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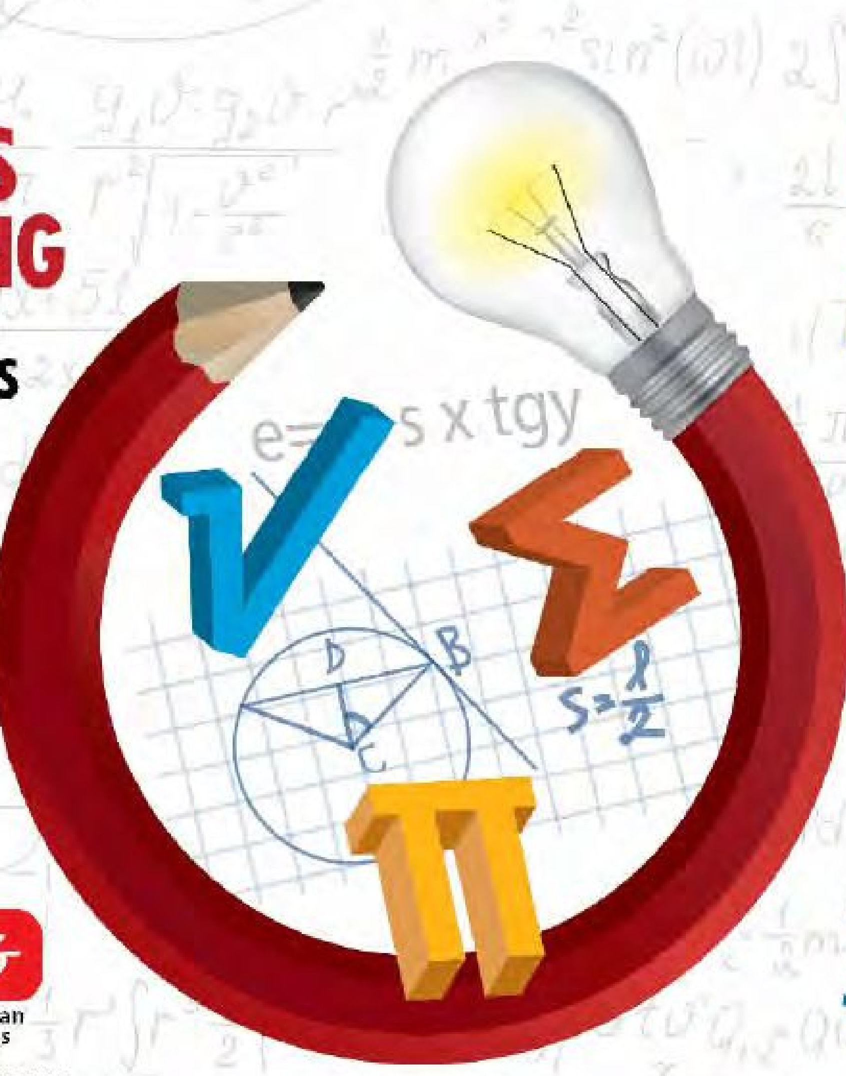

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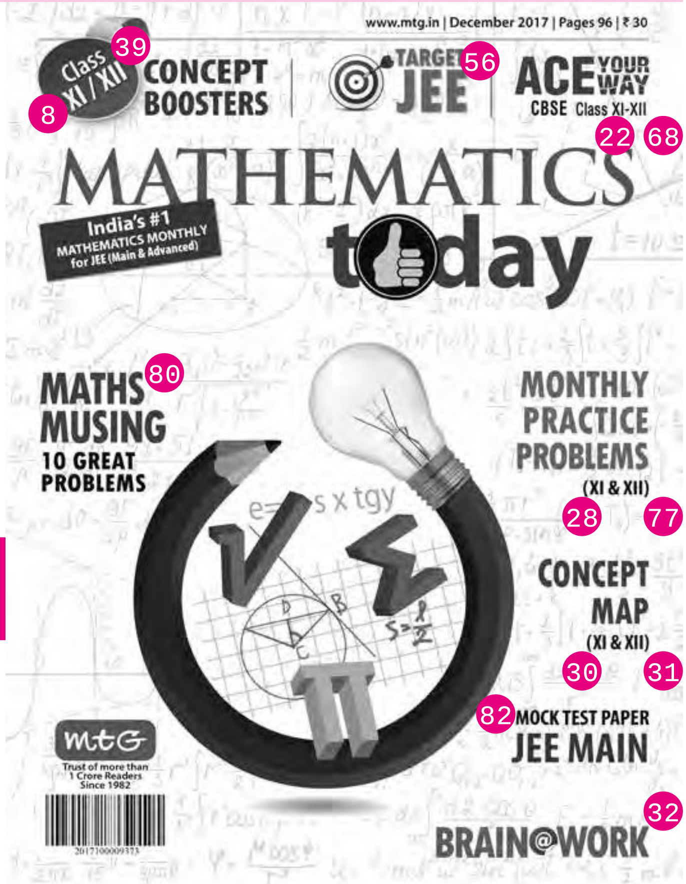

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Editor : Anil Ahlawat

CONCEPT BOOSTERS

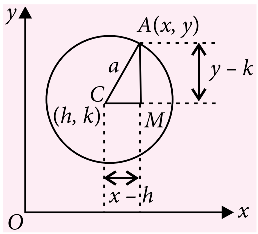
Circles

Class XI

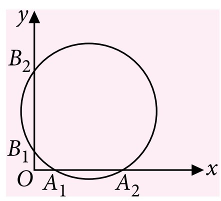
This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

- The circle with centre (h, k) & radius 'a' has the equation;
  $(x - h)^2 + (y - k)^2 = a^2$
  As in triangle ACM,
  $CM^2 + AM^2 = AC^2$
  $\Rightarrow (x - h)^2 + (y - k)^2 = a^2$



- General equation $x^2 + y^2 + 2gx + 2fy + c = 0$, can be written as
  $(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$
  or $(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2+f^2-c}\right)^2$ ...(i)
  Comparing eq (i) with $(x - h)^2 + (y - k)^2 = a^2$,
  we get $h = -g,\ k = -f,\ a = \sqrt{g^2+f^2-c}$
  So equation (i) represents a circle with centre$(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.
  If $g^2 + f^2 - c > 0$ radius of circle is real
  If $g^2 + f^2 - c < 0$ radius of circle is not real & hence it represent a circle with real centre & imaginary radius.
  If $g^2 + f^2 - c = 0$ radius of circle is zero so we can say that circle becomes a point $(-g, -f)$, or we can say that the circle is a point circle.
  - The general equation of a circle contains three arbitrary constants, g, f & c which corresponds to the fact that a unique circle passes through three non collinear points.
  - The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ also represents a circle, if following conditions are satisfied :
    (i) $a = b$ (coefficient of x^2 and y^2 should be same)
    (ii) $h = 0$(coefficient of xy should be zero)
- Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are two ends of diameter of a circle, P is any point on circle with co-ordinates (x, y).
  $\angle APB = 90°$ (as angle made in semi circle is right angle)
  $\therefore$ slope of $AP \times$ slope of $BP = -1$
  $$\Rightarrow \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$
  $\Rightarrow (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$
  So $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is required equation.
- This will be the circle of least radius passing through (x_1, y_1) & (x_2, y_2).
- Ordinates of A_1 & A_2 are zero $(y = 0)$ and abscissa are x_1 and x_2. So equation of circle on which A_1 and A_2 lies get reduce to $x^2 + 2gx + c = 0$.
  Here x_1 and x_2 are roots of above equation.



  $\therefore\ x_1 + x_2 = -2g$ and $x_1 x_2 = c$
  Intercepts on x-axis is $A_1A_2 = x_2 - x_1$
  $$= \sqrt{(x_1+x_2)^2 - 4x_1x_2} = \sqrt{(-2g)^2 - 4c}$$
  $$\therefore\ A_1A_2 = 2\sqrt{g^2 - c}$$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

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- Abscissae of B_1 & B_2 are zero ($x = 0$) and ordinates of B_1 & B_2 are y_1, y_2 respectively as B_1 and B_2 lie on circle so equation of circle get reduced to $y^2 + 2fy + c = 0$ as y_1 & y_2 lie on circle so they are roots of above equation.

  Hence $y_1 + y_2 = -2f$ and $y_1y_2 = c$

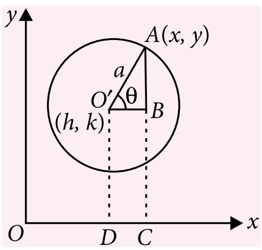
  Hence $B_1B_2 = y_2 - y_1 = \sqrt{(y_1+y_2)^2 - 4y_1y_2}$

  $= \sqrt{(-2f)^2 - 4c} = 2\sqrt{f^2 - c}$

  $\therefore$ Intercept on x-axis $= 2\sqrt{g^2 - c}$

  Intercept on y-axis $= 2\sqrt{f^2 - c}$

  Remarks
  - If $g^2 > c$ then circle cut x-axis at two distinct, real points.
  - If $g^2 < c$ then circle do not cut x-axis at real points.
  - If $g^2 = c$ then circle touches x-axis.
  - If $f^2 > c$ then circle cut y-axis at two distinct, real points.
  - If $f^2 = c$ then circle touches y-axis.
  - If $f^2 < c$ then circle do not cut y-axis at real points.
- The point (x_1, y_1) is inside or on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$ or $= 0$ or > 0
- Let $L = 0$ be a line & $S = 0$ be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then :
  - $p > r \Leftrightarrow$ the line does not meet the circle *i.e.* passes outside the circle.
  - $p = r \Leftrightarrow$ the line touches the circle.
  - $p < r \Leftrightarrow$ the line is a secant of the circle.
  - $p = 0 \Rightarrow$ the line is a diameter of the circle.
- Let equation of given circle is $(x - h)^2 + (y - k)^2 = a^2$. $O'D$ and AC perpendiculars on x-axis $O'B$ is perpendicular on AC.

  Let $\angle BO'A = \theta$ and $O'A = a$

  Hence $O'B = a\cos\theta$, $AB = a\sin\theta$

  Now $x = OC$

  $= OD + DC = OD + O'B$

  $\therefore\ x = h + a\cos\theta$

  Similarly $y = BC + AB$

  $= k + a\sin\theta$

  So $x = h + a\cos\theta$ and $y = k + a\sin\theta$ are parametric equations of circle $(x - h)^2 + (y - k)^2 = a^2$
- If centre of circle is at origin then $h = k = 0$, therefore parametric equations gets reduced to $x = a\cos\theta$, $y = a\sin\theta$.
- The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $xx_1 + yy_1 = a^2$. Hence the equation of a tangent at $(a\cos\alpha, a\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = a$

  The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is

  $$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$
- The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is: $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- $y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.
- If a line is normal / orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$.
- The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

  $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

  where K is a parameter.
- The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.
- The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is given by $S_1 + KS_2 = 0$ $(K \neq -1)$

- The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$
- Family of circles circumscribing a triangle whose sides are given by
  $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ;
  $L_1L_2 + \lambda\, L_2L_3 + \mu\, L_3L_1 = 0$ provided co-efficient of $xy = 0$ & co-efficient of x^2 = coefficient of y^2.

LENGTH OF A TANGENT AND POWER OF A POINT

The length of a tangent from an external point (x_1, y_1) to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$

Square of length of the tangent from the point P is also called the power of point w.r.t. a circle. Power of a point remains constant w.r.t. a circle.

- The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.
- The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.
- The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.
- If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $SS_1 = T^2$
  where $S \equiv x^2 + y^2 + 2gx + 2fy + c$;
  $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
  $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
- Where the two circles neither intersect nor touch each other, there are four common tangents, two of them are transverse & the others are direct common tangents.
- When they intersect there are two common tangents, both of them being direct.

When they touch each other :

- externally, there are three common tangents, two direct and one is the tangent at the point of contact.
- internally, only one common tangent possible at their point of contact.

- Length of an external common tangent & internal common tangent to the two circles is given by :
  $L_{\text{ext}} = \sqrt{d^2 - (r_1 - r_2)^2}$ and
  $L_{\text{int}} = \sqrt{d^2 - (r_1 + r_2)^2}$

Where d = distance between the centres of the two circles, r_1 & r_2 are the radii of two circles.

RADICAL AXIS AND RADICAL CENTRE

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of two circles $S_1 = 0$ & $S_2 = 0$ is given by $S_1 - S_2 = 0$ *i.e.* $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$

- Radical axis is always perpendicular to the line joining the centres of the two circles.
- If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.
- If two circles intersects, then the radical axis is the common chord of the two circles
- Radical axis bisects a common tangent between the two circles.
- The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.
- A system of circles, if each pair have the same radical axis, is called a co-axial system.
- Pairs of circles which do not have radical axis are concentric.
- Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$
- Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

PROBLEMS

Single Correct Answer Type

1. For all values of θ, the locus of the point of intersection of the lines $x \cos\theta + y \sin\theta = a$ and $x \sin\theta - y \cos\theta = b$ is
(a) An ellipse (b) A circle
(c) A parabola (d) A hyperbola

2. The area of the circle whose centre is at (1, 2) and which passes through the point (4, 6) is
(a) 5π (b) 10π
(c) 25π (d) None of these

3. If a circle passes through the point (0, 0), $(a, 0)$, $(0, b)$, then its centre is
(a) (a, b) (b) (b, a)
(c) $\left(\frac{a}{2}, \frac{b}{2}\right)$ (d) $\left(\frac{b}{2}, -\frac{a}{2}\right)$

4. The number of circle having radius 5 and passing through the points $(-2, 0)$ and $(4, 0)$ is
(a) One (b) Two (c) Four (d) Infinite

5. The equation of the circle passing through the origin and cutting intercepts of length 3 and 4 units from the positive axes, is
(a) $x^2 + y^2 + 6x + 8y + 1 = 0$
(b) $x^2 + y^2 - 6x - 8y = 0$
(c) $x^2 + y^2 + 3x + 4y = 0$
(d) $x^2 + y^2 - 3x - 4y = 0$

6. The equation of the circle with centre at $(1, -2)$ and passing through the centre of the given circle $x^2 + y^2 + 2y - 3 = 0$, is
(a) $x^2 + y^2 - 2x + 4y + 3 = 0$
(b) $x^2 + y^2 - 2x + 4y - 3 = 0$
(c) $x^2 + y^2 + 2x - 4y - 3 = 0$
(d) $x^2 + y^2 + 2x - 4y + 3 = 0$

7. For the circle $x^2 + y^2 + 3x + 3y = 0$, which of the following statements is true?
(a) Centre lies on x-axis
(b) Centre lies on y-axis
(c) Centre is at origin
(d) Circle passes through origin

8. The locus of the centre of a circle which always passes through the fixed points $(a, 0)$ and $(-a, 0)$, is
(a) $x = 1$ (b) $x + y = 6$
(c) $x + y = 2a$ (d) $x = 0$

9. The circle passing through points of intersection of the circle $S = 0$ and the line $P = 0$ is
(a) $S \pm \lambda P = 0$ (b) $P - \lambda S = 0$
(c) $\lambda S + P = 0$ (d) All of these

10. Equations to the circles which touch the lines $3x - 4y + 1 = 0$, $4x + 3y - 7 = 0$ and pass through (2, 3) are
(a) $(x - 2)^2 + (y - 8)^2 = 25$
(b) $5x^2 + 5y^2 - 12x - 24y + 31 = 0$
(c) Both (a) and (b)
(d) None of these

11. The equation of the circle in the first quadrant which touches each axis at a distance 5 from the origin is
(a) $x^2 + y^2 + 5x + 5y + 25 = 0$
(b) $x^2 + y^2 - 10x - 10y + 25 = 0$
(c) $x^2 + y^2 - 5x - 5y + 25 = 0$
(d) $x^2 + y^2 + 10x + 10y + 25 = 0$

12. The equation of circle with centre (1, 2) and tangent $x + y - 5 = 0$ is
(a) $x^2 + y^2 + 2x - 4y + 6 = 0$
(b) $x^2 + y^2 - 2x - 4y + 3 = 0$
(c) $x^2 + y^2 - 2x + 4y + 8 = 0$
(d) $x^2 + y^2 - 2x - 4y + 8 = 0$

13. The circle $x^2 + y^2 - 3x - 4y + 2 = 0$ cuts x-axis at
(a) (2, 0), (–3, 0) (b) (3, 0), (4, 0)
(c) (1, 0), (–1, 0) (d) (1, 0), (2, 0)

14. The centre of a circle is (2, –3) and the circumference is 10π. Then the equation of the circle is
(a) $x^2 + y^2 + 4x + 6y + 12 = 0$
(b) $x^2 + y^2 - 4x + 6y + 12 = 0$
(c) $x^2 + y^2 - 4x + 6y - 12 = 0$
(d) $x^2 + y^2 - 4x - 6y - 12 = 0$

15. The limit of the perimeter of the regular n-gons inscribed in a circle of radius R as $n \to \infty$ is
(a) $2\pi R$ (b) πR
(c) $4R$ (d) πR^2

16. For what value of k, the points (0, 0), (1, 3), (2, 4) and $(k, 3)$ are con-cyclic?
(a) 2 (b) 1 (c) 4 (d) 5

17. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is

(a) $x^2 + y^2 + 2x - 2y - 23 = 0$
(b) $x^2 + y^2 - 2x - 2y - 23 = 0$
(c) $x^2 + y^2 + 2x + 2y - 23 = 0$
(d) $x^2 + y^2 - 2x + 2y - 23 = 0$

18. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points such that their abscissa x_1 and x_2 are the roots of the equation $x^2 + 2x - 3 = 0$ while the ordinates y_1 and y_2 are the roots of the equation $y^2 + 4y - 12 = 0$. The centre of the circle with PQ as diameter is
(a) $(-1, -2)$ (b) $(1, 2)$
(c) $(1, -2)$ (d) $(-1, 2)$

19. A circle is drawn to cut a chord of length $2a$ units along X-axis and to touch the Y-axis. The locus of the centre of the circle is
(a) $x^2 + y^2 = a^2$ (b) $x^2 - y^2 = a^2$
(c) $x + y = a^2$ (d) $x^2 - y^2 = 4a^2$

20. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ is
(a) 0 (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/2$

21. If OA and OB be the tangents to the circle $x^2 + y^2 - 6x - 8y + 21 = 0$ drawn from the origin O, then $AB =$
(a) 11 (b) $\frac{4}{5}\sqrt{21}$
(c) $\sqrt{\frac{17}{3}}$ (d) None of these

22. If the line $y = mx + c$ be a tangent to the circle $x^2 + y^2 = a^2$, then the point of contact is
(a) $\left(\frac{-a^2}{c}, a^2\right)$ (b) $\left(\frac{a^2}{c}, \frac{-a^2 m}{c}\right)$
(c) $\left(\frac{-a^2 m}{c}, \frac{a^2}{c}\right)$ (d) $\left(\frac{-a^2 c}{m}, \frac{a^2}{m}\right)$

23. A point inside the circle $x^2 + y^2 + 3x - 3y + 2 = 0$ is
(a) $(-1, 3)$ (b) $(-2, 1)$
(c) $(2, 1)$ (d) $(-3, 2)$

24. The equations of the tangents to the circle $x^2 + y^2 = 50$ at the points where the line $x + 7 = 0$ meets it, are
(a) $7x \pm y + 50 = 0$ (b) $7x \pm y - 5 = 0$
(c) $y \pm 7x + 5 = 0$ (d) $y \pm 7x - 5 = 0$

25. The angle between the tangents to the circle $x^2 + y^2 = 169$ at the points $(5, 12)$ and $(12, -5)$, is
(a) 30° (b) 45°
(c) 60° (d) 90°

26. If the equation of one tangent to the circle with centre at $(2, -1)$ from the origin is $3x + y = 0$, then the equation of the other tangent through the origin is
(a) $3x - y = 0$ (b) $x + 3y = 0$
(c) $x - 3y = 0$ (d) $x + 2y = 0$

27. The equation of the normal to the circle $x^2 + y^2 - 2x = 0$ parallel to the line $x + 2y = 3$ is
(a) $2x + y - 1 = 0$ (b) $2x + y + 1 = 0$
(c) $x + 2y - 1 = 0$ (d) $x + 2y + 1 = 0$

28. If a line passing through origin touches the circle $(x - 4)^2 + (y + 5)^2 = 25$, then its slope should be
(a) $\pm\frac{3}{4}$ (b) 0
(c) ± 3 (d) ± 1

29. The equations of the tangents to the circle $x^2 + y^2 = 13$ at the points whose abscissa is 2, are
(a) $2x + 3y = 13$, $2x - 3y = 13$
(b) $3x + 2y = 13$, $2x - 3y = 13$
(c) $2x + 3y = 13$, $3x - 2y = 13$
(d) None of these

30. If O is the origin and OP, OQ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the circumcentre of the triangle OPQ is
(a) $(-g, -f)$ (b) (g, f)
(c) $(-f, -g)$ (d) None of these

31. The equation of circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is
(a) 1 (b) 2 (c) 3 (d) 6

32. At which point on y-axis the line $x = 0$ is a tangent to circle $x^2 + y^2 - 2x - 6y + 9 = 0$?
(a) $(0, 1)$ (b) $(0, 2)$
(c) $(0, 3)$ (d) $(0, 4)$

33. If the straight line $y = mx + c$ touches the circle $x^2 + y^2 - 4y = 0$, then the value of c will be
(a) $1+\sqrt{1+m^2}$ (b) $1-\sqrt{m^2+1}$
(c) $2(1+\sqrt{1+m^2})$ (d) $2+\sqrt{1+m^2}$

34. Which of the following lines is a tangent to the circle $x^2 + y^2 = 25$ for all values of m?
(a) $y = mx + 25\sqrt{1+m^2}$ (b) $y = mx + 5\sqrt{1+m^2}$
(c) $y = mx + 25\sqrt{1-m^2}$ (d) $y = mx + 5\sqrt{1-m^2}$

35. Square of the length of the tangent drawn from the point (α, β) to the circle $ax^2 + ay^2 = r^2$ is

(a) $a\alpha^2 + a\beta^2 - r^2$ (b) $\alpha^2 + \beta^2 - \frac{r^2}{a}$

(c) $\alpha^2 + \beta^2 + \frac{r^2}{a}$ (d) $\alpha^2 + \beta^2 - r^2$

36. $y - x + 3 = 0$ is the equation of normal at $\left(3 + \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ to which of the following circles?

(a) $\left(x - 3 - \frac{3}{\sqrt{2}}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2 = 9$

(b) $\left(x - 3 - \frac{3}{\sqrt{2}}\right)^2 + y^2 = 6$

(c) $(x - 3)^2 + y^2 = 9$

(d) $(x - 3)^2 + (y - 3)^2 = 9$

37. Length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

(a) $(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)^{1/2}$

(b) $(x_1^2 + y_1^2)^{1/2}$

(c) $[(x_1 + g)^2 + (y_1 + f)^2]^{1/2}$

(d) None of these

38. The points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are

(a) $(-2, -6), (4, 2)$ (b) $(2, 6), (-4, -2)$

(c) $(-2, 6), (-4, 2)$ (d) None of these

39. The equation of the tangent to the circle $x^2 + y^2 = r^2$ at (a, b) is $ax + by - \lambda = 0$, where λ is

(a) a^2 (b) b^2

(c) r^2 (d) None of these

40. The equation of pair of tangents to the circle $x^2 + y^2 - 2x + 4y + 3 = 0$ from $(6, -5)$, is

(a) $7x^2 + 23y^2 + 30xy + 66x + 50y - 73 = 0$

(b) $7x^2 + 23y^2 + 30xy - 66x - 50y - 73 = 0$

(c) $7x^2 + 23y^2 - 30xy - 66x - 50y + 73 = 0$

(d) None of these

41. The gradient of the tangent line at the point $(a \cos\alpha, a \sin\alpha)$ to the circle $x^2 + y^2 = a^2$, is

(a) $\tan\alpha$ (b) $\tan(\pi - \alpha)$

(c) $\cot\alpha$ (d) $-\cot\alpha$

42. The two circles which passes through $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$ will intersect each other at right angle, if

(a) $a^2 = c^2(2m + 1)$ (b) $a^2 = c^2(2 + m^2)$

(c) $c^2 = a^2(2 + m^2)$ (d) $c^2 = a^2(2m + 1)$

43. If the line $3x - 4y = \lambda$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$, then λ is equal to

(a) $-35, -15$ (b) $-35, 15$

(c) $35, 15$ (d) $35, -15$

44. Tangents drawn from origin to the circle $x^2 + y^2 - 2ax - 2by + b^2 = 0$ are perpendicular to each other, if

(a) $a - b = 1$ (b) $a + b = 1$

(c) $a^2 = b^2$ (d) $a^2 + b^2 = 1$

45. Given the circles $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 6x - 2y + 6 = 0$. Let P be a point (α, β) such that the length of tangents from P to both the circles are equal, then

(a) $2\alpha + 10\beta + 11 = 0$ (b) $2\alpha - 10\beta + 11 = 0$

(c) $10\alpha - 2\beta + 11 = 0$ (d) $10\alpha + 2\beta + 11 = 0$

46. If $2x - 4y = 9$ and $6x - 12y + 7 = 0$ are the tangents of same circle, then its radius will be

(a) $\frac{\sqrt{3}}{5}$ (b) $\frac{17}{6\sqrt{5}}$

(c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{17}{3\sqrt{5}}$

47. If $5x - 12y + 10 = 0$ and $12y - 5x + 16 = 0$ are two tangents to a circle, then the radius of the circle is

(a) 1 (b) 2 (c) 4 (d) 6

48. The value of c, for which the line $y = 2x + c$ is a tangent to the circle $x^2 + y^2 = 16$, is

(a) $-16\sqrt{5}$ (b) 20

(c) $4\sqrt{5}$ (d) $16\sqrt{5}$

49. The number of common tangents to circle $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 9 = 0$ is

(a) 1 (b) 3

(c) 2 (d) None of these

50. If line $ax + by = 0$ touches $x^2 + y^2 + 2x + 4y = 0$ and is a normal to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$, then value of (a, b) will be

(a) $(2, 1)$ (b) $(1, -2)$

(c) $(1, 2)$ (d) $(-1, 2)$

51. The locus of a point which moves so that the ratio of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is $2 : 3$, is

(a) $5x^2 + 5y^2 - 60x + 7 = 0$

(b) $5x^2 + 5y^2 + 60x - 7 = 0$

(c) $5x^2 + 5y^2 - 60x - 7 = 0$

(d) $5x^2 + 5y^2 + 60x + 7 = 0$

Assertion and Reason Type

(a) *Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.*
(b) *Statement-1 is true, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.*
(c) *Statement-1 is true, Statement-2 is false.*
(d) *Statement-1 is false, Statement-2 is true.*

52. **Statement-1 :** Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius one unit then $ABCD$ is a square.
Statement-2 : A cyclic quadrilateral is a square if its diagonals are the perpendicular diameters of the circle.

53. **Statement-1 :** Number of circles passing through the points $(1, 2), \left(3, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{5}{2}\right)$ is one.
Statement-2 : Through three non-collinear points in a plane only one circle can be drawn.

54. **Statement-1 :** Equation of a circle through the origin and belonging to the coaxial system, of which the limiting points are (1, 1) and (3, 3) is
$2x^2 + 2y^2 - 3x - 3y = 0$.
Statement-2 : Equation of a circle passing through the points (1, 1) and (3, 3) is
$x^2 + y^2 - 2x - 6y + 6 = 0$.

55. Let $(1 + ax)^n = 1 + 8x + 24x^2 + ...$ and there exists a line through the point (a, n) in the cartesian plane.
Statement-1 : If the line through $P(a, n)$ cuts the circle $x^2 + y^2 = 4$ in A and B then $PA \cdot PB = 16$.
Statement-2 : The point (a, n) lies outside the circle.

56. **Statement-1 :** Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = r^2$ then the locus of the mid points of the secants intercepted by the circle is $x^2 + y^2 = hx + ky$.
Statement-2 : The equation of the chord having mid point at (x_1, y_1) is $xx_1 + yy_1 = x_1^2 + y_1^2$.

Comprehension Type

Paragraph for Q. No. 57-58

An equation of the family of circles passing through a given pair of points (x_1, y_1) and (x_2, y_2) can be taken as

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

k being a real parameter. If a member of this family satisfies some other condition then that enables us to determine k and hence the member.

57. The number of values of $\lambda \in R$ for which there exists exactly one circle passing through the points $(2, -3)$ and $(\lambda, 2\lambda - 1)$ and touching the line $16x - 2y + 27 = 0$, is
(a) 0 (b) 1
(c) 2 (d) Infinitely many

58. There exist exactly two circles that pass through $(3, -5)$ and $(5, -3)$ and touch the line $2x + 2y + 13 = 0$. Let the ratio of radii of the two circles be m/n with $m(> 0)$ and $n(> 0)$ having no common factors except 1. Then $(m + n)$ equals
(a) 2 (b) 3 (c) 5 (d) 7

Paragraph for Q. No. 59 to 61

C_1, C_2 are circles of unit radius with centres at $P(0, 0)$ and $Q(1, 0)$ respectively. C_3 is a circle of unit radius which passes through P and Q and having its centre 'R' above x-axis.

59. The length of a common tangent to the circles C_2 and C_3 is
(a) 2 (b) $\sqrt{3}/2$ (c) 1 (d) 5

60. The equation of a common tangent to C_1 and C_3 which does not intersect C_2 is
(a) $\sqrt{3}x - y + 2 = 0$ (b) $\sqrt{3}x - y - 2 = 0$
(c) $x + \sqrt{3}y - 2 = 0$ (d) None of these

61. The length of the common chord of the circles on PQ and PR as diameters is
(a) 1/2 (b) $\sqrt{3}/2$ (c) 2 (d) 1

Matrix–Match Type

62. Match the following.

	Column-I		Column-II
(A)	A circle cut off an intercept of 8 unit on the x-axis and k units on y-axis. If tangent at (9, 3) is parallel to y-axis, then k is equal to	(p)	12
(B)	If $y = 2[2x - 1] - 1 = 3[2x - 2] + 1$ then the values of $[y + 5x]$ can be, [ ] denotes the G.I. F	(q)	8
(C)	The number of solution of equation $\cos x\sqrt{16\sin^2 x} = 1$ in $(-\pi, \pi)$ is	(r)	7
(D)	If one root of the equation $(a - 6)x^2 - (a - 6)x + 10 = 0$ is smaller than 1 and the other root greater than 2 then the value of a can be	(s)	6
		(t)	4

63. Match the following.

Column-I		Column-II	
(A)	If a circle passes through $A(1, 0)$, $B(0, -1)$ and $C\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$ such that the tangent at B makes an angle θ with line AB then $\tan\theta$ equals	(p)	-4
(B)	From a point $(h, 0)$ common tangents are drawn to the circles $x^2 + y^2 = 1$ and $(x - 2)^2 + y^2 = 4$. The value of h can be	(q)	-2
(C)	If the common chord of the circle $x^2 + y^2 = 8$ and $(x - a)^2 + y^2 = 8$ subtends right angle at the origin then a can be	(r)	1
(D)	If the tangents drawn from $(4, k)$ to the circle $x^2 + y^2 = 10$ are at right angles then k can be	(s)	2
		(t)	4

Integer Answer Type

64. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let 'P' be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangents to C_1 and C passing through 'P' is also a common tangent to C_2 and C. Then the radius of the circle C is

65. If the curves $\frac{x^2}{4} + y^2 = 1$ and $\frac{x^2}{a^2} + y^2 = 1$ for suitable value of a cut on four concyclic points, then find the radius of the smallest circle passing through these 4 points.

66. The number of integral values of α for which the point $(\alpha - 1, \alpha + 1)$ lies in the larger segment of the circle $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$ is

67. Radius of the smallest circle that can be drawn to pass through the point (0, 4) and touching the x–axis is

SOLUTIONS

1. (b) : The point of intersection is
$x = a\cos\theta + b\sin\theta$
$y = a\sin\theta - b\cos\theta$
$\therefore \quad x^2 + y^2 = a^2 + b^2.$
which is an equation of a circle.

2. (c) : Radius $= \sqrt{(1-4)^2 + (2-6)^2} = 5$
Hence the area is given by $\pi r^2 = 25\pi$ sq. units.

3. (c) : Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Now on passing through the points, we get three equations.
$c = 0,\ a^2 + 2ga + c = 0,\ b^2 + 2fb + c = 0$
On solving them, we get $g = -\frac{a}{2},\ f = -\frac{b}{2}$
Hence, the centre is $\left(\frac{a}{2}, \frac{b}{2}\right)$.

4. (b) : Two, centre of each circle lying on the perpendicular bisector of the join of the two points.

5. (d) : Since, the centre of the circle is $\left(\frac{3}{2}, 2\right)$.
$\therefore$ Equation of circle is
$$\left(x - \frac{3}{2}\right)^2 + (y-2)^2 = \left(\frac{5}{2}\right)^2 \Rightarrow x^2 + y^2 - 3x - 4y = 0$$

6. (a) : According to the question, the required circle passes through (0,–1).
$\therefore$ The radius is the distance between the points (0, –1) and (1, –2) *i.e.*, $\sqrt{2}$.
Hence the equation is $(x-1)^2 + (y+2)^2 = (\sqrt{2})^2$
$\Rightarrow \quad x^2 + y^2 - 2x + 4y + 3 = 0$

7. (d) : If $c = 0$; circle passes through origin.

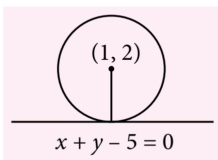
8. (d) : Perpendicular bisector of join of the two points $(a, 0)$ and $(-a, 0)$ *i.e.*, $x = 0$.

9. (d)

10. (c) : Both the circles given in option (a) and (b) satisfy the given conditions.

11. (b) : The centre of the circle which touches each axis in first quadrant at a distance 5, will be (5, 5) and radius will be 5.
$\therefore \quad (x - h)^2 + (y - k)^2 = a^2 \Rightarrow (x - 5)^2 + (y - 5)^2 = (5)^2$
$\Rightarrow \quad x^2 + y^2 - 10x - 10y + 25 = 0$

12. (b) : Radius of circle = perpendicular distance of tangent from the centre of circle
$$\Rightarrow \ r = \left|\frac{1+2-5}{\sqrt{1+1}}\right| = \sqrt{2}$$



Hence the equation of required circle is

$(x-1)^2+(y-2)^2=(\sqrt{2})^2$

$\Rightarrow\ x^2+y^2-2x-4y+3=0.$

13. (d) : Given, equation of circle is
$x^2 + y^2 - 3x - 4y + 2 = 0$ and it cuts the x-axis.
$\therefore\quad x^2 + 0 - 3x + 2 = 0 \quad\Rightarrow\quad x^2 - 3x + 2 = 0$
$\Rightarrow\quad (x-1)(x-2) = 0 \quad\Rightarrow\quad x = 1, 2.$
$\therefore$ The points are (1, 0) and (2, 0).

14. (c) : We have, centre (2, – 3), circumference = 10π
$\Rightarrow\ 2\pi r = 10\pi \ \Rightarrow\ r = 5$
$\therefore\quad (x-2)^2 + (y+3)^2 = 5^2$
$\Rightarrow\quad x^2 + y^2 - 4x + 6y + 13 = 25$
$\Rightarrow\quad x^2 + y^2 - 4x + 6y - 12 = 0,$
which is the required equation of the circle.

15. (a) : As $n \to \infty$, polygon becomes a circle and perimeter of circle = $2\pi R$.

16. (b) : The equation of circle through points (0, 0), (1, 3) and (2, 4) is $x^2 + y^2 - 10x = 0$
Point $(k, 3)$ will be on the circle.
$\therefore\quad k^2 - 10k + 9 = 0$
$\Rightarrow\quad (k-9)(k-1) = 0 \quad\Rightarrow\quad k = 1$ or $k = 9$.

17. (d) : According to question two diameters of the circle are $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$
On solving, we get $x = 1, y = -1$
$\therefore$ Centre of the circle is $(1, -1)$
Given $2\pi r = 10\pi \quad\Rightarrow\quad r = 5$
$\therefore$ Required circle is $(x-1)^2 + (y+1)^2 = 5^2$
or $\quad x^2 + y^2 - 2x + 2y - 23 = 0$

18. (a) : x_1, x_2 are roots of $x^2 + 2x - 3 = 0$

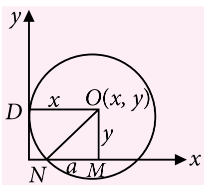
$\Rightarrow\quad x_1 + x_2 = -2 \quad\Rightarrow\quad \dfrac{x_1+x_2}{2}=-1$

y_1, y_2 are roots of $y^2 + 4y - 12 = 0$

$\Rightarrow\quad y_1+y_2=-4 \Rightarrow \dfrac{y_1+y_2}{2}=-2$

Centre of circle $\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)=(-1,-2)$

19. (b) : Since, the perpendicular drawn on chord from centre bisects the chord.



$\therefore\quad NM = a,\ OM = y$
Now, $(ON)^2 = (OM)^2 + (MN)^2$
$\Rightarrow\quad x^2 = y^2 + a^2$
$\Rightarrow\quad x^2 - y^2 = a^2$

20. (d) : Any line through (0, 0) be $y - mx = 0$ and it is a tangent to circle $(x-7)^2 + (y+1)^2 = 25$, if

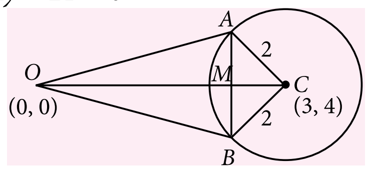
$\dfrac{-1-7m}{\sqrt{1+m^2}}=5\Rightarrow m=\dfrac{3}{4},-\dfrac{4}{3}$

Therefore, the product of both the slopes is –1.

i.e., $\dfrac{3}{4}\times\left(-\dfrac{4}{3}\right)=-1$.

Hence the angle between the two tangents is $\dfrac{\pi}{2}$.

21. (b) : Here the equation of AB (chord of contact) is
$0 + 0 - 3(x+0) - 4(y+0) + 21 = 0$
$\Rightarrow\quad 3x + 4y - 21 = 0$ ...(i)



CM = perpendicular distance from (3, 4) to line (i)

$=\dfrac{3\times3+4\times4-21}{\sqrt{9+16}}=\dfrac{4}{5}$

$AM=\sqrt{AC^2-CM^2}=\sqrt{4-\dfrac{16}{25}}=\dfrac{2}{5}\sqrt{21}$

$\therefore\ AB=2AM=\dfrac{4}{5}\sqrt{21}$

22. (c) : Since intersection of $y = mx + c$ and $x^2 + y^2 = a^2$ is $\left(-\dfrac{a^2m}{c},\dfrac{a^2}{c}\right)$.

Hence, point of contact is $\left(\dfrac{-a^2m}{c},\dfrac{a^2}{c}\right)$.

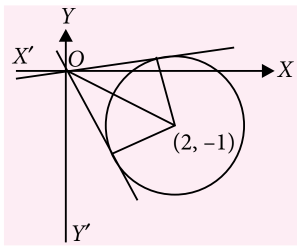
23. (b) : Point is inside, outside or on the circle as S_1 is <, > or = 0. For point (– 2, 1), $S_1 < 0$.

24. (a) : Points where $x + 7 = 0$ meets the circle $x^2 + y^2 = 50$ are (–7, 1) and (–7, –1). Hence equations of tangents at these points are $-7x \pm y = 50$ or $7x \pm y + 50 = 0$.

25. (d) : The equations of tangents to the given circle will be $5x + 12y = 169$ and $12x - 5y = 169$. Obviously the angle between the tangents is 90°.

26. (c) : Centre is (2, –1).

Therefore $r=\left|\dfrac{3(2)-1}{\sqrt{10}}\right|=\dfrac{5}{\sqrt{10}}$



Now draw a perpendicular on $x - 3y = 0$, we get

$r=\left|\dfrac{2-3(-1)}{\sqrt{10}}\right|=\dfrac{5}{\sqrt{10}}$

27. (c) : Any line parallel to $x + 2y = 3$ is $x + 2y + \lambda = 0$ and for this to be a normal to the given circle, it must pass through its centre (1, 0), *i.e.* $\lambda = -1$.
So, equation of normal is $x + 2y - 1 = 0$.

28. (b) : Let equation of line be $y = mx$ or $y - mx = 0$
Then applying condition for tangency,
$$\left|\frac{-5-4m}{\sqrt{1+m^2}}\right| = 5 \Rightarrow 25+16m^2+40m = 25+25m^2$$
$$\Rightarrow\ 9m^2 - 40m = 0 \ \Rightarrow\ m = 0 \text{ or } m = \frac{40}{9}$$

29. (a) : Let the point be $(2, y')$, then
$2^2 + y'^2 = 13 \ \Rightarrow\ y' = \pm 3$
Hence the required tangents are $2x \pm 3y = 13$.

30. (d) : Required circumcentre is the mid-point of $(0, 0)$ and $(-g, -f)$ *i.e.*, $\left(-\frac{g}{2}, -\frac{f}{2}\right)$.

