac{1192}{81} \right)$$

$$\begin{aligned}
& +C_A C_F^2 \left(\left(\left(-\frac{88}{3} \right) \zeta_2 - \frac{310}{3} \right) \ln z + \frac{260}{9} \ln^2 z + \frac{32}{9} \zeta_2 + \frac{10744}{81} \right) \\
& +C_A^2 C_F \left((8\zeta_2 + 40) \ln z + \left(-\frac{88}{9} \right) \ln^2 z + \frac{56}{9} \zeta_2 - \frac{4562}{81} \right) \\
& +C_F^2 N_f T_F \left(\left(-\frac{64}{9} \right) \ln^2 z + 16 \ln z + \frac{128}{9} \zeta_2 - \frac{2384}{81} \right) \\
& +C_F^3 \left(\left(\frac{80}{3} \zeta_2 + \frac{140}{3} \right) \ln z + \left(-\frac{56}{3} \right) \ln^2 z - 32\zeta_2 - 40 \right), \tag{4.14}
\end{aligned}$$

4.2 Numerical fits of the N³LO coefficients

The NLO and NNLO space-like coefficient functions were obtained in Refs.[62, 63, 100, 101]. The present work is devoted to extending these results to higher orders, and for the first time we provide the complete expressions up to N³LO. By comparing our results for the regulator-independent TMD PDFs and TMD FFs with those reported in Ref.[63], we confirm agreement in the $\bar{q} \rightarrow q$ (and $q \rightarrow \bar{q}$) channel. For the $q \rightarrow q$ channel, however, we observe minor discrepancies

$$\begin{aligned}
\delta h_{T,qq}^{(2)} | - \delta h_{T,qq}^{(2)} |_{\text{DIA}} &= -6C_F^2(1-x), \\
\delta g_{T,qq}^{(2)} | - \delta g_{T,qq}^{(2)} |_{\text{DIA}} &= -6C_F^2(1-z) - \frac{2}{3}\pi^4 C_A C_F \delta(1-z). \tag{4.15}
\end{aligned}$$

The analytic expressions of N³LO coefficient functions contain harmonic polylogarithms up to transcendental weight 5. To facilitate straightforward numerical implementation, we provide a numerical fitting to all the coefficient functions. Following Ref. [102], we use the following elementary functions to fit the results,

$$L_x \equiv \ln x, \quad L_{\bar{x}} \equiv \ln(1-x), \quad \bar{x} \equiv 1-x. \tag{4.16}$$

For NNLO and N³LO coefficient functions, we fit the exact results in the region $10^{-6} < x < 1 - 10^{-6}$ (Numerical evaluation of HPLs are made with the HPL package [103]), and we have set the color factor to their numerical values in QCD, i.e.

$$C_F = \frac{4}{3}, \quad C_A = 3, \quad T_F = \frac{1}{2}. \tag{4.17}$$

In more detail, we subtract the $x \rightarrow 0$ and $x \rightarrow 1$ limits and then fit the remaining terms in the region $10^{-6} < x < 1 - 10^{-6}$. Combining the two parts, the fitted results can achieve an accuracy better than 10^{-3} for $0 < x < 1$. We show below the numerical fitting with six significant digits. The full numerical fitting is attached as ancillary files with the arXiv submission.

