BIG 'OH' NOTATION PRACTICE EXERCISES

Definition 1. For a positive function g, we say that $f(x) = O(g(x))$ as $x \to \infty$ if there exists an x_0 and a constant $C > 0$, independent of x_0, such that

$$|f(x)| \leqslant Cg(x)$$

for all $x \geqslant x_0$.

So, the sign of $f(x)$ is irrelevant as are any constants; we are just interested in the dominant behaviour/general size. Throughout, the big 'Oh' terms are for $x \to \infty$.

1. Polynomials and rational functions

Prove the following

(1) $x = O(x)$
(2) $2x = O(x)$
(3) $-x/5 = O(x)$
(4) $2x + 5 = O(x)$
(5) $-3x + 8 = O(x)$
(6) $x = O(x^2)$
(7) $5x + 2 = O(x^2)$
(8) $x^2 - 2x + 10 = O(x^2)$
(9) $-x^2 + 5x + 1 = O(x^2)$
(10) $x^3 + 3x - 1 = O(x^3)$
(11) $-7x^3 - x^2 + x - 3 = O(x^3)$
(12) $x^4 + x^3 + x^2 + x + 1 = O(x^4)$
(13) $16(x + 7)^4 = O(x^4)$

(14) A general polynomial of degree n is $O(x^n)$.

(15) $\dfrac{1}{x} = O(1)$

(16) $\dfrac{1}{2x+1} = O(1/x)$

(17) $\dfrac{1}{x^2} = O(1)$

(18) $\dfrac{1}{x^2} = O(1/x)$

(19) $\dfrac{1}{2x^2+3x+2} = O(1/x^2)$

(20) $\dfrac{x}{x+1} = O(1)$

(21) $\dfrac{x}{2x^2-1} = O(1/x)$

(22) $\dfrac{x}{1-2x^2} = O(1/x)$

(23) $\dfrac{x}{x^3-1} = O(1/x^2)$

(24) $\dfrac{x^2}{2x^4-4x} = O(1/x^2)$

(25) $\dfrac{x^2}{1-x} = O(x)$

(26) $\dfrac{x^4}{x^2+3x-1} = O(x^2)$

(27) Let $R(x) = P(x)/Q(x)$ where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m. Prove that $R(x) = O(x^{n-m})$.

(28) Let $R(x) = P(x)/Q(x)$ where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree $\geqslant n+2$. Suppose $Q(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that

$$\int_{-\infty}^{\infty} R(x)dx$$

converges.

2. Exponentials

(29) (Fundamental) Prove that $x^r = O(e^x)$ for any $r \in \mathbb{R}$.

Prove the following.

(30) $x + e^x = O(e^x)$

(31) $e^x - x^2 = O(e^x)$

(32) $x^3 + e^{-x} - e^{2x} = O(e^{2x})$

(33) $x^2 = O(e^{x/2})$

(34) $x^2 e^{-x} = O(e^{-x/2})$

(35) $x^r e^{-x} = O(e^{-cx})$ $\forall r > 0, c < 1$.

(36) $e^{ax} = O(e^{bx})$ $\forall b \geqslant a$

(37) $e^{100x} = O(e^{x^{1.01}})$

(38) $e^{cx} = O(e^{x^{1+\epsilon}})$ $\forall c, \epsilon > 0$

(39) $\dfrac{1}{e^x + 2} = O(e^{-x})$

(40) $\dfrac{e^x}{e^x + 1} = O(1)$

(41) $\dfrac{e^x + x}{e^x + x^2} = O(1)$

(42) $\dfrac{e^x + x^2}{e^x + x} = O(1)$

(43) $\dfrac{5x^3 + e^x}{e^{2x} - 1} = O(e^{-x})$

3. Logarithms

(44) (Fundamental) Prove that $\log x = O(x^\epsilon)$ for any $\epsilon > 0$. (hint: use a result from the previous section, or write $\log x$ as an integral and compare the integrand with something)

Prove the following

(45) $(\log x)^2 = O(x^{0.0001})$

(46) $(\log x)^r = O(x^\epsilon)$ $\forall r, \epsilon > 0$

(47) $\int_1^\infty \frac{(\log x)^r}{x^{1+\epsilon}} dx$ converges $\forall r, \epsilon > 0$

(48) $\dfrac{x + \log x}{2x + (\log x)^2 + 1} = O(1)$

(49) $\dfrac{2x + (\log x)^2 + 1}{x + \log x} = O(1)$

(50) $\log\log x = O((\log x)^\epsilon)$

(51) $\log\log x = O(\frac{\log x}{\log\log x})$

(52) $\log x = O(x^{1/\log\log x})$

(53) $x^{1/\log\log x} = O(x^\epsilon)$

(54) $(\log x)^{\log\log x} = O(x^\epsilon)$

(55) $\frac{1}{x} = O(e^{-c\sqrt{\log x}})$, $c > 0$

Thus far, we have the following basic chain of orders of growth:

$$C = O(\log x) = O(x^r) = O(e^x), \qquad r > 0$$

The chain is reversed on taking reciprocals. Also, it is easy to find intermediate orders (as in exercises (52),(53)) by writing everything in terms of exponentials. We now seek more accuracy in our formulas by writing them as a dominant term + $O($ something smaller).

4. Asymptotics

Definition 2. We write $f(x) = g(x) + O(h(x))$ if $f(x) - g(x) = O(h(x))$ i.e. if there exists a constant $C > 0$ and an x_0 such that $|f(x) - g(x)| \leqslant Ch(x)$ for all $x \geqslant x_0$.