31. (d) : Centre of circle is of type (c, c) and radius is
$$\left|\frac{4c+3c-12}{5}\right| = \sqrt{c^2} \Rightarrow c = 6$$

32. (c) : For $x = 0$, we get $y^2 - 6y + 9 = 0$ or $(y - 3)^2 = 0$
i.e., point of contact is $(0, 3)$.

33. (c) : Apply for tangency of line, centre being $(0, 2)$ and radius $= 2$
$$\left|\frac{-2+c}{\sqrt{1+m^2}}\right| = 2 \Rightarrow c^2 - 4c + 4 = 4 + 4m^2$$
$$\Rightarrow\ c = \frac{4 \pm \sqrt{16+16m^2}}{2} \quad \text{or } c = 2 \pm 2\sqrt{1+m^2}$$

34. (b) : Line $y = mx + c$ is a tangent if $c = \pm a\sqrt{1+m^2}$
$\therefore\ y = mx + 5\sqrt{1+m^2}$

35. (b) : Length of tangent is $\sqrt{S_1}$.
Equation of circle is $x^2 + y^2 - \frac{r^2}{a} = 0$
Hence $S_1 = \alpha^2 + \beta^2 - \frac{r^2}{a}$

36. (c)

37. (a) : Length of tangent $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

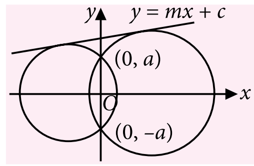
38. (a) : Substituting $x = \frac{3y+10}{4}$ in equation of circle, we get a quadratic in y. Solving, we get two values of y as 2 and $-$ 6 from which we get value of x as 4 and -2 respectively.

39. (c) : Equation of tangent is $ax + by - r^2 = 0$.

40. (a) : $SS_1 = T^2$
$$\Rightarrow\ (x^2+y^2-2x+4y+3)(36+25-12-20+3) = (6x-5y-x-6+2(y-5)+3)^2$$
$$\Rightarrow\ 7x^2+23y^2+30xy+66x+50y-73 = 0$$

41. (d) : Equation of a tangent at $(a\cos\alpha, a\sin\alpha)$ to the circle $x^2 + y^2 = a^2$ is $ax\cos\alpha + ay\sin\alpha = a^2$.
Hence its gradient is $-\frac{a\cos\alpha}{a\sin\alpha} = -\cot\alpha$.

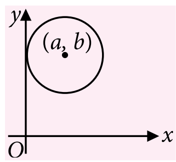
42. (c) : Equation of circles
$[x^2 + (y - a)(y + a)] + \lambda x = 0$
$\Rightarrow\ x^2 + y^2 + \lambda x - a^2 = 0$



and $\sqrt{\left(\frac{\lambda}{2}\right)^2 + a^2} = \left|\frac{\frac{-m\lambda}{2}+c}{\sqrt{1+m^2}}\right|$
$$\Rightarrow\ (1+m^2)\left[\frac{\lambda^2}{4}+a^2\right] = \left(\frac{m\lambda}{2}-c\right)^2$$
$$\Rightarrow\ (1+m^2)\left[\frac{\lambda^2}{4}+a^2\right] = \frac{m^2\lambda^2}{4} - mc\lambda + c^2$$
$$\Rightarrow\ \lambda^2 + 4mc\lambda + 4a^2(1+m^2) - 4c^2 = 0$$
$$\therefore\ \lambda_1\lambda_2 = 4[a^2(1+m^2)-c^2] \Rightarrow g_1g_2 = [a^2(1+m^2)-c^2]$$
and $g_1g_2 + f_1f_2 = \frac{c_1+c_2}{2} \Rightarrow a^2(1+m^2) - c^2 = -a^2$
Hence $c^2 = a^2(2 + m^2)$.

43. (b) : We have, $\frac{3(2)-4(4)-\lambda}{\sqrt{3^2+4^2}} = \pm\sqrt{2^2+4^2+5}$
$\Rightarrow\ -10-\lambda = \pm 25 \Rightarrow \lambda = -35, 15$

44. (c) : Obviously, tangents from origin will be perpendicular if it touches both axes *i.e.*, $a = b$ or $a^2 = b^2$.



45. (c) : Accordingly, $\alpha^2 + \beta^2 - 4\alpha - 5 = \alpha^2 + \beta^2 + 6\alpha - 2\beta + 6$
$\Rightarrow\ 10\alpha - 2\beta + 11 = 0$

46. (b) : Since the tangents are parallel, therefore the distance between these two tangents will be its diameter
i.e., diameter $= \frac{34}{\sqrt{180}} = \frac{34}{6\sqrt{5}}$
Hence, radius $= \frac{17}{6\sqrt{5}}$.

47. (a) : Given tangents are
$5x - 12y + 10 = 0$, $5x - 12y - 16 = 0$
$$\therefore\ \text{Radius} = \frac{c_1 - c_2}{2\sqrt{a^2+b^2}} = \frac{26}{2 \cdot 13} = 1$$

48. (c) : We have, $a = 4$, $m = 2$

$\therefore\ c = \pm 4\sqrt{1+4} = \pm 4\sqrt{5}$ $\quad \left[\because c = \pm a\sqrt{1+m^2}\right]$

49. (c) : $x^2 + y^2 + 2x + 8y - 23 = 0$

$\therefore\ C_1(-1,-4), r_1 = 2\sqrt{10}$

Again $x^2 + y^2 - 4x - 10y + 9 = 0$

$\therefore\ C_2(2,5), r_2 = 2\sqrt{5}$

$\therefore\ C_1C_2 = \sqrt{9+81} = 3\sqrt{10} = 9.486,$

$r_1 + r_2 = 2(\sqrt{10}+\sqrt{5}) = 10.8$

$r_1 - r_2 = 2\sqrt{5}(\sqrt{2}-1) = 2\times 2.2\times 0.4 = 4.4\times 0.4 = 1.76$

Thus, $r_1 - r_2 < C_1C_2 < r_1 + r_2 \Rightarrow$ Two tangents can be drawn.

50. (c) : As the line $ax + by = 0$ touches the circle $x^2 + y^2 + 2x + 4y = 0$, distance of the centre $(-1, -2)$ from the line = radius

$$\Rightarrow \left|\frac{-a-2b}{\sqrt{a^2+b^2}}\right| = \sqrt{(-1)^2+(-2)^2} \Rightarrow (a+2b)^2 = 5(a^2+b^2)$$

$\Rightarrow\ 4a^2 - 4ab + b^2 = 0 \Rightarrow (2a - b)^2 = 0 \Rightarrow b = 2a$

Also, $ax + by = 0$ is a normal to $x^2 + y^2 - 4x + 2y - 3 = 0$, so the centre $(2, -1)$ should lie on $ax + by = 0$.

$\therefore\ 2a - b = 0 \Rightarrow b = 2a$. Hence, $a = 1$, $b = 2$.

51. (d) : Let the point be (x_1, y_1).

According to question, $\dfrac{\sqrt{x_1^2+y_1^2+4x_1+3}}{\sqrt{x_1^2+y_1^2-6x_1+5}} = \dfrac{2}{3}$

Squaring both sides, $\dfrac{x_1^2+y_1^2+4x_1+3}{x_1^2+y_1^2-6x_1+5} = \dfrac{4}{9}$

$\Rightarrow 9x_1^2+9y_1^2+36x_1+27 = 4x_1^2+4y_1^2-24x_1+20$

$\Rightarrow 5x_1^2+5y_1^2+60x_1+7=0$

Hence, locus is $5x^2 + 5y^2 + 60x + 7 = 0$.

52. (d)

53. (d) : The points $(1,2), \left(3,\frac{1}{2}\right), \left(\frac{1}{3},\frac{5}{2}\right)$ are collinear and no circle can be drawn from 3 collinear points. Also through 3 non-collinear points a unique circle can be drawn.

54. (b) : Two members of the system of circles in statement-1 are the circles with centres at the limiting points and radius equal to zero *i.e.*

$(x - 1)^2 + (y - 1)^2 = 0$ and $(x - 3)^2 + (y - 3)^2 = 0$

or $x^2 + y^2 - 2x - 2y + 2 = 0$

and $x^2 + y^2 - 6x - 6y + 18 = 0$

Equation of the coaxial system is

$x^2 + y^2 - 6x - 6y + 18 + \lambda(x^2 + y^2 - 2x - 2y + 2) = 0$

which passes through the origin if $\lambda = -9$ and the equation of the required circle is

$2x^2 + 2y^2 - 3x - 3y = 0$. So that statement-1 is true.

Statement-2 is also true as the circle in it passes through (1, 1) and (3, 3) but does not lead to statement-1.

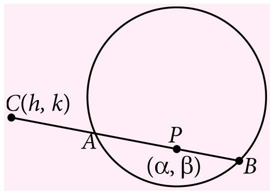
55. (b) : We have, $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$

So, $na = 8$ ...(i) and $\dfrac{n(n-1)}{2}a^2 = 24$ ...(ii)

On solving (i) and (ii), we get $a = 2$ and $n = 4$

$PA\cdot PB = (\sqrt{S_1})^2 = 2^2+4^2-4 = 16$

56. (a) : Equation of AB is $x\alpha + y\beta = \alpha^2 + \beta^2$, it passes through (h, k).



$\therefore\ h\alpha + k\beta = \alpha^2 + \beta^2$

$\therefore$ Locus of (α, β) is

$x^2 + y^2 = hx + ky$

57. (c) : There will exist exactly one circle if the line passing through $A(2, -3)$ and $B(\lambda, 2\lambda - 1)$ is parallel to the given line $16x - 2y + 27 = 0$

Also, if the point $B(\lambda, 2\lambda - 1)$ lies on the line $16x - 2y + 27 = 0$, then we will have exactly one circle.

Thus two values of λ are possible.

58. (a) : The line joining (3, –5) and (5, –3) has slope 1 and thus it is perpendicular to $2x + 2y + 13 = 0$. Hence the two circles will have same radii.

59. (c) : $RP = RQ = 1 \Rightarrow R$ is point of intersection of C_1 and C_2 in first quadrant. C_2, C_3 are circles of equal radii.

$\Rightarrow$ Length of common tangent = QR

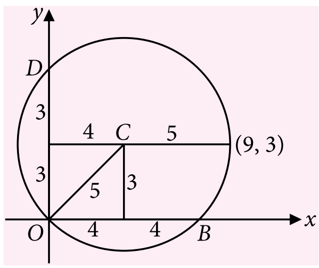
60. (a) : ΔPQR is equilateral $\Rightarrow \angle RPQ = 60°$

Hence, the required equation is $\sqrt{3}x - y + 2 = 0$

61. (b) : Common chord is the altitude through P in ΔPQR.

62. **(A) $\to$ (s), (B) $\to$ (q, r,s), C $\to$ (t), (D) $\to$ (p,q,r)**

(A) $OD = 6$



(B) $y = 2[2x] - 3 = 3[2x] - 5 \Rightarrow [2x] = 2 \Rightarrow y = 1$
$2 \le 2x < 3 \Rightarrow 5 \le 5x \le 7.5 \Rightarrow 6 \le 5x + y \le 8.5$
$[5x + y] = 6, 7$ or 8.
(C) $4|\sin x| \cos x = 1$
Solutions are possible if $\cos x > 0$

$$\Rightarrow \frac{-\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow -\pi \le 2x \le \pi$$

if $x \in \left[-\frac{\pi}{2}, 0\right)$

$$\sin 2x = \frac{-1}{2} \Rightarrow 2x = \frac{-\pi}{6}, \frac{-5\pi}{6} \Rightarrow x = \frac{-\pi}{12}, \frac{-5\pi}{12}$$

if $x \in \left(0, \frac{\pi}{2}\right)$

$$\sin 2x = \frac{1}{2} \Rightarrow 2x = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$$

(D) $(a - 6)f(1) < 0 \Rightarrow a > 6$
$(a - 6)f(2) < 0 \Rightarrow 6 < a < 13$

63. **(A) → (r), (B) → (q), (C) → (p,t), (D) → (q,s)**
(A) Origin is the circumcentre.
$\therefore$ Equation of circle is $x^2 + y^2 = 1 \Rightarrow \theta = \frac{\pi}{4}$

(B) A tangent to $x^2 + y^2 = 1$ is $y = mx \pm \sqrt{1+m^2}$.

It touches $(x - 2)^2 + y^2 = 4$ if $\left|\frac{2m \pm \sqrt{1+m^2}}{\sqrt{1+m^2}}\right| = 2$

$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$. The common tangents are $y = \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$

and $y = -\frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$ which intersect at $(-2, 0)$.
(C) Common chord of the given circles is
$(x^2 + y^2 - 8) - [(x - a)^2 + y^2 - 8] = 0$
$\Rightarrow 2x - a = 0$

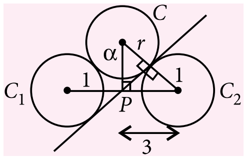
$$\Rightarrow \frac{2x}{a} = 1$$

Homogenising $x^2 + y^2 - 8 = 0 \Rightarrow x^2 + y^2 - 8\left(\frac{2x}{a}\right)^2 = 0$
It represents perpendicular lines

$$\Rightarrow 1 - \frac{32}{a^2} + 1 = 0 \Rightarrow a^2 = 16 \Rightarrow a = \pm 4$$

(D) $(4, k)$ must lie on the director circle of the given circle, which is $x^2 + y^2 = 20$. Thus $16 + k^2 = 20$
$\Rightarrow k = \pm 2$

64. (8) : We have, $(r + 1)^2 = \alpha^2 + 9$
and $r^2 + 8 = \alpha^2$



$\Rightarrow r^2 + 2r + 1 = r^2 + 8 + 9 \Rightarrow 2r = 16 \therefore r = 8$

65. (1) : $\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda\left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$

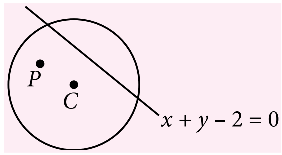
$$\Rightarrow x^2\left(\frac{a^2 + 4\lambda}{4a^2(1+\lambda)}\right) + y^2 = 1$$

Clearly radius is 1 unit.

66. (1) : $S(x, y) = x^2 + y^2 - x - y - 6 = 0$ ...(i)

has centre at $C \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$

According to the required conditions, the given point $P(\alpha - 1, \alpha + 1)$ must lie inside the given circle.
i.e. $S(\alpha - 1, \alpha + 1) < 0$
$\Rightarrow (\alpha - 1)^2 + (\alpha + 1)^2 - (\alpha - 1) - (\alpha + 1) - 6 < 0$
$\Rightarrow \alpha^2 - \alpha - 2 < 0 \Rightarrow (\alpha - 2)(\alpha + 1) < 0$
$\Rightarrow -1 < \alpha < 2$ ...(ii)
Also, P and C must lie on the same side of the line



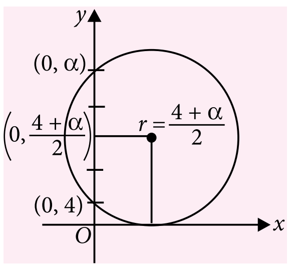
$L(x, y) \equiv x + y - 2 = 0$ ...(iii)
i.e. $L(1/2, 1/2)$ and $L(\alpha - 1, \alpha + 1)$ must have the same sign.

Since $L\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} - 2 < 0$

$\therefore L(\alpha - 1, \alpha + 1) = (\alpha - 1) + (\alpha + 1) - 2 < 0$
i.e., $\alpha < 1$ ...(iv)
Inequalities (ii) and (iv) together give the permissible values of α as $-1 < \alpha < 1$.

67. (2) : $r = \frac{4+\alpha}{2}, \alpha \ge 0$

when $\alpha = 0$, smallest radius $= 2$.



CLASS XI Series 8

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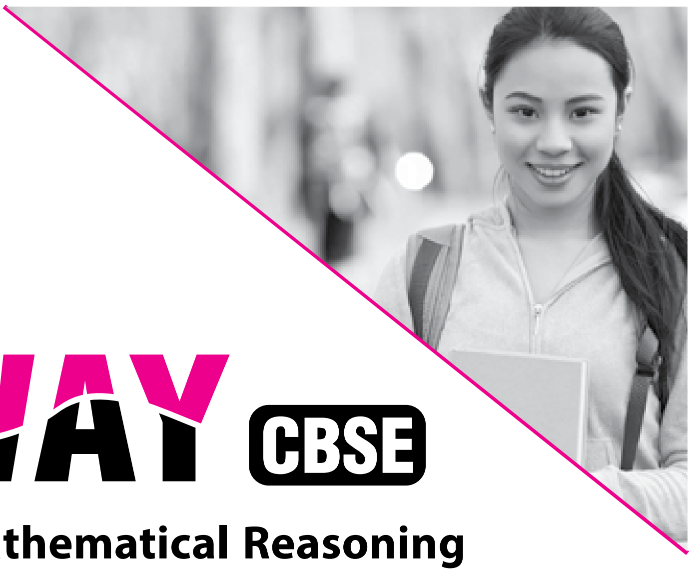


Limits and Derivatives | Mathematical Reasoning

IMPORTANT FORMULAE

LIMITS AND DERIVATIVES

If f(x) and g(x) are two functions such that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist, then*

- $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided* $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a}[f(x)]^{[g(x)]} = \left[\lim_{x\to a} f(x)\right]^{\lim_{x\to a} g(x)}$
- $\lim_{x\to a} \lambda\cdot f(x) = \lambda\cdot \lim_{x\to a} f(x)$, λ *being some real number*
- $\lim_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$, *n being a positive integer*
- $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$
- $\lim_{x\to a}\dfrac{\sin(x-a)}{x-a} = 1$
- $\lim_{x\to 0}\cos x = 1$
- $\lim_{x\to 0}\dfrac{\tan x}{x} = 1$
- $\lim_{x\to a}\dfrac{\tan(x-a)}{x-a} = 1$
- $\lim_{x\to 0}\dfrac{1-\cos x}{x} = 0$
- $\lim_{x\to 0}\dfrac{\log(1+x)}{x} = 1$
- $\lim_{x\to 0}\dfrac{e^x-1}{x} = 1$
- $\lim_{x\to 0}\dfrac{a^x-1}{x} = \log_e a,\ a > 1$
- $\lim_{x\to 0}(1+x)^{1/x} = e.$
- *If* $P = \lim_{x\to a}(f(x))^{g(x)}$, *then* $\log_e P = \lim_{x\to a} g(x)\log(f(x))$
- $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}[f(x)-g(x)] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$
- $\dfrac{d}{dx}[f(x)\cdot g(x)] = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$ *(Product rule)*
- $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2}$, *provided* $g(x) \neq 0$ *(Quotient Rule)*
- $\lim_{x\to\infty} k^x = \begin{cases} \infty, & \text{if } k > 1 \\ 0, & \text{if } 0 \le k < 1 \\ 1, & \text{if } k = 1 \end{cases}$
- $\lim_{x\to\infty} kx = \begin{cases} \infty, & \text{if } k > 0 \\ -\infty, & \text{if } k < 0 \\ 0, & \text{if } k = 0 \end{cases}$
- *If* $f(x) \le g(x)\ \forall\ x$, *then* $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

- $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$
- $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
- $\sim(p \Rightarrow q) \equiv \sim\{\sim p \vee q\} = \{p \wedge (\sim q)\}$
  $\therefore \quad \sim(p \Rightarrow q) \equiv \{p \wedge (\sim q)\}$
- $\sim(p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

STATEMENT

A sentence is called mathematically acceptable statement if it is either true or false but not both. A true statement is also known as a valid statement. If a statement is false, we call it invalid statement.

Negation of a Statement

The denial of a statement p is called its negation and we write it as $\sim p$ and read as 'not p'.

TYPES OF STATEMENT

- **Simple Statement:** A statement which can not be broken into two or more statement is called a simple statement.
- **Compound Statement:** A statement that can be formed by combining two or more simple statements is called a compound statement or compound proposition.

LOGICAL CONNECTIVES

Statements can be connected by using connectives like "and", "or", "implies".

- The compound statement with 'And' is true if all its component statements are true.
- The compound statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).
- A compound statement with 'Or' is true when one component statement is true or both the component statements are true.
- A compound statement with 'Or' is false when both the component statements are false.

QUANTIFIERS

Many mathematical statements contain phrases 'there exists' and 'for all' or 'for every'. These phrases are called quantifiers.

IMPLICATIONS

If any two simple statements can be combined by the word 'if-then', then it is called the implication and it is denoted by the symbol '$\Rightarrow$' or '$\rightarrow$'.

- **If then implication :** A sentence "if p then q" can be written in the following ways :
  - p implies q (denoted by $p \Rightarrow q$)
  - p is sufficient condition for q
  - q is necessary condition for p
  - p only if q
  - $\sim q$ implies $\sim p$

Contrapositive

If p and q are two statements, then the contrapositive of the statement "If p then q" is "if $\sim q$ then $\sim p$".

Converse

If p and q are two statements, then the converse of the implication "if p then q" is "if q then p".

Inverse

If p and q are two statements, then the inverse of "if p then q" is "if $\sim p$ then $\sim q$."

- **If and only if Implication :** If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$.

Conjunction

- If two statements p and q are combined by the connective 'AND' ($\wedge$), then the compound statement $p \wedge q$ so formed is called a conjunction.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- If two statements p and q are combined by the connective 'OR' ($\vee$), then the compound statement $p \vee q$ so formed is called a disjunction.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional (Implication)

- If p and q are two simple statements, then the compound statement 'If p then q' is known as the conditional or implication and is denoted by $p \Rightarrow q$ or $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (Double Implication)

- If p and q are two statements, then the compound statement 'p if and only if q' is known as the biconditional or double implication and is denoted by $p \Leftrightarrow q$ or $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Find $\lim_{x \to -1} (4x^2 + 2)$.
2. Find $\lim_{n \to \infty} \frac{n!}{(n+1)! - n!}$.
3. Evaluate $\lim_{x \to -2^+} \frac{x^2 - 1}{2x + 4}$.
4. Find the derivative of $3x^{10}$ w.r. to x.
5. Let c denote contradiction and p be any statement. Then prove that $p \vee c \equiv p$.

SHORT ANSWER TYPE

6. Evaluate $\lim_{\theta \to 0} \frac{1 - \sqrt{\cos\theta}}{\theta^2}$.
7. Find $\lim_{x \to 0} \frac{x(\cos x + \cos 2x)}{\sin x}$.
8. Evaluate $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$.
9. Evaluate $\lim_{x \to 0} \frac{a^x - b^x}{x}$.
10. Find the derivative of $5\sin x - 11\cos x + \frac{1}{x^2}$, w.r. to x.

LONG ANSWER TYPE - I

11. Find $\lim_{x \to 0} \frac{\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x} \cdot x$
12. Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)}$.
13. Find the derivatives of the following functions.
    (i) $\frac{4 - x^2}{4 + x^2}$ **(ii)** $\frac{2x^4 + x}{3x - 5}$
14. Prove that the following statement is true.
    If $x, y \in Z$ such that x and y are odd, then xy is odd.
15. Are the following statements negation of each other?
    (i) "x is not a rational number."
    "x is not a irrational number."
    (ii) "x is not a real number."
    "x is not an imaginary number."

LONG ANSWER TYPE - II

16. Find $\lim_{x \to -\infty} \left(\sqrt{x^2 + 4x} - \sqrt{x^2 - 4x}\right)$.
17. Differentiate $\frac{x^2 + 2}{x + 2}$ from first principle.
18. Let $f(x) = \begin{cases} 3 - x^2, x \le -2 \\ ax + b, -2 < x < 2 \\ x^2/2, x \ge 2 \end{cases}$
    Find a and b so that $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ exist.
19. For any three statements p, q, r prove that
    (i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
    (ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
20. If α and β be the roots of equation $ax^2 + bx + c = 0$, then find $\lim_{x \to \frac{1}{\alpha}} \sqrt{\frac{1 - \cos(cx^2 + bx + a)}{2(1 - \alpha x)^2}}$.

SOLUTIONS

1. Since $\lim\limits_{x \to -1} (4x^2 + 2)$ is not in indeterminate form:

$\therefore \lim\limits_{x \to -1} (4x^2 + 2) = 4(-1)^2 + 2 = 4 + 2 = 6$

2. $\lim\limits_{n \to \infty} \dfrac{n!}{(n+1)! - n!}$

$$= \lim_{n \to \infty} \frac{\frac{n!}{(n+1)!}}{1 - \frac{n!}{(n+1)!}} = \lim_{n \to \infty} \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{0}{1-0} = 0$$

3. When $x \to -2^+$, x approaches -2 and $x > -2$

$\therefore$ $(x + 2) \to 0^+$ (*i.e.*, $x \to 0$ from right)

and when $x \to -2^+$, $(x^2 - 1) \to 3$

Hence, $\lim\limits_{x \to -2^+} \dfrac{x^2 - 1}{2x + 4} = \lim\limits_{x \to -2^+} \dfrac{x^2 - 1}{2(x+2)} = \infty$

4. Let $y = 3x^{10}$

Now, $\dfrac{dy}{dx} = \dfrac{d}{dx}(3x^{10}) = 3\dfrac{d}{dx}(x^{10}) = 3 \cdot 10x^9 = 30x^9$

5. Truth table for $p \vee c$ is :

p	c	$p \vee c$
T	F	T
F	F	F

Hence from truth table, $p \vee c \equiv p$.

6. $\lim\limits_{\theta \to 0} \dfrac{1 - \sqrt{\cos\theta}}{\theta^2}$ $\left[\frac{0}{0} \text{form}\right]$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2\left(1 + \sqrt{\cos\theta}\right)} = \lim_{\theta \to 0} \frac{2\sin^2\frac{\theta}{2}}{\theta^2\left(1 + \sqrt{\cos\theta}\right)}$$

$$= \lim_{\theta \to 0} \frac{2\left(\dfrac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 \cdot \dfrac{\theta^2}{4}}{\theta^2\left(1 + \sqrt{\cos\theta}\right)} = \frac{1}{4}$$

7. $\lim\limits_{x \to 0} \dfrac{x(\cos x + \cos 2x)}{\sin x} = \lim\limits_{x \to 0} \dfrac{x(\cos x + \cos 2x)}{\frac{\sin x}{x} \cdot x}$

[Here factor ($\cos x + \cos 2x$), does not tends to zero, hence it is not necessary to simplify it.]

$$= \lim_{x \to 0} \frac{\cos x + \cos 2x}{\frac{\sin x}{x}} = \frac{1+1}{1} = 2$$

8. $\lim\limits_{x \to 0} \dfrac{\sin x - 2\sin 3x + \sin 5x}{x}$ $\left[\frac{0}{0} \text{form}\right]$

$$= \lim_{x \to 0} \frac{(\sin x - \sin 3x) - (\sin 3x - \sin 5x)}{x}$$

$$= \lim_{x \to 0} \frac{-2\cos 2x \sin x + 2\cos 8x \sin x}{x}$$

$$= \lim_{x \to 0} 2(\cos 8x - \cos 2x) \cdot \frac{\sin x}{x} = 2(1 - 1) \cdot 1 = 0$$

9. $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} = \lim\limits_{x \to 0} \dfrac{(a^x - 1) - (b^x - 1)}{x}$

$$= \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x} = \log_e a - \log_e b = \log_e\left(\frac{a}{b}\right)$$

10. Let $y = 5\sin x - 11\cos x + \dfrac{1}{x^2}$

Now, $\dfrac{dy}{dx} = 5\dfrac{d}{dx}\sin x - 11\dfrac{d}{dx}\cos x + \dfrac{d}{dx}(x^{-2})$

$= 5\cos x - 11(-\sin x) - 2x^{-3} = 5\cos x + 11\sin x - \dfrac{2}{x^3}$

11. $\lim\limits_{x \to 0} \dfrac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x} \cdot x$ $\left[\frac{0}{0} \text{form}\right]$

$$= \lim_{x \to 0} \frac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{2\sin(\alpha+\beta)x \cdot \sin(\alpha-\beta)x} \cdot x$$

$$= \lim_{x \to 0} \frac{\left[\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x} \cdot (\alpha+\beta)x + \frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x} \cdot (\alpha-\beta)x + \frac{\sin 2\alpha x}{2\alpha x} \cdot 2\alpha x\right]}{2\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x} \cdot (\alpha+\beta)x \cdot \frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x} \cdot (\alpha-\beta)x} \cdot x$$

$$= \lim_{x \to 0} \frac{\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x} \cdot (\alpha+\beta) + \frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x} \cdot (\alpha-\beta) + \frac{\sin 2\alpha x}{2\alpha x} \cdot 2\alpha}{2\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x} \cdot (\alpha+\beta) \times \frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x} \cdot (\alpha-\beta)}$$

$$= \frac{1 \cdot (\alpha+\beta) + 1 \cdot (\alpha-\beta) + 1 \cdot 2\alpha}{2 \cdot 1 \cdot (\alpha+\beta) \cdot 1 \cdot (\alpha-\beta)} = \frac{4\alpha}{2(\alpha^2 - \beta^2)} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

12. Put $x = \dfrac{\pi}{2} + h$. As $x \to \dfrac{\pi}{2}$, $h \to 0$

Now, $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)} = \lim\limits_{h \to 0} \dfrac{2^{-\cos\left(\frac{\pi}{2}+h\right)} - 1}{\left(\frac{\pi}{2} + h\right)\left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)}$

$$= \lim_{h\to 0} \frac{2^{\sin h}-1}{\left(\frac{\pi}{2}+h\right)h} = \lim_{h\to 0} \frac{2^{\sin h}-1}{\sin h}\cdot\frac{\sin h}{h}\cdot\frac{1}{\left(\frac{\pi}{2}+h\right)}$$

$$= \log_e 2\times 1\times\frac{1}{\pi/2} = \frac{2}{\pi}\log_e 2$$

13. **(i)** Let $y = \dfrac{4-x^2}{4+x^2}$

$$\therefore \frac{dy}{dx} = \frac{(4+x^2)\frac{d}{dx}(4-x^2)-(4-x^2)\cdot\frac{d}{dx}(4+x^2)}{(4+x^2)^2}$$

$$= \frac{(4+x^2)(0-2x)-(4-x^2)(0+2x)}{(4+x^2)^2}$$

$$= \frac{-8x-2x^3-8x+2x^3}{(4+x^2)^2} = \frac{-16x}{(4+x^2)^2}$$

(ii) Let $y = \dfrac{2x^4+x}{3x-5}$

$$\therefore \frac{dy}{dx} = \frac{(3x-5)\frac{d}{dx}(2x^4+x)-(2x^4+x)\frac{d}{dx}(3x-5)}{(3x-5)^2}$$

$$= \frac{(3x-5)(8x^3+1)-(2x^4+x)(3)}{(3x-5)^2}$$

$$= \frac{24x^4+3x-40x^3-5-6x^4-3x}{(3x-5)^2} = \frac{18x^4-40x^3-5}{(3x-5)^2}$$

14. Let $p : x, y \in Z$ such that x and y are odd.
And $q : xy$ is odd.
Then, we have to prove that xy is odd.
We assume that p is true and show that q is true.
p is true means x and y are odd integers. Then,
$x = (2m + 1)$ for some integer m
and $y = (2n + 1)$ for some integer n.
$\therefore\ xy = (2m + 1)(2n + 1)$
$= (4mn + 2m + 2n + 1)$
$= 2(2mn + m + n) + 1$, which is clearly odd.
Thus, $p \Rightarrow q$
Hence, the given statement is true.

15. **(i)** Let $p : x$ is not a rational number.
and $q : x$ is not an irrational number.
Then $\sim p : x$ is a rational number.
and $\sim q : x$ is an irrational number.
If x is a rational number, then x is not an irrational number.
$\therefore\ \sim p = q$.
Again if x is an irrational number, then x is not a rational number.
$\therefore\ \sim q = p$.
Thus, p and q are negation of each other.
(ii) Let $p : x$ is not a real number.
and $q : x$ is not an imaginary number.
Then $\sim p : x$ is a real number.
and $\sim q : x$ is an imaginary number.
If x is an imaginary number, then it is definitely not a real number. Therefore $\sim q = p$.
Again if x is a real number, then x is definitely not an imaginary number.
Therefore, $\sim p = q$.
Thus p and q are negation of each other.

16. $\lim_{x\to -\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

$$= \lim_{x\to-\infty}\frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}$$

$$= \lim_{x\to-\infty}\frac{(x^2+4x)-(x^2-4x)}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}$$

$$= \lim_{x\to-\infty}\frac{8x}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}$$

$$= \lim_{x\to-\infty}\frac{8}{\frac{\sqrt{x^2+4x}}{x}+\frac{\sqrt{x^2-4x}}{x}}$$

$$= \lim_{x\to-\infty}\frac{8}{-\sqrt{\frac{x^2+4x}{x^2}}-\sqrt{\frac{x^2-4x}{x^2}}}$$

[Here $x \to -\infty \ \therefore\ x = -\sqrt{x^2}$ for example
$-4 = -\sqrt{(-4)^2} = -\sqrt{16}$]

$$= \lim_{x\to-\infty}\frac{8}{-\sqrt{1+\frac{4}{x}}-\sqrt{1-\frac{4}{x}}} = \frac{8}{-1-1} = \frac{8}{-2} = -4$$

17. Let $y = f(x) = \dfrac{x^2+2}{x+2}$ ...(i)

$$\therefore \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(x+h)^2+2}{x+h+2}-\frac{x^2+2}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(x+2)(x^2+2xh+h^2+2)-(x^2+2)(x+2+h)}{(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(x+2)[x^2+2xh+h^2+2-x^2-2]-h(x^2+2)}{(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \left[\frac{1}{h}\frac{h(2x+h)}{x+h+2} - \frac{x^2+2}{(x+h+2)(x+2)}\right]$$

$$= \frac{2x}{x+2} - \frac{x^2+2}{(x+2)^2} = \frac{2x(x+2)-x^2-2}{(x+2)^2}$$

Hence $\dfrac{dy}{dx} = \dfrac{x^2+4x-2}{(x+2)^2}$

18. $\lim_{x\to 2} f(x)$ exists $\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$$\Rightarrow \lim_{x\to 2}(ax+b) = \lim_{x\to 2}\frac{x^2}{2} \Rightarrow 2a+b = \frac{2^2}{2}$$

$\Rightarrow\ 2a + b = 2$ ...(i)

Again, $\lim_{x\to -2} f(x)$ exists $\Rightarrow \lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$

$$\Rightarrow \lim_{x\to -2}(3-x^2) = \lim_{x\to -2}(ax+b)$$

$\Rightarrow\ -1 = -2a + b$ ...(ii)

Solving (i) and (ii), we get $a = \dfrac{3}{4}, b = \dfrac{1}{2}$

19. **(i)** The truth table for $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ is :

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

From the truth table it follows that
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(ii) The truth table for $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ is :

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

From the truth table it follows that
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

20. Since α, β are the roots of equation $ax^2 + bx + c = 0$, therefore roots of equation $cx^2 + bx + a = 0$ will be $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

$\therefore\ cx^2 + bx + a = c\left(x - \dfrac{1}{\alpha}\right)\left(x - \dfrac{1}{\beta}\right)$

Required limit

$$= \lim_{x\to\frac{1}{\alpha}} \sqrt{\frac{1-\cos\{c(x-1/\alpha)(x-1/\beta)\}}{2(1-\alpha x)^2}}$$

$$= \lim_{x\to\frac{1}{\alpha}} \sqrt{\frac{2\sin^2\{(c/2)(x-1/\alpha)(x-1/\beta)\}}{2(1-\alpha x)^2}}$$

$$= \lim_{x\to\frac{1}{\alpha}} \left|\frac{\sin\left\{\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right\}}{1-\alpha x}\right| \quad \left[\because \sqrt{x^2} = |x|\right]$$

$$= \lim_{x\to\frac{1}{\alpha}} \left|\frac{\sin\left\{\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right\}}{\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)} \cdot \frac{c(\alpha x-1)(\beta x-1)}{2\alpha\beta(1-\alpha x)}\right|$$

$$= \left|\frac{c}{2\alpha\beta}\left(\frac{\beta}{\alpha}-1\right)\right| \quad [\because |-x| = |x|]$$

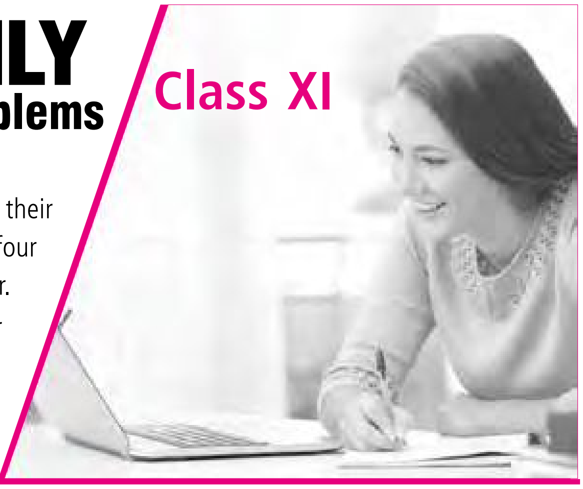
$$= \left|\frac{c}{2\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right|$$

MPP-8 CLASS XII ANSWER KEY

1.	(a)	**2.**	(c)	**3.**	(a)	**4.**	(a)	**5.**	(d)
6.	(a)	**7.**	(b,d)	**8.**	(a,b)	**9.**	(a,b,c,d)	**10.**	(a,b)
11.	(a,b,c)	**12.**	(a,b)	**13.**	(a,d)	**14.**	(b)	**15.**	(d)
16.	(a)	**17.**	(4)	**18.**	(8)	**19.**	(1)	**20.**	(2)

MPP-8 MONTHLY Practice Problems

Class XI



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Straight Lines

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Locus of a point that is equidistant from the lines $x+y-2\sqrt{2}=0$ and $x+y-\sqrt{2}=0$ is

(a) $x+y-5\sqrt{2}=0$ (b) $x+y-3\sqrt{2}=0$
(c) $2x+2y-3\sqrt{2}=0$ (d) $2x+2y-5\sqrt{2}=0$

2. If the algebraic sum of the distances of points (2, 1), (3, 2) and (–4, 7) from the line $y = mx + c$ is zero, then this line will always pass through a fixed point whose co-ordinate is

(a) (1, 3) (b) (1, 10)
(c) (1, 6) (d) $\left(\frac{1}{3}, \frac{10}{3}\right)$

3. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q respectively. Then, the point O divides the segment PQ in the ratio

(a) 1 : 2 (b) 3 : 4
(c) 2 : 1 (d) 4 : 3

4. The equation of a straight line passing through (3, 2) and cutting an intercept of 2 units between the lines $3x + 4y = 11$ and $3x + 4y = 1$ is

(a) $2x + y - 8 = 0$ (b) $3y - 4x + 6 = 0$
(c) $3x + 4y - 17 = 0$ (d) $2x - y - 4 = 0$

5. In triangle ABC, equation of the right bisectors of the sides AB and AC are $x + y = 0$ and $y - x = 0$ respectively. If $A \equiv (5, 7)$ then equation of side BC is

(a) $7y = 5x$ (b) $5x = y$
(c) $5y = 7x$ (d) $5y = x$

6. ABC is a right angled isosceles triangle, right angled at $A(2, 1)$. If the equation of side BC is $2x + y = 3$, then the combined equation of lines AB and AC is

(a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
(b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
(c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
(d) none of these

One or More Than One Option(s) Correct Type

7. If the lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then

(a) $\lambda = 2$ (b) $\lambda = -3$
(c) $\lambda = 4$ (d) $\lambda = -4$

8. If p_1 and p_2 are the lengths of the perpendiculars from the origin to the straight lines $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, then the value of $4p_1^2 + p_2^2$ is

(a) $4a^2$ (b) $2a^2$
(c) a^2 (d) none of these

Exam Corner

Exam	Date
JEE Main (Offline)	8th April, 2018
WB JEE	22nd April, 2018
JEE Advanced	20th May, 2018

9. If one of the lines given by the equation $2x^2 + pxy + 3y^2 = 0$ coincide with one of those given by $2x^2 + qxy - 3y^2 = 0$ and the other lines represented by them are perpendicular, then
(a) $p = 5$ (b) $p = -5$
(c) $q = -1$ (d) $q = 1$

10. The equation $x^3 + x^2y - xy^2 = y^3$ represents
(a) three real straight lines
(b) lines in which two are perpendicular to each other
(c) lines in which two are coincident
(d) none of these

11. The equation of the line passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin is
(a) $\sqrt{3}x + y - \sqrt{3} = 0$ (b) $x + \sqrt{3}y - \sqrt{3} = 0$
(c) $\sqrt{3}x - y - \sqrt{3} = 0$ (d) $x - \sqrt{3}y - \sqrt{3} = 0$

12. Two particles start from the same point (2, –1) one moving 2 unit along the line $x + y = 1$ and the other 5 unit along $x - 2y = 4$. If the particles move towards increasing y, then their new position will be
(a) $(2-\sqrt{2}, \sqrt{2}-1)$ (b) $(2\sqrt{5}+2, \sqrt{5}-1)$
(c) $(2+\sqrt{2}, \sqrt{2}+1)$ (d) $(2\sqrt{5}-2, \sqrt{5}-1)$

13. Angles made with the x-axis by a straight line drawn through (1, 2) so that it intersects $x + y = 4$ at a distance $\sqrt{6}/3$ from (1, 2) are
(a) 105° (b) 75°
(c) 60° (d) 15°

Comprehension Type

The line $6x + 8y = 48$ intersects the co-ordinate axes at A and B, respectively. A line L bisects the area and the perimeter of the triangle OAB where O is the origin.