4.2.1 Numerical fit for TMD PDFs

$$\delta I_{qq}^{(1)}(x) = 0, \tag{4.18}$$

$$\begin{aligned}
\delta I_{qq}^{(2)}(x) = N_f & \left(5.53086 \left[\frac{1}{\bar{x}} \right]_+ + x^3 (0.159353L_x^2 + 1.83868L_x - 3.46249) \right. \\
& + x^2 (0.918768L_x^2 + 3.30005L_x + 1.07696) + x (0.888889L_x^2 + 2.96296L_x + 2.37017) \\
& \left. - 0.0100775x^6 + 0.090776x^5 - 0.657935x^4 + 1.48148\bar{x} - 7.90123 \right) \\
& + 14.9267 \left[\frac{1}{\bar{x}} \right]_+ + x^3 (2.43688L_x^3 - 17.6078L_x^2 + 43.7086L_x - 39.0133) \\
& + x^2 (-2.54756L_x^3 - 13.8299L_x^2 - 7.50098L_x + 63.2888) \\
& + x (-2.66667L_x^3 - 9.33333L_x^2 - 39.1111L_x - 16.4544) + \bar{x}^3 (0.508876L_{\bar{x}}^2 - 0.601064L_{\bar{x}}) \\
& + \bar{x}^2 (1.19128L_{\bar{x}}^2 - 2.40075L_{\bar{x}}) - 7.11111L_{\bar{x}}^2 + 22.2222L_{\bar{x}} \\
& + \bar{x} (3.55556L_{\bar{x}}^2 - 13.3333L_{\bar{x}} + 10.0354) + 0.0762313x^6 - 0.683931x^5 + 3.19561x^4 - 9.851,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\delta I_{q\bar{q}}^{(3)}(x) = N_f & \left(154.257 \left[\frac{1}{\bar{x}} \right]_+ + x^3 (-10.139L_x^4 + 53.293L_x^3 - 362.551L_x^2 + 1069.36L_x - 2080.42) \right. \\
& + x^2 (3.35283L_x^4 + 38.0541L_x^3 + 204.028L_x^2 + 893.7L_x + 1807.17) \\
& + x (3.16049L_x^4 + 27.0398L_x^3 + 95.569L_x^2 + 243.104L_x + 167.326) - 0.197531L_x^2 \\
& - 8.49383L_x + \bar{x}^3 (0.0903308L_{\bar{x}}^3 - 0.534541L_{\bar{x}}^2 + 1.48298L_{\bar{x}}) \\
& + \bar{x}^2 (-0.344655L_{\bar{x}}^3 - 1.88691L_{\bar{x}}^2 - 4.7445L_{\bar{x}}) + 2.107L_{\bar{x}}^3 + 10.3704L_{\bar{x}}^2 - 10.4458L_{\bar{x}} \\
& + \bar{x} (-1.0535L_{\bar{x}}^3 - 11.358L_{\bar{x}}^2 + 3.08951L_{\bar{x}} + 132.) - 0.0981767x^6 - 0.236655x^5 + 65.9262x^4 \\
& - 317.852) + N_f \left(-9.09324 \left[\frac{1}{\bar{x}} \right]_+ + x^3 (0.431032L_x^3 - 3.50262L_x^2 + 9.32163L_x - 8.91467) \right. \\
& + x^2 (-0.62563L_x^3 - 2.59739L_x^2 - 1.44432L_x + 15.2779) \\
& + x (-0.658436L_x^3 - 3.29218L_x^2 - 7.2428L_x - 6.71585) + 0.395062L_x + 0.00672043x^6 \\
& \left. - 0.0760913x^5 + 0.55363x^4 - 6.05761\bar{x} + 15.8093 \right) + 140.136 \left[\frac{1}{\bar{x}} \right]_+ \\
