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Rank 1 JEE ADVANCED Sarvesh Mehtani

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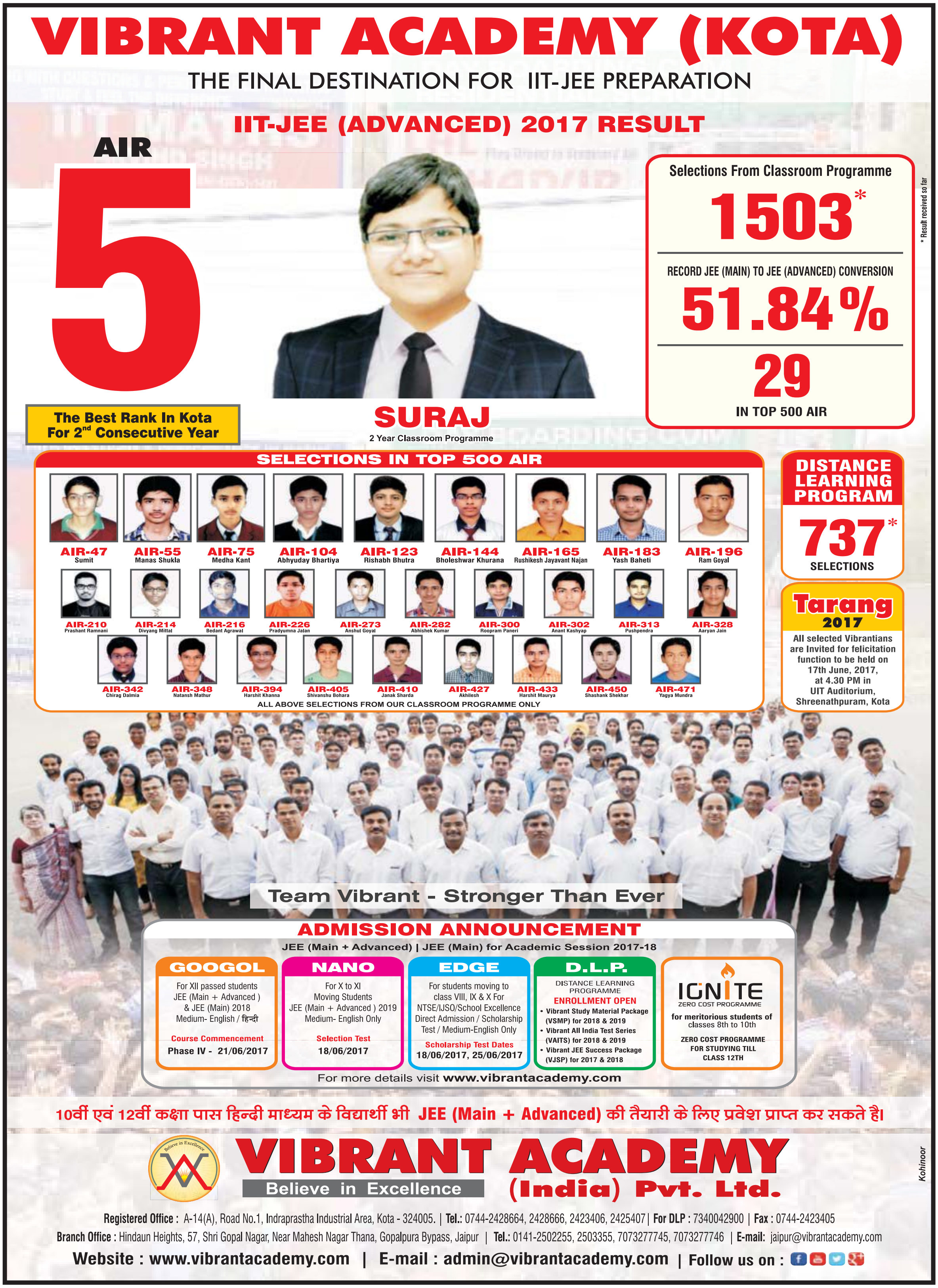

