

# Energy Correlators Resolving Proton Spin

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We investigate the partonic origin of the proton longitudinal spin using spin-dependent energy correlator measured in lepton-hadron collisions with longitudinally polarized proton beams. These observables encode angular correlations in energy flow and are sensitive to the spin-momentum structure of confined partons. Using soft-collinear effective theory (SCET), we analyze the correlation patterns in both nearly back-to-back and forward limits, which establishes a direct correspondence with longitudinally polarized transverse momentum dependent distributions (TMDs) and nucleon energy correlators (NECs). The TMDs and NECs allow consistent matching onto hard radiation region, and provide a comprehensive description of the transition from perturbative parton branching to nonperturbative confinement. Using renormalization group evolution, we obtain joint N<sup>3</sup>LL/NNLL quantitative predictions for spin-dependent energy correlation patterns in the current and target fragmentation regions (CFR and TFR). The framework provides new theoretical insight into how the internal motion and spin of partons contribute to the formation of the proton longitudinal spin, and offers an experimental paradigm for probing the interplay between color confinement and spin dynamics at forthcoming Electron-Ion Collider (EIC).

## I. INTRODUCTION

The origin of the proton spin has long been a central puzzle in QCD. In polarized deep inelastic scattering (DIS), the key observable is the first moment of the spin-dependent structure function

$$\Gamma_1^P(Q^2) = \int_0^1 g_1(x, Q^2) dx. \quad (1)$$

Ellis and Jaffe, assuming vanishing strange-quark polarization, predicted a large positive value,  $\Gamma_1^P \approx 0.186$  [1]. The EMC measurement in 1988 [2, 3], however, reported a much smaller value,  $\Gamma_1^P \approx 0.126$ , in striking disagreement with the prediction, thereby igniting the ‘Proton spin crisis’.

To address this problem, major progress has been made on both theoretical and experimental fronts, see Ref. [4] for an overview. On the experimental side, semi-inclusive DIS (SIDIS) has provided accesses to the flavor dependence of polarized quark distributions [5, 6], while polarized  $pp$  scattering at RHIC has constrained gluon and sea-quark helicities [7–9]. These data have been incorporated into recent global analyses of polarized PDFs [10–12]. Looking ahead, the Electron–Ion Collider (EIC) [13, 14] will provide unprecedented sensitivity to

the quantitative determination of the partonic contributions to the proton spin.

On the theory side, two complementary decompositions of the proton spin have been established: the frame-independent ‘Ji sum rule’ [15, 16] and the partonic Jaffe-Manohar sum rule [17] in infinite momentum frame (IMF)

$$\frac{\hbar}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + \ell_q + \ell_g, \quad (2)$$

where the first two terms are the first moments of quark and gluon helicity distributions, whose scale evolution is known to next-to-next-to-leading order (NNLO) accuracy [18–23]. The last two terms,  $\ell_q$  and  $\ell_g$ , denote higher-twist quark and gluon orbital angular momentum contributions. For example, it has been proposed in quark models that  $\ell_q$  is related to the TMD pretzelosity distribution  $h_{1T}^{\perp q}$  [24–26].

Significant progress has been made in addressing the proton spin puzzle from the lattice QCD perspective. The theoretical foundation was established in Ref. [15] through gauge-invariant decomposition of the nucleon spin, subsequent developments [27, 28] have made it possible to connect this framework with lattice simulations, and recent studies [29–31] have reported direct numerical results for the quark and gluon contributions to the proton spin.

On the small- $x$  front, helicity evolution equations derived by Kovchegov–Pitonyak–Sievert resum the single and double logarithms, leading to quantitative predictions for the small- $x$  asymptotics of the quark/gluon he-

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licity distributions and the potential spin carried at very low  $x$  [32–35].

Perturbative QCD, in addressing the proton spin puzzle through factorization and global analyses that solve the ‘inverse problem’ of determining uncountably many low-energy constants, has now reached the NNLO frontier with the inclusion of higher-order radiative corrections [20, 36–41]. While these approaches rely primarily on polarized DIS and  $pp$  scattering cross sections, it is natural to seek complementary observables offering additional insight into the spin structure. Energy-energy correlators (EECs), originally proposed in [42–45] as a means to investigate QCD dynamics, provide precisely such an opportunity. Over the years, EECs have emerged as a versatile tool for exploring QCD dynamics across different regimes. In particular, they probe transverse-momentum-dependent dynamics through back-to-back jets [46–60] and reveal detailed jet substructure information in the collinear limit [61–77]. EECs have been generalized to the deep inelastic scattering (DIS) framework [78–81]. As collinear-safe observables, they provide a robust probe of QCD dynamics at both perturbative and nonperturbative levels. Their versatility makes them a promising tool for testing the Standard Model with high precision, constraining parton distribution functions, and studying QCD factorization in new kinematic regimes. A comprehensive review can be found in Ref. [82].

In this work, we extend the EEC formalism to the case of a polarized proton target in DIS, defined as follows

$$\langle \mathcal{E}(\hat{n}) \rangle_N = \int d^4x e^{iq \cdot x} \langle P_N, S_{\parallel} | J(x) \mathcal{E}(\hat{n}) J(0) | P_N, S_{\parallel} \rangle, \quad (3)$$

where  $N$  denotes the proton target and  $\hat{n}$  is the generic radial unit vector where the energy-weight is applied. Compared to conventional  $q_T$ -based observables, energy correlators offer several distinct advantages. First, owing to their strict operator definition, they provide a promising way to probe the nonperturbative spin structure of the proton from first principles. Second, as collinear-safe observables, EECs can be consistently defined across the full kinematic regime, from the hard radiation region to the current and target fragmentation regions. This seamless definition enables consistent matching across the dynamical domains, thereby offering a unique opportunity to investigate the transition from perturbative parton branching to nonperturbative confinement.

The remainder of this paper is organized as follows. In Sec. II, we review the definition of Bjorken- $x$  weighted EECs. Section III discusses the factorization and resummation in the current region, while Sec. IV presents the corresponding analysis in the target region. The matching of polarized NECs to polarized PDFs is carried out in Sec. V. Finally, we summarize our findings in Sec. VI.

## II. CONSTRUCTING EEC IN DEEP-INELASTIC-SCATTERING

In this paper, we consider the polar angular  $\theta$  distribution in lepton-hadron colliders with a polarized high-energy lepton beam scattering off a longitudinally polarized nucleon target:

$$\ell(p_\ell, \lambda_\ell) + N(P_N, S_{\parallel}) \rightarrow \ell'(p_{\ell'}, \lambda_{\ell'}) + X(P_X). \quad (4)$$

We will parametrize the events by the angular variable  $\theta$ , defined to be the polar angle between the detected particles (jets) with respect to proton velocity. The leptonic momentum transfer and the invariant mass of the virtual photon are given by  $q = p_\ell - p_{\ell'}$  and  $Q^2 = -q^2$ , respectively. Besides,  $\lambda_\ell/S_{\parallel}$  denotes the chirality of the incoming lepton/target. The standard set of kinematic invariants is given by

$$x = \frac{Q^2}{2P_N \cdot q} \quad y = \frac{P_N \cdot q}{P_N \cdot p_\ell}, \quad z = \frac{P_N \cdot P_h}{P_N \cdot q}. \quad (5)$$

The energy correlator is defined in the Breit frame, where the proton carries the momentum

$$P_N = \frac{Q}{2x} = \frac{Q}{2x}(1, 0, 0, 1). \quad (6)$$

and the virtual photon carries momentum in its  $z$  component

$$q = \frac{Q}{2}(\bar{n} - n) = Q(0, 0, 0, -1). \quad (7)$$

In the back-to-back limit, the angular  $\theta$  correlation is directly related to the conventional  $q_T$  spectrum

$$\sin \theta = \frac{q_T}{Q/2}. \quad (8)$$

We are interested in energy-weighted angular  $\theta$  correlations

$$\begin{aligned} d\sigma^{\text{EC}} &= \frac{16\pi^2\alpha^2}{2E_{\text{cm}}^2 Q^4} \frac{d^3p_{\ell'}}{(2\pi)^3 2E_{\ell'}} L_{\mu\nu}(p_\ell, p_{\ell'}) W_{\text{EC}}^{\mu\nu}(q, P_N, \theta) \\ &= \frac{2\pi\alpha_e^2}{Q^4} y \frac{dQ^2}{E_{\text{cm}}^2} \frac{dx}{x} L_{\mu\nu}(p_\ell, p_{\ell'}) W_{\text{EC}}^{\mu\nu}(q, P_N, \theta), \end{aligned} \quad (9)$$

where  $\alpha = e_R^2/(4\pi)$  denotes the electromagnetic coupling, and instead of the conventional DIS hadronic tensor, the energy-flow operator is inserted between the electromagnetic currents

$$\begin{aligned} W_{\text{EC}}^{\mu\nu}(q, P_N, \theta) &= \prod_X \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^{(4)}(q + P_N - P_X) \\ &\quad \times \langle P_N, S_{\parallel} | J^{\dagger\mu} \hat{\mathcal{E}}(\theta) | X \rangle \langle X | J^\nu | P_N, S_{\parallel} \rangle, \\ &= \int d^4x e^{iq \cdot x} \langle P_N, S_{\parallel} | J^{\dagger\mu}(x) \hat{\mathcal{E}}(\theta) J^\nu | P_N, S_{\parallel} \rangle. \end{aligned} \quad (10)$$

The cumulant energy correlation operator quantifies the energy deposited in the detector for radiation within a radial angle bounded above by  $\theta$

$$\hat{\mathcal{E}}(\theta)|X\rangle = \sum_{h \in X} \frac{E_h}{E_N} \Theta(\theta - \theta_h)|X\rangle. \quad (11)$$

From Poincaré covariance and parity symmetry, the energy correlation hadronic tensor allows decomposition into independent hadronic structure functions [83]

$$W_{\text{EC}}^{\mu\nu}(q, P_N, \theta) = f_T^{\mu\nu} x W_T + f_L^{\mu\nu} x W_L + \Delta f^{\mu\nu} x g_1, \quad (12)$$

where the tensor bases are explicitly given in App. (B). We adopt the commonly used notation in which  $W_T$  denotes the contribution from transverse modes of the virtual photon, while  $W_L$  represents the longitudinal contribution. The polarized hadronic function  $g_1$  contributes to single spin asymmetry  $1/2(\sigma(S_{\parallel} = +) - \sigma(S_{\parallel} = -))$ . Combining the hadronic tensor with the lepton current

$$L^{\mu\nu} = 2\delta_{\lambda_\ell \lambda_{\ell'}} \left[ \left( p_\ell^\mu p_{\ell'}^\nu + p_\ell^\nu p_{\ell'}^\mu - \frac{Q^2}{2} g^{\mu\nu} \right) + i\lambda_\ell \epsilon^{\mu\nu\ell\ell'} \right], \quad (13)$$

we have decomposition as follows

$$\frac{d\sigma^{\text{EC}}}{dQ^2} = \delta_{\lambda_\ell \lambda_{\ell'}} \frac{2\pi\alpha_e^2}{Q^4} dx (\sigma_T W_T + \sigma_L W_L + \lambda_\ell S_{\parallel} \Delta\sigma g_1), \quad (14)$$

where the born level form factors are given by

$$\sigma_T = 1 + (1 - y)^2, \quad \sigma_L = 2 - 2y, \quad \Delta\sigma = 1 - (1 - y)^2. \quad (15)$$

We will also investigate Bjorken  $x$  weighted ECs, defined as follows

$$\begin{aligned} \frac{d\Sigma_N(\theta, \mu)}{dQ^2} &= \sum_h \int x^{N-1} d\sigma(x, Q, P_h) \frac{E_h}{E_N} \Theta(\theta - \theta_h) \\ &= \int x^{N-1} \frac{d\sigma^{\text{EC}}(x, Q, \theta)}{dQ^2}. \end{aligned} \quad (16)$$

The DIS energy correlation exhibits remarkable simplicity in specific kinematic limits, offering novel insights into the mechanisms of color confinement and origin of nucleon spin. For instance, the small spin limit  $N \rightarrow 1$  limit should probe Regge asymptotics of QCD governed by Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution [84, 85] and beyond. The study presented here is devoted to two complementary angular limits: the back-to-back limit  $\theta \rightarrow \pi$  and the collinear limit  $\theta \rightarrow 0$ , where we establish all-order factorization theorems in terms of universal non-perturbative objects. Reliable theoretical predictions are then obtained through renormalization group evolution.

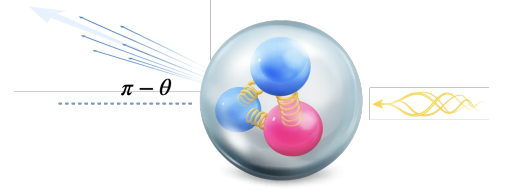


FIG. 1: Current Fragmentation.

