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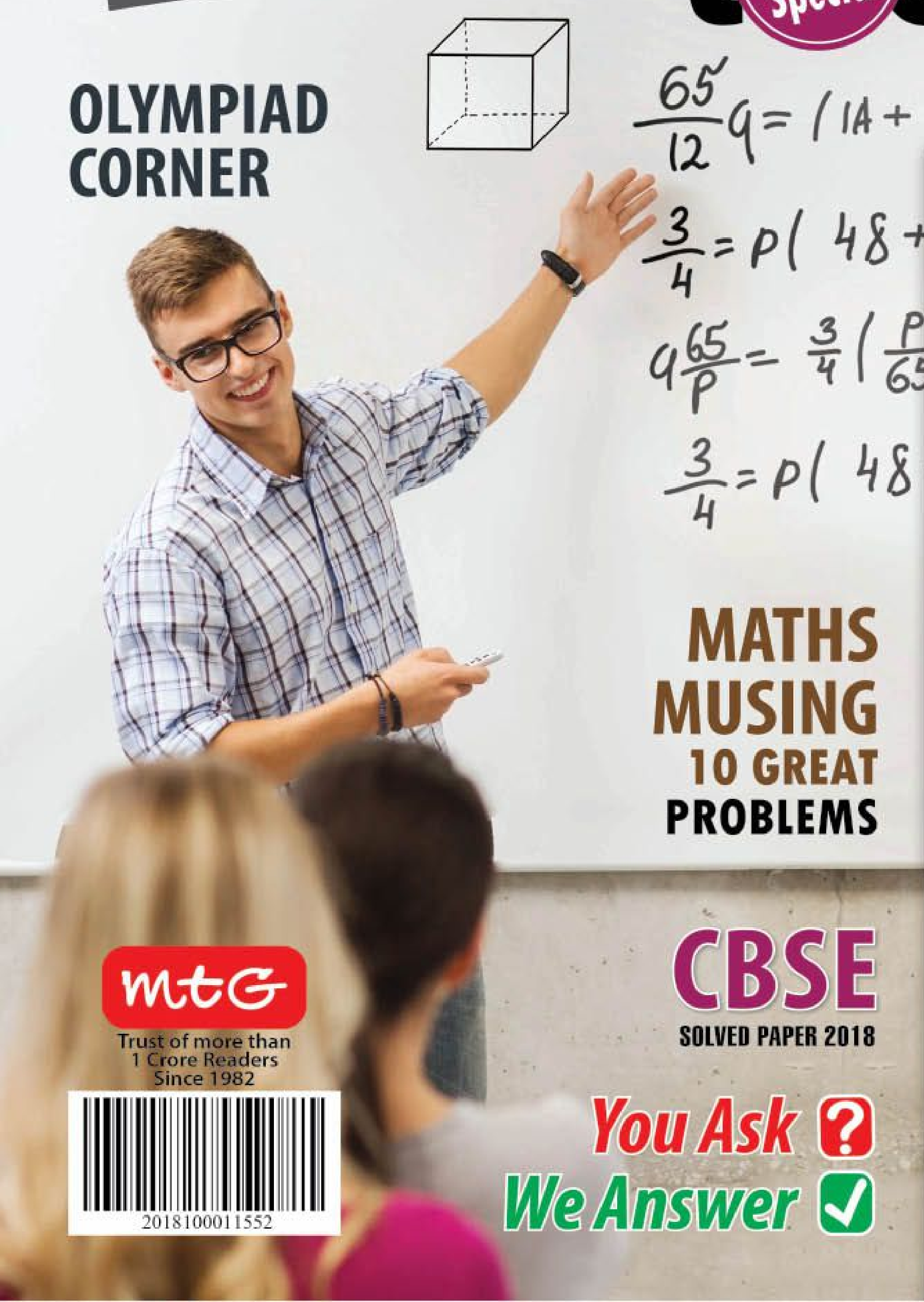

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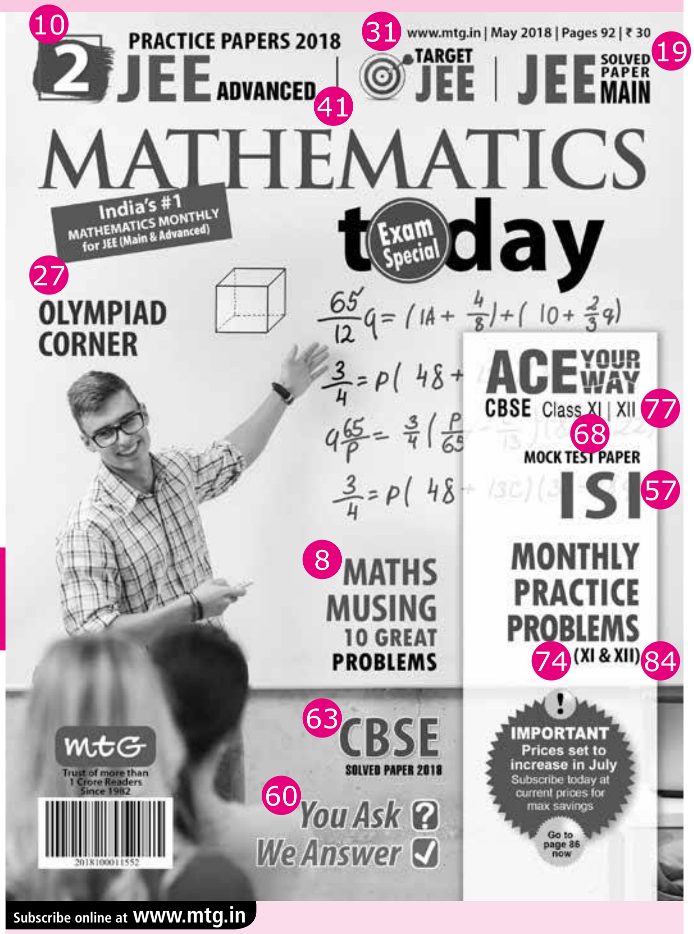

TO OUR READERS

We are happy that intelligent students, teachers and other professionals continue to patronise Mathematics Today, Chemistry Today, Physics For You and Biology Today.

To them, we are addressing this open letter in view of increase in the cost of production and postage in the last seven years. All round spiralling prices have pushed production costs so high, that many in out fraternity find it impossible to continue business. We are compelled to raise the price to ₹ 40 from July 2018 issue.

We understand the pressure of cost on the student-teacher community in general but, we are hoping our readers will understand our problems and that we have no option but to comply with this unavoidable move.

We on our part, will keep up our efforts to improve the magazines in all its aspects.

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 185

JEE MAIN

1. If z and w are two non-zero complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then $\bar{z}w$ is equal to

(a) 1 (b) −1 (c) i (d) $-i$

2. If $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} + \dots\right) + \cos^{-1}(1 + b + b^2 + \dots) = \frac{\pi}{2}$, then

(a) $b = \frac{2a-5}{3a}$ (b) $b = \frac{3a-2}{2a}$

(c) $a = \frac{3}{2-3b}$ (d) $a = \frac{2}{3-2b}$

3. The point on the line $\frac{x+2}{2} = \frac{y+6}{3} = \frac{z-34}{-10}$ which is nearest to the line $\frac{x+6}{4} = \frac{y-7}{-3} = \frac{z-7}{-2}$ is (a, b, c), where $a + b + c =$

(a) 9 (b) 10 (c) 11 (d) 12

4. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \dots + C_{n-r}C_n =$

(a) $\binom{2n}{r}$ (b) $\binom{2n}{r-1}$ (c) $\binom{2n}{r+1}$ (d) $\binom{2n}{n-r}$

5. If $f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$, then $f(x)$ has minimum at $x =$

(a) 1 (b) 7/5 (c) 2 (d) −2

JEE ADVANCED

6. Given that $x \in [0, 1]$ and $y \in [0, 1]$. Let A be the event of (x, y) satisfying $y^2 \le x$ and B be the event of (x, y) satisfying $x^2 \le y$. Then,

(a) $P(A \cap B) = \frac{1}{3}$ (b) A, B are exhaustive

(c) A, B are mutually exclusive

(d) A, B are independent

COMPREHENSION

Mr. X is a teacher of mathematics. His students want to know the ages of his son's S_1 and S_2. He told that their ages are 'a' and 'b' respectively such that $f(x + y) - axy = f(x) + by^2\ \forall\ x, y \in R$. After some time students said that information is insufficient, please give more information. Teacher says that $f(1) = 8$ and $f(2) = 32$.

7. The age of S_1 & S_2 will be respectively

(a) 4, 16 (b) 8, 16 (c) 16, 8 (d) 32, 8

8. The function $f(x)$ is

(a) even (b) odd

(c) neither even nor odd

(d) periodic as well as odd

INTEGER TYPE

9. A triangle is formed by the lines $x + y = 1$ and the two common tangents of the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 - 4x - 2y + 1 = 0$. Its area is

MATRIX MATCH

10. Match the curves given in Column-I with the differential equations satisfied by the curves in Column-II.

	Column-I		Column-II
(P)	The circles which touch the x-axis at the origin	(1)	$(x^2 - y^2)y_1 = 2xy$
(Q)	The circles which touch the y-axis at the origin	(2)	$2xyy_1 = y^2 - x^2$
(R)	The circle of unit radius with centres on the x-axis	(3)	$(1 - x^2)\,y_1^2 = x^2$
(S)	The circles of unit radius with centres on xy-axis	(4)	$(1 + y_1^2)y^2 = 1$

	P	Q	R	S
(a)	1	2	3	4
(b)	1	2	4	3
(c)	3	4	1	2
(d)	4	3	2	1

See Solution Set of Maths Musing 184 on page no. 62

JEE ADVANCED PRACTICE PAPER 2018

Exam on 20th May

*ALOK KUMAR, B.Tech, IIT Kanpur

SINGLE OPTION CORRECT TYPE

1. $f:\left(0, \dfrac{\pi}{2}\right) \to A,\ f(x) = \log_e(\sin x^{\sin x} + 1)$, then the minimum value of $f(x)$ is

(a) $\log_e 2$ (b) $\log_e\left(\left(\dfrac{1}{e}\right)^{1/e} + 1\right)$

(c) $\log_e((e)^2 + 1$ (d) 2

2. If $x^5 - x^3 + x = a$, then minimum value of x^6 is

(a) a (b) $2a - 1$ (c) $a + 2$ (d) $a - 3$

3. If z be a complex number satisfying, $|z|^2 + 2(z + \bar{z}) + 3i(z - \bar{z}) + 4 = 0$ then complex number $z + 3 + 2i$ will lie on

(a) circle with centre $1 - 5i$ and radius 4

(b) circle with centre $1 + 5i$ and radius 4

(c) circle with centre $1 + 5i$ and radius 3

(d) circle with centre $1 - 5i$ and radius 3

4. If $f(x - y) = f(x)g(y) - f(y)g(x)\ \forall\ x, y \in R$, (where $f(x)$ is not identically zero), then

(a) If $f'(0^+)$ exists then $f'(0^-)$ also exists but not equal

(b) If $f'(0^+)$ exists then $f'(0^-)$ does not exist

(c) If $f'(0^+)$ exists then it is equal to $f'(0^-)$

(d) none of these

5. A quadratic polynomial maps from $[-2, 3]$ onto $[0, 3]$ and touches x-axis at $x = 3$, then the polynomial is

(a) $\dfrac{3}{16}(x^2 - 6x + 16)$ (b) $\dfrac{3}{25}(x^2 - 6x + 9)$

(c) $\dfrac{3}{25}(x^2 - 6x + 16)$ (d) none of these

6. Let $f(x) = \displaystyle\int_0^x (t^2 + 2t + 2)\,dt$, where x is set of real numbers satisfying the inequation $\log_2(1 + \sqrt{6x - x^2 - 8}) \ge 0$. If $f(x) \in [\alpha, \beta]$ then maximum value of $|\alpha - \beta|$ is

(a) $\dfrac{103}{3}$ (b) $\dfrac{104}{3}$

(c) $\dfrac{106}{3}$ (d) none of these

7. The set of values of a for which inequation $(a - 1)x^2 - (a + 1)x + a - 1 \ge 0$ is true for all $x \ge 2$ is

(a) $\left(1, \dfrac{7}{3}\right]$ (b) $(-\infty, 1)$

(c) $\left[\dfrac{7}{3}, \infty\right)$ (d) none of these

8. If $x_1, x_2, x_3, ..., x_n$ are the roots of $x^n + ax + b = 0$, then the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)...(x_1 - x_n)$ is

(a) n_1^n (b) $nx_1^{n-1} + a$

(c) nx_1 (d) $nx_1 + a^n$

9. The value of $\displaystyle\lim_{n\to\infty}\left(\frac{n!}{(mn)^n}\right)^{1/n}$ is

(a) em (b) $\dfrac{e}{m}$

(c) $\dfrac{1}{em}$ (d) none of these

10. If one of the roots of the equation $\begin{vmatrix} 7 & 6 & x^2 - 13 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix} = 0$ is $x = 2$, then sum of all other five roots is

(a) $2\sqrt{15}$ (b) -2

(c) $\sqrt{20} + \sqrt{15} - 2$ (d) none of these

11. A unit vector is orthogonal to $5\hat{i} + 2\hat{j} + 6\hat{k}$ and is coplanar to $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - \hat{j} + \hat{k}$, then the vector is

---

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

(a) $\frac{3\hat{j}-\hat{k}}{\sqrt{10}}$ (b) $\frac{2\hat{i}+5\hat{j}}{\sqrt{29}}$

(c) $\frac{6\hat{i}-5\hat{k}}{\sqrt{61}}$ (d) $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$

12. The range of the function $y=\sqrt{2\{x\}-\{x\}^2-\frac{3}{4}}$ is (where {.} denotes fractional part)

(a) $\left[-\frac{1}{4}, \frac{1}{4}\right]$ (b) $\left[0, \frac{1}{2}\right)$

(c) $\left[0, \frac{1}{4}\right]$ (d) $\left[\frac{1}{4}, \frac{1}{2}\right]$

13. $\int \frac{dx}{\sqrt{\cos^3 x \cos(x-\alpha)}}$ is equal to

(a) $2\sqrt{\tan x \sin\alpha + \cot\alpha} + c$

(b) $2\operatorname{cosec}\alpha\sqrt{\tan x \sin\alpha + \cos\alpha} + c$

(c) $2\sqrt{\sin\alpha + \tan x \cos\alpha} + c$

(d) $2\operatorname{cosec}\alpha\sqrt{\tan x \cos\alpha + \sin\alpha} + c$

14. Inside a big circle exactly n small circles each of radius r can be drawn in such a way that each small circle touches the big circle and two small circles. If $n \geq 3$, then the radius of the bigger circle is

(a) $r\operatorname{cosec}\left(\frac{\pi}{n}\right)$ (b) $r\left\{1+\operatorname{cosec}\left(\frac{\pi}{n}\right)\right\}$

(c) $r\left\{1+\operatorname{cosec}\left(\frac{2\pi}{n}\right)\right\}$ (d) $r\left\{1+\operatorname{cosec}\left(\frac{\pi}{2n}\right)\right\}$

15. If m_1 and m_2 be the slopes of two perpendicular chords of equal length passing through origin of circle $(x-1)^2+(y+2)^2=5$, then the value of $m_1^2+m_2^2$ is equal to

(a) $\frac{80}{9}$ (b) $\frac{82}{9}$

(c) $\frac{83}{9}$ (d) none of these

16. If the system of equations $x-ky-z=0$, $kx-y-z=0$, $x+y-z=0$ has a non-zero solutions, then the possible values of k are

(a) $-1, 2$ (b) $0, 1$ (c) $1, 2$ (d) $-1, 1$

17. Period of $|\cos|x|| + |\sin|x||$ is

(a) 2π (b) π

(c) $\frac{\pi}{2}$ (d) none of these

18. If O is the origin and A and B are respectively (a_1, b_1) and (a_2, b_2), then $OA \cdot OB \sin(AOB)$ is equal to

(a) $a_1a_2 + b_1b_2$ (b) $a_1b_2 + a_2b_1$

(c) $b_1a_2 - b_2a_1$ (d) $a_1a_2 - b_1b_2$

19. If $f(x)=[|x|]$, then $\int_0^{100} f(x)dx$ is equal to (where [.] denotes the greatest integer function)

(a) 4950 (b) 5000 (c) 4250 (d) 4590

20. The largest term of $a_n=\frac{n^2}{n^3+200}$, $n \in N$ is

(a) $\frac{29}{453}$ (b) $\frac{49}{543}$ (c) $\frac{43}{553}$ (d) $\frac{41}{451}$

21. If $\vec{r}=\lambda(\vec{a}\times\vec{b})+\mu(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})$ and $[\vec{a}\,\vec{b}\,\vec{c}]=\frac{1}{8}$, then $\lambda+\mu+\gamma$ is

(a) $8(\vec{r}.\vec{a})$ (b) $8(\vec{r}.\vec{b})$

(c) $8(\vec{r}.\vec{c})$ (d) $8\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$

ONE OR MORE THAN ONE OPTION CORRECT TYPE

22. If $[x]$ represents the greatest integer less than or equal to x, then which of the following statement is true?

(a) $\sin[x] = \cos[x]$ has no solution

(b) $\sin[x] = \tan[x]$ has infinitely many solutions

(c) $\sin[x] = \cos[x]$ possess unique solution

(d) $\sin[x] = \tan[x]$ for no value of x

23. If p, q, r are in H.P. and p, q, $-2r$ are in G.P; $(p, q, r > 0)$ then

(a) p^2, q^2, r^2 are in G.P. (b) p^2, q^2, r^2 are in A.P.

(c) $2p, q, 2r$ are in A.P. (d) $p + q + r = 0$

24. The det $\Delta=\begin{vmatrix} d^2+r & de & df \\ de & e^2+r & ef \\ df & ef & f^2+r \end{vmatrix}$ is divisible by

(a) r (b) $(d + e^2 + f^2 + r)$

(c) $(d^2 + e^2 + f^2 + r)$ (d) $(d^2 + e + f^2 + r^2)$

25. If both the roots of equation $ax^2 + x + c - a = 0$ are imaginary and $c > -1$, then

(a) $3a > 2 + 4c$ (b) $a > 0$

(c) $c < a$ (d) None of these

26. The solution of the equation

$3^{\sin 2x+2\cos^2 x}+3^{1-\sin 2x+2\sin^2 x}=28$

(a) $(2n+1)\frac{\pi}{2}, n \in I$ (b) $n\pi-\frac{\pi}{4}, n \in I$

(c) $2n\pi$ (d) none of these

27. If $\sin y = x\cos(a+y)$, then $\frac{dy}{dx}$ is

(a) $\frac{\cos^2(a+y)}{\cos a}$ (b) $\frac{\cos(a+y)}{x\sin(a+y)+\cos y}$

(c) $\frac{\cos a}{3x^2-2x\cos a-1}$ (d) none of these

COMPREHENSION TYPE

Passage for Q.No. 28 to 30

A ray of light is coming along the line $L = 0$ and strikes the plane mirror kept along the plane $P = 0$ at B. $A(2, 1, 6)$ is a point on the line $L = 0$ whose image about $P = 0$ is A'. It is given that $L = 0$ is $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}$ and $P = 0$ is $x + y - 2z = 3$.

28. Co-ordinates of A' are

(a) (6, 5, 2) (b) (6, 5, –2)

(c) (6, –5, –2) (d) none of these

29. Co-ordinates of B are

(a) (5, 10, 6) (b) (10, 15, 11)

(c) (–10, –15, –14) (d) none of these

30. If $L_1 = 0$ is the reflected ray, then its equation is

(a) $\frac{x+10}{4}=\frac{y-5}{4}=\frac{z+2}{3}$

(b) $\frac{x+10}{3}=\frac{y+15}{5}=\frac{z+14}{5}$

(c) $\frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$

(d) none of these

Passage for Q.No. 31 to 33

Given $A(3 + 4i)$, $B(-4 + 3i)$ and $C(4 + 3i)$ be the vertices of ΔABC which is inscribed in a circle $S = 0$. Let AD, BE, CF are altitudes through A, B, C which meet the circle $S = 0$ at $D(z_1)$, $E(z_2)$ and $F(z_3)$ respectively.

31. Complex number z_1 is equal to

(a) $(2\sqrt{6}+i)$ (b) $(1+2\sqrt{6}i)$

(c) $3 - 4i$ (d) $-3 - 4i$

32. Orthocentre of triangle ABC is at

(a) $2 + 9i$ (b) $3 + 10i$

(c) $3 + 11i$ (d) none of these

33. The value of $z_1z_2z_3$ is equal to

(a) $75 + 100i$ (b) $-100 + 75i$

(c) $100 + 75i$ (d) none of these

Passage for Q.No. 34 to 36

Consider an ellipse $\frac{x^2}{4}+y^2=\alpha$, (α is parameter > 0) and a parabola $y^2 = 8x$. A common tangent to the ellipse and the parabola meets the co-ordinate axes at A & B.

34. Locus of mid point of AB is

(a) $y^2 = -2x$ (b) $y^2 = -x$

(c) $y^2=-\frac{x}{2}$ (d) $\frac{x^2}{4}+\frac{y^2}{2}=1$

35. If the eccentric angle of a point on the ellipse where the common tangent meets it is $\left(\frac{2\pi}{3}\right)$, then α is equal to

(a) 4 (b) 5 (c) 26 (d) 36

36. If two of the three normals drawn from the point $(h, 0)$ on the ellipse to the parabola $y^2 = 8x$ are perpendicular, then

(a) $h = 2$ (b) $h = 3$

(c) $h = 4$ (d) $h = 6$

MATRIX MATCH TYPE

37. Match the following :

	List-I		List-II
(A)	$3x^2 + 2(b^2 + 1)x + b^2 - 3b + 2 = 0$ will have roots of opposite sign if $b \in$	(1)	$\left(\frac{2}{3}, 1\right)$
(B)	The probability of a problem being solved by 3 students are $\frac{1}{2}, \frac{1}{3}$ and $\alpha \in (0, 1)$. Then probability that problem will be solved $\in$	(2)	$[\sqrt{2}, \infty)$
(C)	If 3 real numbers satisfy the equation $x + y + z = 5$ and $xy + yz + zx = 8$, then $z \in$	(3)	$\left[1, \frac{7}{3}\right]$
		(4)	(1, 2)

38. In a tournament there are twelve players $S_1, S_2, \ldots, S_{12}$ and divided into six pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assuming all the pairs are of equal strength, then match the following :

	List-I		List-II
(A)	Probability that S_2 is among the losers is	(1)	$\frac{5}{22}$

(B)	Probability that exactly one of S_3 and S_4 is among the losers	(2)	$\frac{10}{11}$
(C)	Probability that exactly one S_2 and S_4 are among the winners is	(3)	$\frac{1}{2}$
		(4)	$\frac{6}{11}$

39. Match the following :

	List-I		List-II
(A)	For a rectangular hyperbola $xy = c^2$ (c is purely imaginary). If a point on it is (a, α), where 'a' is a positive number, then α can take value	(1)	3
(B)	If sides of a triangle are in A.P. and $(a - b + c)s = kb^2$ (where s is the semi perimeter), then k is equal to	(2)	$\frac{1-\sqrt{3}}{2}$
(C)	Radius of the circle having centre (3, 4) and touching the circle $x^2 + y^2 = 4$ can be	(3)	17
		(4)	$\frac{3}{2}$

40. Match the following :

	List-I		List-II
(A)	If $x, y, z \in N$, then number of ordered triplet (x, y, z) satisfying $xyz = 243$ is	(1)	19
(B)	The number of terms in the expansion of $(x + y + z)^6$ is	(2)	28
(C)	If $x \in N$, then number of solutions of $x^2 + x - 400 \le 0$ is	(3)	21
(D)	If $x, y, z \in N$, then number of solution of $x + y + z = 10$	(4)	36

41. Match the following :

	List-I		List-II
(A)	If inequation $x^2 - ax + a > 0\ \forall\ x \in R$, then a belongs to	(1)	[0, 4)
(B)	If $x^3 - 3x + \frac{a}{2} = 0$ has three real and distinct root, then $\|a\|$ belong to	(2)	[0, 3]
(C)	If $x^3 + ax^2 + x + 1 = 0$ is an increasing function, then a^2 may belong to	(3)	(0, 4)
(D)	If quadratic equation $x^2 + 3ax + a^2 - 9 = 0$ has roots of opposite sign, then a belongs to	(4)	(–3, 3)

42. Match the following :

	List-I		List-II
(A)	The set of values of b for which the function $f(x) = x^2 + bx + 5$ is an increasing function on [1, 3] is	(1)	$(-\infty, 0)$
(B)	India plays 4 matches with Sri Lanka and South Africa each. In any match the probability of getting points 0, 1, 2 are 0.15, 0.35, 0.5 respectively assuming the outcomes are independent. The probability of getting at least 7 points is	(2)	0.2375
(C)	The radical centre of 3 circles described on the 3 sides of a triangle ABC as diameter is the ______ of the triangle ABC.	(3)	orthocentre
(D)	The domain of the function $f(x) = \frac{1}{\sqrt{\|x\| - x}}$ is	(4)	$[-2, \infty)$

INTEGER ANSWER TYPE

43. In ΔABC if BC is unity,

$$\sin\frac{A}{2} = x_1,\ \sin\frac{B}{2} = x_2,\ \cos\frac{A}{2} = x_3 \text{ and } \cos\frac{B}{2} = x_4$$

with $\left(\frac{x_1}{x_2}\right)^{2007} - \left(\frac{x_3}{x_4}\right)^{2006} = 0$, then the length of AC is

44. If $I_n = \int_0^\infty e^{-x}(\sin x)^n dx\ (n > 1)$, then the value of $\frac{101 I_{10}}{90 I_8}$ is equal to

45. The A.M., G.M. and H.M. of first and last term of the series 100, 101, 102,...., n are also the term of this series, then the value of $\frac{n}{200}$ $(100 < n \le 500)$ is

SOLUTIONS

1. (b) : Let $h(x) = \log(x^x + 1)$, for $0 < x < 1$, $h'(x) > 0$ if $x > \frac{1}{e}$. If $x > \frac{1}{e}$, for $0 < x < 1$, function will be increasing, then it will be one-one for $x \in (0, 1)$. $h'(x) < 0$, if $x < \frac{1}{e}$, so, at $x = \frac{1}{e}$, there is minimum.

$$f_{\min} = \log_e\left(\left(\frac{1}{e}\right)^{1/e} + 1\right).$$

2. (b) : $x(x^4 - x^2 + 1) = \frac{x(x^6+1)}{x^2+1}$. Suppose $x > 0$, $a > 0$,

$$x^6 + 1 = \frac{a(x^2+1)}{x} = a\left(x + \frac{1}{x}\right) \geq 2a,\ x^6 \geq 2a - 1.$$

If $x \leq 0 \Rightarrow a \leq 0$, $2a - 1 < 0 \leq x^6 \Rightarrow x^6 \geq 2a - 1$.

3. (c) : Given equation of circle is

$|z|^2 + z(2 + 3i) + \bar{z}(2 - 3i) + 4 = 0$ centre $-(2 - 3i)$,

radius $= \sqrt{a\bar{a} - b} = 3$

Let $\omega = z + 3 + 2i = z + 2 - 3i + 1 + 5i$,

$|\omega - 1 - 5i| = |z + 2 - 3i| = 3$

so, ω lies on circle whose centre is $1 + 5i$, radius $= 3$.

4. (c) : $f(x - y) = f(x)g(y) - f(y)g(x)$ ...(i)

Put $x = y$ in (i), $f(0) = 0$,

Put $y = 0$ in (i), $f(x) = f(x)\,g(0) - f(0)\,g(x) \Rightarrow g(0) = 1$

$$\text{Now, } f'(0^+) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{f(0)g(-h) - f(-h)g(0)}{h} = \lim_{h\to 0}\frac{f(-h)}{-h} = f'(0^-)$$

i.e. $f'(0^+) = f'(0^-)$.

5. (b) : Let $f(x) = ax^2 + bx + c$.

As it touches x-axis at $x = 3$

$$\Rightarrow \frac{-b}{2a} = 3 \Rightarrow b = -6a \quad \text{...(i)}$$

also, $9a + 3b + c = 0$ ...(ii)

$4a - 2b + c = 3$ ...(iii)

Solving (i), (ii) and (iii), we get

$$a = \frac{3}{25},\ b = -\frac{18}{25},\ c = \frac{27}{25}$$

$$\Rightarrow f(x) = \frac{3}{25}(x^2 - 6x + 9)$$

6. (b) : $\log_2(1 + \sqrt{6x - x^2 - 8}) \geq 0$

$\Rightarrow 1 + \sqrt{6x - x^2 - 8} \geq 1$

$\Rightarrow 6x - x^2 - 8 \geq 0 \Rightarrow x^2 - 6x + 8 \leq 0$

$\Rightarrow (x - 2)(x - 4) \leq 0 \Rightarrow 2 \leq x \leq 4$

Now $f'(x) = x^2 + 2x + 2 > 0\ \forall\ x \in R$

$\Rightarrow f(x)$ is strictly increasing in $[2, 4]$

$$f(x) = \frac{x^3}{3} + x^2 + 2x,\ \alpha = f(2) = \frac{8}{3} + 4 + 4 = \frac{32}{3}$$

$$\beta = f(4) = \frac{64}{3} + 16 + 8 = \frac{136}{3},\ |\alpha - \beta|_{\max} = \frac{104}{3}$$

7. (c) : Given inequation is $(a - 1)x^2 - (a + 1)x + a - 1 \geq 0$

$\Rightarrow a(x^2 - x + 1) - (x^2 + x + 1) \geq 0$

$$\Rightarrow a \geq \frac{x^2 + x + 1}{x^2 - x + 1} = 1 + \frac{2x}{x^2 - x + 1} = 1 + \frac{2}{x + \frac{1}{x} - 1} \quad \text{...(i)}$$

Let $y = x + \frac{1}{x}$, y is increasing in $[2, \infty)$

$\Rightarrow 1 + \frac{2}{x + \frac{1}{x} - 1} \in \left(1, \frac{7}{3}\right]$ for all $x \geq 2$ equation (i) should be true .

$\Rightarrow a \geq \frac{7}{3}$.

8. (b) : Given $x^n + ax + b = 0 = (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n)$

$$\Rightarrow (x - x_2)(x - x_3)\ldots(x - x_n) = \frac{x^n + ax + b}{x - x_1}$$

$\therefore$ $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots (x_1 - x_n)$

$$= \lim_{x\to x_1}\left(\frac{x^n + ax + b}{x - x_1}\right) = nx_1^{n-1} + a.$$

9. (c) : Let $L = \lim_{n\to\infty}\frac{1}{m}\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\ldots\frac{n}{n}\right)^{1/n}$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\ln\left(\frac{1}{m}\right) + \frac{1}{n}\left(\ln\frac{1}{n} + \ln\frac{2}{n} + \ldots\ldots + \ln\frac{n}{n}\right)\right]$$

$$= -\ln m + \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right)$$

$$= -\ln m + \int_0^1 \ln x\,dx = -\ln m - 1 = \ln\left(\frac{1}{em}\right) \therefore L = \frac{1}{em}.$$

10. (b) : Let $x^2 - 13 = t$, then $t^3 - 67t + 126 = 0$

$\Rightarrow t = -9, 2, 7 \Rightarrow x = \pm 2, \pm\sqrt{20}, \pm\sqrt{15}$.

11. (a) : A vector orthogonal to $\vec{c}$ and coplanar with $\vec{a}$ and $\vec{b}$ is $\vec{c} \times (\vec{a} \times \vec{b})$

Here, $\vec{c} = (5,2,6), \vec{a} = (2,1,1), \vec{b} = (1,-1,1)$

Then $\vec{c} \times (\vec{a} \times \vec{b}) = (0, 27, -9)$

$\therefore$ Unit vector is $\frac{3\hat{j} - \hat{k}}{\sqrt{10}}$

12. (b) : Set $\{x\} = t$ so that $0 \le t < 1$, $y = \sqrt{-t^2 + 2t - \frac{3}{4}}$

The domain of definition is $\frac{1}{2} \le t < 1$

$y\left(\frac{1}{2}\right) = \sqrt{-\frac{1}{4} + 1 - \frac{3}{4}} = 0$; $y(1) = \sqrt{-1 + 2 - \frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

But $\frac{1}{2}$ is not attained, so the range is $\left[0, \frac{1}{2}\right)$.

13. (b) : Let $I = \int \frac{dx}{\sqrt{\cos^3 x(\cos x \cos\alpha + \sin x \sin\alpha)}}$

$= \int \frac{\sec^2 x dx}{\sqrt{\cos\alpha + \tan x \sin\alpha}}$, Put $\tan x \sin\alpha + \cos\alpha = t^2$

$\Rightarrow \sec^2 x dx = \frac{2tdt}{\sin\alpha}$ $\therefore$ $I = \int \frac{2tdt}{\sin\alpha . t} = 2\text{cosec}\alpha . t$

$= 2\text{cosec}\alpha\sqrt{\tan x \sin\alpha + \cos\alpha} + c$

14. (b) : We have, $\sin\left(\frac{\pi}{n}\right) = \frac{r}{R - r} \Rightarrow R - r = r\text{cosec}\left(\frac{\pi}{n}\right)$

$\therefore$ $R = r\left\{1 + \text{cosec}\left(\frac{\pi}{n}\right)\right\}$

15. (b) : Let slope of chord, $OA = m$, $\tan 45° = \left|\frac{m+2}{1-2m}\right|$

$\Rightarrow$ $|m+2| = |1-2m| \Rightarrow m+2 = \pm(1-2m) \Rightarrow m = 3$ or $-\frac{1}{3}$

$\Rightarrow$ $m_1^2 + m_2^2 = (m_1 + m_2)^2 - 2m_1m_2 = \frac{64}{9} + 2 = \frac{82}{9}$

16. (d) : Set of homogeneous equations will have non-trivial solution if $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -k & 0 \\ k & -1 & k-1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \qquad [\text{Applying } C_3 \to C_3 + C_1]$$

$\Rightarrow$ $k^2 - 1 = 0$ $\Rightarrow$ $k = \pm 1$

17. (c) : $f(x) = f\left(\frac{\pi}{2} + x\right)$.

18. (c) : $\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$

$$= \frac{b_1}{\sqrt{a_1^2 + b_1^2}} \cdot \frac{a_2}{\sqrt{a_2^2 + b_2^2}} - \frac{a_1}{\sqrt{a_1^2 + b_1^2}} \cdot \frac{b_2}{\sqrt{a_2^2 + b_2^2}}$$

$\Rightarrow$ $\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\sin(\theta_1 - \theta_2) = b_1a_2 - b_2a_1$

$\Rightarrow$ $OA \cdot OB \sin(AOB) = b_1a_2 - b_2a_1$.

19. (a) : $\int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx + ... + \int_{99}^{100} 99dx = \frac{99 \times 100}{2}$

$= 4950$

20. (b) : Let $f(x) = \frac{x^2}{x^3 + 200}, x \ge 1$

$f'(x) = \frac{x(400 - x^3)}{(x^3 + 200)^2}$

$f'(x) > 0$ if $0 < x < \sqrt[3]{400}$ and $f'(x) < 0$ if $x > \sqrt[3]{400}$

$\Rightarrow$ $f(x)$ has a maximum at $x = \sqrt[3]{400}$, $7 < \sqrt[3]{400} < 8$

$\Rightarrow$ Either a_7 or a_8 is largest term.

$a_7 = \frac{49}{543}$, $a_8 = \frac{8}{89}$, $a_7 > a_8$

$\therefore$ largest term $= \frac{49}{543}$

21. (d) : $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D$$

Given $\vec{r} = \lambda(\vec{a} \times \vec{b}) + \mu(\vec{b} \times \vec{c}) + \gamma(\vec{c} \times \vec{a})$, $[\vec{a}\,\vec{b}\vec{c}] = \frac{1}{8}$

Multiply the above relation with $\vec{a}, \vec{b}, \vec{c}$, we get

$\vec{r} . \vec{a} = \lambda . 0 + \mu[\vec{a}\,\vec{b}\,\vec{c}] + 0 = \frac{\mu}{8}$

$\Rightarrow$ $8(\vec{r} \cdot \vec{a}) = \mu$

Similarly $\gamma = 8(\vec{r} \cdot \vec{b})$, $\lambda = 8(\vec{r} \cdot \vec{c})$

$\therefore$ $\lambda + \mu + \gamma = 8\vec{r} \cdot (\vec{a} + \vec{b} + \vec{c})$

22. (a, b) : Since, according to option (a), we have $\sin[x] = \cos[x]$

$$\Rightarrow \quad [x] = n\pi + \left(\frac{\pi}{4}\right) = \left(n + \frac{1}{4}\right)\pi$$

Which is not possible as LHS $\in Z$ and RHS $\in Q$.
Hence, $\sin[x] = \cos[x]$ has no solution.
However, $\sin[x] = \tan[x] \Rightarrow [x] = n\pi$, $n \in I$ which is possible when $[x] = 0$ and $n = 0$ $\therefore$ $0 \le x < 1$.

23. (b, c) : p, q, r are in H.P. $\Rightarrow$ $q = \dfrac{2pr}{p+r}$

$\Rightarrow$ $q^2(p + r)^2 = (2pr)^2$
Also, $q^2 = -2pr \Rightarrow q^2(p + r)^2 = q^4$
$\Rightarrow$ $q^2[(p + r)^2 - q^2] = 0 \Rightarrow (p + r + q)(p + r - q) = 0,$
$\Rightarrow$ $p + r = q$ $\qquad (\because\ p + q + r \ne 0)$

$\therefore$ $p, \dfrac{q}{2}, r$ are in A.P. or $2p, q, 2r$ are in A.P.

Now, $q^2[p^2 + r^2 + 2pr] = q^4 \Rightarrow q^2[p^2 + r^2 - q^2] = q^4$
$\Rightarrow$ $[p^2 + r^2 - 2q^2] = 0$
or $p^2 + r^2 = 2q^2$ $\therefore$ p^2, q^2, r^2 are in A.P.