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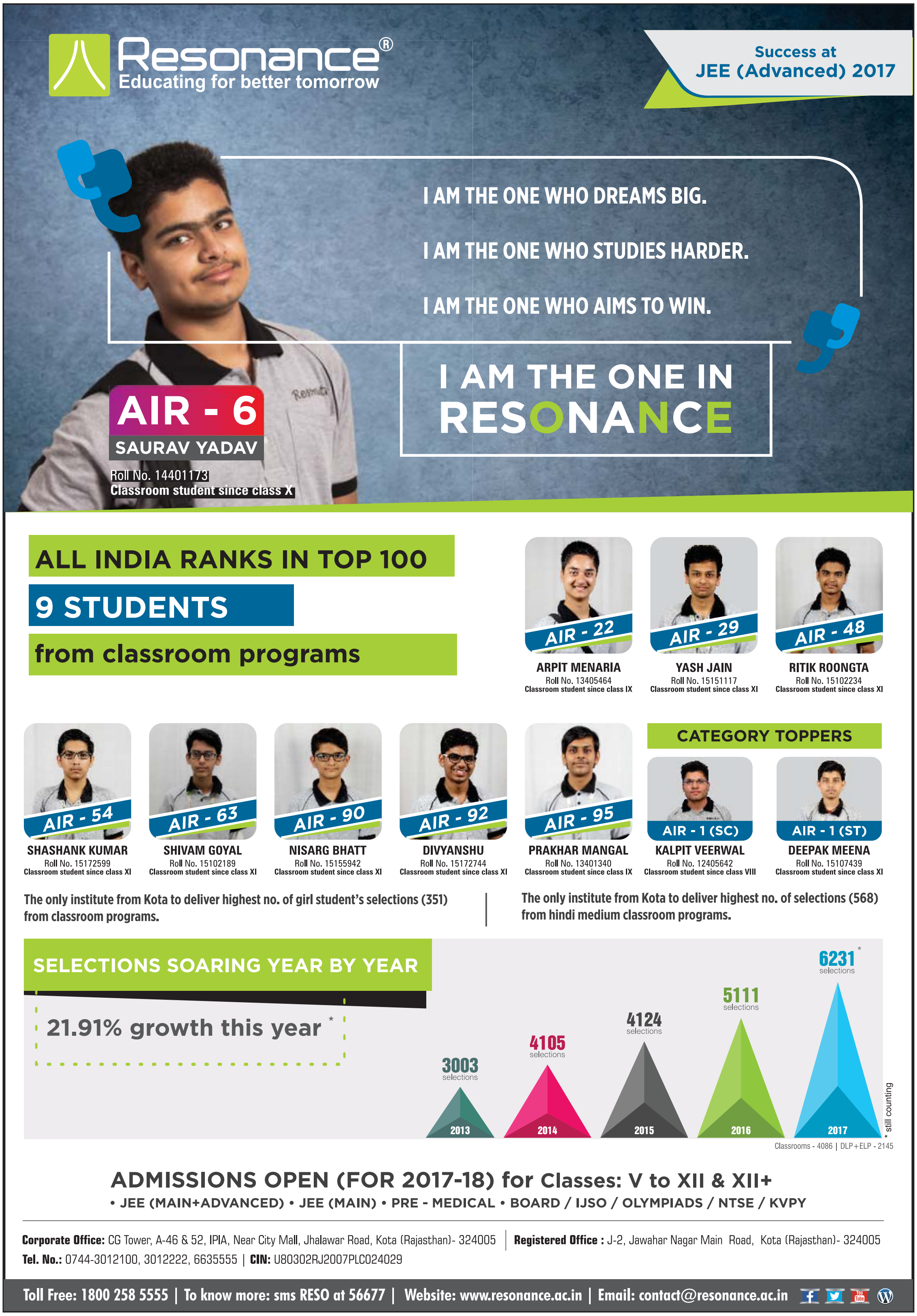
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4124 selections
2015
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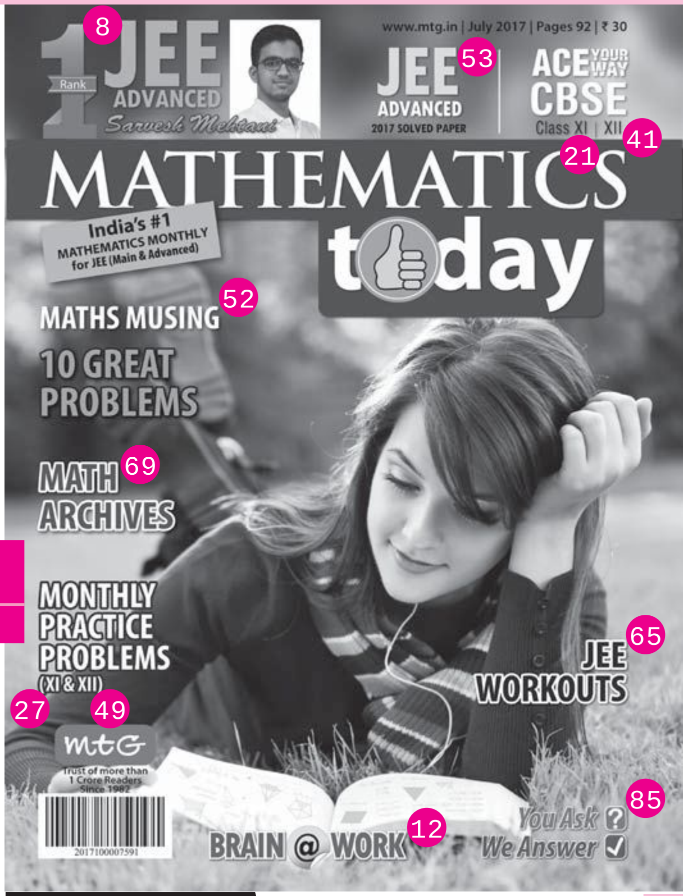

