Power Index Report

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1 Introduction

Typically we see power indices calculate the power value of a party, within a voting system, and then normalise those values to reach a final relative power ranking. This is they way almost all common power indices work[1][2][3] and makes complete sense within the context of the individual index. The primary difference between each of these indices is how they go about with the initial calculations on each parties power value. We shall look at some examples later.

This report details the advancements made since the Coalition Existence Probability Report earlier this month. The mathematics detailed in that report stand alone, however, the mathematics that I shall go through in this one are built on top of the equations of conductance and coalition existence probability from the previous report.
In this report I shall look at what the continuation of the overall algorithm looks like and give a full working run through of the new algorithm, but beforehand let's look at two notable examples in the field already that demonstrate calculating a standard power value.

2 Examples in the Field

2.1 Deegan-Packel (1978)

The Deegan-Packel power index is a lesser known power index but that does not define its utility. It uses minimal winning coalitions when computing the index of a voting system[2], which, of the major power indices, makes it unique in that regard. I will explain how it works by working through and example. Let's take the following voting system:

$$[20|12, 10, 14, 6]$$

Here, 20 is the number of votes needed by a coalition to win. Party A has 12 votes, B has 10 votes and so on. As discussed in my previous report, a minimal winning coalition is one where the removal of any party results in the winning coalition becoming a losing one. With that in mind, all MWCs are as follows;

$$\langle A, B\rangle, \langle B, C\rangle, \langle C, D\rangle, \langle A, C\rangle$$

From here we can then calculate power value of each party by summing the reciprocal of the number of parties in that coalition for all coalitions that party appears in. Taking A for example, there are two parties in both coalitions that it is in. This results in $\frac{1}{2} + \frac{1}{2} = 1$. For B this is also the case. For party D, it only appears in one coalition, made up of two parties, giving it a power value of $\frac{1}{2}$. Finally, C has a power value of $\frac{3}{2}$.

2.2 Shapley-Shubik (1954)

The Shapley-Shubik power index is perhaps the most popular and well known power index in use today. It is unique in that it makes use of ordered coalitions[1] to calculate the power value of each party. This means that when using this particular algorithm, you need all possible permutations of coalitions from the parties in the system. Let's look at the following example:

$$[8|3, 6, 3]$$

This is our voting system. Party A has 3 votes, party B has 6 votes and party C has 3 votes. The target each coalition needs to reach is 8. When using this index, order matters. That is to say $\langle A, B \rangle$ is not the same as $\langle B, A \rangle$. This is because Shapley-Shubik makes use of what are called *Pivotal Parties*. Every time a party is the pivotal party, (a party is the pivotal party when adding that party to a coalition results in the coalition reaching the target) it gains a point, or a tally. It is also important to remember that Shapley-Shubik only uses maximal, or grand coalitions. These are where all parties in each coalition. Given all the possible coalitions are:

$$\langle A, B, C \rangle, \langle A, C, B \rangle, \langle B, A, C \rangle, \langle B, C, A \rangle, \langle C, A, B \rangle, \langle C, B, A \rangle$$

We get the following tally chart of times each party was the pivotal party in a coalition.

Party	Tally
A	1
B	4
C	1

Now, we turn these tallies into fraction by dividing each tally by the number of coalitions used to get the power value of each party.

Party	Power Value
A	0.16666
B	0.66666
C	0.16666

2.3 Normalising the Power Values

We have seen how two power indices work but have not touched on the final part of the Deegan-Packel power index. Although, sneakily, we did it in the Shapley-Shubik index. This final part is the normalisation of the power values. This is what gives us the relative power of a relative power index. Normalising is as follows:

$$\frac{P_i}{\sum_{k=1}^{n} P_k}$$

Where P_i is the parties power value you want normalised and n is the total number of parties in the voting system. Effectively, take the party you want to normalise and divide its power value by the sum of all power values in the system. I will leave it as an exercise for the reader to normalise the example given in the Deegan-Packel section, and also, to explain why normalisation has already occurred in the Shapley-Shubik example.

3 How the Power Index Works

In continuation of the previous report, discussing the conductance ψ and the CEP of a graph, we now move forward with using those two values to calculate the power value and then the power index of each party. First, a reminder of what the conductance of a graph means. The conductance ψ of a graph is the proportion the external pull of a partition is of the sum of the external and internal pull of said partition. That is to say, the higher ψ is, the more external weight there is pulling the coalition apart. With that in mind, the first calculation to make once you have ψ and CEP for each minimal winning coalition is to divide the latter by the former.

$$\Lambda = \frac{CEP}{\psi}$$

Once Λ has been calculated for each MWC, a tally must be made of each time a party appears in an MWC. Second, you must take each tally and find the fraction of all tallies each party has. The Λ of each party is the largest Λ from an MWC that contains that party.

$$\Lambda_i = \frac{Tally_i}{\sum_{k=1}^{n} Tally_k}$$

After the power value of each party has been found, normalise each of them to find the final power index.