### III. CURRENT FRAGMENTATION AND TMD FACTORIZATION

In this section, we are concerned with back-to-back limit  $\theta \rightarrow \pi$  where the outgoing quark fragments into QCD jets, see Fig. (1). This kinematic region is also referred to as current-fragmentation region (CFR), and the appropriate theoretical framework for describing it is TMD factorization. To derive the factorization formulae, it is useful to identify a hadron inside the leading jet in the first place. To this end, we consider differential cross section with one identified hadron in the final-states

$$\frac{d\sigma_{\ell+N \rightarrow \ell'+h+X}}{d\vec{q}_\perp dx dy dz} = \frac{2\pi\alpha^2}{4Q^4} L_{\mu\nu}(P_\ell, P_{\ell'}) W^{\mu\nu}(q, P_N, P_h), \quad (17)$$

where  $\vec{q}_\perp$  is the transverse momentum of the virtual photon in hadron-hadron frame, or is a parameterization of the transverse momentum of the detected leading jet  $\vec{q}_\perp = -\vec{P}_h^\perp/z$  in photon-hadron frame [83]. The hadronic tensor with  $h$  being resolved is given by

$$\begin{aligned} W^{\mu\nu}(q, P_N, P_h) &= \prod_X \frac{d^3 P_X}{(2\pi)^3 2E_X} \delta^{(4)}(q + P_N - P_h - P_X) \\ &\times \langle S_{\parallel}, P_N | J^\dagger | h, X \rangle \langle h, X | J^\nu | N \rangle. \end{aligned} \quad (18)$$

Kinematic constraints in CFR require that all final-state radiation in  $X$  be collimated, with momenta either collinear to the incoming target  $N$  or the observed hadron  $h$ , and the hadronic amplitude is factorized through [86]

$$\begin{aligned} \langle h; X_{Nh}; X_s | J^\mu | N \rangle &\simeq \mathcal{C}(Q^2, \mu) \langle X_s | \mathcal{T} [Y_{\vec{n}}^\dagger Y_n] | 0 \rangle \\ &\times \langle h, X_h | \bar{\chi}_{\vec{n}} | 0 \rangle \gamma^\mu \langle X_N | \chi_n | N \rangle, \end{aligned} \quad (19)$$

where we have absorbed the soft zero-bin subtraction into the normalization of the collinear fields, since matrix element computed with zero-bin subtracted SCET Lagrangian is equivalent to dividing the naive matrix element by vacuum expectation values of zero-bin Wilson lines [86, 87]. The hard amplitude  $\mathcal{C}(Q^2, \mu)$  is insensitive to the infrared (IR) dynamics of soft and collinear degrees of freedom, and thus can be computed at partonic level. It is advantageous to consider  $q + \gamma^* \rightarrow q$ , with no soft radiation present

$$\langle q | J^\mu | q \rangle = \mathcal{C}(Q^2, \mu) \langle q | \bar{\chi}_{\vec{n}} | 0 \rangle \gamma^\mu \langle 0 | \chi_n | q \rangle. \quad (20)$$

In this case, the leading-power collinear expansion is exact (= instead of  $\simeq$ ), as each collinear trajectory contains just one particle-taking the collinear limit simply does nothing. By construction, the IR divergences of the partonic amplitude  $\langle q|J^\mu|q \rangle$  is reproduced by the UV divergences of the hard scattering operators in Soft-Collinear Effective Field Theory (SCET) [88–91]. Indeed, the right-hand side of eq. (20) is

$$\langle q|\mathcal{O}_{\text{scet}}^{\bar{\lambda}}|q \rangle = \langle q|\mathcal{O}_{\text{scet}}^{\bar{\lambda}}|q \rangle_{\text{tree}} (1 + \delta Z_{\bar{\lambda}}(\epsilon_{\text{IR}})) , \quad (21)$$

where  $\mathcal{O}_{\text{scet}}^{\bar{\lambda}} \sim \bar{\chi}_n^\pm \gamma^\mu \chi_n^\pm$ ,  $\lambda = \pm$ . The result is proportional to tree-level 't Hooft-Veltman [92] (tHV) bases because pure virtual-loop corrections in SCET are scaleless and vanish identically. As a result, the IR poles of a helicity amplitude is reproduced by UV renormalization counter-term diagrams provided by  $\delta Z_{\bar{\lambda}} = Z_{\bar{\lambda}} - 1$  [93].

In  $q_T$  factorization, several subtleties need to be addressed. The main issue concerns the validity of factorization itself, namely demonstrating that Glauber effects drop out at the cross-section level, thereby justifying the separation of modes. Once the hard and soft-collinear degrees of freedom are disentangled, the next issue is to

consistently account for soft-collinear particles in a manner free of double counting. Since these modes have the same invariant mass, they are distinguished by their rapidity, and the phase space integral must be regulated with a rapidity cutoff  $\nu$  [94]. Additionally, a zero-bin subtraction is performed to remove the overlapping between collinear and soft sectors, yielding genuine collinear beam functions [95]. With these ingredients in place, we obtain factorization formulae [83, 96–100]

$$W^{\mu\nu} \simeq \frac{4}{z} \sum_f H_f(Q^2, \mu) \int \frac{d^2 \vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} \mathcal{S}_{n\bar{n}}(b_\perp, \mu, \nu) \times \text{Tr}[\mathcal{B}_{f/N}(S_\parallel, x, b_\perp, E_n, \mu, \nu) \gamma^\mu \mathcal{F}_{h/f}(z, b_\perp/z, E_{\bar{n}}, \mu, \nu) \gamma^\nu] . \quad (22)$$

By applying SCET Fierz identity

$$1 \otimes 1 = \frac{1}{2} \left[ \frac{\not{q}}{2} \otimes \frac{\not{q}}{2} - \frac{\not{q}\gamma_5}{2} \otimes \frac{\not{q}\gamma_5}{2} - \frac{\not{q}\gamma_\perp^\mu}{2} \otimes \frac{\not{q}\gamma_{\perp\mu}}{2} \right] , \quad (23)$$

the resulting SIDIS TMD factorization formula is

$$xyz \frac{d\sigma_{\ell+N \rightarrow \ell'+h+X}}{d^2 \vec{q}_\perp dx dy dz} \simeq \sum_f H_f(Q^2, \mu) \int \frac{d^2 \vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} \mathcal{S}_{n\bar{n}}(b_\perp, \mu, \nu) z \mathcal{F}_{h/f}\left(z, \frac{b_\perp}{z}, E_{\bar{n}}, \mu, \nu\right) \times (\sigma_0^U \times x \mathcal{B}_{f/N}(x, b_\perp, E_n, \mu, \nu) + \lambda_\ell S_\parallel \times \sigma_0^L \times x \Delta \mathcal{B}_{f/N}(x, b_\perp, E_n, \mu, \nu)) , \quad (24)$$

where  $H_f(Q^2, \mu) = |\mathcal{C}_f(Q, \mu)|^2$  represents the square of the hard matching coefficient [101–103], and  $E_n$  ( $E_{\bar{n}}$ ) are the energies of struck (fragmented) partons. The born-level unpolarized cross-section  $\sigma_0^U$  and its asymmetry counterpart  $\sigma_0^L$  are

$$\sigma_0^U = 2\pi\alpha^2 \frac{1 + (1-y)^2}{Q^2} , \quad \sigma_0^L = 2\pi\alpha^2 \frac{1 - (1-y)^2}{Q^2} . \quad (25)$$

The TMD beam and fragmentation functions are the genuine collinear contributions with zero-bin subtraction

$$\mathcal{B}_{q/N}(x, b_\perp, E_n, \mu, \nu) \equiv \lim_{\nu \rightarrow \infty} \frac{\mathcal{B}_{q/N}^0(x, b_\perp, E_n, \mu, \nu)}{\mathcal{S}_{n\bar{n}}^0(b_\perp, \mu, \nu)} , \quad (26)$$

the superscript 0 denotes UV-renormalized but unsubtracted collinear beam functions. With an exponential rapidity regulator [94], the zero-bin soft function is identical to the SIDIS TMD soft function, the latter is related to back-to-back EEC soft function [104] by a boost

$$\mathcal{S}_{n\bar{n}}(b_\perp, \mu, \nu) = \mathcal{S}_{\text{EEC}}\left(L_b + \ln \frac{n \cdot \bar{n}}{2}, \mu, \nu\right) . \quad (27)$$

The explicit expression of the TMD soft function can be found in Refs. [48, 104, 105]. In this work we're concerned with energy correlations between the proton and the backward leading jets. To this end, we weight SIDIS TMD factorization formulae eq. (24) with energy weight  $E_h/E_P \simeq xz$ , and sum over all possible hadrons to get

$$y \frac{d\sigma_{\ell+N \rightarrow \ell'+h+X}}{d^2 \vec{q}_\perp dx dy} \simeq \sum_f \int \frac{d^2 \vec{b}_\perp}{(2\pi)^2} e^{i\vec{b}_\perp \cdot \vec{q}_\perp} [H_f(Q^2, \mu) \mathcal{S}_{n\bar{n}}(b_\perp, \mu, \nu)] \mathcal{J}_f(b_\perp, E_{\bar{n}}, \mu, \nu) \times (\sigma_0^U \times x \mathcal{B}_{f/N}(x, b_\perp, E_n, \mu, \nu) + \lambda_\ell S_\parallel \times \sigma_0^L \times x \Delta \mathcal{B}_{f/N}(x, b_\perp, E_n, \mu, \nu)) , \quad (28)$$

where we perform twist-2 matchings of the TMD fragmentation function

$$\mathcal{F}_{h/f}\left(z, \frac{b_\perp}{z}, E_{\bar{n}}, \mu, \nu\right) = \sum_i d_{h/i} \otimes C_{iq} + \mathcal{O}(b_T^2 \Lambda_{\text{QCD}}^2), \quad (29)$$

and use FF sum rules to obtain the EEC TMD jet function  $\mathcal{J}_f$  [48],

$$\begin{aligned} \mathcal{J}_f(b_\perp, E_{\bar{n}}, \mu, \nu) &\equiv \sum_h \int_0^1 dz z \mathcal{F}_{h/f}^{\text{OPE}}\left(z, \frac{b_\perp}{z}, E_{\bar{n}}, \mu, \nu\right) \\ &= \sum_i \int_0^1 d\xi \xi C_{if}\left(\xi, \frac{b_\perp}{\xi}, E_{\bar{n}}, \mu, \nu\right). \end{aligned} \quad (30)$$

Non-perturbative corrections to this function is recently analysed in Refs. [58, 106]. In Fig. (2), we validate our factorization formula eq. (28) against fixed-order calculations at leading-order in QCD. Good agreements are found for both the unpolarized cross section and the spin asymmetry in the large angle limit. The calculations are carried out with the FMNLO program [107, 108] and summed over all possible hadrons using toy FFs, since the results are independent of the exact choice of FFs giving constraint from the momentum sum rule.

For the experimental extraction of the TMDs, we express the factorization formula in terms of physical TMDs and jet functions, which are obtained by multiplying the former by the square root of the TMD soft function

$$\begin{aligned} f_1^q(x, b_\perp, \xi^n, \mu) &= \mathcal{B}_{q/N}(x, b_\perp, E_n, \mu, \nu) \sqrt{\mathcal{S}_{n\bar{n}}}, \\ D_1^q\left(z, \frac{b_\perp}{z}, \xi^{\bar{n}}, \mu\right) &= \mathcal{F}_{h/q}\left(z, \frac{b_\perp}{z}, E_{\bar{n}}, \mu, \nu\right) \sqrt{\mathcal{S}_{n\bar{n}}}. \end{aligned} \quad (31)$$

As a result, the rapidity divergences  $\ln(4E_i^2/\nu^2)$  cancel between collinear and soft sectors, leaving behind physical rapidity logarithms  $\ln(\xi^i/\mu_b^2)$  with  $\mu_b = b_0/b_T$ ,  $b_0 = 2e^{-\gamma_E}$ ,  $\xi^i$  are the Collins-Soper (CS) scales [83, 113, 114] for SIDIS

$$\xi^n = Q^2 \frac{x}{x+y-xy}, \quad \xi^{\bar{n}} = Q^2 \frac{x+y-xy}{x}. \quad (32)$$

The  $\mu$ -evolution of the hard matrix and the TMDs is linear in  $\ln \mu$  to all-loop orders, as can be proved by RG consistency

$$\frac{d}{d \ln \mu} \ln f_i(x_i, b_\perp, \xi^i, \mu) = -\gamma_{\text{cusp}}^i \ln \frac{\xi^i}{\mu^2} - \gamma_H^i. \quad (33)$$

The rapidity evolution of the TMDs is referred to as the

Collins-Soper (CS) kernels

$$K_i(b_\perp, \mu) \equiv \frac{d}{d \ln \sqrt{\xi^i}} \ln f_i(x_i, b_\perp, \xi^i, \mu), \quad (34)$$

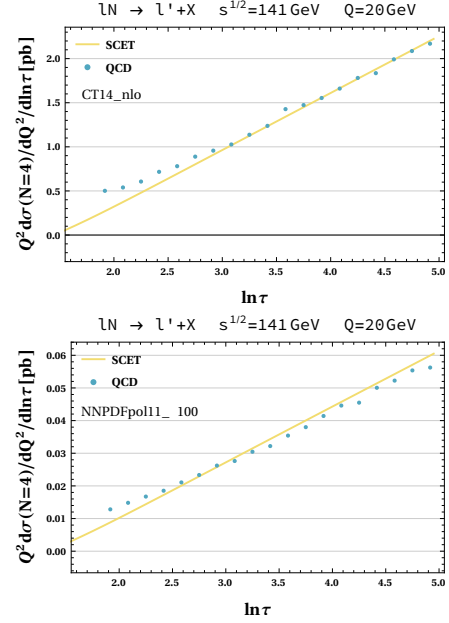


FIG. 2: QCD fixed-order results versus SCET prediction in the large angle limit, using LHAPDF6 [109] PDF datasets CT14nlo [110] and NNPDFpol1.1 [111].