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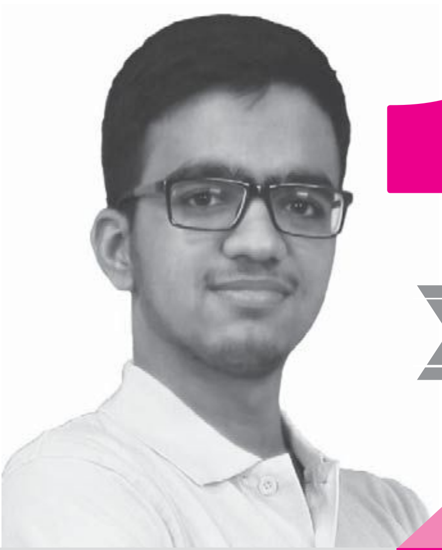
Cracking the JEE Advanced EXAM



Rank 1

Sarvesh Mehtani

- **MTG : Why did you appear for Engineering Entrance?**

  Sarvesh Mehtani : I saw the movie '3-Idiots' in class 8th and was really inspired. I had a keen interest in Science and Mathematics, so appearing for Engineering exams became quite obvious.

- **MTG : What exams have you appeared for and what are your ranks in these exams?**

  Sarvesh : NTSE - both the stages
  KVPY - 31st rank (Stage II)
  IMO - topper (International Rank I) in class 10th conducted by Science Olympiad Foundation, New Delhi
  BITS - 430/450
  JEE Advanced -AIR -1st (Topper)
  JEE Main - AIR - 55th rank

- **MTG : How many hours in a day did you study to prepare for the examination?**

  Sarvesh : The days when I did not go to my coaching, I did 8-10 hours of self study and with coaching 5-6 hours of self study.

- **MTG : On which topics and chapters you laid more stress in each subject?**

  Sarvesh : Mathematics – Trigonometry, Co-ordinate Geometry, Integral Calculus, Vectors and Three Dimensional Geometry.
  Physics – Rotation, Magnetism, EMI, Waves and Mechanics.
  Chemistry – Physical Chemistry - Equilibrium, Surface Chemistry; Organic Chemistry - Aldehydes, Ketones and Carboxylic acids, Hydrocarbons and Inorganic Chemistry - *p*-Block elements, Metallurgy, Qualitative analysis.