24. (a, c) : Multiply R_1, R_2, R_3 by d, e, f and take out d, e, f common from C_1, C_2, C_3, then $\Delta = \begin{vmatrix} d^2+r & d^2 & d^2 \\ e^2 & e^2+r & e^2 \\ f^2 & f^2 & f^2+r \end{vmatrix}$

Operate $R_1 \to R_1 + R_2 + R_3$ and take out $d^2 + e^2 + f^2 + x$ common from R_1, then

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ e^2 & e^2+r & e^2 \\ f^2 & f^2 & f^2+r \end{vmatrix}(d^2+e^2+f^2+r)$$

$$= r^2(d^2+e^2+f^2+r).$$

25. (b) : Here $f(1) = a + 1 + c - a$.
As $c > -1$ $\Rightarrow$ $f(1) > 0$

26. (a, b) : We have,

$$3^{\sin 2x + 2\cos^2 x} + 3^{1-\sin 2x + 2\sin^2 x} = 28,\ y + \frac{27}{y} = 28,$$

$\Rightarrow$ $y^2 - 28y + 27 = 0 \Rightarrow y = 1$
$(\because\ y \ne 27$, since $\sin 2x + \cos 2x = \sqrt{2}$)
$\therefore$ $\sin 2x + 2\cos^2 x = 0 \Rightarrow 2\cos x(\sin x + \cos x) = 0$

$$\Rightarrow \quad x = (2n+1)\frac{\pi}{2} \text{ or } x = n\pi - \frac{\pi}{4}$$

27. (a, b) : $\sin y = x\cos(a+y) \Rightarrow \dfrac{\sin y}{\cos(a+y)} = x$ $\quad$ ...(i)

$$\frac{dx}{dy} = \frac{\cos y\cos(a+y) + \sin y\sin(a+y)}{\cos^2(a+y)} = \frac{\cos a}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a} \qquad \text{...(ii)}$$

$$= \frac{\cos^2(a+y)}{\cos y\cos(a+y) + \sin y\sin(a+y)} = \frac{\cos(a+y)}{\cos y + x\sin(a+y)}$$

[Using (i)]

28. (b) : Let $A'(x_2, y_2, z_2)$ be image of $A(2, 1, 6)$ about mirror $x + y - 2z = 3$, then

$$\frac{x_2-2}{1} = \frac{y_2-1}{1} = \frac{z_2-6}{-2} = \frac{-2(2+1-12-3)}{1^2+1^2+2^2} = 4$$

$\Rightarrow$ $(x_2, y_2, z_2) = (6, 5, -2)$

29. (c) : Let $\dfrac{x-2}{3} = \dfrac{y-1}{4} = \dfrac{z-6}{5} = \lambda,$

$\therefore$ $x = 2 + 3\lambda$, $y = 1 + 4\lambda$, $z = 6 + 5\lambda$, lies on plane $x + y - 2z = 3$
$\therefore$ $2 + 3\lambda + 1 + 4\lambda - 2(6 + 5\lambda) = 3$
$\Rightarrow$ $3 + 7\lambda - 12 - 10\lambda = 3 \Rightarrow -3\lambda = 12 \Rightarrow \lambda = -4$
$\therefore$ Point $B \equiv (-10, -15, -14)$.

30. (c) : Equation of reflected ray $L_1 = 0$ is a line joining $A'(6, 5, -2)$ and $B(-10, -15, -14)$

i.e. $\dfrac{x+10}{16} = \dfrac{y+15}{20} = \dfrac{z+14}{12} \Rightarrow \dfrac{x+10}{4} = \dfrac{y+15}{5} = \dfrac{z+14}{3}$

31. (c) : $z_1 = -\dfrac{(-4+3i)(4+3i)}{3+4i} = 3 - 4i$

32. (b) : Circumcentre of $\Delta ABC = (0, 0)$.

Now, centroid of $\Delta ABC = \dfrac{3+10i}{3}$

$\therefore$ Orthocentre of $\Delta ABC = 3 + 10i$.

33. (a) : $z_1z_2z_3 = -z_az_bz_c = -(3 + 4i)(-4 + 3i)(4 + 3i)$
$= (3 + 4i)(4 - 3i)(4 + 3i) = (3 + 4i)25 = 75 + 100i$.

(34-36) : 34. (b) 35. (d) 36. (d)

Equation of tangent to $y^2 = 8x$ is $yt - x - 2t^2 = 0$ ...(1)

Equation of tangent to ellipse is $\dfrac{x\cos\theta}{2\sqrt{\alpha}} + \dfrac{y\sin\theta}{\sqrt{\alpha}} = 1$ ...(2)

If the tangent meets the coordinate axes at A and B then A is $\left(\dfrac{2\sqrt{\alpha}}{\cos\theta}, 0\right)$ and B is $\left(0, \dfrac{\sqrt{\alpha}}{\sin\theta}\right)$.

Let mid point of AB is (h, k) then,

$$h = \frac{\sqrt{\alpha}}{\cos\theta}, k = \frac{\sqrt{\alpha}}{2\sin\theta}, h = -t^2, k = t \Rightarrow k^2 = -h \text{ or } y^2 = -x$$

$$k^2 = -h \Rightarrow \frac{\alpha}{\sin^2\theta} = \frac{-4\sqrt{\alpha}}{\cos\theta} \Rightarrow \sqrt{\alpha} = -\frac{4\sin^2\theta}{\cos\theta} = 6$$

Any normal to parabola is
$y = mx - 4m - 2m^3 \Rightarrow 2m^3 + (4 - h)m = 0 \Rightarrow h = 6$

37. (A) → (4), (B) → (1), (C) → (3)

(A) The equation will have roots of opposite sign if it has real roots and product of roots is negative

$4(b^2 + 1)^2 - 12(b^2 - 3b + 2) \geq 0$ and $\frac{b^2 - 3b + 2}{3} < 0$

$\Rightarrow \quad 1 < b < 2$

(B) The probability of problem being solved

$$= 1 - P(\bar{A}).P(\bar{B}).P(\bar{C}) = 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)(1 - \alpha)$$

$$= \frac{2}{3} + \frac{\alpha}{3} \in \left(\frac{2}{3}, 1\right)$$

(C) Here, $x = 5 - (y + z)$ ...(i)

Now, $yz + x(y + z) = 8 \Rightarrow yz + (y + z)(5 - (y + z)) - 8 = 0$ [Using (i)]

$\Rightarrow \quad y^2 + y(z - 5) + (z^2 - 5z + 8) = 0$

For real solution, $(z-1)\left(z - \frac{7}{3}\right) \leq 0, 1 \leq z \leq \frac{7}{3}$.

38. (A) → (3), (B) → (4), (C) → (1)

(A) $\frac{^{11}C_5}{^{12}C_6} = \frac{1}{2}$

(B) Let E_1 be the event that S_3 and S_4 are in same group and E_2 be the event that S_3 and S_4 are in different group

$\therefore \quad P(E_1) = \frac{1}{11}, P(E_2) = \frac{10}{11}$

Let E be the event that exactly one of S_3 and S_4 is among the losers, then

$$P(E) = P(E_1)P\left(\frac{E}{E_1}\right) + P(E_2)P\left(\frac{E}{E_2}\right) = \frac{1}{11} \times 1 + \frac{10}{11} \times \left(\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2}\right) = \frac{6}{11}$$

(C) S_2 and S_4 should be in different groups for both being winner

$\therefore \quad$ Required probability $= \frac{10}{11}\left(\frac{1}{2}.\frac{1}{2}\right) = \frac{5}{22}$ or $\frac{^{10}C_4}{^{12}C_6} = \frac{5}{22}$

39. (A) → (2), (B) → (4), (C) → (1)

(A) $xy = c^2 = -\text{ve}$ ($\because$ c is purely imaginary)

$\Rightarrow \quad$ If x is positive y must be negative real $\Rightarrow \frac{1 - \sqrt{3}}{2}$

(B) $2b = a + c$,

Now, $(a + c - b)s = kb^2 \Rightarrow \frac{b(a + b + c)}{2} = kb^2$

$\Rightarrow \quad \frac{3b^2}{2} = kb^2 \Rightarrow k = \frac{3}{2}$

(C) $r = \sqrt{(3-0)^2 + (4-0)^2} - 2 = 3$.

40. (A) → (3), (B) → (2), (C) → (1), (D) → (4)

(A) $xyz = 3^5$ number of solution (x, y, z) is $^{3+5-1}C_5 = 21$

(B) Number of terms $= {}^{6+3-1}C_{3-1} = 28$.

(C) $x^2 + x - 400 \leq 0 \Rightarrow x(x + 1) \leq 400$

$\therefore$ The numbers which satisfied the equation are 1, 2, 3, ..., 19. Thus, number of solutions = 19.

(D) $x + y + z = 10$

number of solutions = coefficient of t^{10} in $(t + t^2 + ... + t^{10})^3$

$= $ coefficient of t^{10} in $t^3\left(\frac{1 - t^{10}}{1 - t}\right)^3$

$=$ coeffieicent of t^7 in $(1 + {}^3C_1 t + {}^4C_2 t^2 + ..) = {}^9C_7 = 36$.

41. (A) → (3), (B) → (1), (C) → (2), (D) → (4)

(A) $x^2 - ax + a > 0 \ \forall \ x \in R, D < 0 \Rightarrow a^2 - 4a < 0$, $a \in (0, 4)$.

(B) Let $f(x) = x^3 - 3x + \frac{a}{2}, f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$\Rightarrow \quad f(1)f(-1) < 0$.

$\Rightarrow \left(1 - 3 + \frac{a}{2}\right)\left(-1 + 3 + \frac{a}{2}\right) < 0 \Rightarrow \left(\frac{a}{2} - 2\right)\left(\frac{a}{2} + 2\right) < 0$

$\Rightarrow -2 < \frac{a}{2} < 2 \Rightarrow a \in (-4, 4) \Rightarrow |a| \in [0, 4)$

(C) Let $f(x) = x^3 + ax^2 + x + 1, f'(x) = 3x^2 + 2ax + 1$, $f'(x) \geq 0 \Rightarrow 4a^2 - 12 \leq 0 \Rightarrow a^2 \leq 3$

(D) $x^2 + 3ax + a^2 - 9 = 0$

Product of roots $= a^2 - 9 < 0 \Rightarrow a^2 < 9 \Rightarrow a \in (-3, 3)$

42. (A) → (4), (B) → (2), (C) → (3), (D) → (1)

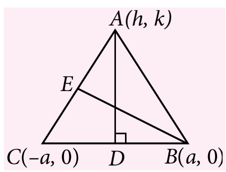
(A) $\frac{dy}{dx} = 2x + b > 0 \forall x \in [1, 3] \Rightarrow 2 + b \geq 0 \Rightarrow b \in [-2, \infty)$

(B) India played 4 matches in which maximum point can be earned is 2. So, maximum point in 4 matches is 8. Getting atleast 7 points means 7 or 8

$\therefore$ Required prob. $= P(7) + P(8)$;

$P(7) = 4C_1(0.35) \times (0.5)^3 = 0.175; P(8) = (0.5)^4 = 0.0625$

$\therefore$ Required prob. = 0.2375



(C) $S_1 = x^2 + y^2 - a^2 = 0$,

$S_2 = (x - h)(x - a) + (y - k)y = 0$

or $\quad x^2 + y^2 - x(h + a) - ky + ah = 0$

Replace a by $-a$, we get

$S_3 = x^2 + y^2 - x(h - a) - ky - ah = 0$

Radical axis of S_1 and S_2 is $S_1 - S_2 = 0$

$x(h + a) + ky - ah - a^2 = 0$

Replacing a by $-a$, radical axis of S_1 and S_3 is

$x(h - a) + ky + ah - a^2 = 0$

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JEE MAIN

Exam held on 8th April

SOLVED PAPER 2018

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1. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(a) $\frac{9}{2}$ (b) 6 (c) $\frac{7}{2}$ (d) 4

2. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}\cdot\vec{b} = 24$, then $|\vec{u}|^2$ is equal to

(a) 336 (b) 315 (c) 256 (d) 84

3. For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then

$$\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\dots+\left[\frac{15}{x}\right]\right)$$

(a) does not exist in R (b) is equal to 0

(c) is equal to 15 (d) is equal to 120

4. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane containing the lines L_1 and L_2 is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{4\sqrt{2}}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

5. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x}\,dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{2}$ (d) 4π

6. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$ is

(a) $\frac{1}{2}(\sqrt{2}-1)$ (b) $\frac{1}{2}(\sqrt{3}-1)$

(c) $\frac{1}{2}(\sqrt{3}+1)$ (d) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$

7. If sum of all the solutions of the equation

$$8\cos x\cdot\left(\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$$

in $[0, \pi]$ is $k\pi$, then k is equal to

(a) $\frac{20}{9}$ (b) $\frac{2}{3}$ (c) $\frac{13}{9}$ (d) $\frac{8}{9}$

8. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is

(a) $2\sqrt{2}$ (b) 3 (c) -3 (d) $-2\sqrt{2}$

9. The integral

$$\int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}dx$$

is equal to

(a) $\frac{-1}{1+\cot^3 x}+C$ (b) $\frac{1}{3(1+\tan^3 x)}+C$

(c) $\frac{-1}{3(1+\tan^3 x)}+C$ (d) $\frac{1}{1+\cot^3 x}+C$

10. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

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He trains IIT and Olympiad aspirants.

(a) $\frac{3}{4}$ (b) $\frac{3}{10}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

11. Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$ respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

(a) $\frac{3\sqrt{5}}{2}$ (b) $\sqrt{10}$ (c) $2\sqrt{10}$ (d) $3\sqrt{\frac{5}{2}}$

12. If the tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is

(a) 95 (b) 195 (c) 185 (d) 85

13. If $\alpha, \beta \in C$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to

(a) 2 (b) -1 (c) 0 (d) 1

14. PQR is a triangular park with $PQ = PR = 200$ m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is

(a) $50\sqrt{2}$ (b) 100

(c) 50 (d) $100\sqrt{3}$

15. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2, ..., x_9$ is

(a) 3 (b) 9 (c) 4 (d) 2

16. The sum of the co-efficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$, $(x > 1)$ is

(a) 2 (b) -1 (c) 0 (d) 1

17. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point $T(0, 3)$, then the area (in sq. units) of ΔPTQ is

(a) $36\sqrt{5}$ (b) $45\sqrt{5}$

(c) $54\sqrt{3}$ (d) $60\sqrt{3}$

18. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is

(a) at least 750 but less than 1000

(b) at least 1000 (c) less than 500

(d) at least 500 but less than 750

19. If the system of linear equations

$x + ky + 3z = 0$

$3x + ky - 2z = 0$

$2x + 4y - 3z = 0$

has a non-zero solutions (x, y, z), then $\frac{xz}{y^2}$ is equal to

(a) 30 (b) -10 (c) 10 (d) -30

20. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$, then the ordered pair (A, B) is equal to

(a) $(4, 5)$ (b) $(-4, -5)$

(c) $(-4, 3)$ (d) $(-4, 5)$

21. Two sets A and B as under :

$A = \{(a, b)\} \in R \times R : |a - 5| < 1$ and $|b - 5| < 1\}$;

$B = \{(a, b)\} \in R \times R : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$.

Then

(a) neither $A \subset B$ nor $B \subset A$

(b) $B \subset A$ (c) $A \subset B$

(d) $A \cap B = \phi$(an empty set)

22. Tangent and normal are drawn at $P(16, 16)$ on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is

(a) $\frac{4}{3}$ (b) $\frac{1}{2}$ (c) 2 (d) 3

23. Let $S = \{t \in R : f(x) = |x - \pi| \cdot (e^{|x|} - 1) \sin|x|$ is not differentiable at $t\}$, then the set S is equal to

(a) $\{0, \pi\}$ (b) ϕ (an empty set)

(c) $\{0\}$ (d) $\{\pi\}$

24. The Boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to

(a) $\sim q$ (b) $\sim p$

(c) p (d) q

25. A straight line through a fixed point $(2, 3)$ intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is

(a) $3x + 2y = 6xy$ (b) $3x + 2y = 6$

(c) $2x + 3y = xy$ (d) $3x + 2y = xy$

26. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + ...$

If $B - 2A = 100\lambda$, then λ is equal to

(a) 496 (b) 232

(c) 248 (d) 464

27. Let $y = y(x)$ be the solution of the differential equation $\sin x \dfrac{dy}{dx} + y\cos x = 4x, x \in (0, \pi)$. If $y\left(\dfrac{\pi}{2}\right) = 0$, then $y\left(\dfrac{\pi}{6}\right)$ is equal to

(a) $-\dfrac{4}{9}\pi^2$ (b) $\dfrac{4}{9\sqrt{3}}\pi^2$

(c) $\dfrac{-8}{9\sqrt{3}}\pi^2$ (d) $-\dfrac{8}{9}\pi^2$

28. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane $x + y + z = 7$ is

(a) $\sqrt{\dfrac{2}{3}}$ (b) $\dfrac{2}{\sqrt{3}}$

(c) $\dfrac{2}{3}$ (d) $\dfrac{1}{3}$

29. Let $S = \{x \in R : x \geq 0$ and $2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S

(a) contains exactly four elements
(b) is an empty set
(c) contains exactly one element
(d) contains exactly two elements

30. Let $a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$.

$a_1^2 + a_2^2 + \ldots\ldots\ldots + a_{17}^2 = 140m$, then m is equal to

(a) 33 (b) 66
(c) 68 (d) 34

SOLUTIONS

1. (a) : Let the curves $y^2 = 6x$ and $9x^2 + by^2 = 16$ intersect at (α, β).

$\beta^2 = 6\alpha$ and $9\alpha^2 + b\beta^2 = 16$

$\left.\dfrac{dy}{dx}\right|_{(\alpha,\beta)}$ for curve $y^2 = 6x$ is $\dfrac{dy}{dx} = \dfrac{3}{y} = \dfrac{3}{\beta}$

$\left.\dfrac{dy}{dx}\right|_{(\alpha,\beta)}$ for curve $9x^2 + by^2 = 16$ is $\dfrac{dy}{dx} = -\dfrac{9x}{by} = -\dfrac{9\alpha}{b\beta}$

As the curves are orthogonal, we have

$$\left(\frac{3}{\beta}\right)\left(-\frac{9\alpha}{b\beta}\right) = -1$$

As $\beta^2 = 6\alpha$, we get $27\alpha = b(\alpha \cdot 6)$

$\Rightarrow\ b = \dfrac{27}{6} = \dfrac{9}{2}$ (as $\alpha \neq 0$)

For $b = 0$ the intersection is non orthogonal. So we can rule out $b = 0$ in the beginning only to conclude $\alpha \neq 0$ in the end.

[Rating : Medium]

2. (a) : Let $\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{u}\cdot\vec{a} = 0$ gives $2x + 3y - z = 0$ ...(i)

$\vec{u}\cdot\vec{b} = 24$ gives $y + z = 24$ ...(ii)

Also $\vec{u}$ is coplanar with $\vec{a}$ and $\vec{b}$, so $[\vec{u}\ \vec{a}\ \vec{b}] = 0$ which yields

$$\begin{vmatrix} x & y & z \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

i.e. $4x - 2y + 2z = 0$ ...(iii)

(ii) and (iii) gives $2x + 2z = 24$ *i.e.* $x + z = 12$

From (i), we get $z = 16$ and thus $x = -4$ and $y = 8$.

Hence, $\vec{u} = -4\hat{i} + 8\hat{j} + 16\hat{k}$

$|\vec{u}| = 4\sqrt{1^2 + 2^2 + 4^2} = 4\sqrt{21}$

$\therefore\ |\vec{u}|^2 = 336$

Alternative solution-1:

As $\vec{u}$ is coplanar with $\vec{a}$ and $\vec{b}$, we have $\vec{u} = \lambda\vec{a} + \mu\vec{b}$ for scalars λ and μ.

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ gives

$\vec{u} = 2\lambda\hat{i} + (3\lambda + \mu)\hat{j} + (\mu - \lambda)\hat{k}$

$\vec{u}\cdot\vec{b} = 24$ gives $\lambda + \mu = 12$

$\vec{u}\cdot\vec{a} = 0$ gives $7\lambda + \mu = 0$

Solving the above, we get $\lambda = -2, \mu = 14$

Hence, $\vec{u} = -4\hat{i} + 8\hat{j} + 16\hat{k}$

$\therefore\ |\vec{u}|^2 = 336$

Alternative solution-2:

This is the most efficient solution, involving the idea of triple product.

A vector that is coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{a}$ can be taken as

$$\begin{aligned}\vec{u} &= m\vec{a}\times(\vec{a}\times\vec{b}) \\ &= m\{|\vec{a}|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\} \\ &= m\{-4\hat{i} + 8\hat{j} + 16\hat{k}\} \\ &= -4m\{\hat{i} - 2\hat{j} - 4\hat{k}\} \\ &= m'\{\hat{i} - 2\hat{j} - 4\hat{k}\}\end{aligned}$$

As $\vec{u}\cdot\vec{b} = 24$ we have $m'(-2 - 4) = 24 \Rightarrow m' = -4$

Thus $\vec{u} = -4(\hat{i} - 2\hat{j} - 4\hat{k})$

As before $|\vec{u}|^2 = 4^2(1^2 + 2^2 + 4^2) = 16 \cdot 21 = 336$

[Rating : Difficult]

3. (d) : Observe that $t - 1 < [t] \le t$

Applying this to numbers $\frac{1}{x}, \frac{2}{x}, ..., \frac{15}{x}$ and summing them, we have

$$\sum_{k=1}^{15} \frac{k}{x} - 15 < \sum_{k=1}^{15} \left[\frac{k}{x}\right] \le \sum_{k=1}^{15} \frac{k}{x}$$

Multiplying throughout by x, we have

$$\sum_{k=1}^{15} k - 15x < x \sum_{k=1}^{15} \left[\frac{k}{x}\right] \le \sum_{k=1}^{15} k$$

Putting the limit $x \to 0^+$, we have

$120 < L \le 120$

As the limit from both sides approaches to 120, we have by sandwich principle, the required limit = 120.

[Rating : Difficult]

4. (c) : A plane passing through the intersection of the given planes is

$(2x - 2y + 3z - 2) + \lambda\,(x - y + z + 1) = 0$

i.e. $(\lambda + 2)x - (2 + \lambda)y + (\lambda + 3)z + (\lambda - 2) = 0$

The plane is having infinite number of solutions with $x + 2y - z - 3 = 0$ and $3x - y + 2z - 1 = 0$.

$$\therefore \begin{vmatrix} (\lambda+2) & -(\lambda+2) & (\lambda+3) \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

$\Rightarrow$ $(\lambda + 2)(4 - 1) + (\lambda + 2)(2 + 3) + (\lambda + 3)(-1 - 6) = 0$

$\Rightarrow$ $\lambda = 5$

$\therefore$ The equation of the plane becomes

$7x - 7y + 8z + 3 = 0$

The perpendicular distance from origin is

$$\frac{3}{\sqrt{7^2 + 7^2 + 8^2}} = \frac{3}{\sqrt{162}} = \frac{3}{9\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

[Rating : Medium]

5. (a) : Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x}\,dx$ ...(i)

Changing x to $-\pi/2 + \pi/2 - x = -x$, we have

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^{-x}}\,dx \quad ...(ii)$$

Adding (i) & (ii), we get

$$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x \left\{\frac{1}{1 + 2^x} + \frac{1}{1 + 2^{-x}}\right\} dx$$

$$= \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$$

$$= 2\int_{0}^{\pi/2} \sin^2 x\,dx \quad \text{(As the integral function is even)}$$

$\therefore$ $I = \frac{\pi}{4}$

[Rating : Easy]

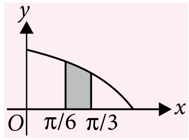
6. (b) : $y = (gof)(x) = \cos\sqrt{x^2} = \cos|x| = \cos x$

$[\because \cos(-x) = \cos x]$

Consider $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow$ $(3x - \pi)(6x - \pi) = 0$

$\Rightarrow$ $x = \frac{\pi}{3}, \frac{\pi}{6}$



$\therefore$ Required area

$$= \int_{\pi/6}^{\pi/3} \cos x\,dx = [\sin x]_{\pi/6}^{\pi/3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{3} - 1)$$

[Rating : Easy]

7. (c) : Note that $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$

We have,

$8\cos x\,\{\cos(\pi/6 + x)\cos(\pi/6 - x) - 1/2\} = 1$

$$\Rightarrow \quad 8\cos x\left\{\cos^2\frac{\pi}{6} - \sin^2 x - 1/2\right\} = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{3}{4} - 1 + \cos^2 x - \frac{1}{2}\right) = 1$$

$$\Rightarrow \quad 8\cos x(\cos^2 x - 3/4) = 1$$

$$\Rightarrow \quad 8\cos x\frac{(4\cos^2 x - 3)}{4} = 1$$

$$\Rightarrow \quad (4\cos^3 x - 3\cos x) = \frac{1}{2}$$

$$\Rightarrow \quad \cos 3x = \frac{1}{2}$$

As $x \in [0, \pi]$ $\Rightarrow$ $3x \in [0, 3\pi]$ which gives

$3x = \pi/3, 5\pi/3, 7\pi/3$

$\therefore$ $x = \pi/9, 5\pi/9, 7\pi/9$

which gives sum of all values of $x = \frac{13\pi}{9}$

$\therefore$ $k = \frac{13}{9}$

[Rating : Medium]

8. (a) : We have, $f(x)=x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2$

And $g(x)=x-\frac{1}{x}$ for $x \in R-\{-1, 0, 1\}$

$$\therefore \quad h(x)=\frac{\left(x-\frac{1}{x}\right)^2+2}{x-\frac{1}{x}}=\left(x-\frac{1}{x}\right)+\frac{2}{\left(x-\frac{1}{x}\right)}$$

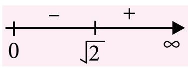
Put $x-\frac{1}{x}=t$

$\therefore \quad H(t)=t+\frac{2}{t}$ for $t\in(-\infty,\infty)-\{0\}$

Consider $H(t)$ on the interval $(0, \infty)$

$$H(t)=t+\frac{2}{t}$$

$$H'(t)=1-\frac{2}{t^2}$$



So, $H(t)$ is decreasing on $(0,\sqrt{2})$ and increasing on $(\sqrt{2},\infty)$. Thus $H(t)$ has a local minimum at $t=\sqrt{2}$.

$\therefore \quad H(\sqrt{2})=2\sqrt{2}$ is the local minimum value of the function at $\sqrt{2}$.

Remark : Observe that

$$t+\frac{2}{t}\geq 2\sqrt{2}$$

thereby again contribute that $2\sqrt{2}$ is a local minimum.

[Rating : Difficult]

9. (c) : Let $I=\int\frac{\sin^2 x\cos^2 x}{(\sin^2 x+\cos^2 x)^2(\sin^3 x+\cos^3 x)^2}dx$

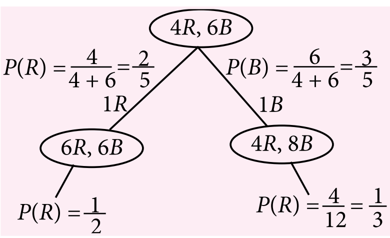
$$=\int\frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}=\int\frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx$$

Put $1+\tan^3 x=t$ so that $3\tan^2 x\sec^2 x\,dx=dt$

$$\therefore \quad I=\frac{1}{3}\int\frac{dt}{t^2}=-\frac{1}{3}\cdot\frac{1}{t}+C=-\frac{1}{3}\cdot\frac{1}{(1+\tan^3 x)}+C$$

[Rating : Difficult]

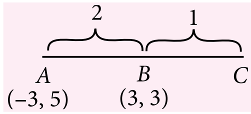
10. (c) : Let's draw a state diagram to understand whole condition



The requested probability is the sum of the product of the probabilities along the two possible paths and is equal to $\frac{2}{5}\cdot\frac{1}{2}+\frac{3}{5}\cdot\frac{1}{3}=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$

[Rating : Medium]

11. (d) : We know that the centroid divides orthocentre and circumcentre in the ratio 2 : 1.



$$AC=\frac{3}{2}AB=\frac{3}{2}\cdot\sqrt{6^2+2^2}=\frac{3}{2}\cdot 2\sqrt{10}=3\sqrt{10}$$

Radius of the circle with AC as diameter $=\frac{3}{2}\sqrt{10}=3\sqrt{\frac{5}{2}}$

[Rating : Difficult]

12. (a) : The equation of tangent at (1, 7) to $x^2=y-6$ is $2x-y+5=0$

The perpendicular distance of centre (–8, –6) to the line $2x-y+5=0$ should be equal to the radius of the circle.

$$\therefore \quad \sqrt{64+36-c}=\frac{|-16+6+5|}{\sqrt{5}}$$

$$\Rightarrow \quad \sqrt{5}=\sqrt{100-c} \Rightarrow c=95$$

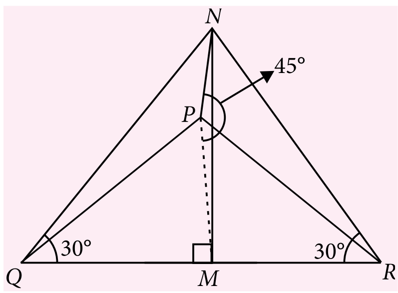
[Rating : Easy]

13. (d) : $x^2-x+1=0$ has its roots $-\omega, -\omega^2$.

Now $(-\omega)^{101}+(-\omega^2)^{107}=-\{\omega^2+\omega^4\}=-(\omega^2+\omega)=1$

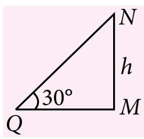
[Rating : Easy]

14. (b) :



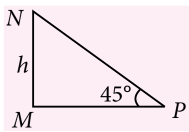
Let the height of tower MN be h.

The triangle NMQ gives

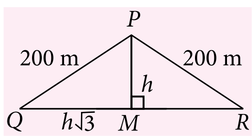


$$QM=h\sqrt{3}, \text{ as } \tan 30°=\frac{1}{\sqrt{3}}$$

The triangle NMP gives



$PM=h$



As ΔPQR is isosceles, PM is also an altitude.

$\therefore$ $PM^2 + QM^2 = PQ^2$ gives $4h^2 = (200)^2 \Rightarrow h = 100$

[Rating : Medium]

15. (d) : The standard deviation is independent of change of origin. So, put $x_i - 5 = y_i$

$\therefore$ Given equations become

$$\sum_{i=1}^{9} y_i = 9 \text{ and } \sum_{i=1}^{9} y_i^2 = 45$$

$$\text{SD} = \sqrt{\frac{1}{n}\sum y_i^2 - \left(\frac{\Sigma y_i}{n}\right)^2}$$

$$= \sqrt{\frac{45}{9} - \left(\frac{9}{9}\right)^2} = \sqrt{5-1} = \sqrt{4} = 2$$

[Rating : Medium]

16. (a) : We have,

$(a + b)^5 + (a - b)^5 = 2\{a^5 + {}^5C_2.a^3b^2 + {}^5C_4\, ab^4\}$

with $a = x,\ b = \sqrt{x^3 - 1}$

$$\therefore \left(x + \sqrt{x^3-1}\right)^5 + \left(x - \sqrt{x^3-1}\right)^5$$

$= 2\{x^5 + 10x^3(x^3 - 1) + 5x(x^3 - 1)^2\}$

Sum of the coeff. of odd degree terms is

$2\{1 - 10 + 5 + 5\} = 2$

[Rating : Easy]

17. (b) : Let the tangent at (α, β) be

$4x\alpha - y\beta = 36$

As (0, 3) lies on the tangent, so we have

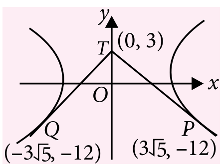
$-3\beta = 36 \Rightarrow \beta = -12$

Now $4\alpha^2 - \beta^2 = 36$ gives $4\alpha^2 - 12^2 = 36$

$\Rightarrow\ 4\alpha^2 = 180 \Rightarrow \alpha^2 = 45 \Rightarrow \alpha = \pm 3\sqrt{5}$

Thus the points P and Q are $P(3\sqrt{5}, -12), Q(-3\sqrt{5}, -12)$

The area of the ΔTQP is given by



$$\Delta = \left\|\frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 3\sqrt{5} & -12 & 1 \\ -3\sqrt{5} & -12 & 1 \end{vmatrix}\right\|$$

$$= \left|\frac{1}{2}[-3\left(6\sqrt{5}\right) - 36\sqrt{5} - 3\,6\sqrt{5}]\right| = \frac{1}{2}\,90\sqrt{5} = 45\sqrt{5}$$

Alternative solution:

The equation of chord of contact from (0, 3) is

$4 \cdot x \cdot 0 - y \cdot 3 = 36 \Rightarrow y = -12$

$\therefore$ The point P is $(3\sqrt{5}, -12)$

Area of triangle $PQT = \frac{1}{2}.PQ.TR$

[Where R is the mid-point of PQ]

$$= \frac{1}{2}.6\sqrt{5}.15 = 45\sqrt{5}$$

[Rating : Difficult]

18. (b) : The number of ways to choose 4 novels out of 6 is ${}^6C_4$.

The number of ways to choose 1 dictionary out of 3 is ${}^3C_1$.

As the place of dictionary is fixed, so total number of ways

$= {}^6C_4\, {}^3C_1 \cdot 4! = 15 \cdot 3 \cdot 24 = 1080$ **[Rating : Easy]**

19. (c) : For non-zero solutions, we have

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$$

$\Rightarrow\ 1(-3k + 8) - k(-9 + 4) + 3(12 - 2k) = 0$

which gives $k = 11$

Now, the system of equations become

$x + 11y + 3z = 0$ ...(i)

$3x + 11y - 2z = 0$ ...(ii)

$2x + 4y - 3z = 0$ ...(iii)

The equation (i) and (iii) gives

$3x + 15y = 0$ *i.e.* $x = -5y$

Putting $x = -5y$ in (i), we have

$-5y + 11y + 3z = 0$

$\Rightarrow\ z = -2y$

$$\text{Now } \frac{xz}{y^2} = \frac{(-5y)(-2y)}{y^2} = 10$$

[Rating : Difficult]

20. (d) : As both sides are polynomial in x, let's set $x = 0$ to obtain

$$\begin{vmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{vmatrix} = A^3$$

which gives $A^3 = -64 \quad \therefore\ A = -4$

Taking x common from all rows of given determinant, we get

$$\begin{vmatrix} 1-\frac{4}{x} & 2 & 2 \\ 2 & 1-\frac{4}{x} & 2 \\ 2 & 2 & 1-\frac{4}{x} \end{vmatrix} = \left(B - \frac{4}{x}\right)\left(1 + \frac{4}{x}\right)^2$$

Take the limit as $x \to \infty$ to obtain

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = B \Rightarrow B = 5$$

Alternative solution:

The most efficient way to obtain the result is to use this result

$$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = (a+2b)(a-b)^2$$

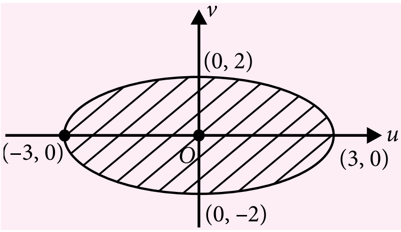
This result gives $A = -4$, $B = 5$

Note that the determinant, when $a = b$, vanish and all the three rows become identical hence $(a - b)^2$ is a factor.

[Rating : Medium]

21. (c) : Let's effect a change of origin

$a - 6 = u$ and $b - 5 = v$



The set B becomes $\dfrac{u^2}{9} + \dfrac{v^2}{4} \le 1$

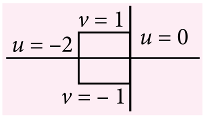
which is the interior of the ellipse.

Now the set A becomes

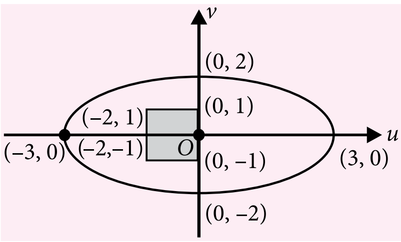
$|u + 1| < 1$ *i.e.* $-2 < u < 0$

$|v| < 1$ *i.e.* $-1 < v < 1$

which is a square.



Drawing the two diagram together, we have



So $A \subset B$ as all the four points $(0, 1)$, $(0, -1)$, $(-2, -1)$ and $(-2, 1)$ lie inside the ellipse.

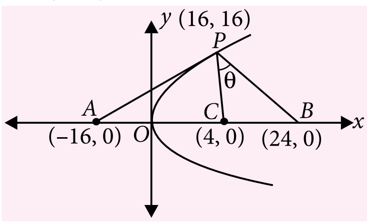
[Rating : Difficult]

22. (c) : The equation of tangent at $P(16, 16)$ is

$x - 2y + 16 = 0$

The equation of normal at $P(16, 16)$ is

$2x + y - 48 = 0$



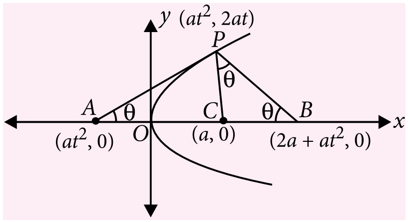
The slope of PC : $m_1 = \dfrac{16}{12} = \dfrac{4}{3}$

The slope of PB : $m_2 = \dfrac{-16}{8} = -2$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{4}{3}+2}{1-\frac{4}{3}(2)}\right| = \left|\frac{\frac{10}{3}}{-\frac{5}{3}}\right| = 2$$

Alternative solution:

Note that for a general point $(at^2, 2at)$ the diagram is as under



$$\text{As } \tan(90^\circ - \theta) = \frac{1}{t} = \frac{1}{2} \Rightarrow \quad \cot\theta = \frac{1}{2}$$

So, $\tan\theta = 2$ **[Rating : Difficult]**

23. (b) : At $x = 0$, we have

$$\text{L.H.D.} = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(\pi+h)(e^h - 1)\sin h}{(-h)} = \pi(0)(-1) = 0$$

$$\text{R.H.D.} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(\pi-h)(e^h-1)\sin h}{h}$$

$= (\pi)(0)(1) = 0$

Let's check at $x = \pi$

$$\text{L.H.D.} = \lim_{h\to 0} \frac{f(\pi-h) - f(\pi)}{-h}$$

$$= \lim_{h\to 0^+} \frac{h(e^{\pi-h} - 1)\sin h}{-h} = (0)(e^{\pi} - 1)(-1) = 0$$

$$\text{R.H.D.} = \lim_{h\to 0} \frac{f(\pi+h) - f(\pi)}{h}$$

$$= \lim_{h\to 0} \frac{h(e^{\pi+h} - 1)(-\sin h)}{h} = (0)(e^{\pi} - 1)(-1) = 0$$

Thus f is differentiable at both $x = 0$ and $x = \pi$.

Remark : This happens as $x = 0$ and $x = \pi$ both are repeated roots of the given function.