again by RG consistency, the  $\mu$ -evolution of  $K_i(b_\perp, \mu)$  is controlled by the cusp anomalous dimension

$$\frac{dK_i(b_\perp, \mu)}{d \ln \mu} = 2\gamma_{\text{cusp}}^i(\alpha_s(\mu)). \quad (35)$$

From above we can solve the RG equation to give

$$K_i(b_\perp, \mu) = -2A_{\text{cusp}}^i(\mu, \mu_b) + \gamma_R^i(\mu_b), \quad (36)$$

the solution has a evolution factor  $A_{\text{cusp}}^i(\mu, \mu_b)$  controlled by the cusp anomalous dimension  $\gamma_{\text{cusp}}^i$ , as shown in eq. (A1), it also leaves a residual boundary term  $\gamma_R^i(\mu_b)$  known as QCD rapidity anomalous dimension. While the cusp anomalous dimension  $\gamma_{\text{cusp}}^i$  is a purely perturbative object which originates from the UV renormalization of the cusped Wilson loops [115, 116], the rapidity anomalous dimension  $\gamma_R^i(\mu_b)$ , on the other hand, has both perturbative part as  $b_T \rightarrow 0$  and genuine non-perturbative part as  $b_T \rightarrow \infty$ . The perturbative accuracy for the CS kernel has reached up to N<sup>4</sup>LL [49, 104, 105], and there has been tremendous progress towards Lattice calculation for the non-perturbative part as well [117–123]. Eq. (33, 34, 36) are referred to as Collins-Soper equations [83, 113, 114], they are fully equivalent to the modern language of SCET rapidity renormalization groups (RRGs) [124, 125] in the context of  $q_T$  factorization. Solving the CS equation, we will have the following RG-improved factorization using physical TMDs



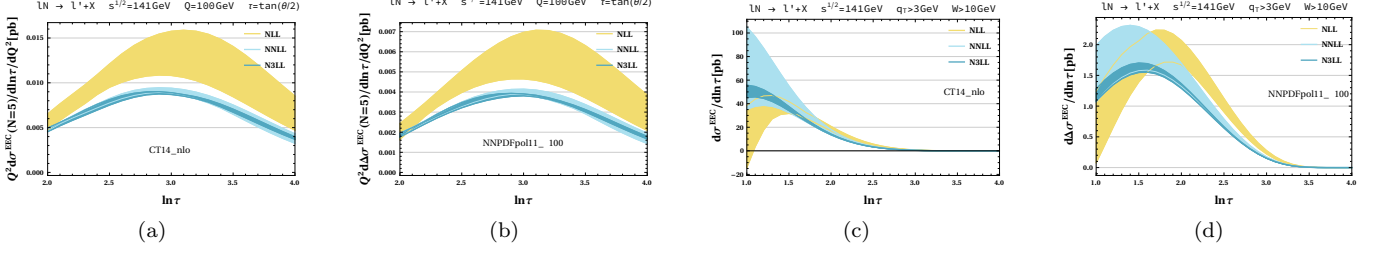


FIG. 3: The energy correlation spectrum parameterized by  $\tau = \tan \theta/2$ . Left (a, b): Bjorken weighted ( $N=5$ ) Energy Correlators with  $b_*$  prescription [112]. Right (c, d): Energy Correlators without  $b_*$  prescription, (c,d) are subject to a  $q_T$  cut to ensure the observable remains in the perturbative regime.

$$\begin{aligned}
 y \frac{d\sigma_{\ell+N \rightarrow \ell'+h+X}}{d^2\vec{q}_\perp dx dy} &\simeq \sum_q \int \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{b}_\perp \cdot \vec{q}_\perp} H_q(Q^2, \mu) J_q(\xi_0^n, \mu_0^n) \times (\sigma_0^U \times x f_1^q(x, b_\perp, \xi_0^n, \mu_0^n) + \lambda_\ell S_\parallel \sigma_0^L \times x g_1^q(x, b_\perp, \xi_0^n, \mu_0^n)) \\
 &\times \prod_i e^{-2K_{\text{cusp}}^i(\mu_0^i, \mu) + A_H^i(\mu_0^i, \mu)} \times \left( \frac{\xi^i}{\mu_0^i} \right)^{A_{\text{cusp}}^i(\mu_0^i, \mu)} \times \left( \frac{\xi^i}{\xi_0^i} \right)^{-A_{\text{cusp}}^i(\mu_0^i, \mu_b) + \frac{1}{2} \gamma_R^i(\mu_b)}. \quad (37)
 \end{aligned}$$

Fig. (3) shows the N<sup>3</sup>LL perturbative resummation of Bjorken weighted energy correlators. In Fig. (3) (a,b), we present Mellin space results by weighting eq. (37) with Bjorken weight  $x^{N-1}$ , we choose  $N = 5$  to suppress the small- $x$  contributions, and we have implemented BLNY18 [126] non-perturbative model therein. While in Fig. (3) (c,d), no  $b_*$  prescription is applied. Instead, a  $q_T$  cut is introduced to maintain control over non-perturbative modes. In the plots, we choose  $q_T^{\text{cut}} = 3\text{GeV}$ , which effectively excludes the genuinely confined modes and places the observable within perturbative regime. To access the genuinely non-perturbative region, one may release the cutoff. This opens a window for studying TMD physics in a model-independent way [127], highlighting a key advantage of the energy correlation observable over the conventional  $q_T$  spectrum.

#### IV. TARGET FRAGMENTATION AND NUCLEON ENERGY CORRELATORS

In this section, we are concerned with collinear limit  $\theta \rightarrow 0$  where the spectator partons fragment into forward QCD jets, see Fig. (4). This kinematic region is also referred to as target-fragmentation region (TFR), and the appropriate theoretical framework for describing it is through fracture functions [128–133]. Fracture functions provide a joint description of the struck parton distribution and the fragmentation of spectator partons into an identified hadron. In Refs [134, 135], the NEC is proposed as an inclusive version of fracture functions by integrating over the momentum fraction of the identified hadrons and summing over all possible species of them,

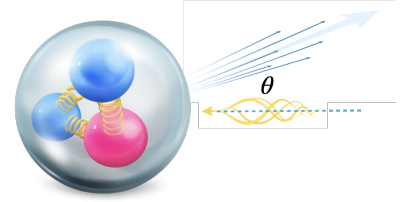


FIG. 4: Target Fragmentation.

the sum rule is discussed in Ref [136]. The NEC has been evaluated at NLL accuracy in Refs [81, 134, 135]. In the present work, we extend the calculation to NNLL accuracy and additionally, include proton longitudinal spin. We will first briefly review the factorization formulae to the leading power of the limit  $\theta \rightarrow 0$ . The complete set of effective operators are obtained in Refs [81, 134] by integrating out hard offshell modes, and are identified as twist-2 operators with the insertion of energy flow operators

$$\begin{aligned}
 O_n^{i;\Gamma}(\omega_\pm, \theta) &= \frac{1}{\omega_+} \left[ \bar{\chi}_{n,\omega_1}^{(i)} \Gamma \hat{\mathcal{E}}(\theta) \chi_{n,\omega_2}^{(i)} \right], \\
 O_n^{g;\Gamma}(\omega_\pm, \theta) &= \frac{-1}{\omega_+^2} \left[ \mathcal{B}_{n,\omega_1}^{\perp\mu} \Gamma_{\mu\nu} \hat{\mathcal{E}}(\theta) \mathcal{B}_{n,\omega_2}^{\perp\nu} \right], \quad (38)
 \end{aligned}$$

where  $\Gamma$ s denote the quark/gluon spin projectors for unpolarized/longitudinally polarized nucleon target

$$\Gamma \in \left\{ \frac{\not{1}}{2}, \frac{\not{1}\gamma_5}{2} \right\}, \quad \Gamma^{\mu\nu} \in \{g_\perp^{\mu\nu}, i\epsilon_\perp^{\mu\nu}\}, \quad (39)$$

and the gauge invariant collinear building blocks with label energy  $\omega$  ( $\omega_{\pm} = \omega_1 \pm \omega_2$ ) is defined by  $\chi_{n,\omega} \equiv [\delta(\omega - \bar{P})\chi_n]$  and  $g\mathcal{B}_{n,\omega}^{\perp\mu} \equiv [g\mathcal{B}_n^{\perp\mu}\delta(\omega - \bar{P}^\dagger)]$ . The hadronic scalar functions in eq. (45) are then matched onto the twist-2 operator bases [137], for instance

$$g_1(x, Q, \theta) \rightarrow \sum_i \int \prod \frac{1}{2} d\omega_{\pm} \mathcal{C}_i^{\Gamma}(\omega_{\pm}, x, Q, \mu) O_n^{i,\Gamma}(\omega_{\pm}, \theta). \quad (40)$$

Here, the spin index  $\Gamma$  should take the right hand side of eq. (39) as anti-symmetric tensors. To proceed, we use the relation between label operators and the conventional NECs, for example

$$\begin{aligned} \langle P_N; S_{\parallel} | \bar{\chi}_{n,\omega} \frac{\not{p} \gamma_5}{2} \hat{\mathcal{E}}(\theta) \chi_{n,\omega} | P_N; S_{\parallel} \rangle &= 4\bar{n} \cdot P_N S_{\parallel} \\ &\times \int_{-1}^1 d\xi \delta(\omega_-) \delta(\omega_+ - 2\xi \bar{n} \cdot P_N) \Delta F_{i/N}^{\text{EC}}(\xi, \theta), \end{aligned} \quad (41)$$

and charge conjugation symmetries of both the NECs and the Wilson coefficients

$$F_{i/N}^{\text{EC}}(\xi, \theta) = -F_{i/N}^{\text{EC}}(-\xi, \theta), \Delta F_{i/N}^{\text{EC}}(\xi, \theta) = \Delta F_{i/N}^{\text{EC}}(-\xi, \theta), \quad (42)$$

to obtain the matching conditions for the unpolarized structure function  $W_T(x, Q, \theta)$ ,  $W_L(x, Q, \theta)$  and the spin-asymmetry counterpart  $g_1(x, Q, \theta)$ . For instance, the leading twist expansion for the spin-asymmetry part is given by a convolution between perturbatively calculable coefficient functions and the non-perturbative NECs

$$g_1(x, Q, \theta) = \sum_{i=1}^{N_f} \mathcal{C}_i^{\Gamma} \otimes (\Delta F_{i/N}^{\text{EC}} + \Delta F_{\bar{i}/N}^{\text{EC}}) + \mathcal{C}_g^{\Gamma} \otimes \Delta F_{g/N}^{\text{EC}}. \quad (43)$$

It proves useful to rewrite the above factorization formula through singlet and non-singlet combinations of the NECs, for instance

$$\begin{aligned} F_{q/N}^{\text{EC}}(\xi, \theta) &= \sum_{i=1}^{N_f} F_{i/N}^{\text{EC}}(\xi, \theta) + F_{\bar{i}/N}^{\text{EC}}(\xi, \theta), \\ F_{i/N}^{\text{EC,NS}}(\xi, \theta) &= F_{i/N}^{\text{EC}}(\xi, \theta) + F_{\bar{i}/N}^{\text{EC}}(\xi, \theta) - \frac{1}{N_f} F_{q/N}^{\text{EC}}(\xi, \theta). \end{aligned} \quad (44)$$

Finally, we gather our factorization formulae as follows

$$\frac{d\sigma^{\text{EC}}}{dQ^2} = \delta_{\lambda_e \lambda_{e'}} \frac{2\pi\alpha_e^2}{Q^4} dx (\sigma_T W_T + \sigma_L W_L + \lambda_e S_{\parallel} \Delta\sigma g_1), \quad (45)$$

where the leading twist expansion of various scalar structure functions is given by

$$W_{\Gamma}(x, Q, \theta) = \sum_i Q_i^2 \mathcal{C}_{i,\text{NS}}^{\Gamma}(Q, \mu) \otimes F_{i/N}^{\text{EC,NS}}(\theta, \mu) + \left( \frac{1}{N_f} \sum_{i=1}^{N_f} Q_i^2 \right) \left[ \mathcal{C}_q^{\Gamma}(Q, \mu) \otimes F_{q/N}^{\text{EC}}(\theta, \mu) + \mathcal{C}_g^{\Gamma}(Q, \mu) \otimes F_{g/N}^{\text{EC}}(\theta, \mu) \right]. \quad (46)$$