- **MTG : How much time does one require for serious preparation for this exam?**

  Sarvesh : It depends on person to person. I had a very regular preparation. The total preparation for JEE Main and JEE Advanced was a little different. I think the preparation should be normally done till 12th standard with the 10th and 12th syllabus and everyone should orient themselves a little towards the JEE Mains because there are little extra topics. But after the JEE Mains are done, you can start preparation for JEE Advanced.

> **"Continuous hard work and following guidance of my teachers."**

- **MTG : How was the preparation for JEE Advanced different from JEE Main?**

  Sarvesh : It was nothing different. Just that for JEE Advanced, there is high amount of pressure as compared to other exams.

- **MTG : Any extra coaching?**

  Sarvesh : I attended (National Institute), Chandigarh for extra guidance.

- **MTG : Which Subjects/Topics you were strong/ weak at?**

  Sarvesh : In Maths, I was weak in Trigonometry and Functions. In Physics, I had more queries in Waves.

Thanks For Making us Proud

$60^{+}/120$ Selections in JEE Advanced-2017

Keshav Gupta
MIT, USA(2017-21)
JEE Adv. AIR-269
JEE Main AIR-232

I am Keshav Gupta, student at KCS Educate during the sessions 2015-16 and 2016-17. The teachers at KCS Educate constantly motivated me and guided me towards the target. In the classroom, the focus always remained on keeping the students interested in the topics. Apart from academics, great emphasis was layed on motivational support for students. It has truly been a catalyst for success.

Keshav
Keshav Gupta

Ashutosh Chaubey
JEE Adv. AIR-402
JEE Main AIR-575
KVPY Qualified

After joining KCS, my performance boosted a lot. The faculty members helped me to strengthen my good areas and to convert my weaknesses into my strengths. The assignments provided helped me to practice questions of various patterns in a really good way. The most important thing that I got from KCS is the motivation to study myself & to know my strengths.
I thank KCS for it's support throughout my JEE preparation.

Ashutosh
ASHUTOSH CHAUBEY

Tushar Agrawal
JEE Adv. AIR-609
JEE Main C.G. Topper
KVPY Qualified

I am Tushar Agrawal. I joined KCS for Two year program. The lectures delievered by Sir were very interacting and motivating. It never felt to me that "I need to study" but like a pleasure flow and eventually I began to enjoy studies. Further the tests and assignments helped me to understand my weak areas and my speed. Avnish Sir motivated at each stage and was ready to help at any time. I want to thank Avnish Sir and his team for their support.

Tushar

Ashish R. Nair
JEE Adv. AIR-829
JEE Main AIR-323
KVPY Qualified

The classes at KCS have been interesting and involving. Avnish sir has been a real support in my preparation. The assignments and tests were useful. The questions in them were according to the current JEE level. Positive thinking does miracles and this is what we were told always by Avnish sir. So, I would like to thank KCS.

Ashish
ASHISH R. NAIR

- **MTG : Which Books/Magazines you read?**

  Sarvesh : I used many subjectwise books. I did not use any special books during the preparation. I also did the sheets given by my institute.

- **MTG : In your words what are the components of an ideal preparation plan?**

  Sarvesh : An ideal preparation plan includes hard work, follow your teachers instructions and continuously work towards your goal in organised manner.

- **MTG : What role did the following play in your success:**

  (a) Parents (b) Teachers (c) School

  Sarvesh : (a) My parents have given me constant support, they have never pressurised me and kept me stress free.

  (b) My teachers taught the complete course and helped me a lot. They have solved all my extra queries.

  (c) In school, focus was on complete NCERT coverage and practicals were conducted quite well.

- **MTG : Your family background?**

  Sarvesh : My father works in Income Tax Department, my mother is a government ITI and my sister is doing her engineering.

- **MTG : What mistake you think you shouldn't have made?**

  Sarvesh : I should have avoided silly mistakes from the beginning. One must focus on not doing silly mistakes along with getting good marks.

- **MTG : How have MTG magazines helped you in your preparation?**

  Sarvesh : MTG Magazines have helped me a lot. I used to read Mathematics Today, Physics For You and Chemistry Today.

- **MTG : Was this your first attempt?**

  Sarvesh : Yes, it is my first attempt.

- **MTG : What do you think is the secret of your success?**

  Sarvesh : Continuous hard work and following guidance of my teachers.