[Rating : Difficult]

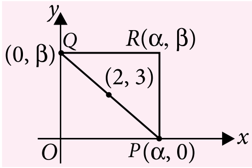
24. (b) : $\sim(p \vee q) \vee (\sim p \wedge q)$

$= (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

$= \sim p \wedge (\sim q \vee q) = \sim p$ **[Rating : Medium]**

25. (d) :
The equation of the given line is

$$\frac{x}{\alpha}+\frac{y}{\beta}=1 \quad ...(i)$$



As (2, 3) lies on (i), $\frac{2}{\alpha}+\frac{3}{\beta}=1$

$\Rightarrow \quad 2\beta + 3\alpha - \alpha\beta = 0$

changing (α, β) to (*x*, *y*) we have the locus of *R* as

$3x + 2y - xy = 0$

[Rating : Medium]

26. (c) : Let $S = 1^2 + 2\cdot2^2 + 3^2 + 2\cdot4^2 + 5^2 + 2\cdot6^2 + ...$

The sum of first 20 terms is

$A = (1^2 + 2^2 + ... + 20^2) + (2^2 + 4^2 + ... + 20^2)$

$$=\frac{20\cdot21\cdot41}{6}+4\cdot\frac{10\cdot11\cdot21}{6}=\frac{20\cdot21}{6}(41+22)=4410$$

$B = 1^2 + 2\cdot2^2 + ... + 2\cdot40^2$

$= (1^2 + 2^2 + ... + 40^2) + (2^2 + 4^2 + ... + 40^2)$

$$=\frac{40\cdot41\cdot81}{6}+\frac{4\cdot20\cdot21\cdot41}{6}=\frac{40\times41}{6}(81+42)$$

$$=\frac{40\cdot41}{6}\times123=33620$$

$B - 2A = 33620 - 8820 = 24800$

$\therefore \quad 100\lambda = 24800 \quad \Rightarrow \quad \lambda = 248$

[Rating : Medium]

27. (d) : $\frac{dy}{dx}+(\cot x)y=4x\operatorname{cosec}x$

$$\text{I.F.}=e^{\int\cot x\,dx}=e^{\log(\sin x)}=\sin x$$

Then the solution is given by

$$y\cdot\sin x=\int 4x\operatorname{cosec}(x)\sin x\,dx+C$$

i.e. $\quad y\sin x = 2x^2 + C$

As $y(\pi/2) = 0$, we have $C = -\pi^2/2$

So, $y\sin x = 2x^2 - \pi^2/2$

$$\therefore \quad y(\pi/6)=2\left\{\frac{2\pi^2}{36}-\frac{\pi^2}{2}\right\}=2\pi^2\left\{\frac{1}{18}-\frac{1}{2}\right\}=-\frac{8}{9}\pi^2$$

[Rating : Medium]

28. (a) : The direction ratios of *AB*, where *A*(5, –1, 4) and *B*(4, –1, 3) are (1, 0, 1)

Let the angle between *AB* and plane is θ, which gives

$$\sin\theta=\frac{2}{\sqrt6} \quad i.e.\ \cos\theta=\frac{1}{\sqrt3}$$

The projection of *AB* on the plane = $AB\cos\theta$

$$=\sqrt2\cdot\frac{1}{\sqrt3}=\sqrt{\frac{2}{3}}$$

[Rating : Medium]

29. (d) : Given $2|\sqrt{x}-3|+\sqrt{x}(\sqrt{x}-6)+6=0$

$$\Rightarrow \quad 2|\sqrt{x}-3|+(\sqrt{x}-3)^2-3=0$$

set $|\sqrt{x}-3|=t$, which gives

$t^2 + 2t - 3 = 0$

$\Rightarrow \quad (t + 3)(t - 1) = 0$

$\Rightarrow \quad t = -3, 1$

As $t \geq 0$ we have $t = 1$

Now $\left|\sqrt{x}-3\right|=1$

$$\Rightarrow \quad \sqrt{x}-3=1 \text{ or } -1 \quad \Rightarrow \quad \sqrt{x}=4,2$$

So, $x = 16, 4$

Thus, there are two solutions.

[Rating : Medium]

30. (d) : $\sum_{k=0}^{12} a_{4k+1}=416$

$\Rightarrow \quad a_1 + a_5 + ... + a_{49} = 416$

$$\Rightarrow \quad \frac{13}{2}(a_1+a_{49})=416$$

As $a_{49} = a_1 + 48d$, so we have

$$\frac{13}{2}(2a_1+48d)=416,$$

$\Rightarrow \quad a_1 + 24d = 32 \quad ...(i)$

Given, $a_9 + a_{43} = 66$

$\Rightarrow \quad a_1 + 8d + a_1 + 42d = 66$

$\Rightarrow \quad a_1 + 25d = 33 \quad ...(ii)$

The equation (i) and (ii) gives

$a_1 = 8,\ d = 1$

Now $a_1^2 + a_2^2 + ... + a_{17}^2 = 8^2 + 9^2 + ... + 24^2$

$= (1^2 + 2^2 + ... + 24^2) - (1^2 + 2^2 + ... + 7^2)$

$$=\frac{24\cdot25\cdot49}{6}-\frac{7\cdot8\cdot15}{6}$$

$= 4\cdot25\cdot49 - 7\cdot20 = 4900 - 140 = 4760 = 34(140)$

$\therefore \quad m = 34$

[Rating : Medium]

Solution Sender of Maths Musing

SET-183

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SET-184

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OLYMPIAD CORNER

1. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum
$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n-1)x}{2n-1} \text{ is positive.}$$

2. The incircle of ABC touches BC, CA and AB at D, E and F respectively. X is a point inside ABC such that the incircle so XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

3. An acute triangle ABC is given.Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A) and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that
$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$
The lines AA_1, BB_1 and CC_1 bound a triangle, and the lines AA_2, BB_2 and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

4. Prove that the average of the numbers $n \sin n°$, $n = 2, 4, 6, \ldots, 180$ is $\cot 1°$.

5. Let p be a prime. Find all solutions in positive integers of the equation :
$$\frac{2}{a} + \frac{3}{b} = \frac{5}{p}.$$

6. Find the value of the continued root :
$$\sqrt{4 + 27\sqrt{4 + 29\sqrt{4 + 31\sqrt{4 + 33\sqrt{\ldots}}}}}\ .$$

7. $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ are positive numbers ($k < n$). suppose that the values of $a_{k+1}, \ldots, a_n$ are fixed. How should one choose the values of $a_1, \ldots, a_n$ in order to minimise $\sum_{i, j, i \neq j} \frac{a_i}{a_j}$?

SOLUTIONS

1. We use mathematical induction.

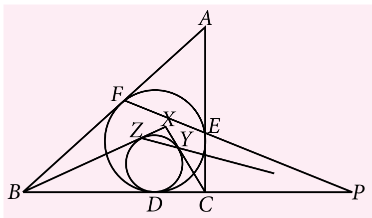
Let $S_n(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{(2k-1)}$.

$S_1(x) = \sin x > 0$ for $x \in (0, \pi)$. Thus the proposed inequality is true for $n = 1$. Let $S_r(x) > 0$ for $r = 1, 2, \ldots, n-1$. We will deduce that $S_n(x) > 0$ for $x \in (0, \pi)$. Suppose that $S_n(x_0) \le 0$ for some $x_0 \in (0, \pi)$, and that $S_n(x)$ attains its minimum at $x = x_0$. Hence $\frac{d}{dx}[S_n(x)]_{x=x_0} = 0$.

That is
$$S_n'(x_0) = \sum_{k=1}^{n} \cos((2k-1)x_0) = 0,$$
so that
$$2 \sin x_0 S_n'(x_0) = \sum_{k=1}^{n} 2\cos((2k-1)x_0)\sin x_0$$
$$= \sum_{k=1}^{n} [\sin(2kx_0) - \sin((2k-2)x_0)]$$
$$= \sin 2nx_0.$$
Thus $S_n'(x_0) = \frac{\sin 2nx_0}{2\sin x_0} = 0$ implying $\sin 2nx_0 = 0$.

Hence
$$x_0 \in \left\{\frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \ldots, \frac{(2n-1)}{2n}\right\}.$$
It is easily verified that at each of these values $S_n(x_0) > 0$, a contradiction. Hence $S_n(x) > 0$ for $x \in (0, \pi)$.

2.



Let P be the intersection of EF with BC. Then by Menelaus' Theorem, we have

$$\frac{BP}{PC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1 \quad ...(1)$$

Since $CE = CD$, $EA = AF$, and $FB = BD$, we get

$$\frac{BP}{PC}\cdot\frac{CD}{BD}=1$$

so that $\frac{BP}{PC}=\frac{BD}{CD}$ ...(2)

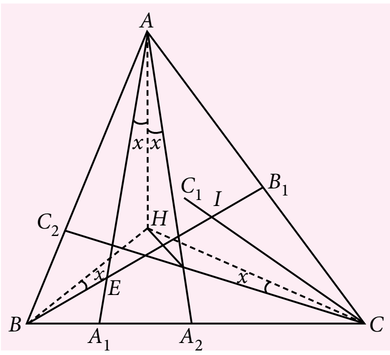
Since $XZ = XY$, $BZ = BD$ and $CY = CD$, we have from (2)

$$\frac{BP}{PC}\cdot\frac{CY}{YX}\cdot\frac{XZ}{ZB}=\frac{BD}{CD}\cdot\frac{CD}{YX}\cdot\frac{XY}{BD}=1$$

Hence by Menelaus' Theorem P, Z and Y are collinear. Since $PF\cdot PE = PD^2$ and $PZ\,.\,PY = PD^2$, we have $PF\cdot PE = PZ\cdot PY$.

Hence $EFZY$ is a cyclic quadrilateral.

3.



Let AA_1, BB_1 meet at the point E; AA_1, CC_2 meet at the point F; and BB_1, CC_1 meet at the point I. Also $\angle A_1AA_2 = \angle B_1BB_2 = \angle C_1CC_2 = 2x$. (1)

The bisectors of the angles at A_1, B_1 and C_1 in triangles ΔA_1AA_2, ΔB_1BB_2 and ΔC_1CC_2 respectively are perpendicular to their respective bases. Hence they are the altitudes of ΔABC. Let H be the orthocentre of ΔABC.

Since $\angle A_1AH = \angle B_1BH = x$ and $\angle A_1AH = \angle C_1CH = x$ each one of the quadrilateral $AHEB$, $AHDC$ is inscribable in a circle.

These two circles have a common chord, the segment AH and since $\angle ABH = \angle ACH = 90° - \angle BAC$, then the circles have equal radii.

Thus, since the inscribed angles $\angle EAH$, $\angle DAH$ are equal, the corresponding chords HE and HD are equal.

Therefore $HE = HD$. Similarly, we prove that $HD = HI$, and so on for all six vertices of these two triangles of the problem.

Thus, all six vertices lie at the same distance from the point H, and the points are concyclic.

4. Let $x = 2\sin 2° + 4\sin 4° + ... + 90\sin 90° + ... + 178\sin 178°$

$= (2 + 178)\sin 2° + (4 + 176)\sin 4° + ...$

$= 180(\sin 2° + \sin 4° + ... + \sin 88°) + 90\sin 90°$

Then

$\bar{x} = \frac{x}{90} = 2\sin 2° + 2\sin 4° + ... + 2\sin 88° + 1$

$\bar{x}\ \sin 1° = 2\sin 2°\sin 1° + 2\sin 4°\sin 1° + ... + 2\sin 88°\sin 1° + \sin 1°$.

Now, $2\sin 2°\sin 1° = \cos 1° - \cos 3°$

$2\sin 4°\sin 1° = \cos 3° - \cos 5°$

...

$2\sin 88°\sin 1° = \cos 87° - \cos 89°$

Hence, $\bar{x}\sin 1° = \cos 1° - \cos 89° + \sin 1° = \cos 1°$.

Thus $\bar{x} = \cot 1°$, as required.

5. We have $p(3a + 2b) = 5ab$; hence there are three cases to consider.

First case : $p = 5$, Then we get

$3a + 2b = ab$ or $(a - 2)(b - 3) = 6 = 1\cdot 6 = 2\cdot 3$.

Hence all pairs of positive integers (a, b) are $(3, 9)$, $(4, 6)$, $(5, 5)$ and $(8, 4)$.

Second case : p divides a. Let $a = a_1p$. Hence $3a_1p = b(5a_1 - 2)$.

Then p divides either b or $5a_1 - 2$. If $b = b_1p$, then $3a_1 = b_1(5a_1 - 2)$ or $(5a_1 - 2)(5b_1 - 3) = 6$ which has only one solution : $(a_1, b_1) = (1, 1)$, hence $(a, b) = (p, p)$. Otherwise, $5a_1 - 2 = a_2p$.

Therefore, $3\frac{a_2p+2}{5}=ba_2$ or $3a_2p + 6 = 5ba_2$.

Hence a_2 divides 6.

If $a_2 = 1$, then $(a, b) = \left(\frac{p(p+2)}{5},\frac{3(p+2)}{5}\right)$, but only if $p \equiv 3 \pmod 5$.

If $a_2 = 2$, then $(a, b) = \left(\frac{2p(p+1)}{5},\frac{3(p+1)}{5}\right)$, but only if $p \equiv 4 \pmod 5$.

If $a_2 = 3$, then $(a, b) = \left(\frac{p(3p+2)}{5},\frac{3p+2}{5}\right)$, but only if $p \equiv 1 \pmod 5$.

If $a_2 = 6$, then $(a, b) = \left(\frac{2p(3p+1)}{5},\frac{3p+1}{5}\right)$, but only if $p \equiv 3 \pmod 5$.

Third case : p divides b. Let $b = b_1 p$. Hence $2b_1 p = a(5b_1 - 3)$.

Then p divides either a or $5b_1 - 3$. If p divides a, then we have the same case (p divides a and b) as already considered above. Again (p, p) is the (only) solution. Otherwise, $5b_1 - 3 = b_2 p$. Therefore,

$2\frac{b_2 p + 3}{5} = ab_2$ or $2b_2 p + 6 = 5ab_2$.

Hence b_2 divides 6.

If $b_2 = 1$, then $(a, b) = \left(\frac{2(p+3)}{5}, \frac{p(p+3)}{5}\right)$, but only if $p \equiv 2$, (mod 5).

If $b_2 = 2$, then $(a, b) = \left(\frac{2p+3}{5}, \frac{p(2p+3)}{5}\right)$, but only if $p \equiv 1$, (mod 5).

If $b_2 = 3$, then $(a, b) = \left(\frac{2(p+1)}{5}, \frac{3p(p+1)}{5}\right)$, but only if $p \equiv 4$, (mod 5).

If $b_2 = 6$, then $(a, b) = \left(\frac{2p+1}{5}, \frac{3p(2p+1)}{5}\right)$, but only if $p \equiv 2$ (mod 5).

[Note that if $p = 2$ or 3, this generates only two distinct solutions : (2, 2), (1, 6) for $p = 2$ and (3, 3), (12, 2) for $p = 3$. If $p > 5$, then the three solutions are all distinct.]

6. **I.** More generally, for any positive integer n, we claim that

$$\sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\ldots}}}} = n+2,$$

where the left side is defined as the limit of

$$F(n,m) = \sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\ldots\sqrt{4+m\sqrt{4}}}}}}$$

as $m \to \infty$ (where m is an integer and $(m-n)$ is even)

If $g(n, m) = F(n, m) - (n + 2)$, we have

$$F(n, m)^2 - (n + 2)^2 = (4 + nF(n + 2, m)) - (4 + n(n + 4))$$

$$= n(F(n + 2, m) - (n + 4)),$$

so, $g(n,m) = \dfrac{n}{F(n,m)+n+2} g(n+2, m).$

Clearly $F(n, m) > 2$, so

$$|g(n,m)| < \frac{n}{n+4}|g(n+2,m)|.$$

By iterating this, we obtain

$$|g(n,m)| < \frac{n(n+2)}{m(m+2)}|g(m,m)| < \frac{n(n+2)}{m}.$$

Therefore $g(n, m) \to 0$ as $m \to \infty$

II. Let

$$S_n = \sqrt{4+(2n-1)\sqrt{4+(2n+1)\sqrt{4+(2n+3)\sqrt{\ldots}}}}$$

S_n satisfies the recurrence relation

$$S_n = \sqrt{4+(2n-1)S_{n+1}}$$

if and only if

$(S_n - 2)(S_n + 2) = (2n - 1)S_{n+1}$

By inspection, this admits $S_n = 2n + 1$ as a solution. We only have to prove that $S_1 = 3$ to make this induction complete. Let

$$T_n = \sqrt{4+\sqrt{4+3\sqrt{\ldots(2n-3)\sqrt{4+(2n-1)\sqrt{(2n+3)}}}}}$$

and

$$U_n = \sqrt{4+\sqrt{4+3\sqrt{\ldots(2n-3)\sqrt{4+(2n-1)(2n+3)}}}} = 3.$$

Clearly $T_n \le U_n$ and the latter is identically equal to 3. Therefore, using the fact that $B \ge A > 0$ implies that $\sqrt{(4+A)/(4+B)} \ge \sqrt{A/B}$

$$1 \ge \frac{T_n}{3} = \frac{T_n}{U_n} = \frac{\sqrt{4+\sqrt{\ldots+(2n-1)\sqrt{(2n+3)}}}}{\sqrt{4+\sqrt{\ldots+(2n-1)(2n+3)}}}$$

$$\ge \frac{\sqrt{\sqrt{\ldots+(2n-1)\sqrt{(2n+3)}}}}{\sqrt{\sqrt{\ldots+(2n-1)(2n+3)}}} \ge \ldots \ge \sqrt[2^{n+1}]{\frac{1}{2n+3}}$$

$$= \frac{1}{(2n+3)^{\left(\frac{1}{2}\right)^{n+1}}} \to 1$$

as $n \to \infty$ [for example, by rewriting as $\exp\{-\ln(2n + 3)/2^{n+1}\}$ and using L'Hospital rule].

This proves that $S_1 = \lim\limits_{n\to\infty} T_n = 3$. The required expression is precisely S_{14} and hence its value is 29.

MPP-1 CLASS XII ANSWER KEY

1. (d)	2. (d)	3. (b)	4. (b)	5. (d)
6. (d)	7. (a,b,c)	8. (c)	9. (b,d)	10. (a,b)
11. (a,b,c)	12. (a,b,c,d)	13. (a,c)	14. (d)	15. (a)
16. (c)	17. (0)	18. (0)	19. (3)	20. (3)

7. To minimise the given rational function, choose

$$a_i = \left(\frac{a_{k+1}+\ldots+a_n}{\frac{1}{a_{k+1}}+\ldots+\frac{1}{a_n}}\right)^{1/2} = (\mathbb{A}\cdot\mathbb{H})^{1/2},\ i = 1, 2, \ldots, k$$

where $\mathbb{A}$ is the arithmetic and $\mathbb{H}$ the harmonic mean of $a_{k+1}, \ldots, a_n$.

To prove this, we will be forgiven if we change notation : let $x_i = a_i$, $i = 1, 2, \ldots, k$ and $b_r = a_{k+r}$, $r = 1, \ldots, m$ with $k + m = n$, and denote the given rational function $F(x_1, \ldots, x_k)$. Then we have $F(x_1, \ldots, x_k) = X + Y + B$, where

$$X = \sum_{1\le i<j\le k}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}\right),\quad Y = \sum_{1\le i\le k}\ \sum_{1\le r\le m}\left(\frac{x_i}{b_r}+\frac{b_r}{x_i}\right),$$

$$B = \sum_{1\le r<s\le m}\left(\frac{b_r}{b_s}+\frac{b_s}{b_r}\right).$$

Note that B is fixed and Y can be improved to

$$Y = \sum_{1\le i\le k}\left(\left(\sum_{1\le r\le m}\frac{1}{b_r}\right)x_i + \left(\sum_{1\le r\le m} b_i\right)\frac{1}{x_i}\right)$$

$$= \sum_i\left(\frac{m}{\mathbb{H}}x_i + \frac{m\mathbb{A}}{x_i}\right)$$

where $\mathbb{A}$ is the arithmetic mean and $\mathbb{H}$ is the harmonic mean of the b_r.

Now we recall that the simple function $\alpha x + \frac{\beta}{x}$ (with α, β, x all positive) assumes its minimum when $\alpha x = \frac{\beta}{x}$; that is $x = \sqrt{\frac{\beta}{\alpha}}$. Thus each of the terms in Y (and so Y itself) assumes its minimum when we choose, for $i = 1, 2, \ldots, k$,

$$x_i = \sqrt{\frac{m\mathbb{A}}{(m/\mathbb{H})}} = \sqrt{\mathbb{AH}},$$

as asserted.

But there is more. It is also known that each term in X, (and so X itself) assumes its minimum when $x_i = x_j$, with $1 \le i < j \le k$. Thus choosing all $x_i = \sqrt{\mathbb{AH}}$ minimises both X and Y and, since B is fixed, minimises $F(x_1, \ldots, x_k)$ as claimed.

◈ ◈

TARGET JEE

Sets, Relations and Functions

SET

A set is a well defined collection of distinct objects (elements or members).

Sets are denoted by capital alphabets and their members or elements/objects by small alphabets like $A = \{a, b, c\}$.

As a is an element of set A, we write it $a \in A$ and read as a belongs to A or a is a member of A. If 'a' does not belong to A, then we write $a \notin A$.

REPRESENTATION OF SETS

There are two ways of representing sets.

- Roster method or Listing method or Tabular method.
- Rule or property or Set builder method

For example, set of even numbers between 7 and 15 can be written as

(a) Roster/Listing/Tabular method :
$A = \{8, 10, 12, 14\}$

(b) Set builder or Rule method :
$A = \{x : x \text{ is an even number}, 7 < x < 15\}$

TYPES OF SETS

- **Empty Set**

(1) A set having no element in it, is called empty or null or void set.

(2) Empty set is subset of every set and every set is subset of itself. We denote it by ϕ or { }.

- **Finite and Infinite Sets**

A set whose elements can be counted. In other words a set which has finite members in it is called finite set and set whose elements cannot be counted is said to be infinite. The set of vowels is finite set, the set of rational numbers between two rationals is an infinite set.

- **Joint and Disjoint Sets**

When two sets A and B having at least one common element between them is said to be joint other wise disjoint sets. In other words, if their intersection is non-empty, then they are called joint *i.e.* $A = \{a, b, c, 4\}$ and $B = \{2, 4, 6\}$ are joint sets and if $A = \{a, b, c\}$, $B = \{2, 4, 6\}$, then A and B are disjoint sets.

- **Pairwise Disjoint Family of Sets**

The family of sets A_1, A_2, A_3 is said to be pairwise disjoint family of sets if no two sets of the family are joint *i.e.* $A_1 \cap A_2 = \phi, A_2 \cap A_3 = \phi, A_3 \cap A_1 = \phi$.

e.g. $A_1 = \{1, 4, 7, 10\}, A_2 = \{2, 5, 8, 11\}$,
$A_3 = \{3, 6, 9, 12\}$ are paiwise disjoint sets.

- **Equal and Equivalent Sets**

(1) Two sets A and B are called equal sets if A and B having identical elements *i.e.* $A = \{a, b, c\}$, $B = \{b, a, c\}$.

(2) The two sets A and B are said to be equivalent if they have equal number of elements *i.e.* $n(A) = n(B)$.

(3) Equal sets are always equivalent but equivalent sets may or may not be equal.

- **Singleton Set**

A set consisting of single element is called a singleton set.

- **Universal Set**

If all the sets under consideration are subsets of a larger set, then this larger set is called universal set which we denote by U. The set of prime numbers $\{2, 3, 5, 7, 11, 13, ...\}$, the set of even numbers $\{2, 4, 6, 8, 10, ...\}$ and the set of odd numbers $\{1, 3, 5, 7, ...\}$ are subsets of $N = \{1, 2, 3, 4, 5, 6, ...\}$, which is universal set which can be denoted by U instead of N. In the plane geometry, the straight lines in a plane are subset of all points in the plane, which is universal set in respect of all straight lines.

ORDER/CARDINAL NUMBER OF A FINITE SET

If a set A having m elements then $n(A) = m$ is known as order of finite set A. The order of the set $A = \{1, 2, 3, 4, 5, 6, 9\}$ is 7 as it contains 7 members.

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SUBSET

For any two sets A and B, the set A is said to be subset of B if every element of set A is also the element of set B.

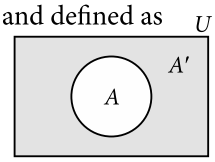
SUPERSET

For any two sets A and B if all elements of set A are present in B but some elements of set B are not present in A, then B is said to be super set of A and A is called subset of B.

COMPLEMENT OF A SET

Let U be the universal set and A is subset of U, then complement of A with respect to U is the set of elements which belongs to U, but not belongs to A. The complement of set A is denoted by A' or $\bar{A}$ or A^c and defined as

$A' = U - A = \{x : x \in U \text{ but } x \notin A\}$



In the venn diagram shaded region is complement of A and in this case we write $x \in A' \Leftrightarrow x \notin A$.

POWER SET

Let A be a set. Then family of all subsets of A is called power set of A, which is denoted by $P(A)$ and defined as $P(A) = \{R \mid R \subset A\}$.

e.g. If $A = \{a, b, c\}$, then

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, A\}$

If a set having n elements then it has 2^n subsets.

As $A = \{a, b, c\}$, then $n(P(A)) = 2^3 = 8$.

INTERVALS AS SUBSETS OF *R*

On real line, various types of subsets are designated as intervals as defined below:

- **Closed Interval**

Let $a, b \in R$ such that $a < b$. Then the set of all real numbers x such that $a \le x \le b$ is called a closed interval which is denoted by $[a, b]$.

i.e. $a \le x \le b$

- **Open Interval**

i.e. $a < x < b$

- **Semi Open or Semi-Closed Interval**

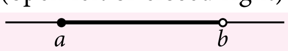
$]a, b]$ or $[a, b[$

(1)

i.e. $a < x \le b$

(open left or closed right)

(2)



i.e. $a \le x < b$ (open right or closed left)

OPERATION ON SETS

- **Union of Sets**

Union of two sets A and B is the set of all those elements which belongs to either A or B or both A and B. The union of two sets A and B is denoted by $A \cup B$ (read as 'A union B'). If $A = \{a, 2, 3, 4\}$, $B = \{2, 3, 7, 9, 11\}$, then

$A \cup B = \{1, 2, 3, 4, 7, 9, 11\}$

- **Intersection of Sets**

The intersection of two sets A and B is the set which contains all those elements which belongs to both the sets A and B. It is denoted by $A \cap B$ (read as 'A intersection B'). For the set A and B given above $A \cap B = \{2, 3\}$

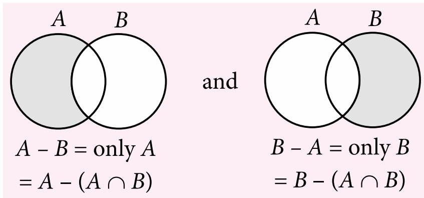
- **Difference of Two Sets**

Let A and B are two sets given above, then the difference from set A to set B denoted by $A - B$ is the set of all those elements which belongs to A but not belongs to B

i.e. $A - B = \{x : x \in A \text{ but } x \notin B\}$

Thus $A - B = \{1, 4\}$ and $B - A = \{7, 9, 11\}$

The Venn diagrams of $A - B$ are shown below.



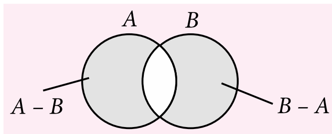
- **Symmetric Difference of two sets**

The symmetric difference of two sets A and B is denoted by $A \Delta B$ and defined as the union of $A - B$ and $B - A$, *i.e.*,

$A \Delta B = (A - B) \cup (B - A)$

$= (A \cup B) - (A \cap B)$

The Venn diagram of $A \Delta B$ is shown by shaded region as



$\therefore$ $x \in A \cup B$ but $x \notin A \cap B$.

PROPERTIES OF UNION AND INTERSECTION (ALGEBRA OF SETS)

(i)	$A \cup A = A$	$A \cap A = A$
(ii)	$A \cup B = B \cup A$	$A \cap B = B \cap A$
(iii)	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
(iv)	$A \cup \phi = A$	$A \cap \phi = \phi$
(v)	$A \cup U = U$	$A \cap U = A$

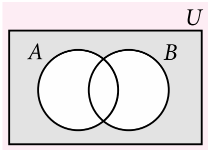
PROPERTIES OF COMPLEMENT OF A SET

- $U' = \{x \in U : x \notin U\} = \phi$
- $\phi' = \{x \in U : x \notin \phi\} = U$
- $(A')' = \{x \in U : x \notin A'\} = \{x \in U : x \in A\} = A$
- $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$
- $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$

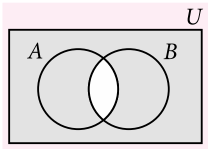
DE-MORGAN'S LAW

If A and B are subsets of universal set X, then we have two following results:

- $(A \cup B)' = A' \cap B' = U - (A \cup B)$ read as complement of the union of two sets is equal to the intersection of their individual complements. The Venn diagram is shown by shaded region.



- $(A \cap B)' = A' \cup B' = U - (A \cap B)$ read as complement of intersection of two sets is equal to the union of their individual complements. The Venn diagram is shown by shaded region



- For any three sets A, B and C, we have

(1) $A - (B \cap C) = (A - B) \cup (A - C)$
(2) $A - (B \cup C) = (A - B) \cap (A - C)$
(3) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(4) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

- If U be a finite universal set and A, B and C are finite (countable) sets, then

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(2) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A$ and B are disjoint non-emtpy sets.
(3) $n(A - B) = n(A) - n(A \cap B)$ ...(i)
$\Rightarrow n(A) = n(A - B) + n(A \cap B)$
(4) $n(A \Delta B)$ = number of elements which belongs to exactly one of A or B
$= n[(A - B) \cup (B - A)] = n(A - B) + n(B - A)$
$= n(A) + n(B) - 2n(A \cap B)$ (Using (i))
$= n(A \cap B') + n(B \cap A')$
(5) $\begin{Bmatrix} n(A \cap B \cap C') = n(A \cap B) - n(A \cap B \cap C) \\ n(B \cap C \cap A') = n(B \cap C) - n(A \cap B \cap C) \\ n(C \cap A \cap B') = n(C \cap A) - n(A \cap B \cap C) \end{Bmatrix}$
(6) $n((A \cup B)') = n(A' \cap B') = n(U) - n(A \cup B)$
(7) $n((A \cap B)') = n(A' \cup B') = n(U) - n(A \cap B)$
(8) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - [n(A \cap B) + n(B \cap C) + n(C \cap A) - n(A \cap B \cap C)]$
(9) $n(\text{only } A) = n(A \cap B' \cap C') = n[A \cap (B \cup C)']$
$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$
(10) None of the elements of A, B and C
$= n(A' \cap B' \cap C') = n(A \cup B \cup C)'$
$= n(U) - n(A \cup B \cup C)$
(11) Number of elements of exactly two of the three sets A, B, C
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
$= n(A \cap B) - n(A \cap B \cap C) + n(B \cap C) - n(A \cap B \cap C) + n(C \cap A) - n(A \cap B \cap C)$
$= n(A \cap B \cap C') + n(B \cap C \cap A') + n(C \cap A \cap B')$
(12) Number of elements in exactly one of the sets A, B and C
$= \Sigma\, n(A) - 2\Sigma n(A \cap B) + 3n(A \cap B \cap C)$

RELATIONS

ORDERED PAIR

A pair of objects in a definite (specific) order is known as ordered pair. In the ordered pair (a_i, b_i), a_i is known as first element and b_i is known as second element. Two ordered pairs (a, b) and (c, d) are said to be equal if and only if $a = c$, $b = d$. A set of two elements $\{a, b\}$ and a ordered pair (a, b) have different meaning *i.e.* $\{a, b\} \neq (a, b)$ and $(a, b) \neq (b, a)$.
Let A and B be two non-empty finite sets consisting of m and n elements (members) respectively, then number of ordered pairs in $A \times B$ are $m \times n$

CARTESIAN PRODUCT

Let A and B be two non-empy sets. The Cartesian product of A and B is the set $A \times B = \{(a, b) : a \in A, b \in B\}$. If $A = \{1, 2\}$ and $B = \{a, b, c\}$.
Then, $A \times B = [(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
In general, $A \times B \neq B \times A$.
$A \times B = B \times A \Leftrightarrow A = B$
For any three sets A, B, C (non-empty)
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(iii) $A \times (B - C) = (A \times B) - (A \times C)$
(iv) If $A \subseteq B$, then $A \times B \subseteq (A \times B) \cap (B \times A)$
(v) If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
(vi) If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any non-empty set C.
(vii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

Note : (1) If A and B are two non-empty sets each having m members, then $n(A \times B) = m^2$.
(2) If A and B be two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

RELATION

- Let A and B be two non-empty sets, then a relation R from A to B is a subset of $A \times B$ and we write $R \subseteq A \times B$.
- If an ordered pair $(a, b) \in R$ then we write $a\,R\,b$ and if $(a, b) \notin R$ then we write $a \not R b$ and we say a is not related to b.
- **Number of relations :** Let A and B be two non-empty finite sets consisting of m and n elements (members) respectively, then number of relations from A to B is $2^{m \times n}$ *i.e.*, $2^{n(A) \times n(B)}$

- **Domain, Range and Codomain of Relation**

As we know that relation from the set A to set B is a subset of $A \times B = \{(a, b) : a \in A, b \in B\}$

Let A and B be two non-empty finite sets consisting of m and n elements (members) respectively, then number of ordered pair in $A \times B$ are $m \times n$

(1) The set of all first elements 'a' of the ordered pairs in R (relation) is called the domain of the relation R, and is denoted by D_R.

(2) The set of all second elements 'b' of the ordered pairs in relation (R) is called the range of the relation R and is denoted by R_R.

(3) The whole set B is called the co-domain of the relation (R).

e.g. If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and a relation R from A to B is $R = \{(a, 2), (b, 3), (a, 3)\}$, then

Domain of R *i.e.* $D_R = \{a, b\}$

Range of relation R *i.e.* $R_R = \{2, 3\}$

and co-domain of the relation = set $B = \{1, 2, 3\}$

TYPES OF RELATIONS

(1) **Void or null or empty relation in a set :** A relation in a set A is subset of $A \times A$, the relation R in set A is called empty relation, if no element of A is related to any element of A *i.e.* $R = \phi \subset A \times A$

(2) **Identity relation :** A relation R is said to be identity relation, if in the set of distinct ordered pairs first and second elements are identical or equal. *i.e.* relation R in a set A is called identity relation if $R = \{(a, a) : a \in A\}$ or $R = \{(a, b) : a, b \in A \text{ and } a = b\}$

(3) **Universal relation :** A relation R in a set A is called universal relation, if each element of A is related to every element of A, *i.e.* $R = A \times A$.

Let us consider the relation R on a set $A = \{1, 2, 3\}$ given by $R = \{(a, b) : a, b \in A, |a - b| \geq 0\}$. Here, all pairs (a, b) in $A \times A$ satisfy $|a - b| \geq 0$. so, R is the whole set of $A \times A$. Therefore R is universal relation.

INVERSE RELATION

Let A and B be two non-empty sets and R be a relation from the set A to the set B, then inverse of R, denoted by R^{-1}, is also a relation from set B to set A and it is denoted by

$R^{-1} = \{(y, x) : (x, y) \in R, x \in A, y \in B\}$

SPECIAL TYPES OF RELATIONS

- **Reflexive Relation**

A relation R in a set A is reflexive if each element of A is related to itself *i.e.* $a\,R\,a\ \forall\ a \in A$.

e.g. $A = \{1, 2, 3\}$, then the relation $R = \{(1, 1), (2, 2), (3, 3), (2, 1)\}$ which is a subset of $A \times A$ is reflexive as $1 \in A$ $\Rightarrow (1, 1) \in R$ and $2 \in A \Rightarrow (2, 2) \in R$ and $3 \in R$ $\Rightarrow (3, 3) \in R$.

Note : Every identity relation is reflexive but every reflexive relation may or may not be an identity relation.

- **Symmetric Relation**

A relation R is said to be symmetric if

$a\,R\,b \Rightarrow b\,R\,a$ *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$.

(1) For symmetric relation, $R^{-1} = R$

(2) Identity relation and universal relation are always symmetric.

(3) The examples of symmetric relations are "is parallel to", "is perpendicular to", "is equal to" etc.

- **Transitive Relation**

A relation R is transitive if $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R\ \forall\ a, b, c \in A$.

e.g. If L be the set of all straight lines in a plane, then the relation 'is parallel to' L is transitive relation as for any l_1, l_2, l_3, we have $l_1 \parallel l_2$ and $l_2 \parallel l_3 \Rightarrow l_1 \parallel l_3$.

- **Equivalence Relation**

The relation R is said to be equivalence if R Satisfies the conditions of reflexivity, symmetricity and transitivity.

(1) An equivalence relation on a set 'A' divide the set into mutually disjoint subsets in such a way that every element in a subset is related to each element in that subset and not related to elements of other sets. If $n(A) = n$ then we have following results :

(2) The number of reflexive relations on A are $2^{n(n+1)}$

(3) The number of symmetric relations on A are $2^{\frac{n(n+1)}{2}}$

(4) The number of relations which satisfies the conditions of reflexivity and symmetricity on set A are $2^{\frac{n(n-1)}{2}}$

(5) The number of divisions (partitions) of A into m disjoint subsets are

$$= \frac{1}{m!}[m^n = C_1^m(m-1)^n + C_2^m(m-2)^n .....]$$

(6) The number of equivalence relations on A are

$$= \sum \frac{1}{m!}(m^n - C_1^m(m-1)^n + C_2^m(m-2)^n ....)$$

- **Anti -Symmetric Relation**

A relation R on set A is said to be an anti-symmetric relation $\Leftrightarrow (a, b) \in R$ and $(b, a) \in R$ gives $a = b\ \ \forall\ a, b \in A$.

FUNCTIONS

Let A and B be two non empty sets, A correspondence between the elements of A and B is said to be a function from A to B if it satisfies the conditions :

- All elements of set A are associated to elements in set B
- An element of set A is assoiciated to one and only one element in set B.

If f is function from A to B, we write it as $f : A \to B$.

- The elements of set A are called pre-images of the elements of set B.
- The elements of set B are called images of the element of set A.
- Every function is a relation but a relation may or may not be function.
- **Domain, Co-Domain and Range of a Function**

(1) Let $f : A \to B$ then set A is known as the domain of f and set B is known as the co-domain of f. The collection of all f-images of elements of A is known as the range of the function f.

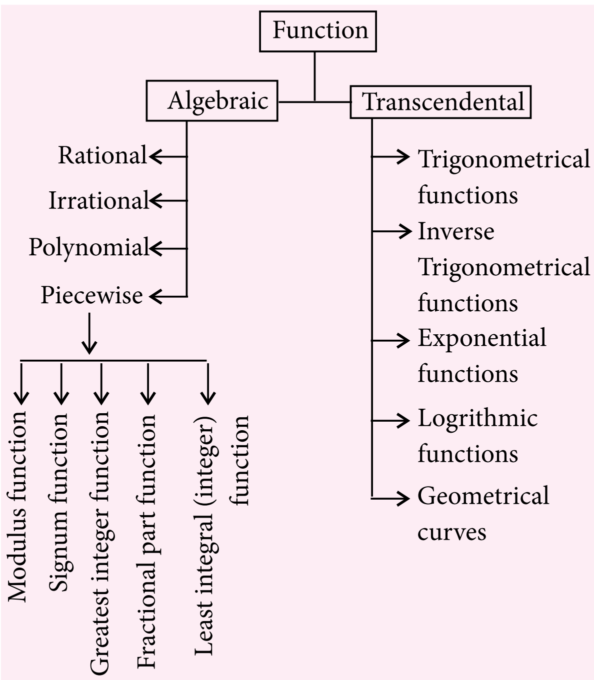
(2) The images of element of set A under f is denoted by $f(A)$. Thus range of $f = f(A) = \{f(a) : a \in A\}$

(3) $f(A)$ is contained in B *i.e.* $f(A) \subseteq B$. *i.e.*, range of f is subset of its co-domain.

(4) Domain and range of a function f are usually abbriviated by D_f and R_f respectively.

TYPES OF FUNCTIONS

Generally the functions can be classified into two category namely (i) Algebraic and (ii) Transcendental.



(1) **Equal function :** Two functions f and g are said to be equal if and only if

(i) Domain of f = Domain of g

(ii) Co-domain of f = codomain of g

(iii) $f(x) = g(x)$ $\forall$ x is an element of their common domain.

e.g., If $f : A \to B$, $A = \{2, 3\}$, $B = \{7, 12\}$, $f(x) = x^2 + 3$ and $g : A \to B$, $g(x) = 5x - 3$, then $f = g$ as the domain and co-domain of f and g both equal and $f(2) = 7 = g(2)$, $f(3) = 12 = g(3)$

(2) **Real valued functions :** A function whose domain and co-domain are subsets of real number, then it is called real valued function.

(3) **Bounded function :** A real function $f(x)$ is said to be bounded function if there exist numbers m and M such that $m \le f(x) \le M$ $\forall$ $x \le D_f$. The examples of bounded function are $\sin x$ and $\cos x$ as $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$.