Prove the following

(56) $x^2 + 3x + 1 = x^2 + O(x)$

(57) $x^2 + \frac{1}{100}x^{3/2} + 3x + 1 = x^2 + O(x^{3/2})$

(58) $e^x + x^r + \log x + C = e^x + O(x^r), \forall r > 0$

(59) $e^x + x^{-r} + \log x + C = e^x + O(\log x)$, $\forall r > 0$

(60) $\dfrac{x}{x+1} = 1 + O(1/x)$

(61) $\dfrac{x}{2x^2 - 1} = \dfrac{1}{2x} + O(1/x^2)$

(62) $\dfrac{e^x + x}{e^x + x^2} = 1 + O(xe^{-x})$

(63) $\dfrac{e^x + x^2}{e^x + x} = 1 + O(x^2 e^{-x})$

(64) If $f(x) = O(\frac{1}{x})$ then $e^{f(x)} = 1 + O(1/x)$

(65) $\log(1 + \frac{1}{x}) = \frac{1}{x} + O(1/x^2)$

(66) $(1 + \frac{1}{x})^x = e + O(1/x)$

(67) If $f(x) = O(\frac{1}{x})$ then $(1 + f(x))^{-1} = 1 + O(1/x)$

(68) $(1 + \frac{1}{x})^{-x} = e^{-1} + O(1/x)$

(69) If $f(x) = g(x) + O(h(x))$ then $f(x)\nu(x) = g(x)\nu(x) + O(h(x)|\nu(x)|)$

(70) $(x + 1 + 1/x)^x = ex^x + O(x^{x-1})$

Now that we know big 'Oh' a little better, we can start abusing it. When we write $O(f(x))$ we really mean a class of functions that grows no greater than $Cf(x)$ in absolute value. For example, the expression $e^{O(1/x)}$ really means $e^{f(x)}$ where $f(x)$ is some function that grows no greater than C/x in absolute value. With this interpretation

$$e^{O(1/x)} = 1 + O(1/x)$$

by exercise (64). As another example, we would think of $x + O(1)$ as $x + f(x)$ where $f(x)$ is some function that is bounded as $x \to \infty$. Prove the following.

(71) $x^3 + O(x^2) = O(x^3)$

(72) $O(x^2) + O(x) = O(x^2)$

(73) $O(1/x) + O(1/x^2) = O(1/x)$

(74)

$O(f(x)) + O(g(x)) = O(\max\{f(x), g(x)\})$

(75) $C + O(1) = O(1)$ (loss of info.)

(76) $1 + 1/x + O(x) = O(x)$ (...and again)

(77) $O(f(x)) \cdot O(g(x)) = O(f(x)g(x))$

(78) $(x + O(1))^2 = x^2 + O(x)$

(79) $(1 + O(1/x))^2 = 1 + O(1/x)$

(80) $(1 + O(1/x))^{-1} = 1 + O(1/x)$

(81) $(x + 1 + O(1/x))^x = ex^x + O(x^{x-1})$

and finally,

(82) Use Stirling's formula

$$n! = \sqrt{2\pi} n^{n+1/2} e^{-n} (1 + O(1/n))$$

to show that the limit $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$ converges at a rate of $O(\log n/n)$.

5. Complex analysis

By virtue of the (complex) absolute value $|\cdot|$, the above definitions also makes sense for complex valued stuff. Prove the following. Throughout, $z = x + iy$ and we choose the branch of $\log z$ with $|\arg z| < \pi$.

(83) $\dfrac{1}{z^2+1} = O(1/|z|^2)$, as $|z| \to \infty$

(84) $\dfrac{z}{z^3+100} = O(1/|z|^2)$

(85) $\dfrac{z^2+1}{z^4+2z^2+2} = O(1/|z|^2)$

(86) $R(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{j=0}^{n} a_j z^j}{\sum_{j=0}^{n+2} b_j z^j} = O(1/|z|^2)$

(87) If $Q(x) \neq 0$, $x \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{\substack{z_j : Q(z_j)=0 \\ \Im(z_j)>0}} \operatorname{res}(R(z), z_j)$$

(88) $\frac{1}{1+iy} = O(1/|y|)$

(89) $\frac{1}{100+iy} = O(1/|y|)$

(90) $\dfrac{1}{z^2} = O(1/|y|^2)$, x fixed, $y \to \pm\infty$

(91) The vertical line integral $\int_{c-i\infty}^{c+i\infty} \frac{e^z}{z^2} dz$, $c > 0$, is absolutely convergent.

(92) $|\sin(z)| = \frac{1}{2}e^y + O(e^{-y})$, uniformly in x as $y \to \pm\infty$.

(93) $\tan(\pi n + iy)$ is bounded for all $y \in \mathbb{R}$, $n \in \mathbb{Z}$.

(94) $\tan z = i + O(e^{-2y})$, uniformly in x as $y \to \infty$.

(95) $\log \tan z = i\pi/2 + O(e^{-2y})$, uniformly in x as $y \to \infty$.

(96) $|z| = x(1 + O(1/x))$ for fixed y, as $x \to \infty$.

(97) $\log z = \log x + O(1/x)$ for fixed y, as $x \to \infty$.

(98) $\log z = \log |y| \pm i\pi/2 + O(1/y)$ for fixed x, as $y \to \pm\infty$.

(99) $|z^w| = |y|^{\Re(w)} e^{\mp\pi\Im(w)/2}(1 + O(1/|y|))$ for fixed x, w, as $y \to \pm\infty$.

(100) $\int_{c-i\infty}^{c+i\infty} \frac{e^z}{z^w} dz$, $c > 0$, is absolutely convergent for $\Re(w) > 1$.