- **MTG : How did you de-stress yourself during the preparation? What are your hobbies? How often could you pursue them?**

  Sarvesh : I de-stress myself by talking to my friends, watching cartoons, listening to music, playing badminton and reading novels.

- **MTG : What do you feel is lacking in our education/ examination system? Is the examination system fair to the student?**

  Sarvesh : There is a lot of theoretical learning rather than practical learning which is deteriorating the overall development of the child's brain. This isn't fair to the students.

- **MTG : Had you not been selected then what would have been your future plan?**

  Sarvesh : I would have joined BITS Pilani as I have got (430/450).

- **MTG : What advice would you like to give our readers who are JEE aspirants?**

  Sarvesh : I would advice to work hard, practice more and more and be focused.

All the Best!☺☺

BRAIN@WORK

TRIGONOMETRY - I

Topics Covered : Measurement of Angles, Trigonometric Functions, Trigonometric Equations, Properties of Triangle

MEASUREMENT OF ANGLES

In Geometry, angles are measured in terms of a right angle. This however, is an inconvenient unit of measurement on account of its size.

There are three systems used for the measurement of angles.

1. Sexagesimal system or English system (degree)
2. Centesimal system or French system (grade)
3. Circular measurement (radian)

1. Sexagesimal System or Degree Measure

In the **Sexagesimal** system of measurement a right angle is divided into 90 equal parts called **Degrees**. Each degree is divided into 60 equal parts called **Minutes**, and each minute into 60 equal parts called **Seconds**.

i.e., 60 Seconds (60'') make One Minute (1'),
60 Minutes (60') make One Degree (1°) and
90 Degree (90°) make One Right Angle.

2. Centesimal System or Grade Measure

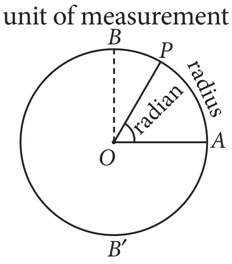
In this system the right angle is divided into 100 equal parts, called **Grades**; each grade is subdivided into 100 **Minutes** and each minute into 100 **Seconds**.

100 Seconds (100'') make One Minute (1')
100 Minutes (100') make One Grade (1^g)
100 Grades (100^g) make One Right Angle

3. Circular Measure or Radian Measure

Circular system: In this system the unit of measurement is radian as defined below:

Radian: One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



Take any circle $APBB'$, whose centre is O, and from any point A measure off an arc AP whose length is equal to the radius of the circle. Join OA and OP.

The angle AOP is the angle which is taken as the unit of circular measurement, *i.e.*, it is the angle in terms of which in this system we measure all others.

This angle is called 'A Radian' and is often denoted by 1^c.

Some Important Theorems on Circular Measure

Theorem 1. Radian is a constant angle. *i.e.*, 1^c = constant

Theorem 2. The number of radians in an angle subtended by an arc of a circle at the centre is equal to $\frac{arc}{\text{radius}}$. *i.e.*, $\theta = \frac{s}{r}$ radians.

Relation between Degrees and Radians

$$1 \text{ radian} = \frac{180^\circ}{\pi}; \quad 1^\circ = \frac{\pi}{180} \text{ radian}$$

Relation between three systems of measurement of angle

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Remark : Where D, G and R be number of degrees, number of grades and number of radians respectively in an angle θ.

1. Radian is the unit to measure angle and it does not means that π stands for 180°, π is a real number. where as π^c stands for 180°.
2. The angle between two consecutive digits in a clock is 30° (= π/6 radians).
3. The hour hand rotates through an angle of 30° in one hour *i.e.* (1/2)° in one minute.
4. The minute hand rotates through an angle of 6° in one minute.

CIRCULAR FUNCTIONS OR TRIGONOMETRIC FUNCTIONS OR TRIGONOMETRIC RATIOS (T-RATIOS)

Let a revolving line starting from OX trace out an angle $XOP = \theta$ in any of the four quadrants. Let M be the foot of perpendicular from P upon $X'OX$. Regarding OM and MP as directed lengths (OP always +ve), the ratios of OM, MP and OP with one another are called circular functions or trigonometrical ratios (briefly t-ratios) of the angle θ.