The NEC provides a joint description of the struck parton distribution and the energy flows of forward QCD jets originating from spectator fragmentations. The associated energy flow operator  $\hat{\mathcal{E}}(\theta)$  acts only on the collinear sector, thus, if  $\hat{\mathcal{E}}(\theta)$  were set to the identity, i.e., in the absence of detectors in the forward region, NECs reduce to standard PDFs and the resulting factorization formula coincides with that of inclusive DIS. From this perspective, the Wilson coefficients are identical to the DIS coefficient functions [20, 138–144], and the UV renormalization of NECs matches that of the conventional PDFs, for instance

$$\begin{bmatrix} F_{q/N}^{\text{EC}} \left( \ln \frac{E_N \sin \theta}{\mu}, \mu \right) \\ F_{g/N}^{\text{EC}} \left( \ln \frac{E_N \sin \theta}{\mu}, \mu \right) \end{bmatrix} = \hat{Z}_S^{\text{PDF}} \otimes \begin{bmatrix} F_{q/N}^{\text{EC0}} \left( \ln \frac{E_N \sin \theta}{\mu}, \mu \right) \\ F_{g/N}^{\text{EC0}} \left( \ln \frac{E_N \sin \theta}{\mu}, \mu \right) \end{bmatrix}, \quad \mathcal{F}_{i/N}^{\text{EC}} \left( N, \ln \frac{E_a \sin \theta}{\mu} \right) = \int_0^1 dz z^{N-1} \mathcal{F}_{i/N}^{\text{EC}} \left( z, \ln \frac{E_a \sin \theta}{\mu} \right), \quad (47)$$

where the NECs are and must be parameterized by hadronic logarithm  $\ln E_N \sin \theta / \mu$ ,  $E_N$  being the energy of the incoming target  $N$ , in DIS  $E_N = Q/(2x)$ . The multiplicative renormalization factor  $\hat{Z}_S^{\text{PDF}}$  is the standard PDF UV renormalization factor for the singlets, the associated Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [145–147] evolution kernel reads

$$\hat{P}_S(x, \alpha_s) = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}. \quad (48)$$

Alternatively, one could consider the Bjorken- $x$  weighted energy correlation in Mellin space, defined in eq. (16). To this end, we first introduce the Mellin moment of NECs,

$$\mathcal{F}_{i/N}^{\text{EC}} \left( N, \ln \frac{E_a \sin \theta}{\mu} \right) = \int_0^1 dz z^{N-1} \mathcal{F}_{i/N}^{\text{EC}} \left( z, \ln \frac{E_a \sin \theta}{\mu} \right), \quad (49)$$

where we reparameterize the NECs by partonic logarithm

$$\mathcal{F}_{i/N}^{\text{EC}}\left(z, \ln \frac{E_a \sin \theta}{\mu}, \mu\right) = F_{i/N}^{\text{EC}}\left(z, \ln \frac{E_N \sin \theta}{\mu}, \mu\right), \quad (50)$$

with  $E_a = zE_N$  as  $z$  denotes the momentum fraction of the struck parton. In DIS, the energy of the active parton is  $E_a = Q/2/\hat{x}$ , where  $\hat{x} = x/z$ . The NECs, when parameterized in terms of a partonic logarithm rather than hadronic variables, obey a modified DGLAP evolution equation

$$\begin{aligned} & \frac{d\mathcal{F}_{i/N}^{\text{EC}}}{d\ln \mu^2}\left(N, \ln \frac{E_a \sin \theta}{\mu}, \mu\right) \\ &= \sum_j \int_0^1 d\xi \xi^{N-1} P_{ij}(\xi, \mu) \mathcal{F}_{j/N}^{\text{EC}}\left(N, \ln \frac{E_a \sin \theta}{\xi\mu}, \mu\right). \end{aligned} \quad (51)$$

The modified DGLAP RG equation can be solved within perturbation theory, the solution consists of a linear term, identical to that of the standard DGLAP evolution, and a nonlinear residual term

$$\begin{aligned} \mathcal{F}_{i/N}^{\text{EC}}\left(N, \ln \frac{E_a \sin \theta}{\mu}, \mu\right) &= \sum_j \mathcal{D}_{ij}^N(\mu, \mu_0) \\ &\times \mathcal{F}_{j/N}^{\text{EC}}\left(N, \ln \frac{E_a \sin \theta}{\mu_0}, \mu_0\right) + \mathcal{R}_i^N(\mu, \mu_0), \end{aligned} \quad (52)$$

where  $\mathcal{D}_{ij}^N(\mu, \mu_0)$  is the standard DGLAP evolution

$$\begin{aligned} \mathcal{D}^N(\mu, \mu_0) &= \exp\left[\int_{\mu_0}^{\mu} d\ln \bar{\mu}^2 P(N, \bar{\mu})\right] \\ &= \exp\left[-2A_{P(N)}(\mu, \mu_0)\right], \end{aligned} \quad (53)$$

expressed through the conventional evolution factor

$$A_\gamma(\mu_0, \mu) = - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \frac{\gamma(\alpha_s)}{\beta(\alpha_s)}. \quad (54)$$

The non-linear modification term is given by

$$\begin{aligned} \mathcal{R}_i^N(\mu, \mu_0) &= \sum_{m=1}^{\infty} \int_{\mu_0}^{\mu} d\ln \bar{\mu}^2 \mathcal{D}^N(\mu, \bar{\mu}) P^{(m)}(N, \bar{\mu}) \mathcal{D}^N(\bar{\mu}, \mu_0) \\ &\times \mathcal{F}_{i/N}^{\text{EC}(m)}\left(N, \ln \frac{E_a \sin \theta}{\mu_0}, \mu_0\right), \end{aligned} \quad (55)$$

where the boundary value of NECs at renormalization scale  $\mu_0$  are reorganized into a power expansion in  $\ln \xi$

$$\begin{aligned} \mathcal{F}_{i/N}^{\text{EC}}\left(N, \ln \frac{E_a \sin \theta}{\xi\mu_0}, \mu_0\right) &= \mathcal{F}_{i/N}^{\text{EC}}\left(N, \ln \frac{E_a \sin \theta}{\mu_0}, \mu_0\right) \\ &+ \sum_{m=1}^{\infty} \ln^m \xi \times \mathcal{F}_{i/N}^{\text{EC}(m)}\left(N, \ln \frac{E_a \sin \theta}{\mu_0}, \mu_0\right), \end{aligned} \quad (56)$$

and the modified splitting functions are defined as

$$P^{(m)}(N, \mu) = \int_0^1 dx x^{N-1} \ln^m x P(x, \mu). \quad (57)$$

Using the Mellin space NECs, we obtain the factorization formula as follows

---


$$\begin{aligned} \frac{d\Sigma_N(\theta, \mu)}{dQ^2} &= \delta_{\lambda_\ell \lambda_{\ell'}} \frac{2\pi\alpha_e^2}{Q^4} \left\{ \int_0^1 d\hat{x} \hat{x}^{N-1-n} \sum_{\Gamma=T,L} \sum_{n=0}^2 \sigma_\Gamma^n(Q^2) \left( \sum_i Q_i^2 \times \mathcal{C}_{i;\text{NS}}^\Gamma(\hat{x}, Q, \mu) \mathcal{F}_{i/N}^{\text{EC,NS}}\left(N-n, \ln \frac{Q/2 \sin \theta}{\hat{x}\mu}\right) \right. \right. \\ &+ \frac{1}{N_f} \sum_{i=1}^{N_f} Q_i^2 \times \left[ \mathcal{C}_q^\Gamma(\hat{x}, Q, \mu) \mathcal{F}_{q/N}^{\text{EC}}\left(N-n, \ln \frac{Q/2 \sin \theta}{\hat{x}\mu}\right) + \mathcal{C}_g^\Gamma(\hat{x}, Q, \mu) \mathcal{F}_{g/N}^{\text{EC}}\left(N-n, \ln \frac{Q/2 \sin \theta}{\hat{x}\mu}\right) \right] \Big) \\ &+ \lambda_\ell S_\parallel \sum_{n=0}^2 \Delta\sigma^n(Q^2) \left( \sum_i Q_i^2 \times \Delta\mathcal{C}_{i;\text{NS}}(\hat{x}, Q, \mu) \Delta\mathcal{F}_{i/N}^{\text{EC,NS}}\left(N-n, \ln \frac{Q/2 \sin \theta}{\hat{x}\mu}\right) + \frac{1}{N_f} \sum_{i=1}^{N_f} Q_i^2 \right. \\ &\times \left[ \Delta\mathcal{C}_q(\hat{x}, Q, \mu) \Delta\mathcal{F}_{q/N}^{\text{EC}}\left(N-n, \ln \frac{Q/2 \sin \theta}{\hat{x}\mu}\right) + \Delta\mathcal{C}_g(\hat{x}, Q, \mu) \Delta\mathcal{F}_{g/N}^{\text{EC}}\left(N-n, \ln \frac{Q/2 \sin \theta}{\hat{x}\mu}\right) \right] \Big) \Big\}, \end{aligned} \quad (58)$$


---

note that the Born-level form factors in Eq. (15) are decomposed according to the weight of Bjorken  $x$ :  $\sigma_\Gamma(y) = \sum_0^2 \sigma_\Gamma^n(Q^2) x^{-n}$ , this step is necessary because the variable  $y = (Q/E_{\text{cm}})^2/x = (Q/E_{\text{cm}})^2/(\hat{x}z)$ . We validate the factorization formula by comparing to the fixed-order

predictions at leading-order in QCD in Fig. (6). Again good agreements are found for both the unpolarized cross section and the spin asymmetry in the small angle limit. In Fig. (5), we present the numerical solution to the modified DGLAP RG equation Eq. (51) to NNLL accuracy.



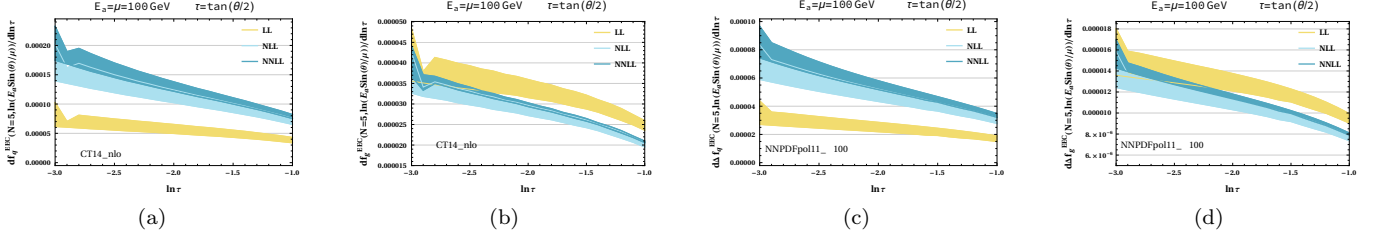


FIG. 5: Unpolarized and polarized NEC singlets renormalized at a perturbative scale with struck parton energy  $E_a = 100\text{GeV}$ . The scale band indicates the variation with respect to the boundary scale  $\mu_0 \sim E_a \sin \theta$ .

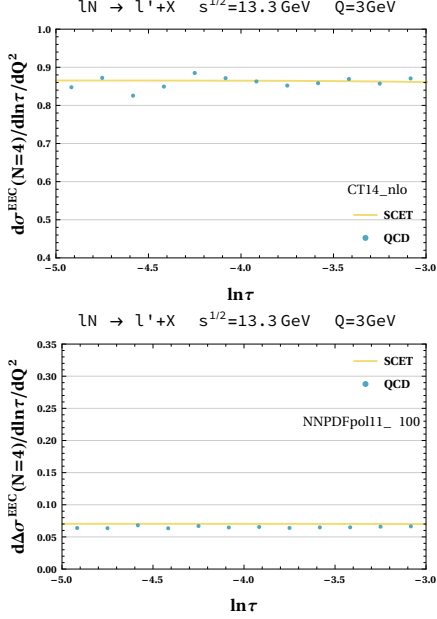


FIG. 6: QCD fixed-order results versus SCET prediction in the small angle limit.

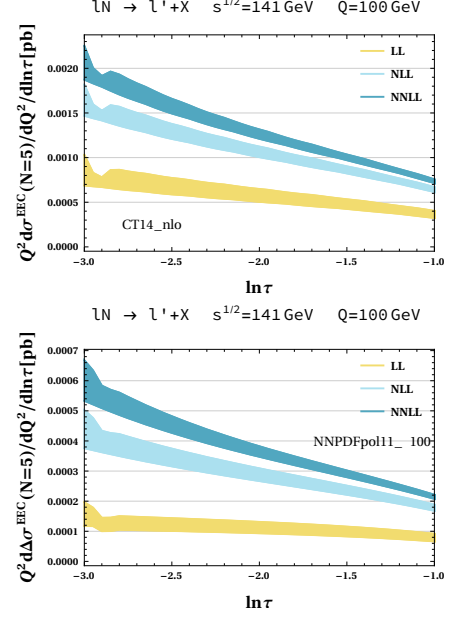


FIG. 7: Unpolarized and polarized energy correlations to NNLL accuracy in the collinear limit.

## V. MATCHING WITH COLLINEAR FACTORIZATION

The NNLL NECs incorporate both the NNLO DGLAP splitting functions [18–23] and the NNLO twist-2 matching coefficients, the latter will be derived in the next section. In addition, the NNLL prediction to the energy correction pattern in the forward direction is shown in Fig. (7), where we vary the initial boundary and factorization scales around  $(\mu_0, \mu_f) \sim (Q/2 \sin \theta, Q)$ . Unlike the CFR, where Sudakov suppression arises due to the recoil of soft gluons, the energy correlator in the forward region is inclusive over soft modes and soft radiation is suppressed in the TFR due to energy weight. As a result, the distribution at small angles is not suppressed. Together with the back-to-back N<sup>3</sup>LL prediction in Fig. (3), the energy correlator has provided the state-of-the-art precision for probing internal structures of the proton across the full range of kinematic regimes, from hard radiation region to current/target fragmentation regions.