& + x^3 (32.8538L_x^5 - 59.8157L_x^4 + 1263.35L_x^3 - 1593.34L_x^2 + 9101.99L_x + 965.588) \\
& + x^2 (-3.03843L_x^5 - 67.7124L_x^4 - 364.242L_x^3 - 1888.44L_x^2 - 2473.02L_x + 836.39) \\
& + x (-2.64691L_x^5 - 37.1111L_x^4 - 137.094L_x^3 - 453.207L_x^2 - 173.141L_x + 1583.11) \\
& + 3.25926L_x^2 + 32.5926L_x + \bar{x}^3 (20.7632L_{\bar{x}}^3 - 16.7047L_{\bar{x}}^2 + 301.823L_{\bar{x}}) \\
& + \bar{x}^2 (6.41987L_{\bar{x}}^3 + 1.21958L_{\bar{x}}^2 + 202.422L_{\bar{x}}) - 34.7654L_{\bar{x}}^3 - 5.09037L_{\bar{x}}^2 \\
& + 637.843L_{\bar{x}} + \bar{x} (-7.90123L_{\bar{x}}^3 + 110.483L_{\bar{x}}^2 - 62.9396L_{\bar{x}} - 863.968) \\
& - 8.32785x^6 + 84.9876x^5 - 3254.01x^4 + 948.568,
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\delta I_{q\bar{q}}^{(2)}(x) = x & (2.14474 - 0.296296L_x^3) + x^3 (15.5499L_x^3 - 48.8464L_x^2 + 242.942L_x - 320.161) \\
& + x^2 (1.20874L_x^3 + 19.8823L_x^2 + 132.773L_x + 344.416) \\
& + 0.243961x^6 - 1.09739x^5 - 25.5468x^4 - 0.444444\bar{x},
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\delta I_{q\bar{q}}^{(3)}(x) = N_f & \left(x^3 (-86.9687L_x^4 + 339.693L_x^3 - 3577.27L_x^2 + 9245.24L_x - 18136.9) \right. \\
& + x^2 (1.88363L_x^4 + 69.5867L_x^3 + 974.23L_x^2 + 6391.29L_x + 16891.9) \\
& + x (0.263374L_x^4 + 1.09739L_x^3 + 5.25509L_x^2 + 2.52335L_x - 14.4093) + 0.197531L_x^2 \\
& + 0.197531L_x + \bar{x}^3 (0.672455L_{\bar{x}}^2 + 0.57305L_{\bar{x}}) + \bar{x}^2 (-0.0270463L_{\bar{x}}^2 - 1.49294L_{\bar{x}}) \\
& + \bar{x} (0.197531L_{\bar{x}}^2 + 1.0535L_{\bar{x}} + 2.87517) + 1.23754x^6 - 39.0649x^5 + 1297.47x^4 - 0.254789) \\
& + x^3 (218.515L_x^5 + 99.2151L_x^4 + 9703.19L_x^3 - 1433.99L_x^2 + 66557.2L_x + 18184.5) \\
& + x^2 (-1.98473L_x^5 - 69.4832L_x^4 - 1029.42L_x^3 - 6944.15L_x^2 - 15843.L_x + 16446.3) \\
& + x (-0.118519L_x^5 - 3.53086L_x^4 - 22.2993L_x^3 - 32.9084L_x^2 + 32.298L_x + 174.427) \\
& - 3.25926L_x^2 - 3.25926L_x + \bar{x}^3 (1.23696L_{\bar{x}}^2 + 9.1761L_{\bar{x}}) + \bar{x}^2 (13.7472L_{\bar{x}} - 0.884454L_{\bar{x}}^2) \\
& + \bar{x} (-3.13186L_{\bar{x}}^2 - 19.7773L_{\bar{x}} - 47.6421) - 56.825x^6 + 1077.22x^5 - 35852.2x^4 + 26.6075, \\
& \left. \right) \tag{4.22}
\end{aligned}$$