Let $OM = x$, $MP = y$ and $OP = r (r > 0)$, we define the various circular functions as follows:

(i) $\dfrac{MP}{OP} = \dfrac{y}{r}$ is called sine of θ, written as $\sin\theta$.

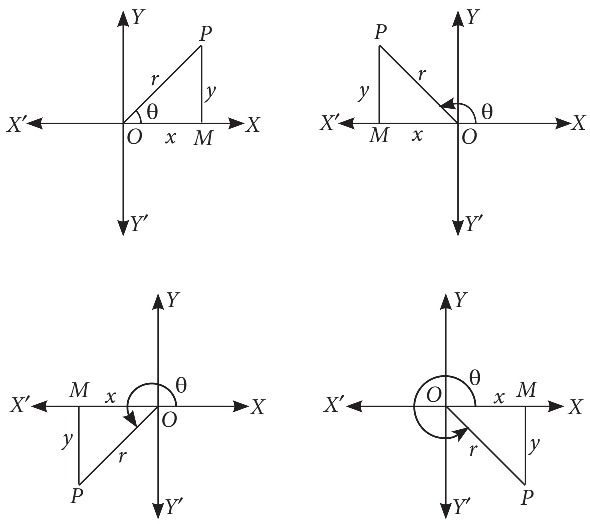
(ii) $\dfrac{OM}{OP} = \dfrac{x}{r}$ is called cosine of θ, written as $\cos\theta$.

(iii) $\dfrac{MP}{OM} = \dfrac{y}{x}$ $(x \neq 0)$ is called tangent of θ, written as $\tan\theta$.

(iv) $\dfrac{OM}{MP} = \dfrac{x}{y} (y \neq 0)$ is called cotangent of θ, written as $\cot\theta$.

(v) $\dfrac{OP}{OM} = \dfrac{r}{x} (x \neq 0)$ is called secant of θ, written as $\sec\theta$.

(vi) $\dfrac{OP}{MP} = \dfrac{r}{y} (y \neq 0)$ is called cosecant of θ, written as $\operatorname{cosec}\theta$.



Relation Between the Trigonometric Functions

From the definition of t-ratios, we have

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \text{and} \quad \sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{and} \quad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}$$

Signs of Trigonometric Ratios in Different Quadrants

The following table describes the signs of various t-ratios in different quadrants.

Quadrant	I	II	III	IV
t-ratios which are +*ve*	All	sin θ cosec θ	tan θ cot θ	cos θ sec θ

Trigonometric Ratios of Allied Angles

Two angles are called allied angles if either

(i) their sum is zero or

(ii) their sum or difference is a multiple of right angle.

t-ratios	–θ	90° – θ	90° + θ	180° – θ	180° + θ	270° – θ	270° + θ	360° – θ	360° + θ
sin θ	– sin θ	cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ
cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	–sin θ	sin θ	cos θ	cos θ
tan θ	– tan θ	cot θ	– cot θ	– tan θ	tan θ	cot θ	– cot θ	– tan θ	tan θ
cosec θ	–cosec θ	sec θ	sec θ	cosec θ	–cosec θ	– sec θ	– sec θ	–cosec θ	cosec θ
sec θ	sec θ	cosec θ	–cosec θ	– sec θ	– sec θ	–cosec θ	cosec θ	sec θ	sec θ
cot θ	– cot θ	tan θ	– tan θ	– cot θ	cot θ	tan θ	– tan θ	– cot θ	cot θ

Trigonometric Ratios of Compound Angles

An angle made up of the sum/difference of the algebraic sum/difference of two or more angles is called a compound angle. Some of the formulae and results regarding compound angles are:

1. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
2. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
3. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
4. $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
5. $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
6. $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot A \pm \cot B}$

Sum of Sines/Cosines in Terms of Products

1. $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
2. $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
3. $\cos A + \cos B = 2\cos\left(\dfrac{A-B}{2}\right).\cos\left(\dfrac{A+B}{2}\right)$
4. $\cos A - \cos B = 2\sin\left(\dfrac{A+B}{2}\right).\sin\left(\dfrac{B-A}{2}\right)$
5. $\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A \cos B}$
6. $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A \cdot \sin B}$
7. $\cot A - \cot B = \dfrac{\sin(B+A)}{\sin A \cdot \sin B}$