(4) **Periodic function :** A function $f(x)$ is said to be periodic function if there exists a fixed positive real number α independent of x which satisfies the condition $f(x + \alpha) = f(x)$ $\forall$ $x \in$ domain and α is called period of the function. In other words, a function is called periodic function if its each value is repeated after a certain interval. The least value of this time interval is called fundament period of the time function.

(i) All trigonometric functions are periodic in their domain. $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are having their fundamental period 2π. The period of $\tan x$ and $\cot x$ is π.

(ii) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\cos^n x$ are periodic function with period 2π or π according as n is add function or even function.

(iii) If $f_1(x), f_2(x), ..., f_n(x)$ are periodic functions with periods $p_1, p_2, ..., p_n$ respectively then period of $h(x) = f_1(x) \pm f_2(x) \pm f_3(x) + ... + f_n(x) =$ L.C.M. of $(p_1, p_2, p_3, ... , p_n)$ if $h(x)$ is not an even function. If $h(x)$ is even function then period of

$$h(x) = \frac{1}{2} \text{ L.C.M. of } (p_1, p_2, p_3, ..., p_n)$$

(iv) If a constant be added, subtracted, multiplied in a periodic function, then period of the function does not get effected.

(v) If $f(x)$ is periodic with period α, then period of $k\, f[c(x + \alpha)]$ is $\frac{\alpha}{c}$. For example period of $2\sin 3x$, $\frac{1}{2}\sin 3x$, $2\sin(3x \pm 2)$, $\frac{1}{2}\sin(3x \pm 2)$ are same and equal to $\frac{2\pi}{3}$.

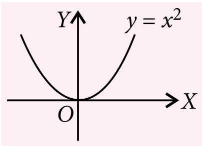
(vi) The of $2^{\sin x}$, $2^{\cos x}$, $2^{\sin x} + 2^{\cos x}$ is 2π

(vii) Every periodic function is many one function.

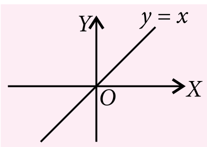
(5) **Even function :** A function $f(x)$ ($x \in$ domain of the function) is said to even function if it satisfies the condition $f(-x) = f(x)$ *e.g.*, $f(x) = e^{2x} + e^{-2x}$, $f(x) = \sin^2 x$, $f(x) = x^4 + x^2 + 2$, $f(x) = x \sin x$ all are even functions.

(6) **Odd function :** If a function $f(x)$ satisfies the condition $f(-x) = -f(x) \forall\ x \in$ the domain of $f(x)$ is called odd function.
e.g., $f(x) = e^{2x} - e^{-2x}$, $f(x) = x^3$, $f(x) = x^3 \cos x$, $f(x) = x + \sin x$ all are odd functions.

- Every even function $y = f(x)$ is symmetric al about y-axis which can be observed by plotting the graph of function. The graph of $y = x^2$ is symmetrical about y-axis therefore it is even function.



- The graph of odd function is symmetrical about the origin. The graph of $f(x) = x, x^3$ .... are symmetrical about origin.



- Every even function is many one function and
- The derivative of odd function is even and derivative of even function is odd.

Remark : Every function can be expressed as the sum of an even and odd function.

$$i.e.\ \ f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

$= h(x) + g(x)$ (say) where $h(x)$ is even and $g(x)$ is odd

(i) The product of two even functions is even.

(ii) The sum and difference of two odd function is an odd function.

(iii) The sum and difference of two even function is an even function.

(iv) The product of two odd functions is an even function.

(v) The sum of an odd and an even function is neither odd or even function.

(vi) Every constant function is both even and odd.

- **Monotonic function :** A function which is either increasing or decreasing in a certain domain, where it is defined is called monotonic function.

(i) A function $f(x)$ is said to be monotonically increasing function if as we increase x, the value of $f(x)$ also increases and vice-versa

(ii) A function $f(x)$ is said to be monotonically decreasing function if as x increases then $f(x)$ also decreases.

(iii) We can check the monotonicity of the function by using derivates the function $y = f(x)$ be a monotonically increasing function if $\frac{dy}{dx} = f'(x) > 0$ in its domain and it is monotonically decreasing if $\frac{dy}{dx} = f'(x) < 0$ in its stated domain.

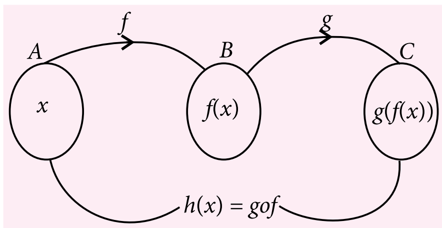
- **Algebraic function :** y is said to be algebraic function of x, if it satisfies an algebraic equation of the form $A_0(x)y^n + A_1(x)y^{n-1} + A_2(x)y^{n-2} + ... + A_{n-1}(x)y + A_n(x) = 0$, where n is natural number and $A_0(x), A_1(x), A_2(x), ..., A_n(x)$ are polynomials in x.
- All polynomial function are algebraic but all algebraic functions need not be a polynomial.
- **Rational function :** If $p(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ and $q(x) = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + ... b_n$ be two polynomials. Then $f(x) = \frac{p(x)}{q(x)}$ where $q(x) \neq 0$ is called rational function.
- **Implicit and explicit function :** Let x be an independent variable and y is dependent on x, the function is said to be explicit if y can be expressed in the form of x directly. *e.g.*, $y = x^2 + 3x - 1$, $y = x^3 + x$ etc., but when y can not be directly expressed in terms of x (*i.e.* in term of independent variable), then function is said to be implicit
e.g., $y = x^2 + y^2 + 2xy$, $xy^2 + yx^2 + x^3 + y^3 = 0$ etc.

COMPOSITION OF FUNCTIONS

If $f : A \to B$ and $g : B \to C$, then the function $h : A \to C$ defined as $h(x) = gof(x)\ g(f(x))\ \forall\ x \in A$ is composition of f and g denoted by gof.

(i) Domain of $gof(x)$ *i.e.*, $g\{f(x)\} = \{x : x \in D_f, f(x) \in D_g\}$

(ii) Domain of $fog(x$; *i.e.*, $f(g(x)) = \{x : x \in D_g, g(x) \in D_f\}$



- **Properties of Composite Function**

(1) If f and g both functions are even, then $h = fog$ is also an even function.

(2) If f and g both are odd functions, then h is also an odd function.

(3) If any one of the function is even/odd, then h is even function.

(4) In general $fog \neq gof$

(5) $(fog)oh = fo(goh)$ (Associative property)

(6) gof exist only if R_f is subset of I_g

(7) If f and g are one-one/ onto, then fog and gof are also one-one/onto.

SOME SPECIAL TYPE OF FUNCTIONS AND THEIR PICTORIAL REPRESENTATION/ DIAGRAM.

• One-one or Injective Function

A mapping f from the set A to set B is said to be one-one if each element in the domain of a function has a distinct image in its co-domain. In other words, if no element of B is the image of more than one element A.

(1) If $x_1, x_2, \in A$ such that $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, then we called the function one-one or injective function.

(2) If $n(A) = m$, $n(B) = n$ and $m \leq n$, then total number of one-one functions from A to B is $^nP_m = \dfrac{n!}{(n-m)!}$

(a) Method of check the injectivity of a function

- Take two arbitrary members a, b in the domain of f.
- Solving $f(a) = f(b)$, if gives $a = b$ only, then we say $f: A \rightarrow B$ is a one-one function, otherwise not.

(b) To check the injectivity by graphical method, plot the graph of $y = f(x)$, if each line drawn parallel to x-axis cuts the graph exactly at one point then we called the function 1-1 function.

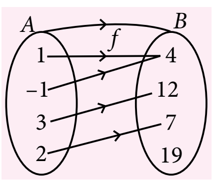
(c) To check the injectivity by using calculus.

(1) If domain of $f(x)$ is continuous and $f'(x) > 0$ or < 0 for all values of x in the stated domain than f is one-one function.

(2) Number of one-one (injective) functions is 0 if $n(A) > n(B)$

• Many One Function

A mapping $f: A \rightarrow B$ is said to be many one if two or more elements of the domain have the same image in its co-domain or $f: A \rightarrow B$ said to be many one function if there exist $a, b \in A$ such that $a \neq b$ but $f(a) = f(b)$.



(1) All even functions are many-one function. *e.g.*, $y = x^2 + 3$, $y = \cos x$.

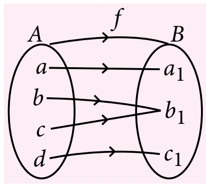
(2) All periodic functions are many-one *e.g.* $y = \sin x$, $\tan x$ *i.e.* all trigonometric functions defined in their domain are many-one.

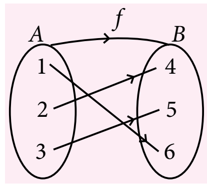
(3) If a line drawn parallel to x-axis cuts the graph of $f(x)$ function at more than one-point then $f(x)$ is said to be many one.

• Onto Function

A mapping $f: A \rightarrow B$ is said to be onto if each element of B is the image of at least one element of A.

or A function $f: A \rightarrow B$ is said to be onto if range of f = co-domain of f.





If A and B are two sets having m & n elements respectively such that $1 \leq n \leq m$, then number of onto function from A to B is

$$\sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r\, r^m = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \ldots + (-1)^{n-1}\binom{n}{1}1^m$$

Note : Total number of functions from A to B are n^m (All Distinct)

• One-One onto function

A function $f: A \rightarrow B$ is said to be one-one onto function if it satisfies the conditions of one one and onto both. It is also called bijective functions. Let f be a bijection from set A to set B if

(i) $f(x) = f(y) \Rightarrow x = y \ \forall\ x, y \in A$ (condition of 1-1)

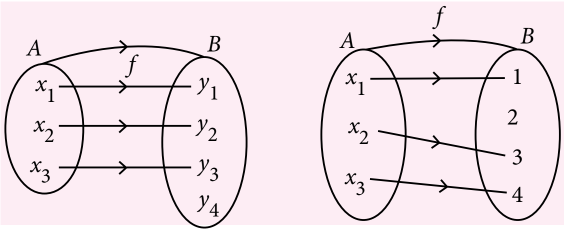
(ii) $\forall\ y \in B$, there exists $x \in A$ such that $f(x) = y$ (condition of onto)

Note : (1) Number of one-one onto functions is $m!$, where $m = n(A) = n(B)$

(2) Number of one-one onto functions is 0 if $n(A) \neq n(B)$

• Into function

A function $f: A \rightarrow B$ is said to be into function if there exist at least one element in B having no pre-image in A. The following arrow diagrams shows into functions



Note : Into function can not be onto function & vice-versa.

- **Many one-into and Many one-onto Function**

A function $f : A \rightarrow B$ is said to be many one into function if it satisfies the condition of many one as well as into function. The similar definition is used for many one-onto function.

INVERSE FUNCTION

Let $f : A \rightarrow B$ be a one-one and onto function, defined by $b = f(a)$, let there exists a unique function $g : B \rightarrow A$ such that for each $b \in B$, $g(b) = a$. The function g so defined is said to be inverse of f and denoted by f^{-1}. It is to be remember that f is inverse of g

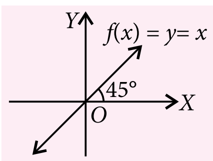
f and f^{-1} are symmetric about the line $y = x$

SOME IMPORTANT FUNCTIONS AND THEIR INVERSE

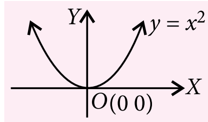
Functions	Inverse functions
1. $f : R \rightarrow (0, \infty)$ defined by $f(x) = a^x$, $a > 0$	1. $f^{-1} : (0, \infty) \rightarrow R$ defined by $f^{-1}(x) = \log_a x$
2. $f : R \rightarrow R$, $f(x) = x^3$	2. $f^{-1} : R \rightarrow R$ defined by $f^{-1}(x) = x^{1/3}$
3. $f : R \rightarrow (0, \infty)$ defined by $f(x) = e^x$	3. $f^{-1} : (0, \infty) \rightarrow R$ defined by $f^{-1}(x) = \log x$
4. $f : R \rightarrow R$, $f(x) = cx + d$, $a \neq 0$	4. $f^{-1} : R \rightarrow R$ given by $f^{-1}(x) = \dfrac{x-d}{c}$
5. $f : [0, \pi] \rightarrow [-1, 1]$ $f(x) = \cos x$	5. $f^{-1} : [-1, 1] \rightarrow [0, \pi]$ defined by $f^{-1}(x) = \cos^{-1} x$
6. $f : \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right) \rightarrow R$ $f(x) = \tan x$	6. $f^{-1} : R \rightarrow \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, $f^{-1}(x) = \tan^{-1} x$

SOME FUNCTIONS AND THEIR GRAPH

- **Identity Function :** A function f defined by $f(x) = x \ \forall \ x \in R$ is called the identity function. Here, $D_f = R_f$ (set of real numbers).

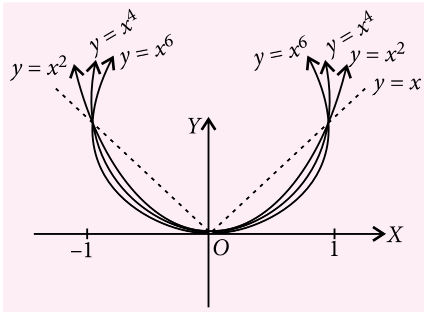


- **Square function :** The function defined as $f(x) = y = x^2$ is called square function, $D_f = R$ (set of all real values)

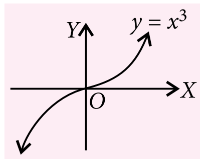


$R_f = R^+$ (set of all positive real numbers) $\cup \{0\}$ which is also called parabolic functions passes through origin and symmetrical about y-axis.

- **Extention of even function :** $f(x) = x^{2n}$, $\forall \ n \in N$, the domain of f is set of all real numbers *i.e.* $D_f = R$ and R_f (range of f) $= R^+ \cup \{0\}$. The graph is symmetrical about y-axis

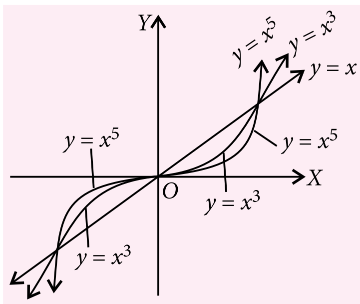


- **Cubical function :** A function $y = x^3$ is called cubic or cubical function passing through origin. $D_f = R_f$ (set of real numbers) Graph of cubic function is symmetrical about opposite quadrant though origin (see graph)



- **Graph of $f(x) = x^{2n-1}$, $n \in N$**

If $n \in N$, then the function $y = f(x) = x^{2n-1}$ is an odd function whose $D_f = R_f =$ set of all real numbers.

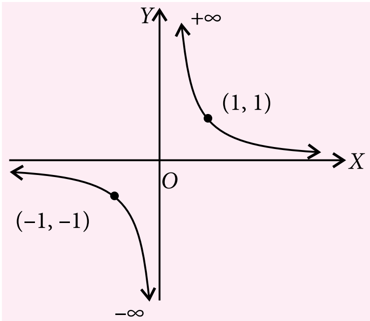


The graph of the function is symmetrical about origin or the opposite quadrants.

Behaviours of $f(x)$ in various intervals is given as

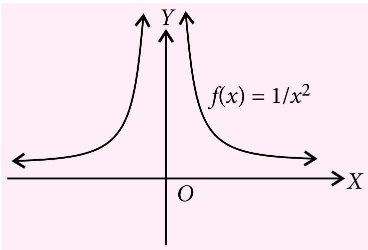
- $x \in (-\infty, -1)$, $x > x^3 > x^5 > ....$
- $x \in (-1, 0)$, $x < x^3 < x^5 < ....$
- $x \in (0, 1)$, $x > x^3 > x^5 > ....$
- $x \in (1, \infty)$, $x < x^3 < x^5 < ....$

- Graph of $f(x) = 1/x$, the function $f(x) = 1/x$, $x \neq 0$ called the reciprocal function or the rectangular function.



  - $D_f = R_f$ is $R - \{0\}$
  - $f(x)$ is odd function, its graph is symmetrical about opposite quadrants.
  - When $x \to 0^+ \Rightarrow f(x) \to +\infty$ and $x \to 0^- \Rightarrow f(x) \to -\infty$.
  - When $x \to \pm\infty, f(x) \to 0$.

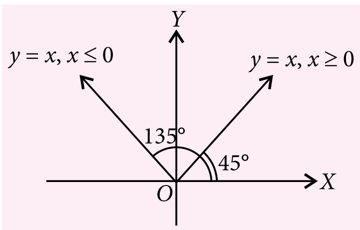
- Graph of $f(x) = 1/x^2$



  - The function $f(x) = 1/x^2$ is an even function.
  - The graph of the function is symmetrical about y-axis.
  - $D_f = R - \{0\}$, $R_f = (0, -\infty)$
  - $y \to \infty$ as $\lim\limits_{x \to 0^+} f(x)$ or $\lim\limits_{x \to 0^-} f(x)$

- **Modulus function :** A function $f(x)$ is said to be modulus function if for all values of x the values of y are positive. It is denoted by $f(x) = |x|$ and defined as

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$



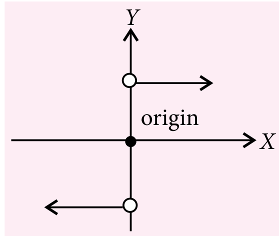
$D_f = R$
$R_f = [0, \infty)$

The modulus function is nowhere discontinuous. It is many one function.

- **Signum function :** The signum function is denoted by $y = \text{sgn}(x)$ and defined by

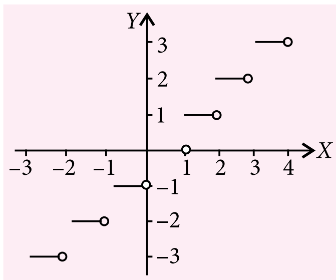
$$y = \text{sgn}(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$



(i) If $f(x) = R$ and range of $f(x) = \{-1, 0, 1\}$
(ii) Signum function is discontinuous and many one function.

- **Greatest integer function or floor function or step function**
If $f(x) = k \ \forall \ x \in [k, k + 1]$ where $k \in I$, then f is called greatest integer function usually denoted by $f(x) = [x]$.



$D_f = R$ (set of real numbers)
R_f = set of integers, the function is discontinuous.

- $f(x) = [x]$ could be expressed as

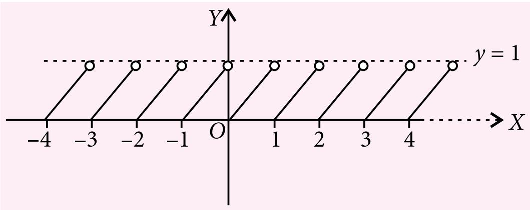
$D_f = x$	$R_f = [x]$
For $-1 \leq x < 0$	-1
For $0 \leq x < 1$	0
For $1 \leq x < 2$	1 and so on

- **Fractional Part Function**

The fractional part function is denoted by $y = \{x\}$ where { } denotes the fractional part of x.
We know that, $x = [x] + \{x\}$
= integral part + fractional part
$\therefore \quad y = \{x\} = x - [x]$ where $0 \leq \{x\} < 1$

Table to draw the graph

x	$\{x\}$
$-2 \le x < -1$	$x + 2$
$-1 \le x < 0$	$x + 1$
$0 \le x < 1$	x
$1 \le x < 2$	$x - 1$
$2 \le x < 3$	$x - 2$



- **Properties of fractional part of x**

(1) $\{x\} = x$ if $0 \le x < 1$

(2) $\{x\} = 0$ if $x \in$ integer thus

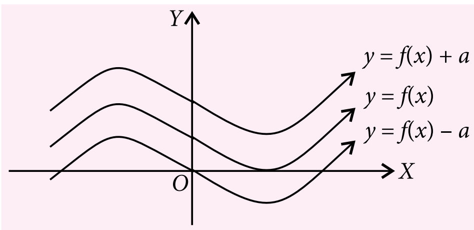
$$\{x\}+\{-x\}=\begin{cases}0, & x\in I\\ 1, & x\notin I\end{cases}$$

(3) $\{-x\} = 1 - \{x\}$; $x \notin$ integer

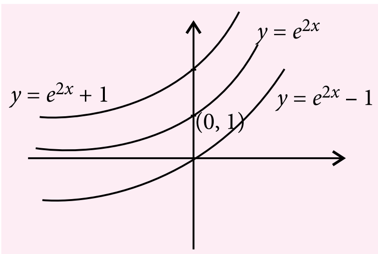
GRAPHICAL TRANSFORMATION

When $f(x)$ transforms to $f(x) \pm a$, $a > 0$. We need to draw the graph of $f(x) + a$ and $f(x) - a$.

- For the graph of $f(x) + a$, shift the graph of $f(x)$ upward through a unit and for the graph of $f(x) - a$, shift the graph of $f(x)$ downward through unit a.



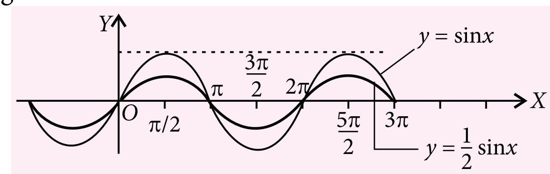
e.g. $y = e^{2x}$, then $e^{2x} + 1$ and $e^{2x} - 1$ are shown as



Graph of $e^{2x} + 1$ is obtained by shifting the graph of $y = e^{2x}$ upwards by 1 unit and the graph of $e^{2x} - 1$ is obtained by shifting the graph downwards by 1 unit.

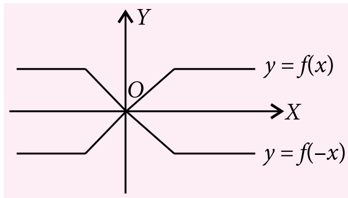
- Drawing the graph of $y = a\,f(x)$ from the graph of $y = f(x)$. In such a graph the abscissa point (x-coordinates) remains same but points of ordinates have their ratio $1 : a$.

The graph of $\frac{1}{2}\sin x$ with the help of $y = \sin x$ is given as

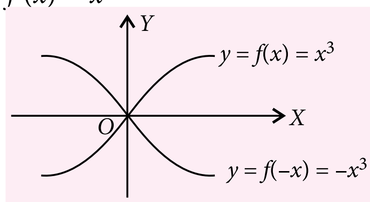


- When $f(x)$ transforms to $f(-x)$ *i.e.* $f(x) \to f(-x)$

To draw the graph of $y = f(-x)$, take the image of the graph of the curve $y = f(x)$ as y-axis on plane mirror *i.e.* turn the graph of $f(x)$ by 180° about y-axis Graphically

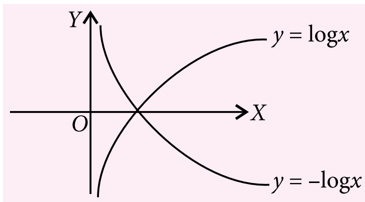


e.g. $y = f(x) = x^3$

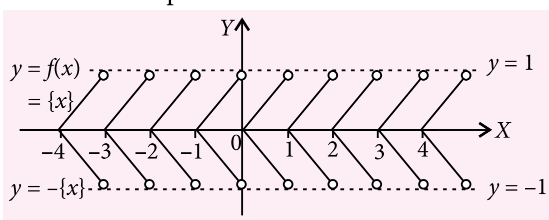


- When $f(x)$ transforms $-f(x)$ *i.e.* $f(x) \to -f(x)$. To draw the graph of $y = -f(x)$ take image of y as the x-axis on plane mirror, which means turn the graph of $f(x)$ by 180° about x-axis.

e.g. To draw $y = -\log x$ where $y = \log x$ is known.



e.g. To draw the graph of $y = -\{x\}$ where { } denotes the fractional part of x.



Exam on 20th May

JEE ADVANCED
PRACTICE PAPER 2018

PAPER-1

SECTION-1

ONE OR MORE THAN ONE OPTION CORRECT TYPE

1. Let f : [– 1, 1] onto [3, 5] be a linear polynomial. Which of the following can be true?

(a) $f\left(\frac{-1}{2}\right)=\frac{7}{2}$ (b) $f^{-1}\left(\frac{15}{4}\right)=\frac{1}{4}$

(c) $f(0) \neq 4$ (d) $f\left(\frac{1}{2}\right)+f\left(\frac{-1}{2}\right)=8$

2. A normal coin is tossed four times. Two events E and F are defined as

E : no two consecutive heads occur in 4 tosses.

F : At least 2 consecutive heads occur in 4 tosses.

The events E and F are

(a) equally likely

(b) mutually exclusive

(c) exhaustive

(d) such that one is twice as likely to occur as other.

3. If the expression $(1 + r\,i)^3$ is of the form of $s(1 + i)$ for some real 's' where 'r' is also real and $i=\sqrt{-1}$, then the value of 'r' can be

(a) $\cot\frac{\pi}{8}$ (b) $\sec \pi$

(c) $\tan\frac{\pi}{12}$ (d) $\tan\frac{5\pi}{12}$

4. Given $f(x) = \sum_{r=1}^{n}(x^r + x^{-r})^2$; $x \neq \pm 1$ and

$$g(x)=\begin{cases}\lim\limits_{n\to\infty}\left((f(x)-2n)x^{-2n-2}(1-x^2)\right); & x\neq \pm 1\\ -1, & ; x=\pm 1\end{cases}$$

then $g(x)$

(a) is discontinuous at x = – 1

(b) is continuous at x = 2

(c) has a removable discontinuity at x = 1

(d) has an irremovable discontinuity at x = 1.

5. The expression $(\alpha \tan\gamma + \beta \cot\gamma)(\alpha \cot\gamma + \beta \tan\gamma) - 4\alpha\beta \cot^2 2\gamma$ is

(a) independent of α, β

(b) independent of γ

(c) dependent on γ

(d) dependent on α, β

6. The parabola $x = y^2 + ay + b$ intersect the parabola $x^2 = y$ at (1, 1) at right angle. Which of the following is/are correct?

(a) a = 4 and b = – 4

(b) a = 2 and b = – 2

(c) Equation of the directrix for the parabola $x = y^2 + ay + b$ is $4x + 1 = 0$.

(d) Area enclosed by the parabola $x = y^2 + ay + b$ and its latus rectum is 1/24.

7. The position vectors of the vertices A, B and C of a tetrahedron are (1, 1, 1), (1, 0, 0) and (3, 0, 0) respectively. The altitude from the vertex D to the opposite face ABC meets the median line through A of the ΔABC at a point E. If the length of side AD is 4 and volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, then the correct statement(s) is/are

(a) The altitude from the vertex D is 2.

(b) There is exactly one position for the point E.

(c) There can be two positions for the point E.

(d) Vector $\hat{j} - \hat{k}$ is normal to the plane ABC.

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SECTION-2

INTEGER ANSWER TYPE

8. Let α, β, γ be distinct real numbers such that

$a\alpha^2 + b\alpha + c = (\sin\theta)\,\alpha^2 + (\cos\theta)\alpha$

$a\beta^2 + b\beta + c = (\sin\theta)\,\beta^2 + (\cos\theta)\beta$

$a\gamma^2 + b\gamma + c = (\sin\theta)\,\gamma^2 + (\cos\theta)\gamma$

(where $a, b, c, \in R$.)

Then the maximum value of the expression

$\dfrac{a^2+b^2}{a^2+3ab+5b^2}$ is

9. The integer satisfying the inequation

$\sqrt{\log_{1/2}^2 x + 4\log_2\sqrt{x}} < \sqrt{2}(4-\log_{16} x^4)$ is

10. In a sequence of circles $C_1, C_2, C_3, \ldots\ldots\, C_n$; the centres lie along positive x-axis with abscissae forming an arithmetic sequence of first term unity and common difference 3. The radius of these circles are in geometric sequence with first term unity and common ratio 2. If the tangent lines with slope m_1 and m_2 of C_3 are intersected at the centre of C_5, then compute the value of $5\,|m_1\, m_2|$.

11. For a positive constant t, let α, β be the roots of the quadratic equation $x^2 + t^2x - 2t = 0$. If the minimum value of $\displaystyle\int_{-1}^{2}\left(\left(x+\frac{1}{\alpha^2}\right)\left(x+\frac{1}{\beta^2}\right)+\frac{1}{\alpha\beta}\right)dx$ is $\sqrt{\dfrac{a}{b}}+c$ where $a, b, c \in N$, then find the least value of $(a+b+c)/10$.

12. Let $A=\begin{bmatrix}1 & \frac{3}{2}\\ 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}4 & -3\\ -2 & 2\end{bmatrix}$

and $C_r=\begin{bmatrix}r\cdot 3^r & 2^r\\ 0 & (r-1)3^r\end{bmatrix}$

be given matrices. If $\displaystyle\sum_{r=1}^{50} tr.((AB)^r C_r)=3+a\cdot 3^b$ where $tr.(A)$ denotes trace of matrix A, then find the value of $\dfrac{1}{20}(a+b)$. [Where a and b are relatively prime]

SECTION-3

MATRIX MATCH TYPE

Answer Q.13, Q.14, Q.15 by information given below:

Column I contains information about the family of curves

Column II contains the differential equation of the family of curve.

Column III contains relation between the order (O) and degree (D) of differential equation of the family of curve. where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constant.

Column I		Column II		Column III	
(A)	$y = c_1\sin^2 x + c_2\cos^2 x + c_3\sin 2x + c_4\cos 2x$	(I)	$\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-y=0$	(i)	$O^D + D^O = 3$
(B)	$y = (c_1 + c_2)\sin(x + c_3) - c_4e^{x+c_5+c_6}$	(II)	$\dfrac{1}{(1+x^2)}+\dfrac{1}{(1+y^2)}\dfrac{dy}{dx}=0$	(ii)	$O^D + D^O = 4$
(C)	$\sqrt{1+x^2}+\sqrt{1+y^2}=c_1(x\sqrt{1+y^2}+y\sqrt{1+x^2})$	(III)	$\dfrac{d^3y}{dx^3}+4\dfrac{dy}{dx}=0$	(iii)	$O^D + D^O = 2$
(D)	$y = c_1e^{3x} + c_2e^{5x}$	(IV)	$\dfrac{d^2y}{dx^2}-8\dfrac{dy}{dx}+15y=0$	(iv)	$2O^D + 3D^O = 9$

13. Which of the following options is the only correct combination?
(a) (A) - (III) - (ii)
(b) (B) - (III) - (i)
(c) (C) - (I) - (iv)
(d) (D) - (II) - (i)

14. Which of the following options is the only correct combination?
(a) (A) - (II) - (ii)
(b) (B) - (III) - (iv)
(c) (C) - (III) - (iv)
(d) (D) - (IV) - (i)

15. Which of the following options is the only incorrect combination?
(a) (B) - (I) - (iv) (b) (C) - (II) - (iii)
(c) (D) - (I) - (i) (d) (A) - (III) - (ii)

Answer Q.16, Q.17, Q.18 by information given below:
Column I contains information about the incident ray from a point on a curve at given point of the curve.
Column II contains the equation of reflected ray after the first reflection.
Column III contains the equation of reflected ray after the second reflection.

Column I		Column II		Column III	
(A)	A ray emanating from $(0, -\sqrt{5})$ incident on the curve $9x^2 + 4y^2 = 36$ at the point with abscissa 2.	(I)	$2x+y\sqrt{5}+2\sqrt{5}=0$	(i)	$x+2y+\sqrt{5}=0$
(B)	A ray emanating from $(\sqrt{5}, 0)$ incident on the curve $4x^2 + 9y^2 = 36$ at the point with ordinate -2	(II)	$2x+y-2\sqrt{5}=0$	(ii)	$4\sqrt{5}x+y+\sqrt{5}=0$
(C)	A ray emanating from $(-\sqrt{5}, 0)$ incident on the curve $4x^2 + 5y^2 = 100$ at the point with ordinate $2\sqrt{5}$	(III)	$\sqrt{5}x+2y-2\sqrt{5}=0$	(iii)	$x+4\sqrt{5}y-\sqrt{5}=0$

16. Which of the following options is the only correct combination?
(a) (A) - (I) - (ii) (b) (A) - (III) - (ii)
(c) (A) - (III) - (i) (d) (A) - (II) - (ii)

17. Which of the following options is the only correct combination?
(a) (B) - (II) - (i) (b) (B) - (I) - (iii)
(c) (B) - (III) - (i) (d) (B) - (II) - (ii)

18. Which of the following options is the correct combination?
(a) (C) - (II) - (iii) (b) (C) - (I) - (ii)
(c) (C) - (III) - (iii) (d) (C) - (II) - (i)

PAPER-2

SECTION-1

SINGLE OPTION CORRECT TYPE

1. Number of 4 digit numbers of the form $N = abcd$ which satisfy following three conditions
(i) $4000 \le N < 6000$
(ii) N is a multiple of 5
(iii) $3 \le b < c \le 6$ is equal to
(a) 12 (b) 18 (c) 24 (d) 48

2. The radii of the escribed circles of ΔABC are r_a, r_b and r_c respectively.
If $r_a + r_b = 3R$ and $r_b + r_c = 2R$, then the smallest angle of the triangle is
(a) $\tan^{-1}(\sqrt{2}-1)$ (b) $\frac{1}{2}\tan^{-1}(\sqrt{3})$
(c) $\frac{1}{2}\tan^{-1}(\sqrt{2}+1)$ (d) $\tan^{-1}(2-\sqrt{3})$

3. The angle between pair of tangents drawn to the curve $7x^2 - 12y^2 = 84$ from $M(1, 2)$ is
(a) $2\tan^{-1}\frac{1}{2}$ (b) $2\tan^{-1}2$
(c) $2\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}\right)$
(d) $2\tan^{-1}3$

4. The position vector of a point in which a line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $\vec{r}.(3\hat{i}-5\hat{j}+2\hat{k})=6$, is
(a) $(1, -1, -1)$ (b) $(-1, -1, 2)$
(c) $(4, 2, 2)$ (d) $\left(\frac{4}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$

5. Let $\binom{n}{k}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum
$\binom{99}{97}+\binom{98}{96}+\binom{97}{95}+....+\binom{3}{1}+\binom{2}{0}$ equals
(a) $\binom{99}{97}$ (b) $\binom{100}{98}$ (c) $\binom{99}{98}$ (d) $\binom{100}{97}$

6. Let $\alpha, \beta \in R$. If α, β^2 be the roots of quadratic equation $x^2 - px + 1 = 0$ and α^2, β be the roots of quadratic equation $x^2 - qx + 8 = 0$, then the value of 'r' if $\frac{r}{8}$ be arithmetic mean of p and q, is

(a) $\frac{83}{8}$ (b) $\frac{83}{4}$ (c) $\frac{83}{2}$ (d) 83

7. Let a, b, c, d are non-zero real numbers such that $6a + 4b + 3c + 3d = 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ has
(a) atleast one root in $[-2, 0]$
(b) atleast one root in $[0, 2]$
(c) atleast two roots in $[-2, 2]$
(d) no root in $[-2, 2]$

SECTION-2

ONE OR MORE THAN ONE OPTION CORRECT TYPE

8. Consider the function $f(x) = \sin^5 x + \cos^5 x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$. Which of the following is/are correct?

(a) f is monotonic increasing in $\left(0, \frac{\pi}{4}\right)$.

(b) f is monotonic decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(c) $\exists$ some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$.

(d) The equation $f(x) = 0$ has two roots in $\left[0, \frac{\pi}{2}\right]$.

9. Which of the following statement(s) is/are correct?
(a) Rolle's theorem is applicable to the function $F(x) = 1 - \sqrt[5]{x^6}$ on the interval $[-1, 1]$.
(b) The domain of definition of the function
$F(x) = \frac{\log_4(5-[x-1]-[x]^2)}{x^2+x-2}$ is $(-3, -2) \cup (-2, 1) \cup (1, 2)$
(where $[x]$ denotes the largest integer less than or equal to x)
(c) The value of a for which the function $F(\theta) = a \sin \theta + \frac{1}{3}\sin 3\theta$ has an extremum at $\theta = \pi/3$ is -2.
(d) The value of $\sum_{k=1}^{2010} \frac{\{x+k\}}{2010}$ is $\{x\}$
(where $\{x\}$ denotes the fractional part of x).

10. Let $I_n = \int_0^{\sqrt{3}} \frac{dx}{1+x^n}$ $(n = 1, 2, 3 \ldots\ldots)$ and $\lim_{n\to\infty} = I_0$ (say), then which of the following statement(s) is/are correct?

(a) $I_1 > I_0$ (b) $I_2 < I_0$
(c) $I_0 + I_1 + I_2 > 3$ (d) $I_0 + I_1 > 2$

11. A circle having its centre at (2, 3) is cut orthogonally by the parabola $y^2 = 4x$. The possible intersection point(s) of these curves, can be
(a) $(3, 2\sqrt{3})$ (b) $(2, 2\sqrt{2})$
(c) (1, 2) (d) (4, 4)

12. If $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 + (\sin^{-1}z)^2 = \frac{3\pi^2}{4}$, then the value of $(x - y + z)$ can be
(a) 1 (b) -1 (c) 3 (d) -3

13. Which of the following is/are true?

The circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$ are such that
(a) they do not intersect.
(b) they touch each other.
(c) their exterior common tangents are parallel.
(d) their interior common tangents are perpendicular.

14. A line L passing through the point $P(1, 4, 3)$, is perpendicular to both the lines
$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{4}$ and $\frac{x+2}{3} = \frac{y-4}{2} = \frac{z+1}{-2}$
If the position vector of point Q on L is (a_1, a_2, a_3) such that $(PQ)^2 = 357$, then $(a_1 + a_2 + a_3)$ can be
(a) 16 (b) 15 (c) 2 (d) 1

SECTION-3

COMPREHENSION TYPE

Paragraph for Question No. 15 and 16

Let $f(x) = 4x^4 - 24x^3 + 31x^2 + 6x - 8$ be a polynomial function. Suppose $f(x) = 0$ has four roots $\alpha < \beta < \gamma < \delta$. If sum of two of its roots vanishes, then

15. $\int \left(\frac{x-\delta}{x-\gamma}\right)^{\alpha+\beta+\delta} dx$ is

(a) $x - 8\ln|x-2| - \frac{24}{x-2} + \frac{16}{(x-2)^2} - \frac{16}{3(x-2)^3} + C$

(b) $x - 16\ln|x-2| - \frac{24}{(x-2)} + \frac{16}{(x-2)^2} - \frac{16}{(x-2)^3} + C$

(c) $x + 16\ln|x-2| - \frac{24}{x-2} + \frac{16}{(x-2)^2} - \frac{16}{3(x-2)^3} + C$

(d) $x + 8\ln|x-2| - \frac{24}{x-2} - \frac{16}{(x-2)^2} + \frac{16}{3(x-2)^3} + C$

16. $\int_{2\alpha}^{2\beta} \frac{x^{\delta+1} - 5x^{\gamma+1} + 2\beta|x| + 1}{x^2 + 4\beta|x| + 1} dx$ is

(a) ln2 (b) 2ln 2

(c) $\frac{1}{2}\ln 2$ (d) None of these

Paragraph for Question No. 17 and 18

Let a, b, c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$. Also the $a(1-x^2)+2bx+c(1+x^2)=0$ has two equal roots.