4.2.2 Numerical fit for TMD FFs

$$\begin{aligned}
\delta C_{qq}^{(1)}(z) = z^3 & (6.88266L_z - 7.0639) + z^2 (10.467L_z - 0.842818) + z (10.6667L_z + 5.3312) \\
& - 0.128696z^6 + 0.818297z^5 - 3.44742z^4 + 5.33333\bar{z} - 5.33333, \tag{4.23}
\end{aligned}$$

$$\begin{aligned}
\delta C_{q\bar{q}}^{(2)}(z) = N_f & \left(5.53086 \left[\frac{1}{\bar{z}} \right]_+ + z^3 (1.70397L_z^2 - 4.64799L_z + 10.359) \right. \\
& + z^2 (0.881779L_z^2 - 8.99257L_z - 0.401235) + z (0.888889L_z^2 - 8.88889L_z - 3.55556) \\
& \left. - 0.00933333z^6 + 0.104686z^5 - 1.16427z^4 - 4.44444\bar{z} - 1.97531 \right) \\
& + 14.9267 \left[\frac{1}{\bar{z}} \right]_+ + z^3 (-9.54698L_z^3 - 53.3203L_z^2 + 15.1648L_z - 156.471) \\
& + z^2 (-22.9556L_z^3 - 10.3136L_z^2 - 68.1265L_z + 154.308) \\
& + z (-23.4074L_z^3 + 12.L_z^2 - 17.7527L_z - 99.8513) - 0.888889L_z \\
& + \bar{z}^3 (-0.799338L_{\bar{z}}^2 - 1.59409L_{\bar{z}}) + \bar{z}^2 (2.65626L_{\bar{z}} - 1.16785L_{\bar{z}}^2) + 7.11111L_{\bar{z}}^2 - 22.2222L_{\bar{z}} \\
& + \bar{z} (-3.55556L_{\bar{z}}^2 + 51.5556L_{\bar{z}} + 149.849) + 0.905556z^6 - 8.00011z^5 + 91.7047z^4 - 149.664, \\
& \left. \right) \tag{4.24}
\end{aligned}$$

$$\begin{aligned}
\delta C_{qq}^{(3)}(z) = & N_f \left(154.257 \left[\frac{1}{\bar{z}} \right]_+ + z^3 (-19.7868L_z^4 + 33.0601L_z^3 - 442.868L_z^2 + 663.837L_z - 1896.52) \right. \\
& + z^2 (-1.97195L_z^4 + 32.7379L_z^3 + 194.866L_z^2 + 441.17L_z + 1895.42) \\
& + z (-2.13992L_z^4 + 17.8217L_z^3 + 86.4094L_z^2 - 297.674L_z - 60.271) - 0.790123L_z^2 \\
& - 11.2593L_z + \bar{z}^3 (-0.141494L_z^3 + 2.07513L_z^2 - 9.48914L_z) \\
& + \bar{z}^2 (0.342792L_z^3 + 1.84641L_z^2 - 25.477L_z) - 2.107L_z^3 - 10.3704L_z^2 + 13.8356L_z \\
& + \bar{z} (1.0535L_z^3 - 7.30864L_z^2 - 71.9318L_z - 212.901) + 0.577185z^6 - 9.72231z^5 + 302.016z^4 \\
& \left. + 42.6462 \right) + N_f^2 \left(-9.09324 \left[\frac{1}{\bar{z}} \right]_+ + z^3 (1.13739L_z^3 - 1.11292L_z^2 + 8.49447L_z + 0.132844) \right. \\
& + z^2 (0.911883L_z^3 - 3.47057L_z^2 + 5.15991L_z - 2.6942) \\
& + z (0.921811L_z^3 - 3.29218L_z^2 + 5.92593L_z - 0.131794) + 0.395062L_z - 0.0139459z^6 \\