Conversely

1. $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Trigonometric Ratios of Multiple Angles

1. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

   $= 2\cos^2\theta - 1 = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
3. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
4. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
5. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
6. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Trigonometric Ratios of Sub-Multiple Angles

1. $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
2. $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1$

   $= 1 - 2\sin^2\dfrac{\theta}{2} = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
3. $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

T-Ratios of the Sum of Three or More Angles

1. $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

   or $\sin(A+B+C) = \cos A \cos B \cos C(\tan A + \tan B + \tan C - \tan A \tan B \tan C)$
2. $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

   $\cos(A+B+C) = \cos A \cos B \cos C(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$
3. $\tan(A+B+C)$

   $= \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

Maximum and Minimum Values of Trigonometric Functions

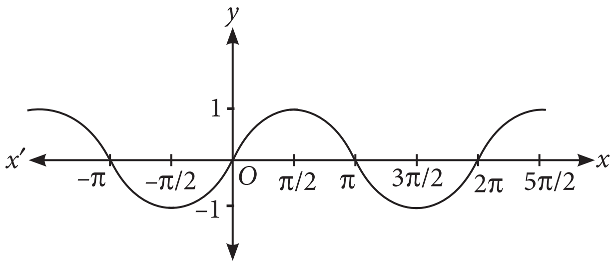
The maximum and minimum values of trigonometric functions of the form $a\sin x + b\cos x$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

Graphs and Other Useful Data of Trigonometric Functions

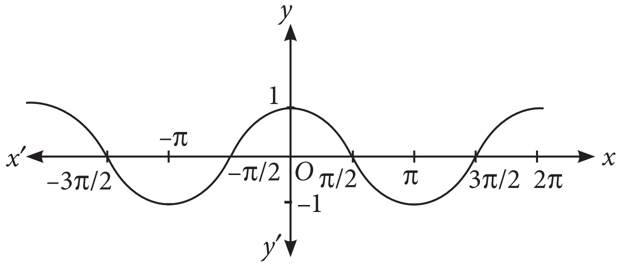
1. $y = f(x) = \sin x$

   Domain $= R$, Range $= [-1, 1]$, Period $= 2\pi$

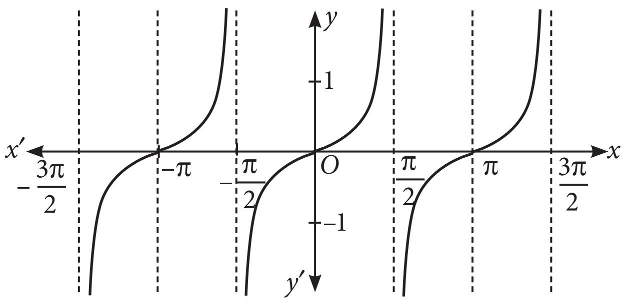
   $\sin^2 x, |\sin x| \in [0, 1]$



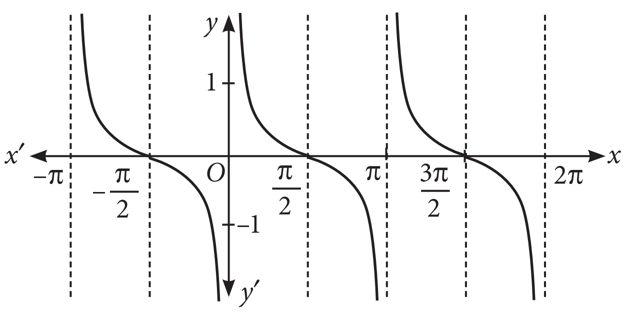
2. $y = f(x) = \cos x$
Domain = R, Range = $[-1, 1]$, Period = 2π
$\cos^2 x$, $|\cos x| \in [0, 1]$