17. Measure of angle C is

(a) 30° (b) 45° (c) 60° (d) 90°

18. The value of $(\sin A+\sin B+\sin C)$ is equal to

(a) $\frac{5}{2}$ (b) $\frac{12}{5}$ (c) $\frac{8}{3}$ (d) 2

SOLUTIONS

PAPER-1

1. (a, b, d) : Let $f(x)=ax+b$

Case-I: f is increasing

$f(-1)=3$ and $f(1)=5$

$\Rightarrow -a+b=3$ and $a+b=5$

$\Rightarrow f(x)=x+4 \Rightarrow f^{-1}(x)=x-4$

Case-II: f is decreasing

$f(-1)=5$ and $f(1)=3$

$\Rightarrow -a+b=5$ and $a+b=3$

$\Rightarrow f(x)=4-x \Rightarrow f^{-1}(x)=4-x$

Now we can verify the options.

2. (a, b, c) : Sample space S is given by

$\{HTHH, HTTH, HTHT, HTTT,$
$THHH, THTH, THHT, THTT,$
$TTHH, TTTH, TTHT, TTTT,$
$HHHH, HHTH, HHHT, HHTT\}$

$\therefore \quad P(E)=\frac{8}{16}=\frac{1}{2}=P(F)$

3. (b, c, d) : $(1+ri)^3=s(1+i)$

$1+3ri+3r^2i^2+r^3i^3=s(1+i)$

$1-3r^2+i(3r-r^3)=s+si$

$\Rightarrow 1-3r^2=s=3r-r^3$

Hence $1-3r^2=3r-r^3$

$\Rightarrow r^3-3r^2-3r+1=0$

$\Rightarrow (r^3+1)-3r(r+1)=0$

$\Rightarrow (r+1)(r^2+1-r-3r)=0$

$\Rightarrow r=-1$ or $r^2-4r+1=0$

$\Rightarrow r=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm2\sqrt{3}}{2}$

$\Rightarrow r=2+\sqrt{3}$ or $2-\sqrt{3}$

4. (a, b, d) : $f(x)=\sum_{r=1}^{n}\left(x^{2r}+\frac{1}{x^{2r}}+2\right)$

$$=\sum_{r=1}^{n}x^{2r}+\sum_{r=1}^{n}\frac{1}{x^{2r}}+2n$$

$$=(x^2+x^4+....+x^{2n})+\left(\frac{1}{x^2}+\frac{1}{x^4}+.....+\frac{1}{x^{2n}}\right)+2n$$

$$=\frac{x^2(1-x^{2n})}{(1-x^2)}+\frac{1}{x^2}\frac{\left(1-\frac{1}{x^{2n}}\right)}{1-\frac{1}{x^2}}+2n$$

$$f(x)=\frac{x^2(1-x^{2n})}{1-x^2}+\frac{(1-x^{2n})}{(1-x^2)x^{2n}}+2n$$

$$f(x)=\frac{(1-x^{2n})}{1-x^2}\left(x^2+\frac{1}{x^{2n}}\right)+2n,\ x\neq\pm1$$

$$\therefore\ (f(x)-2n)(1-x^2)=(1-x^{2n})\left(x^2+\frac{1}{x^{2n}}\right)$$

Now consider $g(x)=\lim_{n\to\infty}\left((f(x)-2n)\,x^{-2n-2}(1-x^2)\right)$ for $x\neq\pm1$

$$=\lim_{n\to\infty}(1-x^{2n})\left(x^2+\frac{1}{x^{2n}}\right)\cdot\frac{1}{x^{2n+2}};\ x\neq\pm1$$

$$=\lim_{n\to\infty}\frac{(1-x^{2n})(x^{2n+2}+1)}{(x^{2n})(x^{2n+2})}$$

$$=\lim_{n\to\infty}\left(-1+\frac{1}{x^{2n}}\right)\left(1+\frac{1}{x^{2n+2}}\right)$$

Now, $g(x)=\begin{cases}-1 & ,|x|>1\\ \infty & ,|x|<1\\ -1 & ,x=\pm1\end{cases}$

Now clearly g has non removable infinite type of discontinuity at $x=1$ and -1

g is continuous at $x=2 \Rightarrow$ a, b, d

5. (b, d) : We have,

$(\alpha\tan\gamma+\beta\cot\gamma)(\alpha\cot\gamma+\beta\tan\gamma)-4\alpha\beta\cot^2 2\gamma$

$$=\alpha^2+(\tan^2\gamma+\cot^2\gamma)\alpha\beta+\beta^2-4\alpha\beta\frac{\cos^2 2\gamma}{\sin^2 2\gamma}$$

$$=\alpha^2+\beta^2+\alpha\beta\left[\left(\frac{\sin^2\gamma}{\cos^2\gamma}+\frac{\cos^2\gamma}{\sin^2\gamma}\right)-\frac{4(\cos^2\gamma-\sin^2\gamma)^2}{4\sin^2\gamma\cos^2\gamma}\right]$$

$$=\alpha^2+\beta^2+\alpha\beta\left[\frac{(\sin^4\gamma+\cos^4\gamma)-(\cos^4\gamma+\sin^4\gamma-2\sin^2\gamma\cos^2\gamma)}{\sin^2\gamma\cos^2\gamma}\right]$$

CONCEPT MAP

COMPLEX NUMBERS

Class XI

DEFINITION

Number of the form
$z = a + ib$ where $a, b \in R$,
a = Real part and
b = Imaginary part

BASIC TERMS

For complex number $z = x + iy$

- **Conjugate :** $\bar{z} = x - iy$
- **Modulus :** $|z| = \sqrt{x^2 + y^2}$
- **Argument :** $\tan^{-1}\left(\frac{y}{x}\right)$

Properties

For $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

- $\overline{z_1 + z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $(\overline{z^n}) = (\bar{z})^n$
- $|z| = |\bar{z}| = |-z|$
- $z\bar{z} = |z|^2$
- $|z^n| = |z|^n$
- $|z_1 + z_2 + z_3 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n|$
- $\arg|z_1 z_2| = \arg z_1 + \arg z_2$
- $\arg|z_1 / z_2| = \arg z_1 - \arg z_2$
- $\arg|z^n| = n\arg(z)$.

SOME FACTS ABOUT LOCUS

If z is a variable point and z_1, z_2 are two fixed points in the Argand plane :

- $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the segment joining z_1 and z_2.
- $|z - z_1| + |z - z_2| = K$ (a fixed quantity > 0) ...(i)
  - If $K > |z_1 - z_2|$ then (i) represents an ellipse.
  - If $K = |z_1 - z_2|$ then (i) represents the segment joining z_1 and z_2.
  - If $K < |z_1 - z_2|$ then (i) does not represent any curve in the Argand plane.
- $|z - z_1| - |z - z_2| = K(> 0)$
  - If $K \ne |z_1 - z_2|$ then $|z - z_1| - |z - z_2| = K$ represent a hyperbola with foci at z_1 and z_2.
  - If $K = |z_1 - z_2|$ then $|z - z_1| - |z - z_2| = K$ represents a straight line joining z_1 and z_2 but excluding the segment joining z_1 and z_2.
- $|z - z_1|^2 + |z - z_2|^2 = K = |z_1 - z_2|^2$ represent a circle with affixes z_1 and z_2 the extremities of a diameter and $K \ge \frac{1}{2}|z_1 - z_2|^2$
- $|z - z_1| = K|z - z_2|$, $(K \ne 1)$, then locus of z is a circle, *i.e.* $\left|\frac{z - z_1}{z - z_2}\right| = K \ (K \ne 0)$ represent a circle, for $K = 1$, represent a straight line.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (a fixed quantity) then locus of z is a circle.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$, then locus of z is a circle as z_1 and z_2 are vertices of the end point of the diameter.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ or π then locus of z is a line passing through the points z_1 and z_2.

GEOMETRICAL APPLICATIONS

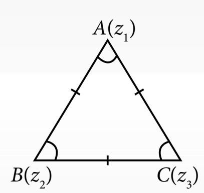
Triangle

Triangle ABC with vertices $A(z_1)$, $B(z_2)$, $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

$$\Leftrightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Leftrightarrow \begin{vmatrix} 1 & z_1 & z_2 \\ 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \end{vmatrix} = 0$$



Circle

- The equation of circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = R$.

or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - R^2 = 0$

$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where $a = -z_0$ and $b = |z_0|^2 - R^2$

$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a circle having centre $-a$ and radius $R = \sqrt{|a|^2 - b}$.

CONTINUITY

Class XII

CONCEPT MAP

CONTINUITY AT A POINT

- A function $f(x)$ is said to be continuous at a point $x = a$ in its domain iff $\lim_{x \to a} f(x)$ exist finitely, $f(a)$ is a finite number and $\lim_{x \to a} f(x) = f(a)$.
- **In open interval :** In an open interval (a, b), $f(x)$ is continuous if it is continuous at every point between (a, b).
- **In closed interval :** In a closed interval $[a, b]$, $f(x)$ is continuous if
  - $f(x)$ is continuous in (a, b)
  - $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$
- **Continuity everywhere :** A function is said to be continuous everywhere if it is continuous on the entire real number line $(-\infty, \infty)$.

DISCONTINUITY AT A POINT

A function $f(x)$ which is not continuous at point say $(x = a)$, then it is discontinuous at $x = a$.

Types

- **Removable discontinuity :**
  Discontinuity at $x = a$ $\lim_{x \to a} f(x) \ne f(a)$. It is called removable because it can be made continuous by redefining it at point a.
- **Discontinuity of 1st kind :** $\lim_{x \to a^-} f(x) \ne \lim_{x \to a^+} f(x)$
- **Discontinuity of 2nd kind :** $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+}$ or both do not exist.

ALGEBRA ON CONTINUOUS FUNCTIONS

Let $f(x)$ & $g(x)$ be two continuous functions on their common domain D & α be any real number then

- $\alpha f(x)$ is continuous
- $f \pm g$ is continuous
- fg is continuous
- f/g is continuous provided $g(x) \ne 0$ for any $x \in D$
- $f^n(x)$, $\forall n \in N$ is continuous.

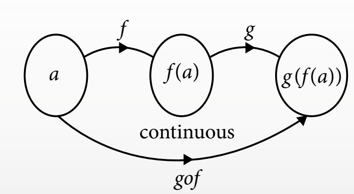
RESULTS

Identity function, Modulus function, Constant function, Exponential function, Logarithmic function, Polynomial functions are continuous.

The largest (greatest) integer function $[x]$ is continuous at all points except at integer (integral) points.

All trigonometric function are continuous in their respective domains like $\sin x$, $\cos x$ are continuous $\forall x R$.

If f, g are continuous functions, then fog & gof are continuous. If f is continuous at a point $x = a$ & g is continuous at $f(a)$, then gof is continuous at $x = a$.



$= \alpha^2 + \beta^2 + 2\alpha\beta = (\alpha + \beta)^2$, which is independent of γ and dependent on α, β.

6. (c, d) : Given curves, $x = y^2 + ay + b$; $x^2 = y$ passes through (1, 1)

$\therefore \quad 1 = 1 + a + b \Rightarrow a + b = 0$

Also, $1 = 2y\dfrac{dy}{dx} + a\dfrac{dy}{dx}$

$\left.\dfrac{dy}{dx}\right]_{(1,1)} = \dfrac{1}{2+a}$; For $y = x^2$; $\dfrac{dy}{dx} = 2x$; $\left.\dfrac{dy}{dx}\right]_{(1,1)} = 2$

$\because$ Given curves intersect at right angles

$\therefore \quad \dfrac{2}{2+a} = -1$

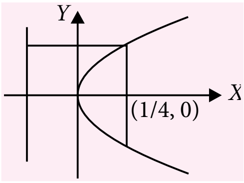
$\Rightarrow \quad -2 = 2 + a \quad \Rightarrow a = -4$ and $b = 4$

Hence parabola is

$y^2 - 4y + 4 = x$

$\Rightarrow \quad (y - 2)^2 = x$

$\Rightarrow \quad Y^2 = X$ [Putting $y - 2 = Y$]



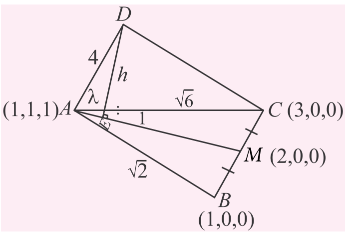
$\therefore \quad \text{Area} = \dfrac{1}{4}\cdot\dfrac{8}{3}\times\dfrac{1}{16} = \dfrac{1}{24}$

and directrix : $x = -\dfrac{1}{4} \Rightarrow 4x + 1 = 0$

7. (a, c, d) : Given $V = \dfrac{2\sqrt{2}}{3}$

Now $\dfrac{1}{3}\cdot\dfrac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}\right\| h = \dfrac{2\sqrt{2}}{3}$

$\Rightarrow h\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}\right\| = 4\sqrt{2}$



[Note ABC is a right triangle $\to$ Area $= \dfrac{1}{2}(2)(\sqrt{2}) = \sqrt{2}$]

$h|\hat{i}(-1+1) + 2(\hat{j} - \hat{k})| = 4\sqrt{2}$

$\Rightarrow \quad h|\hat{j} - \hat{k}| = 2\sqrt{2} \Rightarrow h = 2$

Let E divides AM in the ratio $\lambda : 1$

Hence $E : \left(\dfrac{2\lambda+1}{\lambda+1}, \dfrac{1}{\lambda+1}, \dfrac{1}{\lambda+1}\right)$

Now, $\quad (AE)^2 + (DE)^2 = (AD)^2$

$\Rightarrow \quad \left(\dfrac{2\lambda+1}{\lambda+1} - 1\right)^2 + \left(1 - \dfrac{1}{\lambda+1}\right)^2 + \left(1 - \dfrac{1}{\lambda+1}\right)^2 + 4 = 16$

$\left(\dfrac{\lambda}{\lambda+1}\right)^2 + 2\left(\dfrac{\lambda}{\lambda+1}\right)^2 = 12 \Rightarrow \left(\dfrac{\lambda}{\lambda+1}\right)^2 = 4$

$\Rightarrow \quad \dfrac{\lambda}{\lambda+1} = 2 \text{ or } -2$

$\therefore$ These are two positions for E which are $(-1, 3, 3)$ and $(3, -1, -1)$.

8. (2) : Let the equation be

$ax^2 + bx + c = (\sin\theta)\, x^2 + (\cos\theta)\, x$

$\Rightarrow \quad x^2 (a - \sin\theta) + x (b - \cos\theta) + c = 0$

Then roots of equation are α, β, γ

Since quadratic has three roots then it must be an identity so.

$a - \sin\theta = 0$, $b - \cos\theta = 0$ and $c = 0$

$\therefore \quad a = \sin\theta$, $b = \cos\theta$, $c = 0$.

$\therefore$ Required value of $\dfrac{a^2 + b^2}{a^2 + 3ab + 5b^2}$

$= \dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta + 3\sin\theta\cdot\cos\theta + 5\cos^2\theta}$

$\Rightarrow \quad \dfrac{2}{6 + 4\cos 2\theta + 3\sin 2\theta}$

For maximum value, denominator should be minimum

$\therefore \quad 6 + 4\cos 2\theta + 3\sin 2\theta = 3 \pm 5$

$\Rightarrow \quad 6 - 5 = 1$

$\therefore$ Maximum value of expression is 2.

9. (3) : $\sqrt{\log_{1/2}^2 x + 4\log_2\sqrt{x}} < \sqrt{2}\,(4 - \log_{16} x^4)$

$= \sqrt{\log_2^2 x + 2\log_2 x} < \sqrt{2}(4 - \log_2 x)$

Put $\log_2 x = t \quad \Rightarrow \sqrt{t^2 + 2t} < \sqrt{2}(4 - t)$ ...(i)

$\Rightarrow \quad t^2 + 2t \geq 0 \quad \Rightarrow \quad t(t + 2) \geq 0$

$\Rightarrow \quad t \leq -2$ or $t \geq 0$ ...(ii)

$\Rightarrow \quad 4 - t \geq 0 \Rightarrow t \leq 4$ ...(iii)

Squaring (i), we get

$t^2 + 2t < 2\,(16 + t^2 - 8t)$

$\Rightarrow \quad 0 < t^2 - 18t + 32$

$\Rightarrow \quad (t - 16)(t - 2) > 0$

$\Rightarrow \quad t < 2$ or $t > 16$ ...(iv)

Using (ii), (iii) & (iv) we get $t \leq -2$ or $0 \leq t < 2$

$\Rightarrow \quad \log_2 x \leq -2$ or $0 \leq \log_2 x < 2$

$\Rightarrow \quad x \leq \dfrac{1}{4}$ or $1 \leq x < 4$

$\Rightarrow \quad x \in \left(0, \dfrac{1}{4}\right] \cup [1, 4)$

The integral value of x is 3.

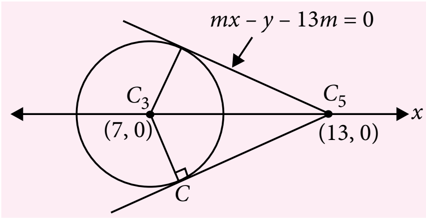
10. (4) : Centre: $C_n = 1 + (n - 1)\,3$; $C_n = 3n - 2$

$C_5 = 13$; $C_3 = 7$; $C_5(13, 0)$; $C_3(7, 0)$

Radius: $R_n = ar^{n-1} = 2^{n-1} \quad \therefore R_3 = 4$

Line AB:

$y - 0 = m\,(x - 13)$; $mx - y - 13m = 0$



$$\Rightarrow \left|\frac{-6m}{\sqrt{m^2+1}}\right| = 4 \ \Rightarrow 9m^2 = 4m^2 + 4$$

$$\Rightarrow m^2 = \frac{4}{5} \Rightarrow m_1 = \frac{2}{\sqrt{5}} \text{ and } m_2 = \frac{-2}{\sqrt{5}}$$

Hence $5|m_1 m_2| = 5 \times \frac{4}{5} = 4$

11. (2) : If α and β are the roots of $x^2 + t^2x - 2t = 0$, then we have $\alpha + \beta = -t^2$ and $\alpha\beta = -2t$.

So $\dfrac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \dfrac{t^2}{4} + \dfrac{1}{t}$

and $\dfrac{1}{(\alpha\beta)^2} + \dfrac{1}{\alpha\beta} = \dfrac{1}{4t^2} - \dfrac{1}{2t}$

Now $I = \displaystyle\int_{-1}^{2}\left(\left(x+\frac{1}{\alpha^2}\right)\left(x+\frac{1}{\beta^2}\right)+\frac{1}{\alpha\beta}\right)dx$

$$= \int_{-1}^{2}\left(x^2 + \left(\frac{t^2}{4}+\frac{1}{t}\right)x + \left(\frac{1}{4t^2}-\frac{1}{2t}\right)\right)dx = \frac{3t^2}{8} + \frac{3}{4t^2} + 3$$

Differentiating I w.r.t. 't', we get $\dfrac{dI}{dt} = \dfrac{3t}{4} - \dfrac{3}{2t^3} = 0$

So, we get $t = \pm\sqrt[4]{2}$, and since t is taken to be positive then

$$I_{\min} = I(\sqrt[4]{2}) = \frac{3\sqrt{2}}{4} + 3 = \sqrt{\frac{18}{16}} + 3 = \sqrt{\frac{9}{8}} + 3 = \sqrt{\frac{a}{b}} + c$$

$\therefore$ Least value of $(a + b + c)/10$ is 2.

12. (5) : $AB = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ $(AB)^1C_1 = C_1$, $(AB)^2C_2 = C_2$ and so on.

Also, $tr(C_r) = r\cdot 3^r + (r-1)\cdot 3^r = (2r-1)\cdot 3^r$

Now, $\displaystyle\sum_{r=1}^{50} tr((AB)^r C_r) = tr((AB)^1C_1) + tr((AB)^2C_2)$

$+ \ldots\ldots + tr((AB)^{50}C_{50}) = S$ (Let)

$\therefore$ $S = tr(C_1) + tr(C_2) + \ldots + tr(C_{50})$

$S = 1\cdot 3^1 + 3\cdot 3^2 + 5\cdot 3^3 + \ldots\ldots + 99\cdot 3^{50}$

$3S = 1\cdot 3^2 + 3\cdot 3^3 + \ldots\ldots + 97\cdot 3^{50} + 99\cdot 3^{51}$

$-2S = 1\cdot 3 + 2\cdot 3^2 + 2\cdot 3^3 + \ldots\ldots + 2\cdot 3^{50} - 99\cdot 3^{51}$

$= -3 + 2\cdot 3 + 2\cdot 3^2 + \ldots\ldots + 2\cdot 3^{50} - 99\cdot 3^{51}$

$= -3 + 2\cdot\dfrac{3\cdot(3^{50}-1)}{3-1} - 99\cdot 3^{51} = -3 + 3^{51} - 3 - 99\cdot 3^{51}$

$= -6 - 98\cdot 3^{51} \Rightarrow S = 3 + 49\cdot 3^{51}$

$\therefore$ $a + b = 100$

Hence $\dfrac{1}{20}(a+b) = 5$

(13 – 15) : 13. (a) 14. (d) 15. (c)

The correct set of combination from the above table is : (A) - (III) - (ii); (B) - (I) - (iv) ; (C) - (II) - (iii) ; (D) - (IV) - (i)

(A) $\because$ $y = c_1\left(\dfrac{1-\cos 2x}{2}\right) + c_2\left(\dfrac{1+\cos 2x}{2}\right)$

$+ c_3 \sin 2x + c_4 \cos 2x = A + B \sin 2x + C \cos 2x$

$\therefore$ $\dfrac{dy}{dx} = 2B\cos 2x - 2C\sin 2x$ ...(i)

$\Rightarrow$ $\dfrac{d^2y}{dx^2} = -4B\sin 2x - 4C\cos 2x$

$\Rightarrow$ $\dfrac{d^3y}{dx^3} = -8B\cos 2x + 8C\sin 2x = -4\dfrac{dy}{dx}$ [From Eq. (i)]

$\Rightarrow$ $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 0$ $\quad\therefore$ $O = 3$, $D = 1$

(B) $\because$ $y = (c_1 + c_2)\sin(x + c_3) - c_4 e^{c_5+c_6}.e^x$

or $y = A\sin(x + B) + Ce^x$ ......(i)

$\therefore$ $\dfrac{dy}{dx} = A\cos(x+B) + Ce^x$ .....(ii)

Subtracting Eq. (i) from Eq. (ii), we get

$\dfrac{dy}{dx} - y = A\cos(x + B) - A\sin(x + B)$ ....(iii)

$\Rightarrow$ $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = -A\sin(x + B) - A\cos(x + B)$

$\Rightarrow$ $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} = -A\cos(x + B) + A\sin(x + B)$

$= -\left(\dfrac{dy}{dx} - y\right)$ [from Eq. (iii)]

$\Rightarrow$ $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$ $\therefore$ $O = 3$, $D = 1$

(C) Put $x = \tan\theta$, $y = \tan\phi$

Then, $(\sec\theta + \sec\phi) = c_1(\tan\theta\sec\phi + \tan\phi\sec\theta)$

$\Rightarrow$ $\left(\dfrac{\cos\theta+\cos\phi}{\cos\theta\cos\phi}\right) = c_1\left(\dfrac{\sin\theta+\sin\phi}{\cos\theta\cos\phi}\right)$

$\Rightarrow$ $2\cos\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$

$= c_1 . 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$

$\Rightarrow \cot\left(\frac{\theta+\phi}{2}\right) = c_1 \Rightarrow \frac{\theta+\phi}{2} = \cot^{-1} c_1$

or $\frac{1}{(1+x^2)} + \frac{1}{(1+y^2)}\frac{dy}{dx} = 0$ $\quad\therefore O = 1,\ D = 1$

(D) $y = c_1e^{3x} + c_2e^{5x}$ or $c_1e^{3x} + c_2e^{5x} - y = 0$ ...(i)

$\therefore \frac{dy}{dx} = 3c_1e^{3x} + 5c_2e^{5x}$

or $3c_1e^{3x} + 5c_2e^{5x} - \frac{dy}{dx} = 0$ ...(ii)

Again differentiating both sides w.r.t. x,

then $9c_1e^{3x} + 25c_2e^{5x} - \frac{d^2y}{dx^2} = 0$ ...(iii)

From Eqs. (i), (ii) and (iii) by eliminating c_1, c_2, c_3, we get

$$\begin{vmatrix} 1 & 1 & y \\ 3 & 5 & \frac{dy}{dx} \\ 9 & 25 & \frac{d^2y}{dx^2} \end{vmatrix} = 0$$

Expanding w.r.t. R_1, then

$$\left(5\frac{d^2y}{dx^2} - 25\frac{dy}{dx}\right) - 1\left(3\frac{d^2y}{dx^2} - 9\frac{dy}{dx}\right) + y(75-45) = 0$$

$$\Rightarrow 2\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 30y = 0 \ \therefore \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

$\therefore \quad O = 2,\ D = 1$

(16 - 18) : 16. (b) 17. (b) 18. (d)

The correct set of combination from the above table is : (A) - (III) - (ii); (B) - (I) - (iii); (C) - (II) - (i)

(A) Given $9x^2 + 4y^2 = 36$ or $\frac{x^2}{4} + \frac{y^2}{9} = 1$

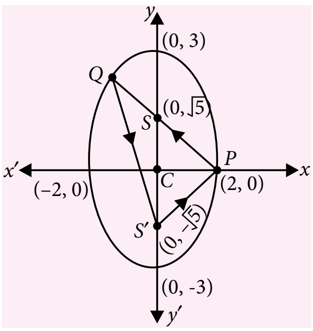
i.e., ellipse along y-axis. If e be the eccentricity, then $4 = 9(1 - e^2)$

$\Rightarrow \quad e = \frac{\sqrt{5}}{3}$

Then, foci $= (0, \pm\sqrt{5})$ and $P \equiv (2, 0)$

$\therefore$ Equation of first reflected ray PQ is

$x\sqrt{5} + 2y - 2\sqrt{5} = 0$ ...(i)



Solving (i) and $9x^2 + 4y^2 = 36$,

we get $Q = \left(-\frac{4}{7}, \frac{9\sqrt{5}}{7}\right)$

$\therefore$ Equation of second reflected ray is

$$y + \sqrt{5} = \frac{\frac{9\sqrt{5}}{7} + \sqrt{5}}{-\frac{4}{7} - 0}(x - 0) \text{ (pass through } Q, S')$$

or $4\sqrt{5}x + y + \sqrt{5} = 0$

(B) $4x^2 + 9y^2 = 36$

or $\frac{x^2}{9} + \frac{y^2}{4} = 1$

i.e., ellipse along x-axis.

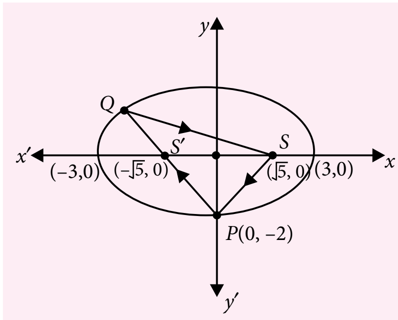
If e be the eccentricity, then $4 = 9(1 - e^2)$

or $e = \frac{\sqrt{5}}{3}$

Then,

foci $= (\pm\sqrt{5}, 0)$

and $P = (0, -2)$



$\therefore$ Equation of first reflected ray PQ is $\frac{x}{-\sqrt{5}} + \frac{y}{-2} = 1$

or $2x + y\sqrt{5} + 2\sqrt{5} = 0$ ...(ii)

Solving (ii) and $4x^2 + 9y^2 = 36$,

we get $Q \equiv \left(-\frac{9\sqrt{5}}{7}, \frac{4}{7}\right)$

$\therefore$ Equation of the second reflected ray is

$$y - 0 = \frac{\frac{4}{7} - 0}{\frac{-9\sqrt{5}}{7} - \sqrt{5}}(x - \sqrt{5}) \text{ (Pass through } Q, S)$$

$\Rightarrow y = -\frac{1}{4\sqrt{5}}(x - \sqrt{5})$ or $x + 4\sqrt{5}y - \sqrt{5} = 0$

(C) $\because$ Given $4x^2 + 5y^2 = 100$ or $\frac{x^2}{25} + \frac{y^2}{20} = 1$

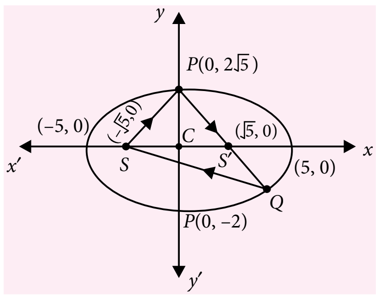
i.e., ellipse along x-axis.

If e be the eccentricity then

$20 = 25(1 - e^2)$

$\Rightarrow \frac{4}{5} = 1 - e^2$

$\therefore e = \frac{1}{\sqrt{5}}$



Then, foci $(\pm\sqrt{5}, 0)$ and $P \equiv (0, 2\sqrt{5})$

$\therefore$ Equation of first reflected ray PQ is

$$\frac{x}{\sqrt{5}}+\frac{y}{2\sqrt{5}}=1 \quad \text{or } 2x + y - 2\sqrt{5} = 0 \qquad \text{...(iii)}$$

Now, solving (iii) and $4x^2 + 5y^2 = 100$

we get $Q \equiv \left(\frac{5\sqrt{5}}{3}, -\frac{4\sqrt{5}}{3}\right)$

$\therefore$ Equation of the second reflected ray is

$$y-0=\frac{\frac{-4\sqrt{5}}{3}-0}{\frac{5\sqrt{5}}{3}+\sqrt{5}}(x+\sqrt{5}) \text{ (pass through } Q, S)$$

$$\Rightarrow \quad y=-\frac{1}{2}(x+\sqrt{5})$$

or $x+2y+\sqrt{5}=0 \quad \therefore \; S: x+2y+\sqrt{5}=0$

PAPER-2

1. (c) : We have $N = \boxed{a}\boxed{b}\boxed{c}\boxed{d}$

First place a can be filled in 2 ways *i.e.* 4, 5 $(4000 \le N < 6000)$

For b and c, total possibilities are '6' $(3 \le b < c \le 6)$ *i.e.* 34, 35, 36, 45, 46, 56

Last place d can be filled in 2 ways *i.e.* 0, 5 (N is a multiple of 5)

Hence total numbers = $2 \times 6 \times 2 = 24$

2. (b) : We have $r_a + r_b = 3R$

$$\Rightarrow \quad \frac{\Delta}{s-a}+\frac{\Delta}{s-b}=3R=\frac{3abc}{4\Delta} \quad \left(\because R=\frac{abc}{4\Delta}\right)$$

$$\Rightarrow \quad \frac{\Delta(s-b+s-a)}{(s-a)(s-b)}=\frac{3abc}{4\Delta}$$

$$\Rightarrow \quad \frac{c\Delta}{(s-a)(s-b)}=\frac{3abc}{4\Delta} \quad \Rightarrow \quad \frac{\Delta^2}{(s-a)(s-b)}=\frac{3ab}{4}$$

$\Rightarrow \quad 4s(s - c) = 3ab \quad \Rightarrow (a + b + c)(a + b - c) = 3ab$

$\Rightarrow \quad (a + b)^2 - c^2 = 3ab \quad \Rightarrow \quad a^2 + b^2 - c^2 = ab$

$\Rightarrow \quad c^2 = a^2 + b^2 - ab$

$\Rightarrow \quad a^2 + b^2 - 2ab \cos C = a^2 + b^2 - ab$

$(\because \; c^2 = a^2 + b^2 - 2ab \cos C)$

$$\Rightarrow \quad \cos C=\frac{1}{2} \quad \Rightarrow \angle C = 60° \qquad \text{...(1)}$$

Similarly from $r_b + r_c = 2R$

$$\Rightarrow \quad \frac{\Delta}{s-b}+\frac{\Delta}{s-c}=2R \quad \Rightarrow \quad \frac{\Delta(2s-b-c)}{(s-b)(s-c)}=\frac{2abc}{4\Delta}$$

$$\Rightarrow \quad \frac{2\Delta^2}{(s-b)(s-c)}=bc \quad \Rightarrow 2s(s-a) = bc$$

$\Rightarrow \quad (b + c + a)(b + c - a) = 2bc$

$\Rightarrow \quad (b + c)^2 - a^2 = 2bc \quad \Rightarrow \quad b^2 + c^2 = a^2$

$\Rightarrow \quad \angle A = 90° \quad \Rightarrow \quad \angle B = 30°$

3. (c) : The director circle of given hyperbola $\frac{x^2}{12}-\frac{y^2}{7}=1$, is $x^2 + y^2 = 5$ and given point $M(1, 2)$ lies on it.

$\Rightarrow$ The angle between pair of tangents $= \frac{\pi}{2}$.

$$\text{As } \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\tan^{-1}1=\frac{\pi}{4}$$

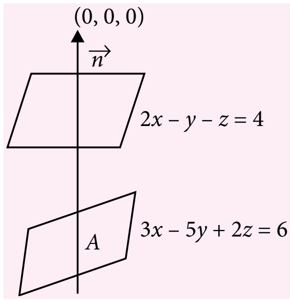
$$\Rightarrow \quad 2\left(\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right)=\frac{\pi}{2}$$

4. (d) : Vector perpendicular to $2x - y - z = 4$ is $\vec{n}=2\hat{i}-\hat{j}-\hat{k}$

Also line is collinear with $\vec{n}$.

So, equation of line is

$$\vec{r}=\lambda(2\hat{i}-\hat{j}-\hat{k}) \qquad \text{...(1)}$$



Now line (1) meet the plane

$$\vec{r}\cdot(3\hat{i}-5\hat{j}+2\hat{k})=6$$

So, $\lambda(6 + 5 - 2) = 6$

$$\Rightarrow \quad \lambda=\frac{2}{3}$$

Hence P.V. of the point 'A' is $\frac{2}{3}(2\hat{i}-\hat{j}-\hat{k})$

5. (d) : $S = {}^2C_0 + {}^3C_1 + {}^4C_2 + {}^5C_3 + \text{.........} + {}^{99}C_{97}$

$$\Rightarrow \quad S=\underbrace{{}^3C_0+{}^3C_1}+{}^4C_2+{}^5C_3+\text{..........}+{}^{99}C_{97}$$

$$=\underbrace{{}^4C_1+{}^4C_2}+{}^5C_3+\text{.........}+{}^{99}C_{97}={}^5C_2+{}^5C_3+\text{..........}$$

$$\therefore \quad S = {}^{100}C_{97} \quad \Rightarrow \quad S=\frac{100\cdot 99\cdot 98}{3\cdot 2}=161700$$

6. (d) : For the equation $x^2 - px + 1 = 0$, the product of roots, $\alpha\beta^2 = 1$

and for the equation $x^2 - qx + 8 = 0$, the product of roots $\alpha^2\beta = 8$

Hence, $(\alpha\beta^2)(\alpha^2\beta) = 8 \Rightarrow \alpha^3\beta^3 = 8 \Rightarrow \alpha\beta = 2$

$\therefore$ From $\alpha\beta^2 = 1$, we have $\beta = \frac{1}{2}$ and from $\alpha^2\cdot\beta = 8$, we have $\alpha = 4$

Now, $p = \alpha + \beta^2 = 4 + \frac{1}{4}=\frac{17}{4}$

and $q=\alpha^2+\beta=16+\frac{1}{2}=\frac{33}{2}$

$\because \; \frac{r}{8}$ is arithmetic mean of p and q

$\therefore \frac{r}{8} = \frac{p+q}{2} \Rightarrow r = 4(p+q) = 4\left(\frac{17}{4} + \frac{33}{2}\right) = 17 + 66 = 83$

7. (b) : We have $ax^3 + bx^2 + cx + d = 0$

Let $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + e \quad \therefore \; f(0) = e$

$f(2) = 4a + \frac{8b}{3} + 2c + 2d + e$

$= \frac{(12a + 8b + 6c + 6d)}{3} + e = \frac{2}{3}(6a + 4b + 3c + 3d) + e = 0$

$\Rightarrow f(2) = e$

$\therefore$ By Rolle's theorem, there exist atleast one value of $x \in (0, 2)$ such that $f'(x) = 0$

$\Rightarrow$ The equation $ax^3 + bx^2 + cx + d = 0$ has atleast one real root in $[0, 2]$

8. (c, d) : We have $f'(x) = 5 \sin^4 x \cos x - 5 \cos^4 x \sin x$
$= 5 \sin x \cos x(\sin x - \cos x)(1 + \sin x \cos x)$

$\therefore \; f'(x) = 0$ at $x = \frac{\pi}{4}$. Also $f(0) = f\left(\frac{\pi}{2}\right) = 0$

Hence $\exists$ some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$
(By Rolle's Theorem)

Also in $\left(0, \frac{\pi}{4}\right)$ f is decreasing and in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ f is increasing

$\Rightarrow$ minimum at $x = \frac{\pi}{4}$

As $f(0) = f\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow$ 2 roots

9. (a, d) : (a) We have, $F(x) = \left(1 - x^{6/5}\right)$

Now, $F'(x) = \frac{-6}{5} x^{\frac{1}{5}}$ exist $\forall \, x \in (-1, 1)$

Also, $F(-1) = 0 = F(1)$

Hence Rolle's theorem is applicable to the function $F(x)$.

(b) For domain of $F(x)$,

$5 - [x] + 1 - [x]^2 > 0$ and $x^2 + x - 2 \neq 0$

$\Rightarrow \; (x + 2)(x - 1) \neq 0 \quad \Rightarrow \; x \neq -2, 1$

Now $[x]^2 + [x] - 6 < 0 \quad \Rightarrow ([x] + 3)([x] - 2) < 0$

$\Rightarrow \; -3 < [x] < 2 \; \Rightarrow -2 \le x < 2$

$\therefore$ Domain $= (-2, 1) \cup (1, 2)$

(c) We have, $F(\theta) = a \sin\theta + \frac{1}{3} \sin 3\theta$

As $F(\theta)$ has an extremum at $\theta = \frac{\pi}{3}$, so

$\Rightarrow \; a \cos\theta + \cos 3\theta = 0$ at $\theta = \frac{\pi}{3} \Rightarrow \frac{a}{2} - 1 = 0$

$\Rightarrow \; \frac{a}{2} - 1 \Rightarrow a = 2$

(d) We have $\sum_{k=1}^{2010} \frac{\{x+k\}}{2010} = \frac{\{x+1\}}{2010} + \frac{\{x+2\}}{2010} + \ldots\ldots$

$+ \frac{\{x + 2010\}}{2010} = \frac{2010\{x\}}{2010} = \{x\}$

10. (a, c, d) : $I_1 = \ln(1 + \sqrt{3})$

$I_2 = \frac{\pi}{3}$

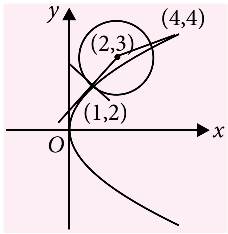
$$I_0 = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left[\int_0^1 \frac{dx}{1+x^n} + \underbrace{\int_1^{\sqrt{3}} \frac{dx}{1+x^n}}_{zero}\right] = \int_0^1 dx = 1.$$

Hence $I_0 = 1$. Now verify all alternatives.

