Coalition Existence Probability Report

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1 Introduction

The primary start point for my new power index is to build off the work done by J. Deegan and E.W. Packel in 1979. More specifically, I will use the work they did on Minimal Winning Coalitions[5] as inspiration. From there, I shall add on new calculations in order to generate a more realistic approach to generating relative power indices. In this report, I shall look at the concepts and calculations that I shall be implementing and evaluating the positives and negatives of such a system. Before doing so, however, we must first define some prerequisite concepts.

1.1 Coalition Existence Probability

CEP stands for Coalition Existence Probability. This is the likelihood that two parties will form a coalition with each other and is modelled as a complete weighted graph where each edge is the percentage chance the two nodes connected will join together in a coalition. All edges in a coalition must be multiplied together to find the complete CEP of the coalition.

1.2 Conductance

The conductance of a graph G, where $G = (V, E)$, is a measure of how strong a partition of the graph is compared to the other.[1] Typically the partitions are made by a single cut, resulting in two partitions $(S, \bar{S})$.[3] Conductance can also be thought of as a numerical measure of if a graph is bottle-necked or how many it might relatively have.[2] The conductance of a partition is calculated using two properties of the partition. The first we shall call the internal strength X of the partition. The second of which is the external pull Y of the partition. The conductance of G is calculated using the following formula:

$$\psi = \frac{Y}{X + Y}$$

Where

$$Y = \sum_{i \in S, j \in \bar{S}} a_{ij}$$

and

$$X + Y = \min(a(S), a(\bar{S}))$$

Where

$$a(S) = \sum_{i \in S} \sum_{j \in V} a_{ij}$$

$$a(\bar{S}) = \sum_{i \in \bar{S}} \sum_{j \in V} a_{ij}$$

This results in X being two times the sum of all the weights of edges solely within the partition, since by summing the weight of every edge from an element of S to an element of V you will count the edges where i and $j \in S$ twice. Y is then equal to the sum of the weights of all edges that go between S and $\bar{S}$. This leaves $a(S)$ being equal to the sum of all edges that connect the nodes in one of the partitions to any other node in G but counts those solely within the partition twice.

1.3 Minimal Winning Coalitions

In order to define a Minimal Winning Coalition, some other more basic concepts should be defined first. Firstly, the set we are calculating over shall be defined as the set P where $\{p_1, p_2, \ldots p_n\}$ are the number of votes each party in a system received. These shall all be positive integers. Secondly a target T shall be defined as the number of votes needed for a party or coalition to form a government. This then means that a winning coalition C where $w \in C$ and $C \subseteq P$ is defined as:

$$\sum_{i \in C}^{n} w_i \geq T$$

From these definitions we can then say that an MWC C exists when the removal of any w results in the sum of the remaining elements being less than T

$$\sum_{i \in C}^{n} w_i - \min(w_i) < T$$

2 Modelling the System

The fundamental idea behind this system is that the political system of a region or entity can be modelled as a graph. More specifically, the chances of each party forming a coalition with the other parties can be modelled as a graph. These chances are the CEPs defined above. Since we need the CEPs for each party and every other party in the system, we end up with a complete graph. This is a graph in which all nodes are connected to all other nodes in said graph. Another definition is that the degree of every node in the graph is $n-1$ where n is the number of nodes in the graph. An arbitrary complete weighted graph of four nodes may look something like Figure 1

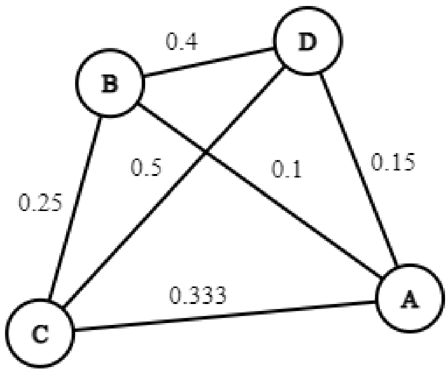

Figure 1: A complete weighted graph with four nodes.

It is from these graphs that we can then compute the conductance of each MWC.

2.1 The Data Required

Now we are aware of how we are modelling this problem, the next thing we should know is what data we need to create a working model and successfully compute the conductance of each MWC. Firstly, and most critically for the graphical model, we need to know how many parties are in a system and the likelihoods that each party will join a coalition with each of the others. For real world examples I intend to use historical trends for this. In fictional examples, these probabilities will be arbitrary. The second piece of data needed relates to the votes themselves. This is the set P discussed earlier. Once we have these two pieces of data we are then able to compute the conductance and CEPs

2.2 An Example

To begin we must first find all MWCs in the voting system, these will be our partitions we create in the graph later. Once the MWCs are found, the next step is to generate the complete weighted graph for the voting system. After the graph is generated we can compute the conductance, as defined above, for each MWC. In addition to this we must also multiply all the edges used in calculating X together to find the CEP of the corresponding MWC Lets follow these steps using an example:

$$[12|4, 6, 3, 9, 2];$$

$$\begin{array}{cccccc} & a & b & c & d & e \\ a & \backslash & 0.4 & 0.2 & 0.2 & 0.7 \\ b & & \backslash & 0.3 & 0.5 & 0.3 \\ c & & & \backslash & 0.5 & 0.7 \\ d & & & & \backslash & 0.1 \\ e & & & & & \backslash \end{array}$$

The MWCs in this example are:

$$M = \{\langle a, d\rangle, \langle b, d\rangle, \langle c, d\rangle, \langle a, b, c\rangle, \langle a, b, e, \rangle\}$$

Figure 2 displays the graph that is produced from these MWCs.