& \left. + 0.212281z^5 - 3.95786z^4 + 0.526749\bar{z} + 9.22493 \right) + 140.136 \left[\frac{1}{\bar{z}} \right]_+ \\
& + z^3 (47.894L_z^5 + 302.009L_z^4 + 1727.78L_z^3 + 2836.22L_z^2 + 18851.7L_z - 138.226) \\
& + z^2 (13.272L_z^5 + 26.3528L_z^4 - 155.496L_z^3 - 1693.73L_z^2 + 10289.1L_z - 417.803) \\
& + z (13.3136L_z^5 - 22.963L_z^4 - 302.296L_z^3 + 602.58L_z^2 + 5948.2L_z + 9896.07) \\
& + 7.11111L_z^2 + 84.0824L_z + \bar{z}^3 (29.1314L_z^3 - 45.6048L_z^2 + 386.789L_z) \\
& + \bar{z}^2 (-4.9028L_z^3 - 96.5395L_z^2 + 243.669L_z) + 34.7654L_z^3 + 5.09037L_z^2 \\
& - 1826.42L_z + \bar{z} (-42.6667L_z^3 + 133.837L_z^2 + 2290.36L_z + 473.618) \\
& - 23.3915z^6 + 335.792z^5 - 11339.4z^4 - 583.13,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\delta C_{qq}^{(2)}(z) = & z^3 (48.286L_z^3 - 13.4154L_z^2 + 671.421L_z - 392.838) \\
& + z^2 (-1.24024L_z^3 + 25.0531L_z^2 + 243.89L_z + 653.385) \\
& + z (2.66667L_z^3 - 0.888889L_z + 6.41883) + 0.888889L_z \\
& - 1.49899z^6 + 20.8743z^5 - 286.341z^4 - 0.444444\bar{z},
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\delta C_{qq}^{(3)}(z) = & N_f (z^3 (37.0364L_z^4 - 346.932L_z^3 + 1818.28L_z^2 - 7637.09L_z + 11385.8) \\
& + z^2 (-3.3539L_z^4 - 52.2866L_z^3 - 733.542L_z^2 - 4732.76L_z - 11873.) \\
& + z (1.54733L_z^4 - 5.48697L_z^3 - 3.1357L_z^2 - 2.86412L_z - 25.5883) + 0.790123L_z^2 \\
& - 1.77778L_z + \bar{z}^3 (-2.41513L_z^2 - 2.27567L_z) + \bar{z}^2 (0.096473L_z^2 + 3.48785L_z) \\
& + \bar{z} (0.197531L_z^2 + 1.0535L_z + 0.92562) + 8.42153z^6 - 80.4256z^5 + 585.108z^4 - 0.254789) \\
& + z^3 (-1978.47L_z^5 + 2860.25L_z^4 - 82483.7L_z^3 + 98475.2L_z^2 - 514733.L_z - 98159.7) \\
& + z^2 (37.7554L_z^5 + 1076.04L_z^4 + 14126.6L_z^3 + 94540.7L_z^2 + 239007.L_z - 89211.8) \\
& + z (-8.17778L_z^5 - 7.95062L_z^4 + 244.41L_z^3 + 90.8704L_z^2 - 547.94L_z - 606.336) \\
& - 7.11111L_z^2 + 42.9687L_z + \bar{z}^3 (53.1005L_z^2 + 66.5415L_z) + \bar{z}^2 (-1.30965L_z^2 - 58.0375L_z) \\
& + \bar{z} (-3.13186L_z^2 - 19.7773L_z - 16.7318) - 59.6703z^6 - 2084.85z^5 + 190130.z^4 - 8.07101,
\end{aligned} \tag{4.27}$$