14. The number of such possible lines is
(a) 1 (b) 2
(c) 3 (d) more than 3

15. The line L does not intersect
(a) AB (b) OA
(c) OB (d) none of these

Matrix Match Type

16. Match the following.

	Column I		Column II
P.	The number of integral values of 'a' for which the point $P(a, a^2)$ lies completely inside the triangle formed by the lines $x = 0$, $y = 0$ and $x + 2y = 3$	1.	1
Q.	If the points $(\lambda + 1, 1)$, $(2\lambda + 1, 3)$ and $(2\lambda + 2, 2\lambda)$ are collinear, then the value of λ is	2.	4
R.	The reflection of the point $(t - 1, 2t + 2)$ in a line is $(2t + 1, t)$ then the line has slope equals to	3.	0
S.	In a ΔABC the bisector of angles B and C lie along the lines $x = y$ and $y = 0$. If A is (1, 2), then $\sqrt{10}d(A, BC)$ is (where $d(A, BC)$ represents distance of point A from side BC)	4.	$-\frac{1}{2}$

	P	Q	R	S
(a)	3	4	1	2
(b)	2	4	1	3
(c)	4	2	3	1
(d)	4	1	3	2

Integer Answer Type

17. If the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represents a pair of straight lines, then the value of $2\lambda^2 - 5$ is equal to

18. If (α, β) be the co-ordinates of incentre of the triangle whose vertices are (4, –2), (–2, 4) and (5, 5), then the value of $\alpha + \beta$ must be

19. If the angle between the pair of straight lines $y^2\sin^2\theta - xy\sin^2\theta + x^2(\cos^2\theta - 1) = 0$ is ϕ, then the value of $\sin^3\phi + 3\sin\phi + 2$ must be

20. If G be the centroid and I be the incentre of the triangle with vertices $A(-36, 7)$, $B(20, 7)$ and $C(0, -8)$ and $GI = \frac{\sqrt{(205)}\,\lambda}{27}$, then the value of λ must be

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT MAP

LINEAR INEQUALITIES

Class XI

INEQUALITY

A statement involving the symbols '>', '<', '≥', '≤' is called an inequality.

TYPES OF INEQUALITY

Slack Inequality
Inequalities involving the symbols '≥' or '≤'.

Strict Inequality
Inequalities involving the symbols '>' or '<'.

Quadratic Inequality
An equation of the form $ax^2 + bx + c < 0$ or $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$ or $ax^2 + bx + c \geq 0$.

Literal Inequality
Inequalities which involve variables.

Linear Inequality
An equation of the form $ax + b < 0$ or $ax + by > 0$ or $ax + b \geq 0$ or $ax + by \leq 0$.

Numerical Inequality
Inequalities which do not involve variables.

Linear Inequality in One Variable

- A linear inequality which has only one variable, is called linear inequality in one variable.

- **Solution of Linear Inequality in One Variable** : Any solution of an inequality in one variable is a value of variable which makes it a true statement. The set of all solutions of an inequality, is called the solution set of the inequality.

Graphical Representation of Intervals on the Real Number Line (x-axis)

$-\infty$ … −2 −1 0 1 2 … ∞

(i) 3 … ∞, $x \in [3, \infty)$, $3 \leq x < \infty$

(ii) $-\infty$ … 4, $x \in (-\infty, 4)$, $-\infty < x < 4$

(iii) −1 … 5, $x \in [-1, 5)$, $-1 \leq x < 5$

(iv) −1 … 0, $x \in (-1, 0)$, $-1 < x < 0$

(v) $-\infty$ … 2 … 6 … ∞, $x \in (-\infty, 2) \cup [6, \infty)$

Linear Inequality in Two Variable

- An inequality of the form $ax + by + c > 0$, $ax + by + c < 0$, $ax + by + c \geq 0$ or $ax + by + c \leq 0$, where $a \neq 0$ and $b \neq 0$, is called a linear inequality in two variables x and y.
- The region containing all the solutions of an inequality, is called the **solution region**.

Algorithm

- Convert the inequality $ax + by + c \leq 0$, into equation $ax + by + c = 0$
- Draw the straight line $ax + by + c = 0$ which divides the plane into two half planes as $ax + by + c < 0$ (or) $ax + by + c > 0$
- Choose a point not on the line if possible (0, 0) and substitute in the inequation.
- If the point satisfies the inequation $ax + by + c < 0$, then the half plane containing the origin represents the inequation and the other plane represents $ax + by + c > 0$

Note : In case of strict inequality, draw the dotted straight line, otherwise draw thick line.

DETERMINANTS

Class XII

CONCEPT MAP

DETERMINANTS

Corresponding to every square matrix A, there exists a number called the determinant of A and denoted by $|A|$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then,

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Properties

- The value of a determinant remains unaltered if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by -1.
- If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.
- If the elements of a row (or column) of a determinant are multiplied by any scalar, then the value of the new determinant is equal to same scalar times the value of the original determinant.
- If each element of any row (or column) of a determinant is the sum of two numbers, then the determinant is expressible as the sum of two determinants of the same order.

Note : (i) If $|A| = 0$, then the matrix is singular.
(ii) If $|A| \neq 0$, then the matrix is non-singular.

Minors and Cofactors

- For any matrix $A = [a_{ij}]_{n\times n}$, if we leave the row and the column of the element a_{ij}, then the value of determinant thus obtained is called the minor of a_{ij} and it is denoted by M_{ij}.
- The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij} and denoted by A_{ij}.

$\therefore\ A_{ij} = (-1)^{i+j} M_{ij}$

Adjoint of a Matrix

Let $B = [A_{ij}]$ be the matrix of cofactors of matrix $A = [a_{ij}]$. Then the transpose of B is called the adjoint of matrix A.

Properties of adj (A)

- $A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_n$
- $\text{adj }(AB) = (\text{adj } B)\cdot(\text{adj } A)$
- $|\text{adj } A| = |A|^{n-1}$, where n is the order of A.
- $\text{adj }(\text{adj } A) = |A|^{n-2} A$
  $\Rightarrow |\text{adj }(\text{adj } A)| = |A|^{(n-1)^2}$

Inverse of a Matrix

For any square matrix A, inverse of A is defined as $A^{-1} = \frac{1}{|A|}(\text{adj } A)$, $|A| \neq 0$

Properties

- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Area of a Triangle

Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then area of ΔABC is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Solution of System of Linear Equations

Let $AX = B$ be the given system of equations :

- If $|A| \neq 0$, the system is consistent and has one unique solution.
- If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system is inconsistent and hence it has no solution.
- If $|A| = 0$ and $(\text{adj } A)B = O$, then the system may be either consistent or inconsistent according as the system has either infinitely many solutions or no solution.

BRAIN@WORK

VECTOR ALGEBRA AND THREE DIMENSIONAL GEOMETRY

1. Any four non-zero vector will always be
(a) linearly dependent (b) linearly independent
(c) either (a) or (b) (d) none of these

2. If $\vec{a}=x\hat{i}+(x-1)\hat{j}+\hat{k}$ and $\vec{b}=(x+1)\hat{i}+\hat{j}+a\hat{k}$ always make an acute angle with each other for every value of $x \in R$, then
(a) $a \in (-\infty, 2)$ (b) $a \in (2, \infty)$
(c) $a \in (-\infty, 1)$ (d) $a \in (1, \infty)$

3. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{x}$, $\vec{a}\cdot\vec{x}=1$, $\vec{b}\cdot\vec{x}=\frac{3}{2}$, $|\vec{x}|=2$, then angle between $\vec{c}$ and $\vec{x}$ is
(a) $\cos^{-1}\left(\frac{1}{4}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$
(c) $\cos^{-1}\left(\frac{3}{8}\right)$ (d) $\cos^{-1}\left(\frac{5}{8}\right)$

4. Let $\vec{a},\vec{b},\vec{c}$ be three unit vectors such that $\vec{a}+5\vec{b}+3\vec{c}=\vec{0}$, then $\vec{a}\cdot(\vec{b}\times\vec{c})$ is equal to
(a) $\vec{a}\cdot\vec{b}$ (b) $\vec{a}\cdot(\vec{b}+2\vec{c})$
(c) $\vec{b}\cdot(\vec{a}+\vec{c})$ (d) none of these

5. A, B, C and D are any four points in the space. If $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|=\lambda\Delta_{ABC}$, where Δ_{ABC} is the area of triangle ABC, then λ is equal to
(a) 2 (b) 1/2 (c) 4 (d) 1/4

6. Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$ is always equal to
(a) $[\vec{a},\vec{b},\vec{c}]$ (b) $2[\vec{a},\vec{b},\vec{c}]$
(c) zero (d) none of these

7. Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]$ is also equal to
(a) $[\vec{a},\vec{b},\vec{c}]^2$ (b) $-[\vec{a},\vec{b},\vec{c}]^2$
(c) zero (d) none of these

8. $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$ is always equal to
(a) $\vec{a}$ (b) $-\vec{a}$ (c) $2\vec{a}$ (d) $-2\vec{a}$

9. Condition for the equations $\vec{r}\times\vec{a}=\vec{b}$ and $\vec{r}\times\vec{c}=\vec{d}$ to be consistent, is
(a) $\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{d}=0$ (b) $\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{d}$
(c) $\vec{b}\cdot\vec{c}=\vec{a}\cdot\vec{d}=0$ (d) $\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{d}=0$

10. If the vector $\hat{i}-3\hat{j}+5\hat{k}$ bisects the angle between $\hat{a}$ and $-\hat{i}+2\hat{j}+2\hat{k}$, where $\hat{a}$ is a unit vector, then
(a) $\hat{a}=\frac{1}{105}(41\hat{i}+88\hat{j}-40\hat{k})$
(b) $\hat{a}=\frac{1}{105}(41\hat{i}+88\hat{j}+40\hat{k})$
(c) $\hat{a}=\frac{1}{105}(-41\hat{i}+88\hat{j}-40\hat{k})$
(d) $\hat{a}=\frac{1}{105}(41\hat{k}-88\hat{j}-40\hat{k})$

11. Distance of $P(\vec{p})$ from the plane $\vec{r}\cdot\vec{n}=0$ is
(a) $|\vec{p}\cdot\vec{n}|$ (b) $\frac{|\vec{p}\times\vec{n}|}{|\vec{n}|}$
(c) $\frac{|\vec{p}\cdot\vec{n}|}{|\vec{n}|}$ (d) none of these

12. Distance of $P(\vec{p})$ from the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is
(a) $\left|(\vec{a}-\vec{p})+\frac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$
(b) $\left|(\vec{b}-\vec{p})+\frac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

(c) $\left|(\vec{a}-\vec{p})+\dfrac{((\vec{p}-\vec{b})\cdot b)\vec{b}}{|\vec{b}|^2}\right|$

(d) none of these

13. The line $\vec{r}=\vec{a}+\lambda\vec{b}$ will not meet the plane $\vec{r}\cdot\vec{n}=q$ provided

(a) $\vec{b}\cdot\vec{n}=0,\ \vec{a}\cdot\vec{n}=q$ (b) $\vec{b}\cdot\vec{n}\neq 0,\ \vec{a}\cdot\vec{n}\neq q$

(c) $\vec{b}\cdot\vec{n}=0,\ \vec{a}\cdot\vec{n}\neq q$ (d) $\vec{b}\cdot\vec{n}\neq 0,\ \vec{a}\cdot\vec{n}=q$

14. The plane $\vec{r}\cdot\vec{n}=q$ will contain the line $\vec{r}=\vec{a}+\lambda\vec{b}$ provided

(a) $\vec{b}\cdot\vec{n}\neq 0,\ \vec{a}\cdot\vec{n}\neq q$ (b) $\vec{b}\cdot\vec{n}=0,\ \vec{a}\cdot\vec{n}\neq q$

(c) $\vec{b}\cdot\vec{n}=0,\ \vec{a}\cdot\vec{n}=q$ (d) $\vec{b}\cdot\vec{n}\neq 0,\ \vec{a}\cdot\vec{n}=q$

15. If the projection of point $P(\vec{p})$ on the plane $\vec{r}\cdot\vec{n}=q$ is the point $S(\vec{s})$, then

(a) $\vec{s}=\dfrac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ (b) $\vec{s}=\vec{p}+\dfrac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

(c) $\vec{s}=\vec{p}-\dfrac{(\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ (d) $\vec{s}=\vec{p}-\dfrac{(\vec{q}-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

16. Let $\vec{a}$ and $\vec{b}$ be unit vectors and are perpendicular to each other, then $[\vec{a}+(\vec{a}\times\vec{b}),\vec{b}+(a\times\vec{b}),\vec{a}\times\vec{b}]$ will always be equal to

(a) 1 (b) zero

(c) –1 (d) none of these

17. If $\vec{a}=\vec{b}+\vec{c},\ \vec{b}\times\vec{d}=\vec{0},\ \vec{c}\cdot\vec{d}=0$ then the vector $\dfrac{\vec{d}\times(\vec{a}\times\vec{d})}{|\vec{d}|^2}$ is always equal to

(a) $\vec{a}$ (b) $\vec{d}$ (c) $\vec{b}$ (d) $\vec{c}$

18. Let $\vec{a}$ and $\vec{b}$ are two unit vectors that are mutually perpendicular. A unit vector that is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ is

(a) $\dfrac{1}{\sqrt{2}}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$ (b) $\dfrac{1}{2}(\vec{a}\times\vec{b}+\vec{a}+\vec{b})$

(c) $\dfrac{1}{\sqrt{3}}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$ (d) $\dfrac{1}{3}(\vec{a}\times\vec{b}+\vec{a}+\vec{b})$

19. $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=\hat{j}-\hat{k},\ \vec{c}$ be a vector such that $\vec{a}\cdot\vec{c}=3,\ \vec{a}\times\vec{c}=\vec{b}$, then

(a) $\vec{c}=\dfrac{1}{2}(\hat{k}+4\hat{j}+\hat{k})$ (b) $\vec{c}=\dfrac{1}{3}(-\hat{i}+8\hat{j}+\hat{k})$

(c) $\vec{c}=\dfrac{1}{2}(4\hat{i}+\hat{j}+\hat{k})$ (d) $\vec{c}=\dfrac{1}{3}(5\hat{i}+2\hat{j}+2\hat{k})$

20. For any arbitrary vector $\vec{a}$ the expression $(\vec{a}\cdot\hat{i})(\vec{a}\times\hat{i})+(\vec{a}\cdot\hat{j})(\vec{a}\times\hat{j})+(\vec{a}\cdot\hat{k})(\vec{a}\times\hat{k})$ is always equal to

(a) $\vec{a}$ (b) $2\vec{a}$

(c) $3\vec{a}$ (d) none of these

21. If $\vec{a}$ satisfies the equation $\vec{a}\times(\hat{i}+2\hat{j}+\hat{k})=\hat{i}-\hat{k}$, then $\vec{a}$ is equal to

(a) $\lambda\hat{i}+(2\lambda-1)\hat{j}+\lambda\hat{k},\ \lambda\in R$

(b) $\lambda\hat{i}+(1-2\lambda)\hat{j}+\lambda\hat{k},\ \lambda\in R$

(c) $\lambda\hat{i}+(1+2\lambda)\hat{j}+\lambda\hat{k},\ \lambda\in R$

(d) $\lambda\hat{i}-(1+2\lambda)\hat{j}+\lambda\hat{k},\ \lambda\in R$

22. Let $\vec{a}$ and $\vec{b}$ are two non collinear vectors such that $|\vec{a}|=1$. The angle of a triangle whose two sides are represented by the vector $\sqrt{3}(\vec{a}\times\vec{b})$ and $\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}$ are

(a) $\dfrac{\pi}{2},\dfrac{\pi}{4},\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2},\dfrac{\pi}{3},\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{2},\dfrac{5\pi}{12},\dfrac{\pi}{12}$ (d) none of these

23. Let $\vec{b}$ and $\vec{c}$ are unit vectors, then for any arbitrary vector $\vec{a}$, $(((\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b}-\vec{c})$ is always equal to

(a) $|\vec{a}|$ (b) $\dfrac{1}{2}|\vec{a}|$ (c) $\dfrac{1}{3}|\vec{a}|$ (d) 0

24. Angle between $\hat{i}$ and the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}+\hat{k})=0$ is equal to

(a) $\cos^{-1}\left(\dfrac{1}{3}\right)$ (b) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

(c) $\cos^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$ (d) none of these

25. If $Q(\vec{q})$ is the image of $P(\hat{i}+3\hat{j}+\hat{k})$ in the plane $\vec{r}\cdot(-2\hat{i}+\hat{j}-4\hat{k})=3$ then $\vec{q}$ is

(a) $\dfrac{1}{7}(\hat{i}-25\hat{j}-12\hat{k})$ (b) $\dfrac{1}{7}(\hat{i}-25\hat{j}+12\hat{k})$

(c) $\dfrac{1}{7}(\hat{i}+25\hat{j}-12\hat{k})$ (d) $\dfrac{1}{7}(-\hat{i}+25\hat{j}-9\hat{k})$

26. 'I' is the incentre of triangle ABC, whose corresponding sides are a, b, c respectively. $a\,\overrightarrow{IA}+b\,\overrightarrow{IB}+c\,\overrightarrow{IC}$ is always equal to

(a) $\vec{0}$ (b) $(a+b+c)\overrightarrow{BC}$

(c) $(\vec{a}+\vec{b}+\vec{c})\overrightarrow{AC}$ (d) $(a+b+c)\overrightarrow{AB}$

27. $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and a unit vector $\vec{c}$ are coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then

(a) $\vec{c}=\frac{1}{\sqrt{3}}(-\hat{i}+\hat{j}+\hat{k})$ (b) $\vec{c}=\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

(c) $\vec{c}=\frac{1}{\sqrt{5}}(\hat{i}-2\hat{j})$ (d) $\vec{c}=\frac{1}{\sqrt{5}}(\hat{i}-2\hat{k})$

28. A directed line segment makes angles α, β, γ with the coordinate axes. The value of $\Sigma\cos 2\alpha$ is always equal to

(a) -1 (b) 1 (c) -2 (d) 2

29. The locus represented by $xy + xz = 0$ is

(a) a pair of perpendicular lines
(b) a pair of parallel lines
(c) a pair of parallel planes
(d) a pair of perpendicular planes

30. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(a)$ with x-axis. The value of 'a' is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{7}$ (d) $\frac{3}{7}$

31. The equation of the right bisecting plane of the line segment joining the points $A \equiv (a, a, 2a)$ and $B \equiv (-a, -a, a)$ is

(a) $x + y + z = a$ (b) $4x + 2y + z = 3a$
(c) $4x - 2y + z = 3a$ (d) $4x + 4y + 2z = 3a$

32. The equation of a plane passing through $(1, 2, -3)$, $(0, 0, 0)$ and perpendicular to the plane $3x - 5y + 2z = 11$, is

(a) $3x + y + \frac{5}{3}z = 0$ (b) $4x + y + 2z = 0$

(c) $3x - y + \frac{z}{3} = 0$ (d) $x + y + z = 0$

33. The equation of a plane passing through the line of intersection of the planes $x + y + z = 5$, $2x - y + 3z = 1$ and parallel to the line $y = z = 0$ is

(a) $3x - z = 0$ (b) $3y - z = 9$
(c) $x - 3z = 9$ (d) $y - 3z = 9$

34. A plane meets the coordinate axes at the points A, B, C respectively. If the centroid of triangle ABC always remains (a, b, c), then equation of plane is

(a) $ax + by + cz = 3$ (b) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

(c) $ax + by + cz = 1$ (d) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

35. Centroid of the tetrahedron $OABC$, where $A \equiv (a, 2, 3)$, $B \equiv (1, b, 2)$, $C \equiv (2, 1, c)$ and O is the origin is $(1, 2, 3)$. The value of $a^2 + b^2 + c^2$ is equal to

(a) 75 (b) 80
(c) 121 (d) none of these

36. The straight lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$, will intersect provided

(a) $k = \{3, -3\}$ (b) $k = \{0, -1\}$
(c) $k = \{-1, 1\}$ (d) $k = \{0, -3\}$

37. Equation of the plane passing through $(-1, 1, 4)$ and containing the line $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z}{5}$, is

(a) $9x - 22y + 2z + 23 = 0$
(b) $x + 22y + z = 25$
(c) $9x + 22y - 3z = 1$
(d) $22y - 9x + z = 35$

38. The distance of the point $(-1, 2, 6)$ from the line $\frac{x-2}{6}=\frac{y-3}{3}=\frac{z+4}{-4}$, is equal to

(a) 7 units (b) 9 units
(c) 10 units (d) 12 units

39. The shortest distance between the line $x + y + 2z - 3 = 2x + 3y + 4z - 4 = 0$ and the z-axis is

(a) 1 unit (b) 2 units
(c) 3 units (d) 4 units

40. Reflection of the line $\frac{x-1}{-1}=\frac{y-2}{3}=\frac{z-4}{1}$ in the plane $x + y + z = 7$ is

(a) $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-4}{1}$ (b) $\frac{x-1}{-3}=\frac{y-2}{-1}=\frac{z-4}{1}$

(c) $\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-4}{-1}$ (d) $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-4}{1}$

SOLUTIONS

1. (a) : Four or more than four non-zero vectors are always linearly dependent.

2. (b) : $\vec{a}\cdot\vec{b}=(x\hat{i}+(x-1)\hat{j}+\hat{k})\cdot((x+1)\hat{i}+\hat{j}+a\hat{k})$

$= x(x + 1) + x - 1 + a = x^2 + 2x + a - 1$

We must have $\vec{a}\cdot\vec{b}>0 \quad \forall\ x \in R$

$\Rightarrow \quad x^2 + 2x + a - 1 > 0 \quad \forall\ x \in R$

$\Rightarrow \quad 4 - 4(a - 1) < 0 \quad \Rightarrow \quad a > 2$

3. (b) : $\vec{a}+\vec{b}+\vec{c}=\vec{x}$

Taking dot with $\vec{x}$ on both sides, we get

$$\vec{x}\cdot\vec{a}+\vec{x}\cdot\vec{b}+\vec{x}\cdot\vec{c}=\vec{x}\cdot\vec{x}=|\vec{x}|^2=4$$

$$\Rightarrow \quad 1+\frac{3}{2}+\vec{x}\cdot\vec{c}=4 \quad \Rightarrow \quad \vec{x}\cdot\vec{c}=\frac{3}{2}$$

If 'θ' be the angle between $\vec{c}$ and $\vec{x}$, then

$$|\vec{x}||\vec{c}|\cos\theta=\frac{3}{2} \Rightarrow \cos\theta=\frac{3}{4} \Rightarrow \theta=\cos^{-1}\left(\frac{3}{4}\right)$$

4. (d) : $\vec{a}+5\vec{b}+3\vec{c}=\vec{0}$

Thus $\vec{a}, \vec{b}, \vec{c}$ are coplanar. Hence $\vec{a}\cdot(\vec{b}\times\vec{c})=0$

5. (c) : Let P.V. of A, B, C and D be $\vec{a},\vec{b},\vec{c}$ and $\vec{0}$

$\Rightarrow \quad \overrightarrow{AB}\times\overrightarrow{CD}=(\vec{b}-\vec{a})\times-\vec{c},\ \overrightarrow{BC}\times\overrightarrow{AD}=(\vec{c}-\vec{b})\times-\vec{a}$

and $\overrightarrow{CA}\times\overrightarrow{BD}=(\vec{a}-\vec{c})\times-\vec{b}$

$\Rightarrow \quad \overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}$

$=\vec{c}\times\vec{b}+\vec{a}\times\vec{c}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{a}+\vec{c}\times\vec{b}$

$=2(\vec{c}\times\vec{b}+\vec{b}\times\vec{a}+\vec{a}\times\vec{c})=2[\vec{c}\times(\vec{b}-\vec{a})-\vec{a}\times(\vec{b}-\vec{a})]$

$=2((\vec{c}-\vec{a})\times(\vec{b}-\vec{a}))=2(\overrightarrow{AC}\times\overrightarrow{AB})$

$\Rightarrow \quad |\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$=4\left|\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AB})\right|=4\Delta_{ABC}$

6. (b) : $[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]$

$=(\vec{a}+\vec{b})\cdot((\vec{b}+\vec{c})\times(\vec{c}+\vec{a}))=(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a})$

$=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$

7. (a) : $[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot((\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}))$

$=(\vec{a}\times\vec{b})\cdot([\vec{b}\ \vec{c}\ \vec{a}]\vec{c}-[\vec{b}\ \vec{c}\ \vec{c}]\vec{a})=[\vec{a}\ \vec{b}\ \vec{c}]^2$

8. (c) : $\hat{i}\times(\vec{a}\times\hat{i})=(\hat{i}\cdot\hat{i})\vec{a}-(\vec{a}\cdot\hat{i})\hat{i}=\vec{a}-(\vec{a}\cdot\hat{i})\hat{i}$

Similarly, $\hat{j}\times(\vec{a}\times\hat{j})=\vec{a}-(\vec{a}\cdot\hat{j})\hat{j}$

and $\hat{k}\times(\vec{a}\times\hat{k})=\vec{a}-(\vec{a}\cdot\hat{k})\hat{k}$

$\Rightarrow \quad \hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$=3\vec{a}-((\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k})=2\vec{a}$

9. (a) : We have, $\vec{r}\times\vec{a}=\vec{b}$,

Taking the cross with $\vec{d}$, we get $\vec{d}\times(\vec{r}\times\vec{a})=\vec{d}\times\vec{b}$

$\Rightarrow \quad (\vec{a}\cdot\vec{d})\vec{r}-(\vec{d}\cdot\vec{r})\vec{a}=\vec{d}\times\vec{b}$ ...(i)

Also, $\vec{r}\times\vec{c}=\vec{d}$

Taking the cross with $\vec{b}$, we get $\vec{b}\times(\vec{r}\times\vec{c})=\vec{b}\times\vec{d}$

$\Rightarrow \quad (\vec{b}\cdot\vec{c})\vec{r}-(\vec{b}\cdot\vec{r})\vec{c}=\vec{b}\times\vec{d}$ ...(ii)

Adding (i) and (ii), we get

$$(\vec{a}\cdot\vec{d}+\vec{b}\cdot\vec{c})\vec{r}-(\vec{d}\cdot\vec{r})\vec{a}-(\vec{b}\cdot\vec{r})\vec{c}=0$$

Now, $\vec{r}\cdot\vec{d}=0$ and $\vec{b}\cdot\vec{r}=0$ as $\vec{d}$ and $\vec{r}$ as well as $\vec{b}$ and $\vec{r}$ are mutually perpendicular.

Thus, $(\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{d})\vec{r}=\vec{0} \quad \Rightarrow \quad \vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{d}=0$

10. (d) : We must have $\lambda(\hat{i}-3\hat{j}+5\hat{k})=\hat{a}+\dfrac{2\hat{k}+2\hat{j}-\hat{i}}{3}$

$\Rightarrow \quad 3\hat{a}=3\lambda(\hat{i}-3\hat{j}+5\hat{k})-(2\hat{k}+2\hat{j}-\hat{i})$

$=\hat{i}(3\lambda+1)-\hat{j}(2+9\lambda)+\hat{k}(15\lambda-2)$

$\Rightarrow \quad 3|\hat{a}|=\sqrt{(3\lambda+1)^2+(2+9\lambda)^2+(15\lambda-2)^2}$

$\Rightarrow \quad 9=(3\lambda+1)^2+(2+9\lambda)^2+(15\lambda-2)^2$

$\Rightarrow \quad 315\lambda^2-18\lambda=0 \quad \Rightarrow \quad \lambda=0, \frac{2}{35}$

For $\lambda=0$, $\hat{a}=\hat{i}-2\hat{j}-2\hat{k}$ (not acceptable)

For $\lambda=\frac{2}{35}$, $\hat{a}=\frac{41}{105}\hat{i}-\frac{88}{105}\hat{j}-\frac{40}{105}\hat{k}$

11. (c) : Let $Q(\vec{q})$ be the foot of altitude drawn from P to the plane $\vec{r}\cdot\vec{n}=0$.

$\Rightarrow \quad \vec{q}-\vec{p}=\lambda\vec{n} \quad \Rightarrow \quad \vec{q}=\vec{p}+\lambda\vec{n}$

Also $\vec{q}\cdot\vec{n}=0 \quad \Rightarrow \quad (\vec{p}+\lambda\vec{n})\cdot\vec{n}=0$

$\Rightarrow \quad \lambda=-\dfrac{(\vec{p}\cdot\vec{n})}{|\vec{n}|^2} \quad \Rightarrow \quad \vec{q}-\vec{p}=-\dfrac{(\vec{p}\cdot\vec{n})}{|\vec{n}|^2}\vec{n}$

$\therefore$ Required distance $=|\vec{q}-\vec{p}|=\dfrac{|\vec{p}\cdot\vec{n}|}{|\vec{n}|}$

12. (a) : Let $Q(\vec{q})$ be the foot of altitude drawn from $P(\vec{p})$ to the line $\vec{r}=\vec{a}+\lambda\vec{b}$,

$\Rightarrow \quad (\vec{q}-\vec{p})\cdot\vec{b}=0$ and $\vec{q}=\vec{a}+\lambda\vec{b}$,

$\Rightarrow \quad (a+\lambda\vec{b}-\vec{p})\cdot\vec{b}=0 \quad \Rightarrow \quad (\vec{a}-\vec{p})\cdot\vec{b}+\lambda|\vec{b}|^2=0$

$\Rightarrow \quad \lambda=\dfrac{(\vec{p}-\vec{a})\cdot\vec{b}}{|\vec{b}|^2} \quad \Rightarrow \quad \vec{q}-\vec{p}=\vec{a}+\dfrac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}-\vec{p}$

$\Rightarrow \quad |\vec{q}-\vec{p}|=\left|\vec{a}-\vec{p}+\dfrac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

13. (c) : We must have, $\vec{b}\cdot\vec{n}=0$ and $\vec{a}\cdot\vec{n}\neq q$

14. (c) : We have, $\vec{b}\cdot\vec{n}=0$ and $\vec{a}\cdot\vec{n}=q$

15. (b) : We have $\vec{s}-\vec{p}=\lambda\vec{n}$ and $\vec{s}\cdot\vec{n}=q$

$\Rightarrow \quad (\lambda\vec{n}+\vec{p})\cdot\vec{n}=q \quad \Rightarrow \quad \vec{s}=\vec{p}+\dfrac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

16. (a) : $[\vec{a}+(\vec{a}\times\vec{b}),\ \vec{b}+(\vec{a}\times\vec{b}),\ \vec{a}\times b]$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot((\vec{b}+(\vec{a}\times\vec{b}))\times(\vec{a}\times\vec{b}))$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot((\vec{b}\times(\vec{a}\times\vec{b}))=(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{a}-(\vec{a}\cdot\vec{b})\vec{b})$

$=\vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{a}\times\vec{b})=1 \qquad (\text{as } \vec{a}\cdot\vec{b}=0,\ \vec{a}\cdot(\vec{a}\times\vec{b})=0)$

17. (d) : $\vec{a}=\vec{b}+\vec{c},\ \vec{b}\times\vec{d}=\vec{0},\ \vec{c}\cdot\vec{d}=0$

$\Rightarrow \quad \vec{a}\times\vec{d}=\vec{b}\times\vec{d}+\vec{c}\times\vec{d}=\vec{c}\times\vec{d}$

$\Rightarrow \quad \vec{d}\times(\vec{a}\times\vec{d})=\vec{d}\times(\vec{c}\times\vec{d})=(\vec{d}\cdot\vec{d})\vec{c}-(\vec{c}\cdot\vec{d})\vec{d}=|\vec{d}|^2\vec{c}$

$\Rightarrow \quad \dfrac{\vec{d}\times(\vec{a}\times\vec{d})}{|\vec{d}|^2}=\vec{c}$

18. (c) : Let required vector $\vec{r}$ is such that

$$\vec{r} = x_1\vec{a} + x_2\vec{b} + x_3(\vec{a}\times\vec{b})$$

We must have, $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot(\vec{a}\times\vec{b})$

$$\vec{r}\cdot\vec{a} = x_1, \vec{r}\cdot\vec{b} = x_2, \vec{r}\cdot(\vec{a}\times\vec{b}) = x_3$$

$\Rightarrow \quad \vec{r} = \lambda(\vec{a}+\vec{b}+(\vec{a}\times\vec{b}))$. Also, $\vec{r}\cdot\vec{r} = 1$

$\Rightarrow \quad \lambda^2(\vec{a}+\vec{b}+(\vec{a}\times\vec{b}))\cdot(\vec{a}+\vec{b}+(\vec{a}\times\vec{b})) = 1$

$\Rightarrow \quad \lambda^2(\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot(\vec{a}\times\vec{b})$

$$+(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{a}\times\vec{b})\cdot\vec{b}+(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = 1$$

$\Rightarrow \quad \lambda^2 = \frac{1}{3} \Rightarrow \lambda = \pm\frac{1}{\sqrt{3}} \Rightarrow \vec{r} = \pm\frac{1}{\sqrt{3}}(\vec{a}+\vec{b}+(\vec{a}\times\vec{b}))$

19. (d) : $\vec{a}\times\vec{c} = \vec{b}$

$\Rightarrow \quad \vec{a}\times(\vec{a}\times\vec{c}) = \vec{a}\times\vec{b} \Rightarrow (\vec{a}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{a})\vec{c} = \vec{a}\times\vec{b}$

Now, $\quad \vec{a}\cdot\vec{a} = |\hat{i}+\hat{j}+\hat{k}|^2 = 3,$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\hat{i}+\hat{j}+\hat{k}$$

$\Rightarrow \quad 3\vec{a} - 3\vec{c} = \vec{a}\times\vec{b} = -2\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow \quad 3\vec{c} = 3\hat{i}+3\hat{j}+3\hat{k}+2\hat{i}-\hat{j}-\hat{k} = 5\hat{i}+2\hat{j}+2\hat{k}$

$\Rightarrow \quad \vec{c} = \frac{1}{3}(5\hat{i}+2\hat{j}+2\hat{k})$

20. (d) : Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\Rightarrow \quad \vec{a}\cdot\hat{i} = a_1, \vec{a}\cdot\hat{j} = a_2, \vec{a}\cdot\hat{k} = a_3$

and $\quad \vec{a}\times i = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times i = -a_2\hat{k}+a_3\hat{j}$

Similarly, $\vec{a}\times\hat{j} = a_1\hat{k}-a_3\hat{i}$, $\vec{a}\times\hat{k} = -a_1\hat{j}+a_2\hat{i}$

$\Rightarrow \quad (\vec{a}\cdot\hat{i})(\vec{a}\times\hat{i})+(\vec{a}\cdot\hat{j})(\vec{a}\times\hat{j})+(\vec{a}\cdot\hat{k})(\vec{a}\times\hat{k})$

$= -a_1a_2\hat{k}+a_1a_3\hat{j}+a_1a_2\hat{k}-a_3a_2\hat{i}+a_3a_2\hat{i}-a_3a_1\hat{j} = \vec{0}$

21. (c) : $\vec{a}\times(\hat{i}+2\hat{j}+\hat{k}) = \hat{i}-\hat{k} = (\hat{j}\times(\hat{i}+2\hat{j}+\hat{k}))$

$\Rightarrow \quad (\vec{a}-\hat{j})\times(\hat{i}+2\hat{j}+\hat{k}) = \vec{0} \Rightarrow \vec{a}-\hat{j} = \lambda(\hat{i}+2\hat{j}+\hat{k})$

$\Rightarrow \quad \vec{a} = \lambda\hat{i}+(2\lambda+1)\hat{j}+\lambda\hat{k}, \ \lambda\in R$

22. (b) : $\vec{r}_1 = \sqrt{3}(\vec{a}\times\vec{b}), \vec{r}_2 = \vec{b}-(\vec{a}\cdot\vec{b})\vec{a}$

Clearly $\vec{r}_1$ and $\vec{r}_2$ are mutually perpendicular as $\vec{r}_2$ is coplanar with $\vec{a}$ and $\vec{b}$ and $\vec{r}_1$ is at right angle to the plane of $\vec{a}$ and $\vec{b}$

$|\vec{r}_1| = \sqrt{3}|\vec{a}\times\vec{b}| \quad \Rightarrow \quad |\vec{r}_1|^2 = 3(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$

$$= 3((\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})^2) = 3(|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2)$$

Also, $\quad |\vec{r}_2|^2 = (\vec{b}-(\vec{a}\cdot\vec{b})\vec{a})\cdot(\vec{b}-(\vec{a}\cdot\vec{b})\vec{a})$

$$= |\vec{b}|^2 - 2(\vec{a}\cdot\vec{b})^2+(\vec{a}\cdot\vec{b})^2(\vec{a}\cdot\vec{a}) = |\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$$

$\Rightarrow \quad \frac{|\vec{r}_1|}{|\vec{r}_2|} = \sqrt{3}$. Thus angles are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$.

23. (d) : $((\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c})$

$$= (\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})+(\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c})$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]\vec{b}-[\vec{a}\ \vec{b}\ \vec{b}]\vec{c}+[\vec{a}\ \vec{c}\ \vec{c}]\vec{b}-[\vec{a}\ \vec{c}\ \vec{b}]\vec{c}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b}+\vec{c})$$

$\Rightarrow \quad (((\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b}-\vec{c})$

$$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b}+\vec{c})\cdot(\vec{b}-\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}](|\vec{b}|^2-|\vec{c}|^2) = 0$$

24. (d) : Line of intersection of $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}+\hat{k}) = 0$ will be parallel to $(3\hat{i}+3\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}+3\hat{k})$, *i.e.*, $7\hat{i}-8\hat{j}+3\hat{k}$.

If the required angle is θ, then

$$\cos\theta = \frac{7}{\sqrt{49+64+9}} = \frac{7}{\sqrt{122}}$$

25. (d) : If $Q(\vec{q})$ is the image of $P(\vec{p})$ in the plane $\vec{r}\cdot\vec{n} = s$, then $\vec{q}-\vec{p} = \lambda\vec{n}$ and $\left(\frac{\vec{q}+\vec{p}}{2}\right)\cdot\vec{n} = s$

$\Rightarrow \quad (2\vec{p}+\lambda\vec{n})\cdot\vec{n} = 2s \quad \Rightarrow \quad \lambda = \frac{2s-2\vec{p}\cdot\vec{n}}{|\vec{n}|^2}$

$\Rightarrow \quad \vec{q} = \vec{p}+\frac{(2s-2p\cdot\vec{n})}{|\vec{n}|^2}\vec{n}$

$$= \hat{i}+3\hat{j}+\hat{k}+\frac{(6-2(-2+3-4))}{21}(-2\hat{i}+\hat{j}-4\hat{k})$$

$$= \frac{1}{7}(-\hat{i}+25\hat{j}-9\hat{k})$$

26. (a)

27. (b) : Let $\vec{c} = x_1\vec{a}+x_2\vec{b}$

Since, $\vec{a}\cdot\vec{c} = 0$ therefore $0 = x_1|\vec{a}|^2+x_2\vec{a}\cdot\vec{b}$

$\Rightarrow \quad 0 = x_1(6) + x_2(2 + 2 - 1) \Rightarrow x_2 + 2x_1 = 0$

Also $|\vec{c}| = 1$

$\Rightarrow \quad (x_1\vec{a}+x_2\vec{b})\cdot(x_1\vec{a}+x_2\vec{b}) = 1$

$\Rightarrow \quad x_1^2|\vec{a}|^2+2x_1x_2\vec{a}\cdot\vec{b}+x_2^2|\vec{b}|^2 = 1$

$\Rightarrow \quad 6x_1^2 + 6x_1x_2 + 6x_2^2 = 1 \Rightarrow 6x_1^2 - 12x_1^2 + 24x_1^2 = 1$

$\Rightarrow \quad x_1^2 = \frac{1}{18} \quad \Rightarrow \quad x_1 = \pm\frac{1}{3\sqrt{2}}$

$\Rightarrow \quad x_2 = \mp\frac{2}{3\sqrt{2}} \Rightarrow \vec{c} = \frac{1}{3\sqrt{2}}(2\hat{i}+\hat{j}+\hat{k})-\frac{2}{3\sqrt{2}}(\hat{i}+2\hat{j}-\hat{k})$

$= \frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

or $\vec{c} = -\frac{1}{3\sqrt{2}}(2\hat{i}+\hat{j}+\hat{k})+\frac{2}{3\sqrt{2}}(\hat{i}+2\hat{j}-\hat{k}) = \frac{1}{\sqrt{2}}(\hat{j}-\hat{k})$

28. (a) : $\sum\cos 2\alpha = \sum(2\cos^2\alpha - 1) = 2\sum l^2 - 3 = -1$

29. (d) : $xy + yz = 0$

$\Rightarrow \quad x(y + z) = 0 \quad \Rightarrow \; x = 0, y + z = 0$

Thus it represents a pair of planes

$x = 0, \; y + z = 0$

that are clearly mutually perpendicular.