The factorization formulae in the CFR (24) and TFR (46, 58) do not require a strict scale hierarchy between  $q_T \sim Q/2 \sin \theta$  and  $\Lambda_{\text{QCD}}$ . Nevertheless, in the perturbative regime  $Q \gg q_T \gg \Lambda_{\text{QCD}}$ , consistency between the CFR/TFR factorization and the standard collinear factorization implies that the TMDs and NECs admit matching onto conventional PDFs. The N<sup>3</sup>LO twist-2 matching of the TMDs are obtained in [23, 148–152]. On the other hand, it was shown in Refs. [134, 135] that the NECs allow OPE onto energy-weighted collinear PDFs as follows

$$\begin{aligned} \mathcal{F}_{i/N}^{\text{EC}} \left( z, \ln \frac{E_a \sin \theta}{\mu}, \mu \right) &= f_i(z, \mu) - \int_z^1 \frac{d\xi}{\xi} \\ &\times \mathcal{I}_{ij} \left( \frac{z}{\xi}, \ln \frac{E_a \sin \theta}{\mu}, \alpha_s(\mu) \right) \xi f_j(\xi, \mu) + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}}{q_T} \right). \end{aligned} \quad (59)$$

The matching coefficients  $\mathcal{I}_{ij}$  is insensitive to the hadronic state  $N$ , thus can be computed from partonic NECs, and renormalized according to eq. (47). After UV renormalization, the NECs are reparametrized by the partonic logarithm through replacement  $\ln \frac{E_N}{\mu} \rightarrow \ln \frac{E_a}{\mu} - \ln z$ , and matched onto energy densities  $\xi f_j(\xi, \mu)$ . The renormalization group equation governing the coefficient function follows directly from Eqs. (51) and (59)

$$\begin{aligned} \frac{d\mathcal{I}_{ij}}{d \ln \mu^2} \left( z, \ln \frac{E_a \sin \theta}{\mu} \right) &= \int_z^1 \frac{d\xi}{\xi} \left[ P_{ik}(\xi) \right. \\ &\times \mathcal{I}_{kj} \left( \frac{z}{\xi}, \ln \frac{E_a \sin \theta}{\xi \mu} \right) - \mathcal{I}_{ik} \left( \frac{z}{\xi}, \ln \frac{E_a \sin \theta}{\mu} \right) \xi P_{kj}(\xi) \Big]. \end{aligned} \quad (60)$$

The first term in eq. (60) originates from RG flow with respect to the UV cut-off from above, while the second term arises from RG flow toward the IR, the solution up to NNLO reads

$$\begin{aligned} \mathcal{I}_{ij} \left( z, \ln \frac{E_a \sin \theta}{\mu} \right) &= \delta_{ij} \delta(1-z) + a_s(\mu) \left( \mathcal{I}_{ij}^{(1)}(z) \right. \\ &+ L_q \left( p_{ij}^{(0)}(z) - P_{ij}^{(0)}(z) \right) \Big) + a_s^2(\mu) \left( \mathcal{I}_{ij}^{(2)}(z) + L_q^2 \right. \\ &\times \left[ \frac{p_{ik}^{(0)} \otimes p_{kj}^{(0)}(z)}{2} + \frac{P_{ik}^{(0)} \otimes P_{kj}^{(0)}(z)}{2} - P_{ik}^{(0)} \otimes p_{kj}^{(0)}(z) \right. \\ &- \frac{1}{2} \beta_0 \left( p_{ij}^{(0)}(z) - P_{ij}^{(0)}(z) \right) \Big] + L_q \left[ \mathcal{I}_{ik}^{(1)} \otimes p_{kj}^{(0)}(z) \right. \\ &- P_{ik}^{(0)} \otimes \mathcal{I}_{kj}^{(1)}(z) - \beta_0 \mathcal{I}_{ij}^{(1)}(z) + p_{ij}^{(1)}(z) - P_{ij}^{(1)}(z) \\ &\left. \left. + 2\tilde{P}_{ik}^{(0)} \otimes (p_{kj}^{(0)} - P_{kj}^{(0)})(z) \right] \right) + \mathcal{O}(a_s^3(\mu)), \end{aligned} \quad (61)$$

where  $a_s(\mu) = \alpha_s(\mu)/(4\pi)$  and  $L_q = 2 \ln \frac{E_a \sin \theta}{\mu}$ . The modified splitting functions are given by

$$p_{ij}^{(m)}(z) = z P_{ij}^{(m)}(z), \quad \tilde{P}_{ij}^{(0)}(z) = \ln z P_{ij}^{(0)}(z). \quad (62)$$

The computation of polarized NECs requires careful treatment of the genuinely four-dimensional objects  $\gamma_5$  and the Levi-Civita tensor  $\epsilon^{\mu\nu\rho\sigma}$ , whose definitions must be consistently extended to  $D = 4 - 2\epsilon$  dimensions. However, the anti-commutativity of  $\gamma_5$  and the cyclicity of the Dirac trace cannot be simultaneously preserved in dimensional regularization. One approach is to retain the four-dimensional Dirac algebra of  $\gamma_5$  by evaluating the trace from a prescribed ‘reading point’ [153, 154], thereby avoiding ambiguities associated with non-cyclic traces. This method is employed in Refs. [155, 156] to obtain the polarized DIS coefficient functions and DGLAP splitting functions in  $\overline{\text{MS}}$  scheme. The ‘reading point’ prescription is re-examined in the course of computing helicity TMDs [23], where we reproduce the two-loop scheme transformation factor  $Z_{\text{ps}}^{(2)}$ , introduced below. Another approach is to abandon the anti-commutativity of  $\gamma_5$ , and instead read in  $\gamma_5$  from the effective vertex. This leads

to the use of the HVBM scheme [92, 157], or the closely related **Larin** prescription [158–160], for the consistent treatment of  $\gamma_5$  in dimensional regularization. In HVBM scheme, one proceeds as [36]

1. Evaluate integrals first
  - a) Use multilinear property  $\text{Tr}[l_1 l_2 \dots \gamma_5] = l_1^{\mu_1} l_2^{\mu_2} \times \dots \text{Tr}[\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma_5]$
  - b) Evaluate tensor-like Feynman integrals and phase space integrals in  $D$ -dimensions
2. Begin with the explicit definition of  $\gamma_5$ 
  - c) Replace the  $\gamma_5$ -matrix by
 
$$\gamma_\mu \gamma_5 = \frac{i}{6} \epsilon_{\mu\rho\sigma\tau} \gamma^\rho \gamma^\sigma \gamma^\tau \quad \text{or} \quad \gamma_5 = \frac{i}{24} \epsilon_{\mu\rho\sigma\tau} \gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\tau$$
  - d) Compute trace of Dirac matrix in  $D$ -dimensions
  - e) Contract the Levi-Civita tensors in four-dimensions

In computing polarized NECs, we employ a novel  $\gamma_5$  scheme—the **Larin**<sup>+</sup> prescription introduced by the authors of Ref. [161]. This particular scheme is well-suited when computing TMDs-like objects, as it avoids the explicit appearance of the Levi-civita tensor  $\epsilon^{\mu\nu\rho\sigma}$ , where one replaces the  $\gamma_5$ -matrix by

$$\gamma^+ \gamma_5 \rightarrow \frac{i \epsilon_{\perp}^{\alpha\beta}}{2} \gamma_\alpha \gamma_\beta, \quad (63)$$

and supplement it by the D-dimensional relation

$$\epsilon_{\perp}^{\alpha_1 \beta_1} \epsilon_{\perp}^{\alpha_2 \beta_2} = -g_{\perp}^{\alpha_1 \alpha_2} g_{\perp}^{\beta_1 \beta_2} + g_{\perp}^{\alpha_1 \beta_2} g_{\perp}^{\beta_1 \alpha_2}. \quad (64)$$

As a matter of fact, we have verified that the HVBM and **Larin**<sup>+</sup> prescriptions yield identical results for helicity TMDs at N<sup>3</sup>LO [23]. Both the HVBM and **Larin**<sup>+</sup> schemes break the anti-commutativity of the  $\gamma_5$ -matrix, which in turn leads to a violation of the Adler–Bardeen theorem for the non-renormalization of the axial anomaly beyond one loop [162]. Therefore, renormalization in prescribed  $\gamma_5$  scheme require additional evanescent counterterms. To extract the required evanescent counterterms, it is necessary to renormalize physical quantities consistently within both Kreimer’s approach and the HVBM or **Larin**<sup>+</sup> schemes to the same perturbative order, and define scheme transformations as ratios between them. This procedure also allows one to fix the scheme transformation factors, whose matrix elements are nontrivial only in the  $q \rightarrow q$  entry

$$\begin{aligned} Z_5 &= 1 + \sum_n \left( \frac{\alpha_s}{4\pi} \right)^n Z_5^{(n)}, \\ (Z_5^{(n)})_{ik} &= \delta_{iq} \delta_{kq} Z_{qq}^{(n)} = \delta_{iq} \delta_{kq} \left\{ Z_{\text{ns},+}^{(n)} + Z_{\text{ps}}^{(n)} \right\}. \end{aligned} \quad (65)$$

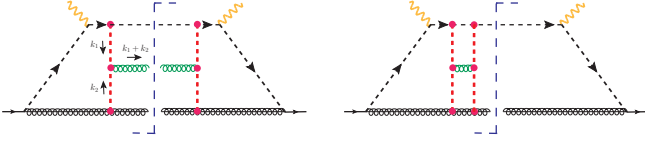


FIG. 8: Spectator-active interactions through Glauber modes, with Lipatov vertex off of the Glauber ladders.

Though 2-loops we have

$$Z_{ik}^s = \delta_{ik} + \delta_{iq}\delta_{kq} \left( \left( \frac{\alpha_s}{4\pi} \right) Z_{ns,+}^{(1)} + \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ Z_{ns,+}^{(2)} + Z_{ps}^{(2)} \right\} \right). \quad (66)$$

Owing to the factorization structure, the polarized NECs are expected to be subjected to a scheme transformation as follows

$$\Delta\mathcal{I}^{\overline{\text{MS}}}(\cdot, L_q) = [Z_5 \otimes \Delta I^{\text{HVB}}(\cdot, L_P)] \odot z_5^{-1}, \quad (67)$$

where  $I^{\text{HVB}}(z, L_P)$  is the matching coefficient parameterized by the hadronic logarithm  $L_P = 2 \ln \frac{E_N \sin \theta}{\mu}$  and  $z_5(z) = z \times Z_5(z)$ . The modified convolution algebra is defined by

$$I(\cdot, L_P) \odot z_5^{-1} \equiv \mathcal{I}(\cdot, L_q) \otimes z_5^{-1}, \quad \mathcal{I}(z, L_q) = I(z, L_P). \quad (68)$$

Given that  $Z_5$  has non-trivial components exclusively in the  $q \rightarrow q$  channel, eq. (67) implies

$$\begin{aligned} \Delta\mathcal{I}_{qq}^{\overline{\text{MS}}} &= [Z_5 \otimes \Delta I_{qq}^{\text{HVB}}] \odot z_5^{-1}, & \Delta\mathcal{I}_{qg}^{\overline{\text{MS}}} &= Z_5 \otimes \Delta I_{qg}^{\text{HVB}} \\ \Delta\mathcal{I}_{gq}^{\overline{\text{MS}}} &= \Delta I_{gq}^{\text{HVB}} \odot z_5^{-1}, & \Delta\mathcal{I}_{gg}^{\overline{\text{MS}}} &= \Delta I_{gg}^{\text{HVB}}. \end{aligned} \quad (69)$$

After applying the scheme transformations, we confirm that  $\Delta\mathcal{I}^{\overline{\text{MS}}}$  obeys the RG solution of eq. (61) with NLO splitting functions in  $\overline{\text{MS}}$  scheme [141, 163, 164]. The complete analytic expression for the polarized coefficient functions are provided as computer-readable files in the ancillary material of the arXiv submission, with numerical routines implemented in `PolyLogsTools` [165]. The logarithmic enhancements in the Regge limit  $z \rightarrow 0$  are collected in App. (C). There, it is found that the polarized coefficient functions scale as  $z^0$ , in contrast to the unpolarized coefficient functions, which exhibit a  $1/z$  scaling behavior in the high-energy limit.

## VI. SUMMARY AND OUTLOOK

In this work, we investigate spin-dependent energy correlations in DIS from current to target fragmentation region. In the CFR, where the dynamics are governed by TMDs, we showed that energy correlators provide a clean and model-independent probe of TMD physics. Indeed, by employing a perturbative  $q_T \simeq 3\text{GeV}$  cut-off, we have obtained N<sup>3</sup>LL renormalization-group-improved predictions for the wide-angle correlations without relying on

a  $b_*$  prescription, and we argue subleading power corrections in  $b_T \Lambda_{\text{QCD}}$  can be systematically restored by lowering the  $q_T$  cut accordingly. In the TFR, where the dynamics are controlled by NECs, we computed the twist-2 matching at NNLO and achieved NNLL level description for spectator fragmentation. Altogether, our study delivers state-of-the-art precision for spin-dependent energy-correlation patterns across a broad range of kinematic regimes, offering an alternative approach to the proton spin decomposition and proton three-dimensional tomography at the forthcoming EIC.