11. (c, d) : Any tangent to the parabola $y^2 = 4x$, is

$yt = x + t^2$ at $(t^2, 2t)$

If it passes through the centre $(2, 3)$ of the circle, then $t^2 - 3t + 2 = 0 \Rightarrow t = 1, 2$



$\therefore$ The point can be $(1, 2)$ or $(4, 4)$

(one circle will be of radius $\sqrt{2}$ and other will be of radius $\sqrt{5}$)

12. (a, b, c, d) :

$\because \; \frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \; \forall -1 \le x \le 1$

$\therefore \; 0 \le (\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 \le \frac{3\pi^2}{4}$

$\therefore \; (\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \frac{3\pi^2}{4}$ is possible

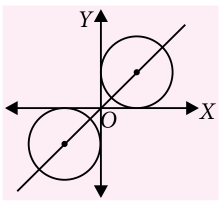
if $x, y, z \in \{-1, 1\}$

$\therefore$ Possible values of $x - y + z$ from the ordered triplet (x, y, z) are as follows

(x, y, z)	$x - y + z$	(x, y, z)	$x - y + z$
$(-1, -1, -1)$	-1	$(1, 1, 1)$	1
$(-1, 1, 1)$	-1	$(1, -1, -1)$	1
$(1, -1, 1)$	3	$(-1, 1, -1)$	-3
$(1, 1, -1)$	-1	$(-1, -1, 1$	1

$\therefore$ Set of values of $x - y + z$ is $\{\pm 1, \pm 3)$

13. (a, c, d) : Centre $(3, 3)$ with radius $= 3$ and centre $(-3, -3)$ with radius $= 3$.



14. (b, d): Equation of the line passing through $P(1, 4, 3)$ and direction ratio's a, b & c is

$$\frac{x-1}{a}=\frac{y-4}{b}=\frac{z-3}{c} \quad \text{...(i)}$$

Since (i) is perpendicular to $\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{4}$

and $\frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}$

Hence $2a + b + 4c = 0$ and $3a + 2b - 2c = 0$

$$\therefore \frac{a}{-2-8}=\frac{b}{12+4}=\frac{c}{4-3} \Rightarrow \frac{a}{-10}=\frac{b}{16}=\frac{c}{1}$$

Hence the equation of the line is

$$\frac{x-1}{-10}=\frac{y-4}{16}=\frac{z-3}{1} \quad \text{...(ii)}$$

Now any point Q on (2) can be taken as $(1 - 10\lambda, 16\lambda + 4, \lambda + 3)$

$\therefore$ Distance of Q from $P(1, 4, 3) = (10\lambda)^2 + (16\lambda)^2 + \lambda^2 = 357$

$\Rightarrow (100 + 256 + 1)\lambda^2 = 357$

$\Rightarrow \lambda = 1$ or -1

$\therefore$ Q is $(-9, 20, 4)$ or $(11, -12, 2)$

Hence, $a_1 + a_2 + a_3 = 15$ or 1

(15 – 16) : 15. (a) 16. (b)

Sum of all four roots = 6; Sum of two roots = 0

$\Rightarrow$ sum of other two roots = 6

The factors corresponding to these roots are of the type $x^2 - 0x + p$ and $x^2 - 6x + q$ *i.e.*, $x^2 + p$ and $x^2 - 6x + q$

Thus, we have $4x^4 - 24x^3 + 31x^2 + 6x - 8 \equiv 4(x^2 + p)(x^2 - 6x + q)$

Equating like powers of x, we get

$$31 = 4(p + q) \Rightarrow p + q = \frac{31}{4}$$

Also, $6 = 4(-6p) \Rightarrow p = -\frac{1}{4}$ $\quad \therefore q = 8$

$\therefore$ Equation $f(x) = 0$ can be written as

$$\left(x^2-\frac{1}{4}\right)(x^2-6x+8)=0 \Rightarrow x=\pm\frac{1}{2} \text{ or } x=2, 4$$

$$\therefore \alpha=-\frac{1}{2},\ \beta=\frac{1}{2},\ \gamma=2,\ \delta=4$$

(i) Now $\displaystyle\int\left(\frac{x-\delta}{x-\gamma}\right)^{\alpha+\beta+\delta} dx=\int\left(\frac{x-4}{x-2}\right)^{-\frac{1}{2}+\frac{1}{2}+4} dx$

$$=\int\left(\frac{x-4}{x-2}\right)^4 dx$$

Let $x - 2 = t$, then

$$\int\frac{(t-2)^4}{t^4}dt=\int\frac{t^4-8t^3+24t^2-32t+16}{t^4}dt$$

$$=\int\left(1-\frac{8}{t}+\frac{24}{t^2}-\frac{32}{t^3}+\frac{16}{t^4}\right)dt$$

$$=t-8\ln|t|-\frac{24}{t}+\frac{16}{t^2}-\frac{16}{3t^3}+C$$

$$=x-8\ln|x-2|-\frac{24}{x-2}+\frac{16}{(x-2)^2}-\frac{16}{3(x-2)^3}+C$$

(ii) $\displaystyle\int_{2\alpha}^{2\beta}\frac{x^{\delta+1}-5x^{\gamma+1}+2\beta|x|+1}{x^2+4\beta|x|+1}dx$

(putting α, β, γ and δ)

$$=\int_{-1}^{1}\frac{x^5-5x^3+|x|+1}{x^2+2|x|+1} \quad (\because\ x^5-5x^3 \text{ are odd function.})$$

$$=\int_{-1}^{1}\frac{|x|+1}{(|x|+1)^2}=2\int_0^1\frac{dx}{(|x|+1)}=2\int_0^1\frac{dx}{1+x}$$

$$= 2[\ln|(1+x)|]_0^1 = 2\ln 2$$

(17 – 18) : 17. (d) 18. (b)

Given $\log\left(\frac{a+c}{a}\right)+\log\left(\frac{a}{b}\right)=\log 2$

$\Rightarrow \log\left(\frac{a+c}{b}\right)=\log 2 \Rightarrow a + c = 2b$ ...(i)

Also, $a - ax^2 + 2bx + c + cx^2 = 0$ has equal roots

$\Rightarrow (c - a)x^2 + 2bx + (c + a) = 0$ has equal roots

$\therefore D = 0 \Rightarrow 4b^2 - 4(c^2 - a^2) = 0$

$\therefore b^2 = c^2 - a^2$ ...(ii)

$\Rightarrow$ Triangle is right angled at $C \Rightarrow \angle C = 90°$

From (i) and (ii), $b^2 = (c - a)(c + a) = (c - a)\,2b$

$\Rightarrow 2(c - a) = b$ ...(iii)

As $\angle C = 90° \Rightarrow \angle A + \angle B = 90°$

From (iii) using Sine law,

$2(\sin C - \sin A) = \sin B$ ...(iv)

Now, $C = 90° \Rightarrow \sin C = 1$

Also, $A + B = 90° \Rightarrow B = 90° - A$

$\therefore$ (iv) becomes $2(1 - \sin A) = \sin(90 - A) = \cos A$

Squaring both sides, we get

$4(1 - \sin A)^2 = \cos^2 A = (1 - \sin^2 A)$

$\Rightarrow 4(1 - \sin A) = (1 + \sin A) \Rightarrow 3 = 5 \sin A$

$\Rightarrow \sin A = 3/5$

Now, $B = 90 - A$

$\Rightarrow \sin B = \cos A = 4/5$ and $\sin C = 1$

$$\therefore \sin A+\sin B+\sin C=\frac{3}{5}+\frac{4}{5}+1=\frac{12}{5}$$

◈◈

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $1/6\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P., then θ is

(a) $2n\pi \pm \frac{\pi}{3}, n \in I$ (b) $2n\pi \pm \frac{\pi}{6}, n \in I$

(c) $n\pi \pm (-1)^n \frac{\pi}{3}, n \in I$ (d) $n\pi \pm \frac{\pi}{3}, n \in I$

2. P is a point on the line $y + 2x = 1$ and Q and R are two points on the line $3y + 6x = 6$ such that triangle PQR is an equilateral triangle. The length of the side of the triangle is

(a) $\frac{2}{\sqrt{15}}$ (b) $\frac{3}{\sqrt{5}}$ (c) $\frac{4}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{5}}$

3. The equation of a circle which has normals $(x - 1)(y - 2) = 0$ and a tangent $3x + 4y = 6$ is

(a) $x^2 + y^2 - 2x - 4y + 4 = 0$

(b) $x^2 + y^2 - 2x - 4y + 5 = 0$

(c) $x^2 + y^2 = 5$ (d) $(x - 3)^2 + (y - 4)^2 = 5$

4. The mean of n items is $\bar{X}$. If the first term, second term, third term .... is increased by 1, 2, 3... and so on. Then new mean is

(a) $\bar{X}+\frac{n+1}{2}$ (b) $\bar{X}+\frac{n}{2}$

(c) $\bar{X}+n$ (d) $\bar{X}+\frac{n-1}{2}$

5. Let $u(x)$ and $v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)}=7$. If $\frac{u'(x)}{v'(x)}=p$ and $\left(\frac{u(x)}{v(x)}\right)'=q$, then $\frac{p+q}{p-q}$ has the value equal to (u' is equal to derivative of 'u')

(a) 1 (b) 0 (c) 7 (d) -7

6. The left hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k is an integer is

(a) $(-1)^k (k - 1)\pi$ (b) $(-1)^{k-1} (k - 1)\pi$

(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k$

7. $A = \{\phi, \{\phi\}\}$ then power set of A is

(a) A (b) $\{\phi, \{\phi\}, A\}$

(c) $\{\phi, \{\phi\}, \{\{\phi\}\}\}$ (d) $\{\phi, \{\phi\}, \{\{\phi\}\}, A\}$

8. The rate of change of the volume of a sphere w.r.t its surface area, when the radius is 2 cm, is

(a) 1 (b) 2 (c) 3 (d) 4

9. A tower stands at the centre of a circular park. 'A' and 'B' are two points on the boundary of the park such that AB $(= a)$ subtends an angle 60° at the foot of the tower, and angle of elevation of the top of the tower from A and B is 30°. The height of the tower is

(a) $\frac{2a}{\sqrt{3}}$ (b) $2a\sqrt{3}$ (c) $\frac{a}{\sqrt{3}}$ (d) $a\sqrt{3}$

10. $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ is equal to

(a) $\sin 2x + c$ (b) $-\frac{1}{2}\sin 2x + c$

(c) $\frac{1}{2}\sin 2x + c$ (d) $-\sin 2x + c$

SOLUTIONS

1. **(a) :** $\cos^2\theta = \frac{1}{6}\sin\theta\tan\theta$

$\Rightarrow 6\cos^3\theta = 1 - \cos^2\theta$

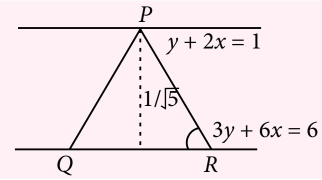
$\Rightarrow 6\cos^3\theta + \cos^2\theta - 1 = 0$

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

$\Rightarrow$ (2cosθ – 1)(3cos²θ + 2cosθ + 1) = 0

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$$

2. (a) :



The given lines are $y + 2x = 1$ and $y + 2x = 2$

The distance between the lines = $\frac{2-1}{\sqrt{5}} = \frac{1}{\sqrt{5}}$

Side length of the triangle = $\frac{1}{\sqrt{5}}\text{cosec}\,60° = \frac{2}{\sqrt{15}}$

3. (a) : The two normal's are $x = 1$ and $y = 2$

Their point of intersection, (1,2) is the centre of the required circle radius $\frac{|3+8-6|}{5} = 1$

Therefore, required circle is $(x - 1)^2 + (y - 2)^2 = 1$

i.e., $x^2 + y^2 - 2x - 4y + 4 = 0$

4 (a) : $a_1 + 1, a_2 + 2, a_3 + 3, \ldots, a_n + n$

$$\text{New mean} = \frac{(a_1 + a_2 + \ldots + a_n) + (1 + 2 + \ldots + n)}{n}$$

$$= \bar{X} + \frac{1 + 2 + 3 + \ldots + n}{n} = \bar{X} + \frac{n(n+1)}{2n}$$

5. (a) : $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x) \Rightarrow p = 7$ (given)

$$\text{Again } \frac{u(x)}{v(x)} = 7 \Rightarrow \frac{u(x)}{v(x)} = 0$$

$$\Rightarrow q = 0. \text{ Now, } \frac{p+q}{p-q} = \frac{7+0}{7-0} = 1$$

6. (a) : If x is just less than k, then $[x] = k - 1$

$\therefore$ $f(x) = (k - 1) \sin\pi x$

$$\text{LHD of } f(x) = \lim_{x \to k} \frac{(k-1)\sin\pi x - k\sin\pi k}{x - k}$$

$$= \lim_{x \to k} \frac{(k-1)\sin\pi x}{x - k}, \text{ where } x = k - h$$

$$= \lim_{h \to 0} \frac{(k-1)\sin\pi(k-h)}{-h} = (k-1)(-1)^k\pi$$

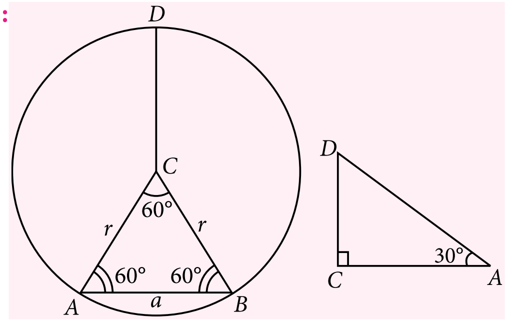
7. (d) : Conceptual.

8. (a) : $V = \frac{4}{3}\pi r^3, S = 4\pi r^2$

$$\frac{dV}{dr} = 4\pi r^2, \frac{dS}{dr} = 8\pi r \Rightarrow \frac{dV}{dS} = \frac{dV/dr}{dS/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$

$$\text{When } r = 2, \frac{dV}{dS} = \frac{2}{2} = 1$$

9. (c) :



CD = height of tower = h.

$AC = BC = AB = a = r$

$$\tan 30° = \frac{h}{a} \Rightarrow h = \frac{a}{\sqrt{3}}$$

10 (b) : $\int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}dx$

$$= \int (\sin^4 x - \cos^4 x)dx$$

$$= -\int(\cos^2 x - \sin^2 x)dx = \int -\cos 2x\, dx = -\frac{\sin 2x}{2} + c$$

MOCK TEST PAPER

ISI

Indian Statistical Institute

Exam on 13th May 2018

1. Consider $a_n = \sin\left(\frac{n\pi}{2}\right)$. Establish the limiting behaviour of each of the following sequences.

(i) $(b_n)_{n\geq 1}$ with $b_n = n\cdot a_n$

(ii) (c_n) with $c_n = \frac{a_n}{n}$, $n \geq 1$

(iii) (d_n) with $d_n = a_n \cos\left(\frac{n\pi}{2}\right)$

2. Consider the function

$$f(x) = \begin{cases} \alpha + x\sin^2 x, x \leq 0 \\ \beta[\sin x + \cos x]^{-1}, x > 0 \end{cases}$$

(i) Find the value of $\alpha, \beta \in R$, if any, for which f has a primitive function in $[-\pi/2, \pi/3]$.

(ii) Find the value of $\alpha, \beta \in R$, if any, for which f is integrable in $[-\pi/2, \pi/3]$.

3. What is the remainder when the given sum S is divided by 4 : $S = 1^5 + 2^5 + 3^5 + \ldots.. + 100^5$

4. Let F be the midpoint of side BC of ΔABC. Construct isosceles right triangles ABD and ACE externally on sides AB and AC respectively with 90° at D and E. Show that ΔDEF is an isosceles right triangle.

5. Find all real solutions of the equation $\sqrt{x^2 - t} + 2\sqrt{x^2 - 1} = x$ for each real value of t.

6. If $C_0, C_1, C_2, \ldots\ldots C_n$ are real numbers satisfying $C_0 + C_1 + C_2 + \ldots\ldots + C_n = 0$. Prove that $C_0 + 2C_1 + 3C_2n^2 + (n + 1)C_nx^n = 0$ has atleast one real root.

7. Find $\sum_{r=1}^{6N-1} \min.\left(\left\langle\frac{r}{3N}\right\rangle, \left\langle\frac{r}{6N}\right\rangle\right)$

where $<x>$ denotes min. $(x - [x], [x] - x + 1)$, $[\cdot]$ denotes G.I.F.

8. Let $a \geq 2$ be a real number. Let α, β be roots of the equation $x^2 - ax + 1 = 0$ and Let $T_n = \alpha^n + \beta^n$, $n = 1, 2, \ldots.$

(i) Prove that the sequence $\left\{\frac{T_n}{T_{n+1}}\right\}_{n=1}^{\infty}$ is decreasing.

(ii) Find all 'a' such that $\frac{T_1}{T_2} + \frac{T_2}{T_3} + \ldots. + \frac{T_n}{T_{n+1}} > n - 1$, for any $n = 1, 2, \ldots\ldots$

9. Let M be the centroid of ΔABC with $\angle AMB = 2\angle ACB$. Prove that

(i) $AB^4 = AC^4 + BC^4 - AC^2 \cdot BC^2$

(ii) $\angle ACB \geq 60°$

10. If $a_1, a_2, \ldots\ldots a_n$ are real numbers, show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} ij\cos(a_i - a_j) \geq 0$$

11. Evaluate $\sum_{r=0}^{\infty} \frac{{}^nC_r}{{}^{2n-1}C_r}$

12. Let (a_n) be the last digit of the number $1^1 + 2^2 + 3^3 + n^n$. Prove that the sequence $(a_n)_n$ is periodic with period 100.

13. The number 1, 2, ... $5n$ are divided into two disjoint sets. Prove that these sets contain atleast n pairs (x, y), $(x > y)$, such that the number $x - y$ is also an element of the set which contains the pair.

14. Find the locus of point P in the plane of square $ABCD$ such that max. $\{PA, PC\} = \frac{1}{\sqrt{2}}(PB + PD)$.

15. Let M be a point in the interior of ΔABC. Lines AM, BM, CM intersect the sides BC, CA, AB at points A_1, B_1, C_1, respectively. Denote the $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6$ respectively the areas of triangles MA_1B, MA_1C, MB_1C, MB_1A, MC_1A, MC_1B. Prove that if $\frac{\Delta_1}{\Delta_2} + \frac{\Delta_3}{\Delta_4} + \frac{\Delta_5}{\Delta_6} = 3$ then M is the centroid of ΔABC

By : Tapas Kr. Yogi, Visakhapatnam Mob : 09533632105

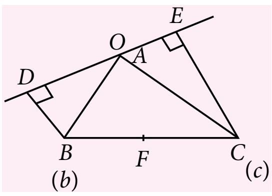
SOLUTIONS

1. (i) Consider the subsequence of b_n corresponding to odd integral, *i.e.* $b_{2n+1} = (-1)^n (2n+1)$.
This does not have a limit. Hence b_n also has no limit.
(ii) c_n sequence is product of a bounded sequence (a_n) with sequence $\left(\frac{1}{n}\right)_{n\geq 1}$ which tends to 0. So, $\lim\limits_{n\to\infty} c_n = 0$.
(iii) $d_n = a_n \cos\left(\frac{n\pi}{2}\right) = \frac{1}{2}\sin(n\pi) = 0$ for all $n \in N$.
So, (d_n) is actually constant and converges to 0.

2. (i) The function f is continuous in $[-\pi/2, \pi/3] - \{0\}$. Further $f(x) \to \beta$ as $x \to 0^+$ and $f(x) \to \alpha$ as $x \to 0^-$. Thus if $\alpha = \beta$ then f is continuous in $[-\pi/2, \pi/3]$ and in this interval has a primitive function. If $\alpha \neq \beta$, then f has a jump discontinuity at $x = 0$ and so, does not admit primitive function in $[-\pi/2, \pi/3]$.
(ii) f is continuous in $[-\pi/2, \pi/3] - \{0\}$ and bounded, hence integrable in $[-\pi/2, \pi/3]$ for every α and β.

3. Notice that $2^5 + 4^5 + ... + 100^5 \equiv 0 \pmod 4$ ...(1)
and $1^5 \equiv 1 \pmod 4$, $3^5 \equiv -1 \pmod 4$, $5^5 \equiv 1 \pmod 4$, $7^5 \equiv -1 \pmod 4$.
So, $1^5 + 3^5 + 5^5 + ... + 99^5 = [1 + (-1) + 1 - 1 + ...] \pmod 4 = 0 \pmod 4$ (50 terms) ...(2)
Hence, $1^5 + 2^5 + 3^5 + 4^5 + ... + 99^5 + 100^5 = 0 \pmod 4$

4. Using complex nos.



$F \equiv \frac{b+c}{2}$

$d = b - b\alpha$, $e = c\alpha$

where $\alpha = \frac{1+i}{2}$ then $d - f = i(e - f)$

$\Rightarrow$ ΔDEF is right isosceles.

5. After simplifying, we have
$16(x^2 - t)(x^2 - 1) = (4x^2 - t - 4)^2$

i.e. $\frac{|t-4|}{\sqrt{16-8t}} = x$ ...(i)

For x to exist, $16 - 8t > 0$ *i.e.*, $t < 2$
Putting (i) in place of x in equation, we have on simplification $|3t - 4| + 2\,|t| = 4 - t$

If $t \geq \frac{4}{3}$ then equation becomes $6t = 8 \Rightarrow t = 4/3$.

If $0 \leq t \leq \frac{4}{3}$ the equation becomes identity and for $t \leq 0$, we have $t = 0$. Hence, solution exists iff $t \in [0, 4/3]$.

6. Let $f(x) = C_0 + 2C_1x + 3C_2x^2 + ... + (n+1)C_nx^n$

Then $\int_0^1 f(x)dx = C_0 + C_1 + C_2 + ... + C_n = 0$

So, by MVT there exists a λ on $(0 < \lambda < 1)$ such that $f(\lambda) = 0$. *i.e.* $f(x) = 0$ has atleast one real root.

7. Middle term of the sequence is $\left\langle \frac{3N}{3N} \right\rangle = 0$

and $\left\langle \frac{6N-r}{3N} \right\rangle = \left\langle \frac{r}{3N} \right\rangle$ and $\left\langle \frac{6N-r}{6N} \right\rangle = \left\langle \frac{r}{6N} \right\rangle$

So, required sum $= 2\sum\limits_{r=1}^{3N-1} \min.\left(\left\langle \frac{r}{3N} \right\rangle, \left\langle \frac{r}{6N} \right\rangle\right)$

$$= 2\left[\sum_{r=1}^{2N}\left\langle \frac{r}{6N}\right\rangle + \sum_{2n+1}^{3N-1}\left\langle \frac{r}{3N}\right\rangle\right]$$

$$= 2\left[\sum_{r=1}^{2N}\frac{r}{6N} + \sum_{r=1}^{N-1}\frac{(N-1)N}{2}\right] = N$$

8. (i) Given, $a \geq 2$, $\alpha\beta = 1$, we have

$$\frac{T_{n-1}}{T_n} - \frac{T_n}{T_{n+1}} = \frac{(\alpha\beta)^{n-1}(\alpha-\beta)^2}{(\alpha^n+\beta^n)(\alpha^{n+1}+\beta^{n+1})} \geq 0$$

So, $\frac{T_{n-1}}{T_n} \geq \frac{T_n}{T_{n+1}} \Rightarrow \left\{\frac{T_n}{T_{n+1}}\right\}$ seq. is decreasing.

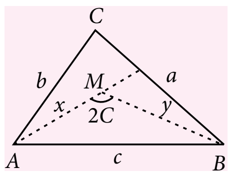
(ii) From the above relation, we have

$$n\cdot\frac{T_1}{T_2} \geq \frac{T_1}{T_2} + \frac{T_2}{T_3} + ... + \frac{T_n}{T_{n-1}} > n-1$$

i.e., $\frac{T_1}{T_2} > 1 - \frac{1}{n}$ As $n \to \infty$, $\frac{T_1}{T_2} \geq 1$

So, $\frac{a}{a^2-2} \geq 1$ *i.e.*, $a \leq 2$ $\because a \leq 2$ $\therefore$ $a = 2$.

9. (i) In ΔAMB, $\cos 2C = \frac{x^2+y^2-c^2}{2xy}$ ...(i)



and $\text{Ar.}(\Delta AMB) = \frac{1}{3}\text{Ar.}(\Delta ABC)$

$\Rightarrow$ $\frac{1}{2}xy\sin 2C = \frac{1}{3}\Delta$

$\Rightarrow$ $xy = \frac{2\Delta}{3\sin 2C}$ ...(ii)

And $x = \frac{2}{3} \times$ (length of median from A)

$$= \frac{2}{3} \times \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2} \qquad \text{...(iii)}$$

and $$y = \frac{2}{3} \times \frac{1}{2} \cdot \sqrt{2c^2 + 2a^2 - b^2} \qquad \text{...(iv)}$$

Simplifying (i), (ii), (iii) and (iv), we have

$c^4 = a^4 + b^4 - a^2b^2$

(ii) It is clear that $a^4 + b^4 - a^2b^2 \geq (a^2 + b^2 - ab)^2$

So, $a^2 + b^2 - 2ab \cos C = c^2 \geq a^2 + b^2 - ab$

$$\Rightarrow \cos C \leq \frac{1}{2} \quad \text{i.e., } C \geq 60°$$

10. The given summation is equivalent to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (ij \cos a_i \cos a_j + ij \sin a_i \sin a_j)$$

$$= \sum_{i=1}^{n} i \cos a_i \sum_{j=1}^{n} j \cos a_j + \sum_{i=1}^{n} i \sin a_i \sum_{j=1}^{n} j \sin a_j$$

$$= \left(\sum i \cos a_i\right)^2 + \left(\sum i \sin a_i\right)^2 \geq 0$$

11. Notice that $$\frac{{}^nC_r}{{}^{2n-1}C_r} = 2\left[\frac{{}^nC_r}{{}^{2n}C_r} - \frac{{}^nC_{r+1}}{{}^{2n}C_{r+1}}\right]$$

So, $$\sum_{r=0}^{\infty} \frac{{}^nC_r}{{}^{2n-1}C_r} = 2\sum_{r=0}^{\infty}\left[\frac{{}^nC_r}{{}^{2n}C_r} - \frac{{}^nC_{r+1}}{{}^{2n}C_{r+1}}\right]$$

$$= 2 \times \frac{{}^nC_0}{{}^{2n}C_0} = 2$$

12. Let us denote the last digit of a number by $l(n)$. The sequence $\{l(n)\}$ is clearly periodic of period 10. Also, for a fixed a $(\in N)$, the sequence $\{l(a^n)\}$ is periodic and period is equal to 1 if 'a' ends in (0, 1, 5, 6), 2 if 'a' ends in (4, 9) and 4 if 'a' ends in (2, 3, 7, 8). Since period is either 1, 2 or 4 or 10. Now L.C.M. of 10 and 4 is 20. So, if we set $m = (n+1)^{n+1} + (n+2)^{n+2} + \ldots.. + (n+20)^{n+20}$ then $l(m)$ does not depend on n.

So last digit of $1^1 + 2^2 + 3^3 + \ldots.. + 20^{20} = 1 + 4 + 7 + 6 + 5 + 6 + \ldots\ldots.. \equiv 4$ on simplification

So, last of given sum of form

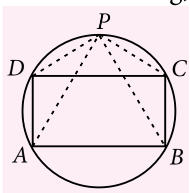
$(n+1)^{n+1} + (n+2)^{n+2} + \ldots.. + (n+100)^{n+100}$ is $l(4 \cdot 5) = l(20) = 0$.

Hence $\{d_n\}$ sequence has periodic with period = 100

13. Let's prove this by contradiction. Suppose these are two sets A and B such that $A \cup B = \{1, 2, 3, \ldots.. 5n\}$. $A \cap B = \phi$ and the sets contain together less than n pairs (x, y), $x > y$ with the desired property. Let λ be a given number $\lambda = 1, 2, \ldots n$. If λ and 2λ are in the same set A or B then so is $2\lambda - \lambda = \lambda$. The same can be said about 2λ or 4λ. Consider the case when λ and 4λ are elements of A and 2λ is an element of B. If 3λ is an element of A then $4\lambda - 3\lambda = \lambda \in A$. Now if $5\lambda \in A$ then $5\lambda - 4\lambda = \lambda \in A$ and if $5\lambda \in B$ then $5\lambda - 3\lambda = 2\lambda \in B$.

So, among the numbers λ, 2λ, 3λ, 4λ, 5λ there is atleast a pair with the required property. Because $\lambda = 1, 2, 3, \ldots.$ it follows that there is atleast n pairs with the required property.

14. Let l be the side length of square $ABCD$. Assuming, max. $\{PA, PC\} = PA$; we have,



$$\sqrt{2}PA = PB + PD$$

$$l \cdot \sqrt{2}PA = l \cdot PB + l \cdot PD$$

So, $BD \cdot PA = AD \cdot PB \; AB \cdot PD$

Hence from converse of ptolemy's theorem, if follows that $PDAB$ is cyclic quad. *i.e.*, locus of P is circumcircle of square $ABCD$.

15. Using, Ceva's theorem, we have

$$\frac{C_1A}{C_1B} \cdot \frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} = 1$$

i.e., $$\frac{\Delta_5}{\Delta_6} \cdot \frac{\Delta_1}{\Delta_2} \cdot \frac{\Delta_3}{\Delta_4} = 1$$

So, given condition becomes

$$\frac{\Delta_1}{\Delta_2} + \frac{\Delta_3}{\Delta_4} + \frac{\Delta_5}{\Delta_6} = 3 \cdot \sqrt[3]{\frac{\Delta_1}{\Delta_2} \cdot \frac{\Delta_3}{\Delta_4} \cdot \frac{\Delta_5}{\Delta_6}}$$

i.e., A.M. = G.M. $\Rightarrow$ Nos. are all equal.

i.e., $$\frac{\Delta_1}{\Delta_2} = \frac{\Delta_3}{\Delta_4} = \frac{\Delta_5}{\Delta_6} = 1 \quad \text{i.e.,} \quad \frac{C_1A}{C_1B} = \frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = 1$$

i.e., A_1, B_1, C_1 are midpoints and hence M is the centroid of the triangle.

YOU ASK
WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. There are n intermediate stations on a railway line from one terminus to another. In how many ways can the train stop at 3 of these intermediate stations if

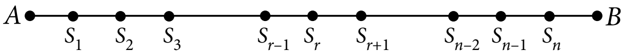
(i) all the three stations are consecutive?

(ii) at least two of the stations are consecutive?

(iii) no two of these stations are consecutive?

(Ritu Jain, Gujarat)

Ans.



(i) The number of triplets of consecutive stations, viz. $S_1S_2S_3$, $S_2S_3S_4$, $S_3S_4S_5$, ..., $S_{n-2}S_{n-1}S_n$ is $(n-2)$.

(ii) The total number of consecutive pair of stations, viz.

S_1S_2, S_2S_3, ..., $S_{n-1}\,S_n$ is $(n-1)$.

Each of the above pair can be associated with a third station in $(n-2)$ ways. Thus, choosing a pair of stations and any third station can be done in $(n-1)(n-2)$ ways. The above count also includes the case of three consecutive stations. However, we can see that each such case has been counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways in which three stations can be chosen so that at least two of them are consecutive is

$$(n-1)(n-2) - (n-2) = (n-2)^2.$$

(iii) Without restriction, the train can stop at any three stations in ${}^nC_3$ ways. Hence, the number of ways the train can stop so that no two stations are consecutive is

$${}^nC_3 - (n-2)^2 = \frac{n(n-1)(n-2)}{1\cdot2\cdot3} - (n-2)^2$$

$$= (n-2)\left(\frac{n^2-n-6n+12}{6}\right) = \frac{(n-2)(n-3)(n-4)}{6} = {}^{n-2}C_3.$$

2. If a, b, c are in G.P. and the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

(Priyansh Sharma, Delhi)

Ans. We have, $b^2 = ac$ ...(i)

The equation $ax^2 + 2bx + c = 0$

gives $x = \dfrac{-2b \pm \sqrt{4b^2 - 4ac}}{2a} = \dfrac{-b}{a}$ [using (i)]

and $dx^2 + 2ex + f = 0 \Rightarrow x = \dfrac{-e \pm \sqrt{e^2 - df}}{d}$

According to the given condition, we have

$$\frac{-b}{a} = \frac{-e \pm \sqrt{e^2-df}}{d} \quad \textit{i.e.} \quad \left(\frac{e}{d} - \frac{b}{a}\right)^2 = \frac{e^2-df}{d^2}$$

i.e. $\dfrac{2be}{ad} = \dfrac{b^2}{a^2} + \dfrac{f}{d} = \dfrac{c}{a} + \dfrac{f}{d}$ [using (i)]

i.e. $\dfrac{2be}{ac} = \dfrac{d}{a} + \dfrac{f}{c}$ $\left[\text{multiplying both sides by } \dfrac{d}{c}\right]$

i.e. $\dfrac{2e}{b} = \dfrac{d}{a} + \dfrac{f}{c} \Rightarrow \dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in A.P.

3. Find the values of a, b, c if the following limit exists :

$$\lim_{x\to0} \frac{ae^x - b\ln(1+x) + cxe^{-x}}{x^2 \sin x} = 2$$

(Rajnish, Jharkhand)

Ans. Let $L = \displaystyle\lim_{x\to0} \frac{ae^x - b\ln(1+x) + cxe^{-x}}{x^2\sin x}$

$$= \lim_{x\to0} \frac{ae^x + cxe^{-x} - b\ln(1+x)}{x^3} \cdot \frac{x}{\sin x}$$

$$= \lim_{x\to0} \frac{ae^x + cxe^{-x} - b\ln(1+x)}{x^3}$$

$$= \lim_{x\to0} \frac{a\left(1 + x + \frac{x^2}{2!} + \ldots\right) + cx\left(1 - x + \frac{x^2}{2!} - \ldots\right) - b\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)}{x^3}$$

$$= \lim_{x\to0} \frac{a + (a+c-b)x + (a-2c+b)\frac{x^2}{2} + \left(\frac{a}{3!} + \frac{c}{2!} - \frac{b}{3}\right)x^3 + \ldots}{x^3}$$

For a finite non-zero limit to exist, we have

$a = 0$ ...(i) $\quad a + c - b = 0$ ...(ii)

$a - 2c + b = 0$ ...(iii)

and $\dfrac{a}{6} + \dfrac{c}{2} - \dfrac{b}{3} = 2$ [according to the given condition]

i.e. $a + 3c - 2b = 12$ ...(iv)

Solving (i), (ii), (iii) and (iv), we can see that there is no value of a, b, c which satisfies all the equations.

◈◈

MATHS MUSING

SOLUTION SET-184

1. (c) : $\because$ a, b, c are in H.P. $\quad\therefore\quad b = 2ac/(a+c)$ ...(i)

b, c, d are in G.P. $\quad\therefore\quad c^2 = bd$ ...(ii)

and c, d, e are in A.P. $\quad\therefore\quad d = (c+e)/2$ ...(iii)

From (i), $ab + bc = 2ac \Rightarrow c = ab/(2a-b)$ ...(iv)

Also, $d = \dfrac{1}{2}\left(\dfrac{ab}{2a-b} + e\right)$ (Using (iii) and (iv)) ...(v)

On putting the values of c and d from (iv) and (v) in (ii), we get

$$\frac{a^2b^2}{(2a-b)^2} = \frac{b}{2}\left[\frac{ab}{2a-b} + e\right] \Rightarrow e = \frac{ab^2}{(2a-b)^2}$$

2. (d) : Every number between 100 and 1000 is a 3-digit number. We first have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^6P_3$. But, these permutations will include those numbers also where 0 is at the 100th place. To get the number of such numbers, we fix 0 at the 100th place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^5P_2$.

So, the required number $= {}^6P_3 - {}^5P_2 = \dfrac{6!}{3!} - \dfrac{5!}{3!} = 100$

3. (d) : $z = ki, k \in R \Rightarrow -(\cos\alpha + i\sin\alpha)k^2 + ki + 1 = 0$

$\Rightarrow$ $k^2\cos\alpha = 1$, $k\sin\alpha = 1$.

Eliminating k, we get $\cos\alpha = \dfrac{\sqrt{5}-1}{2}, \tan\alpha = \sqrt{\dfrac{\sqrt{5}+1}{2}}$

4. (c) : $\displaystyle\int_1^4 \frac{2e^{\sin x^2}}{x}dx = \int_1^{16}\frac{e^{\sin z}}{z}dz \quad \left[\begin{array}{l}\text{Put } x^2 = z\\ \Rightarrow 2dx = \dfrac{dz}{x}\end{array}\right]$

$= \displaystyle\int_1^{16} d(F(z)) \quad \left[\because \frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}, x > 0\right]$

$= [F(z)]_1^{16} = F(16) - F(1)$ $\therefore$ $F(16) = F(k)$

Hence, one of the possible values of $k = 16$.

5. (d) : Let $g(x) = f(x) - x^2$

Using Rolle's Theorem, we get

$g'(x) = 0$ for at least one $x \in (1, 2)$

Also, $g'(x) = 0$ for at least one $x \in (2, 3)$

$\therefore$ $g''(x) = 0$ for at least one $x \in (1, 3)$

$\therefore$ $f''(x) = 2$ for some $x \in (1, 3)$

6. (b) : $(f'(x))^2 - k^2(f(x))^2 \le 0$

$\Rightarrow$ $(f'(x) - kf(x))(f'(x) + kf(x)) \le 0$

$\Rightarrow$ $(f(x)e^{-kx})(f(x)e^{kx}) \le 0$

$\Rightarrow$ Exactly one of the functions $g_1(x) = f(x)e^{-kx}$ or $g_2(x) = f(x)e^{kx}$ is non decreasing.

But $f(0) = 0 \Rightarrow$ both function g_1 and g_2 have a value zero at $x = 0$.

7. (d) : $f_1oF = f_4 \Rightarrow F = f_1^{-1}of_4 = f_1of_4 \Rightarrow F = f_4$

8. (c) : $f_3oJof_2 = f_4 \Rightarrow J = f_3^{-1}of_4of_2^{-1} = f_3of_4of_2$

$\therefore$ $J(x) = f_3of_4\left(\dfrac{1}{x}\right) = f_3\left(\dfrac{1}{1-\frac{1}{x}}\right)$

$= f_3\left(\dfrac{x}{x-1}\right) = 1 - \left(\dfrac{x}{x-1}\right) = f_4(x)$

9. (2) : Let $\sin x = t \Rightarrow L = \displaystyle\lim_{t\to 1}\left(\frac{t - t^t}{1 - t + \ln t}\right)$

$= \displaystyle\lim_{t\to 1}\frac{1 - t^t(1+\ln t)}{-1+\frac{1}{t}}$ (By L.H. Rule)

$= \displaystyle\lim_{t\to 1}\frac{-t^t(1+\ln t)^2 - t^{t-1}}{-\frac{1}{t^2}} = \frac{-1-1}{-1} = 2$

10. (b) : (P) $\to$ 3; (Q) $\to$ 1; (R) $\to$ 4; (S) $\to$ 2

P. Here, $(\vec{c} - \vec{a})\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -5 & -3 & 4\\ 6 & 1 & -2\end{vmatrix} = 2\hat{i} + 14\hat{j} + 13\hat{k}$

$\therefore$ $d = \dfrac{|2\hat{i}+14\hat{j}+13\hat{k}|}{|6\hat{i}+\hat{j}-2\hat{k}|} = \sqrt{\dfrac{369}{41}} = 3$

Q. $z = \omega, \omega^2$

$\therefore$ $\displaystyle\sum_{r=1}^{6}\left(\omega^r + \frac{1}{\omega^r}\right)^2 = 1+1+4+1+1+4 = 12$

R. $\dfrac{3}{1+x^3} = \dfrac{1}{x+1} - \dfrac{(2x-1-3)}{2(x^2-x+1)}$

$\therefore$ $\displaystyle\int_0^1 \frac{3dx}{1+x^3} = \ln\left.\frac{x+1}{\sqrt{x^2-x+1}}\right|_0^1 + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left.\left(\frac{2x-1}{\sqrt{3}}\right)\right|_0^1$

$= \ln 2 + \sqrt{3}\dfrac{\pi}{3} = \ln 2 + \dfrac{\pi}{\sqrt{3}}$

S. $t = \sin 2x + 2\cos^2 x \Rightarrow 1 - \sin 2x + 2\sin^2 x = 3 - t$

The equation becomes $3^t + \dfrac{27}{3^t} = 28 \Rightarrow 3^t = 1, 27$

$\therefore$ $t = 0, 3$. Now $\sin 2x + 2\cos^2 x < 3$

$\therefore$ $\sin 2x + 2\cos^2 x = 0 \Rightarrow \sin 2x + \cos 2x = -1$

$\cos\left(2x - \dfrac{\pi}{4}\right) = \cos\dfrac{3\pi}{4}$

$\therefore$ $2x - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{3\pi}{4} \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

CBSE BOARD SOLVED PAPER 2018

CLASS XII

Units	VSA(1 mark)	SA(2 marks)	VBQ(4 marks)	LA I(4 marks)	LA II(6 marks)	Total
Relations and Functions	1(1)	---	---	---	6(1)*	7(2)
Inverse Trigonometric Functions	1(1)	2(1)	---	---	---	3(2)
Matrices	1(1)	---	---	---	---	1(1)
Determinants	---	2(1)	---	4(1)	6(1)*	12(3)
Continuity and Differentiability	---	2(1)	---	8(2)*	---	10(3)
Application of Derivatives	---	2(1)	4(1)	4(1)*	---	10(3)
Integrals	---	2(1)	---	4(1)	6(1)*	12(3)
Application of Integrals	---	---	---	---	6(1)	6(1)
Differential Equations	---	2(1)	---	4(1)*	---	6(2)
Vector Algebra	1(1)	2(1)	---	4(1)	---	7(3)
Three Dimensional Geometry	---	---	---	4(1)	6(1)	10(2)
Linear Programming	---	---	---	---	6(1)	6(1)
Probability	---	2(1)	---	8(2)	---	10(3)
Total	**4(4)**	**16(8)**	**4(1)**	**40(10)**	**36(6)**	**100(29)**

* It is choice based question.