30. (c) : Let θ be the angle between them, then

$\sin\theta = \dfrac{2}{\sqrt{4+9+36}} = \dfrac{2}{7} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\dfrac{2}{7}\right)$

31. (d) : Direction ratios of segment AB are $(2a, 2a, a)$.

Mid-point of AB is $\left(0, 0, \dfrac{3a}{2}\right)$.

Required plane will be in the form

$x \cdot 2a + y \cdot 2a + z \cdot a = p$

It should pass through $\left(0, 0, \dfrac{3a}{2}\right) \quad \Rightarrow \quad \dfrac{3a^2}{2} = p$

Thus, equation of the plane is $2x + 2y + z = \dfrac{3a}{2}$

i.e, $4x + 4y + 2z = 3a$.

32. (d) : Let the required plane be $ax + by + cz = 0$

We have $3a - 5b + 2c = 0$, $a + 2b - 3c = 0$

$\Rightarrow \quad \dfrac{a}{15-4} = \dfrac{b}{2+9} = \dfrac{c}{6+5} \; \Rightarrow a : b : c = 11 : 11 : 11$

Thus plane is $x + y + z = 0$

33. (b) : Plane will be in the form

$(x + y + z - 5) + a(2x - y + 3z - 1) = 0$

i.e., $x(1 + 2a) + y(1 - a) + z(1 + 3a) = 5 + a$

It is parallel to the line $y = z = 0$.

$\Rightarrow \quad (1 + 2a) = 0 \quad \Rightarrow \quad a = -1/2$

Thus required plane is $\dfrac{3}{2}y - \dfrac{1}{2}z = \dfrac{9}{2}$ *i.e.,* $3y - z = 9$

34. (b) : Let the plane be $\dfrac{x}{x_1} + \dfrac{y}{y_1} + \dfrac{z}{z_1} = 1$

$\Rightarrow \quad A \equiv (x_1, 0, 0), B \equiv (0, y_1, 0), C \equiv (0, 0, z_1)$

Since the centroid of triangle ABC is (a, b, c)

$\therefore \quad 3a = x_1, 3b = y_1, 3c = z_1$

Thus, equation of plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3$

35. (a) : We have, $4 = a + 1 + 2 + 0 \;\Rightarrow\; a = 1$,

$8 = 2 + b + 1 + 0 \quad \Rightarrow \quad b = 5$,

$12 = 3 + 2 + c + 0 \;\; \Rightarrow \quad c = 7$

$\therefore \quad a^2 + b^2 + c^2 = 1 + 25 + 49 = 75$

36. (d) : Any point on the first line can be takes as

$P_1 \equiv (r_1 + 2, r_1 + 3, -kr_1 + 4)$

Similarly any point on second line can be taken as

$P_2 \equiv (kr_2 + 1, 2r_2 + 4, r_2 + 5)$.

These lines will intersect if for some r_1 and r_2 we have $r_1 + 2 = kr_2 + 1$, $r_1 + 3 = 2r_2 + 4$, $-kr_1 + 4 = r_2 + 5$.

$\therefore \quad r_1 - kr_2 + 1 = 0, r_1 = 2r_2 + 1$

$\Rightarrow \quad r_2 = \dfrac{2}{k-2}, \quad r_1 = \dfrac{k+2}{k-2}$

On substituting, we get $k^2 + 3k = 0 \;\Rightarrow\; k = \{-3, 0\}$

37. (d) : Equation of any plane containing the line $\dfrac{x-1}{3} = \dfrac{y-2}{1} = \dfrac{z}{5}$ will be $a(x - 1) + b(y - 2) + cz = 0$

where, $3a + b + 5c = 0$ .... (i)

It is given that plane passes through $(-1, 1, 4)$.

$\therefore \quad -2a - b + 4c = 0$ ... (ii)

From (i) and (ii), we get $\dfrac{a}{-9} = \dfrac{b}{22} = \dfrac{c}{1}$

Thus the equation of required plane is,

$-9(x - 1) + 22(y - 2) + z = 0$

i.e., $22y - 9x + z = 35$

38. (a) : Any point on the line is

$P \equiv (6r_1 + 2, 3r_1 + 3, -4r_1 - 4)$

Direction ratio of the line segment PQ, where

$Q \equiv (-1, 2, 6)$, are $6r_1 + 3, 3r_1 + 1, -4r_1 - 10$.

If 'P' be the foot of altitude drawn from Q to the given line, then $6(6r_1 + 3) + 3(3r_1 + 1) + 4(4r_1 + 10) = 0$

$\Rightarrow \quad r_1 = -1$.

Thus, $P \equiv (-4, 0, 0)$

$\therefore \quad$ Required distance $= \sqrt{9+4+36} = 7$ units

39. (b) : We have, $x + y + 2z - 3 = 0$,

$x + 2z - 2 + \dfrac{3}{2}y = 0$

Solving these equations, we get $y = -2$

Thus required shortest distance is 2 units.

40. (c) : Given line passes through (1, 2, 4) and this point also lies on the given plane.

Thus required line will be in the form of

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-4}{n}$$

Any point on the given line is $(-r_1 + 1, 3r_1 + 2, r_1 + 4)$

If $r_1 = 1$, this point becomes $P \equiv (0, 5, 5)$.

Let $Q(a, b, c)$ be the reflection of 'P' in the given plane, then

$\dfrac{a}{2}\cdot 1 + \dfrac{b+5}{2}\cdot 1 + \dfrac{5+c}{2}\cdot 1 = 7$ *i.e.,* $a + b + c = 4$,

and $\dfrac{a}{1} = \dfrac{b-5}{1} = \dfrac{c-5}{1} = \lambda$ (say)

Thus, $Q \equiv (-2, 3, 3)$.

$\therefore \quad$ Required equation is $\dfrac{x-1}{-3} = \dfrac{y-2}{1} = \dfrac{z-4}{-1}$

CONCEPT BOOSTERS

Vector Algebra and Three Dimensional Geometry

Class XII

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

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VECTORS

- Vector quantities are specified by definite magnitude and definite directions. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.
- A vector of zero magnitude is a zero vector. *i.e.* which has the same initial & terminal point, is called a zero vector. It is denoted by O. The direction of zero vector is indeterminate.
- A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.
- Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.
- Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called parallel vectors. If they have the same direction they are named as like vectors otherwise unlike vectors.

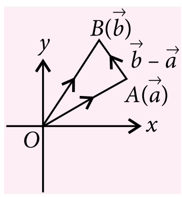
  Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$.

  Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$
- A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note : "Two vectors are always coplanar".
- If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram $OACB$.

Algebra of vectors

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)
- $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$
- $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$
- $|\vec{a} - \vec{b}| \ge ||\vec{a}| - |\vec{b}||$
- $|\vec{a} \pm \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.
- If $\vec{a}$ is a vector and m is a scalar, then $m\,\vec{a}$ is a vector parallel to whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called scalar multiplication.
- Let O be the origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then, $\overrightarrow{AB} = \vec{b} - \vec{a}$

  = p.v. of B – p.v. of A.



Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is $AB = |\vec{a} - \vec{b}|$

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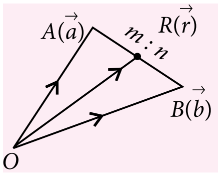
If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio $m : n$ is given by : $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.



Note : p.v. of mid point of $AB = \dfrac{\vec{a}+\vec{b}}{2}$.

Dot product

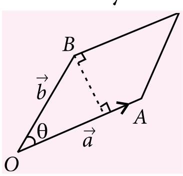
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1; \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.
- $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$ $(0 \le \theta \le \pi)$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

  Note : If θ is acute then $\vec{a}\cdot\vec{b} > 0$ and if θ is obtuse then $\vec{a}\cdot\vec{b} < 0$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = \vec{a}^2$
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)
- $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)
- $(m\vec{a})\cdot\vec{b} = \vec{a}\cdot(m\vec{b}) = m(\vec{a}\cdot\vec{b})$ (associative) where m is scalar.
- $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ (where, $\vec{a} \ne 0, \vec{b} \ne 0$)
- Maximum value of $\vec{a}\cdot\vec{b}$ is $|\vec{a}|\,|\vec{b}|$.
- Minimum value of $\vec{a}\cdot\vec{b}$ is $-|\vec{a}|\,|\vec{b}|$.
- Any vector $\vec{a}$ can be written as,

  $\vec{a} = (\vec{a}\cdot\hat{i})\,\hat{i} + (\vec{a}\cdot\hat{j})\,\hat{j} + (\vec{a}\cdot\hat{k})\,\hat{k}$

Vector or cross product

- If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system.
- Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

  $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (not commutative)
- $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$, (associative) where m is a scalar.
- $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (distributive)
- $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a} \ne 0, \vec{b} \ne 0)$ *i.e.* $\vec{a} = K\vec{b}$, where K is a scalar.
- Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
- A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.
- If $\vec{a}, \vec{b}$ & $\vec{c}$ are the p.v.'s of 3 points A, B & C then the vector area of triangle

  $ABC = \dfrac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}]$.

  The points A, B & C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$.
- Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$.
- Lagrange's Identity: For any two vectors $\vec{a}$ & $\vec{b}$;

  $$(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$

Scalar Triple product

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as : $\vec{a}\times\vec{b}\cdot\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also written as $[\vec{a}\;\vec{b}\;\vec{c}]$ and spelled as box product.
- Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ *i.e.* $V = [\vec{a}\;\vec{b}\;\vec{c}]$.

- In a scalar triple product the position of dot & cross can be interchanged *i.e.* $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ or $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$
- $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$ *i.e.* $[\vec{a}\,\vec{b}\,\vec{c}]=-[\vec{a}\,\vec{c}\,\vec{b}]$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ & $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then $[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$.
- If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow[\vec{a}\ \vec{b}\ \vec{c}]=0$.
- Scalar product of three vectors, two of which are equal or parallel is 0 *i.e.*, $[\vec{a}\,\vec{b}\,\vec{c}]=0$, where $\vec{a}=\vec{b}$ or $\vec{b}=\vec{c}$ or $\vec{c}=\vec{a}$.
- $[K\vec{a}\ \vec{b}\ \vec{c}]=K[\vec{a}\ \vec{b}\ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}]=[\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$
- The volume of the tetrahedron *OABC* with *O* as origin & the p.v.'s of *A*, *B* and *C* being $\vec{a}$, $\vec{b}$ & $\vec{c}$ respectively is given by $V=\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$
- The position vector of the centroid of a tetrahedron if the p.v.'s of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

Vector triple product

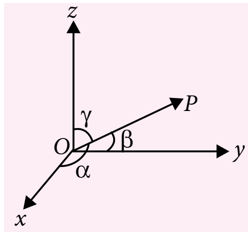
- Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector and is called a vector triple product.
- $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$

LINE

- The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=r$. This form is called symmetric form. A general point on the line is given by (x_1+ar, y_1+br, z_1+cr).
- Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$ where λ is a scalar.
- The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is
$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$
- Vector equation of a straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.
- Reduction of cartesion form of equation of a line to vector form & vice versa $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$
$\Leftrightarrow\ \vec{r}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$

PLANE

- Direction cosines : Let α, β, γ be the angles, which a directed line makes with the positive directions of the axes of *x*, *y* and *z* respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line.



The direction cosines are usually denoted by (l, m, n). Thus $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$.
- If l, m, n be the direction cosines of a line, then $l^2+m^2+n^2=1$
- Direction ratios : Let *a*, *b*, *c* be proportional to the direction cosines *l*, *m*, *n* then *a*, *b*, *c* are called the direction ratios.
- If *l*, *m*, *n* be the direction cosines and *a*, *b*, *c* be the direction ratios of a vector, then
$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\frac{b}{\sqrt{a^2+b^2+c^2}},\ n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$
or
$$l=\frac{-a}{\sqrt{a^2+b^2+c^2}},\ m=\frac{-b}{\sqrt{a^2+b^2+c^2}},\ n=\frac{-c}{\sqrt{a^2+b^2+c^2}}$$
- If $OP=r$, where *O* is the origin and the direction cosines of *OP* are *l*, *m*, *n* then the coordinates of *P* are (lr, mr, nr).
- If direction cosines of the line *AB* are *l*, *m*, *n*, $|AB|=r$ and the coordinates of *A* is (x_1, y_1, z_1) then the coordinates of *B* is given as (x_1+rl, y_1+rm, z_1+rn)
- If the coordinates of *P* and *Q* are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then the direction ratios of line *PQ* are, $a=x_2-x_1$, $b=y_2-y_1$ & $c=z_2-z_1$ and the direction cosines of line *PQ* are $l=\frac{x_2-x_1}{|PQ|}$, $m=\frac{y_2-y_1}{|PQ|}$ and $n=\frac{z_2-z_1}{|PQ|}$

- Direction cosines of axes : Since the positive x-axis makes angles 0°, 90°, 90° with axes of x, y and z respectively. Therefore,
  Direction cosines of x-axis are (1, 0, 0)
  Direction cosines of y-axis are (0, 1, 0)
  Direction cosines of z-axis are (0, 0, 1)
- If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively then we can consider two vectors parallel to the lines as $a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$ and angle between them can be given as $\cos\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
- The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- The lines will be parallel if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$
- If the coordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then the projection of the line segment PQ on a line having direction cosines l, m, n is $|l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|$
- Normal form of the equation of a plane is $lx + my + nz = p$, where, l, m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.
- General form : $ax + by + cz + d = 0$ is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.
- The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where a, b, c are the direction ratios of the normal to the plane.
- Plane through three points: The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is
$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0$$
- Intercept Form: The equation of a plane cutting intercepts a, b, c on the x, y, z axes respectively is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$
- Vector form: The equation of a plane passing through a point having position vector $\vec{a}$ & normal to vector $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n}=0$ or $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$
- Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given as $\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

- Coplanarity of four points :
  The points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are coplanar then
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}=0$$
  The points $A(\vec{r}_1)$, $B(\vec{r}_2)$, $C(\vec{r}_3)$ and $D(\vec{r}_4)$ are coplanar if $[\vec{r}_4-\vec{r}_1\ \vec{r}_4-\vec{r}_2\ \vec{r}_4-\vec{r}_3]=0$
- Distance of the point (x', y', z') from the plane $ax+by+cz+d=0$ is given by $\left|\dfrac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}\right|$.
- The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}\cdot\vec{n}=d$ is given by $p=\dfrac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$.
- The coordinates of the foot of perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are given by
$$\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$
- The coordinate of the image of point (x_1, y_1, z_1) w.r.to the plane $ax + by + cz + d = 0$ are given by
$$\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}$$
$$=-2\frac{|(ax_1+by_1+cz_1+d)|}{a^2+b^2+c^2}$$
- Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a', b', c') respectively, hence, the angle θ between them, is given by
$$\cos\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}$$
  Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$.
- The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

- Any plane passing through the line of intersection of non-parallel planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by
$a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$

PROBLEMS

Single Correct Answer Type

1. If the length of a vector be 21 and direction ratios be 2, – 3, 6 then its direction cosines are
(a) $\frac{2}{21},\frac{-1}{7},\frac{2}{7}$ (b) $\frac{2}{7},\frac{-3}{7},\frac{6}{7}$
(c) $\frac{2}{7},\frac{3}{7},\frac{6}{7}$ (d) none of these

2. If the co-ordinates of the points P and Q be (1, –2, 1) and (2, 3, 4) and O be the origin, then
(a) $OP = OQ$ (b) $OP \perp OQ$
(c) $OP \parallel OQ$ (d) none of these

3. xy-plane divides the line joining the points (2, 4, 5) and (– 4, 3, – 2) in the ratio
(a) 3 : 5 (b) 5 : 2
(c) 1 : 3 (d) 3 : 4

4. If the co-ordinates of A and B be (1, 2, 3) and (7, 8, 7), then the projections of the line segment AB on the co-ordinate axes are
(a) 6, 6, 4 (b) 4, 6, 4
(c) 3, 3, 2 (d) 2, 3, 2

5. A line makes angles α, β, γ with the co-ordinate axes. If $\alpha+\beta=90°$, then $\gamma =$
(a) 0 (b) 90°
(c) 180° (d) none of these

6. The co-ordinates of a point P are (3, 12, 4) with respect to origin O, then the direction cosines of OP are
(a) 3, 12, 4 (b) $\frac{1}{4},\frac{1}{3},\frac{1}{2}$
(c) $\frac{3}{\sqrt{13}},\frac{1}{\sqrt{13}},\frac{2}{\sqrt{13}}$ (d) $\frac{3}{13},\frac{12}{13},\frac{4}{13}$

7. A line makes angles of 45° and 60° with the positive axes of X and Y respectively. The angle made by the same line with the positive axis of Z, is
(a) 30° or 60° (b) 60° or 90°
(c) 90° or 120° (d) 60° or 120°

8. If $P(3, 4, 5)$, $Q(4, 6, 3)$, $R(-1, 2, 4)$, $S(1, 0, 5)$ then the projection of RS on PQ is
(a) – 2/3 (b) – 4/3 (c) 1/2 (d) 2

9. The projection of a line on a co-ordinate axes are 2, 3, 6. Then the length of the line is
(a) 7 (b) 5 (c) 1 (d) 11

10. A line which makes angle 60° with y-axis and z-axis, then the angle which it makes with x-axis is
(a) 45° (b) 60° (c) 75° (d) 30°

11. The distance of the point (4, 3, 5) from the y-axis is
(a) $\sqrt{34}$ (b) 5 (c) $\sqrt{41}$ (d) $\sqrt{15}$

12. The angle between the lines $\frac{x+4}{1}=\frac{y-3}{2}=\frac{z+2}{3}$ and $\frac{x}{3}=\frac{y-1}{-2}=\frac{z}{1}$ is
(a) $\sin^{-1}\left(\frac{1}{7}\right)$ (b) $\cos^{-1}\left(\frac{2}{7}\right)$
(c) $\cos^{-1}\left(\frac{1}{7}\right)$ (d) none of these

13. The straight line $\frac{x-3}{3}=\frac{y-2}{1}=\frac{z-1}{0}$ is
(a) Parallel to x-axis (b) Parallel to y-axis
(c) Parallel to z-axis (d) Perpendicular to z-axis

14. The magnitudes of mutually perpendicular forces $\vec{a},\vec{b}$ and $\vec{c}$ are 2, 10 and 11 respectively. Then the magnitude of its resultant is
(a) 12 (b) 15 (c) 9 (d) 6

15. If the position vectors of A and B are $\hat{i}+3\hat{j}-7\hat{k}$ and $5\hat{i}-2\hat{j}+4\hat{k}$, then the direction cosine of $\overrightarrow{AB}$ along y-axis is
(a) $\frac{4}{\sqrt{162}}$ (b) $-\frac{5}{\sqrt{162}}$
(c) – 5 (d) 11

16. Let α, β, γ be distinct real numbers. The points with position vectors
$\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}, \beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}, \gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$
(a) are collinear
(b) form an equilateral triangle
(c) form a scalene triangle
(d) form a right angled triangle

17. What should be added in vector $\vec{a}=3\hat{i}+4\hat{j}-2\hat{k}$ to get its resultant a unit vector?
(a) $-2\hat{i}-4\hat{j}+2\hat{k}$ (b) $-2\hat{i}+4\hat{j}-2\hat{k}$
(c) $2\hat{i}+4\hat{j}-2\hat{k}$ (d) none of these

18. The sum of two forces is 18N and resultant whose direction is at right angles to the smaller force is 12N. The magnitude of the two forces are
(a) 13, 5 (b) 12, 6 (c) 14, 4 (d) 11, 7

19. If $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}, \vec{b}=2\hat{i}-4\hat{j}-3\hat{k}$ and $\vec{c}=-\hat{i}+2\hat{j}+2\hat{k}$, then $\vec{a}+\vec{b}+\vec{c}$ is

(a) $3\hat{i}-4\hat{j}$ (b) $3\hat{i}+4\hat{j}$
(c) $4\hat{i}-4\hat{j}$ (d) $4\hat{i}+4\hat{j}$

20. The position vectors of A and B are $\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}-\hat{j}+3\hat{k}$. The position vector of the middle point of the line AB is

(a) $\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}+\hat{k}$ (b) $2\hat{i}-\hat{j}+\frac{5}{2}\hat{k}$
(c) $\frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}$ (d) none of these

21. P is the point of intersection of the diagonals of the parallelogram $ABCD$. If O is any point, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=$

(a) $\overrightarrow{OP}$ (b) $2\overrightarrow{OP}$ (c) $3\overrightarrow{OP}$ (d) $4\overrightarrow{OP}$

22. If C is the middle point of AB and P is any point outside AB, then

(a) $\overrightarrow{PA}+\overrightarrow{PB}=\overrightarrow{PC}$ (b) $\overrightarrow{PA}+\overrightarrow{PB}=2\overrightarrow{PC}$
(c) $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}=0$ (d) $\overrightarrow{PA}+\overrightarrow{PB}+2\overrightarrow{PC}=0$

23. The position vector of the points which divides internally in the ratio 2 : 3 the join of the points $2\vec{a}-3\vec{b}$ and $3\vec{a}-2\vec{b}$, is

(a) $\frac{12}{5}\vec{a}+\frac{13}{5}\vec{b}$ (b) $\frac{12}{5}\vec{a}-\frac{13}{5}\vec{b}$
(c) $\frac{3}{5}\vec{a}-\frac{2}{5}\vec{b}$ (d) none of these

24. If the position vectors of the points A, B, C be $\vec{a}, \vec{b}, 3\vec{a}-2\vec{b}$ respectively, then the points A, B, C are

(a) collinear (b) non-collinear
(c) form a right angled triangle
(d) none of these

25. If $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{i}+\hat{k}$, then a unit vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{a}$ is

(a) $\hat{i}$ (b) $\hat{j}$
(c) $\hat{k}$ (d) none of these

26. If the position vectors of the points A and B be $2\hat{i}+3\hat{j}-\hat{k}$ and $-2\hat{i}+3\hat{j}+4\hat{k}$ then the line AB is parallel to

(a) xy-plane (b) yz-plane
(c) zx-plane (d) none of these

27. If the vectors $3\hat{i}+2\hat{j}-\hat{k}$ and $6\hat{i}-4x\hat{j}+y\hat{k}$ are parallel, then the value of x and y will be

(a) – 1, – 2 (b) 1, – 2
(c) – 1, 2 (d) 1, 2

28. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. The value of x for which the vectors $\vec{c}=(x-2)\vec{a}+\vec{b}$ and $\vec{d}=(2x+1)\vec{a}-\vec{b}$ are collinear, is

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{5}$

29. If a = (1, –1) and b = (–2, m) are two collinear vectors, then m =

(a) 4 (b) 3 (c) 2 (d) 0

30. If $\vec{p}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{q}=3\hat{i}+\hat{j}+2\hat{k}$, then a vector along $\vec{r}$ which is linear combination of $\vec{p}$ and $\vec{q}$ and also perpendicular to $\vec{q}$ is

(a) $\hat{i}+5\hat{j}+4\hat{k}$ (b) $\hat{i}-5\hat{j}+4\hat{k}$
(c) $-\frac{1}{2}(\hat{i}+5\hat{j}-4\hat{k})$ (d) none of these

31. If $|\vec{a}|=3, |\vec{b}|=1, |\vec{c}|=4$ and $\vec{a}+\vec{b}+c=0$, then $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=$

(a) – 13 (b) – 10 (c) 13 (d) 10

32. A, B, C, D are any four points, then $\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{BC}\cdot\overrightarrow{AD}+\overrightarrow{CA}\cdot\overrightarrow{BD}=$

(a) $2\overrightarrow{AB}\cdot\overrightarrow{BC}\cdot\overrightarrow{CD}$ (b) $\overrightarrow{AB}\cdot\overrightarrow{BC}+\overrightarrow{CD}$
(c) $5\sqrt{3}$ (d) 0

33. $(\vec{a}\cdot\vec{b})\vec{c}$ and $(\vec{a}\cdot\vec{c})\vec{b}$ are

(a) two like vectors (b) two equal vectors
(c) two vectors in direction of $\vec{a}$
(d) none of these

34. If $\vec{a}$ and $\vec{b}$ are adjacent sides of a rhombus, then

(a) $\vec{a}\cdot\vec{b}=0$ (b) $\vec{a}\times\vec{b}=0$
(c) $\vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b}$ (d) none of these

35. If $\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})$, then $\vec{a}=$

(a) $\hat{i}$ (b) $\hat{k}$ (c) $\hat{j}$ (d) $\hat{i}+\hat{j}$

36. If a unit vector lies in yz–plane and makes angles of 30° and 60° with the positive y-axis and z-axis respectively, then its components along the co-ordinate axes will be

(a) $\frac{\sqrt{3}}{2}, \frac{1}{2}, 0$ (b) $0, \frac{\sqrt{3}}{2}, \frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}, 0, \frac{1}{2}$ (d) $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$

37. If $|\vec{a}|=3, |\vec{b}|=4$ and the angle between $\vec{a}$ and $\vec{b}$ be $|4\vec{a}+3\vec{b}|=$ then

(a) 25 (b) 12 (c) 13 (d) 7

38. A unit vector which is coplanar to vector $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $i+j+k$ is

(a) $\frac{\hat{i}-\hat{j}}{\sqrt{2}}$ (b) $\pm\left(\frac{\hat{j}-\hat{k}}{\sqrt{2}}\right)$

(c) $\frac{\hat{k}-\hat{i}}{\sqrt{2}}$ (d) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

39. $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors with magnitude $|\vec{a}|=4, |\vec{b}|=4, |\vec{c}|=2$ and such that $\vec{a}$ is perpendicular to $(\vec{b}+\vec{c}), \vec{b}$ is perpendicular to $(\vec{c}+\vec{a})$ and $\vec{c}$ is perpendicular to $(\vec{a}+\vec{b})$. It follows that $|\vec{a}+\vec{b}+\vec{c}|$ is equal to

(a) 9 (b) 6 (c) 5 (d) 4

40. If the position vectors of the points A, B, C, D be $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$, then the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

41. The angle between the vectors $\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ is

(a) $\cos^{-1}\left(\frac{1}{\sqrt{15}}\right)$ (b) $\cos^{-1}\left(\frac{4}{\sqrt{15}}\right)$

(c) $\cos^{-1}\left(\frac{4}{15}\right)$ (d) $\frac{\pi}{2}$

42. If $A(-1, 2, 3)$, $B(1, 1, 1)$ and $C(2, -1, 3)$ are points on a plane. A unit vector normal to the plane ABC is

(a) $\pm\left(\frac{2\hat{i}+2\hat{j}+\hat{k}}{3}\right)$ (b) $\pm\left(\frac{2\hat{i}-2\hat{j}+\hat{k}}{3}\right)$

(c) $\pm\left(\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}\right)$ (d) $\pm\left(\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}\right)$

43. For any two vectors $\vec{a}$ and $\vec{b}$, $(\vec{a}\times\vec{b})^2$ is equal to

(a) $\vec{a}^2-\vec{b}^2$ (b) $\vec{a}^2+\vec{b}^2$

(c) $\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2$ (d) none of these

44. The sine of the angle between the two vectors $3\hat{i}+2\hat{j}-\hat{k}$ and $12\hat{i}+5\hat{j}-5\hat{k}$ will be

(a) $\frac{\sqrt{115}}{\sqrt{14}\sqrt{194}}$ (b) $\frac{51}{\sqrt{14}\sqrt{144}}$

(c) $\frac{\sqrt{64}}{\sqrt{14}\sqrt{194}}$ (d) none of these

Assertion & Reason Type

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

45. Statement-1 : $L_1 : \frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2}$, $L_2 : \frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}$.

The unit vector perpendicular to both L_1 and L_2 is $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$.

Statement-2 : The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is $23/5\sqrt{3}$.

46. Statement-1 : The lines $\frac{x-1}{1}=\frac{y}{-1}=\frac{z+1}{1}$ and $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ are coplanar and equation of the plane containing them is $5x + 2y - 3z - 8 = 0$.

Statement-2 : The line $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ is perpendicular to the plane $3x + 6y + 9z - 8 = 0$ and parallel to the plane $x + y - z = 0$.

47. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Statement-1 : The parametric equation of the line of intersection of the given planes is $x = 3 + 14t$, $y = 1 + 2t$, $z = 15t$, t being the parameter.

Statement-2 : The vector $14\hat{i}+2\hat{j}+15\hat{k}$ is parallel to line of intersection of the given planes.

48. Consider the lines $L_1 : \frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $L_2 : \frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$.

Statement-1 : L_1 and L_2 are non-coplanar

Statement-2 : L_1 and L_2 intersect.

49. **Statement-1 :** If $\vec{u}$ and $\vec{v}$ are unit vectors inclined at an angle α and $\vec{x}$ is a unit vector bisecting the angle between them, then $\vec{x} = \frac{\vec{u}+\vec{v}}{2\cos\frac{\alpha}{2}}$.

Statement-2 : If ΔABC is an isosceles triangle with $AB = AC = 1$, then vector representing bisector of angle A is given by $\overrightarrow{AD} = \frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}$.

50. **Statement-1 :** For the real numbers α, β, γ; $(\cos\alpha + \cos\beta + \cos\gamma)^2 \le 3(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$.

Statement-2 : For two non–zero vectors $\vec{A}$ and $\vec{B}$, $(\vec{A}\cdot\vec{B})^2 \le |\vec{A}|^2 |\vec{B}|^2$.

Comprehension Type

Paragraph for Q. No. 51 - 52

L is the line of intersection of two non-parallel planes π_1, π_2. L_1 is a straight line which is perpendicular to L and points on L_1 are equidistant from the planes π_1, π_2. Equation of π_1 is $2x + 3y + z = 1$ and equations of L_1 are $6x = 3y = 2z$.

51. The direction ratios of L are

(a) (6, –3, 0) (b) (7, –5, 1)
(c) (5, –1, –1) (d) (11, –1, –3)

52. The direction ratios of normal to the plane containing L, L_1 are

(a) (12, –3, 4) (b) (17, 20, –19)
(c) (13, –2, –3) (d) (14, –5, 2)

Paragraph for Q. No. 53 to 55

Intersection of a sphere by a plane is called circular section.

(i) If the plane intersects the sphere in more than one different points, then the section is called a circle.

(ii) If the circle of section is of greatest, possible radius, then the circle is called great circle.

(iii) If the radius of circular section is zero, then the section is a point circle.

(iv) If the plane does not meet the sphere at all, then the section is an imaginary circle.

53. Sphere $x^2 + y^2 + z^2 = 4$ is intersected by the plane $2x + 3y + 6z + 7 = 0$ in

(a) a great circle
(b) a real circle but not great
(c) a point circle (d) an imaginary circle

54. Sphere $x^2 + y^2 + z^2 - 2x + 4y + 6z - 17 = 0$ is intersected by the plane $3x - 4y + 2z - 5 = 0$ in

(a) a great circle (b) a real circle but not great
(c) a point circle (d) an imaginary circle

55. The sphere $x^2 + y^2 + z^2 + 2x + 6y - 8z - 1 = 0$ is intersected by the plane $x + 2y - 3z - 7 = 0$ in

(a) a great circle (b) a real circle but not great
(c) a point circle (d) an imaginary circle

Paragraph for Q. No. 56-57

Let $\vec{r}$ be the variable point satisfying $\vec{r}\cdot\vec{n}_1 = d_1$, $\vec{r}\cdot\vec{n}_2 = d_2$, $\vec{r}\cdot\vec{n}_3 = d_3$, where $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ are non– coplanar vectors. Then

56. The position vector of the point of intersection of three planes, is

(a) $\frac{1}{[\vec{n}_1\vec{n}_2\vec{n}_3]}[d_3(\vec{n}_1\times\vec{n}_2)+d_1(\vec{n}_2\times\vec{n}_3)+d_2(\vec{n}_3\times\vec{n}_1)]$

(b) $\frac{4}{[\vec{n}_1\vec{n}_2\vec{n}_3]}[d_3(\vec{n}_1\times\vec{n}_2)+d_1(\vec{n}_2\times\vec{n}_3)+d_2(\vec{n}_1\times\vec{n}_3)]$

(c) $\frac{-4}{[\vec{n}_1\vec{n}_2\vec{n}_3]}[d_3(\vec{n}_1\times\vec{n}_2)+d_1(\vec{n}_2\times\vec{n}_3)+d_2(\vec{n}_1\times\vec{n}_3)]$

(d) none of these

57. If the planes $\vec{r}\cdot\vec{n}_1 = d_1$, $\vec{r}\cdot\vec{n}_2 = d_2$ and $\vec{r}\cdot\vec{n}_3 = d_3$, have a common line of intersection, then $d_1(\vec{n}_2\times\vec{n}_3)+d_2(\vec{n}_3\times\vec{n}_1)+d_3(\vec{n}_1\times\vec{n}_2)$ is

(a) $[\vec{n}_1\vec{n}_2\vec{n}_3]$ (b) $4[\vec{n}_1\vec{n}_2\vec{n}_3]$
(c) $2[\vec{n}_1\vec{n}_2\vec{n}_3]$ (d) none of these

Paragraph for Q. No. 58-59

The vertices of a ΔABC are $A(1, 0, 2)$, $B(-2, 1, 3)$ and $C(2, -1, 1)$. If D is the foot of the perpendicular drawn from A on BC, then

58. The equation of median of ΔABD passing through the vertex A, is

(a) $\vec{r} = (\hat{i}+2\hat{k})+\frac{\lambda}{3}(-5\hat{i}+\hat{j}+\hat{k})$

(b) $\vec{r} = (\hat{i}-2\hat{k})+\frac{\lambda}{3}(-5\hat{i}+\hat{j}+\hat{k})$

(c) $\vec{r} = (\hat{i}+2\hat{k})+\frac{\lambda}{3}(5\hat{i}-\hat{j}+\hat{k})$

(d) none of these

59. The vector equation of the bisector of $\angle A$, is given by

(a) $\vec{r} = (\hat{i}+2\hat{j})+\lambda\left(\frac{-3\hat{i}+\hat{j}+\hat{k}}{\sqrt{11}}+\frac{-\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}\right)$

(b) $\vec{r} = (\hat{i}+2\hat{k})+\lambda\left(\frac{-3\hat{i}+\hat{j}+\hat{k}}{\sqrt{11}}+\frac{-\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}\right)$

(c) $\vec{r} = (\hat{i} + 2\hat{j}) + \lambda\left(\frac{3\hat{i} + \hat{j} + \hat{k}}{\sqrt{11}} + \frac{-\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}\right)$

(d) none of these

Matrix–Match Type

60. Match the following.

	Column-I		Column-II
(A)	$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-1}{3} = \frac{y-3}{4} = \frac{z-5}{5}$ are	(p)	Coincident
(B)	$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$ are	(q)	Parallel and different
(C)	$\frac{x-2}{5} = \frac{y+3}{4} = \frac{5-z}{2}$ and $\frac{x-7}{5} = \frac{y-1}{4} = \frac{z-2}{-2}$ are	(r)	Skew
(D)	$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-4}{5}$ and $\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-7}{5}$ are	(s)	Intersecting in a point
		(t)	coplanar

61. A variable plane cuts the x-axis, y-axis and z-axis at the points A, B and C respectively such that the volume of the tetrahedron $OABC$ remains constant equal to 32 cubic unit and O is the origin of the coordinate system. Then

	Column-I		Column-II
(A)	The locus of the centroid of the tetrahedron is	(p)	$xyz = 24$
(B)	The locus of the point equidistant from O, A, B and C is	(q)	$(x^2 + y^2 + z^2)^3 = 192xyz$
(C)	The length of the perpendicular from origin to the plane is	(r)	$xyz = 3$
(D)	If PA, PB and PC are mutually perpendicular then the locus of P is	(s)	$(x^2 + y^2 + z^2)^3 = 1536xyz$

62. Match the following.

	Column-I		Column-II
(A)	$\vec{a}$ and $\vec{b}$ are unit vectors and $\vec{a} + 2\vec{b}$ is $\perp$ to $5\vec{a} - 4\vec{b}$, then $2(\vec{a}\cdot\vec{b})$ is equal to	(p)	0
(B)	The points (1, 0, 3), (–1, 3, 4) and (k, 2, 5) are coplanar when k is equal to	(q)	–1
(C)	The vectors (1, 1, m), (1, 1, m + 1) and (1, –1, m) are coplanar then the number of values of m is	(r)	1
(D)	$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$ is equal to	(s)	2

Integer Answer Type

63. The plane $2x + 2y + z = 3$ is rotated about the line where it cuts the xy plane by an acute angle α. If the new position of plane contains the point (3, 1, 1) then $9\cos\alpha$ equal to

64. The shortest distance between the z-axis and the line, $x + y + 2z - 3 = 0$, $2x + 3y + 4z - 4 = 0$ is

65. ABC is any triangle and O is any point in the plane of the same. If AO, BO and CO meet the sides BC, CA and AB in D, E, F respectively, then $\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} =$

66. Find the distance of the point $\hat{i} + 2\hat{j} + 3\hat{k}$ from the plane $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 5$ measured parallel to the vector $2\hat{i} + 3\hat{j} - 6\hat{k}$.

67. Non-zero vectors satisfy $\vec{a}\cdot\vec{b} = 0, (\vec{b} - \vec{a})\cdot(\vec{b} + \vec{c}) = 0$ and $2|\vec{b} + \vec{c}| = |\vec{b} - \vec{a}|$. If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the value of μ is

SOLUTIONS

1. **(b) :** D.c.'s are $\frac{2}{\sqrt{2^2 + (-3)^2 + 6^2}}$, $\frac{-3}{\sqrt{2^2 + (-3)^2 + (6)^2}}$ and $\frac{6}{\sqrt{2^2 + (-3)^2 + (6)^2}}$ *i.e.*, $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$

2. **(b) :** $a_1a_2 + b_1b_2 + c_1c_2 = 0$, so, $OP \perp OQ$

3. **(b) :** Required ratio $= -\left(\frac{5}{-2}\right) = \frac{5}{2}$ *i.e.*, $5:2$.

4. **(a) :** Here, $x_2 - x_1 = 6$, $y_2 - y_1 = 6$, $z_2 - z_1 = 4$ and d.c.'s of x, y, z-axes are (1,0,0), (0,1,0), (0, 0, 1) respectively.

Now projection = $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$

$\therefore$ Projections of line AB on co-ordinate axes are 6, 6, 4 respectively.

5. (b) : Here, $\cos^2\alpha + \cos^2(90° - \alpha) + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma + 1 = 1 \Rightarrow \gamma = 90°$

6. (d) : Required direction cosines are

$$\frac{3}{\sqrt{3^2+12^2+4^2}}, \frac{12}{\sqrt{3^2+12^2+4^2}}, \frac{4}{\sqrt{3^2+12^2+4^2}}$$

i.e., $\frac{3}{13}, \frac{12}{13}, \frac{4}{13}$.

7. (d) : Given $\alpha = 45°$, $\beta = 60°$

$\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow \gamma = 60° \text{ or } 120°.$

8. (b) : Here, $x_2 - x_1 = 1, y_2 - y_1 = 2, z_2 - z_1 = -2$

$\therefore$ l, m, n for PQ are $\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}$.

Now, for RS, $x_2 - x_1 = 2$, $y_2 - y_1 = -2$, $z_2 - z_1 = 1$

$\therefore$ Projection of RS on PQ = $\sum l(x_2 - x_1)$

$$= \frac{2}{3} - \frac{4}{3} - \frac{2}{3} = \frac{-4}{3}.$$

9. (a) : Let d be the length of line, then projection on x-axis = dl = 2, projection on y-axis = dm = 3, Projection on z-axis = dn = 6

Now $d^2(l^2 + m^2 + n^2) = 4 + 9 + 36$

$\Rightarrow d^2(1) = 49 \Rightarrow d = 7$

10. (a) : $\cos^2\alpha = 1 - \cos^2 60^o - \cos^2 60^o$

$$= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} \Rightarrow \alpha = 45^o.$$

11. (c) : Distance from y-axis is $\sqrt{x^2+z^2}$

$= \sqrt{4^2+5^2} = \sqrt{16+25} = \sqrt{41}.$

12. (c) : Angle between two lines,

$$\cos\theta = \frac{1\times3 + 2\times(-2) + 3\times1}{\sqrt{1^2+2^2+3^2}\sqrt{3^2+(-2)^2+1^2}} = \frac{2}{\sqrt{14}\sqrt{14}}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{7}\right).$

13. (d) : (3, 1, 0) · (0, 0, 1) = (0, 0, 0). So the line is perpendicular to z-axis.

14. (b) : $R = \sqrt{4+100+121} = 15.$

15. (b) : $\overrightarrow{AB} = 4\hat{j} - 5\hat{j} + 11\hat{k}$

Direction cosine along y-axis $= \frac{-5}{\sqrt{16+25+121}} = \frac{-5}{\sqrt{162}}.$

16. (b) : Let P, Q and R be points having position vectors $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, $\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$ and $\gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$ respectively.

Then, $|\overrightarrow{PQ}| = |\overrightarrow{QR}| = |\overrightarrow{RP}| = \sqrt{(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2}$

Hence ΔPQR is an equilateral triangle.