Looking ahead, several further improvements are worth pursuing. First, we assume Glauber effects drop out for the establishment of factorization. Although there were substantial arguments [166] for this to be correct, the mechanism responsible for factorization within effective field theories is still missing. For example, Glauber gluons mediate spectator-active interactions, such as the H diagram in Fig. (8). Depending on the observable, their contributions may be absorbed into collinear Wilson lines [167], remain as genuine Glauber contributions [60], or they cancel out between cut and uncut diagrams [168]. A full picture is still unfolding.

Second, the unpolarized NECs exhibit divergences at the Mellin moment  $N = 1 \sim x^{-1} \ln x$ , accessing the small- $x$  regime therefore requires an all-order resummation of the corresponding logarithms, a program which in turn demands a precise definition of the Glauber gluons. In contrast, the polarized NECs diverge at the Mellin moment  $N = 0 \sim x^0 \ln x$ . We expect the polarized Regge dynamics are dominated by subleading fermionic Glauber operators [169, 170], since the leading  $t$ -channel Reggeon/Glauber exchanges are insensitive to the proton spin and they cancel in spin asymmetries. Consequently, contributions that are subleading in the unpolarized case become the leading effects in the polarized small- $x$  regime. It is also instructive to relate the EFT perspective to the shock-wave formalism with polarized Wilson lines [33, 35, 171], which provides a complementary framework for small- $x$  helicity evolution.

Third, the present framework can be extended to T-odd operators, thereby enabling dedicated probes of the process-dependences of the TMDs and NECs [172]. For instance, the T-odd Boer-Mulders [173] and the Sivers function [174], whose existence is attributed to initial-state or final-state interactions between the active partons and the spectators, are known to exhibit a sign reversal when crossing from SIDIS to the Drell-Yan process [175–179]. A precise study of these phenomena once again requires a careful definition of the Glauber gluons.

Addressing these points will pave the way toward extending our framework into Multi-Regge Kinematics (MRK) and higher-twist domains. We expect this to substantially broaden the phenomenological reach of energy correlators, particularly for EIC kinematics.

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## Appendix A: RG Factor

We introduce two conventional integrals in the context of Sudakov problems, their perturbative expansions can be found, e.g., in Ref. [101]

$$\begin{aligned} K_\Gamma(\mu_0, \mu) &= - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \frac{\Gamma(\alpha_s)}{\beta(\alpha_s)} \int_{\alpha_s(\mu_0)}^{\alpha_s} d\alpha'_s \frac{1}{\beta(\alpha'_s)}, \\ A_\gamma(\mu_0, \mu) &= - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s \frac{\gamma(\alpha_s)}{\beta(\alpha_s)}. \end{aligned} \quad (\text{A1})$$

## Appendix B: Form Factors

To isolate contributions from each of the photon polarization modes, the following bases are chosen for the

decomposition of the hadronic tensor

$$\begin{aligned} f_T^{\mu\nu} &= \frac{1}{2x} \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) \\ &+ \left( P_N^\mu + \frac{1}{2x} q^\mu \right) \left( P_N^\nu + \frac{1}{2x} q^\nu \right) \frac{1}{P_N \cdot q} \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \Delta f^{\mu\nu} &= i \epsilon^{\mu\nu\alpha\beta} \frac{q_\alpha S_\beta}{x P_N \cdot q}, \\ f_L^{\mu\nu} &= \left( P_N^\mu + \frac{1}{2x} q^\mu \right) \left( P_N^\nu + \frac{1}{2x} q^\nu \right) \frac{x}{P_N \cdot q}. \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \epsilon_\mu^0(\epsilon_\nu^0)^* f_T^{\mu\nu} &= 0, \quad \epsilon_\mu^0(\epsilon_\nu^0)^* f_L^{\mu\nu} = \frac{1}{2}, \\ \sum \epsilon_\mu^\pm(\epsilon_\nu^\pm)^* f_T^{\mu\nu} &= \frac{1}{x}, \quad \sum \epsilon_\mu^\pm(\epsilon_\nu^\pm)^* f_L^{\mu\nu} = 0. \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} L_{\mu\nu} f_T^{\mu\nu} &= \frac{E_{\text{cm}}^2}{y} \sigma_T \delta_{\lambda_\ell \lambda_{\ell'}}, \quad L_{\mu\nu} f_L^{\mu\nu} = \frac{E_{\text{cm}}^2}{y} \sigma_L \delta_{\lambda_\ell \lambda_{\ell'}}, \\ L_{\mu\nu} \Delta f^{\mu\nu} &= \lambda_\ell S_\parallel \frac{E_{\text{cm}}^2}{y} \Delta \sigma \delta_{\lambda_\ell \lambda_{\ell'}}. \end{aligned} \quad (\text{B4})$$

Note that  $\lambda_\ell, S_\parallel = \pm$  denotes the helicity index of the lepton and proton beams.

## Appendix C: Regge limit

The NECs coefficient functions in eq. (61) exhibit logarithmic divergences in the small- $z$  limit, collected below.

$$\begin{aligned} \Delta \mathcal{I}_{qq}^{(1)}(z) &\simeq -2C_F(1 - 2 \ln z), \quad \Delta \mathcal{I}_{qg}^{(1)}(z) \simeq -4N_f(1 + \ln z), \\ \Delta \mathcal{I}_{gq}^{(1)}(z) &\simeq 4C_F(1 + 2 \ln z), \quad \Delta \mathcal{I}_{gg}^{(1)}(z) \simeq 8C_A(1 + 2 \ln z). \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} \Delta \mathcal{I}_{qq}^{(2)}(z) &\simeq \left[ \frac{22}{3} C_A C_F - 19 C_F^2 + \frac{26}{3} C_F N_f \right] \ln^3 z + \left[ \frac{79}{2} C_A C_F - 17 C_F^2 + 68 C_F N_f \right] \ln^2 z \\ &+ \left[ C_A C_F \left( \frac{656}{9} - 32 \zeta_2 \right) + C_F N_f \left( \frac{1375}{9} - 16 \zeta_2 \right) + C_F^2 (-24 + 24 \zeta_2) \right] \ln z \\ &+ C_A C_F \left( \frac{332}{27} - 44 \zeta_2 - 28 \zeta_3 \right) + C_F^2 \left( \frac{110}{3} + \frac{100}{3} \zeta_2 + 12 \zeta_3 \right) + C_F N_f \left( \frac{3805}{27} - \frac{16}{3} \zeta_2 \right). \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} \Delta \mathcal{I}_{qg}^{(2)}(z) &\simeq \left[ \frac{74}{3} C_A N_f + \frac{13}{3} C_F N_f \right] \ln^3 z + \left[ 131 C_A N_f - \frac{23}{2} C_F N_f \right] \ln^2 z \\ &+ \left[ C_A N_f (276 - 32 \zeta_2) + C_F N_f (-72 + 8 \zeta_2) \right] \ln z \\ &+ C_A N_f \left( \frac{716}{3} - \frac{92}{3} \zeta_2 + 4 \zeta_3 \right) + C_F N_f \left( -\frac{239}{3} + \frac{68}{3} \zeta_2 + 12 \zeta_3 \right). \end{aligned} \quad (\text{C3})$$

$$\begin{aligned}
\Delta\mathcal{I}_{gq}^{(2)}(z) \simeq & \left[ -\frac{148}{3}C_A C_F - \frac{26}{3}C_F^2 \right] \ln^3 z + \left[ -\frac{472}{3}C_A C_F + 6C_F^2 - \frac{32}{3}C_F N_f \right] \ln^2 z \\
& + \left[ -\frac{160}{9}C_F N_f + C_F^2(49 - 48\zeta_2) + C_A C_F \left( -\frac{3266}{9} + 96\zeta_2 \right) \right] \ln z \\
& - \frac{272}{27}C_F N_f + C_F^2(115 - 80\zeta_2 - 64\zeta_3) + C_A C_F \left( -\frac{10612}{27} + 100\zeta_2 + 32\zeta_3 \right). \quad (C4)
\end{aligned}$$

$$\begin{aligned}
\Delta\mathcal{I}_{gg}^{(2)}(z) \simeq & \left[ -\frac{296}{3}C_A^2 + \frac{26}{3}C_F N_f \right] \ln^3 z + \left[ -\frac{701}{3}C_A^2 - \frac{46}{3}C_A N_f + 29C_F N_f \right] \ln^2 z \\
& + \left[ -\frac{142}{3}C_A N_f + C_F N_f(74 - 16\zeta_2) + C_A^2 \left( -\frac{1349}{3} + 64\zeta_2 \right) \right] \ln z \\
& + C_A^2 \left( -\frac{20285}{54} + \frac{20}{3}\zeta_2 - 64\zeta_3 \right) + C_A N_f \left( -\frac{2653}{54} + \frac{40}{3}\zeta_2 \right) + C_F N_f(88 + 8\zeta_2). \quad (C5)
\end{aligned}$$

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- [1] J. R. Ellis and R. L. Jaffe, *Phys. Rev. D* **9**, 1444 (1974), [Erratum: *Phys.Rev.D* 10, 1669 (1974)].
  - [2] J. Ashman *et al.* (European Muon), *Phys. Lett. B* **206**, 364 (1988).
  - [3] J. Ashman *et al.* (European Muon), *Nucl. Phys. B* **328**, 1 (1989).
  - [4] X. Ji, F. Yuan, and Y. Zhao, *Nature Rev. Phys.* **3**, 27 (2021), [arXiv:2009.01291 \[hep-ph\]](#).
  - [5] A. Airapetian *et al.* (HERMES), *Phys. Rev. D* **71**, 012003 (2005), [arXiv:hep-ex/0407032](#).
  - [6] M. Alekseev *et al.* (COMPASS), *Phys. Lett. B* **680**, 217 (2009), [arXiv:0905.2828 \[hep-ex\]](#).
  - [7] E.-C. Aschenauer *et al.* (RHIC SPIN), (2023), [arXiv:2302.00605 \[nucl-ex\]](#).
  - [8] C. A. Aidala, S. D. Bass, D. Hasch, and G. K. Mallot, *Rev. Mod. Phys.* **85**, 655 (2013), [arXiv:1209.2803 \[hep-ph\]](#).
  - [9] A. Deur, S. J. Brodsky, and G. F. De Téramond, *Rept. Prog. Phys.* **82**, 076201 (2019), [arXiv:1807.05250 \[hep-ph\]](#).
  - [10] I. Borsa, M. Stratmann, W. Vogelsang, D. de Florian, and R. Sassot, *Phys. Rev. Lett.* **133**, 151901 (2024), [arXiv:2407.11635 \[hep-ph\]](#).
  - [11] V. Bertone, A. Chiefa, and E. R. Nocera (MAP (Multi-dimensional Analyses of Partonic distributions)), *Phys. Lett. B* **865**, 139497 (2025), [arXiv:2404.04712 \[hep-ph\]](#).
  - [12] J. Cruz-Martinez, T. Hasenack, F. Hekhorn, G. Magni, E. R. Nocera, T. R. Rabemananjara, J. Rojo, T. Sharma, and G. van Seeventer, *JHEP* **07**, 168 (2025), [arXiv:2503.11814 \[hep-ph\]](#).
  - [13] R. Abdul Khalek *et al.*, *Nucl. Phys. A* **1026**, 122447 (2022), [arXiv:2103.05419 \[physics.ins-det\]](#).
  - [14] R. Abdul Khalek *et al.*, (2022), [arXiv:2203.13199 \[hep-ph\]](#).
  - [15] X.-D. Ji, *Phys. Rev. Lett.* **78**, 610 (1997), [arXiv:hep-ph/9603249](#).
  - [16] X.-D. Ji, *Phys. Rev. D* **58**, 056003 (1998), [arXiv:hep-ph/9710290](#).
  - [17] R. L. Jaffe and A. Manohar, *Nucl. Phys. B* **337**, 509 (1990).
  - [18] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Nucl. Phys. B* **889**, 351 (2014), [arXiv:1409.5131 \[hep-ph\]](#).
  - [19] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Phys. Lett. B* **748**, 432 (2015), [arXiv:1506.04517 \[hep-ph\]](#).
  - [20] J. Blümlein, P. Marquard, C. Schneider, and K. Schönwald, *JHEP* **11**, 156 (2022), [arXiv:2208.14325 \[hep-ph\]](#).
  - [21] J. Blümlein, P. Marquard, C. Schneider, and K. Schönwald, *Nucl. Phys. B* **971**, 115542 (2021), [arXiv:2107.06267 \[hep-ph\]](#).
  - [22] J. Blümlein, P. Marquard, C. Schneider, and K. Schönwald, *JHEP* **01**, 193 (2022), [arXiv:2111.12401 \[hep-ph\]](#).
  - [23] Y. J. Zhu, (2025), [arXiv:2509.01655 \[hep-ph\]](#).
  - [24] J. She, J. Zhu, and B.-Q. Ma, *Phys. Rev. D* **79**, 054008 (2009), [arXiv:0902.3718 \[hep-ph\]](#).
  - [25] H. Avakian, A. V. Efremov, P. Schweitzer, O. V. Teryaev, F. Yuan, and P. Zavada, *Mod. Phys. Lett. A* **24**, 2995 (2009), [arXiv:0910.3181 \[hep-ph\]](#).
  - [26] H. Avakian, A. V. Efremov, P. Schweitzer, and F. Yuan, *Phys. Rev. D* **81**, 074035 (2010), [arXiv:1001.5467 \[hep-ph\]](#).
  - [27] X. Ji, J.-H. Zhang, and Y. Zhao, *Phys. Lett. B* **743**, 180 (2015), [arXiv:1409.6329 \[hep-ph\]](#).
  - [28] Y. Zhao, K.-F. Liu, and Y. Yang, *Phys. Rev. D* **93**, 054006 (2016), [arXiv:1506.08832 \[hep-ph\]](#).
  - [29] C. Alexandrou, S. Bacchio, M. Constantinou, J. Finkenrath, K. Hadjiyiannakou, K. Jansen, G. Koutsou, H. Panagopoulos, and G. Spanoudes, *Phys. Rev. D* **101**, 094513 (2020), [arXiv:2003.08486 \[hep-lat\]](#).
  - [30] C. Alexandrou, M. Constantinou, K. Hadjiyiannakou, K. Jansen, C. Kallidonis, G. Koutsou, A. Vaquero Avilés-Casco, and C. Wiese, *Phys. Rev. Lett.* **119**, 142002 (2017), [arXiv:1706.02973 \[hep-lat\]](#).
  - [31] Y.-B. Yang, R. S. Sufian, A. Alexandru, T. Draper, M. J. Glatzmaier, K.-F. Liu, and Y. Zhao, *Phys. Rev. Lett.* **118**, 102001 (2017), [arXiv:1609.05937 \[hep-ph\]](#).
  - [32] Y. V. Kovchegov, D. Pitonyak, and M. D. Sievert, *Phys. Rev. Lett.* **118**, 052001 (2017), [arXiv:1610.06188 \[hep-ph\]](#).