Time Allowed : 3 hours **Maximum Marks : 100**

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **29** questions.
(iii) Questions **1-4** in Section-A are very short-answer type questions carrying **1** mark each.
(iv) Questions **5-12** in Section-B are short-answer type questions carrying **2** marks each.
(v) Questions **13-23** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Questions **24-29** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

1. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$.

2. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of 'a' and 'b'.

3. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

4. If $a * b$ denotes the larger of 'a' and 'b' and if $a \ o \ b = (a * b) + 3$, then write the value of $(5) \ o \ (10)$, where $*$ and o are binary operations.

SECTION - B

5. Prove that : $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

6. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

7. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

8. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

9. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

10. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants.

11. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin\theta$.

12. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

SECTION - C

13. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$$

14. If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$

OR

If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

15. If $y = \sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0.$$

16. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

OR

Find the intervals in which the function

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12 \text{ is}$$

(a) strictly increasing, (b) strictly decreasing

17. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

18. Find : $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$

19. Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x)\sec^2 y\, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$.

OR

Find the particular solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$.

20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$.

21. Find the shortest distance between the lines

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) \text{ and}$$

$$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k}).$$

22. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die?

23. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

SECTION - D

24. Let $A = \{x \in Z : 0 \le x \le 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

OR

Show that the function $f : R \to R$ defined by $f(x) = \frac{x}{x^2+1}, \forall x \in R$ is neither one-one nor onto. Also, if $g : R \to R$ is defined as $g(x) = 2x - 1$, find $fog(x)$.

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use it to solve the system of equations

$2x - 3y + 5z = 11$
$3x + 2y - 4z = -5$
$x + y - 2z = -3$.

OR

Using elementary row transformations, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

26. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

27. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$

OR

Evaluate : $\int_1^3 (x^2 + 3x + e^x)dx$, as the limit of the sum.

28. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

29. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws 'A' while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of ₹ 1. Assuming that he can sell all the screws be manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit? Formulate the above LPP and solve it graphically and find the maximum profit.

SOLUTIONS

1. Since, $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

$$= \tan^{-1}\sqrt{3} - (\pi - \cot^{-1}\sqrt{3})$$

$$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3} = \frac{\pi}{3} - \pi + \frac{\pi}{6} = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

2. A square matrix A is said to be skew symmetric matrix if $A = -A'$ ...(i)

Now, $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ $\therefore$ $A' = \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$

From (i), $A + A' = O$

$$\Rightarrow \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 0 & 2+a & b-3 \\ a+2 & 0 & 0 \\ b-3 & 0 & 0 \end{bmatrix} = O$$

$\Rightarrow$ $a + 2 = 0$ & $b - 3 = 0$
$\therefore$ $a = -2$ & $b = 3$

3. We are given that, $|\vec{a}| = |\vec{b}|, \theta = 60°$ and $\vec{a} \cdot \vec{b} = \frac{9}{2}$

Now, $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \cos 60° = \frac{9/2}{|\vec{a}|^2} \Rightarrow \frac{1}{2} = \frac{9/2}{|\vec{a}|^2}$

$\Rightarrow |\vec{a}|^2 = 9 \Rightarrow |\vec{a}| = 3$ So, $|\vec{b}| = 3$

4. $(5)\ o\ (10) = (5 * 10) + 3$
$= \text{larger of } (5, 10) + 3 = 10 + 3 = 13$

5. Put $\sin^{-1}x = \theta$. Then $x = \sin\theta$

Now, $\sin 3\theta = (3\sin\theta - 4\sin^3\theta) = (3x - 4x^3)$
$\Rightarrow$ $3\theta = \sin^{-1}(3x - 4x^3)$
$\Rightarrow$ $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$ $[\because \theta = \sin^{-1}x]$

6. $|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$

So, A is a non-singular matrix and therefore, it is invertible.

$\text{Adj }(A) = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\text{Adj}(A) = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow \; 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ ...(i)

Now, $9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = 2A^{-1}$ [From (i)]

7. Let $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$

$= \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\tan\left(\frac{\pi}{2} - \frac{x}{2}\right) = \frac{\pi}{2} - \frac{x}{2}$

$\therefore \; \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x}{2}\right) = -\frac{1}{2}.$

8. We have $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$ ...(i)

Differentiating (i) w.r.t. x, we have

$MC = \frac{dC(x)}{dx} = \frac{d(0.005x^3 - 0.02x^2 + 30x + 5000)}{dx}$

$= 3 \times 0.005x^2 - 2 \times (0.02)x + 30$

$= 0.015x^2 - 0.04x + 30$ ...(i)

$\therefore \; \left(\frac{dc}{dx}\right)_{x=3} = 0.015(3)^2 - 0.04(3) + 30 = 30.015$

Hence, required marginal cost when $x = 3$ is 30.015

9. Let $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

$= \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x}dx = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x}dx$

$= \int \frac{1}{\cos^2 x}dx = \int \sec^2 x\,dx = \tan x + C$

10. We have, $y = a\,e^{bx+5}$ ...(i)

$\therefore \; \frac{dy}{dx} = a e^{bx+5} \cdot b \;\Rightarrow\; e^{bx+5} = \frac{1}{ab}\frac{dy}{dx}$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$b \cdot e^{bx+5} = \frac{1}{ab}\frac{d^2y}{dx^2} \;\Rightarrow\; e^{bx+5} = \frac{1}{ab^2}\frac{d^2y}{dx^2}$ ...(iii)

From (ii) & (iii), we have

$\frac{1}{ab}\frac{dy}{dx} = \frac{1}{ab^2}\frac{d^2y}{dx^2} \;\Rightarrow\; \frac{d^2y}{dx^2} = b\frac{dy}{dx}$ ...(iv)

From (i) and (ii), we have $\frac{1}{y}\frac{dy}{dx} = b$ ...(v)

Solving (iv) and (v), we get $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

11. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \; (\hat{i} - 2\hat{j} + 3\hat{k})\cdot(3\hat{i} - 2\hat{j} + \hat{k}) = \sqrt{(1)^2 + (-2)^2 + (3)^2}$

$\times \sqrt{(3)^2 + (-2)^2 + (1)^2}\cos\theta$

$\Rightarrow \; 3 + 4 + 3 = \sqrt{14} \times \sqrt{14}\cos\theta \;\Rightarrow\; \cos\theta = \frac{10}{14}$

$\therefore \; \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{100}{196}} = \sqrt{\frac{96}{196}} = \frac{2\sqrt{6}}{7}$

12. E : 'a total of 8' and F : 'red die resulted in a number less than 4'

i.e., $E = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

and $F = \{(x, y) : x \in \{1, 2, 3, 4, 5, 6\}, y \in \{1, 2, 3\}\}$

i.e., $F = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2),$
$(2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (5, 1),$
$(5, 2), (5, 3), (6, 1), (6, 2), (6, 3)\}$

Hence, $E \cap F = \{(5, 3), (6, 2)\}$, $P(E) = 5/36$,
$P(F) = 18/36$, $P(E \cap F) = 2/36$

$\therefore$ Required probability $= P(E \mid F)$

$= \frac{P(E \cap F)}{P(F)} = \frac{2/36}{18/36} = \frac{1}{9}.$

13. L.H.S. $= \begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$\text{L.H.S.} = \begin{vmatrix} 1 & 0 & 3x \\ 1+3y & -3y & -3y \\ 1 & 3z & 0 \end{vmatrix}$

Taking out 3 common from both C_2 and C_3, we get

$\text{L.H.S.} = 9\begin{vmatrix} 1 & 0 & x \\ 1+3y & -y & -y \\ 1 & z & 0 \end{vmatrix}$

Expanding along R_1, we get

L.H.S. $= 9[1(0 + zy) - 0 + x(z + 3yz + y)]$

$= 9[zy + xz + 3xyz + xy]$

$= 9(3xyz + xy + yz + zx) = $ R.H.S. Hence proved

14. We have, $(x^2 + y^2)^2 = xy$

Differentiating both sides, we get

$$2(x^2+y^2)\left\{2x+2y\frac{dy}{dx}\right\}=x\frac{dy}{dx}+y\cdot 1$$

$$\Rightarrow \quad 4(x^2+y^2)\left(x+y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 4x(x^2 + y^2) - y = (x - 4y(x^2 + y^2))\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx}=\frac{4x^3+4xy^2-y}{x-4yx^2-4y^3}$$

OR

We have, $x = a(2\theta - \sin 2\theta)$ ...(i)

and $y = a(1 - \cos 2\theta)$ ...(ii)

Differentiating (i) and (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta),\ \frac{dy}{d\theta} = 2a\sin 2\theta$$

$$\text{Now, } \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2a\sin 2\theta}{a(2-2\cos 2\theta)}=\frac{\sin 2\theta}{1-\cos 2\theta}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=\frac{\sin 2\left(\frac{\pi}{3}\right)}{1-\cos\frac{2\pi}{3}}=\frac{\sin\left(\pi-\frac{\pi}{3}\right)}{1-\cos\left(\pi-\frac{\pi}{3}\right)}$$

$$=\frac{\sin\left(\frac{\pi}{3}\right)}{1+\cos\left(\frac{\pi}{3}\right)}=\frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}}=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$$

15. We have, $y = \sin(\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\cos(\sin x)\cdot\cos x$$

Again differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cdot\cos x.\cos x + (-\sin x)\cos(\sin x)$$

$$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$$

$$\text{Now, L.H.S.} = \frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x$$

$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$
$\quad + \tan x(\cos x\cdot\cos(\sin x)) + \cos^2 x\cdot\sin(\sin x)$

$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x) + \sin x\cdot\cos(\sin x)$
$\quad + \cos^2 x\cdot\sin(\sin x)$

$= 0 = $ R.H.S.

16. Given curve is, $16x^2 + 9y^2 = 145$ ...(i)

Differentiating (i) both sides w.r.t. x, we get

$$32x + 18yy' = 0 \Rightarrow y'=\frac{-32x}{18y}=\frac{-16}{9}\frac{x}{y} \quad ...(ii)$$

Now, (i) passes through (x_1, y_1)

$\therefore \quad 16x_1^2 + 9y_1^2 = 145$

$\Rightarrow \quad 16(2)^2 + 9y_1^2 = 145 \ (\because x_1 = 2)$

$\Rightarrow \quad y_1^2 = 9 \Rightarrow y_1 = \pm 3$

But $y_1 > 0 \ \therefore y_1 = 3$

$\therefore$ From (ii), we get

$$y'=\frac{-16}{9}\frac{x_1}{y_1} \Rightarrow y'=\frac{-16}{9}\times\frac{2}{3}=\frac{-32}{27}$$

So, equation of tangent passing through (2, 3) and having slope $\left(\frac{-32}{27}\right)$ is

$$y-3=\frac{-32}{27}(x-2) \Rightarrow 32x + 27y - 145 = 0$$

Also, equation of normal passing through (2, 3) is $(y - 3)$

$$= \frac{-1}{\left(\frac{-32}{27}\right)}(x-2) \Rightarrow 27x - 32y + 42 = 0$$

OR

We have $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ ...(i)

$f(x)$ being a polynomial function is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$$f'(x)=\frac{4x^3}{4}-3x^2-10x+24 = x^3 - 3x^2 - 10x + 24$$

$$= (x - 2)(x^2 - x - 12) = (x - 2)(x - 4)(x + 3)$$

(a) For increasing, $f'(x) > 0$

$\Rightarrow \quad (x - 2)(x - 4)(x + 3) > 0$

$\Rightarrow \quad x \in (4, \infty) \cup (-3, 2)$

(b) For decreasing, $f'(x) < 0$

$\Rightarrow \quad (x - 2)(x - 4)(x + 3) < 0$

$\Rightarrow \quad x \in (2, 4) \cup (-\infty, -3)$

For complete solutions refer to MTG CBSE Champion Mathematics

MPP-1 CLASS XI ANSWER KEY

1.	(d)	**2.**	(c)	**3.**	(b)	**4.**	(d)	**5.**	(b)
6.	(c)	**7.**	(a, c)	**8.**	(a)	**9.**	(c)	**10.**	(a, c, d)
11.	(a, b)	**12.**	(a, d)	**13.**	(b, c)	**14.**	(d)	**15.**	(c)
16.	(c)	**17.**	(4)	**18.**	(1)	**19.**	(6)	**20.**	(7)

ACE YOUR WAY CBSE

Sets

Set is the well defined collection of objects.

REPRESENTATION OF SETS

- **Roster/Tabular form:** The elements are separated by commas and are enclosed within braces { }.
- **Set builder form :** The elements are represented by using a variable x followed by ':' and after ':' we write the characteristic or property possessed by the elements. The whole description is enclosed within braces { }.

TYPES OF SETS

- **Empty Set** - Set which does not contain any element. It is denoted by ϕ or { }.
- **Finite Set** - Set which is empty or having finite number of elements.
- **Infinite Set** - Set which is not a finite set.
- **Equal Sets** - Two given sets having exactly the same elements.

SUBSET

For any two sets P and Q,

- P is said to be the subset of Q, *i.e.* $P \subseteq Q$ if every element of P is also an element of Q.
- If $P \subset Q$ and $P \neq Q$, then P is called proper subset of Q and Q is called superset of P.

INTERVALS AS SUBSETS OF R

Let $a, b \in R$ and $a < b$, then

- **Closed interval :** $[a, b] = \{x : a \leq x \leq b\}$
- **Semi closed or semi open interval :** $[a, b) = \{x : a \leq x < b\}$ and $(a, b] = \{x : a < x \leq b\}$
- **Open interval :** $(a, b) = \{x : a < x < b\}$

POWER SET

- Set of all the subsets of a set is called power set of a set which is denoted by $P(A)$ *i.e.*, $P(A) = \{S : S \subset A\}$, where S is subset of set A.

UNIVERSAL SET

- If all sets under consideration are subsets of a larger set, then this larger set is called universal set, denoted by U.

COMPLEMENT OF A SET

- The complement of a given set is the set which contains all those members of the universal set that does not belong to the given set.
- The complement of the set A is denoted by A' or by $A^c = \{x : x \in U, x \notin A\}$.

OPERATIONS ON SETS

- **Union of Sets :** For two sets A and B, union of sets written as $A \cup B$ is the set of all those elements which belong to A or B or both.
Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- **Intersection of Sets :** For two sets A and B, intersection of sets written as $A \cap B$ is the set of all those elements which belong to A and B both.
Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

- **Difference of Sets :** For two sets A and B, difference of sets written as $A - B$ is the set of elements which are in A but not in B.
  Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$
  Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$

IMPORTANT PROPERTIES

- **Properties of union**
  (i) $A \cup B = B \cup A$ (Commutative law)
  (ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
  (iii) $A \cup \phi = A$ (Law of identity element)
  (iv) $A \cup A = A$ (Idempotent law)
  (v) $U \cup A = U$ (Law of U)
- **Properties of intersection**
  (i) $A \cap B = B \cap A$ (Commutative law)
  (ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)
  (iii) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and U)
  (iv) $A \cap A = A$ (Idempotent law)
- **Distributive laws**
  (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **Complement laws**
  (i) $A \cup A' = U$
  (ii) $A \cap A' = \phi$
- **De Morgan's law**
  (i) $(A \cup B)' = A' \cap B'$
  (ii) $(A \cap B)' = A' \cup B'$
- **Law of double complementation**
  $(A')' = A$
- **Laws of empty set and universal set**
  $\phi' = U$ and $U' = \phi$

VENN DIAGRAM

- A diagram used to illustrate relationship between the sets.

IMPORTANT FORMULAE BASED ON NUMBER OF ELEMENTS IN SETS

For two finite sets A and B, we have

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if $A \cap B \neq \phi$
- $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Write the set $A = \{x : x \in Z, x^2 < 20\}$ in the roster form.
2. If X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.
3. Are the sets $A = \{x : x$ is a letter in the word "$LOYAL$"$\}$, $B = \{x : x$ is a letter of the word "$ALLOY$"$\}$ equal?
4. List all the elements of sets:
   (i) $C = \{x : x = 2n, n \in N \text{ and } n \leq 5\}$
   (ii) $H = \{x : x = n^2, n \in N, 2 \leq n \leq 5\}$
5. Write the set $X = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...\right\}$ in the set-builder form.

SHORT ANSWER TYPE

6. State which of the following sets are finite and which are infinite:
   (i) $A = \{x : x \in Z \text{ and } x^2 - 5x + 6 = 0\}$
   (ii) $B = \{x : x \in Z \text{ and } x^2 \text{ is even}\}$
   (iii) $C = \{x : x \in Z \text{ and } x^2 = 36\}$
   (iv) $D = \{x : x \in Z \text{ and } x > -10\}$
7. Express each of following sets as an interval:
   (i) $A = \{x : x \in R, -4 < x < 0\}$
   (ii) $B = \{x : x \in R, 0 \leq x < 3\}$
   (iii) $C = \{x : x \in R, 2 < x \leq 6\}$
   (iv) $D = \{x : x \in R, -5 \leq x \leq 2\}$
8. Describe the following sets in Roster form:
   (i) The set of all letters in the word 'MATHEMATICS'
   (ii) The set of squares of integers.
9. A survey shows that 73% of the Indians like apples, whereas 65% like oranges. What percentage of Indians like both apples and oranges?
10. Write the following sets in Roster form :
    (i) $A = \{a_n : n \in N, a_{n+1} = 3a_n \text{ and } a_1 = 1\}$
    (ii) $B = \{a_n : n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = a_2 = 1\}$

LONG ANSWER TYPE - I

11. Write the following subsets of R as intervals:
    (i) $\{x : x \in R, -4 < x \leq 6\}$
    (ii) $\{x : x \in R, -12 < x < -10\}$
    (iii) $\{x : x \in R, 0 \leq x < 7\}$
    (iv) $\{x : x \in R, 3 \leq x \leq 4\}$
    Also, find the length of each interval.
12. For any sets A and B, show that $A - B = A \cap B'$
13. Let A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.
14. In a survey of 100 students, the number of students studying the various languages is found as: English only 18; English but not Hindi 23; English and Sanskrit 8; Sanskrit and Hindi 8; English 26; Sanskrit 48 and no language 24.

(i) How many students are studying Hindi?

(ii) How many students are studying English and Hindi both?

15. If $A = \{x : x = N, x$ is a factor of $6\}$ and $B = \{x \in N : x$ is a factor of $8\}$, then find

(i) $A \cup B$ (ii) $A \cap B$

(iii) $A - B$ (iv) $B - A$

LONG ANSWER TYPE - II

16. A college awarded 38 medals for Honesty, 15 for Punctuality and 20 for Obedience. If these medals were bagged by a total of 58 students and only 3 students got medals for all three values, how many students received medals for exactly two of the three values?

17. There are 240 students in class XI of a school, 130 play cricket, 100 play football, 75 play volleyball, 30 of these play cricket and football, 25 play volleyball and cricket, 15 play football and volleyball. Also each student plays atleast one of the three games. How many students play all the three games?

18. In a town of 10000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspaper, find the number of families which buy

(i) A only (ii) B only

(iii) None of A, B and C.

19. A survey of 500 television viewers produced the following information; 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?

20. A survey shows that 63% of the Americans like cheese whereas 76% like apples. If x% of the Americans like both cheese and apples, find the value of x.

SOLUTIONS

1. The integers whose squares are less than 20 are 0, $\pm 1, \pm 2, \pm 3, \pm 4$.

$\therefore$ Set A in roster form is $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

2. We have, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$\Rightarrow$ $38 = 17 + 23 - n(X \cap Y)$

$\Rightarrow$ $n(X \cap Y) = 40 - 38 = 2$

3. We have, $A = \{L, O, Y, A, L\} = \{L, O, Y, A\}$

and $B = \{A, L, L, O, Y\} = \{L, O, Y, A\}$

Clearly, $A = B$.

4. (i) $C = \{2, 4, 6, 8, 10\}$

(ii) $H = \{4, 9, 16, 25\}$

5. The elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set builder form is $X = \left\{\frac{1}{n^2} : n \in N\right\}$.

6. (i) $A = \{x : x \in Z$ and $x^2 - 5x + 6 = 0\} = \{2, 3\}$

So, A is a finite set.

(ii) $B = \{x : x \in Z$ and x^2 is even$\}$

$= \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$

Clearly, B is an infinite set.

(iii) $C = \{x : x \in Z$ and $x^2 = 36\} = \{6, -6\}$

Clearly, C is a finite set.

(iv) $D = \{x : x \in Z$ and $x > -10\} = \{-9, -8, -7, ...\}$

Clearly, D is an infinite set.

7. (i) $A = (-4, 0)$ (ii) $B = [0, 3)$

(iii) $C = (2, 6]$ (iv) $D = [-5, 2]$

8. (i) We observe that distinct letters in the word 'MATHEMATICS' are:

M, A, T, H, E, I, C, S

Since the order in which the elements of a set are written is immaterial and the repetition of elements has no effect. So, required set can be described as follows :

{M, A, T, H, E, I, C, S}.

(ii) Since square of a negative integer is same as the square of its absolute value. Therefore, squares of integers are 0, 1, 4, 9, 16, 25,...... Hence, required set is {0, 1, 4, 9, 16,....}.

9. Let A = set of Indians who like apples

and B = set of Indians who like oranges.

Then, $n(A) = 73$, $n(B) = 65$ and $n(A \cup B) = 100$

$\because$ $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 73 + 65 - 100 = 38$.

Hence, 38% of the Indians like both apples and oranges.

10. (i) We have, $a_1 = 1$ and $a_{n+1} = 3a_n$ for all $n \in N$

Putting $n = 1, 2, 3, 4, 5,.........$ in $a_{n+1} = 3a_n$, we get

$a_2 = 3a_1 = 3 \times 1 = 3$ $[\because a_1 = 1]$

$a_3 = 3a_2 = 3 \times 3 = 3^2$ $[\because a_2 = 3]$

$a_4 = 3a_3 = 3 \times 3^2 = 3^3$ $[\because a_3 = 3^2]$

$a_5 = 3a_4 = 3 \times 3^3 = 3^4$, $a_6 = 3a_5 = 3 \times 3^4 = 3^5$ and so on.

Hence, $A = \{a_1, a_2, a_3, a_4, a_5, a_6, ...\}$

$= \{1, 3, 3^2, 3^3, 3^4, 3^5, ...\}$

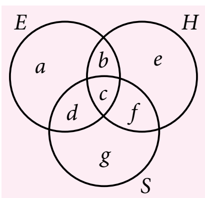
(ii) We have, $a_1 = 1, a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$.
Putting $n = 1, 2, 3, 4, ...$ in $a_{n+2} = a_{n+1} + a_n$, we get $a_3 = a_2 + a_1 = 1 + 1 = 2$; $a_4 = a_3 + a_2 = 2 + 1 = 3$; $a_5 = a_4 + a_3 = 3 + 2 = 5$; $a_6 = a_5 + a_4 = 5 + 3 = 8$ and so on.
Hence, $B = \{a_1, a_2, a_3, a_4, a_5, a_6, ...\}$
$= \{1, 1, 2, 3, 5, 8, ...\}$

11. (i) $\{x : x \in R, -4 < x \leq 6\} = (-4, 6]$
Length $= 6 - (-4) = 10$
(ii) $\{x : x \in R, -12 < x < -10\} = (-12, -10)$
Length $= -10 - (-12) = 2$
(iii) $\{x : x \in R, 0 \leq x < 7\} = [0, 7)$
Length $= 7 - 0 = 7$
(iv) $\{x : x \in R, 3 \leq x \leq 4\} = [3, 4]$
Length $= 4 - 3 = 1$

12. Let $x \in A - B$. Then,
$x \in A - B \Rightarrow x \in A$ and $x \notin B$
$\Rightarrow \; x \in A$ and $x \in B' \;\; \Rightarrow \; x \in A \cap B'$.
$\therefore \; (A - B) \subseteq (A \cap B')$ ...(i)
Again, let $y \in (A \cap B')$. Then,
$y \in (A \cap B') \Rightarrow y \in A$ and $y \in B'$
$\Rightarrow \; y \in A$ and $y \notin B$
$\Rightarrow \; y \in (A - B)$.
$\therefore \; (A \cap B') \subseteq (A - B)$ ...(ii)
From (i) and (ii), we get $(A - B) = (A \cap B')$.

13. We have, $A \cup B = A \cup C$
$\Rightarrow \; (A \cup B) \cap C = (A \cup C) \cap C$
$\Rightarrow \; (A \cap C) \cup (B \cap C) = C \;\; [\because (A \cup C) \cap C = C]$
$\Rightarrow \; (A \cap B) \cup (B \cap C) = C$ ...(i)
Again, $A \cup B = A \cup C$
$\Rightarrow \; (A \cup B) \cap B = (A \cup C) \cap B$
$\Rightarrow \; B = (A \cap B) \cup (C \cap B) \;\; [\because (A \cup B) \cap B = B]$
$\Rightarrow \; B = (A \cap B) \cup (B \cap C)$ ...(ii)
From (i) and (ii), we get $B = C$.

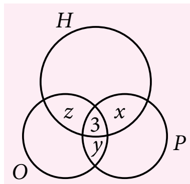
14. Let E, H, S denote the sets of students studying English, Hindi and Sanskrit respectively.
In the adjoining Venn diagram, let a, b, c, d, e, f and g denote the number of students in the respective regions. According to given information, we have



$a = 18, a + d = 23, c + d = 8,$
$c + f = 8, a + b + c + d = 26,$
$c + d + f + g = 48,$
$a + b + c + d + e + f + g$
$= 100 - 24 = 76.$
$\therefore \; a = 18, d = 5, c = 3,$
$f = 5$ and $b = 0$.
$\therefore \; g = 48 - (3 + 5 + 5) = 35$
and $e = 76 - (18 + 0 + 3 + 5 + 5 + 35) = 10$.
(i) Number of students studying Hindi
$= (b + c + e + f) = (0 + 3 + 10 + 5) = 18$.
(ii) Number of students studying English and Hindi both $= (b + c) = (0 + 3) = 3$.

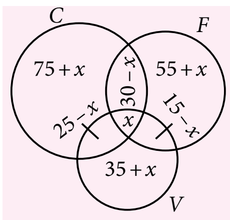
15. We have,
$A = \{x : x \in N, x$ is a factor of $6\} = \{1, 2, 3, 6\}$
and $B = \{x : x \in N, x$ is a factor of $8\} = \{1, 2, 4, 8\}$
(i) $A \cup B = \{1, 2, 3, 6\} \cup \{1, 2, 4, 8\} = \{1, 2, 3, 4, 6, 8\}$
(ii) $A \cap B = \{1, 2, 3, 6\} \cap \{1, 2, 4, 8\} = \{1, 2\}$
(iii) $A - B = \{1, 2, 3, 6\} - \{1, 2, 4, 8\} = \{3, 6\}$
(iv) $B - A = \{1, 2, 4, 8\} - \{1, 2, 3, 6\} = \{4, 8\}$

16. Let H, P and O be the sets of students who received medals in Honesty, Punctuality and Obedience respectively. Then,
$n(H) = 38, n(P) = 15, n(O) = 20, n(H \cup P \cup O) = 58$
and $n(H \cap P \cap O) = 3$.
Now, $n(H \cup P \cup O) = n(H) + n(P) + n(O)$
$- n(H \cap P) - n(P \cap O) - n(H \cap O) + n(H \cap P \cap O)$
$\Rightarrow \; 58 = 38 + 15 + 20 - n(H \cap P) - n(P \cap O)$
$- n(H \cap O) + 3$
$\Rightarrow \; n(H \cap P) + n(P \cap O) + n(H \cap O) = 18$ ...(i)
In the adjoining Venn diagram, let x denote the number of students who got medals in Honesty and Punctuality only, y denote the number of students who got medals in Punctuality and Obedience only, z denote the number of students who got medals in Honesty and Obedience only. 3 students got medals in all the three values.



Using (i) and Venn diagram, we get
$(x + 3) + (y + 3) + (z + 3) = 18$
$\Rightarrow \; x + y + z = 9$.
Hence, the number of students who received medals in exactly two values out of the three values = 9.

17. Let C, F and V be the sets of students who play cricket, football and volleyball respectively.
Let x be the number of students who play all the three games, then the number of students according to given information in the question are shown in different regions of the adjoining Venn diagram.



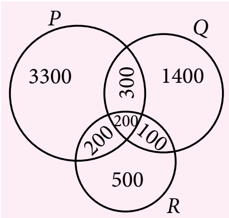
As each student plays atleast one of three games, $n(C \cup F \cup V) = 240$.
From the Venn diagram, we have
$(75 + x) + (30 - x) + (55 + x) + (15 - x) + (35 + x) + (25 - x) + x = 240$

$\Rightarrow \quad 75 + 30 + 55 + 15 + 35 + 25 + x = 240$

$\Rightarrow \quad 235 + x = 240 \Rightarrow x = 5.$

Hence, 5 students play all the three games.

18. Let P, Q and R denote the sets of families who buy newspapers A, B and C respectively, then



$n(P) = 40\%$ of $10000 = 4000$,

$n(Q) = 20\%$ of $10000 = 2000$,

$n(R) = 10\%$ of $10000 = 1000$,

$n(P \cap Q) = 5\%$ of $10000 = 500$,

$n(Q \cap R) = 3\%$ of $10000 = 300$,

$n(P \cap R) = 4\%$ of $10000 = 400$ and

$n(P \cap Q \cap R) = 2\%$ of $10000 = 200$

From the Venn diagram, we get

(i) Number of families who buy newspaper A only = 3300.

(ii) Number of families who buy newspaper B only = 1400.

(iii) Number of families who buy newspaper A, B or C *i.e.* atleast one of the newspaper

$= 4000 + 1400 + 100 + 500 = 6000$

$\therefore$ Number of families who do not buy any of newspaper A, B and $C = 10000 - 6000 = 4000$.

19. Let N = Total number of television viewers = 500, $n(F) = 285, n(H) = 195, n(B) = 115, n(F \cap B) = 45, n(F \cap H) = 70, n(H \cap B) = 50, n(F' \cap H' \cap B') = 50$

Now, $n(F' \cap H' \cap B') = 50$

$\Rightarrow \quad n[(F \cup H \cup B)'] = 50$

$\Rightarrow \quad N - n(F \cup H \cup B) = 50$

$\Rightarrow \quad 500 - [n(F) + n(H) + n(B) - n(F \cap H) - n(F \cap B) - n(H \cap B) + n(F \cap H \cap B)] = 50$

$\Rightarrow \quad n(F \cap H \cap B) = 500 - 285 - 195 - 115 + 70 + 50 + 45 - 50 = 20.$

Hence, number of viewers who watch all the three games = 20

Number of viewers who watch exactly one of the three games $= n(F \cap H' \cap B') + n(F' \cap H' \cap B) + n(F' \cap H \cap B')$

$= n(F) + n(H) + n(B) - 2[n(F \cap H) + n(H \cap B) + n(B \cap F)] + 3n(F \cap H \cap B)$

$= 285 + 195 + 115 - 2\,(70 + 50 + 45) + 3(20)$

$= 595 - 2(165) + 60 = 595 - 330 + 60 = 325$

20. Let A denote the set of Americans who like cheese, B denote those who like apples and x denote those who like both cheese and apples. Let the population of America be 100. Then, $n(A) = 63$, $n(B) = 76$.

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$\Rightarrow \quad n(A \cap B) = 63 + 76 - n(A \cup B) = 139 - n(A \cup B)$

But, $n(A \cup B) \le 100$.

$\Rightarrow \quad -n(A \cup B) \ge -100$

$\Rightarrow \quad 139 - n(A \cup B) \ge 139 - 100$

$\Rightarrow \quad 139 - n(A \cup B) \ge 39$

$\Rightarrow \quad n(A \cap B) \ge 39 \qquad \text{...(i)}$

Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\Rightarrow \quad n(A \cap B) \le n(A)$ and $n(A \cap B) \le n(B)$

$\Rightarrow \quad n(A \cap B) \le 63$ and $n(A \cap B) \le 76$

$\Rightarrow \quad n(A \cap B) \le 63 \qquad \text{...(ii)}$

From (i) and (ii), we obtain

$39 \le n(A \cap B) \le 63$

$\Rightarrow \quad 39 \le x \le 63.$

INDIA'S BEST EDUCATORS

The Indian Institute of Science, Bangalore, has again been declared the best educational Institution by the HRD ministry's National Institutional Ranking Framework. India Rankings 2018 show Delhi's Miranda House is the best college

▲ Rank went up ▼ Rank slipped ● Same rank as 2017

OVERALL RANKINGS

Rank 2018	Institution		Rank 2017
1	**Indian Institute of Science, Bangalore**	●	1
2	IIT Madras	●	2
3	IIT Bombay	●	3
4	IIT Delhi	▲	5
5	IIT Kharagpur	▼	4
6	Jawaharlal Nehru University	●	6
7	IIT Kanpur	●	7
8	IIT Roorkee	▲	9
9	Banaras Hindu University, Varanasi	▲	10
10	Anna University, Chennai	▲	13

Participation must for public institutes

■ 2,809 institutions participated in nine categories. Some colleges, including St. Stephen's and Hindu College, did not participate in the ranking process last year but are part of the latest edition

■ The HRD ministry has made participation by public institutes mandatory from 2019.

Methodology

The rankings are ascertained on the basis of teaching, learning and resources, research and professional practices, graduation outcomes, outreach and inclusivity and perception.

The system of announcing the rankings every year was introduced by the HRD ministry in 2016

UNIVERSITIES

Rank 2018	Institution		Rank 2017
1	**Indian Institute of Science, Bangalore**	●	1
2	Jawaharlal Nehru University, New Delhi	●	2
3	Banaras Hindu University, Varanasi	●	3
4	Anna University, Chennai	▲	6
5	University of Hyderabad	▲	7
6	Jadavpur University, Kolkata	▼	5
7	University of Delhi, New Delhi	▲	8
8	Amrita Vishwa Vidyapeetham, Coimbatore	▲	9
9	Savitribai Phule Pune University	▲	10
10	Aligarh Muslim University, Aligarh	▲	11

COLLEGES

Rank 2018	Institution		Rank 2017
1	**Miranda House, Delhi University**	●	1
2	St. Stephens, Delhi University		n/a
3	Bishop Heber College, Tiruchirappalli	▲	4
4	Hindu College, Delhi		n/a
5	Presidency College, Chennai		n/a
6	Loyola College, Chennai	▼	2
7	Shri Ram College for Commerce, Delhi	▼	3
8	Lady Shri Ram College for Women, Delhi	▼	7
9	Ramakrishna Mission Vidyamandira, Howrah		n/a
10	Madras Christian College, Chennai	▲	12

ENGINEERING COLLEGES

Rank 2018	Institution		Rank 2017
1	**IIT Madras**	●	1
2	IIT Bombay	●	2
3	IIT Delhi	▲	4
4	IIT Kharagpur	▼	3
5	IIT Kanpur	●	5
6	IIT Roorkee	●	6
7	IIT Guwahati	●	7
8	Anna University, Chennai	●	8
9	IIT Hyderabad	▲	10
10	Institute of Chemical Technology, Mumbai	▲	14

MPP-1 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Sets, Relations and Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If $f(x) = 4x^3 + 3x^2 + 3x + 4$, then $x^3 f\left(\frac{1}{x}\right)$ is equal to
   (a) $f(-x)$ (b) $\frac{1}{f(x)}$
   (c) $\left(f\left(\frac{1}{x}\right)\right)^2$ (d) $f(x)$

2. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then which of the following is a function from A to B?
   (a) $\{(1, 2), (2, 3), (3, 4), (2, 2)\}$
   (b) $\{(1, 2), (2, 3), (1, 3)\}$
   (c) $\{(1, 3), (2, 3), (3, 3)\}$
   (d) $\{(1, 1), (2, 3), (3, 4)\}$

3. If $f(x)$ is a polynomial function satisfying $f(x)\,f(y) = f(x) + f(y) + f(xy) - 2$ for all real x and y and $f(3) = 10$, then $f(4)$ is equal to
   (a) 16 (b) 17 (c) 18 (d) 19

4. If $f(x) = \cos(\log x)$, then $f(x) f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value
   (a) -1 (b) $\frac{1}{2}$ (c) -2 (d) 0

5. In a survey of 200 people of a town, it was found that 120 like tea, 90 like coffee and 70 like cold-drink, 40 like tea and coffee, 30 like coffee and cold-drink, 50 like cold-drink and tea and 20 none of these beverages. The number of people who like all the three beverages is
   (a) 30 (b) 20
   (c) 22 (d) 25

6. If $R = \{(x, y): x, y \in I, x^2 + y^2 \leq 4\}$ is a relation in I, then domain of R is
   (a) $\{0, 1, 2\}$ (b) $\{-2, -1, 0\}$
   (c) $\{-2, -1, 0, 1, 2\}$ (d) None of these

One or More Than One Option(s) Correct Type

7. If $X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$, then
   (a) the smallest set X is $\{3, 5, 9\}$
   (b) the smallest set X is $\{2, 3, 5, 9\}$
   (c) the largest set X is $\{1, 2, 3, 5, 9\}$
   (d) the largest set X is $\{2, 3, 4, 9\}$

8. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to
   (a) $B \cap C'$ (b) $A \cap C$
   (c) $B' \cap C'$ (d) None of these

9. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x,\ x \neq 0$, and $S = \{x \in R : f(x) = f(-x)\}$; then S
   (a) is an empty set
   (b) contains exactly one element
   (c) contains exactly two elements
   (d) contains more than two elements

10. In a group of 50 students, the number of students studying French, English, Sanskrit were found to be as follows.
    French = 17, English = 13, Sanskrit = 15
    French and English = 09, English and Sanskrit = 4
    French and Sanskrit = 5, English, French and Sanskrit = 3. The number of students who study
    (a) French only is 6
    (b) Sanskrit only is 8
    (c) French and Sanskrit but not English is 2
    (d) atleast one of the three languages is 30

11. The domain and range of the function f given by $f(x) = 2 - |x - 5|$ is

(a) Domain = R (b) Range = $(-\infty, 2]$
(c) Domain = $R - \{0\}$ (d) Range = $(-\infty, 2)$

12. For the function $f(x) = \left[\frac{1}{[x]}\right]$, where $[x]$ denotes the greatest integer less than or equal to x, which of the following statements is/are false?