17. (a) : Let $\vec{b}$ should be added, then $\vec{a} + \vec{b} = \hat{i}$

$\Rightarrow \vec{b} = \hat{i} - \vec{a} = \hat{i} - (3\hat{i} + 4\hat{j} - 2\hat{k}) = -2\hat{i} - 4\hat{j} + 2\hat{k}.$

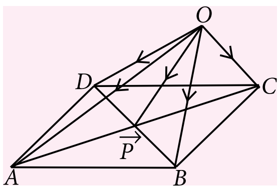
18. (a)

19. (c) : $\vec{a} + \vec{b} + \vec{c} = (3+2-1)\hat{i} + (-2-4+2)\hat{j} + (1-3+2)\hat{k}$

$$= 4\hat{i} - 4\hat{j}$$

20. (b) : $\frac{\vec{a}+\vec{b}}{2} = 2\hat{i} - \hat{j} + \frac{5}{2}\hat{k}.$

21. (d) : We know that P will be the mid point of AC and BD.



$\therefore \overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OP}$ ...(i)

and $\overrightarrow{OB} + \overrightarrow{OD} = 2\overrightarrow{OP}$ ...(ii)

Adding (i) and (ii), we get

$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}.$

22. (b)

23. (b) : Position vectors of the points which divides internally is, $\frac{3(2\vec{a}-3\vec{b})+2(3\vec{a}-2\vec{b})}{5} = \frac{12\vec{a}-13\vec{b}}{5}.$

24. (a) : Here $\overrightarrow{AB} = \vec{b} - \vec{a}$

and $\overrightarrow{AC} = (3\vec{a} - 2\vec{b}) - (\vec{a}) = -2(\vec{b} - \vec{a})$

$\therefore \overrightarrow{AB} = m\overrightarrow{AC}.$

Hence A, B, C are collinear.

25. (d) : Let $\vec{c} = \lambda\vec{a} + \mu\vec{b} = (\lambda+\mu)\hat{i} - \lambda\hat{j} + \mu\hat{k}$

Now, $\vec{c}\cdot\vec{a} = 0 \Rightarrow 2\lambda + \mu = 0 \Rightarrow \mu = -2\lambda$

Therefore, $\vec{c} = -\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k} = (\sqrt{6})(-\lambda)\left[\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}\right]$

Hence, unit vector $= \frac{(\hat{i}+\hat{j}+2\hat{k})}{\sqrt{6}}$

26. (c) : $\overrightarrow{AB} = -4\hat{i} + 5\hat{k}$, which is a vector lying in a plane parallel to zx-plane.

27. (a) : Obviously, $\frac{3}{6} = \frac{2}{-4x} = -\frac{1}{y} \Rightarrow x = -1$ and $y = -2$

28. (c)

29. (c) : Condition for collinearity, $\vec{b} = \lambda\vec{a}$

$\Rightarrow \quad (-2\hat{i} + m\hat{j}) = \lambda(\hat{i} - \hat{j})$

Comparing coefficient, we get
$\lambda = -2$ and $-\lambda = m$. So, $m = 2$

30. (c) : $\vec{r} = \vec{p} + \lambda\vec{q} \Rightarrow \vec{r}.\vec{q} = \vec{p}.\vec{q} + \lambda\vec{q}.\vec{q}$

$\Rightarrow \quad 0 = 7 + 14\lambda \quad \Rightarrow \lambda = -\frac{1}{2}$

Therefore, $\vec{r} = -\frac{1}{2}(\hat{i} + 5\hat{j} - 4\hat{k})$.

31. (a) : $(\vec{a} + \vec{b} + \vec{c})^2 = 0$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a} = 0$

$\Rightarrow \quad 9 + 1 + 16 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{26}{2} = -13.$

32. (d)

33. (d) : They are different vectors.

34. (c) : $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos\theta = |\vec{a}|^2, \quad (\because \theta = 0^o)$
and $\vec{b}\cdot\vec{b} = |\vec{b}||\vec{b}|\cos\theta = |\vec{b}|^2, \quad$ (Here $\theta = 0°$)

Also, since $\vec{a}$ and $\vec{b}$ are sides of rhombus
$\therefore \quad |\vec{a}| = |\vec{b}|$. Hence $\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b}$.

35. (a) : Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$.
Then $\vec{a}\cdot\hat{i} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{i} = x$
and $\vec{a}\cdot(\hat{i} + \hat{j}) = x + y$ and $\vec{a}\cdot(\hat{i} + \hat{j} + \hat{k}) = x + y + z$
$\because$ Given that $x = x + y = x + y + z$
Now $x = x + y \Rightarrow y = 0$
and $x + y = x + y + z \Rightarrow z = 0$
Hence $x = 1 \Rightarrow \vec{a} = \hat{i}$.

36. (b) **37. (b)** **38. (b)**

39. (b) : Here $|\vec{a}| = 4, \ |\vec{b}| = 4, |\vec{c}| = 2$

and $\vec{a}\cdot(\vec{b} + \vec{c}) = 0 \Rightarrow \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0$ ...(i)

$\vec{b}\cdot(\vec{c} + \vec{a}) = 0 \Rightarrow \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} = 0$ ...(ii)

$\vec{c}\cdot(\vec{a} + \vec{b}) = 0 \Rightarrow \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0$ ...(iii)

Adding (i), (ii) and (iii), we get, $2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\therefore \quad |\vec{a} + \vec{b} + \vec{c}|$

$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})}$

$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2} = \sqrt{16 + 16 + 4} = \sqrt{36}$

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = 6.$

40. (d) : $\overrightarrow{AB} = \hat{i} + 4\hat{j} - \hat{k}, \overrightarrow{CD} = -2\hat{i} - 8\hat{j} + 2\hat{k}$

$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|} = \frac{-2 - 32 - 2}{\sqrt{18}\cdot\sqrt{72}} \Rightarrow \theta = \pi.$

41. (d) : $(\hat{i} - \hat{j} + \hat{k})\cdot(\hat{i} + 2\hat{j} + \hat{k}) = \sqrt{3}\sqrt{6}\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{0}{\sqrt{3}\sqrt{6}} \Rightarrow \theta = \frac{\pi}{2}.$

42. (a) : $\overrightarrow{AB} = 2\hat{i} - \hat{j} - 2\hat{k}, \ \overrightarrow{AC} = 3\hat{i} - 3\hat{j} + 0\hat{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 3 & -3 & 0 \end{vmatrix} = (-6\hat{i} - 6\hat{j} - 3\hat{k})$$

Hence unit vector $= \pm\left(\frac{2\hat{i} + 2\hat{j} + \hat{k}}{3}\right)$.

43. (c) : $(\vec{a} \times \vec{b})^2 = (\vec{a} \times \vec{b})\cdot(\vec{a} \times \vec{b})$
$= (ab\sin\theta\hat{n})(ab\sin\theta\hat{n}) = a^2b^2\sin^2\theta = a^2b^2(1 - \cos^2\theta)$
$= a^2b^2 - a^2b^2\cos^2\theta = a^2b^2 - (\vec{a}\cdot\vec{b})^2$.

44. (a) **45. (c)**

46. (b) : $\begin{vmatrix} 1-2 & 0+1 & -1-0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$

$\Rightarrow$ Given lines are coplanar

Equation of the plane is $\begin{vmatrix} x-1 & y & z+1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$

i.e., $5x + 2y - 3z - 8 = 0$

Since $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} \Rightarrow \frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{3}$ is perpendicular to the plane $3x + 6y + 9z - 8 = 0$

And also $1(1) + 2(1) + 3(-1) = 0$

$\Rightarrow \quad \frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{3}$ is parallel to $x + y - z = 0$

47. (d)

48. (d) : Any point on the line L_1 is
$(2r_1 + 3, -3r_1 - 1, r_1 - 2)$
Any point on the line L_2 is $(-3r_2 + 7, r_2, 2r_2 - 7)$
Let the lines L_1 and L_2 intersect at P.

$\therefore \quad 2r_1 + 3 = -3r_2 + 7 \Rightarrow 2r_1 + 3r_2 = 4$ ...(i)
Also $-3r_1 - 1 = r_2 \Rightarrow -3r_1 - r_2 = 1$ ...(ii)
and $r_1 - 2 = 2r_2 - 7 \Rightarrow r_1 - 2r_2 = -5$ ...(iii)
Solving (i) & (iii), we get $r_1 = -1$, $r_2 = 2$
Clearly $r_1 = -1$ and $r_2 = 2$ satisfy equation (ii)
$\therefore$ Lines L_1 and L_2 intersect $\Rightarrow L_1$ and L_2 are coplanar.

49. (a)

50. (a) : Consider $\vec{A} = \hat{i} + \hat{j} + \hat{k}$,
$\vec{B} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$

$(\vec{A}\cdot\vec{B})^2 \le |\vec{A}|^2 |\vec{B}|^2$

$\Rightarrow \quad (\cos\alpha + \cos\beta + \cos\gamma)^2 \le (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)\,(3).$

51. (b) **52. (b)**

Vector parallel to L is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 7i - 5\hat{j} + \hat{k}$

Equation of plane containing L, L_1 is $\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 7 & -5 & 1 \end{vmatrix} = 0$

i.e., $17x + 20y - 19z = 0$

53. (b) : Distance of the centre (0, 0, 0) from the plane is $\dfrac{7}{\sqrt{4+9+36}} = 1 < 2$

$\therefore$ The section is a real circle but not great.

54. (a) : Distance of the centre (1, –2, –3) from the plane is $3 + 8 - 6 - 5 = 0$
$\therefore$ The section is a great circle.

55. (d) : Distance of the centre (–1, –3, 4) from the plane is $\left|\dfrac{-1-6-12-7}{\sqrt{1+4+9}}\right| = \dfrac{26}{\sqrt{14}} > 5$ (radius)

$\therefore$ The section is an imaginary circle.

56. (a) **57. (d)**

$\vec{n}_1, \vec{n}_2, \vec{n}_3$ are non-coplanar vectors. Therefore vectors $\vec{n}_1 \times \vec{n}_2, \vec{n}_2 \times \vec{n}_3$ and $\vec{n}_3 \times \vec{n}_1$ are also non-coplanar.
Let $\vec{\alpha}$ be the position vector of the point of intersection of the given planes. Then,
$\vec{\alpha}\cdot\vec{n}_1 = d_1, \vec{\alpha}\cdot\vec{n}_2 = d_2$ and $\vec{\alpha}\cdot\vec{n}_3 = d_3$
We know that any vector in space can be written as the linear combination of three non-coplaner vectors. So, let

$\Rightarrow \quad \vec{\alpha} = x(\vec{n}_1 \times \vec{n}_2) + y(\vec{n}_2 \times \vec{n}_3) + z(\vec{n}_3 \times \vec{n}_1)$ ...(i)

Now, $\vec{\alpha}\cdot\vec{n}_1 = d_1$

$\Rightarrow \quad \{x(\vec{n}_1 \times \vec{n}_2) + y(\vec{n}_2 \times \vec{n}_3) + z(\vec{n}_3 \times \vec{n}_1)\}\cdot\vec{n}_1 = d_1$

$\Rightarrow \quad y\{(\vec{n}_2 \times \vec{n}_3)\cdot\vec{n}_1\} = d_1 \Rightarrow y = \dfrac{d_1}{[n_1\ n_2\ n_3]}$

Similarly, we have $\vec{\alpha}\cdot\vec{n}_2 = d_2$ and $\vec{\alpha}\cdot\vec{n}_3 = d_3$

$\Rightarrow \quad z = \dfrac{d_2}{[\vec{n}_1\vec{n}_2\vec{n}_3]}$ and $x = \dfrac{d_3}{[\vec{n}_1\vec{n}_2\vec{n}_3]}$

$\therefore$ Positive vector of the point of intersection of three planes, is $\dfrac{1}{[\vec{n}_1\vec{n}_2\vec{n}_3]}\{d_1(\vec{n}_2 \times \vec{n}_3) + d_2(\vec{n}_3 \times \vec{n}_1) + d_3(\vec{n}_1 \times \vec{n}_2)\}$

Also, the equation of a plane passing through the line of intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(\vec{n}_1 + \vec{n}_2\lambda) = d_1 + \lambda d_2$, where λ is parameter since, three planes have a common line of intersection.
$\therefore$ The above equation should be identical to $\vec{r}\cdot\vec{n}_3 = d_3$ for some value of λ.
Thus for some value of λ, we have

$\vec{n}_1 + \lambda\vec{n}_2 = \mu\vec{n}_3$ ...(ii) and $\vec{d}_1 + \lambda\vec{d}_2 = \mu d_3$ ...(iii)

Now, $\vec{n}_1 + \lambda\vec{n}_2 = \mu\vec{n}_3$

$\Rightarrow \quad (\vec{n}_1 + \lambda\vec{n}_2) \times \vec{n}_3 = \mu(\vec{n}_3 \times \vec{n}_3)$

$\Rightarrow \quad \vec{n}_1 \times \vec{n}_3 + \lambda(\vec{n}_2 \times \vec{n}_3) = 0$ ...(iv)

$\Rightarrow \quad \lambda = -\dfrac{(\vec{n}_1 \times \vec{n}_3)}{(\vec{n}_2 \times \vec{n}_3)}$

Again, $\vec{n}_1 + \lambda\vec{n}_2 = \mu\vec{n}_3$

$\Rightarrow \quad (\vec{n}_1 + \lambda\vec{n}_2) \times \vec{n}_2 = \mu(\vec{n}_3 \times \vec{n}_2)$

$\Rightarrow \quad \mu(\vec{n}_2 \times \vec{n}_3) = -(\vec{n}_1 \times \vec{n}_2)$ ...(v)

Now, $d_1 + \lambda d_2 = \mu d_3$

$\Rightarrow \quad (d_1 + \lambda d_2)(\vec{n}_2 \times \vec{n}_3) = d_3\{\mu(\vec{n}_2 \times \vec{n}_3)\}$

$\Rightarrow \quad d_1(\vec{n}_2 \times \vec{n}_3) + d_2(\vec{n}_3 \times \vec{n}_1) + d_3(\vec{n}_1 \times \vec{n}_2) = 0$
{using (iv) and (v)}

58. (a) **59. (b)**

60. (A) $\to$ (s, t), (B) $\to$ (p, t), (C) $\to$ (q), (D) $\to$ (r)

(A) Both the lines pass through the point (7, 11, 15)

(B) < 2, 3, 4 > are direction ratios of both the lines. Also the point (1, 2, 3) is common to both lines.
$\therefore$ The lines are coincident.

(C) < 5, 4 – 2 > are direction ratios of both the lines.
$\therefore$ The lines are parallel.
Also $x = 2 + 5\lambda$, $y = -3 + 4\lambda$, $z = 5 - 2\lambda$

$\therefore \quad \dfrac{2+5\lambda-7}{5} = \dfrac{-3+4\lambda-1}{4} = \dfrac{5-2\lambda-2}{-2}$

i.e., $\lambda - 1 = \lambda - 1 = \dfrac{3-2\lambda}{-2}$

$\therefore$ No value of λ exists.
Thus the lines are parallel and different.

(D) < 2, 3, 5> and < 3, 2, 5 > are direction ratios of $\frac{x-3}{2}=\frac{y+2}{3}=\frac{z-4}{5}$ and $\frac{x-3}{3}=\frac{y-2}{2}=\frac{z-7}{5}$ line respectively.

$\therefore$ The lines are not parallel

Now, $x = 3 + 2\lambda$, $y = -2 + 3\lambda$, $z = 4 + 5\lambda$

and $x = 3 + 3\mu$, $y = 2 + 2\mu$, $z = 7 + 5\mu$

are parametric equations of the lines.

Solving $3 + 2\lambda = 3 + 3\mu$ and $-2 + 3\lambda = 2 + 2\mu$,

we get $\lambda=\frac{12}{5}, \mu=\frac{8}{5}$

Now substituting these values in $4 + 5\lambda = 7 + 5\mu$ we get $4 + 12 = 7 + 8$ *i.e.*, $16 = 15$ which is not true.

$\therefore$ The lines do not intersect.

Hence the lines are skew.

61. (A) → (r), (B) → (p), (C) → (q), (D) → (s)

62. (A) → (r), (B) → (q), (C) → (p), (D) → (p)

(A) $(\vec{a}+2\vec{b})\cdot(5\vec{a}-4\vec{b})=0$

$\Rightarrow \quad \vec{a}\cdot\vec{b}=\frac{1}{2}$ or $2(\vec{a}\cdot\vec{b})=1$

(B) $\vec{a}=\hat{i}+3\hat{k}, \vec{b}=-\hat{i}+3\hat{j}+4\hat{k}$,

$\vec{c}=\hat{i}+2\hat{j}+\hat{k}, \vec{d}=k\hat{i}+2\hat{j}+5\hat{k}$ are coplanar.

$\therefore \quad [\vec{d}\ \vec{b}\ \vec{c}]+[\vec{d}\ \vec{c}\ \vec{a}]+[\vec{d}\ \vec{a}\ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow \quad k=-1$

(C) For no values of m the vectors are coplanar.

(D) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$\vec{b}\times(\vec{c}\times\vec{a})=(\vec{b}\cdot\vec{a})\vec{c}-(\vec{c}\cdot\vec{b})\vec{a}$

$\vec{c}\times(\vec{a}\times\vec{b})=(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$

$\therefore \quad \vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

63. (7) : Let equation of new plane $2x + 2y + z - 3 + \lambda z = 0$

Point (3, 1, 1) lie on it $\Rightarrow \lambda = -2$

Hence equation of new plane is $2x - 2y - z = 3$

$\cos\alpha=\frac{4+4-1}{3\cdot 3}=\frac{7}{9}$

64. (2) : The equation of any plane containing the given line is $(x + y + 2z - 3) + \lambda(2x + 3y + 4z - 4) = 0$

$\Rightarrow \quad (1 + 2\lambda)x + (1 + 3\lambda)y + (2 + 4\lambda)z - (3 + 4\lambda) = 0$ ...(i)

If the plane is parallel to z-axis whose direction cosines are 0, 0, 1 ; then the normal to the plane will be perpendicular to z-axis.

$\therefore \quad (1+2\lambda)(0)+(1+3\lambda)(0)+(2+4\lambda)(1)=0$

$\Rightarrow \quad \lambda=-\frac{1}{2}$

Putting value of λ in (i), we get required plane

$(x+y+2z-3)-\frac{1}{2}(2x+3y+4z-4)=0$

$\Rightarrow \quad y+2=0$ ...(ii)

$\therefore$ S.D. = distance of any point say (0, 0, 0) on z-axis from plane (ii) $=\frac{2}{\sqrt{(1)^2}}=2$

65. (1)

66. (7) : The distance of the point 'a' from the plane $\vec{r}\cdot\vec{n}=q$ measured in the direction of the unit vector $\hat{b}$ is $=\frac{q-\vec{a}\cdot\vec{n}}{\hat{b}\cdot\vec{n}}$

Here $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}, \vec{n}=\hat{i}+\hat{j}+\hat{k}$ and $q = 5$

Also, $\hat{b}=\frac{2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{(2)^2+(3)^2+(-6)^2}}=\frac{2\hat{i}+3\hat{j}-6\hat{k}}{7}$

$\therefore$ The required distance $=\frac{5-(\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}{\frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}$

$=\frac{5-(1+2+3)}{\frac{1}{7}(2+3-6)}=7$

67. (0) : $\vec{c}=\frac{\vec{a}-\mu\vec{b}}{4}$ and $\vec{a}\cdot\vec{b}=0$

Now, $(\vec{b}-\vec{a})\cdot(\vec{b}+\vec{c})=0$

$\Rightarrow \quad (\vec{b}-\vec{a})\cdot\left(\vec{b}+\frac{\vec{a}-\mu\vec{b}}{4}\right)=0$

$\Rightarrow \quad (4-\mu)b^2=a^2$ ...(i)

$\therefore \quad \mu<4$

Again $4|\vec{b}+\vec{c}|^2=|\vec{b}-\vec{a}|^2$

$\Rightarrow \quad 4\left|\frac{(4-\mu)\vec{b}+\vec{a}}{4}\right|^2=|\vec{b}-\vec{a}|^2$

$\Rightarrow \quad 4\left(\frac{4-\mu}{4}\right)^2 b^2+\frac{a^2}{4}=b^2+a^2$

$\Rightarrow \quad ((4-\mu)^2-4)b^2=3a^2$ ...(ii)

From (i) and (ii), we get $\frac{(4-\mu)^2-4}{4-\mu}=\frac{3}{1}\Rightarrow \mu^2-5\mu=0$

$\Rightarrow \quad \mu = 0$ or 5 but as $\mu < 4$, so, $\mu = 0$.

TARGET JEE

Integration

Basic formulas of indefinite integral

- $\int x^n dx = \frac{x^{n+1}}{n+1} + k, n \neq -1$
- $\int \frac{1}{x} dx = \log x + k$
- $\int a^x dx = \frac{a^x}{\log_e a} + k$
- $\int e^{ax} dx = \frac{e^{ax}}{a} + k$
- $\int \tan x dx = \log \sec x + k = -\log \cos x + k$
- $\int \cot x \, dx = \log \sin x + k = -\log(\text{cosec } x) + k$
- $\int \frac{\sin x}{\cos^2 x} dx = \sec x + k$
- $\int \sec x dx = \log(\sec x + \tan x) + k$
- $\int \frac{\cos x}{\sin^2 x} dx = -\text{cosec } x + k$
- $\int \frac{\sec^2 mx}{\tan mx} dx = \frac{\log(\tan mx)}{m} + k$
- $\int \frac{\text{cosec}^2 mx}{\cot mx} dx = \frac{-\log(\cot mx)}{m} + k$
- $\int \text{cosec } x \, dx = -\log(\text{cosec } x + \cot x) + k$
- $\int \sec^3 x dx = \frac{1}{2} \frac{d}{dx}(\sec x) + \frac{1}{2} \int \sec x dx$

Some standard results

- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + k$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + k$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + k$
- $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log(x + \sqrt{x^2 + a^2}) + k$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + k$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}) + k$
- $\int \sqrt{x^2 + a^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log(x + \sqrt{x^2 + a^2}) + k$
- $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log(x + \sqrt{x^2 - a^2}) + k$
- $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + k$

Results using methods of integration

- $\int e^{ax+b}[af(x) + f'(x)]dx = e^{ax+b} f(x) + k$
- $\int f^n(x) f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + k$
- $\int \frac{f'(x)}{f^n(x)} dx = \begin{cases} \log|f(x)| + k, \text{ if } n = 1 \\ \int t^{-n} dt, \text{ where } t = f(x) \end{cases}$
- $\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\log_e a} + k, \; a \neq 1, \; a > 0$
- $\int e^{f(x)} f'(x) dx = e^{f(x)} + k$

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Reducing algebraic functions into trigonometrical function by method of substitution.

- $\int\sqrt{\frac{a-x}{a+x}}dx, \int\sqrt{\frac{a+x}{a-x}}dx, \int\frac{a+x}{a-x}dx, \int\frac{a-x}{a+x}dx$

  Put $x = a\cos\theta$ or $x = a\sin\theta$ or $x = a\cos2\theta$ or $x = a\sin2\theta$

- $\int\sqrt{2ax-x^2}\,dx, \int\frac{dx}{2ax-x^2}, \int\frac{dx}{\sqrt{2ax-x^2}}$

  Put $x = 2a\sin^2\theta$ or $2a\sin^2 2\theta$

  or $x = 2a\cos^2\theta$ or $2a\cos^2 2\theta$

- $\int\sqrt{(x-\alpha)(x-\beta)}dx, \int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}, \int\frac{x-\alpha}{\beta-x}dx$

  Put $x = \alpha\sec^2\theta - \beta\tan^2\theta$ or $x = \alpha\,\text{cosec}^2\theta - \beta\cot^2\theta$

- $\int\sqrt{(x-\alpha)(\beta-x)}dx, \int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}, \int\sqrt{\frac{x-\alpha}{\beta-x}}dx$

  Put $x = \alpha\sin^2\theta + \beta\cos^2\theta$

  or $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Integration of special trigonometrical functions (expressions)

In such type of expressions first reduce them into the simplest form and then integrate.

- $I=\int\frac{d\theta}{\tan\theta+\cot\theta+\sec\theta+\text{cosec}\,\theta}$

  $=\int\frac{\cos\theta+\sin\theta-1}{2}d\theta=\frac{\sin\theta-\cos\theta-\theta}{2}+k$

- $I=\int\frac{d\theta}{1+\sin\theta\cos\theta}=\int\frac{\sec^2\theta}{\tan^2\theta+\tan\theta+1}d\theta$

  Put $\tan\theta = t$, we get

- $I=\int\frac{dt}{t^2+t+1}=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta+1}{\sqrt{3}}\right)+k$

Some tricky integration

1. $\int\frac{e^x\log x}{x^x}dx=\int\left(\frac{e}{x}\right)^x\log x\,dx=-\left(\frac{e}{x}\right)^x+k$
2. $I=\int\frac{b+a\cos x}{(a+b\cos x)^2}dx=\frac{\sin x}{a+b\cos x}+k$
3. $\int\frac{b+a\sin x}{(a+b\sin x)^2}dx=\frac{\cos x}{a+b\sin x}+k$
4. $\int(\tan x+\sec x)^4\sec^2 x\,dx$

   $=\frac{1}{2}\left(\frac{t^5}{5}+\frac{t^3}{3}\right)+k$, where $t = \sec x + \tan x$
5. $\int x^x(1+\log x)dx = x^x + k$
6. (i) $\int(\sin x)^{\cos x}\left[\frac{d}{dx}(\cos x\cdot\log\sin x)\right]dx$

   $=(\sin x)^{\cos x}+k$

   (ii) $\int(\cos x)^{\sin x}\left[\frac{d}{dx}(\sin x\cdot\log\cos x)\right]dx$

   $=(\cos x)^{\sin x}+k$
7. $\int\lambda^{\lambda^{\lambda^x}}\cdot\lambda^{\lambda^x}\cdot\lambda^x dx=\frac{\lambda^{\lambda^{\lambda^x}}}{(\log\lambda)^3}+k$

DEFINITE INTEGRATION

Definition : $\int_a^b f(x)dx$ is the integration of $f(x)$ w.r.t. x with $x = a$ as lower limit and $x = b$ as upper limit.

$\int_a^b f(x)dx=F(b)-F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

Properties of definite integral

1. $\int_a^b f(x)dx=\int_a^b f(t)dt$
2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$
3. (i) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$, where $a < c < b$

   (ii) $\int_a^b f(x)dx=\int_a^{a_1} f(x)dx+\int_{a_1}^{a_2} f(x)dx$

   $+\int_{a_2}^{a_3} f(x)dx+........+\int_{a_n}^{b} f(x)dx$

   where $a < a_1 < a_2 < .... < a_n < b$.
4. (i) $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

   (ii) $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$
5. (i) $\int_{-a}^{a} f(x)dx=\int_0^a[f(x)+f(-x)]dx$

(ii) $\displaystyle\int_{-a}^{a} f(x)dx = \begin{cases} 2\displaystyle\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} \end{cases}$

6. $\displaystyle\int_0^{2a} f(x)dx = \begin{cases} 2\displaystyle\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0, \text{ if } f(2a-x) = -f(x) \end{cases}$

7. $\displaystyle\int_0^{b-a} f(x+a)dx = \int_a^b f(x)dx$

8. $\displaystyle\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$

9. $\displaystyle\int_a^b f(x)dx = (b-a)\int_0^1 [f(a+(b-a)x]dx$

10. If $f(t)$ is an odd function, then $g(x) = \displaystyle\int_a^x f(t)dt$ is an even function.

11. If $f(t)$ is an even function, then $g(x) = \displaystyle\int_0^x f(t)dt$ is an odd function.

12. If $f(x)$ is a continuous on $]\alpha, \beta[$, then there exist a number 'γ' in $]\alpha, \beta[$such that $\displaystyle\int_\alpha^\beta f(x)dx = f(\gamma)(\beta-\alpha)$

where $f(\gamma) = \dfrac{1}{\beta-\alpha}\displaystyle\int_\alpha^\beta f(x)dx$ is called mean value of the function $f(x)$ on $[\alpha, \beta]$

13. If $f(x)$ is a continuous function on $[\alpha, \infty)$ then $\displaystyle\int_\alpha^\infty f(x)dx$ is known as improper integral, we define improper integral as $\displaystyle\int_\alpha^\infty f(x)dx = \lim_{\beta\to\infty}\int_\alpha^\beta f(x)dx$

14. If $\displaystyle\int_a^b |f(x)|dx = 0$ then $\displaystyle\int_a^b (f(x))^2 dx = 0$

15. (i) $\displaystyle\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$

(ii) $\displaystyle\int_a^{\pi-a} xf(\sin x)dx = \frac{\pi}{2}\int_a^{\pi-a} f(\sin x)dx$

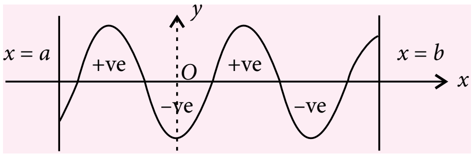
16. $\displaystyle\int_0^{n\pi} f(\cos^2 x)dx = n\int_0^\pi f(\cos^2 x)dx, n \in I$

17. If m, n are positive integer then

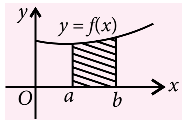
$$\int_0^\pi \sin mx \sin nx dx = \begin{cases} \dfrac{\pi}{2} \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$$

Geometrical meaning of definite integral and its applications

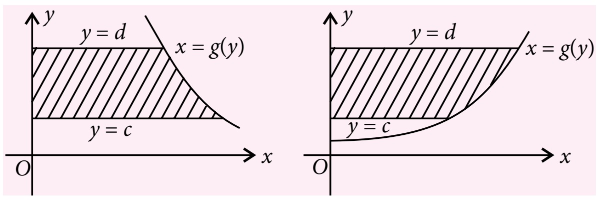
- Let $f(x)$ be a function defined in the closed interval $[a, b]$ then $\displaystyle\int_a^b f(x)dx$ represents the algebraic sum of the area of the region bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$, the value of the definite integral may be positive, negative or zero.



$\therefore$ Area $= \displaystyle\int_a^b |f(x)|dx$

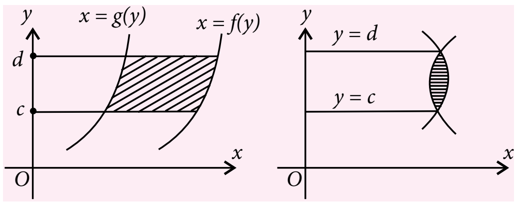


- Let $x = g(y)$ be a continuous function defined on $[a, b]$. Area bounded by the curve $x = g(y)$, y-axis and the lines $y = c$ and $y = d$ is given by



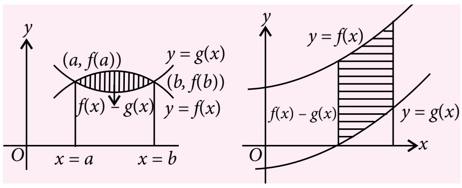
$$A = \int_c^d |x|dy = \int_c^d |g(y)|dy$$

- The area bounded by two curves $x = f(y)$, $x = g(y)$ and the lines $y = c$ and $y = d$ where $(f(c), c)$ and $(f(d), d)$ are points of intersection of both the curves is given by



$$A = \left|\int_c^d f(y)dy - \int_c^d g(y)dy\right|$$

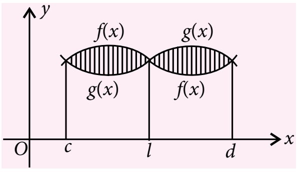
- The area bounded by two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by



$$\text{Area} = A = \left| \int_a^b f(x)dx - \int_a^b g(x)dx \right|$$

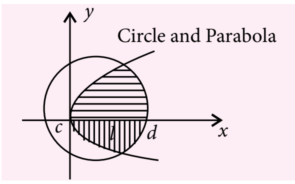
where $a, f(a), b, f(b)$ are the points of intersection of the curves $y = f(x)$ and $y = g(x)$.

Note : If $f(x) \geq g(x)$ in $[c, l]$ and $f(x) \leq g(x)$ in $[l, d]$ where $c < l < d$ then area bounded by the curves is given by



$$\int_c^l [f(x) - g(x)]dx + \int_l^d [g(x) - f(x)]dx$$

- If the curve lies below x-axis then the area be negative, then take its modulus value *i.e.*, count the area with +ve sign,



Formula's on definite integral of periodic functions

- $\displaystyle\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in I$
- $\displaystyle\int_{aT}^{bT} f(x)dx = (b-a)\int_0^T f(x)dx, a, b \in I$
- $\displaystyle\int_0^{nT} f(x)dx = \sum_{r=1}^{r=n} \int_{(r-1)}^{rT} f(x)dx$ and $f(r-1)T + y = f(y), n \in I$
- $\displaystyle\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx,\ n \in I$
- If $f(-x) = -f(x)$ *i.e.*, $f(x)$ is an odd periodic function defined on the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ where T is period of $f(x)$, then

  $g(x) = \displaystyle\int_a^x f(t)dt$ is periodic with period T

  i.e., $g(x + T) = g(x)$
- If $f(x)$ be an odd periodic function with period T then $f(x) = \displaystyle\int_a^x f(t)dt$ is an even function with period T
- If $f(x)$ be a periodic function with period T then $\displaystyle\int_a^{a+T} f(x)dx$ is independent of a *i.e.*, if $\displaystyle\int_a^{a+T} f(x)dx = \phi(a)$ then $\dfrac{d}{da}(\phi(a)) = 0$ *i.e.*, $\phi(a)$ is a constant function.
- If $f(x)$ satisfies $f(x + a) + f(x) = 0\ \forall\ x \in R$ and 'a' be constant, then period of $f(x)$ is $2a$.
- $x - [x]$ is a periodic function with period 1 where [.] represents greatest integer function.

Leibnitz theorem (Leibnitz rule for differentiation under the integral sign).

Let $\phi(x)$ and $\psi(x)$ be two functions defined on $[\alpha, \beta]$ and differentiable at a point $x \in\ ]\alpha, \beta[$ and $f(x, t)$ is continuous function then,

1. $$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{\partial}{dx} f(x,t)dt + \frac{d}{dx}(\psi(x)) f(x, \psi(x))$$

   $-\dfrac{d}{dx}(\phi(x)) f(x, \phi(x))$ where $f(x, t)$ is an implicit function.
2. For explicit function (Leibnitz rule)

   If functions $\phi(x)$ and $\psi(x)$ are defined on $[\alpha, \beta]$ and differentiable at a point $x \in\ ]\alpha, \beta[$and $f(t)$ is continuous on $[\phi(\alpha), \phi(\beta)]$ then

   $$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)dt\right] = \psi'(x) f(\psi(x)) - \phi'(x) f(\phi(x))$$

Approximation in definite integral

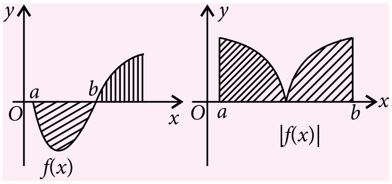
If $h(x) \leq f(x) \leq g(x) \forall x \in [a, b]$ then

$$\int_a^b h(x)dx \leq \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

- If the absolute maximum and minimum value of $f(x)$ are M and $m\ \forall\ x \in [a, b]$

  then $m(b-a) \leq \displaystyle\int_a^b f(x)dx \leq M(b-a)$

- $\int_a^b f(x)dx \le \int_a^b |f(x)|dx$ as shown by graphs



Definite integral as the limit of a sum

1. $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+.....+f(a+(n-1)h)]$

   $= \lim_{n\to\infty} h\sum f(a+rh)$

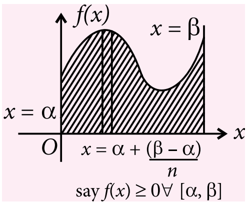
   $= \lim_{n\to 0} h[f(a+h)+f(a+2h)+.....+f(a+nh)],$

   where $h = \dfrac{b-a}{n}$

2. $\int_\alpha^\beta f(x)dx = (\beta-\alpha)\lim_{n\to\infty}$

   $\dfrac{1}{n}\sum_{r=1}^{r=n} f\left[\alpha+(\beta-\alpha)\dfrac{r}{n}\right]$

   $= (\beta-\alpha)\int_0^1 f[\alpha+(\beta-\alpha)x]dx$



In general, term $\dfrac{1}{n}\left[f\left(\alpha+(\beta-\alpha)\dfrac{r}{n}\right)\right]$ of the sum of the series $\dfrac{1}{n}\sum f\left[\alpha+(\beta-\alpha)\dfrac{r}{n}\right]$ is a function of $\dfrac{r}{n}$ such that various terms in the series are obtained by giving values 1, 2, 3, ..., n to r and the new integrand $f[\alpha + (\beta - \alpha)x]$ is obtained by changing $\left(\dfrac{r}{n}\right)$ to x.

The particular cases of the above result are

- $\lim_{n\to\infty}\sum_{r=1}^{r=n}\dfrac{1}{n}f\left(\dfrac{r}{n}\right)$ or $\lim_{n\to\infty}\sum_{r=0}^{r=n-1}\dfrac{1}{n}f\left(\dfrac{r}{n}\right) = \int_0^1 f(x)dx$
- $\lim_{n\to\infty}\sum_{r=1}^{r=np}\dfrac{1}{n}f\left(\dfrac{r}{n}\right) = \int_\alpha^\beta f(x)dx = \int_0^p f(x)dx$

  where $\alpha = \lim_{n\to\infty}\dfrac{r}{n} = 0$ as $r = 1$

  and $\beta = \lim_{n\to\infty}\dfrac{r}{n} = p$ (as $r = np$)

- The method to evaluate the integral, as the limit of the sum of an infinite series is called as integration by first principle.

PROBLEMS

Single Correct Answer Type

1. If $\int \dfrac{(\sqrt{x})^7}{(\sqrt{x})^9 + x^8}dx = a\log\left(\dfrac{x^k}{x^k+1}\right)+c$ then the value of $k - \dfrac{1}{a}$ equals

(a) 0 (b) 1 (c) 2 (d) 3

2. If A is a square matrix and $e^A = I + A + \dfrac{A^2}{2!} + ...\infty$ $= \dfrac{1}{2}\begin{bmatrix} f(x) & g(x) \\ g(x) & f(x)\end{bmatrix}$, where $A = \begin{bmatrix} x & x \\ x & x\end{bmatrix}$ $\forall\, x \in (0, 1)$ and I is identity matrix of order 2. Then

$\int_{1/4}^{1/2} (f(x)-1)(g(x)+1)\,dx$ is equal to

(a) $\dfrac{1}{4}(e^2+e)$ (b) $\dfrac{1}{4}(e^2-e)$

(c) $\dfrac{1}{2}(e^2+e)$ (d) $\dfrac{1}{2}(e^2-e)$

3. $\int \dfrac{3x^{16}+5x^{14}}{(x^5+x^2+1)^4}dx$ equals

(a) $\dfrac{x^{15}}{(x^5+x^2+1)^3}+c$ (b) $\dfrac{x^{10}}{(x^5+x^2+1)^3}+c$

(c) $\dfrac{x^{15}}{3(x^5+x^2+1)^3}+c$

(d) $\log|x^5+x^2+1| + \sqrt{3x^{16}+5x^{14}}+c$

4. $\int 4^{4^{4^{4^x}}}\cdot 4^{4^{4^x}}\cdot 4^{4^x}\cdot 4^x\,dx$ equals

(a) $4^{4^x}(\log 2)^4 + c$ (b) $\dfrac{4^{4^{4^{4^x}}}}{16(\log 2)^4}+c$

(c) $\dfrac{4^{4^{4^x}}}{(\log 4)^3}+c$ (d) none of these

5. $\int (x^m+2x^{2m}+3x^{3m})(1+x^m+x^{2m})^{1/m}\,dx$ equals

(a) $\dfrac{1}{m}(1+x^m+x^{2m})^{\frac{m+1}{m}}+c$

(b) $\left(\frac{1}{m+1}\right)(1+x^m+x^{2m})^{\frac{m+1}{m}}+c$

(c) $\frac{1}{m}\left(x^m+x^{2m}+x^{3m}\right)^{\frac{m+1}{m}}+c$

(d) $\left(\frac{1}{m+1}\right)(x^m+x^{2m}+x^{3m})^{\frac{m+1}{m}}+c$

6. The value of $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}dt$ equals

(a) $\pi/2$ (b) $\pi/4$ (c) $\pi/3$ (d) π

7. The value of $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+a^x} dx$ equals

(a) 1 (b) 2 (c) 3 (d) 4

8. Let the equation of the curve passing through the point (0, 1) be given by $y=\int x^3 e^{x^4} dx$. If the equation of the curve is written in the form $x = f(y)$, then $f(y)$ equals

(a) $\log|(4y-3)|$ (b) $\sqrt[4]{\log|(4y-3)|}$

(c) $\sqrt[4]{\log\left|\frac{(3-4y)}{4}\right|}$ (d) $\log\left|\frac{4y-3}{4}\right|$

9. If $\int \sin^4 x dx = Ax + B\sin 2x + C\sin 4x + D$, then $\frac{3}{A}+\frac{1}{B}+\frac{1}{C}$ equals

(a) 44 (b) 40 (c) 36 (d) 20

10. $\int \frac{\sec^2 x - 2017}{\sin^{2017} x} dx$ equals

(a) $-\frac{\cot x}{(\sin x)^{2017}}+c$ (b) $-\frac{\tan x}{(\sin x)^{2018}}+c$

(c) $\frac{\tan x}{(\sin x)^{2018}}+c$ (d) $\frac{\tan x}{(\sin x)^{2017}}+c$

More than One Correct Answer Type

11. If $\int x^{6a-1}(1+x^{3a})^{2/3}dx=\frac{1}{a}(1+x^{3a})^{5/3}[A(1+x^{3a})+B]$ then

(a) $A+B=\frac{-3}{40}$ (b) $A-B=\frac{13}{40}$

(c) $\frac{A}{B}=\frac{-5}{8}$ (d) $\frac{A}{B}=\frac{5}{8}$

12. If $\int\left(\cos x\frac{d}{dx}(\operatorname{cosec} x)\right)dx = f(x)+g(x)+c$, then $f(x)\cdot g(x)$ equals

(a) $x \cot x$ (b) $x \cos x \operatorname{cosec} x$

(c) $\frac{x}{\tan x}$ (d) $x \tan x$

13. If the primitive of $\cos(\log x)$ is $f(x)\{\cos g(x) + \sin h(x)\} + c$, $c \in R$, then

(a) $\lim_{x\to 4} f(x)=2$ (b) $\lim_{x\to 2}\frac{g(x)}{h(x)}=1$

(c) $h(e^4) = 4$ (d) $g(e^7) = 7$

14. If $f(x)=\int_x^1 \frac{dt}{1+t^2}$ and $g(x)=\int_0^{1/x}\frac{dt}{1+t^2}$ $\forall\, x>0$ then

(a) $g(x)=\cot^{-1}x-\frac{\pi}{4}$ (b) $f(x) = g(x)$

(c) $f(x) > g(x)$ (d) $g(x) > f(x)$

15. The value of the integral $\int_0^1 e^{x^2} dx$ is

(a) $< e$ (b) $> e$ (c) < 1 (d) > 1

16. Let $f(x)=\int e^{2x}(x-1)(x-2)(x-3)dx$, then

(a) $f(x)$ increases for $x > 4$

(b) $f(x)$ decreases in $2 < x < 3$

(c) $f(x)$ decreases for $x < 1$

(d) $f(x)$ increases for $1 < x < 2$

17. The area bounded by the curves $y = x^2$ and $x = y^2$ is equal to

(a) $2\int_0^1 (x-x^2)\,dx$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$

(d) Area of the region, $\{(x, y) = x^2 \le y \le |x|\}$

Comprehension Type

Paragraph for Q.No. 18-19

Let $f(x) = x^2 - 5x + 6$, $x \in R$, the graph of $f(x)$ meet x-axis at (2, 0) and (3, 0) and y-axis at (0, 6).