- [33] Y. V. Kovchegov, D. Pitonyak, and M. D. Sievert, *JHEP* **01**, 072 (2016), [Erratum: *JHEP* 10, 148 (2016)], [arXiv:1511.06737 \[hep-ph\]](#).
- [34] Y. V. Kovchegov, A. Tarasov, and Y. Tawabutr, *JHEP* **03**, 184 (2022), [arXiv:2104.11765 \[hep-ph\]](#).
- [35] F. Cougoulic, Y. V. Kovchegov, A. Tarasov, and Y. Tawabutr, *JHEP* **07**, 095 (2022), [Erratum: *JHEP* 09, 052 (2024)], [arXiv:2204.11898 \[hep-ph\]](#).
- [36] V. Ravindran, J. Smith, and W. L. van Neerven, *Nucl. Phys. B* **682**, 421 (2004), [arXiv:hep-ph/0311304](#).
- [37] R. Boughezal, H. T. Li, and F. Petriello, *Phys. Lett. B* **817**, 136333 (2021), [arXiv:2101.02214 \[hep-ph\]](#).
- [38] I. Borsa, D. de Florian, and I. Pedron, *Phys. Rev. Lett.* **125**, 082001 (2020), [arXiv:2005.10705 \[hep-ph\]](#).
- [39] I. Borsa, D. de Florian, and I. Pedron, *Phys. Rev. D* **103**, 014008 (2021), [arXiv:2010.07354 \[hep-ph\]](#).
- [40] I. Borsa, D. de Florian, and I. Pedron, *Eur. Phys. J. C* **82**, 1167 (2022), [arXiv:2210.12014 \[hep-ph\]](#).
- [41] I. Borsa, D. de Florian, and I. Pedron, *Phys. Rev. D* **107**, 054027 (2023), [arXiv:2212.06625 \[hep-ph\]](#).
- [42] C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, *Phys. Rev. D* **17**, 2298 (1978).
- [43] C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, *Phys. Rev. Lett.* **41**, 1585 (1978).
- [44] C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, *Phys. Rev. D* **19**, 2018 (1979).
- [45] C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, *Phys. Lett. B* **85**, 297 (1979).
- [46] J. C. Collins and D. E. Soper, *Phys. Rev. Lett.* **48**, 655 (1982).
- [47] D. de Florian and M. Grazzini, *Nucl. Phys. B* **704**, 387 (2005), [arXiv:hep-ph/0407241](#).
- [48] I. Moulton and H. X. Zhu, *JHEP* **08**, 160 (2018), [arXiv:1801.02627 \[hep-ph\]](#).
- [49] C. Duhr, B. Mistlberger, and G. Vita, *Phys. Rev. Lett.* **129**, 162001 (2022), [arXiv:2205.02242 \[hep-ph\]](#).
- [50] U. G. Aglietti and G. Ferrera, *Phys. Rev. D* **110**, 114004 (2024), [arXiv:2403.04077 \[hep-ph\]](#).
- [51] G. P. Korchemsky, *JHEP* **01**, 008 (2020), [arXiv:1905.01444 \[hep-th\]](#).
- [52] H. Chen, X. Zhou, and H. X. Zhu, *JHEP* **10**, 132 (2023), [arXiv:2301.03616 \[hep-ph\]](#).
- [53] I. Moulton, G. Vita, and K. Yan, *JHEP* **07**, 005 (2020), [arXiv:1912.02188 \[hep-ph\]](#).
- [54] A. Kardos, S. Kluth, G. Somogyi, Z. Tulipánt, and A. Verbitskyi, *Eur. Phys. J. C* **78**, 498 (2018), [arXiv:1804.09146 \[hep-ph\]](#).
- [55] Z. Tulipánt, A. Kardos, and G. Somogyi, *Eur. Phys. J. C* **77**, 749 (2017), [arXiv:1708.04093 \[hep-ph\]](#).
- [56] R. von Kuk, J. K. L. Michel, and Z. Sun, *JHEP* **07**, 129 (2024), [arXiv:2404.08622 \[hep-ph\]](#).
- [57] U. G. Aglietti and G. Ferrera, *Eur. Phys. J. C* **85**, 272 (2025), [arXiv:2412.02629 \[hep-ph\]](#).
- [58] Z.-B. Kang, J. Penttala, and C. Zhang, (2024), [arXiv:2410.21435 \[hep-ph\]](#).
- [59] A. Gao, H. T. Li, I. Moulton, and H. X. Zhu, *Phys. Rev. Lett.* **123**, 062001 (2019), [arXiv:1901.04497 \[hep-ph\]](#).
- [60] A. Gao, H. T. Li, I. Moulton, and H. X. Zhu, *JHEP* **09**, 072 (2024), [arXiv:2312.16408 \[hep-ph\]](#).
- [61] K. Lee, I. Moulton, and X. Zhang, *JHEP* **05**, 129 (2025), [arXiv:2409.19045 \[hep-ph\]](#).
- [62] K. Lee, I. Moulton, and X. Zhang, (2024), [arXiv:2410.01902 \[hep-ph\]](#).
- [63] X. Liu, W. Vogelsang, F. Yuan, and H. X. Zhu, *Phys. Rev. Lett.* **134**, 151901 (2025), [arXiv:2410.16371 \[hep-ph\]](#).
- [64] Y. L. Dokshitzer, G. Marchesini, and B. R. Webber, *JHEP* **07**, 012 (1999), [arXiv:hep-ph/9905339](#).
- [65] K. Lee, A. Pathak, I. W. Stewart, and Z. Sun, *Phys. Rev. Lett.* **133**, 231902 (2024), [arXiv:2405.19396 \[hep-ph\]](#).
- [66] H. Chen, P. F. Monni, Z. Xu, and H. X. Zhu, *Phys. Rev. Lett.* **133**, 231901 (2024), [arXiv:2406.06668 \[hep-ph\]](#).
- [67] K. Konishi, A. Ukawa, and G. Veneziano, *Phys. Lett. B* **80**, 259 (1979).
- [68] K. Konishi, A. Ukawa, and G. Veneziano, *Nucl. Phys. B* **157**, 45 (1979).
- [69] K. Konishi, A. Ukawa, and G. Veneziano, *Phys. Lett. B* **78**, 243 (1978).
- [70] L. J. Dixon, I. Moulton, and H. X. Zhu, *Phys. Rev. D* **100**, 014009 (2019), [arXiv:1905.01310 \[hep-ph\]](#).
- [71] H. Chen, M.-X. Luo, I. Moulton, T.-Z. Yang, X. Zhang, and H. X. Zhu, *JHEP* **08**, 028 (2020), [arXiv:1912.11050 \[hep-ph\]](#).
- [72] H. Chen, *JHEP* **01**, 035 (2024), [arXiv:2311.00350 \[hep-ph\]](#).
- [73] E. Craft, K. Lee, B. Meçaj, and I. Moulton, (2022), [arXiv:2210.09311 \[hep-ph\]](#).
- [74] H. Chen, I. Moulton, and H. X. Zhu, *JHEP* **08**, 233 (2022), [arXiv:2104.00009 \[hep-ph\]](#).
- [75] H. Chen, I. Moulton, and H. X. Zhu, *Phys. Rev. Lett.* **126**, 112003 (2021), [arXiv:2011.02492 \[hep-ph\]](#).
- [76] H. Chen, H. Ruan, and H. X. Zhu, (2025), [arXiv:2505.16753 \[hep-ph\]](#).
- [77] M.-S. Gao, Z.-B. Kang, W. Li, and D. Y. Shao, (2025), [arXiv:2509.15809 \[hep-ph\]](#).
- [78] H. T. Li, I. Vitev, and Y. J. Zhu, *JHEP* **11**, 051 (2020), [arXiv:2006.02437 \[hep-ph\]](#).
- [79] A. Ali, G. Li, W. Wang, and Z.-P. Xing, *Eur. Phys. J. C* **80**, 1096 (2020), [arXiv:2008.00271 \[hep-ph\]](#).
- [80] H. T. Li, Y. Makris, and I. Vitev, *Phys. Rev. D* **103**, 094005 (2021), [arXiv:2102.05669 \[hep-ph\]](#).
- [81] H. Cao, H. T. Li, and Z. Mi, *Phys. Rev. D* **109**, 096004 (2024), [arXiv:2312.07655 \[hep-ph\]](#).
- [82] I. Moulton and H. X. Zhu, (2025), [arXiv:2506.09119 \[hep-ph\]](#).
- [83] J. Collins, *Foundations of Perturbative QCD*, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology, Vol. 32 (Cambridge University Press, 2023).
- [84] I. I. Balitsky and L. N. Lipatov, *Sov. J. Nucl. Phys.* **28**, 822 (1978).
- [85] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Sov. Phys. JETP* **45**, 199 (1977).
- [86] I. Feige and M. D. Schwartz, *Phys. Rev. D* **90**, 105020 (2014), [arXiv:1403.6472 \[hep-ph\]](#).
- [87] C. Lee and G. F. Sterman, *Phys. Rev. D* **75**, 014022 (2007), [arXiv:hep-ph/0611061](#).
- [88] C. W. Bauer, S. Fleming, and M. E. Luke, *Phys. Rev. D* **63**, 014006 (2000), [arXiv:hep-ph/0005275](#).
- [89] C. W. Bauer, S. Fleming, D. Pirjol, and I. W. Stewart, *Phys. Rev. D* **63**, 114020 (2001), [arXiv:hep-ph/0011336](#).
- [90] C. W. Bauer and I. W. Stewart, *Phys. Lett. B* **516**, 134 (2001), [arXiv:hep-ph/0107001](#).
- [91] C. W. Bauer, D. Pirjol, and I. W. Stewart, *Phys. Rev. D* **65**, 054022 (2002), [arXiv:hep-ph/0109045](#).
- [92] G. 't Hooft and M. J. G. Veltman, *Nucl. Phys. B* **44**, 189 (1972).