5 Conclusion

In this work we presented the first N³LO twist-2 matching coefficients for TMD quark transversity and extracted the complete NNLO DGLAP splitting functions in both space-like and time-like regimes. We verified that our NLO space-like results are consistent with the literature and rederived the NNLO TMD transversity PDFs and FFs, noting minor discrepancies compared to the work of Ref. [63]. These results place transversity on the same theoretical footing as unpolarized and helicity sectors and provide key inputs for precision study of SIDIS azimuthal asymmetries at the forthcoming EIC.

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A QCD Beta Function

The QCD beta function is defined as

$$\frac{d\alpha_s}{d\ln\mu} = \beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \beta_n, \quad (\text{A.1})$$

with [104]

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F N_f, \\ \beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F N_f - 4C_F T_F N_f, \\ \beta_2 &= \left(\frac{158C_A}{27} + \frac{44C_F}{9}\right) N_f^2 T_F^2 + \left(-\frac{205C_A C_F}{9} - \frac{1415C_A^2}{27} + 2C_F^2\right) N_f T_F + \frac{2857C_A^3}{54}. \end{aligned} \quad (\text{A.2})$$

B Anomalous dimensions

For all the anomalous dimensions entering the renormalization group equations of various TMD functions, we define the perturbative expansion in α_s according to

$$\gamma(\alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \gamma_n, \quad (\text{B.1})$$

where the coefficients for quark are given by

$$\begin{aligned} \Gamma_0^{\text{cusp}} &= 4C_F, \\ \Gamma_1^{\text{cusp}} &= \left(\frac{268}{9} - 8\zeta_2\right) C_A C_F - \frac{80C_F T_F N_f}{9}, \\ \Gamma_2^{\text{cusp}} &= \left[\left(\frac{320\zeta_2}{9} - \frac{224\zeta_3}{3} - \frac{1672}{27}\right) C_A C_F + \left(64\zeta_3 - \frac{220}{3}\right) C_F^2\right] N_f T_F \end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{1072\zeta_2}{9} + \frac{88\zeta_3}{3} + 88\zeta_4 + \frac{490}{3} \right) C_A^2 C_F - \frac{64}{27} C_F N_f^2 T_F^2, \\
\gamma_0^S &= 0, \\
\gamma_1^S &= \left[\left(-\frac{404}{27} + \frac{11\zeta_2}{3} + 14\zeta_3 \right) C_A + \left(\frac{112}{27} - \frac{4\zeta_2}{3} \right) T_F N_f \right] C_F, \\
\gamma_2^S &= \left(-\frac{88}{3} \zeta_3 \zeta_2 + \frac{6325\zeta_2}{81} + \frac{658\zeta_3}{3} - 88\zeta_4 - 96\zeta_5 - \frac{136781}{1458} \right) C_A^2 C_F + \left(\frac{80\zeta_2}{27} - \frac{224\zeta_3}{27} \right. \\
& \quad \left. + \frac{4160}{729} \right) C_F N_f^2 T_F^2 + \left(-\frac{2828\zeta_2}{81} - \frac{728\zeta_3}{27} + 48\zeta_4 + \frac{11842}{729} \right) C_A C_F N_f T_F \\
& \quad + \left(-4\zeta_2 - \frac{304\zeta_3}{9} - 16\zeta_4 + \frac{1711}{27} \right) C_F^2 N_f T_F. \\
\gamma_0^R &= 0, \\
\gamma_1^R &= \left[\left(-\frac{404}{27} + 14\zeta_3 \right) C_A + \frac{112}{27} T_F N_f \right] C_F, \\
\gamma_2^R &= \left[\left(-\frac{824\zeta_2}{81} - \frac{904\zeta_3}{27} + \frac{20\zeta_4}{3} + \frac{62626}{729} \right) C_A N_f T_F + \left(-\frac{88}{3} \zeta_3 \zeta_2 + \frac{3196\zeta_2}{81} + \frac{6164\zeta_3}{27} \right. \right. \\
& \quad \left. \left. + \frac{77\zeta_4}{3} - 96\zeta_5 - \frac{297029}{1458} \right) C_A^2 + \left(-\frac{304\zeta_3}{9} - 16\zeta_4 + \frac{1711}{27} \right) C_F N_f T_F + \left(-\frac{64\zeta_3}{9} \right. \right. \\
& \quad \left. \left. - \frac{3712}{729} \right) N_f^2 T_F^2 \right] C_F. \tag{B.2}
\end{aligned}$$

Since cusp and soft and rapidity anomalous dimensions exhibit Casimir scaling, the corresponding anomalous dimensions for gluon could be obtained by multiplying in above with C_A/C_F .

The beam anomalous dimensions do not exhibit Casimir scaling, thus should be listed separately. The beam anomalous dimensions for quark are