18. The area bounded by the curve $f(|x|)$ between $2 \le |x| \le 3$ and x-axis (in square units) is

(a) $\frac{28}{3}$ (b) $\frac{29}{3}$ (c) $\frac{14}{3}$ (d) $\frac{15}{4}$

19. The area bounded by the curve $|f(|x|)|$ between $2 \le |x| \le 3$ and x-axis (in square units) is

(a) 2/3 (b) 16/3 (c) 1/6 (d) 1/3

Paragraph for Q.No. 20 to 22

Let $f(x)=\dfrac{\sin 2x\sin\left(\dfrac{\pi}{2}\cos x\right)}{2x-\pi}$

20. $\int_0^{\pi} f(x)dx=$

(a) $\dfrac{4}{\pi}$ (b) $\dfrac{8}{\pi}$ (c) $\dfrac{8}{\pi^2}$ (d) 0

21. $\int_0^{\pi} x\, f(x)dx=$

(a) 0 (b) $\dfrac{8}{\pi^2}$ (c) $\dfrac{16}{\pi^2}$ (d) $\dfrac{8}{\pi}$

22. $\int_0^{\pi} x^2 f(x)dx=$

(a) $\dfrac{8}{\pi}$ (b) $\dfrac{16}{\pi^2}$ (c) $\dfrac{4}{\pi^2}$ (d) 0

Matrix-Match Type

23. Match the following.

Column-I		Column-II	
A.	$\int_0^{\pi} x\log(\sin x)\,dx=$	1.	$\dfrac{\pi}{8}\log 2$
B.	$\int_0^{\infty}\dfrac{\log(x+x^{-1})}{(1+x^2)}\,dx=$	2.	$-\dfrac{\pi^2}{2}\log 2$
C.	$\int_0^{\pi/4}\log(1+\tan x)\,dx=$	3.	$-\pi\log 2$
D.	$\int_0^{\pi}\log(1-\cos x)\,dx=$	4.	$\pi\log 2$

24. Match the following.

Column-I		Column-II	
A.	$\lim\limits_{x\to 0}\dfrac{1}{x}\left(\int_y^a e^{\sin^2 t}dt-\int_{x+y}^a e^{\sin^2 t}dt\right)=$ where $y,\ a\in R$	1.	1
B.	$\lim\limits_{x\to\infty}\dfrac{\left(\int_0^{x+y} e^{t^2}\,dt\right)^2}{\int_0^{x+y} e^{2t^2}\,dt}=$, where $y\in R$	2.	$e^{\sin^2 y}$
C.	$\lim\limits_{x\to 0}\dfrac{\left(\int_0^{x^2}\sin\sqrt{x}\,dx\right)}{x^3}=$	3.	0
D.	$\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\cos t^2 dt}{x\sin x}=$, where $y\in R$	4.	2/3

Integer Type

25. If $I=\int_0^{\pi} x[\sin^2(\sin x)+\cos^2(\cos x)]\,dx$, then $[I]$ = (where $[\cdot]$ denotes greatest integer function).

26. If $\int\sqrt{x+\sqrt{x^2+2}}\,dx=k\left(x+\sqrt{x^2+2}\right)^{p/2}-\dfrac{2}{\left(x+\sqrt{x^2+2}\right)^{q/2}}+c$

then $[3k+p+q]$ = (where $[x]$ is greatest integer function).

27. If a curve $g(x)=\int x^{27}(1+x+x^2)^6\,(6x^2+5x+4)dx$ is passes through origin and $g(1)=\dfrac{3^k}{k}$ then the value of k is

28. If $\int\dfrac{\cos 13x-\cos 14x}{1+2\cos 9x}dx=\dfrac{\sin 4x}{a}-\dfrac{\sin 5x}{b}+c$ and $a^b=2^k$ then sum of digits of k is

29. The area bounded by the curve $y=x^6(\pi-x)^{10}$ is $\dfrac{\pi^{a+b+1}\,a!\,b!}{(a+b+1)!}$, then $\dfrac{a+b}{8}=$

30. If $I_{m,p}=\int x^m(a+bx^n)^p\,dx$

$=\dfrac{x^{m+1}(a+bx^n)^p}{m+1}-f(m,n,p,b)I_{m+n,\,p-1}$

and $f(1,2,3,4)=-\dfrac{1}{2}(\lambda!)$, then λ =

SOLUTIONS

1. **(a) :** Let $I=\int\dfrac{(\sqrt{x})^7}{(\sqrt{x})^9+x^8}dx=\int\dfrac{(\sqrt{x})^7}{(\sqrt{x})^9+(\sqrt{x})^{16}}dx$

$=\int\dfrac{dx}{(\sqrt{x})^2+(\sqrt{x})^9}=\int\dfrac{dx}{(\sqrt{x})^9\left(1+\left(\dfrac{1}{\sqrt{x}}\right)^7\right)}$

$$\Rightarrow\ -7(\sqrt{x})^{-8}\left(\frac{1}{2\sqrt{x}}\right)dx=dt\ \ (\text{Putting } (\sqrt{x})^{-7}=t)$$

$$\Rightarrow\ (\sqrt{x})^{-9}\,dx=-\frac{2}{7}\,dt$$

$$\therefore\ I=-\frac{2}{7}\int\frac{dt}{1+t}=-\frac{2}{7}\log|1+t|+c$$

$$=-\frac{2}{7}\log\left(1+\frac{1}{(\sqrt{x})^7}\right)+c=\frac{2}{7}\log\left(\frac{(\sqrt{x})^7}{(\sqrt{x})^7+1}\right)+c$$

$$=\frac{2}{7}\log\left(\frac{x^{7/2}}{x^{7/2}+1}\right)+c=a\log\left(\frac{x^k}{x^k+1}\right)+c$$

$$\therefore\ k=\frac{7}{2}\ \text{ and }\ a=\frac{2}{7}\ \Rightarrow\ k-\frac{1}{a}=0$$

2. (b)

3. (c) : Let $I=\int\frac{3x^{16}+5x^{14}}{(x^5+x^2+1)^4}dx=\int\frac{\frac{3x^{16}+5x^{14}}{x^{20}}}{\left(\frac{x^5+x^2+1}{x^5}\right)^4}dx$

$$=\int\frac{\frac{3}{x^4}+\frac{5}{x^6}}{\left(1+\frac{1}{x^3}+\frac{1}{x^5}\right)^4}dx$$

Putting $1+\frac{1}{x^3}+\frac{1}{x^5}=t\ \Rightarrow\ \left(\frac{3}{x^4}+\frac{5}{x^6}\right)dx=-dt$

$$\therefore\ I=-\int\frac{dt}{t^4}=\frac{1}{3t^3}+c$$

$$=\frac{1}{3\left(1+\frac{1}{x^3}+\frac{1}{x^5}\right)^3}+c=\frac{x^{15}}{3(x^5+x^2+1)^3}+c$$

4. (b) : Let $I=\int 4^{4^{4^{4^x}}}\cdot 4^{4^{4^x}}\cdot 4^{4^x}\cdot 4^x\,dx$

Putting $4^{4^{4^{4^x}}}=t$

$$\Rightarrow\ (\log 4)^4(4^{4^{4^{4^x}}}\cdot 4^{4^{4^x}}\cdot 4^{4^x}\cdot 4^x)dx=dt$$

$$\therefore\ I=\frac{1}{(\log 4)^4}\int dt=\frac{t}{(\log 4)^4}+c$$

$$=\frac{4^{4^{4^{4^x}}}}{(\log 2^2)^4}+c=\frac{4^{4^{4^{4^x}}}}{16(\log 2)^4}+c$$

5. (d) : Let $I=\int(x^m+2x^{2m}+3x^{3m})(1+x^m+x^{2m})^{1/m}\,dx$

$$=\int x(x^{m-1}+2x^{2m-1}+3x^{3m-1})(1+x^m+x^{2m})^{1/m}dx$$

$$=\int(x^{m-1}+2x^{2m-1}+3x^{3m-1})(x^m)^{1/m}(1+x^m+x^{2m})^{1/m}\,dx$$

$(\because x=(x^m)^{1/m})$

$$=\int(x^{m-1}+2x^{2m-1}+3x^{3m-1})(x^m+x^{2m}+x^{3m})^{1/m}\,dx$$

Putting $x^m+x^{2m}+x^{3m}=t$

$$\Rightarrow\ (x^{m-1}+2x^{2m-1}+3x^{3m-1})dx=\frac{dt}{m}$$

$$\therefore\ I=\frac{1}{m}\int t^{1/m}\,dt=\frac{1}{m}\left(\frac{m}{m+1}\right)t^{\frac{m+1}{m}}+c$$

$$=\frac{1}{(m+1)}(x^m+x^{2m}+x^{3m})^{\frac{m+1}{m}}+c$$

6. (b) : Let $f(x)=\int_0^{\sin^2 x}\sin^{-1}\sqrt{t}\,dt+\int_0^{\cos^2 x}\cos^{-1}\sqrt{t}\,dt$

$$\therefore\ f'(x)=\left[\frac{d}{dx}(\sin^2 x)\right]\sin^{-1}\sqrt{\sin^2 x}+\left[\frac{d}{dx}(\cos^2 x)\right]\cos^{-1}\sqrt{\cos^2 x}$$

$=(2\sin x\cos x)x+(-2\sin x\cos x)x=0$

$\therefore\ f'(x)=0\ \Rightarrow\ f(x)=c\ \Rightarrow\ f(\pi/4)=c$

$$\therefore\ \int_0^{1/2}\sin^{-1}\sqrt{t}\,dt+\int_0^{1/2}\cos^{-1}\sqrt{t}\,dt=c$$

$$\Rightarrow\ \int_0^{1/2}(\sin^{-1}\sqrt{t}+\cos^{-1}\sqrt{t})dt=c$$

$$\Rightarrow\ \int_0^{1/2}\frac{\pi}{2}dt=\frac{\pi}{2}(t)_0^{1/2}=\frac{\pi}{4}=c\ \Rightarrow\ f\left(\frac{\pi}{4}\right)=\frac{\pi}{4}$$

7. (a)

8. (b) : The curve is given by

$y=\int x^3e^{x^4}dx$ which passes through (0, 1)

$$\therefore\ y=\int x^3e^{x^4}dx\ \Rightarrow\ y=\frac{1}{4}\int 4x^3e^{x^4}dx$$

$\Rightarrow\ y=\frac{1}{4}e^{x^4}+k\Rightarrow k=\frac{3}{4}$ as curve passes through (0, 1)

$$\therefore\ y=\frac{1}{4}e^{x^4}+\frac{3}{4}\ \Rightarrow\ e^{x^4}=4\left(y-\frac{3}{4}\right)$$

$\Rightarrow\ e^{x^4}=4y-3\ \Rightarrow\ x^4=\log(4y-3)$

$\Rightarrow\ x=\sqrt[4]{\log(4y-3)}=f(y)$

9. (c) : $\because\ \int\sin^4 x\,dx=\frac{1}{4}\int(2\sin^2 x)^2dx$

$$=\frac{1}{4}\int(1-\cos 2x)^2dx=\frac{1}{4}\int(1-2\cos 2x+\cos^2 2x)dx$$

$$= \frac{1}{4}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{8}\int(3-4\cos 2x+\cos 4x)\,dx$$

$$= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+D$$

$= Ax + B\sin 2x + C\sin 4x + D \therefore A=\frac{3}{8}, B=-\frac{1}{4}, C=\frac{1}{32}$

Thus, $\frac{3}{A}+\frac{1}{B}+\frac{1}{C}=8-4+32=36$

10. (d) : $\int \frac{\sec^2 x-2017}{\sin^{2017} x}dx$

$$= \int(\sin x)^{-2017}\sec^2 x dx \;-2017\int\frac{dx}{\sin^{2017} x}$$

$$=(\sin x)^{-2017}\tan x-\int[(-2017)(\sin x)^{-2018}(\cos x)\tan x]dx -\int\frac{2017}{\sin^{2017} x}dx$$

$$=(\sin x)^{-2017}\tan x+\int 2017\cdot(\sin x)^{-2017}dx-2017\int\frac{dx}{\sin^{2017} x}$$

$$=\frac{\tan x}{(\sin x)^{2017}}+c$$

11. (a, b, c) : Let $I=\int x^{6a-1}(1+x^{3a})^{2/3}dx$

Putting $1+x^{3a}=t^2 \Rightarrow x^{3a-1}\,dx=\frac{1}{3a}2t\cdot dt$

$$\therefore\; I=\frac{2}{3a}\int 2t(t^2-1)t^{4/3}dt=\frac{2}{3a}\int t^{7/3}(t^2-1)\,dt$$

$$=\frac{2}{3a}\int(t^{13/3}-t^{7/3})\,dt=\frac{2}{3a}\left(\frac{3}{16}t^{16/3}-\frac{3}{10}t^{10/3}\right)$$

$$=\frac{2}{3a}\left(\frac{3}{16}t^2-\frac{3}{10}\right)t^{10/3}=\frac{1}{a}(1+x^{3a})^{5/3}\left(\frac{1}{8}(1+x^{3a})-\frac{1}{5}\right)$$

$$=\frac{1}{a}(1+x^{3a})^{5/3}[A(1+x^{3a})+B]\Rightarrow A=\frac{1}{8},\; B=-\frac{1}{5}$$

$$\therefore\; A+B=\frac{1}{8}-\frac{1}{5}=-\frac{3}{40},\; A-B=\frac{13}{40},\; \frac{A}{B}=-\frac{5}{8}$$

12. (a, b, c) : $\int\left(\cos x\frac{d}{dx}(-\text{cosec}x)\right)dx$

$$=\int(\cos x\cdot\text{cosec}x\cdot\cot x)\,dx=\int\cot^2 x\,dx$$

$$=\int(\text{cosec}^2 x-1)dx=-\cot x-x+c$$

$= f(x)+g(x)+c$ or $g(x)+f(x)+c$

$\therefore\; f(x)\cdot g(x)=x\cot x=\frac{x}{\tan x}$ = $x\cos x\ \text{cosec}x$

13. (a, b, c, d) : Let $I=\int\cos(\log x)\,dx$

$=\int e^t\cos t\,dt$ (on putting $\log x = t$)

$$=\frac{e^t}{2}(\cos t+\sin t)+c=\frac{x}{2}(\cos(\log x)+\sin(\log x))+c$$

$$=f(x)\{\cos g(x)+\sin h(x)\}+c$$

$\therefore\; f(x)=\frac{x}{2},\; g(x)=h(x)=\log x \;\therefore\; \lim_{x\to 4}f(x)=\frac{4}{2}=2$

$\lim_{x\to 2}\frac{g(x)}{h(x)}=1$ (as $g(x)=h(x)$)

and $h(x)=\log x \;\therefore\; h(e^4)=\log e^4=4$

Also $g(x)=\log x \;\therefore\; g(e^7)=\log e^7=7$

14. (a, b) : $\because\; f(x)=\int_x^1\frac{dt}{1+t^2}=\left(\tan^{-1}t\right)_x^1$

$$=\tan^{-1}1-\tan^{-1}x=\frac{\pi}{4}-\left(\frac{\pi}{2}-\cot^{-1}x\right)$$

$$=\cot^{-1}x-\frac{\pi}{4} \text{ and } g(x)=\int_1^{1/x}\frac{dt}{1+t^2}=(\tan^{-1}t)_1^{1/x}$$

$=\tan^{-1}(1/x)-\tan^{-1}(1)=\cot^{-1}x-\pi/4=f(x)$

Thus $f(x)=g(x)$ and $g(x)=\cot^{-1}x-\frac{\pi}{4}$

15. (a, d) : From the integral we are given

$0<x<1 \;\Rightarrow\; 0<x^2<1$

$$\Rightarrow e^0<e^{x^2}<e^1\Rightarrow 1<e^{x^2}<e \;\therefore\; \int_0^1 1\,dx<\int_0^1 e^{x^2}dx<\int_0^1 e^1\,dx$$

$$\Rightarrow\; 1<\int_0^1 e^{x^2}dx<e(1-0) \quad \left(\because \int_0^1 e^1dx=e(x)_0^1\right)$$

$$\therefore\; \int_0^1 e^{x^2}dx<e \text{ and } \int_0^1 e^{x^2}dx>1$$

16. (a, b, c, d) : $\because\; f(x)=\int e^{2x}(x-1)(x-2)(x-3)dx$

$\Rightarrow f'(x)=e^{2x}(x-1)(x-2)(x-3)$

$\therefore\; f'(x)=0 \;\Rightarrow\; x=1, 2, 3$

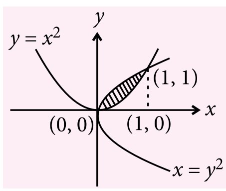
Consider $x<1$, $1<x<2$, $2<x<3$, $x>4$ etc.

For $x<1$ and $2<x<3$, $f(x)$ decreases as $f'(x)<0$ when $\begin{cases}-\infty<x<1\\ 2<x<3\end{cases}$ and for $1<x<2$ and $x>4$, $f'(x)>0$, so $f(x)$ increases.

17. (a, b, d) : The given curves are $y=x^2$ and $x=y^2$

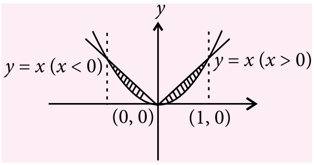
$\therefore$ Enclosed area

$$=\int_0^1(\sqrt{x}-x^2)dx=\frac{1}{3} \text{ sq. units}$$



Again, $2\int_0^1 (x-x^2)dx = \left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$

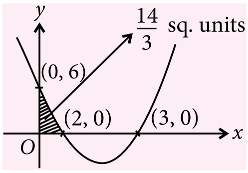
Also, the area of the region bounded by $\{(x, y) : x^2 \le y \le |x|\}$ is given by $2\int_0^1 (x-x^2)dx = \frac{1}{3}$



18. (a) : Given $f(x) = x^2 - 5x + 6$

$\therefore\ f(|x|) = |x^2| - 5|x| + 6$

Now, the points at which $y = f(|x|)$ meet the x-axis are obtained by putting $y = 0$.

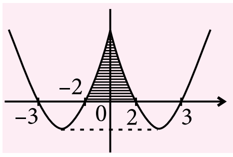


$\therefore\ |x|^2 - 5|x| + 6 = 0$

$\Rightarrow\ (|x| - 2)(|x| - 3) = 0$

$\Rightarrow\ x = \pm 2,\ x = \pm 3$

$\therefore$ Points are $(2, 0)$, $(3, 0)$, $(-2, 0)$, $(-3, 0)$



Required area $= 2\int_0^2 y\,dx = 2\int_0^2 (x^2 - 5x + 6)dx$

$= 2\times\left(\frac{x^3}{3}-\frac{5x^2}{2}+6x\right)_0^2 = \frac{28}{3}$ sq. units

19. (d)

20. (d) : Let $I = \int_0^\pi f(x)dx = \int_0^\pi \frac{\sin 2x \sin\left(\frac{\pi}{2}\cdot\cos x\right)}{2x-\pi}dx$

$= \int_0^\pi \frac{\sin 2(\pi - x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right)}{2(\pi - x)-\pi}dx$

$= \int_0^\pi \frac{\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{\pi - 2x}dx = -I \Rightarrow 2I = 0.\ \therefore\ I = 0$

21. (b) : From given, we have

$\int_0^\pi x\,f(x)dx = \int_0^\pi \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

$= \frac{1}{2}\int_0^\pi \frac{(2x-\pi+\pi)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

$= \frac{1}{2}\int_0^\pi \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)dx + \frac{\pi}{2}\int_0^\pi f(x)dx$

$= \frac{1}{2}\int_0^\pi \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)dx + 0 \quad \left(\because \int_0^\pi f(x)dx = 0\right)$

$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{2t}{\pi}\sin t\,dt$

(By putting $\frac{\pi}{2}\cos x = t$ and changing the limit)

$= \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t\sin t\,dt = \frac{8}{\pi^2}\int_0^{\pi/2} t\sin t\,dt = \frac{8}{\pi^2}(1) = \frac{8}{\pi^2}$

22. (a) : Since $\int_0^\pi f(x)dx = \int_0^{\pi/2}(f(x)+f(\pi - x))\,dx$

$\therefore\ \int_0^\pi x^2 f(x)dx = \pi\int_0^{\pi/2}\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)dx$

$= 2\pi\int_0^1 t\sin\left(\frac{\pi}{2}t\right)dt$

(By putting $\cos x = t$ and change limits then use the property $\int_a^b f(x)dx = -\int_b^a f(x)dx$)

$= 2\pi\left[t\cos\left(\frac{\pi}{2}t\right)\left(\frac{-2}{\pi}\right)\right]_0^1 - 2\pi\int_0^1 1\cdot\cos\left(\frac{\pi}{2}t\right)\left(\frac{-2}{\pi}\right)dt$

$= 0 + \frac{2\pi\times 2}{\pi}\int_0^1 \cos\left(\frac{\pi}{2}t\right)dt$

$= 4\left(\sin\left(\frac{\pi}{2}t\right)\frac{2}{\pi}\right)_0^1 = \frac{8}{\pi}(1-0) = \frac{8}{\pi}$

23. A $\to$ 2, B $\to$ 4, C $\to$ 1, D $\to$ 3

A. Let $I = \int_0^\pi x\log\sin x\,dx$

$\Rightarrow\ I = \int_0^\pi (\pi - x)\log\,\sin(\pi - x)dx \Rightarrow I = \pi\int_0^\pi \log\,\sin x dx - I$

$\Rightarrow\ I = \frac{\pi}{2}\int_0^\pi \log(\sin x)\,dx = \frac{\pi}{2}(-\pi\log 2) = -\frac{\pi^2}{2}\log 2$

B. $\int_0^\infty \frac{\log(x+x^{-1})}{(1+x^2)}dx$ (Putting $x = \tan\theta$)

$= \int_0^{\pi/2}\log(\tan\theta+\cot\theta)\,d\theta = \int_0^{\pi/2}\log\left(\frac{2}{\sin 2\theta}\right)d\theta$

$$= \int_0^{\pi/2} \log 2\, d\theta - \int_0^{\pi/2} \log \sin 2\theta\, d\theta$$

$$= \frac{\pi}{2}\log 2 - \int_0^{\pi/2} \log \sin 2\theta\, d\theta = \frac{\pi}{2}\log 2 - \left(-\frac{\pi}{2}\log 2\right)$$

$$\therefore \int_0^{\pi/2} \log \sin 2t dt = -\frac{\pi}{2}\log 2 = \pi \log 2$$

C. Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$$I = \frac{\pi}{4}\log 2 - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2 \ \therefore \int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2$$

D. $I = \int_0^{\pi} \log(1-\cos x)\,dx = \int_0^{\pi} \log\left(2\sin^2\frac{x}{2}\right)dx$

$$= \int_0^{\pi} \log 2\, dx + 2\int_0^{\pi} \log\left(\sin\frac{x}{2}\right)dx$$

$$I = \pi \log 2 + 4\int_0^{\pi/2} \log(\sin t)\,dt \quad \left(\begin{array}{l}\text{Put } \frac{x}{2} = t \Rightarrow dx = 2dt \\ x = 0, t = 0, x = \pi, t = \frac{\pi}{2}\end{array}\right)$$

$$= \pi \log 2 + 4\int_0^{\pi/2} \log(\sin x)\,dx \quad \text{(changing the variable)}$$

$$= \pi \log 2 + 4\left(-\frac{\pi}{2}\log 2\right) = \pi \log 2 - 2\pi \log 2 = -\pi \log 2$$

24. A → 2, B → 3, C → 4, D → 1

A. $\lim_{x\to 0} \frac{1}{x}\left(\int_y^a e^{\sin^2 t} dt - \int_{x+y}^a e^{\sin^2 t} dt\right)$

$$= \lim_{x\to 0} \frac{1}{x}\left(\int_y^a e^{\sin^2 t} dt + \int_a^{x+y} e^{\sin^2 t} dt\right)$$

$$= \lim_{x\to 0} \frac{1}{x}\int_y^{x+y} e^{\sin^2 t} dt \text{ or } \lim_{x\to 0} \frac{\int_y^{x+y} e^{\sin^2 t} dt}{x} \quad (0/0 \text{ form})$$

$$= \lim_{x\to 0} \frac{\left[\frac{d}{dx}(x+y)\right] e^{\sin^2(x+y)}}{1}$$

(treating y as constant) or consider as variable

$$= \lim_{x\to 0} (1+0)\, e^{\sin^2(x+y)} = e^{\sin^2 y}$$

B. $\lim_{x\to\infty} \frac{\left(\int_0^{x+y} e^{t^2} dt\right)^2}{\int_0^{x+y} e^{2t^2} dt} = \lim_{x\to\infty} \frac{2\left(\int_0^{x+y} e^{t^2} dt\right)\cdot e^{(x+y)^2}}{e^{2(x+y)^2}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to\infty} \frac{2\int_0^{x+y} e^{t^2} dt}{e^{(x+y)^2}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{2\cdot e^{(x+y)^2}}{e^{(x+y)^2}}\cdot\frac{1}{2(x+y)} = \lim_{x\to\infty} \frac{1}{x+y} = 0$$

C. $\lim_{x\to 0} \frac{\int_0^{x^2} \sin\sqrt{x}\, dx}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{2x\sin x}{3x^2} = \frac{2}{3}\lim_{x\to 0}\frac{\sin x}{x} = \frac{2}{3}$$

D. $\lim_{x\to 0} \frac{\int_0^{x^2} \cos t^2 dt}{x\sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

Using L-Hospital rule, we get

$$\lim_{x\to 0} \frac{\frac{d}{dx}(x^2)\cos x^4}{x\frac{d}{dx}(\sin x) + \sin x} = \frac{2(\cos 0^4)}{1+1} = \cos 0 = 1$$

25. (4) : $\because\ I = \int_0^{\pi} x[\sin^2(\sin x) + \cos^2(\cos x)]\,dx$

$$\Rightarrow\ I = \int_0^{\pi} (\pi - x)\left\{\sin^2(\sin x) + \cos^2(\cos x)\right\} dx$$

$$\Rightarrow\ 2I = \pi\int_0^{\pi} [\sin^2(\sin x) + \cos^2(\cos x)]dx$$

$$\Rightarrow\ I = \frac{\pi}{2}\int_0^{\pi} dx = \frac{\pi^2}{2} \ \therefore\ [I] = \left[\frac{\pi^2}{2}\right] = [4.93] = 4$$

26. (5) : Let $I = \int \sqrt{x+\sqrt{x^2+2}}\, dx$

Putting $x + \sqrt{x^2+2} = t$ ...(i)

$$\Rightarrow \frac{1}{x+\sqrt{x^2+2}} = \frac{1}{t} \Rightarrow \frac{x-\sqrt{x^2+2}}{-2} = \frac{1}{t} \quad \text{...(ii)}$$

$\therefore\ t-\frac{2}{t}=2x$ (On adding (i) and (ii))

$\therefore\ \left(1+\frac{2}{t^2}\right)dt=2dx$ or $dx=\frac{1}{2}\left(1+\frac{2}{t^2}\right)dt$

$\therefore\ I=\frac{1}{2}\int t^{1/2}\left(1+\frac{2}{t^2}\right)dt$

$$=\frac{1}{2}\int(t^{1/2}+2t^{-3/2})\,dt=\frac{1}{2}\left[\frac{t^{3/2}}{3/2}+\frac{2t^{-1/2}}{-1/2}\right]$$

$$=\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2}-\frac{2}{\left(x+\sqrt{x^2+2}\right)^{1/2}}+c$$

$$=k\left(x+\sqrt{x^2+2}\right)^{p/2}-\frac{2}{\left(x+\sqrt{x^2+2}\right)^{q/2}}+c$$

$\Rightarrow\ k=\frac{1}{3},\ p=3,\ q=1$

$\therefore\ 3k+p+q=1+3+1=5$

$\therefore\ [3k+p+q]=[5]=5$

27. (7) : $\because\ g(x)=\int x^{27}(1+x+x^2)^6(6x^2+5x+4)dx$

$$=\int x^3(x^4)^6(1+x+x^2)^6(6x^2+5x+4)dx$$

$$=\int x^3(6x^2+5x+4)(x^4+x^5+x^6)^6dx$$

(Putting $x^4+x^5+x^6=t \Rightarrow (4x^3+5x^4+6x^5)dx=dt$
$\Rightarrow\ x^3(6x^2+5x+4)dx=dt$)

$$=\int t^6dt=\frac{t^7}{7}+c$$

$\therefore\ g(x)=\frac{(x^4+x^5+x^6)^7}{7}+c\quad\because\ g(0)=0\ \Rightarrow\ c=0$

$\therefore\ g(1)=\frac{3^7}{7}=\frac{3^k}{k}$ (given) $\therefore\ k=7$

28. (1) : Let $I=\int\frac{\cos 13x-\cos 14x}{1+2\cos 9x}dx$

$$=\int\frac{2\sin\frac{27}{2}x\cdot\sin\frac{x}{2}}{1+2\left(1-2\sin^2\frac{9x}{2}\right)}dx=\int\frac{2\sin\frac{27}{2}x\cdot\sin\frac{x}{2}}{3-4\sin^2\frac{9x}{2}}dx$$

$$=\int\frac{\left(2\sin\frac{27}{2}x\cdot\sin\frac{x}{2}\right)}{3\sin\frac{9x}{2}-4\sin^3\frac{9x}{2}}\sin\frac{9x}{2}dx$$

$$=\int\frac{2\sin\frac{27}{2}x\cdot\sin\frac{x}{2}\cdot\sin\frac{9x}{2}}{\sin\frac{27x}{2}}dx$$

$$=\int 2\sin\frac{x}{2}\sin\frac{9x}{2}dx=\int(\cos 4x-\cos 5x)\,dx$$

$$=\frac{\sin 4x}{4}-\frac{\sin 5x}{5}+c=\frac{\sin 4x}{a}-\frac{\sin 5x}{b}+c$$

$\therefore\ a=4,\ b=5\ \therefore\ a^b=4^5=2^{10}=2^k$

$\therefore\ k=10\Rightarrow$ Sum of digits $=1+0=1$

29. (2) : When $y=0\ \Rightarrow\ x^6(\pi-x)^{10}=0$

$\Rightarrow\ x=0$ and $x=\pi$

$\therefore$ Required area $=\int_0^{\pi}x^6(\pi-x)^{10}dx$

Putting $x=\pi t\ \Rightarrow\ dx=\pi dt$

When $x=0,\ t=0$ and $x=\pi,\ t=1$

$\therefore$ Area $=\int_0^1\pi^{6+10}\,t^6(1-t)^{10}(\pi\,dt)=\int_0^1\pi^{16}t^6(1-t)^{10}(\pi)dt$

$$=\frac{\pi^{17}\times 6!\ 10!}{17!}\left[\text{Using}\int_0^1 x^m(1-x)^n dx=\frac{m!\ n!}{(m+n+1)!}\right]$$

$\therefore\ a+b=6+10=16$ and $\frac{a+b}{8}=2$

30. (4) : Let $t=x^{(m+1)}(a+bx^n)^p$

$$\therefore\ \frac{dt}{dx}=x^{m+1}p(a+bx^n)^{p-1}nbx^{n-1}+(a+bx^n)^p(m+1)x^m$$

$$\Rightarrow\ \frac{dt}{dx}=(a+bx^n)^{p-1}(npbx^{m+n})+(m+1)x^m(a+bx^n)^p$$

$$\Rightarrow\ \int dt=\int(a+bx^n)^{p-1}(npbx^{m+n})dx+(m+1)\int x^m(a+bx^n)^p dx$$

$t=npb\,I_{m+n,\,p-1}+(m+1)I_{m,\,p}$

$$\therefore\ x^{m+1}(a+bx^n)^p=npb\,I_{m+n,p-1}+(m+1)I_{m,\,p}$$

$$\therefore\ I_{m,p}=\frac{x^{m+1}(a+bx^n)^p}{m+1}-\frac{npb}{m+1}I_{m+n,\,p-1}$$

$$\Rightarrow\ \frac{x^{m+1}(a+bx^n)^p}{m+1}-f(m,n,p,b)I_{m+n,\,p-1}$$

$$=\frac{x^{m+1}(a+bx^n)^p}{m+1}-\frac{npb}{m+1}I_{m+n,p-1}$$

$$\Rightarrow\ f(m,n,p,b)=\frac{npb}{m+1}$$

$$\Rightarrow\ f(1,2,3,4)=\frac{2\cdot 3\cdot 4}{2}=\frac{1}{2}(4!)=\frac{1}{2}(\lambda!)\ \Rightarrow\ \lambda=4$$

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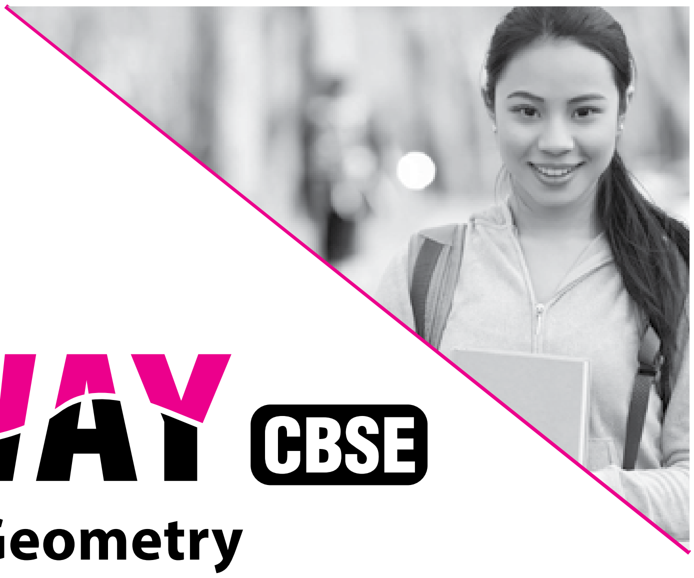
CLASS XII Series 8

ACE YOUR WAY CBSE



Three Dimensional Geometry

Important Formulae

- *If a, b, c are the direction ratios of a line then its direction cosines are*
$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$
- *If* $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$ *then the direction ratios of* $\vec{r}$ *are a, b, c.*
- *Let PQ be a line joining P* (x_1, y_1, z_1) *and Q* (x_2, y_2, z_2). *Then the direction ratios of the line PQ are* $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$.
- *If* θ *is the angle between two lines* L_1 *and* L_2 *whose d.c.'s are* l_1, m_1, n_1 *and* l_2, m_2, n_2, *then the following holds true.*
- $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$
- $\sin\theta = \sqrt{\sum (m_1n_2 - m_2n_1)^2}$
- *Lines* L_1 *and* L_2 *are perpendicular* $\Leftrightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$
- *Lines* L_1 *and* L_2 *are parallel* $\Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$
- *The vector equation of a line through a point with p.v.* $\vec{r}_1$ *and parallel to* $\vec{m}$ *is* $\vec{r}=\vec{r}_1+\lambda\vec{m}$
- *The cartesian equation of a line with d.r's a, b, c and passing through A* (x_1, y_1, z_1) *is*
$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$
- *The vector equation of a line through two points with p.v.'s* $\vec{r}_1$ *and* $\vec{r}_2$ *is* $\vec{r}=\vec{r}_1+\lambda(\vec{r}_2-\vec{r}_1)$
- *The cartesian equation of a line through* $A(x_1, y_1, z_1)$ *and* $B(x_2, y_2, z_2)$ *is*
$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$
- *A* (x_1, y_1, z_1), *B* (x_2, y_2, z_2) *and C* (x_3, y_3, z_3) *are collinear if*
$$\frac{x_3-x_1}{x_2-x_1}=\frac{y_3-y_1}{y_2-y_1}=\frac{z_3-z_1}{z_2-z_1}$$
- *Three points A, B, C with position vectors* $\vec{a}, \vec{b}, \vec{c}$ *respectively are collinear if and only if there exist scalars* μ_1, μ_2, μ_3 *not all zero, such that*
$$\mu_1\vec{a}+\mu_2\vec{b}+\mu_3\vec{c}=\vec{0} \text{ and } \mu_1+\mu_2+\mu_3=0.$$
- *Let* θ *be the angle between the lines*
$\vec{r}=\vec{r}_1+\lambda\vec{m}_1$ and $\vec{r}=\vec{r}_2+\mu\vec{m}_2$, *then*
$$\therefore \quad \cos\theta = \frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1|\cdot|\vec{m}_2|}.$$
- *Cartesian equations of two given lines be*
$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$
then $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{(\sqrt{a_1^2+b_1^2+c_1^2})(\sqrt{a_2^2+b_2^2+c_2^2})}.$
- *The shortest distance between two skew lines* $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ *and* $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ *is given by*

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|.$$

- *If the lines* $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ *and* $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ *intersect. Then, the shortest distance between them is zero.*
- *The shortest distance between the skew lines* $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ *and* $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ *is given by*

$$S.D. = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}}$$

where $D = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}$.