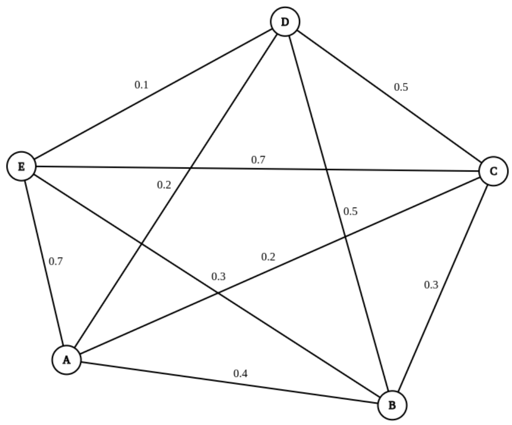

Figure 2: The graph produced from the set of MWCs M.

For each MWC, the conductance and CEP are as follows:

$$\psi_{m_1} = \frac{0.1 + 0.5 + 0.5 + 0.4 + 0.2 + 0.7}{2(0.2) + 0.1 + 0.5 + 0.5 + 0.4 + 0.2 + 0.7} = \frac{6}{7};$$

$$CEP_{m_1} = 0.2$$

$$\psi_{m_2} = \frac{9}{14};$$

$$CEP_{m_2} = 0.5$$

$$\psi_{m_3} = \frac{2}{3};$$

$$CEP_{m_3} = 0.5$$

$$\psi_{m_4} = \frac{29}{47};$$

$$CEP_{m_4} = 0.2 \times 0.3 \times 0.4 = 0.024$$

$$\psi_{m_5} = \frac{15}{43};$$
$$CEP_{m_5} = 0.084$$

Now we have all the results we need to continue on to calculate the relative power of each coalition, however that is outside the scope of this report, and will be detailed at a later time.

3 Evaluation

When creating a new system or idea, it is important to evaluate its merits and drawbacks compared to other systems within the field. This allows for a fair analysis of the system which can be used to determine its usability and functionality.

3.1 Positives

Firstly, let's look at the pros of the method that I have detailed in this report. Due to the use of MWCs the number of coalitions and therefore the number of calculations required are inherently lower than in more prominent power indices like Shapley-Shubik or Banzhaf. [7][8] For example, the running time of the Shapley-Shubik Power Index is $\Theta(n!)$ [7] while the worst case for this system is $O(2^n)$. This is due to the fact that, in a worst case example, all parties would have the same number of votes and the target would be a simple majority.

$$[30|10, 10, 10, 10, 10, 10]$$

Figure 3: An example of a worst case scenario.

In this particular example, the number of MWCs that need to be calculated over is 20. In general, in the case where all parties receive the same amount of votes, and the target is the sum of half the votes of all parties, then the number of MWCs that exist is:

$$\binom{n}{\frac{n}{2}}$$

Where n is the number of parties in the system. Since this number grows at an exponential rate, we can say that the worst case running time is as fast, if not faster, than the two most commonly used power indices.

Another benefit of this system is that it provides a more realistic approach to modelling the power of political parties. It does this by doing what no other power index does: taking the likelihood of coalitions actually existing into account. The CEPs calculated within the system provide a more detailed analysis of power within a voting system, as parties that are more willing to join with other parties are somewhat more likely to be in the final winning coalition. In other words, parties able to put aside their differences are more likely to join forces and win the vote. This is what, to some extent, my system captures.

3.2 Negatives

The largest issue with this system is the amount of data needed. There needs to be a sufficient amount of data in order to add weights to the edges of the graph. While the graph should, in theory, be a complete graph but the weights can still be zero. Issues will occur however if most of the weightings are zero, resulting in an inaccurate power index. While these probabilities are not incalculable, they can be hard for the average person to find, and this should be taken into account when considering the usability of this method.

4 Conclusion

It is my belief that the system I intend to use has a lot of great benefits that drastically improve the realism of power indices. While the challenge of data collection is one of non-insignificant proportions, it seems to be one that is outweighed by the advantages of running time and real world accuracy. As such I will be moving forward with my new power index, with the intent to utilise the systems and ideas discussed in this report, hopefully with great success.

5 References

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