$$\begin{aligned}
\gamma_0^B &= 3C_F, \\
\gamma_1^B &= \left[\left(\frac{3}{2} - 12\zeta_2 + 24\zeta_3 \right) C_F + \left(\frac{17}{6} + \frac{44\zeta_2}{3} - 12\zeta_3 \right) C_A + \left(-\frac{2}{3} - \frac{16\zeta_2}{3} \right) T_F N_f \right] C_F, \\
\gamma_2^B &= \left[\left(-\frac{2672\zeta_2}{27} + \frac{400\zeta_3}{9} + 4\zeta_4 + 40 \right) C_A C_F + \left(\frac{40\zeta_2}{3} - \frac{272\zeta_3}{3} + \frac{232\zeta_4}{3} - 46 \right) C_F^2 \right] N_f T_F \\
& \quad + \left(16\zeta_3 \zeta_2 - \frac{410\zeta_2}{3} + \frac{844\zeta_3}{3} - \frac{494\zeta_4}{3} + 120\zeta_5 + \frac{151}{4} \right) C_A C_F^2 + \left(\frac{320\zeta_2}{27} - \frac{64\zeta_3}{9} - \frac{68}{9} \right) \\
& \quad \times C_F N_f^2 T_F^2 + \left(\frac{4496\zeta_2}{27} - \frac{1552\zeta_3}{9} - 5\zeta_4 + 40\zeta_5 - \frac{1657}{36} \right) C_A^2 C_F + \left(-32\zeta_3 \zeta_2 + 18\zeta_2 \right. \\
& \quad \left. + 68\zeta_3 + 144\zeta_4 - 240\zeta_5 + \frac{29}{2} \right) C_F^3. \tag{B.3}
\end{aligned}$$

The beam anomalous dimensions for gluon are

$$\begin{aligned}
\gamma_0^B &= \frac{11}{3} C_A - \frac{4}{3} T_F N_f, \\
\gamma_1^B &= C_A^2 \left(\frac{32}{3} + 12\zeta_3 \right) + \left(-\frac{16}{3} C_A - 4C_F \right) N_f T_F,
\end{aligned}$$

$$\begin{aligned}
\gamma_2^B = & C_A^3 \left(-80\zeta_5 - 16\zeta_3\zeta_2 + \frac{55}{3}\zeta_4 + \frac{536}{3}\zeta_3 + \frac{8}{3}\zeta_2 + \frac{79}{2} \right) \\
& + C_A^2 N_f T_F \left(-\frac{20}{3}\zeta_4 - \frac{160}{3}\zeta_3 - \frac{16}{3}\zeta_2 - \frac{233}{9} \right) + \frac{58}{9} C_A N_f^2 T_F^2 - \frac{241}{9} C_A C_F N_f T_F \\
& + 2C_F^2 N_f T_F + \frac{44}{9} C_F N_f^2 T_F^2.
\end{aligned} \tag{B.4}$$

The cusp anomalous dimension Γ^{cusp} can be found in [105]. The beam anomalous dimension γ^B is related to the soft anomalous dimension γ^S [106] and the hard anomalous dimensions γ^H [107–109] by renormalization group invariance condition $\gamma^B = \gamma^S - \gamma^H$. The rapidity anomalous dimension γ^R can be found in [88, 96]. Note that the normalization here differ from those in [88] by a factor of 1/2.

C Renormalization Constants

The following constants are needed for the renormalization of zero-bin subtracted [110] TMD PDFs through N³LO, see e.g. Ref. [57, 86]. The first three-order corrections to Z^B and Z^S are