- *If a plane makes intercepts of lengths a, b, c with the x-axis, y-axis and z-axis respectively, the equation of the plane is* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
- *If* $\hat{n}$ *is a unit vector normal to a given plane and p is the length of the perpendicular drawn from the origin to the plane then the vector equation of the plane is* $\vec{r}\cdot\hat{n} = p$.
- *The equation of a plane through the intersection of two planes* $\vec{r}\cdot\vec{n}_1 = q_1$ *and* $\vec{r}\cdot\vec{n}_2 = q_2$ *is given by* $\vec{r}\cdot(\vec{n}_1 + \lambda\vec{n}_2) = q_1 + \lambda\, q_2$.
- *The equation of a plane through the intersection of two planes* $a_1x + b_1y + c_1z + d_1 = 0$ *and* $a_2x + b_2y + c_2z + d_2 = 0$ *is given by* $(a_1x + b_1y + c_1z + d_1) + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0$.
- *The vector equation of a plane passing through a point A with position vector* $\vec{a}$ *and perpendicular to a given vector* $\vec{n}$ *is* $(\vec{r} - \vec{a})\cdot\vec{n} = 0$.
- *The Cartesian equation of a plane passing through a point A* (x_1, y_1, z_1) *and perpendicular to a line having direction ratios* n_1, n_2, n_3 *is* $(x - x_1)n_1 + (y - y_1)n_2 + (z - z_1)n_3 = 0$.
- *The vector equation of a plane passing through a given point with position vector* $\vec{a}$ *and parallel to two given vectors* $\vec{b}$ *and* $\vec{c}$ *is* $(\vec{r} - \vec{a})\cdot(\vec{b} \times \vec{c}) = 0$.
- *The equation of a plane passing through a given point A* (x_1, y_1, z_1) *and parallel to two given lines having direction ratios* b_1, b_2, b_3 *and* c_1, c_2, c_3 *is*

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

- *The vector equation of a plane passing through three non-collinear points with position vectors* $\vec{a}, \vec{b}, \vec{c}$ *is* $(\vec{r} - \vec{a})\cdot[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$.
- *The equation of a plane passing through three given non-collinear points* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)$ *is given by*

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

- *If* θ *be the angle between two planes* $\vec{r}\cdot\vec{n}_1 = q_1$ *and* $\vec{r}\cdot\vec{n}_2 = q_2$, *then*
  $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_1|}$.
- *Two planes* $\vec{r}\cdot\vec{n}_1 = q_1$ *and* $\vec{r}\cdot\vec{n}_2 = q_2$ *are perpendicular to each other*
  $\Leftrightarrow$ $\vec{n}_1 \perp \vec{n}_2$.
  $\Leftrightarrow$ $\vec{n}_1\cdot\vec{n}_2 = 0$.
- *Two planes* $\vec{r}\cdot\vec{n}_1 = q_1$ *and* $\vec{r}\cdot\vec{n}_2 = q_2$ *are parallel to each other*
  $\Leftrightarrow$ $\vec{n}_1 \parallel \vec{n}_2$.
  $\Leftrightarrow$ $\vec{n}_1 = \lambda\vec{n}_2$ *for some scalar* λ.
- *Let* θ *be the angle between two planes* $a_1x + b_1y + c_1z + d_1 = 0$ *and* $a_2x + b_2y + c_2z + d_2 = 0$
  Then, $\cos\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$.
- *Two planes* $a_1x + b_1y + c_1z + d_1 = 0$ *and* $a_2x + b_2y + c_2z + d_2 = 0$ *are perpendicular to each other*
  $\Leftrightarrow$ *their normals are perpendicular to each other.*
  $\Leftrightarrow$ *lines with direction ratios* a_1, b_1, c_1, *and* a_2, b_2, c_2 *are perpendicular to each other.*
  $\Leftrightarrow$ $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
- *Two planes* $a_1x + b_1y + c_1z + d_1 = 0$ *and* $a_2x + b_2y + c_2z + d_2 = 0$ *are parallel to each other,*
  $\Leftrightarrow$ *their normals are parallel to each other.*
  $\Leftrightarrow$ *lines with direction ratios* a_1, b_1, c_1, *and* a_2, b_2, c_2 *are parallel to each other.*

$\Leftrightarrow \quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}.$

- *If θ is the angle between the line* $\vec{r} = \vec{a} + \lambda\,\vec{b}$ *and the plane* $\vec{r}\cdot\vec{n} = q$ *then*

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}.$$

- *The line* $\vec{r} = \vec{a} + \lambda\,\vec{b}$ *and the plane* $\vec{r}\cdot\vec{n} = q$ *are*
  - *perpendicular if* $\vec{b} = t\vec{n}$ *for some scalar t.*
  - *parallel if* $\vec{b}\cdot\vec{n} = 0$.
- *If the line* $\vec{r} = \vec{a} + \lambda\,\vec{b}$ *and the plane* $\vec{r}\cdot\vec{n} = q$ *are parallel then the distance between them is* $\dfrac{|\vec{a}\cdot\vec{n} - q|}{|\vec{n}|}$.
- *If θ is the angle between the line*

$\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ *and the plane* $a_2x + b_2y + c_2z + d = 0$, *then*

$$\sin\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{(\sqrt{a_1^2 + b_1^2 + c_1^2})\,(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

- *The line* $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ *is perpendicular to the plane* $a_2x + b_2y + c_2z + d = 0$ *only if* $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.
- *The line* $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ *is parallel to the plane* $a_2x + b_2y + c_2z + d = 0$ *only if*

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

- *The length p of the perpendicular drawn from a point with position vector* $\vec{a}$ *to the plane* $\vec{r}\cdot\vec{n} = d$ *is given by*

$$p = \frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}.$$

- *The length p of the perpendicular from a point A* (x_1, y_1, z_1) *to the plane* $ax + by + cz + d = 0$ *is given by*

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

- *The two lines* $\vec{r} = \vec{a_1} + \lambda\,\vec{b_1}$ *and* $\vec{r} = \vec{a_2} + \mu\,\vec{b_2}$ *are coplanar if*

$(\vec{a_2} - \vec{a_1})\cdot(\vec{b_1} \times \vec{b_2}) = 0$, *i.e.,* $[\vec{a_2} - \vec{a_1}\ \ \vec{b_1}\ \ \vec{b_2}] = 0$.

Also, the equation of the plane containing both these lines is

$(\vec{r} - \vec{a_1})\cdot(\vec{b_1} \times \vec{b_2}) = 0$ *or* $(\vec{r} - \vec{a_2})\cdot(\vec{b_1} \times \vec{b_2}) = 0$.

- *The lines* $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ *and* $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ *are coplanar*

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

And, the equation of the plane containing both these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or}$$

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Find the equation of the plane which is parallel to the plane $2x - 3y + z + 8 = 0$ and which passes through the point $(-1, 1, 2)$.
2. Write the vector equation of $\dfrac{x - 5}{3} = \dfrac{y + 4}{7} = \dfrac{6 - z}{2}$.
3. Find the angle between the pair of lines $\dfrac{-x + 2}{-2} = \dfrac{y - 1}{7} = \dfrac{z + 3}{-3}$ and $\dfrac{x + 2}{-1} = \dfrac{2y - 8}{4} = \dfrac{z - 5}{4}$.
4. For what values of m, the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $m\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear?
5. Find the direction cosines of the line passing through the following points $(-2, 4, -5)$, $(1, 2, 3)$.

SHORT ANSWER TYPE

6. Find the value of λ for which the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear.

7. Find the vector and Cartesian equations of the plane which passes through the point (5, 2, –4) and perpendicular to the line with direction ratios 2, 3, –1.

8. Find the angle between the lines $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$.

9. Find the direction cosines of the unit vector, perpendicular to the given plane $\vec{r}\cdot(3\hat{i} - 5\hat{j} + 3\hat{k}) + 3 = 0$, passing through the origin.

10. Find the angle between the two planes $3x - 6y + 2z - 7 = 0$ and $2x + 2y - 2z - 5 = 0$.

LONG ANSWER TYPE - I

11. Find the shortest distance between the lines L_1 and L_2 whose vector equations are

$L_1 : \vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$

$L_2 : \vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

12. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}\cdot(3\hat{i} - \hat{j} - 4\hat{k}) = 0$ whose perpendicular distance from the origin is unity.

13. Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

14. Find the equation of a line passing through the point (1, 2, –4) and perpendicular to two lines

$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ and

$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

15. Find the angle between two diagonals of a cube.

LONG ANSWER TYPE - II

16. Find the equations of the line passing through the points $A(0, 6, -9)$ and $B(-3, -6, 3)$. If D is the foot of the perpendicular drawn from a point $C(7, 4, -1)$ on the line AB then find the coordinates of D and the equations of the line CD.

17. Find the length and the equation of the line of shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

18. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r}\cdot(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and passing through the point (2, 1, 3).

19. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also, find their point of intersection.

20. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of parallelogram $ABCD$. Find the vector equations of sides AB and BC and also find coordinates of point D.

SOLUTIONS

1. Any plane parallel to the given plane is
$2x - 3y + z + k = 0$ for some constant k.
If it passes through the point $A(-1, 1, 2)$, then
$2 \times (-1) - 3 \times 1 + 2 + k = 0$
$\Rightarrow (-2 - 3 + 2) + k = 0 \Rightarrow -3 + k = 0 \Rightarrow k = 3$
Hence, required equation of plane is
$2x - 3y + z + 3 = 0$.

2. The line passes through point (5, –4, 6) and dr's of the line are 3, 7, –2.

$\therefore$ Vector equation is $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$

3. D.r.'s of lines are 2, 7, –3 and –1, 2, 4
As $2 \times (-1) + 7 \times 2 - 3 \times 4 = 0$, so lines are perpendicular.
Hence, required angle is 90°.

4. Let given vectors are $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = m\hat{i} + 6\hat{j} - 8\hat{k}$.

We know that, vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are said to be collinear, if $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$.

$\therefore \frac{m}{2} = \frac{6}{-3} = \frac{-8}{4} \Rightarrow \frac{m}{2} = -2 \Rightarrow m = -4$

5. D.r.'s are 1+ 2, 2 – 4, 3 + 5, *i.e.*, 3, –2, 8.
Dividing by $\sqrt{9+4+64} = \sqrt{77}$; dc's are $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

6. The equation of the line AB is

$$\frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2}$$

$$\Rightarrow \quad \frac{x+1}{-3}=\frac{y-3}{-1}=\frac{z-2}{-4} \qquad ...(i)$$

Since the points A, B and C are collinear, so the point $C(5, 5, \lambda)$ lies on (i).

$$\therefore \quad \frac{5+1}{-3}=\frac{5-3}{-1}=\frac{\lambda-2}{-4} \Rightarrow \frac{\lambda-2}{-4}=-2$$

$$\Rightarrow \lambda - 2 = 8 \quad \Rightarrow \quad \lambda = 10.$$

7. The plane passes through the point $A(5, 2, -4)$ whose position vector is $\vec{a}=(5\hat{i}+2\hat{j}-4\hat{k})$ and the normal vector $\vec{n}$ perpendicular to the plane is $\vec{n}=(2\hat{i}+3\hat{j}-\hat{k})$.

So, the vector equation of the plane is $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$$\Rightarrow \vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})=(5\hat{i}+2\hat{j}-4\hat{k})\cdot(2\hat{i}+3\hat{j}-\hat{k})$$
$$= (10 + 6 + 4) = 20$$

Hence, the required vector equation of the plane is

$\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})=20$ ...(i)

Cartesian Form :

Let $\vec{r}=(x\hat{i}+y\hat{j}+z\hat{k})$, then from eqn. (i)

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+3\hat{j}-\hat{k})=20 \Rightarrow 2x+3y-z=20.$

Hence, the required Cartesian equation is $2x + 3y - z = 20$.

8. Given lines are $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$

and $\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$

On comparing with $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$.

we get $\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$ and $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ The angle between the lines is given by

$$\cos\theta=\left|\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|=\left|\frac{(3\hat{i}+2\hat{j}+6\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{3^2+2^2+6^2}\sqrt{1^2+2^2+2^2}}\right|$$

$$=\left|\frac{3+4+12}{\sqrt{9+4+36}\sqrt{1+4+4}}\right|=\frac{19}{\sqrt{49}\sqrt{9}}=\frac{19}{21}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{19}{21}\right)$$

9. Given equation of plane can be written as

$\vec{r}\cdot(-3\hat{i}+5\hat{j}-3\hat{k})=3$ $\quad [\because d>0]$...(i)

Now, $|-3\hat{i}+5\hat{j}-3\hat{k}|=\sqrt{(-3)^2+(5)^2+(-3)^2}$

$$=\sqrt{9+25+9}=\sqrt{43}$$

On dividing both sides of Eq. (i) by $\sqrt{43}$, we get

$$\vec{r}\cdot\frac{(-3\hat{i}+5\hat{j}-3\hat{k})}{\sqrt{43}}=\frac{3}{\sqrt{43}}$$

$$\Rightarrow \quad \vec{r}\cdot\left(\frac{-3}{\sqrt{43}}\hat{i}+\frac{5}{\sqrt{43}}\hat{j}-\frac{3}{\sqrt{43}}\hat{k}\right)=\frac{3}{\sqrt{43}}$$

which is of the form $\vec{r}\cdot\hat{n}=d$, where $\hat{n}$ is a unit vector perpendicular to the plane through the origin.

$$\therefore \quad \hat{n}=\left(-\frac{3}{\sqrt{43}}\hat{i}+\frac{5}{\sqrt{43}}\hat{j}-\frac{3}{\sqrt{43}}\hat{k}\right)$$

Hence, direction cosines of $\hat{n}$ are

$$\left(-\frac{3}{\sqrt{43}},\frac{5}{\sqrt{43}},-\frac{3}{\sqrt{43}}\right).$$

10. Given equations of planes are $3x - 6y + 2z - 7 = 0$ and $2x + 2y - 2z - 5 = 0$

On comparing with $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, we get

$a_1 = 3$, $b_1 = -6$, $c_1 = 2$ and $a_2 = 2$, $b_2 = 2$, $c_2 = -2$

Now, the angle between two planes is

$$\theta=\cos^{-1}\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$=\cos^{-1}\left|\frac{3\times2+(-6)\times2+2\times(-2)}{\sqrt{3^2+(-6)^2+(2)^2}\sqrt{2^2+2^2+(-2)^2}}\right|$$

$$=\cos^{-1}\left|\frac{6-12-4}{\sqrt{9+36+4}\sqrt{4+4+4}}\right|=\cos^{-1}\left|\frac{-10}{\sqrt{49}\sqrt{12}}\right|$$

$$=\cos^{-1}\left(\frac{10}{7\times2\sqrt{3}}\right)=\cos^{-1}\left(\frac{5\sqrt{3}}{21}\right)$$

11. Comparing the given equations with the standard equations $\vec{r}=\vec{a}_1+\lambda\,\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\,\vec{b}_2$, we have

$\vec{a}_1=(\hat{i}+\hat{j}),\ \vec{b}_1=(2\hat{i}-\hat{j}+\hat{k})$

$\vec{a}_2=(2\hat{i}+\hat{j}-\hat{k})$ and $\vec{b}_2=(3\hat{i}-5\hat{j}+2\hat{k})$

$$\therefore \quad (\vec{a}_2-\vec{a}_1)=(2\hat{i}+\hat{j}-\hat{k})-(\hat{i}+\hat{j})=(\hat{i}-\hat{k})$$

and $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 3 & -5 & 2\end{vmatrix}=(-2+5)\hat{i}-(4-3)\hat{j}+(-10+3)\hat{k}$

$$=(3\hat{i}-\hat{j}-7\hat{k})$$

$$\therefore \quad |\vec{b}_1\times\vec{b}_2|=\sqrt{3^2+(-1)^2+(-7)^2}=\sqrt{59}$$

$$\therefore \quad \text{Shortest distance}=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$$=\frac{|(\hat{i}-\hat{k})\cdot(3\hat{i}-\hat{j}-7\hat{k})|}{\sqrt{59}}$$

$$=\frac{|3-0+7|}{\sqrt{59}}=\frac{10\sqrt{59}}{59}\text{ units.}$$

12. Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ then the given equation of planes are

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+3\hat{j})-6=0 \Rightarrow x+3y-6=0$ ...(i)

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}-4\hat{k})=0 \Rightarrow 3x-y-4z=0$ ...(ii)

The equation of any plane passing through the line of intersection of the given planes is given by $(x + 3y - 6) + \lambda(3x - y - 4z)$ for some real number λ.

$\Rightarrow (1 + 3\lambda)x + (3 - \lambda)y - 4\lambda z - 6 = 0.$ ...(iii)

Length of perpendicular from origin to plane (iii) is given as 1.

$$\therefore \frac{|0+0-0-6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}=1$$

$\Rightarrow (1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2 = 36$

$\Rightarrow 26\lambda^2 = 26 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$

Putting $\lambda = 1$ in (iii), we get

$4x + 2y - 4z - 6 = 0 \Rightarrow 2x + y - 2z - 3 = 0.$

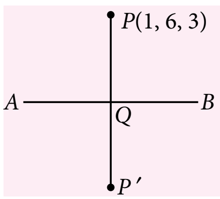
Putting $\lambda = -1$ in (iii), we get

$-2x + 4y + 4z - 6 = 0 \Rightarrow x - 2y - 2z + 3 = 0.$

Hence, the required equations of the plane are

$2x + y - 2z - 3 = 0$ and $x - 2y - 2z + 3 = 0$

13. Let the given point be $P(1, 6, 3)$ and the given line be AB whose equation is



$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$$

Draw $PQ \perp AB$ and produce it to P' such that $QP' = PQ$.

Then P' is the image of P in the line AB. Coordinates of Q are $(r, 2r + 1, 3r + 2)$ for some value of r.

D.r.'s of PQ are $r - 1, 2r + 1 - 6, 3r + 2 - 3$

i.e., $r - 1, 2r - 5, 3r - 1$

D.r.'s of AB are 1, 2, 3.

Since $PQ \perp AB$

$\therefore 1(r - 1) + 2(2r - 5) + 3(3r - 1) = 0$

$\Rightarrow 14r - 14 = 0 \Rightarrow r = 1$

$\therefore$ Coordinates of Q are (1, 3, 5).

Let the coordinates of P' be (α, β, γ).

Since Q is the mid-point of PP',

$$\therefore \frac{\alpha+1}{2}=1, \frac{\beta+6}{2}=3, \frac{\gamma+3}{2}=5$$

$\Rightarrow \alpha = 1, \beta = 0, \gamma = 7$

Hence, the coordinates of P' are (1, 0, 7)

14. Let line through the point (1, 2, –4) be

$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda'\vec{m}$, where λ' is a scalar ...(i)

Line (i) is perpendicular to lines

$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$

and $\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$

$\therefore (3\hat{i}-16\hat{j}+7\hat{k})\cdot\vec{m}=0$ and $(3\hat{i}+8\hat{j}-5\hat{k})\cdot\vec{m}=0$

$$\Rightarrow \vec{m}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}=24\hat{i}+36\hat{j}+72\hat{k}$$

$\therefore$ From (i) line is

$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda'(24\hat{i}+36\hat{j}+72\hat{k})$

or $\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda''(2\hat{i}+3\hat{j}+6\hat{k})$,

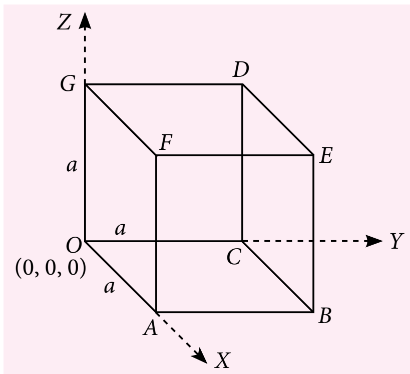
where $\lambda'' = 12\lambda'$ is a scalar

15. Let each side of the cube be a. Let us take O as the origin and OA, OC, OG as the x, y, z axis respectively. The coordinates of O and E are respectively (0, 0, 0) and (a, a, a).

Therefore the direction cosines of OE are

$$\left(\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}\right)$$

i.e., $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



The coordinates of A and D are respectively $(a, 0, 0)$ and $(0, a, a)$.

Therefore the direction cosines of AD are

$$\left(\frac{-a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}\right)$$

i.e., $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

If θ be the angle between the diagonals OE and AD, then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$$= \frac{1}{\sqrt{3}} \cdot \left(-\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

Hence $\theta = \cos^{-1}\left(\frac{1}{3}\right)$

Thus, the angle between any two diagonals is $\cos^{-1}\left(\frac{1}{3}\right)$.

16. The equation of line AB is

$$\frac{x-0}{-3-0} = \frac{y-6}{-6-6} = \frac{z+9}{3+9} \Rightarrow \frac{x}{-3} = \frac{y-6}{-12} = \frac{z+9}{12}$$

$$\Rightarrow \frac{x}{1} = \frac{y-6}{4} = \frac{z+9}{-4} = a\,(\text{say}) \qquad \text{...(i)}$$

From (i), we get $x = a$, $y = 4a + 6$ and $z = -4a - 9$.

So, the coordinates of D are $D(a, 4a + 6, -4a - 9)$ for some particular value of a.

D.r.'s of AB are 1, 4, –4.

D.r.'s of CD are $(a - 7)$, $(4a + 2)$, $(-4a - 8)$.

Since $AB \perp CD$, we have

$1(a - 7) + 4(4a + 2) - 4(-4a - 8) = 0$

$\Rightarrow\ 33a = -33 \quad \Rightarrow \quad a = -1.$

Putting $a = -1$, we get the coordinates of D as $D(-1, 2, -5)$.

The equation of the line CD is

$$\frac{x-7}{-1-7} = \frac{y-4}{2-4} = \frac{z+1}{-5+1} \Rightarrow \frac{x-7}{-8} = \frac{y-4}{-2} = \frac{z+1}{-4}$$

$$\Rightarrow \frac{x-7}{4} = \frac{y-4}{1} = \frac{z+1}{2}.$$

17. The equations of the given lines are

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = \lambda\,(\text{say}) \qquad \text{...(i)}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \mu\,(\text{say}) \qquad \text{...(ii)}$$

Any point on (i) is $P(7\lambda - 1, -6\lambda - 1, \lambda - 1)$.

Any point on (ii) is $Q(\mu + 3, -2\mu + 5, \mu + 7)$.

The direction ratios of PQ are

$(\mu - 7\lambda + 4, -2\mu + 6\lambda + 6, \mu - \lambda + 8)$.

Now, PQ will be the shortest distance between (i) and (ii) only when PQ is perpendicular to both (i) and (ii).

$\therefore\ 7(\mu - 7\lambda + 4) - 6(-2\mu + 6\lambda + 6) + 1\cdot(\mu - \lambda + 8) = 0$

and $1\cdot(\mu - 7\lambda + 4) - 2(-2\mu + 6\lambda + 6) + 1\cdot(\mu - \lambda + 8) = 0$

$\Rightarrow\ 20\mu - 86\lambda = 0$ and $6\mu - 20\lambda = 0$

$\Rightarrow\ 10\mu - 43\lambda = 0$ and $3\mu - 10\lambda = 0$

$\Rightarrow\ \lambda = 0$ and $\mu = 0$.

$\therefore\ PQ$ will be the line of shortest distance when $\lambda = 0$ and $\mu = 0$.

Putting $\lambda = 0$ and $\mu = 0$, we get the points $P(-1, -1, -1)$ and $Q(3, 5, 7)$.

$\therefore$ Shortest distance $= |PQ|$

$$= \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$$

$$= \sqrt{(4)^2 + (6)^2 + (8)^2} = \sqrt{16 + 36 + 64}$$

$= \sqrt{116} = 2\sqrt{29}$ units.

Clearly, the equation of the line of shortest distance is the equation of line PQ given by

$$\frac{x-3}{3+1} = \frac{y-5}{5+1} = \frac{z-7}{7+1}$$

$$\Rightarrow \frac{x-3}{4} = \frac{y-5}{6} = \frac{z-7}{8} \text{ or } \frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$$

18. The equations of the given planes are $\vec{r}\cdot(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r}\cdot(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$.

These are of the form $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$, where $\vec{n}_1 = (2\hat{i} + 2\hat{j} - 3\hat{k})$, $\vec{n}_2 = (2\hat{i} + 5\hat{j} + 3\hat{k})$, $q_1 = 7$ and $q_2 = 9$.

The equation of the plane through the intersection of the given planes is given by

$\vec{r}\cdot(\vec{n}_1 + \lambda\vec{n}_2) = (q_1 + \lambda q_2)$ for some real number λ

$$\Rightarrow\ \vec{r}\cdot[(2\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k})] = (7 + 9\lambda)$$

$$\Rightarrow\ \vec{r}\cdot[(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (-3 + 3\lambda)\hat{k}] = (7 + 9\lambda) \qquad \text{...(i)}$$

If the plane (i) passes through the point $A(2, 1, 3)$, then $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k})$ must satisfy it.

$$\therefore \quad (2\hat{i}+\hat{j}+3\hat{k})\cdot[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j} +(-3+3\lambda)\hat{k}]=(7+9\lambda)$$

$$\Rightarrow \quad 2(2+2\lambda)+1\cdot(2+5\lambda)+3(-3+3\lambda)=7+9\lambda$$

$$\Rightarrow \quad (4+4\lambda)+(2+5\lambda)+(-9+9\lambda)=7+9\lambda$$

$$\Rightarrow \quad 9\lambda = 10 \quad \Rightarrow \quad \lambda = \frac{10}{9}$$

Putting $\lambda = \frac{10}{9}$ in (i), we get the required equation of the plane as

$$\vec{r}\cdot\left[\left(2+\frac{20}{9}\right)\hat{i}+\left(2+\frac{50}{9}\right)\hat{j}+\left(-3+\frac{30}{9}\right)\hat{k}\right]=(7+10)$$

$$\Rightarrow \quad \vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153.$$

19. Let the given lines are

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=\lambda \qquad \text{...(i)}$$

and $$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=\mu \qquad \text{...(ii)}$$

Then, any point on line (i) is

$P(3\lambda - 1, 5\lambda - 3, 7\lambda - 5)$ ...(iii)

and any point on line (ii) is

$Q(\mu + 2, 3\mu + 4, 5\mu + 6)$ ...(iv)

If lines (i) and (ii) intersect, then these points must coincide.

$\therefore \quad 3\lambda - 1 = \mu + 2,\ 5\lambda - 3 = 3\mu + 4$ and $7\lambda - 5 = 5\mu + 6$

$\Rightarrow \quad 3\lambda - \mu = 3,$ ...(v)

$5\lambda - 3\mu = 7$ ...(vi)

and $7\lambda - 5\mu = 11$ ...(vii)

On multiplying Eq. (v) by 3 and then subtracting Eq. (vi) from it, we get

$9\lambda - 3\mu - 5\lambda + 3\mu = 9 - 7$

$$\Rightarrow \quad 4\lambda = 2 \ \Rightarrow \ \lambda = \frac{1}{2}$$

On putting the value of λ in Eq. (v), we get

$$3\times\frac{1}{2}-\mu=3$$

$$\Rightarrow \quad \frac{3}{2}-\mu=3 \Rightarrow \mu=-\frac{3}{2}$$

On putting the values of λ and μ in Eq. (vii), we get

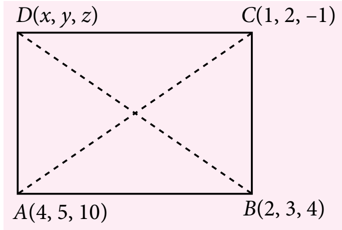
$$7\times\frac{1}{2}-5\left(-\frac{3}{2}\right)=11 \ \Rightarrow \ \frac{7}{2}+\frac{15}{2}=11 \ \Rightarrow \ \frac{22}{2}=11$$

$\Rightarrow \quad 11 = 11$, which is true.

Hence, lines (i) and (ii) intersect and their point of intersection is

$$P\left(3\times\frac{1}{2}-1, 5\times\frac{1}{2}-3, 7\times\frac{1}{2}-5\right) \ i.e. \ P\left(\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\right)$$

20. Firstly, we find vector equation of AB, where $A(4, 5, 10)$ and $B(2, 3, 4)$.



We know that, vector equation of a line passing through two points is given by

$$\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a}) \qquad \text{...(i)}$$

where $\vec{a}$ and $\vec{b}$ are the position vectors of points through which the line is passing.

Here, position vectors are

$$\vec{a}=\overrightarrow{OA}=4\hat{i}+5\hat{j}+10\hat{k}$$

and $\vec{b}=\overrightarrow{OB}=2\hat{i}+3\hat{j}+4\hat{k}$

From Eq. (i), the required equation of line AB is

$$\vec{r}=(4\hat{i}+5\hat{j}+10\hat{k})+\lambda[2\hat{i}+3\hat{j}+4\hat{k}-(4\hat{i}+5\hat{j}+10\hat{k})]$$

$$\Rightarrow \quad \vec{r}=(4\hat{i}+5\hat{j}+10\hat{k})-\lambda(2\hat{i}+2\hat{j}+6\hat{k})$$

Similarly, vector equation of line BC, where $B(2, 3, 4)$ and $C(1, 2, -1)$ is

$$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+\mu[\hat{i}+2\hat{j}-\hat{k}-(2\hat{i}+3\hat{j}+4\hat{k})]$$

$$\Rightarrow \quad \vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})-\mu(\hat{i}+\hat{j}+5\hat{k})$$

Now, Mid-point of diagonal BD = Mid-point of diagonal AC

[$\because$ diagonal of a parallelogram bisect each other]

$$\therefore \quad \left(\frac{x+2}{2},\frac{y+3}{2},\frac{z+4}{2}\right)=\left(\frac{4+1}{2},\frac{5+2}{2},\frac{10-1}{2}\right)$$

On comparing corresponding coordinates, we get

$$\frac{x+2}{2}=\frac{5}{2},\ \frac{y+3}{2}=\frac{7}{2} \text{ and } \frac{z+4}{2}=\frac{9}{2}$$

$\Rightarrow \quad x = 3, y = 4$ and $z = 5$

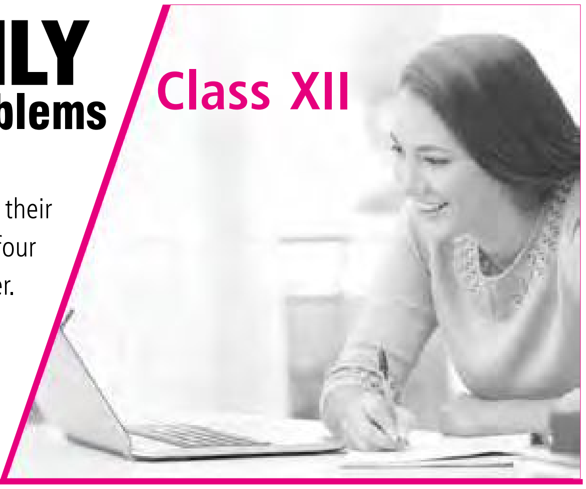
$\therefore \quad$ Coordinates of point $D = (x, y, z) = (3, 4, 5)$

MPP-8 CLASS XI ANSWER KEY

1.	(c)	2.	(d)	3.	(b)	4.	(b)	5.	(a)
6.	(d)	7.	(a,d)	8.	(c)	9.	(a,b,c,d)	10.	(a,b,c)
11.	(a,c)	12.	(a, b)	13.	(b,d)	14.	(a)	15.	(b)
16.	(a)	17.	(3)	18.	(5)	19.	(6)	20.	(9)

MPP-8 MONTHLY Practice Problems

Class XII



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Differential Equations

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Solve : $\dfrac{ds}{dx}+x^2=x^2\,e^{3s}$

(a) $(1-e^{-3s})=Ae^{x^3}$ (b) $(1+e^{-3s})=Ae^{x^3}$

(c) $(1-e^{3s})=Ae^{x^3}$ (d) $(1-e^{-3s})=Ae^{-x^3}$

2. Solution of the differential equation

$\dfrac{dy}{dx}=\dfrac{y(x-y\log y)}{x(x\log x-y)}$ is

(a) $\dfrac{\log x}{x}-\dfrac{\log y}{y}=c$ (b) $\dfrac{\log x}{x}+\dfrac{\log y}{y}=c$

(c) $\dfrac{x\log x+y\log y}{xy}=c$ (d) $\dfrac{x\log x-y\log y}{xy}=c$

3. Solve the differential equation.

$\left[xe^{y/x}-y\sin\dfrac{y}{x}\right]dx+x\sin\dfrac{y}{x}dy=0$

(a) $\dfrac{1}{2}e^{-y/x}\left(\sin\dfrac{y}{x}+\cos\dfrac{y}{x}\right)=\log kx$

(b) $\dfrac{1}{2}e^{y/x}\left(\sin\dfrac{y}{x}+\cos\dfrac{y}{x}\right)=\log kx$

(c) $e^{y/x}\left(\sin\dfrac{y}{x}+\cos\dfrac{y}{x}\right)=\log kx$

(d) $e^{-y/x}\left(\sin\dfrac{y}{x}+\cos\dfrac{y}{x}\right)=\log kx$

4. Solve : $(2x+y+3)dx=(2y+x+1)dy$

(a) $(x+y+4/3)\,(x-y+2)^3=c$

(b) $(x+y+4/3)\,(x+y+2)^3=c$

(c) $(x-y+4/3)\,(x-y+2)^3=c$

(d) $(x-y-4/3)\,(x-y-2)^3=c$

5. Solve : $\dfrac{dy}{dx}+\dfrac{y}{(1-x^2)^{3/2}}=\dfrac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

(a) $y=\dfrac{x}{1-x^2}+ce^{x/\sqrt{1-x^2}}$

(b) $y=\dfrac{-x}{\sqrt{1-x^2}}+ce^{x/\sqrt{1-x^2}}$

(c) $y=\dfrac{x}{\sqrt{1-x^2}}-ce^{x\sqrt{1-x^2}}$

(d) none of these

6. Solve the differential equation

$\cos^2 x\dfrac{dy}{dx}-(\tan 2x)\,y=\cos^4 x,\ |x|<\dfrac{\pi}{4}$

where $y\left(\dfrac{\pi}{6}\right)=\dfrac{3\sqrt{3}}{8}$.

(a) $y=\dfrac{1}{2}\cdot\dfrac{\sin 2x}{1-\tan^2 x}$ (b) $y=\dfrac{1}{2}\cdot\dfrac{\sin 2x}{1+\tan^2 x}$

(c) $y=\dfrac{\sin 2x}{1-\tan^2 x}$ (d) $y=\dfrac{\sin 2x}{1+\tan^2 x}$

One or More Than One Option(s) Correct Type

7. The solution of $y+px=p^2x^4,\ p=\dfrac{dy}{dx}$, is equal to

(a) $y+cx=c^2x^4$

(b) $xy=c^2\,x-c$

(c) obtained by putting $p=\dfrac{c}{x}$ in the given equation.

(d) obtained by putting $p=\dfrac{c}{x^2}$ in the given equation.

8. The equation of the curve satisfying the differential equation, $y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-x=0$ can be a

(a) circle (b) straight line
(c) parabola (d) ellipse

9. The solution of the equation $\frac{dy}{dx}$ – $3y$ = sin$2x$ is equal to

(a) $ye^{-3x}=-\frac{1}{13}e^{-3x}(2\cos 2x+3\sin 2x)+c$

(b) $y=-\frac{1}{13}(2\cos 2x+3\sin 2x)+ce^{3x}$

(c) $y=\left\{-\frac{1}{\sqrt{13}}\right\}\cos\left(2x-\tan^{-1}\left(\frac{3}{2}\right)\right)+ce^{3x}$

(d) $y=\left\{-\frac{1}{\sqrt{13}}\right\}\sin\left(2x+\tan^{-1}\left(\frac{2}{3}\right)\right)+ce^{3x}$

(where c is arbitrary constant)

10. Solution of the differential equation

$x\cos x\left(\frac{dy}{dx}\right)+y(x\sin x+\cos x)=1$, is equal to

(a) $xy = \sin x + c\cos x$
(b) $xy\sec x = \tan x + c$
(c) $xy + \sin x + c\cos x = 0$
(d) none of these
(where c is arbitrary constant)

11. The solutions of $v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$, where $u = y$ and $v = xy$ are equal to

(a) $y = 0$ (b) $y = -4x$
(c) $xy = cy + c^2$ (d) $x^2y = cy + c^2$

12. The solution of $\frac{dy}{dx}+x=xe^{(n-1)y}$ is

(a) $\frac{1}{n-1}\log\left(\frac{e^{(n-1)y}-1}{e^{(n-1)y}}\right)=\frac{x^2}{2}+c$

(b) $e^{(n-1)y}=ce^{(n-1)y+(n-1)\frac{x^2}{2}}+1$

(c) $\log\left(\frac{e^{(n-1)y}-1}{(n-1)e^{(n-1)y}}\right)=x^2+c$

(d) $e^{(n-1)y}=ce^{(n-1)y\frac{2}{2+x}}+1$

13. The solution of $\frac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$, is equal to

(a) $\sqrt{x^2+y^2}=a\left(\sin\left(\tan^{-1}\frac{y}{x}\right)+c\right)$

(b) $\sqrt{x^2+y^2}=a\left(\cos\left(\tan^{-1}\frac{y}{x}\right)+c\right)$

(c) $\sqrt{x^2+y^2} = a\left(\tan\left(\sin^{-1}\frac{y}{x}\right)+c\right)$

(d) $y = x\tan\left(c+\sin^{-1}\frac{1}{a}\sqrt{x^2+y^2}\right)$

Comprehension Type

$y = xp + f(p)$, where $p = \frac{dy}{dx}$, is known as Clairaut's Equation. To solve given equation, differentiate it w.r. to x, which gives either

$\frac{dp}{dx} = 0 \Rightarrow p = c$ or $x + f'(p) = 0$

14. The number of solutions of the equation $f(x) = -1$ and the singular solutions of the equation

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 \text{ is}$$

(a) 1 (b) 2
(c) 4 (d) 0

15. The singular solution of the differential equation $y = mx + m - m^3$ where $m = \frac{dy}{dx}$ passes through the point

(a) (0, 0) (b) (0, 1)
(c) (1, 0) (d) (–1, 0)

Matrix Match Type

16. Match the differential equations with their solution.

	Column I		Column II
P.	$(1+x\sqrt{x^2+y^2})dx$ $+(-1+\sqrt{x^2+y^2})ydy$ $= 0$	1.	$xe^{\sin\frac{y}{x}} = c$
Q.	$(x - 1)\, dy + y\, dx$ $= x(x-1)\, y^{1/3}\, dx$	2.	$x - \frac{y^2}{2} + \frac{1}{3}(x^2+y^2)^{3/2} = c$
R.	$(y\,(1+x^{-1}) + \sin y)dx$ $+ (x + \log x$ $+ x\cos y)dy =$	3.	$y^{2/3}(x-1)^{2/3} = \frac{(x-1)^{8/3}}{4}$ $+\left(\frac{2}{5}\right)(x-1)^{5/3} + c$
S.	$x\cos\frac{y}{x}dy$ $= \left(y\cos\frac{y}{x} - x\right)dx$	4.	$xy + y\log x + x\sin y = c$

	P	Q	R	S
(a)	2	3	4	1
(b)	1	3	4	2
(c)	3	1	2	4
(d)	4	2	1	3

Integer Answer Type

17. If the curve satisfying $ydx + (x + x^2 y)dy = 0$ passes through (1, 1), then $-4\log y(4) + \frac{1}{y(4)}$ is equal to

18. A particle moves in a straight line with a velocity given by $\frac{dx}{dt} = x+1$ (x is the distance travelled). If the time taken by a particle to travel a distance of 99 m is λ, then the value of $4\lambda \log_{10} e$ must be

19. The largest value of $\frac{2}{\pi}c$ such that there exists a differentiable function $h(x)$ for $-c < x < c$ that is a solution of $y_1 = 1 + y^2$ with $h(0) = 0$ is

20. An object falling from rest in air is subject not only to the gravitational force but also to air resistance. Assume that the air resistance is proportional to the velocity with constant of proportionality as $k > 0$, and acts in a direction opposite to motion ($g = 9.8$ m/s^2). If the velocity cannot exceed A m/s, then the value of $\frac{10}{49}Ak$ is

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK	Check your score! If your score is	
No. of questions attempted	> 90%	EXCELLENT WORK ! You are well prepared to take the challenge of final exam.
No. of questions correct	90-75%	GOOD WORK ! You can score good in the final exam.
Marks scored in percentage	74-60%	SATISFACTORY ! You need to score more next time.
	< 60%	NOT SATISFACTORY! Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 180

JEE MAIN

1. Let S be the sum of all two-digit numbers each of which contains one odd and one even digits. Then S is divisible by

(a) 2 (b) 3 (c) 4 (d) 6

2. $OABC$ is a tetrahedron, $OA = 3$, $BC = 2$ and the shortest distance between OA and BC is 2. If the angle between OA and BC is 30°, then its volume is

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

3. The minimum value of

$f(x) = 2^{(\log_8 3)\cos^2 x} + 3^{(\log_8 2)\sin^2 x}$ is

(a) $2^{1-\log_8 \sqrt{3}}$ (b) $2^{\log_8 \sqrt{2}}$

(c) $3^{\log_8 \sqrt{2}}$ (d) $2^{1+\log_8 \sqrt{3}}$

4. If $f(z)=\dfrac{z+i}{z-i}$, $z \neq i$, $z_n = f(z_{n-1})$, $n \in N$ and $z=\dfrac{1}{5}+i$, then $z_{2011} = a + bi$, where $a + b =$

(a) 11 (b) 101 (c) 1011 (d) 2011

5. In triangle ABC, $AB = AC = m$ and D is a point on BC such that $BD = 9$, $DC = 21$ and $AD = n$. If m and n are integers then $m - n$ can be

(a) 2 (b) 5 (c) 6 (d) 7

JEE ADVANCED

6. Two sides of an isosceles triangle are parallel to the coordinate axes. If m_1 and m_2 are the slopes of the bisectors of the acute angles of the triangle, then $\dfrac{m_1}{m_2} =$

(a) $\sqrt{2}-1$ (b) 1

(c) $3-2\sqrt{2}$ (d) none of these

COMPREHENSION

The hexagon $ABCDEF$ is inscribed in a circle. If $AB = 11$, $BC = CD = DE = EF = FA = 4$, then

7. $AE =$

(a) 10 (b) $2\sqrt{14}$ (c) $3\sqrt{10}$ (d) $\sqrt{19}$

8. $AD =$

(a) 10 (b) $2\sqrt{14}$ (c) $3\sqrt{14}$ (d) $\sqrt{20}$

INTEGER TYPE

9. If $C_r = \dbinom{5}{r}$ and $N = \sum\limits_{r=2}^{5}(r-1)r\cdot C_{r-1}C_r$, then the sum of digits of N is

MATRIX MATCH

10. Match the following.

	List-I		List-II
P.	The value of x for the expression $12^x + 12^{x+1} = 3^x + 3^{x+1} + 3^{x+2}$ is	1.	0
Q.	If $\int_{\sqrt{2}}^{x}\dfrac{dt}{t\sqrt{t^2-1}} = \dfrac{\pi}{12}$, then x is	2.	2
R.	The number of roots of $5\cos 2\theta + 2\cos^2\dfrac{\theta}{2} + 1 = 0$ in $(0, \pi/2)$ is	3.	6
S.	If $n \in N$, then the highest integer m such that 2^m divides $3^{2n+2} - 8n - 9$ is	4.	1

	P	Q	R	S
(a)	4	2	1	3
(b)	2	4	3	1
(c)	1	2	4	3
(d)	2	3	1	4

See Solution Set of Maths Musing 179 on page no 88

JEE Main 2018

MOCK TEST PAPER

Series-6

Time: 1 hr 15 min.