(a) The domain is $(-\infty, \infty)$.
(b) The range is $\{0\} \cup \{-1\} \cup \{1\}$.
(c) The domain is $(-\infty, 0) \cup [1, \infty)$.
(d) The range is $\{0\} \cup \{1\}$.

13. Consider the following relations :

I. $A - B = A - (A \cap B)$
II. $A = (A \cap B) \cup (A - B)$
III. $A - (B \cup C) = (A - B) \cup (A - C)$

Which of these is/are correct?

(a) I and III (b) II
(c) I (d) II and III

Comprehension Type

Out of 800 boys in a school, 224 played Cricket, 240 played Hockey and 336 played Basketball. Of the total, 64 played both Basketball and Hockey; 80 played Cricket and Basketball and 40 played Cricket and Hockey; 24 played all the three games.

14. The number of boys who did not play any game is

(a) 128 (b) 216 (c) 240 (d) 160

15. The number of boys who played only Cricket is

(a) 240 (b) 220 (c) 128 (d) 320

Matrix Match Type

16. Match the following.

	Column I		Column II
P.	The domain of definition of the function $f(x) = \dfrac{\log(2x-3)}{\sqrt{x-1}} + \sqrt{5-2x}$ is	1.	$(-\infty, -1) \cup (1, 4]$
Q.	The domain of the function f defined by $f(x) = \sqrt{4-x} + \dfrac{1}{\sqrt{x^2-1}}$ is equal to	2.	R – {2, 6}
R.	If R is the set of all real numbers and if $f : R - \{2\} \to R$ is defined by $f(x) = \dfrac{2+x}{2-x}$ for $x \to R - \{2\}$, then the range of f is	3.	$\left(\dfrac{3}{2}, \dfrac{5}{2}\right]$
S.	The domain of the function $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$ is	4.	R – {–1}

	P	Q	R	S
(a)	1	3	2	4
(b)	2	1	3	4
(c)	3	1	4	2
(d)	3	4	1	2

Integer Answer Type

17. If domain of $\sqrt{\log\left(\dfrac{5x - x^2}{4}\right)}$ is $[1, x]$ then x is equal to

18. If F is function such that $F(0) = 2$, $F(1) = 3$, $F(x + 2) = 2F(x) - F(x + 1)$ for $x \geq 0$, then unit place in $F(5)$ is equal to

19. Let Z denote the set of all integers and $A = \{(a, b) : a^2 + 3b^2 = 28, a, b \in Z\}$ and $B = \{(a, b) : a > b, a, b \in Z\}$, then the number of elements in $A \cap B$ is

20. If $A = \{x : x = n^2, n = 1, 2, 3\}$, then number of proper subsets is

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

ACE YOUR WAY CBSE

Relations and Functions

RELATIONS

Let A and B be two non-empty sets. The relation R between A and B is a subset of $A \times B$. Symbolically, we write the relation between A and B as $R : A \to B$ if and only if $R \subseteq A \times B$

Types of Relations

- **Reflexive Relation :** A relation R on a set A is reflexive, if $(a, a) \in R$, $\forall\ a \in A$.
- **Symmetric Relation :** A relation R on a set A is symmetric if $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R\ \forall\ a_1, a_2 \in A$
- **Transitive Relation :** A relation R on a set A is transitive if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R \Rightarrow (a_1, a_3) \in R\ \forall\ a_1, a_2, a_3 \in A$.
- **Equivalence Relation :** A relation R is an equivalence relation if R is reflexive, symmetric and transitive.

FUNCTIONS

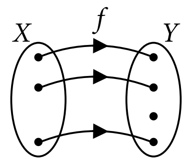
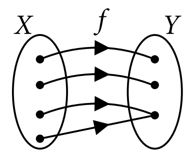
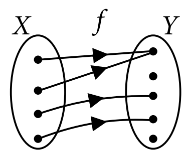
Let A and B be two non-empty sets. If there exists a correspondence by which each element $x \in A$ is related to a unique element $y \in B$, then such correspondence is called the function from A to B. It is written as $f : A \to B$ such that $x \in A$ and $y \in B$, where $y = f(x)$.

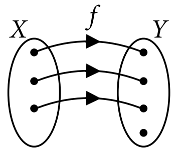
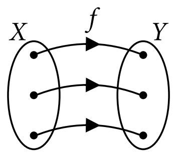
The element $y \in B$ is called the image of x under f and x is called the pre-image of y under f.

Domain, Co-domain, Range of Function $f(x)$

Domain	The set A is called the domain of function f.
Co-domain	The set B is called the co-domain of function f.
Range	The set $\{f(x) : x \in A$ and $f(x) \in B\}$ for all values taken by f is called the range of f. Obviously, it is a subset of set B.

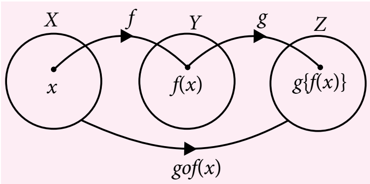
Types of Functions

Type	Definition	Representation
One-one (Injective) Function	A function $f : X \to Y$ is one-one, if different elements of X have different images in Y under f.	
Onto (Surjective) Function	A function $f : X \to Y$ is onto, if every element of Y is the image of some element of X under f.	
Many-one Function	A function $f : X \to Y$ is many-one, if two or more than two elements of X have the same image in Y.	

Into Function	A function $f: X \to Y$ is into, if there exists a single element in Y having no pre-image in X.	
Bijective Function	A function $f: X \to Y$ is bijective, if it is both one-one and onto.	

COMPOSITION OF FUNCTIONS

For two functions $f: X \to Y$ and $g: Y \to Z$, composition of functions is denoted by gof defined as
$gof(x) = g(f(x))\ \forall\ x \in X$.



INVERSE OF A FUNCTION

- Inverse of a function $f: A \to B$ is defined as $f^{-1}: B \to A$ s.t. $f^{-1}(y) = x \Leftrightarrow f(x) = y$
- A function f is invertible if and only if f is one-one onto.

BINARY OPERATION

Let A be a non-empty set, then the function from $A \times A$ into A is called a binary operation. Symbolically, a function '$*$' which is $* : A \times A \to A$ is called a binary operation. The image of any $(a, b) \in A \times A$ under '$*$' is denoted by $a * b$.

(i) A binary operation '$*$' over a set A is said to be **commutative** if $a * b = b * a\ \forall\ a, b \in A$

(ii) A binary operation '$*$' over a set A is said to be **associative** if $(a * b) * c = a * (b * c)\ \forall\ a, b, c \in A$

(iii) Let $*$ be a binary operation on a set A. An element $e \in A$ is said to be an **identity element** for the binary operation $*$ if $a * e = a = e * a\ \forall\ a \in A$

(iv) Let $*$ be a binary operation on a set A and let e be the identity element of the set A for this operation $*$. An element $b \in A$ is said to be the **inverse of an element** $a \in A$ if $a * b = e = b * a$.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. On Q, the set of all rational numbers, a binary operation $*$ is defined by $a * b = \dfrac{ab}{5}$ for all $a, b \in Q$. Find the identity element for $*$ in Q.

2. Let R be the set of all real numbers. Let $f: R \to R$ such that $f(x) = \sin x$ and $g: R \to R$ such that $g(x) = x^2$. Prove that $gof \neq fog$.

3. Let N be the set of natural numbers and relation R on N be defined by
$R = \{(x, y); x, y \in N, x + 4y = 10\}$
Determine whether the above relation is reflexive, symmetric.

4. Let $f: R \to R$ be defined as $f(x) = x^2 + 1$. Find :
(i) $f^{-1}(-5)$ (ii) $f^{-1}\{10, 37\}$

5. If the binary operation '$*$' on I is defined by $a * b = 2a - 3b$, then find
(i) $3 * 7$ (ii) $7 * 3$

SHORT ANSWER TYPE

6. If $f(x) = \dfrac{1}{2x+1}, x \neq -\dfrac{1}{2}$, then show that $f(f(x)) = \dfrac{2x+1}{2x+3}$, provided that $x \neq -\dfrac{1}{2}, -\dfrac{3}{2}$.

7. Show that the function $f: N \to N$ defined by $f(x) = x^3$ is injective but not surjective.

8. Let '$*$' be a binary operation on N given by $a * b = \text{HCF}(a, b)$ for all $a, b \in N$.
(i) Find $12 * 4$, $18 * 24$, $7 * 5$.
(ii) Check the commutativity and associativity of '$*$' on N.

9. Is $f: R \to R$ defined by $f(x) = |x| + x$ is one-one or onto? Also find the range of f.

10. Prove that the inverse of a bijection is unique.

LONG ANSWER TYPE - I

11. Let $A = Q \times Q$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for (a, b), $(c, d) \in A$. Then with respect to $*$ on A.
(i) Find the identity element in A.
(ii) Find the invertible elements of A.

12. Let A be the set of all lines in a plane and let R be a relation in A defined by
$R = \{(L_1, L_2) : L_1 \perp L_2\}$.
Show that R is symmetric but neither reflexive nor transitive.

13. If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$, show that $(fof)(x) = x$ for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

14. Let R^+ be the set of all positive real numbers. Let $f: R^+ \to [4, \infty)$ such that $f(x) = x^2 + 4$. Show that f is invertible and find f^{-1}.

15. Let N be the set of all natural numbers and let R be a relation in N, defined by

$R = \{(a, b) : a$ is a multiple of $b\}$

Show that R is reflexive and transitive but not symmetric.

LONG ANSWER TYPE - II

16. Let S be the set of all points in a plane and R be a relation on S defined as

$R = \{(P, Q) :$ Distance between P and Q is less than 2 units$\}$.

Show that R is reflexive and symmetric but not transitive.

17. Show that $f : N \to N$, defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

is a many-one onto function.

18. Let $f : W \to W : f(n) = \begin{cases} (n-1), & \text{when } n \text{ is odd} \\ (n+1), & \text{when } n \text{ is even.} \end{cases}$

Show that f is invertible. Find f^{-1}.

19. Let $A = R - \left\{\frac{3}{5}\right\}$ and $B = R - \left\{\frac{7}{5}\right\}$.

Let $f : A \to B : f(x) = \frac{7x+4}{5x-3}$ and

$g : B \to A : g(y) = \frac{3y+4}{5y-7}$.

Show that $(gof) = I_A$ and $(fog) = I_B$.

20. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let R_1 be a relation on X given by $R_1 = \{(x, y) : x - y$ is divisible by 3$\}$ and R_2 be another relation on X given by $R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$.

SOLUTIONS

1. Let e be the identity element. Then,

$a * e = a = e * a$, for all $a \in Q$

$\Rightarrow \frac{ae}{5} = a$ and $\frac{ea}{5} = a$, for all $a \in Q$

$\Rightarrow e = 5$

Thus, 5 is the identity element for the binary operation $*$ defined on Q.

2. Let x be an arbitrary real number, then

$(gof)(x) = g\{f(x)\} = g(\sin x) = (\sin x)^2$

$(fog)(x) = f\{g(x)\} = f(x^2) = \sin x^2$

Clearly, $(\sin x)^2 \neq \sin x^2$.

Hence, $gof \neq fog$.

3. Given, $R = \{(x, y); x, y \in N, x + 4y = 10\}$

$\therefore \quad R = \{(2, 2), (6, 1)\}$

R is not reflexive because $(1, 1) \notin R$.

R is not symmetric, because $(6, 1) \in R$ but $(1, 6) \notin R$.

4. **(i)** Let $f^{-1}(-5) = x$. Then,

$f(x) = -5 \Rightarrow x^2 + 1 = -5 \Rightarrow x^2 = -6 \Rightarrow x = \pm\sqrt{-6}$, which is not in R.

So, $f^{-1}(-5) = \phi$.

(ii) $f^{-1}\{10, 37\} = \{x \in R : f(x) = 10$ or $f(x) = 37\}$

$= \{x \in R : x^2 + 1 = 10$ or $x^2 + 1 = 37\}$

$= \{x \in R : x^2 = 9$ or $x^2 = 36\} = \{3, -3, 6, -6\}$

5. **(i)** $3 * 7 = 2(3) - 3(7) = 6 - 21 = -15$

(ii) $7 * 3 = 2(7) - 3(3) = 14 - 9 = 5$

6. We have, $f(x) = \frac{1}{2x+1}$

Clearly, domain $(f) = R - \left\{-\frac{1}{2}\right\}$

Let $y = \frac{1}{2x+1}$,

$\Rightarrow 2x + 1 = \frac{1}{y} \Rightarrow x = \frac{1-y}{2y}$

Since x is a real number distinct from $-\frac{1}{2}$.

Therefore, y can take any non-zero real value. So, Range $(f) = R - \{0\}$.

$\Rightarrow$ Range $(f) = R - \{0\} \not\subseteq$ Domain $(f) = R - \left\{-\frac{1}{2}\right\}$

$\therefore$ Domain $(fof) = \{x : x \in$ Domain (f) and $f(x) \in$ Domain $(f)\}$

$\Rightarrow$ Domain $(fof) = \left\{x : x \neq -\frac{1}{2} \text{ and } \frac{1}{2x+1} \neq -\frac{1}{2}\right\}$

$\Rightarrow$ Domain $(fof) = \left\{x : x \neq -\frac{1}{2} \text{ and } x \neq -\frac{3}{2}\right\}$

$$= R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\}$$

Also, $fof(x) = f(f(x)) = f\left(\frac{1}{2x+1}\right)$

$$= \frac{1}{2\left(\frac{1}{2x+1}\right)+1} = \frac{2x+1}{2x+3}$$

Thus, $fof : R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\} \to R$ is defined by $fof(x) = \frac{2x+1}{2x+3}$.

7. Let $x_1, x_2 \in N$ be such that $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 = x_2^3 \Rightarrow x_1^3 - x_2^3 = 0$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow \quad x_1 - x_2 = 0 \quad (\because x_1, x_2 \in N, \text{ so } x_1^2 + x_1x_2 + x_2^2 > 0)$

$\Rightarrow \quad x_1 = x_2.$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one *i.e.*, injective.

Since, $2 \in N$ (codomain of f) and there does not exist any $x \in N$ (domain of f) such that $f(x) = 2$ *i.e.*, $x^3 = 2$.

So, f is not onto *i.e.*, f is not surjective.

8. **(i)** $12 * 4 = \text{HCF}(12, 4) = 4$, $18 * 24 = \text{HCF}(18, 24) = 6$ and $7 * 5 = \text{HCF}(7, 5) = 1$

(ii) Commutativity : For any $a, b \in N$, we have

$a * b = \text{HCF}(a, b) = \text{HCF}(b, a) = b * a$

So, '$*$' is commutative on N.

Associativity: For any $a, b, c \in N$, we have

$(a * b) * c = \text{HCF}(a, b) * c = \text{HCF}(a, b, c)$

and $a * (b * c) = a * \text{HCF}(b, c) = \text{HCF}(a, b, c)$

$\therefore \quad (a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

So, '$*$' is associative on N.

9. When $x \ge 0$, $|x| = x \quad \Rightarrow f(x) = x + x = 2x$;

when $x < 0$, $|x| = -x \quad \Rightarrow f(x) = -x + x = 0$

Since, $f(x) = 0$ for all $x < 0 \Rightarrow f(-1) = 0$ and $f(-2) = 0$

So, the two different elements $-1, -2 \in R$ (domain of f) have same image.

$\Rightarrow \quad f$ is not one-one.

When $x \ge 0$, $f(x) = 2x \ge 0$ and when $x < 0$, $f(x) = 0$

$\Rightarrow \quad$ Range of $f = [0, \infty)$.

Since, Range of $f = [0, \infty)$, which is a proper subset of R (codomain of f)

$\Rightarrow \quad f$ is not onto.

10. Let $f : A \to B$ be a bijection. If possible, let $g : B \to A$ and $h : B \to A$ be two inverses of f.

We have to prove that $g = h$. In order to prove this it is sufficient to show that $g(y) = h(y)$ for all $y \in B$.

Let y be an arbitrary element of B.

Let $g(y) = x_1$ and $h(y) = x_2$. Then,

$g(y) = x_1 \Rightarrow f(x_1) = y \qquad [\because g \text{ is inverse of } f]$

and $h(y) = x_2 \Rightarrow f(x_2) = y \qquad [\because h \text{ is inverse of } f]$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \qquad [\because f \text{ is one-one}]$

$\Rightarrow \quad g(y) = h(y)$

Thus, $g(y) = h(y)$ for all $y \in B$. Hence, $g = h$.

11. **(i)** Let (x, y) be the identity element in A. Then,

$(a, b) * (x, y) = (a, b) = (x, y) * (a, b)$ for all $(a, b) \in A$

$\Rightarrow \quad (ax, b + ay) = (a, b) = (xa, y + bx)$ for all $a, b \in A$

$\Rightarrow \quad (ax, b + ay) = (a, b)$ and $(a, b) = (xa, y + bx)$ for all $a, b \in A$

$\Rightarrow \quad ax = a$ and $b + ay = b$ and $xa = a$, $y + bx = b$ for all $a, b \in A$

$\Rightarrow \quad x = 1, y = 0$

Clearly, $(1, 0) \in Q \times Q = A$

So, $(1, 0)$ is the identity element in A.

(ii) Let (a, b) be an invertible element of A. Then there exists $(c, d) \in A$ such that

$(a, b) * (c, d) = (1, 0) = (c, d) * (a, b)$

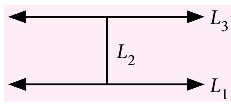
$\Rightarrow \quad (ac, b + ad) = (1, 0)$ and $(ca, d + bc) = (1, 0)$

$\Rightarrow \quad ac = 1, b + ad = 0$ and $ca = 1, d + bc = 0$

$\Rightarrow \quad c = \frac{1}{a}$ and $d = -\frac{b}{a}$, if $a \ne 0$.

Thus, (a, b) is an invertible element of A, if $a \ne 0$ and in such a case the inverse of (a, b) is $\left(\frac{1}{a}, \frac{-b}{a}\right)$.

12. Clearly, any line L cannot be perpendicular to itself.



$\therefore \quad (L, L) \notin R$ for any $L \in A$.

So, R is not reflexive.

Again, let $L_1, L_2 \in A$. Then,

$(L_1, L_2) \in R \Rightarrow L_1 \perp L_2$

$\Rightarrow \quad L_2 \perp L_1 \Rightarrow (L_2, L_1) \in R$.

$\therefore \quad R$ is symmetric.

Now, let $L_1, L_2, L_3 \in A$ such that $L_1 \perp L_2$ and $L_2 \perp L_3$.

Clearly L_1 is not perpendicular to L_3.

Thus, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$, but $(L_1, L_3) \notin R$.

$\therefore \quad R$ is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

13. Given $f(x) = \frac{4x+3}{6x-4}$, $x \ne \frac{2}{3}$

$\Rightarrow \quad$ Domain of function $f = R - \left\{\frac{2}{3}\right\}$.

Let $\quad y = \frac{4x+3}{6x-4} \Rightarrow 6xy - 4y = 4x + 3$

$\Rightarrow x = \frac{4y+3}{6y-4}$ but $x \in R \Rightarrow 6y - 4 \ne 0$

$\Rightarrow \quad y \ne \frac{2}{3} \Rightarrow$ range of $f = R - \left\{\frac{2}{3}\right\}$

Thus, f is a function from $R - \left\{\frac{2}{3}\right\}$ to $R - \left\{\frac{2}{3}\right\}$.

Therefore, the composite function fof exists.

$fof : R - \left\{\frac{2}{3}\right\} \to R - \left\{\frac{2}{3}\right\}$ and

$(fof)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right)$

$$= \frac{4 \cdot \frac{4x+3}{6x-4} + 3}{6 \cdot \frac{4x+3}{6x-4} - 4} = \frac{16x + 12 + 18x - 12}{24x + 18 - 24x + 16} = \frac{34x}{34} = x$$

$\Rightarrow \quad fof =$ Identity function on $R - \left\{\frac{2}{3}\right\}$

$\Rightarrow \quad f$ is invertible and $f^{-1} = f$.

14. Let $f(x_1) = f(x_2) \Rightarrow x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow \quad x_1^2 = x_2^2 \Rightarrow x_1^2 - x_2^2 = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1 + x_2) = 0 \Rightarrow x_1 - x_2 = 0$

$[\because (x_1 + x_2) \neq 0]$

$\Rightarrow \quad x_1 = x_2.$

$\therefore \quad f$ is one-one.

Now, $y = x^2 + 4 \Rightarrow x = \sqrt{y-4}$

For each $y \in [4, \infty)$ there exists $x = \sqrt{y-4}$ in R^+ such that $f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4$ $= (y - 4) + 4 = y.$

$\therefore \quad f$ is onto.

Thus, f is one-one onto and therefore invertible.

Now, $y = f(x) \Rightarrow y = x^2 + 4$

$\Rightarrow \quad x = \sqrt{y-4}$

$\Rightarrow \quad f^{-1}(y) = \sqrt{y-4}.$

$\therefore \quad f^{-1} : [4, \infty) \rightarrow R^+$ such that $f^{-1}(y) = \sqrt{y-4}.$

15. (i) *Reflexivity* : Let a be an arbitrary element of N. Then, $a = (a \times 1)$ shows that a is a multiple of a.

$\therefore \quad (a, a) \in R \ \forall \ a \in N.$

So, R is reflexive.

(ii) *Symmetry* : Clearly, 6 and 2 are natural numbers and 6 is a multiple of 2.

$\therefore \quad (6, 2) \in R.$

But, 2 is not a multiple of 6.

$\therefore \quad (2, 6) \notin R.$

Thus, $(6, 2) \in R$ but $(2, 6) \notin R$.

Hence, R is not symmetric.

(iii) *Transitivity* : Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad$ (a is a multiple of b) and (b is a multiple of c)

$\Rightarrow \quad a = bd$ and $b = ce$ for some $d, e \in N$

$\Rightarrow \quad a = (ce)d \Rightarrow a = c(ed)$

$\Rightarrow \quad a$ is a multiple of c

$\Rightarrow \quad (a, c) \in R$

$\therefore \quad (a, b) \in R$ and $(b, c) \in R \quad \Rightarrow (a, c) \in R.$

Hence, R is transitive.

16. (i) *Reflexivity* : For any point P in set S, we find that distance between P and itself is 0 which is less than 2 units.

Thus, $(P, P) \in R$ for all $P \in S$.

So, R is reflexive on S.

(ii) *Symmetry* : Let P and Q be two points in S such that $(P, Q) \in R$.

$\Rightarrow \quad$ Distance between P and Q is less than 2 units.

$\Rightarrow \quad$ Distance between Q and P is less than 2 units

$\Rightarrow \quad (Q, P) \in R$

So, R is symmetric on S.

(iii) *Transitivity* : Consider points P, Q and T having coordinates (0, 0), (1.5, 0) and (3.2, 0). Now, the distance between P and Q is 1.5 units which is less than 2 units and the distance between Q and T is 1.7 units which is also less than 2 units. But, the distance between P and T is 3.2 units which is not less than 2 units. Thus, $(P, Q) \in R$ and $(Q, T) \in R$ but $(P, T) \notin R$. So, R is not transitive on S.

17. We have

$$f(1) = \frac{(1+1)}{2} = \frac{2}{2} = 1 \text{ and } f(2) = \frac{2}{2} = 1$$

Thus, $f(1) = f(2)$, while $1 \neq 2$.

$\therefore \quad f$ is many-one.

In order to show that f is onto, consider an arbitrary element $n \in N$.

If n is odd, then $(2n - 1)$ is odd and

$$f(2n-1) = \frac{(2n-1+1)}{2} = \frac{2n}{2} = n.$$

If n is even, then $2n$ is even and

$$f(2n) = \frac{2n}{2} = n.$$

Thus, for each $n \in N$ (whether even or odd) there exists its pre-image in N.

$\therefore \quad f$ is onto.

Hence, f is many-one onto.

18. Let $f(n_1) = f(n_2)$.

Case 1 : When n_1 is odd and n_2 is even, then

$f(n_1) = f(n_2)$

$\Rightarrow \quad n_1 - 1 = n_2 + 1$

$\Rightarrow \quad n_1 - n_2 = 2$

If n_1 is odd and n_2 is even, then $(n_1 - n_2) \neq 2$.

Thus, we arrive at a contradiction.

So, in this case, $f(n_1) \neq f(n_2)$.

Similarly, when n_1 is even and n_2 is odd, then $f(n_1) \neq f(n_2)$.

Case 2 : When n_1 and n_2 both are odd, then

$f(n_1) = f(n_2)$

$\Rightarrow \quad n_1 - 1 = n_2 - 1$

$\Rightarrow \quad n_1 = n_2.$

Case 3 : When n_1 and n_2 both are even, then

$f(n_1) = f(n_2)$

$\Rightarrow \quad n_1 + 1 = n_2 + 1$

$\Rightarrow \quad n_1 = n_2.$

Thus, from all the cases, we get $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$

$\therefore \quad f$ is one-one.

Now, we show that f is onto.

Let $n \in W$.

Case 1 : When n is odd

In this case, $(n - 1)$ is even

and $f(n - 1) = (n - 1) + 1 = n$. ...(i)

Case 2 : When n is even

In this case, $(n + 1)$ is odd

and $f(n + 1) = (n + 1) - 1 = n$. ...(ii)

Thus, each $n \in W$ has its pre-image in W.

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

Clearly, we have

$$f^{-1}(n) = \begin{cases} (n-1), \text{ when } n \text{ is odd} \\ (n+1), \text{ when } n \text{ is even.} \end{cases} \quad \text{[using (i) and (ii)]}$$

19. Let $x \in A$. Then,

$$(gof)(x) = g[f(x)] = g\left(\frac{7x+4}{5x-3}\right)$$

$$= \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} = \frac{(21x+12+20x-12)}{(35x+20-35x+21)}$$

$$= \frac{41x}{41} = x = I_A(x).$$

$\therefore$ $(gof) = I_A$.

Again, let $y \in B$. Then,

$$(fog)(y) = f[g(y)] = f\left(\frac{3y+4}{5y-7}\right)$$

$$= \frac{7\left(\frac{3y+4}{5y-7}\right)+4}{5\left(\frac{3y+4}{5y-7}\right)-3}$$

$$= \frac{(21y+28+20y-28)}{(15y+20-15y+21)}$$

$$= \frac{41y}{41} = y = I_B(y).$$

$\therefore$ $(fog) = I_B$.

Hence, $(gof) = I_A$ and $(fog) = I_B$.

20. Clearly, R_1 and R_2 are subsets of $X \times X$. In order to prove that $R_1 = R_2$, it is sufficient to show that $R_1 \subset R_2$ and $R_2 \subset R_1$.

Now, the difference between any two elements of each of the sets $\{1, 4, 7\}$, $\{2, 5, 8\}$ and $\{3, 6, 9\}$ is a multiple of 3.

Let (x, y) be an arbitrary element of R_1. Then,

$(x, y) \in R_1$

$\Rightarrow$ $x - y$ is divisible by 3.

$\Rightarrow$ $x - y$ is a multiple of 3.

$\Rightarrow$ $\{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}$

$\Rightarrow$ $(x, y) \in R_2$

So, $R_1 \subset R_2$ ...(i)

Now, let (a, b) be an arbitrary element of R_2. Then,

$(a, b) \in R_2$

$\Rightarrow$ $\{a, b\} \subset \{1, 4, 7\}$ or $\{a, b\} \subset \{2, 5, 8\}$ or $(a, b) \subset \{3, 6, 9\}$

$\Rightarrow$ $a - b$ is divisible by 3.

$\Rightarrow$ $(a, b) \in R_1$

So, $R_2 \subset R_1$ ...(i)

From (i) and (ii), $R_1 = R_2$

Contd. from Page no. 18

Solving we get the radial centre as $\left(h, \frac{a^2-h^2}{k}\right)$.

Now in order to find the orthocentre we have to find the equation of altitudes AD and BE.
AD is $x = h$ and BE is

$$y-0=-\frac{h+a}{k}(x-a)$$

$$\left[\because BE \perp AC, \text{ slope of } AC \text{ is } \frac{k}{h+a}\right]$$

Solving, we get orthocentre $\left(h, \frac{a^2-h^2}{k}\right)$.

(D) The function is defined for all values of x for $|x| - x > 0 \Rightarrow |x| > x$. If $x > 0$, $|x| = x$
$\therefore$ All positive values of x shall not be in domain of x. But if x is negative, then $|x| = -x > x$
$\Rightarrow$ $2x < 0$, $x < 0$ $\therefore$ $(-\infty, 0)$.

43. (1) : In given ΔABC both $\frac{A}{2}$ and $\frac{B}{2}$ lie strictly between $\left(0, \frac{\pi}{2}\right)$ and $\sin x$ is always increasing in $\left(0, \frac{\pi}{2}\right)$ whereas $\cos x$ is always decreasing throughout $\left(0, \frac{\pi}{2}\right)$.

So if $\frac{A}{2} > \frac{B}{2} \Rightarrow \sin\frac{A}{2} > \sin\frac{B}{2} \Rightarrow x_1 > x_2$

and $\frac{1}{x^3} > \frac{1}{x^4}$. Now, $x_1^{2007}x_4^{2006} = x_2^{2007}x_3^{2006}$ is not valid.

Similarly for $\frac{A}{2} < \frac{B}{2}$

$\sin\frac{A}{2} < \sin\frac{B}{2} \Rightarrow x_1 < x_2$ and $\frac{1}{x_3} < \frac{1}{x_4}$. Again equality is not possible.

Therefore only possible case is when $\frac{A}{2} = \frac{B}{2}$

$\Rightarrow$ $x_1 = x_2$ and $\frac{1}{x_3} = \frac{1}{x_4}$.

Hence in this case ΔABC is isosceles with $\angle ABC = \angle CAB$

44. (1) : $I_n = \left(\frac{e^{-x}(\sin x)^n}{-1}\right)_0^\infty + \int_0^\infty n(\sin x)^{n-1}\cos x e^{-x}dx$

$$= 0 + n\left(\frac{(\sin x)^{n-1}\cos x e^{-x}}{-1}\right)_0^\infty$$

$$+n\int_0^\infty \left[(\sin x)^{n-1}(-\sin x)+(\cos x)(n-1)\cos x(\sin x)^{n-2}\right]e^{-x}dx$$

$$= n\int_0^\infty \left[-(\sin x)^n+(n-1)(1-\sin^2 x)(\sin x)^{n-2}\right]e^{-x}dx$$

$$= \frac{n(n-1)}{n^2+1}I_{n-2}, \frac{101}{90}\frac{I_{10}}{I_8} = \frac{101}{90}\times\frac{10\times 9}{10^2+1} = 1$$

45. (2) : A.M. $= \frac{100+n}{2}$, G.M. $= \sqrt{100\times n}$,

H.M. $= \frac{200n}{100+n}$

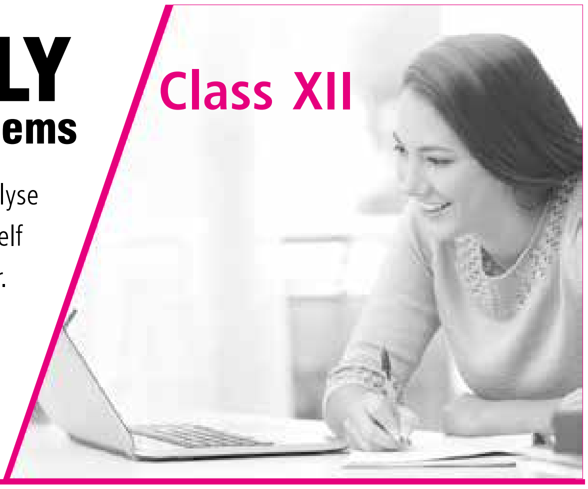
Let $n = 4k^2$ as for A.M and G.M we need a even perfect square

$\therefore$ H.M. $= \frac{800k^2}{100+4k^2} = \frac{200k^2}{25+k^2}$

Now, $100 < \frac{200k^2}{25+k^2} \le 500 \Rightarrow 2500 + 100k^2 \le 200k^2$
$\Rightarrow$ $2500 \le 100\,k^2$
and $200k^2 \le 12500 + 500k^2 \Rightarrow -300k^2 \le 12500$
$\Rightarrow$ $k^2 = 25, 36, 49, 64, 81, 100$, H.M. is an integer for $k^2 = 100$ only
$\Rightarrow$ $n = 4k^2 = 4 \times 100 = 400$.

MPP-1 MONTHLY Practice Problems

Class XII



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Relations and Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2}-1\right) + \log \cos x$ is defined as
   (a) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$
   (c) $[0, \pi]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

2. Let $f : R \to R$, $g : R \to R$ be two functions such that $f(x) = 2x - 3$, $g(x) = x^3 + 5$. The function $(fog)^{-1}(x)$ is equal to
   (a) $\left(\frac{x+7}{2}\right)^{1/3}$ (b) $\left(x-\frac{7}{2}\right)^{1/3}$
   (c) $\left(\frac{x-2}{7}\right)^{1/3}$ (d) $\left(\frac{x-7}{2}\right)^{1/3}$

3. If $f(x) = \frac{3x-2}{2x-3}$, then $f(f(x)) =$
   (a) $\frac{1}{x}$ (b) x (c) x^2 (d) $\frac{1}{x^2}$

4. $f : R \to R$, $f(x) = x|x|$ is
   (a) one-one but not onto
   (b) one-one onto
   (c) onto but not one-one
   (d) none of these

5. Let $A = \{7, 8, 9, 10\}$ and $R = \{(8, 8), (9, 9), (10, 10), (7, 8)\}$ be a relation on A, then R is
   (a) transitive (b) reflexive
   (c) symmetric (d) none of these

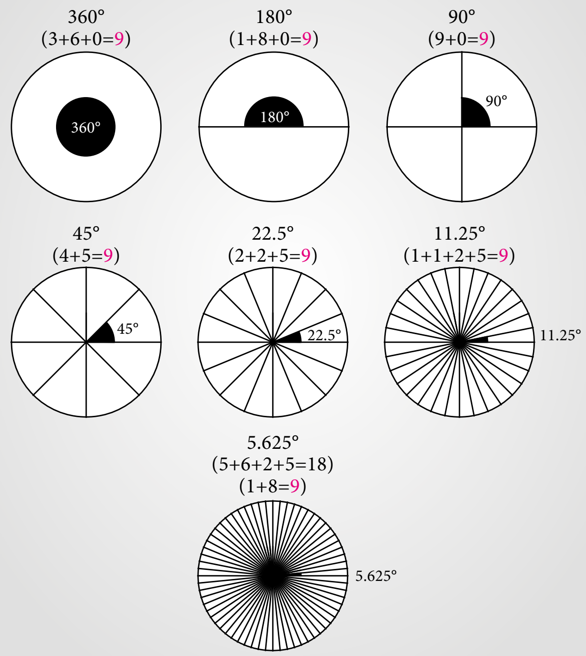
6. Let $*$ be a binary operation on N defined by $a * b = a + b + 10$ for all $a, b \in N$. The identity element for $*$ in N is
   (a) -10 (b) 0
   (c) 10 (d) Does not exist

One or More Than One Option(s) Correct Type

7. Let n be a fixed positive integer. Let a relation R be defined in I (the set of all integers) as follows: aRb iff $n/(a - b)$, that is, iff $a - b$ is divisible by n. Then, the relation R is

FUN FACT

Do you think this is arbitrary?
The resulting angle always reduces to 9.



(a) reflexive (b) symmetric
(c) transitive (d) none of these

8. Let $f(x) = x^3$ and $g(x) = 3^x$. The values of a such that $g(f(a)) = f(g(a))$ are
(a) 0, 2 (b) 1, 3 (c) $0, \pm\sqrt{3}$ (d) 1, ± 2

9. Which of the following functions $f : R \to R$ are bijective ?
(a) $f(x) = x \sin x$ (b) $f(x) = x - \sin^2 x$
(c) $f(x) = x + \sqrt{x^2}$ (d) $f(x) = -x + \cos^2 x$

10. Let '$*$' be a binary operation on Q_0 (set of all non-zero rational numbers) defined by $a * b = \dfrac{ab}{4}$ for all $a, b \in Q_0$, then
(a) identity element of $*$ is 4
(b) inverse of an element a in Q_0 is $\dfrac{16}{a}$
(c) identity element of $*$ is 8
(d) inverse of an element a in Q_0 is $\dfrac{4}{a}$

11. Let '$*$' be a binary operation on N given by $a * b =$ L.C.M (a, b) for all $a, b \in N$, then
(a) $*$ is commutative (b) $*$ is associative
(c) identity element of $*$ is 1
(d) identity element of $*$ is 2

12. The function $f(x) = \dfrac{|x-1|}{x^2}$ is
(a) one-one in $(2, \infty)$ (b) one-one in $(0, 1)$
(c) one-one in $(-\infty, 0)$ (d) one-one in $(1, 2)$

13. With reference to a universal set, the inclusion of a subset in another, is a relation which is
(a) non Symmetric (b) equivalence
(c) reflexive (d) none of these

Comprehension Type

Consider the function $f : R \to (0, \infty)$ defined by $f(x) = 2^x + 2^{|x|}$

14. $f(x)$ is
(a) one-one onto (b) one-one into
(c) many-one onto (d) many-one into

15. The number of solutions of the equation $f(x) = \ln \pi$ is
(a) 0 (b) 1 (c) 2 (d) 3

Matrix Match Type

16. Let $f, g : R \to R$ be the function defined by $f(x) = x^2 + 1$ and $g(x) = 2[x] - 1$, where $[x]$ is the largest integer $\le x$. Then match the following:

	Column-I		Column-II
P.	$(gof)\left(\frac{1}{2}\right)$	1.	3
Q.	$(fog)\left(\frac{8}{3}\right)$	2.	10
R.	$(fogof)\left(\frac{3}{4}\right)$	3.	2
S.	$(gofog)\left(\frac{2}{3}\right)$	4.	1

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	3	1
(c)	4	2	3	1
(d)	3	1	2	4

Integer Answer Type

17. If $f(x)$ is an even function and satisfies the relation $x^2 f(x) - 2f\left(\dfrac{1}{x}\right) = g(x)$, where $g(x)$ is an odd function, then $f(10)$ equals

18. If $f : R \to R$, $g : R \to R$ and $h : R \to R$ is such that $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$, then the value of $[ho(gof)]\,(x)$, if $x = \dfrac{\sqrt{\pi}}{2}$ will be

19. If f is an invertible function defined as $f(x) = \dfrac{3x-4}{5}$, then $f^{-1}(1)$ is

20. Let $f : N \to R : f(x) = \dfrac{(2x-1)}{2}$ and $g : Q \to R : g(x) = x + 2$ be two functions, then $(gof)\left(\dfrac{3}{2}\right)$ is

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

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