$$\begin{aligned}
Z_1^B &= \frac{1}{2\epsilon} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q), \\
Z_2^B &= \frac{1}{8\epsilon^2} \left((\Gamma_0^{\text{cusp}} L_Q - 2\gamma_0^B)^2 + 2\beta_0 (\Gamma_0^{\text{cusp}} L_Q - 2\gamma_0^B) \right) + \frac{1}{4\epsilon} (2\gamma_1^B - \Gamma_1^{\text{cusp}} L_Q), \\
Z_3^B &= \frac{1}{48\epsilon^3} (2\gamma_0^B - \Gamma_0^{\text{cusp}} L_Q) \left(8\beta_0^2 + 6\beta_0 (-2\gamma_0^B + \Gamma_0^{\text{cusp}} L_Q) + (-2\gamma_0^B + \Gamma_0^{\text{cusp}} L_Q)^2 \right) \\
&+ \frac{1}{24\epsilon^2} \left(\beta_1 (-8\gamma_0^B + 4\Gamma_0^{\text{cusp}} L_Q) + (4\beta_0 - 6\gamma_0^B + 3\Gamma_0^{\text{cusp}} L_Q) (-2\gamma_1^B + \Gamma_1^{\text{cusp}} L_Q) \right) \\
&+ \frac{1}{6\epsilon} (2\gamma_2^B - \Gamma_2^{\text{cusp}} L_Q) \\
Z_1^S &= \frac{1}{\epsilon^2} \Gamma_0^{\text{cusp}} + \frac{1}{\epsilon} (-2\gamma_0^S - \Gamma_0^{\text{cusp}} L_\nu), \\
Z_2^S &= \frac{1}{2\epsilon^4} (\Gamma_0^{\text{cusp}})^2 - \frac{1}{4\epsilon^3} \left(\Gamma_0^{\text{cusp}} (3\beta_0 + 8\gamma_0^S) + 4(\Gamma_0^{\text{cusp}})^2 L_\nu \right) - \frac{1}{2\epsilon} (2\gamma_1^S + \Gamma_1^{\text{cusp}} L_\nu) \\
&+ \frac{1}{4\epsilon^2} \left(\Gamma_1^{\text{cusp}} + 2(2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu)(\beta_0 + 2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu) \right), \\
Z_3^S &= \frac{1}{6\epsilon^6} (\Gamma_0^{\text{cusp}})^3 - \frac{1}{4\epsilon^5} (\Gamma_0^{\text{cusp}})^2 (3\beta_0 + 4\gamma_0^S + 2\Gamma_0^{\text{cusp}} L_\nu) + \frac{1}{36\epsilon^4} \Gamma_0^{\text{cusp}} \left(22\beta_0^2 + 45\beta_0 (2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu) \right. \\
&+ 9 \left(\Gamma_1^{\text{cusp}} + 2(2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu)^2 \right) + \frac{1}{36\epsilon^3} \left(-16\beta_1 \Gamma_0^{\text{cusp}} - 12\beta_0^2 (2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu) \right. \\
&- 2\beta_0 (5\Gamma_1^{\text{cusp}} + 9(2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu)^2) - 3 \left[\Gamma_1^{\text{cusp}} (6\gamma_0^S + 9\Gamma_0^{\text{cusp}} L_\nu) \right. \\
&+ 2 \left(8(\gamma_0^S)^3 + 6\Gamma_0^{\text{cusp}} \gamma_1^S + 12\Gamma_0^{\text{cusp}} (\gamma_0^S)^2 L_\nu + 6(\Gamma_0^{\text{cusp}})^2 \gamma_0^S L_\nu^2 + (\Gamma_0^{\text{cusp}})^3 L_\nu^3 \right) \left. \right] \left. \right) \\
&+ \frac{1}{18\epsilon^2} \left(2\Gamma_2^{\text{cusp}} + 3(2\beta_1 (2\gamma_0^S + \Gamma_0^{\text{cusp}} L_\nu) + (2\beta_0 + 6\gamma_0^S + 3\Gamma_0^{\text{cusp}} L_\nu) (2\gamma_1^S + \Gamma_1^{\text{cusp}} L_\nu)) \right)
\end{aligned}$$

$$-\frac{2\gamma_2^S + \Gamma_2^{\text{cusp}} L_\nu}{3\epsilon}. \quad (\text{C.1})$$

Keep in mind that the anomalous dimensions appeared above depends on the flavor, they should be replaced by the corresponding values in Sec B. We also remind the reader that the renormalization constants are formally identical for TMD PDFs and TMD FFs, the logarithms appeared above should be replaced by their corresponding values in each case, and we have

$$L_\perp = \ln \frac{b_T^2 \mu^2}{b_0^2}, \quad L_\nu = \ln \frac{\nu^2}{\mu^2}, \quad (\text{C.2})$$

with $b_0 = 2e^{-\gamma_E}$ for both TMD PDFs and TMD FFs. On the other hand, we have for TMD PDFs

$$L_Q = 2 \ln \frac{x P_+}{\nu}, \quad (\text{C.3})$$

while for TMD FFs,

$$L_Q = 2 \ln \frac{P_+}{z \nu}. \quad (\text{C.4})$$

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