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

Unit No. 6	Topic	Syllabus In Detail
	Differential calculus	Derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions. Derivative of implicit functions. derivative up to order two,
	Trigonometry	Inverse trigonometrical functions and their properties. Height and distance
	Vector Algebra	Vectors and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.
	Matrices and Determinants	Matrices as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of square matrix of order up to three, Properties of determinants, area of triangles using determinants , inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables

1. If $x=\sin t-\operatorname{cosec} t$, $y=\sin^5 t-\operatorname{cosec}^5 t$, then

$(x^2+4)\dfrac{d^2y}{dx^2}=$

(a) $x\dfrac{dy}{dx}-5y$ (b) $-x\dfrac{dy}{dx}+5y$

(c) $x\dfrac{dy}{dx}-25y$ (d) $-x\dfrac{dy}{dx}+25y$

2. The derivative of the function

$$f(x)=\cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x-3\sin x)\right\}+\sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\sin x+3\cos x)\right\}$$

with respect to $\sqrt{1+x^2}$ is

(a) $2x$ (b) $2\sqrt{1+x^2}$

(c) $\dfrac{2}{x}\sqrt{1+x^2}$ (d) $\dfrac{2x}{\sqrt{1+x^2}}$

3. Let y be an implicit function of x defined by $x^{2x}-2x^x\cot y-1=0$. Then $y'(1)=$

(a) -1 (b) 1 (c) ln2 (d) $-$ln2

4. If $f(x)=|x-2|$ and $g(x)=f(f(x))$, then $\sum\limits_{r=0}^{3}g'(2r-1)=$

(a) 0 (b) 1 (c) -1 (d) 2

5. Let $g(x)$ be the inverse of an invertible function $f(x)$, which is differentiable for all real x, then $g''(x)=$

(a) $-\dfrac{f''(x)}{(f'(x))^3}$ (b) $\dfrac{f'(x)f''(x)-(f'(x))^2}{f'(x)}$

(c) $\dfrac{f'(x)f''(x)-(f'(x))^2}{(f'(x))^2}$

(d) none of these

6. If $t=\dfrac{2\sqrt{2}-(1+\sqrt{3})}{\sqrt{3}-1}$ and $f(x)=\dfrac{2x}{1-x^2}$,

$g(x)=\dfrac{3x-x^3}{1-3x^2}$, then $\dfrac{d}{dt}(f(g(t)))$ at $t=R$ is

By : Sankar Ghosh, S.G.M.C, Mob : 09831244397.

(a) 1 (b) 2
(c) 0 (d) none of these

7. If $y = \log_x(\log_e x)(\log_e x)$, then $\frac{dy}{dx} =$

(a) $\frac{2}{x\log_x x\log_x x}$ (b) $\frac{1}{x\log_e x}$
(c) 0 (d) none of these

8. If $xy = e - e^y$, then $\left(\frac{d^2y}{dx^2}\right)_{x=0} =$

(a) $\frac{1}{e}$ (b) $\frac{1}{e^3}$
(c) $\frac{1}{e^2}$ (d) none of these

9. If $y^{y^{y^{\cdot^{\cdot^{\cdot\infty}}}}} = \log_e(x+\log(x+...))$, then $\frac{dy}{dx}$ at $(x = e^2 - 2, y = \sqrt{2})$ is

(a) $\frac{\log\left(\frac{e}{2}\right)}{2\sqrt{2}(e^2-1)}$ (b) $\frac{\log 2}{2\sqrt{2}(e^2-1)}$
(c) $\frac{\sqrt{2}\log\frac{e}{2}}{(e^2-1)}$ (d) none of these

10. If $P(x)$ is a polynomial of degree ≤ 2 such that $P(0) = 0$, $P(1) = 1$ and $P'(x) > 0\ \forall\ x \in [0, 1]$ then S, the set of such polynomials is
(a) ϕ (b) $\{(1-b)x^2 + bx, 0 < b < 3\}$
(c) $\{(1-b)x^2 + bx, 0 < b < \infty\}$
(d) none of these

11. If $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} + ...\right) + \cos^{-1}(1 + b + b^2 + ...) = \frac{\pi}{2}$, then
(a) $a = -3$ and $b = 1$ (b) $a = 1$ and $b = -1/3$
(c) $a = 1/6$ and $b = 1/2$ (d) none of these

12. If $\alpha = \sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{3}$ and $\beta = \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{3}$, then
(a) $\alpha > \beta$ (b) $\alpha = \beta$
(c) $\alpha < \beta$ (d) $\alpha + \beta = 2\pi$

13. If $[\cot^{-1}x] + [\cos^{-1}x] = 0$ where 'x' is non-negative real number and $[\cdot]$ denotes the greatest integer function, then complete set of x is
(a) (cos1, 1] (b) (cos1, cot1)
(c) (cot1, 1] (d) none of these

14. The set of the values of x satisfying
$2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ is

(a) [0, 1] (b) $\left[\frac{1}{\sqrt{2}}, 1\right]$
(c) $\left[0, \frac{1}{\sqrt{2}}\right]$ (d) [–1, 1]

15. If $A = 2\tan^{-1}(2\sqrt{2}-1)$ and $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$, then
(a) $A = B$ (b) $A < B$
(c) $A > B$ (d) none of these

16. The value of
$$\sin^{-1}\left[\cot\left(\sin^{-1}\sqrt{\left(\frac{2-\sqrt{3}}{4}\right)} + \cos^{-1}\frac{\sqrt{12}}{4}\right) + \sec^{-1}\sqrt{2}\right]$$
is
(a) 0 (b) $\pi/4$ (c) $\pi/6$ (d) $\pi/2$

17. If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, where $[\cdot]$ denotes the greatest function, then x belongs to the interval
(a) [tan sin cos 1, tan sin cos sin 1]
(b) (tan sin cos 1, tan sin cos sin 1)
(c) [1, –1]
(d) [sin cos tan 1, sin cos sin tan 1]

18. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ and $f(1) = 1$, $f(p + q) = f(p) \cdot f(q)\ \forall\ p, q \in R$ then
$$x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{x+y+z}{x^{f(1)} + y^{f(2)} + z^{f(3)}} =$$
(a) 0 (b) 1 (c) 2 (d) 3

19. Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining their feet and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and as seen from B are 60° and 45°. If $AB = 30$ m, then the distance between the flagstaffs (in metres) is
(a) $30 + 15\sqrt{3}$ (b) $45 + 15\sqrt{3}$
(c) $60 - 15\sqrt{3}$ (d) $60 + 15\sqrt{3}$

20. A pole 50 m high stands on a building 250 m high. To an observer at a height of 300 m, the building and the pole subtend equal angles. The horizontal distance of the observer from the pole is
(a) 25 m (b) 50 m
(c) $25\sqrt{6}$ m (d) $25\sqrt{3}$ m

21. For three vectors $\vec{a}, \vec{b}, \vec{c}$
$\left[\vec{a}\times(\vec{b}\times\vec{c}), \vec{b}\times(\vec{c}\times\vec{a}), \vec{c}\times(\vec{a}\times\vec{b})\right] =$
(a) 2 (b) 4 (c) 8 (d) 0

22. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that each is equally inclined to the other at an angle θ, then θ ∈
(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[\frac{\pi}{2}, \pi\right]$
(c) $\left[0, \frac{2\pi}{3}\right]$ (d) $[0, \pi]$

23. The number of unit vectors perpendicular to the vectors $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$ is
(a) 1 (b) 2
(c) 3 (d) infinite

24. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is
(a) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (b) $2\hat{i} + 3\hat{j} + 3\hat{k}$
(c) $-2\hat{i} + \hat{j} + 5\hat{k}$ (d) $2\hat{i} + \hat{j} + 5\hat{k}$

25. A vector of magnitude $\sqrt{2}$ coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to $\hat{i} + \hat{j} + \hat{k}$ is
(a) $-\hat{j} + \hat{k}$ (b) $\hat{i} - \hat{k}$
(c) $\hat{i} - \hat{j}$ (d) $\hat{i} - 2\hat{j} + \hat{k}$

26. If $f(x) = \begin{vmatrix} x & \sin x & \cos x \\ x^2 & \tan x & -x^3 \\ 2x & \sin 2x & 5x \end{vmatrix}$, then $\lim\limits_{x\to 0} \frac{f'(x)}{x}$ is
(a) 1 (b) 4 (c) 3 (d) −4

27. If $0 \le [x] < 2, -1 \le [y] < 1$ and $1 \le [z] < 3$, ([·] denotes the greatest integer function) then the maximum value of determinant $\Delta = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is
(a) 2 (b) 6
(c) 4 (d) none of these

28. The value of α for which the system of equations $\alpha x - 3y + z = 0$; $x + \alpha y + 3z = 1$; $3x + y + 5z = 2$ does not have unique solutions are
(a) $-1, \frac{11}{5}$ (b) $-1, \frac{-11}{5}$
(c) $1, \frac{-11}{5}$ (d) $1, \frac{11}{5}$

29. If for a square matrix $A = [a_{ij}]$, $a_{ij} = i^2 - j^2$ of even order, then
(a) A is a skew-symmetric and $|A| = 0$
(b) A is symmetric and $|A|$ is a square
(c) A is symmetric and $|A| = 0$
(d) none of these

30. The set of linear equations
$$x + 2\omega y + 3\omega^2 z = (1+\omega)(1-\omega)$$
$$2x + 3\omega y + \omega^2 z = \omega(\omega - 1)$$
$$3x + \omega y + 2\omega^2 z = \omega - 1$$
where $\omega \ne 1$ is a cube root of unity, has unique solution
(a) (1, 1, 1) (b) (1, −1, 1)
(c) (1, −1, −1) (d) (−1, −1, −1)

SOLUTIONS

1. **(d) :** Here $\frac{dy}{dx} = \frac{5(\sin^4 t \cdot \cos t + \operatorname{cosec}^5 t \cdot \cot t)}{\cos t + \operatorname{cosec} t \cdot \cot t}$
$$\frac{dy}{dx} = \frac{5(\sin^5 t + \operatorname{cosec}^5 t)}{\sin t + \operatorname{cosec} t} = y'$$
Now, $(y')^2 = \frac{25[(\sin^5 t - \operatorname{cosec}^5 t)^2 + 4]}{(\sin t - \operatorname{cosec} t)^2 + 4} = \frac{25(y^2+4)}{(x^2+4)}$
$\Rightarrow (x^2 + 4)(y')^2 = 25(y^2 + 4)$
Differentiating with respect to x and dividing both sides by $2 \cdot \frac{dy}{dx}$ *i.e.* $2y'$, we get $(x^2 + 4)y'' + xy' = 25y$

2. **(c) :** Let $u = \cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)\right\}$
$$+\sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\sin x + 3\cos x)\right\}$$
Again let $\cos\theta = \frac{2}{\sqrt{13}}$ and $\sin\theta = \frac{3}{\sqrt{13}}$
∴ $u = \cos^{-1}(\cos\theta \cos x - \sin\theta \sin x)$
$+ \sin^{-1}(\cos\theta \sin x + \sin\theta \cos x)$
$= \cos^{-1}\cos(\theta + x) + \sin^{-1}\sin(\theta + x)$
$\Rightarrow u = 2x + 2\theta \Rightarrow \frac{du}{dx} = 2$
Let $v = \sqrt{1+x^2} \Rightarrow \frac{dv}{dx} = \frac{1 \times 2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$
∴ $\frac{du}{dv} = \frac{2\sqrt{1+x^2}}{x}$

3. **(a)**

4. **(a) :** Given that $f(x) = |x - 2|$ and $g(x) = f(f(x))$
Now, $g(x) = ||x - 2| - 2|$

$$\therefore \quad g(x)=\begin{cases} |x| & \text{when} \quad x\le 2 \\ 4-x & \text{when} \quad 2<x\le 4 \\ x-4 & \text{when} \quad 4<x \end{cases}$$

Now, $\sum_{r=0}^{3} g'(2r-1)=g'(-1)+g'(1)+g'(3)+g'(5)$

$= -1 + 1 - 1 + 1 = 0$

5. (a) : Given that $g^{-1}(x) = f(x)$

$\Rightarrow x = g(f(x)) \Rightarrow g'(f(x))\, f'(x) = 1$

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow g''(f(x))\cdot f'(x) = -\dfrac{1}{\{f'(x)\}^2}\times f''(x)$

$\therefore g''(f(x)) = -\dfrac{f''(x)}{\{f'(x)\}^3}$

6. (c) : Here, $t=\dfrac{2\sqrt{2}-(1+\sqrt{3})}{\sqrt{3}-1}$

$=\dfrac{\{2\sqrt{2}-(1+\sqrt{3})\}(\sqrt{3}+1)}{3-1}$

$=\dfrac{2\sqrt{6}+2\sqrt{2}-4-2\sqrt{3}}{2}=\sqrt{6}+\sqrt{2}-2-\sqrt{3} = \tan 7\dfrac{1°}{2}$

Now, $f(x)=\dfrac{2x}{1-x^2} \Rightarrow f(t)=\dfrac{2t}{1-t^2} = \tan 15° = 2-\sqrt{3}$

$g(x)=\dfrac{3x-x^3}{1-3x^2} \Rightarrow g(t)=\dfrac{3t-t^3}{1-3t^2} = \tan 22\dfrac{1°}{2} = \sqrt{2}-1$

Now, $f(g(t)) = f(\sqrt{2}-1) = \tan 2\left(22\dfrac{1°}{2}\right) = \tan 45°$

$f(g(t)) = 1$

$\therefore (f(g(t)))' = 0.$

7. (b) : $y = \log_x(\log_e x)(\log_e x) = \dfrac{\log_e(\log_e x)}{\log_e x}\times \log_e x$

$\Rightarrow y = \log_e(\log_e x)$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{x\log_e x}$

8. (c) : $xy = e - e^y$ ... (i)

$\Rightarrow y + x\dfrac{dy}{dx} = -e^y\dfrac{dy}{dx}$

$\Rightarrow e^y\dfrac{dy}{dx}+x\dfrac{dy}{dx}+y=0$ ... (ii)

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{e}$ [$\because$ from (i), we get $y = 1$ if $x = 0$]

Differentiating (ii) again with respect to x, we get

$$e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2+x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Putting $x = 0$, $y = 1$, $\dfrac{dy}{dx}=-\dfrac{1}{e}$, we get $\dfrac{d^2y}{dx^2}=\dfrac{1}{e^2}$

9. (a) : Let $y^{y^{y^{\cdot^{\cdot^{\cdot^{\infty}}}}}} = \log_e(x+\log_e(x+....)) = v$

$\therefore y^v = \log_e(x + v) = v$

$\therefore y = v^{\frac{1}{v}}$ and $x = e^v - v$

Now, $\dfrac{dy}{dv}=v^{\frac{1}{v}}\left(\dfrac{1}{v}\cdot\dfrac{1}{v}-\dfrac{1}{v^2}\log v\right)=v^{\frac{1}{v}-2}(1-\log v)$

$\dfrac{dx}{dv}=e^v-1$

$\therefore \dfrac{dy}{dx}=\dfrac{v^{\frac{1}{v}-2}(1-\log v)}{e^v-1}$

Now, $v^{\frac{1}{v}}=\sqrt{2}$ and $e^v-v=e^2-2$

$\therefore v = 2$, $e^v - v = e^2 - 2 = x$, given, so true.

$\left(\dfrac{dy}{dx}\right)_{x=e^2-2,\ y=\sqrt{2}}=\left(\dfrac{dy}{dx}\right)_{v=2}=\dfrac{1-\log 2}{2\sqrt{2}(e^2-1)}$

10. (d) : Let $P(x) = ax^2 + bx + c$

$P(0) = 0 \Rightarrow c = 0$, $P(1) = 1 \Rightarrow a + b = 1$

$\therefore P(x) = (1 - b)x^2 + bx$

$P'(x) = 2(1 - b)x + b > 0$ for $0 < x < 1$

11. (b) : Given that

$$\sin^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}+....\right)+\cos^{-1}(1+b+b^2+...)=\frac{\pi}{2}$$

This is possible when $a-\dfrac{a^2}{3}+\dfrac{a^3}{9}+...=1+b+b^2+...$

Also, $-1\le a-\dfrac{a^2}{3}+\dfrac{a^3}{9}+...\le 1$

and $-1 \le 1 + b + b^2 + ... \le 1$

$\Rightarrow |b|<1 \Rightarrow |a|<3$

$\therefore \dfrac{a}{1+\frac{a}{3}}=\dfrac{1}{1-b} \Rightarrow \dfrac{3a}{a+3}=\dfrac{1}{1-b}$

There are infinitely many solution. But in the given option it satisfies option only when $a = 1$ and $b = -1/3$.

12. (c) : Given that $\alpha=\sin^{-1}\dfrac{\sqrt{3}}{2}+\sin^{-1}\dfrac{1}{3}$

and $\beta=\cos^{-1}\dfrac{\sqrt{3}}{2}+\cos^{-1}\dfrac{1}{3}$

$\therefore \alpha + \beta = \pi$ [$\because \sin^{-1}x + \cos^{-1}x = \pi/2\ \forall\ x$]

Also, $\alpha=\frac{\pi}{3}+\sin^{-1}\frac{1}{3}<\frac{\pi}{3}+\sin^{-1}\frac{1}{2}$

$\Rightarrow\ \alpha<\frac{\pi}{3}+\frac{\pi}{6}$ [$\because$ sinθ is increasing in $[0, \pi/2]$]

$\Rightarrow\ \alpha<\frac{\pi}{2}\ \Rightarrow\ \beta>\frac{\pi}{2}>\alpha$

13. (c) : $[\cot^{-1}x]+[\cos^{-1}x]=0$
$\Rightarrow$ $[\cot^{-1}x]=0$ and $[\cos^{-1}x]=0$
$\Rightarrow$ $x\in(\cot 1, \infty)$ and $x\in(\cos 1, 1]\Rightarrow x\in(\cot 1, 1]$

14. (b) : Let $\cos^{-1}x=\theta$
L.H.S. $=2\theta$ when $0\le\theta\le\pi$ ... (i)
R.H.S. $=\sin^{-1}(2\sin\theta\cos\theta)=\sin^{-1}(\sin 2\theta)=2\theta$

when $-\frac{\pi}{2}<(2\theta)<\frac{\pi}{2}\ \Rightarrow\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$ ...(ii)

By (i) and (ii), $0\le\theta\le\frac{\pi}{4}\ \Rightarrow\ 0\le\cos^{-1}x=\frac{\pi}{4}$

$\Rightarrow\ \frac{1}{\sqrt{2}}\le x\le 1\ \Rightarrow\ x\in\left[\frac{1}{\sqrt{2}}, 1\right]$

15. (c) : We have $A=2\tan^{-1}(2\sqrt{2}-1)=2\tan^{-1}(1.828)$
$>2\tan^{-1}\sqrt{3}$

$\Rightarrow\ A>\frac{2\pi}{3}$

We have $\sin^{-1}\left(\frac{1}{3}\right)<\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}\Rightarrow 3\sin^{-1}\left(\frac{1}{3}\right)<\frac{\pi}{2}$

Now, $3\sin^{-1}\left(\frac{1}{3}\right)=\sin^{-1}\left(3\frac{1}{3}-4\left(\frac{1}{3}\right)^3\right)=\sin^{-1}(0.852)$

$\Rightarrow\ 3\sin^{-1}\left(\frac{1}{3}\right)<\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

Now, $\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}(0.6)<\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow\ \sin^{-1}\left(\frac{3}{5}\right)<\frac{\pi}{3}$

Hence, $B=3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)<\frac{\pi}{3}+\frac{\pi}{3}<\frac{2\pi}{3}$

Hence $A>B$.

16. (a) : $\sin^{-1}\left[\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}\right)+\sec^{-1}\sqrt{2}\right]$

$=\sin^{-1}\left[\cot\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{1}{\sqrt{2}}\right)\right]$

$=\sin^{-1}(\cot(15^\circ+30^\circ+45^\circ))=\sin^{-1}(\cot 90^\circ)=0$

17. (a) : We have $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x]=1$

$\Rightarrow\ 1\le\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x\le\frac{\pi}{2}$

$\Rightarrow\ \sin 1\le\cos^{-1}\sin^{-1}\tan^{-1}x\le 1$

$\Rightarrow$ $\cos\sin 1\ge\sin^{-1}\tan^{-1}x\ge\cos 1$
$\Rightarrow$ $\sin\cos\sin 1\ge\tan^{-1}x\ge\sin\cos 1$
$\Rightarrow$ $\tan\sin\cos\sin 1\ge x\ge\tan\sin\cos 1$
Hence $x\in$ [tan sin cos1, tan sin cos sin1]

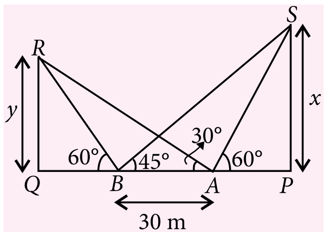
18. (c) : We know $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$

$\therefore\ \sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

$\Rightarrow\ \sin^{-1}x=\sin^{-1}y=\sin^{-1}z=\frac{\pi}{2}$

$\Rightarrow$ $x=y=z=1$
Also, $f(p+q)=f(p)\cdot f(q)\ \forall\ p, q\in R$ ...(i)
Given, $f(1)=1$
From (i), $f(1+1)=f(1)\cdot f(1)\ \Rightarrow\ f(2)=1^2=1$ ...(ii)
From (ii), $f(2+1)=f(2)\cdot(1)\ \Rightarrow f(3)=1^2\cdot 1=1^3=1$
Now, the given expression $=3-1=2$

19. (d) : Let x and y be the height of the flagstaffs respectively.



Then $AP=x\cot 60^\circ=\frac{x}{\sqrt{3}}$

$AQ=y\cot 30^\circ=y\sqrt{3},\ BP=x\cot 45^\circ=x$

$BQ=y\cot 60^\circ=\frac{y}{\sqrt{3}}$

$AB=BP-AP=x-\frac{x}{\sqrt{3}}\ \Rightarrow\ 30=x\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$

$\Rightarrow\ x=15(3+\sqrt{3})$

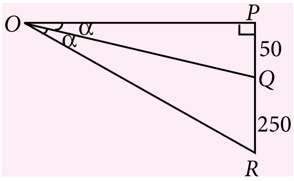
Similarly, $AB=AQ-BQ=y\sqrt{3}-\frac{y}{\sqrt{3}}$

$\Rightarrow\ 30=\frac{y(3-1)}{\sqrt{3}}\ \Rightarrow\ y=15\sqrt{3}$

So, $PQ=BP+BQ=x+\frac{y}{\sqrt{3}}=15(3+\sqrt{3})+15$

$=(60+15\sqrt{3})$ metres

20. (c) : Let PQ be the pole on the building QR and O be the observer, then
PQ = 50 m, QR = 250 m.



$\therefore\quad \tan\alpha=\frac{50}{x}$ and $\tan 2\alpha=\frac{300}{x}$

So, $\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{300}{x} \Rightarrow \frac{100}{x}\times\frac{x^2}{x^2-2500}=\frac{300}{x}$

$\Rightarrow x^2 = 3x^2 - 7500 \Rightarrow x^2 = 3750 \Rightarrow x=25\sqrt{6}$ m

21. (d) : $\vec a\times(\vec b\times\vec c)=(\vec a\cdot\vec c)\vec b-(\vec a\cdot\vec b)\vec c$

$\vec b\times(\vec c\times\vec a)=(\vec a\cdot\vec b)\vec c-(\vec b\cdot\vec c)\vec a$

$\vec c\times(\vec a\times\vec b)=(\vec b\cdot\vec c)\vec a-(\vec c\cdot\vec a)\vec b$

Taking the scalar triple product, we get

$(\vec a\cdot\vec c)(\vec a\cdot\vec b)(\vec b\cdot\vec c)[\vec a\ \vec b\ \vec c]-(\vec a\cdot\vec b)(\vec b\cdot\vec c)(\vec c\cdot\vec a)[\vec a\ \vec b\ \vec c]=0$

Since $[\vec c\ \vec a\ \vec b]=[\vec a\ \vec b\ \vec c]$

22. (c) : $[\vec a\ \vec b\ \vec c]^2=\begin{vmatrix} a^2 & ab\cos\theta & ac\cos\theta \\ ab\cos\theta & b^2 & bc\cos\theta \\ ca\cos\theta & bc\cos\theta & c^2\end{vmatrix}$

$=a^2b^2c^2\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1\end{vmatrix}$

$= a^2b^2c^2\,(1 - 3\cos^2\theta + 2\cos^3\theta)$

$= a^2b^2c^2\,(1 - \cos\theta)^2(1 + 2\cos\theta) \geq 0$

$\therefore\ \cos\theta\geq-\frac{1}{2} \Rightarrow \theta\in\left[0,\ \frac{2\pi}{3}\right]$

23. (b) : $\vec a\times\vec b=(\hat i+\hat j)\times(\hat j+\hat k)=\hat k-\hat j+\hat i$

$\Rightarrow \pm\frac{\vec a\times\vec b}{|\vec a\times\vec b|}=\pm\frac{\hat i-\hat j+\hat k}{\sqrt3}$

24. (a) : A vector in the plane of $\vec b$ and $\vec c$ is $\vec b+\lambda\vec c$

i.e., $(1+\lambda)\hat i+(2+\lambda)\hat j-(1+2\lambda)\hat k$

Its projection on $\vec a$ is $\sqrt{\frac{2}{3}}$

$\therefore\quad \frac{2(1+\lambda)-(2+\lambda)-(1+2\lambda)}{\sqrt6}=\pm\sqrt{\frac{2}{3}}$

$\therefore\quad \lambda + 1 = \pm 2 \Rightarrow \lambda = 1, -3$

Thus, $\lambda = 1 \Rightarrow$ Required vector $= 2\hat i+3\hat j-3\hat k$

and $\lambda = -3 \Rightarrow$ Required vector $= -2\hat i-\hat j+5\hat k$.

25. (a) : A vector coplanar with $\hat i+\hat j+2\hat k$ and $\hat i+2\hat j+\hat k$ is $\hat i+\hat j+2\hat k+\lambda(\hat i+2\hat j+\hat k)$

$=(1+\lambda)\hat i+(1+2\lambda)\hat j+(2+\lambda)\hat k$.

It is perpendicular to $\hat i+\hat j+\hat k$

$\therefore\quad 1 + \lambda + 1 + 2\lambda + 2 + \lambda = 0 \Rightarrow \lambda = -1$

$\therefore$ The vector is $-\hat j+\hat k$.

26. (d) : $f'(x)=\begin{vmatrix} 1 & \cos x & -\sin x \\ x^2 & \tan x & -x^3 \\ 2x & \sin 2x & 5x\end{vmatrix}$

$+\begin{vmatrix} x & \sin x & \cos x \\ 2x & \sec^2 x & -3x^2 \\ 2x & \sin 2x & 5x\end{vmatrix}+\begin{vmatrix} x & \sin x & \cos x \\ x^2 & \tan x & -x^3 \\ 2 & 2\cos 2x & 5\end{vmatrix}$

Applying $R_2\to\frac{1}{x}R_2, R_3\to\frac{1}{x}R_3$ and $R_2\to\frac{1}{x}R_2$ on 1st, 2nd and 3rd determinants respectively, we get

$\frac{f'(x)}{x}=\begin{vmatrix} 1 & \cos x & -\sin x \\ x & \frac{\tan x}{x} & -x^2 \\ 2x & \sin 2x & 5x\end{vmatrix}+\begin{vmatrix} x & \sin x & \cos x \\ 2x & \sec^2 x & -3x^2 \\ 2 & \frac{\sin 2x}{x} & 5\end{vmatrix}$

$+\begin{vmatrix} x & \sin x & \cos x \\ x & \frac{\tan x}{x} & -x^2 \\ 2 & 2\cos 2x & 5\end{vmatrix}$

$\therefore\quad \lim_{x\to0}\frac{f'(x)}{x}=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{vmatrix}+\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 5\end{vmatrix}+\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 5\end{vmatrix}=-4$

27. (c) : $\Delta=\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1\end{vmatrix}=\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ [x] & [y] & [z]+1\end{vmatrix}$

$\{R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3\}$

$= 1([z] + 1 + [y]) + 1([x]) = [x] + [y] + [z] + 1$

$\therefore$ Maximum value is 1 + 0 + 2 + 1 = 4.

28. (a) **29. (a)**

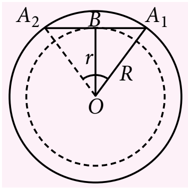
30. (d) : The sum of the coefficients in the first equation $= 1 + 2\omega + 3\omega^2 = \omega^2 - 1 = -$R.H.S

Similarly, for the other two equations also the results holds so that (–1, –1, –1) is the unique solution.

MATHS MUSING

SOLUTION SET-179

1. (a): Let A_1A_2 be the sides and O be the centre. Let $OB \perp A_1A_2$. Clearly, $OA_1 = R$ and $OB = r$.



Also, $\angle A_1OA_2 = \dfrac{2\pi}{n}$ and $\angle A_1OB = \dfrac{\pi}{n}$.

In ΔA_1OB, $\dfrac{r}{R} = \cos \angle A_1OB = \cos\dfrac{\pi}{n}$ ...(i)

and $\dfrac{A_1B}{OB} = \tan\dfrac{\pi}{n}$, *i.e.*, $A_1B = r\tan\dfrac{\pi}{n}$

$\therefore \quad A_1A_2 = 2A_1B = 2r\tan\dfrac{\pi}{n}$ ...(ii)

Now, $2(R+r)\tan\dfrac{\pi}{2n} = 2\left(\dfrac{r}{\cos\frac{\pi}{n}} + r\right)\tan\dfrac{\pi}{2n}$, [from (i)]

$$= 2r\left(\frac{1+\cos\frac{\pi}{n}}{\cos\frac{\pi}{n}}\right)\cdot\frac{1-\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} \qquad \left\{\because \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}\right\}$$

$$= 2r\cdot\frac{1-\cos^2\frac{\pi}{n}}{\cos\frac{\pi}{n}\cdot\sin\frac{\pi}{n}} = 2r\cdot\frac{\sin^2\frac{\pi}{n}}{\cos\frac{\pi}{n}\cdot\sin\frac{\pi}{n}}$$

$= 2r\tan\dfrac{\pi}{n} = A_1A_2$, [from (ii)].

2. (a): Here the form is 1^∞.

$$\text{Limit} = e^{\lim\limits_{x\to\infty}\log\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+\ldots+n^{1/x}}{n}\right)^{nx}} \qquad \text{...(i)}$$

Now, $\lim\limits_{x\to\infty}\log\left(\dfrac{1^{1/x}+2^{1/x}+3^{1/x}+\ldots+n^{1/x}}{n}\right)^{nx}$

$$= \lim_{x\to\infty} nx\cdot\log\frac{1^{1/x}+2^{1/x}+3^{1/x}+\ldots+n^{1/x}}{n}$$

$$= \lim_{y\to 0} n\cdot\frac{\log(1^y+2^y+\ldots+n^y)-\log n}{y} \qquad \left\{\text{putting } y = \frac{1}{x}\right\}$$

$$= n\cdot\lim_{y\to 0}\frac{\left[\frac{1}{1^y+2^y+\ldots+n^y}\cdot\{2^y\log 2 + 3^y\log 3+\ldots+n^y\log n\}-0\right]}{1}$$

{using L'Hospital's rule}

$$= n\cdot\frac{1}{1^0+2^0+\ldots+n^0}\{\log 2+\log 3+\ldots+\log n\}$$

$= \log n!$

$\therefore$ by (i), limit $= e^{\log n!} = n!$.

3. (d): $I = \displaystyle\int_0^{\sqrt{e-1}} 1\cdot dx + \int_{\sqrt{e-1}}^{2}\ln(1+x^2)\,dx$

$$= \sqrt{e-1} + x\ln(1+x^2)\Big|_{\sqrt{e-1}}^{2} - 2\int_{\sqrt{e-1}}^{2}\left(1-\frac{1}{1+x^2}\right)dx$$

$$= \sqrt{e-1} + 2\ln 5 - \sqrt{e-1} - 4 + 2\sqrt{e-1} + 2\tan^{-1}2 - 2\tan^{-1}\sqrt{e-1}$$

$\therefore$ Required value = –4

4. (d): $11^{2012} + 23^{2012} - 3^{2012}$

$= (1+10)^{2012} + (3+20)^{2012} - 3^{2012}$

$$= \left\{1 + {}^{2012}C_1\cdot 10 + {}^{2012}C_2\cdot 10^2 + \ldots..\right\} + \left\{3^{2012} + {}^{2012}C_1\cdot 3^{2011}\cdot 20 + \ldots..\right\} - 3^{2012}$$

$$= \left\{{}^{2012}C_1\cdot 10 + {}^{2012}C_2\cdot 10^2 + \ldots..\right\} + \left\{{}^{2012}C_1\cdot 3^{2011}\cdot 20 + {}^{2012}C_2\cdot 3^{2010}\cdot 20^2 + \ldots..\right\} + 1$$

= (All terms multiple of 10) + 1

$\therefore$ Unit place digit = 1 $\Rightarrow a = 1$

Let $I = \displaystyle\int_{a-1}^{a}\frac{dx}{\sqrt{1-x^2}-x+\frac{1}{x}} = \int_0^1\frac{xdx}{x\sqrt{1-x^2}+1-x^2}$

$$= \int_0^1\frac{xdx}{\left(x+\sqrt{1-x^2}\right)\sqrt{1-x^2}}$$

Putting $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

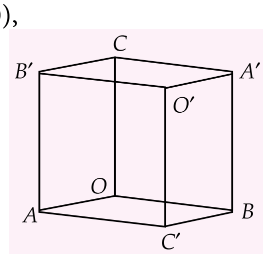
$$= \int_0^{\pi/2}\frac{\sin\theta\cos\theta d\theta}{(\sin\theta+\cos\theta)\cos\theta} \qquad \text{...(i)}$$

Again $I = \displaystyle\int_0^{\pi/2}\frac{\sin\theta\ d\theta}{\sin\theta+\cos\theta} = \int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-\theta\right)\,d\theta}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}$

$$= \int_0^{\pi/2}\frac{\cos\theta\ d\theta}{\cos\theta+\sin\theta} \qquad \text{...(ii)}$$

On adding (i) & (ii), $2I = \displaystyle\int_0^{\pi/2}d\theta \quad \therefore \quad I = \frac{\pi}{4}$

5. (d): Vertices are O (0, 0, 0), O' (1, 1, 1), A (1, 0, 0), A' (0, 1, 1), B (0, 1, 0), B' (1, 0, 1), C (0, 0, 1), C' (1, 1, 0).



The number of all triangles is $\dbinom{8}{3} = 56$

The number of right angled isosceles triangles like ΔOAB, each of area $\frac{1}{2}$ is 24. The number of right angled scalene triangles like $\Delta OCO'$, each of area $\frac{1}{\sqrt{2}}$ is 24.

The number of equilateral triangles like $O'BC$, each of area $\frac{\sqrt{3}}{2}$ is 8.

$\therefore$ The total area $= 12+12\sqrt{2}+4\sqrt{3}$

$= 12+\sqrt{288}+\sqrt{48}$

$\Rightarrow$ $m = 12$, $n = 288$, $p = 48$ $\therefore$ $m + n + p = 348$

6. (a, d) : $z^{2r-1} = \cos(2r-1)1° + i\sin(2r-1)1°$

$= \cos(2r-1)° + i\sin(2r-1)°$

$$\therefore \quad A=\sum_{r=1}^{45}\text{Re}(z^{2r-1})=\sum_{r=1}^{45}\cos(2r-1)°$$

$= \cos1° + \cos3° + \cos5° + \ldots\ldots + \cos89°$

$$=\frac{\cos\left(\frac{1°+89°}{2}\right)\sin 45(1°)}{\sin 1°}=\frac{1}{2\sin 1°}$$

$$\left[\because \cos\theta + \cos(\theta+\beta) + \cos(\theta+2\beta) + \ldots + \cos(\theta+(n-1)\beta)=\frac{\cos\left(\frac{\theta+\theta+(n-1)\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$\therefore \quad B=\sum_{r=1}^{45}\text{Im}(z^{2r-1})=\sum_{r=1}^{45}\sin(2r-1)°$$

$= \sin1° + \sin3° + \sin5° + \ldots\ldots + \sin89°$

$$=\frac{\sin\left(\frac{1°+89°}{2}\right)\cdot\sin 45(1°)}{\sin 1°}=\frac{1}{2\sin 1°}$$

$$\left[\because \sin\theta + \sin(\theta+\beta) + \sin(\theta+2\beta) + \ldots + 2\sin\{\theta+(n-1)\beta\}=\frac{\sin\left(\frac{\theta+\theta+(n-1)\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$\therefore$ $A = B$, but, $A^2+B^2=1$ & $\frac{1}{A^2}+\frac{1}{B^2}=1$ are not true.

Further, $\bar{z}=\cos 1°-i\sin 1°$

$\Rightarrow$ $i(\bar{z}-z)=i(-2i\sin 1°)=2\sin 1°=\frac{1}{A}$

7. (c) : $g(x+2)-g(2)=\int_0^{x+2} f(t)dt-\int_0^2 f(t)dt$

$$=\int_0^x f(t)dt+\int_0^2 f(t)dt-\int_0^2 f(t)dt=\int_0^x f(t)dt=g(x).$$

8. (a) : $g(2)=\int_0^2 f(t)dt=\int_0^1 f(t)dt+\int_1^2 f(t)dt$

In second integral put $t = 2 + y$

$t\to 1 \Rightarrow y\to -1,\ t\to 2\Rightarrow y\to 0$

$$\Rightarrow \int_0^1 f(t)dt+\int_{-1}^0 f(2+y)dy=\int_0^1 f(t)dt+\int_{-1}^0 f(y)dy$$

$$=\int_0^1 f(t)dt+\int_{-1}^0 f(t)dt=\int_{-1}^1 f(t)dt=2\int_0^1 f(t)dt$$

$= 2g(1) = 2a$, similarly $g(5) = 5a$

9. (3): We have, $\frac{x}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$

$\Rightarrow$ $x = t$, $y = 1 + 2t$, $z = -(1 + 2t)$

$\therefore$ Image in the plane $x + y + z - 1 = 0$ is

$$\frac{x-t}{1}=\frac{y-(1+2t)}{1}=\frac{z+1+2t}{1}=\frac{-2}{3}(t-1)$$

$$\Rightarrow x=\frac{t+2}{3},\ y=\frac{4t+5}{3},\ z=\frac{-(8t+1)}{3}$$

Given plane meets at x-z plane

So, $y=0 \Rightarrow t=-\frac{5}{4}$ $\quad\therefore\ x=\frac{1}{4},\ z=3$

$\therefore$ $(a, b, c) = \left(\frac{1}{4}, 0, 3\right)$

10. (b): P. $5^{2+4+6+\ldots\ldots+2x} = (25)^{28}$

$\Rightarrow$ $5^{x(x+1)} = 5^{56}$ $\Rightarrow$ $x^2+x-56=0 \Rightarrow x=7$ as $x>0$

Q. $2\log_5 x=\log_{\sqrt{5}}\left(\frac{1/4}{1-1/2}\right)\log_5(0.2)$

$$=\log_{\sqrt{5}}\left(\frac{1}{2}\right)\log_5\left(\frac{1}{5}\right)=-\frac{\log_5\left(\frac{1}{2}\right)}{\log_5\sqrt{5}}=\log_5 4\Rightarrow x=2$$

R. $\log x=\log_{5/2}\left(\frac{1/3}{1-1/3}\right)\log(0.16)$

$= \log_{5/2}(1/2)\log(2/5)^2 = \log 4 \Rightarrow x = 4$

S. $3^x\left[\frac{(1/3)}{1-1/3}\right]=\frac{2(5^2)}{1-1/5} \Rightarrow \frac{1}{2}(3^x)=\frac{1}{2}(5^3)$

$\Rightarrow$ $x = 3\log_3 5$

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