4 A Complete Example

Let's walk through, from start to end, an example. Let's use the voting system:

$$[15|3, 4, 7, 9, 6]$$

The likelihood of each party joining another in a coalition is:

	a	b	c	d	e
a		0.7	0.4	0.1	0.2
b			0.6	0.2	0.1
c				0.6	0.8
d					0.7
e					

The first step is to find all the minimal winning coalitions. In this case they are as follows:

$$\langle D, E\rangle, \langle C, D\rangle, \langle B, C, E\rangle, \langle A, B, D\rangle, \langle A, C, E\rangle$$

The graph from the matrix is presented in Figure 1

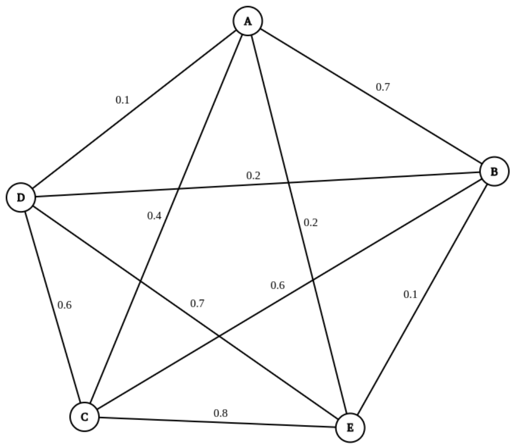

Figure 1: The likelihood matrix modelled as a complete weighted graph.

From the graph in Figure 1 we can calculate the conductance ψ and the CEP of each MWC.

$$\psi_1 = \frac{2}{3.4}, CEP_1 = 0.7$$

$$\psi_2 = \frac{2.8}{4}, CEP_2 = 0.6$$

$$\psi_3 = \frac{2.8}{5.8}, CEP_3 = 0.048$$

$$\psi_4 = \frac{2.6}{4.6}, CEP_4 = 0.014$$

$$\psi_5 = \frac{2.8}{5.6}, CEP_5 = 0.064$$

Now we must calculate Λ by dividing CEP by ψ

$$\Lambda_1 = 1.1900$$

$$\Lambda_2 = 0.8571$$

$$\Lambda_3 = 0.0994$$

$$\Lambda_4 = 0.2477$$

$$\Lambda_5 = 0.1280$$

Once Λ has been calculated for each MWC, we need to make a tally of each time a party appears in an MWC. That tally chart looks like this:

Party	Tally	Fraction
A	2	$\frac{2}{13}$
B	2	$\frac{2}{13}$
C	3	$\frac{3}{13}$
D	3	$\frac{3}{13}$
E	3	$\frac{3}{13}$

We now have everything we need to find the final power index of each party, we just need to select the correct Λ for each of them. The correct Λ will be the largest Λ from an MWC that a party was a part of. For A, that will be Λ_4. For B, it will be Λ_4. C will use Λ_2, and D and E will use Λ_1. Adding this on to our table we get:

Party	Tally	Fraction	Λ
A	2	$\frac{2}{13}$	0.2477
B	2	$\frac{2}{13}$	0.2477
C	3	$\frac{3}{13}$	0.8571
D	3	$\frac{3}{13}$	1.1900
E	3	$\frac{3}{13}$	1.1900

Multiplying the fractions with the respective Λ we get power values of

$$0.0381, 0.0381, 0.1978, 0.2746 \text{ and } 0.2746 \text{ for A, B, C, D and E respectively}$$

Normalising these values gives us our final power index which is displayed in figure 2

Party	Power Index
A	0.0463
B	0.0463
C	0.2403
D	0.3336
E	0.3336

Figure 2: The final power index of the example.

5 Conclusion

To conclude, the complete algorithm takes the work done by Deegan and Packel and adds the idea that coalitions should be given some form of probability of all the parties in them working together. It is this probability in combination with the alternatives available to each coalition that provides a weighting to be applied to the Deegan-Packel index. Once the weighting is applied and the values normalised, it provides the final power index for the voting system.

6 References

[1] Shapley, L. and Shubik, M., 1954. A Method for Evaluating the Distribution of Power in a Committee System. American Political Science Review, 48(3), pp.787-792.

[2] Deegan, J. and Packel, E., 1978. A new index of power for simple n-person games. International Journal of Game Theory, 7(2).

[3] Banzhaf, J., 1965. Weighted Voting Doesn't Work: A Mathematical Analysis. Rutgers Law Review.