- [93] I. Moulst, I. W. Stewart, F. J. Tackmann, and W. J. Waalewijn, *Phys. Rev. D* **93**, 094003 (2016), [arXiv:1508.02397 \[hep-ph\]](#).
- [94] Y. Li, D. Neill, and H. X. Zhu, *Nucl. Phys. B* **960**, 115193 (2020), [arXiv:1604.00392 \[hep-ph\]](#).
- [95] A. V. Manohar and I. W. Stewart, *Phys. Rev. D* **76**, 074002 (2007), [arXiv:hep-ph/0605001](#).
- [96] R.-b. Meng, F. I. Olness, and D. E. Soper, *Nucl. Phys. B* **371**, 79 (1992).
- [97] R. Meng, F. I. Olness, and D. E. Soper, *Phys. Rev. D* **54**, 1919 (1996), [arXiv:hep-ph/9511311](#).
- [98] P. M. Nadolsky, D. R. Stump, and C. P. Yuan, *Phys. Rev. D* **61**, 014003 (2000), [Erratum: *Phys. Rev. D* **64**, 059903 (2001)], [arXiv:hep-ph/9906280](#).
- [99] X.-d. Ji, J.-P. Ma, and F. Yuan, *Phys. Lett. B* **597**, 299 (2004), [arXiv:hep-ph/0405085](#).
- [100] Y. Koike, J. Nagashima, and W. Vogelsang, *Nucl. Phys. B* **744**, 59 (2006), [arXiv:hep-ph/0602188](#).
- [101] T. Becher, M. Neubert, and B. D. Pecjak, *JHEP* **01**, 076 (2007), [arXiv:hep-ph/0607228](#).
- [102] P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, *Phys. Rev. Lett.* **102**, 212002 (2009), [arXiv:0902.3519 \[hep-ph\]](#).
- [103] T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli, and C. Studerus, *JHEP* **11**, 102 (2010), [arXiv:1010.4478 \[hep-ph\]](#).
- [104] Y. Li and H. X. Zhu, *Phys. Rev. Lett.* **118**, 022004 (2017), [arXiv:1604.01404 \[hep-ph\]](#).
- [105] I. Moulst, H. X. Zhu, and Y. J. Zhu, *JHEP* **08**, 280 (2022), [arXiv:2205.02249 \[hep-ph\]](#).
- [106] A. B. Cuervo, I. Scimemi, and A. Vladimirov, (2025), [arXiv:2507.17478 \[hep-ph\]](#).
- [107] C. Liu, X. Shen, B. Zhou, and J. Gao, *JHEP* **09**, 108 (2023), [arXiv:2305.14620 \[hep-ph\]](#).
- [108] J. Gao, C. Liu, X. Shen, H. Xing, and Y. Zhao, *Phys. Rev. D* **110**, 114019 (2024), [arXiv:2407.04422 \[hep-ph\]](#).
- [109] A. Buckley, J. Ferrando, S. Lloyd, K. Nordström, B. Page, M. Rüfenacht, M. Schönherr, and G. Watt, *Eur. Phys. J. C* **75**, 132 (2015), [arXiv:1412.7420 \[hep-ph\]](#).
- [110] S. Dulat, T.-J. Hou, J. Gao, M. Guzzi, J. Huston, P. Nadolsky, J. Pumplin, C. Schmidt, D. Stump, and C. P. Yuan, *Phys. Rev. D* **93**, 033006 (2016), [arXiv:1506.07443 \[hep-ph\]](#).
- [111] E. R. Nocera, R. D. Ball, S. Forte, G. Ridolfi, and J. Rojo (NNPDF), *Nucl. Phys. B* **887**, 276 (2014), [arXiv:1406.5539 \[hep-ph\]](#).
- [112] J. Collins and T. Rogers, *Phys. Rev. D* **91**, 074020 (2015), [arXiv:1412.3820 \[hep-ph\]](#).
- [113] J. C. Collins and D. E. Soper, *Nucl. Phys. B* **193**, 381 (1981), [Erratum: *Nucl. Phys. B* **213**, 545 (1983)].
- [114] J. C. Collins and D. E. Soper, *Ann. Rev. Nucl. Part. Sci.* **37**, 383 (1987).
- [115] R. A. Brandt, F. Neri, and M.-a. Sato, *Phys. Rev. D* **24**, 879 (1981).
- [116] R. A. Brandt, A. Gocksch, M. A. Sato, and F. Neri, *Phys. Rev. D* **26**, 3611 (1982).
- [117] M.-H. Chu *et al.* (Lattice Parton (LPC)), *Phys. Rev. D* **106**, 034509 (2022), [arXiv:2204.00200 \[hep-lat\]](#).
- [118] P. Shanahan, M. Wagman, and Y. Zhao, *Phys. Rev. D* **102**, 014511 (2020), [arXiv:2003.06063 \[hep-lat\]](#).
- [119] P. Shanahan, M. Wagman, and Y. Zhao, *Phys. Rev. D* **104**, 114502 (2021), [arXiv:2107.11930 \[hep-lat\]](#).
- [120] A. Avkhadiev, P. E. Shanahan, M. L. Wagman, and Y. Zhao, *Phys. Rev. Lett.* **132**, 231901 (2024), [arXiv:2402.06725 \[hep-lat\]](#).
- [121] M. Schlemmer, A. Vladimirov, C. Zimmermann, M. Engelhardt, and A. Schäfer, *JHEP* **08**, 004 (2021), [arXiv:2103.16991 \[hep-lat\]](#).
- [122] Y. Li *et al.*, *Phys. Rev. Lett.* **128**, 062002 (2022), [arXiv:2106.13027 \[hep-lat\]](#).
- [123] Q.-A. Zhang *et al.* (Lattice Parton), *Phys. Rev. Lett.* **125**, 192001 (2020), [arXiv:2005.14572 \[hep-lat\]](#).
- [124] J.-y. Chiu, A. Jain, D. Neill, and I. Z. Rothstein, *Phys. Rev. Lett.* **108**, 151601 (2012), [arXiv:1104.0881 \[hep-ph\]](#).
- [125] J.-Y. Chiu, A. Jain, D. Neill, and I. Z. Rothstein, *JHEP* **05**, 084 (2012), [arXiv:1202.0814 \[hep-ph\]](#).
- [126] P. Sun, J. Isaacson, C. P. Yuan, and F. Yuan, *Int. J. Mod. Phys. A* **33**, 1841006 (2018), [arXiv:1406.3073 \[hep-ph\]](#).
- [127] M. A. Ebert, J. K. L. Michel, I. W. Stewart, and Z. Sun, *JHEP* **07**, 129 (2022), [arXiv:2201.07237 \[hep-ph\]](#).
- [128] L. Trentadue and G. Veneziano, *Phys. Lett. B* **323**, 201 (1994).
- [129] A. Berera and D. E. Soper, *Phys. Rev. D* **53**, 6162 (1996), [arXiv:hep-ph/9509239](#).
- [130] M. Grazzini, L. Trentadue, and G. Veneziano, *Nucl. Phys. B* **519**, 394 (1998), [arXiv:hep-ph/9709452](#).
- [131] D. de Florian and R. Sassot, *Phys. Rev. D* **56**, 426 (1997), [arXiv:hep-ph/9703228](#).
- [132] F. A. Ceccopieri and L. Trentadue, *Phys. Lett. B* **655**, 15 (2007), [arXiv:0705.2326 \[hep-ph\]](#).
- [133] M. Anselmino, V. Barone, and A. Kotzinian, *Phys. Lett. B* **699**, 108 (2011), [arXiv:1102.4214 \[hep-ph\]](#).
- [134] X. Liu and H. X. Zhu, *Phys. Rev. Lett.* **130**, 091901 (2023), [arXiv:2209.02080 \[hep-ph\]](#).
- [135] H. Cao, X. Liu, and H. X. Zhu, *Phys. Rev. D* **107**, 114008 (2023), [arXiv:2303.01530 \[hep-ph\]](#).
- [136] K.-B. Chen, J.-P. Ma, and X.-B. Tong, *JHEP* **08**, 227 (2024), [arXiv:2406.08559 \[hep-ph\]](#).
- [137] C. W. Bauer, S. Fleming, D. Pirjol, I. Z. Rothstein, and I. W. Stewart, *Phys. Rev. D* **66**, 014017 (2002), [arXiv:hep-ph/0202088](#).
- [138] J. A. M. Vermaseren, A. Vogt, and S. Moch, *Nucl. Phys. B* **724**, 3 (2005), [arXiv:hep-ph/0504242](#).
- [139] S. Moch, M. Rogal, and A. Vogt, *Nucl. Phys. B* **790**, 317 (2008), [arXiv:0708.3731 \[hep-ph\]](#).
- [140] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Nucl. Phys. B* **813**, 220 (2009), [arXiv:0812.4168 \[hep-ph\]](#).
- [141] E. B. Zijlstra and W. L. van Neerven, *Nucl. Phys. B* **417**, 61 (1994), [Erratum: *Nucl. Phys. B* **426**, 245 (1994), Erratum: *Nucl. Phys. B* **773**, 105–106 (2007), Erratum: *Nucl. Phys. B* **501**, 599–599 (1997)].
- [142] A. Vogt, S. Moch, M. Rogal, and J. A. M. Vermaseren, *Nucl. Phys. B Proc. Suppl.* **183**, 155 (2008), [arXiv:0807.1238 \[hep-ph\]](#).
- [143] E. B. Zijlstra and W. L. van Neerven, *Nucl. Phys. B* **383**, 525 (1992).
- [144] S. Moch and J. A. M. Vermaseren, *Nucl. Phys. B* **573**, 853 (2000), [arXiv:hep-ph/9912355](#).
- [145] V. N. Gribov and L. N. Lipatov, *Sov. J. Nucl. Phys.* **15**, 438 (1972).
- [146] L. N. Lipatov, *Yad. Fiz.* **20**, 181 (1974).
- [147] G. Altarelli and G. Parisi, *Nucl. Phys. B* **126**, 298 (1977).
- [148] M.-X. Luo, X. Wang, X. Xu, L. L. Yang, T.-Z. Yang, and H. X. Zhu, *JHEP* **10**, 083 (2019), [arXiv:1908.03831](#)

- [hep-ph].
- [149] M.-X. Luo, T.-Z. Yang, H. X. Zhu, and Y. J. Zhu, *JHEP* **01**, 040 (2020), [arXiv:1909.13820 \[hep-ph\]](#).
  - [150] M.-x. Luo, T.-Z. Yang, H. X. Zhu, and Y. J. Zhu, *JHEP* **06**, 115 (2021), [arXiv:2012.03256 \[hep-ph\]](#).
  - [151] M. A. Ebert, B. Mistlberger, and G. Vita, *JHEP* **07**, 121 (2021), [arXiv:2012.07853 \[hep-ph\]](#).
  - [152] M. A. Ebert, B. Mistlberger, and G. Vita, *JHEP* **09**, 146 (2020), [arXiv:2006.05329 \[hep-ph\]](#).
  - [153] J. G. Korner, D. Kreimer, and K. Schilcher, *Z. Phys. C* **54**, 503 (1992).
  - [154] D. Kreimer, (1993), [arXiv:hep-ph/9401354](#).
  - [155] Y. Matiounine, J. Smith, and W. L. van Neerven, *Phys. Rev. D* **58**, 076002 (1998), [arXiv:hep-ph/9803439](#).
  - [156] P. J. Rijken and W. L. van Neerven, *Nucl. Phys. B* **523**, 245 (1998), [arXiv:hep-ph/9711335](#).
  - [157] P. Breitenlohner and D. Maison, *Commun. Math. Phys.* **52**, 11 (1977).
  - [158] S. A. Larin and J. A. M. Vermaseren, *Phys. Lett. B* **259**, 345 (1991).
  - [159] E. B. Zijlstra and W. L. van Neerven, *Phys. Lett. B* **297**, 377 (1992).
  - [160] S. A. Larin, *Phys. Lett. B* **303**, 113 (1993), [arXiv:hep-ph/9302240](#).
  - [161] D. Gutiérrez-Reyes, I. Scimemi, and A. A. Vladimirov, *Phys. Lett. B* **769**, 84 (2017), [arXiv:1702.06558 \[hep-ph\]](#).
  - [162] S. L. Adler and W. A. Bardeen, *Phys. Rev.* **182**, 1517 (1969).
  - [163] W. Vogelsang, *Nucl. Phys. B* **475**, 47 (1996), [arXiv:hep-ph/9603366](#).
  - [164] W. Vogelsang, *Phys. Rev. D* **54**, 2023 (1996), [arXiv:hep-ph/9512218](#).
  - [165] C. Duhr and F. Dulat, *JHEP* **08**, 135 (2019), [arXiv:1904.07279 \[hep-th\]](#).
  - [166] J. C. Collins, *Phys. Rev. D* **57**, 3051 (1998), [Erratum: *Phys. Rev. D* **61**, 019902 (2000)], [arXiv:hep-ph/9709499](#).
  - [167] I. Z. Rothstein and I. W. Stewart, *JHEP* **08**, 025 (2016), [arXiv:1601.04695 \[hep-ph\]](#).
  - [168] M. D. Schwartz, K. Yan, and H. X. Zhu, *Phys. Rev. D* **97**, 096017 (2018), [arXiv:1801.01138 \[hep-ph\]](#).
  - [169] L. N. Lipatov and M. I. Vyazovsky, *Nucl. Phys. B* **597**, 399 (2001), [arXiv:hep-ph/0009340](#).
  - [170] I. Moutl, M. P. Solon, I. W. Stewart, and G. Vita, *JHEP* **02**, 134 (2018), [arXiv:1709.09174 \[hep-ph\]](#).
  - [171] Y. V. Kovchegov and M. D. Sievert, *Phys. Rev. D* **99**, 054032 (2019), [arXiv:1808.09010 \[hep-ph\]](#).
  - [172] X. Liu and H. X. Zhu, (2024), [arXiv:2403.08874 \[hep-ph\]](#).
  - [173] D. Boer and P. J. Mulders, *Phys. Rev. D* **57**, 5780 (1998), [arXiv:hep-ph/9711485](#).
  - [174] D. W. Sivers, *Phys. Rev. D* **41**, 83 (1990).
  - [175] J. C. Collins, *Phys. Lett. B* **536**, 43 (2002), [arXiv:hep-ph/0204004](#).
  - [176] S. J. Brodsky, D. S. Hwang, and I. Schmidt, *Nucl. Phys. B* **642**, 344 (2002), [arXiv:hep-ph/0206259](#).
  - [177] S. J. Brodsky, D. S. Hwang, and I. Schmidt, *Phys. Lett. B* **530**, 99 (2002), [arXiv:hep-ph/0201296](#).
  - [178] S. J. Brodsky, D. S. Hwang, Y. V. Kovchegov, I. Schmidt, and M. D. Sievert, *Phys. Rev. D* **88**, 014032 (2013), [arXiv:1304.5237 \[hep-ph\]](#).
  - [179] A. V. Belitsky, X. Ji, and F. Yuan, *Nucl. Phys. B* **656**, 165 (2003), [arXiv:hep-ph/0208038](#).