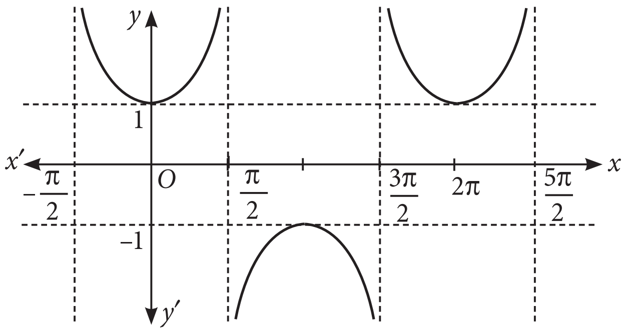
3. $y = f(x) = \tan x$
Domain = $R \sim (2n + 1)\pi/2$, $n \in I$, Range = $(-\infty, \infty)$
Period = π
Discontinuous at $x = (2n + 1)\pi/2$, $n \in I$
$\tan^2 x$, $|\tan x| \in [0, \infty)$
$\tan x = 0 \Rightarrow x = n\pi$, $n \in I$



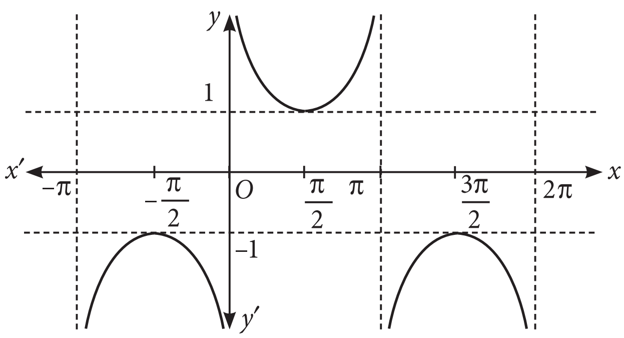
4. $y = f(x) = \cot x$
Domain = $R \sim n\pi$, $n \in I$; Range = $(-\infty, \infty)$
Period = π
Discontinuous at $x = n\pi$, $n \in I$
$\cot^2 x$, $|\cot x| \in [0, \infty)$



5. $y = f(x) = \sec x$
Domain = $R \sim (2n + 1)\pi/2$, $n \in I$
Range = $(-\infty, -1] \cup [1, \infty)$
Period = 2π, $\sec^2 x$, $|\sec x| \in [1, \infty)$



6. $y = f(x) = \text{cosec } x$
Domain = $R \sim n\pi$, $n \in I$;
Range = $(-\infty, -1] \cup [1, \infty)$
Period = 2π, $\text{cosec}^2 x$, $|\text{cosec } x| \in [1, \infty)$



TRIGONOMETRIC EQUATIONS

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

Solution or Root of a Trigonometric Equation

The value of an unknown angle which satisfies the given trigonometric equations is called a solution or root of the equation.

Principal Solution of a Trigonometric Equation

The solution of a trigonometric equation lying in the interval $[0, 2\pi)$ is called its principal solution.

General Solution of a Trigonometric Equation

It is known that trigonometric ratios are periodic functions. In fact, $\sin x$, $\cos x$, $\sec x$ and $\text{cosec } x$ are periodic functions with a period 2π, and $\tan x$ and $\cot x$ are periodic functions with a period π. Therefore, solution of trigonometric equations can be generalized with the help of period of trigonometric functions.

The solution consisting of all possible solutions of a trigonometric equation is called its general solution. General solution of a trigonometric equation will involve integral $n \in Z$.

General Solution of Some Equations

Equation	Solution
$\sin\theta = 0$	$\theta = n\pi,\ n \in Z$
$\cos\theta = 0$	$\theta = (2n+1)\dfrac{\pi}{2},\ n \in Z$
$\tan\theta = 0$	$\theta = n\pi,\ n \in Z$
$\sin\theta = 1$	$\theta = (4n+1)\dfrac{\pi}{2},\ n \in Z$
$\sin\theta = -1$	$\theta = (4n-1)\dfrac{\pi}{2},\ n \in Z$
$\cos\theta = 1$	$\theta = 2n\pi,\ n \in Z$
$\cos\theta = -1$	$\theta = (2n+1)\pi,\ n \in Z$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha,\ n \in Z$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha,\ n \in Z$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$, where $n \in Z$
$\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha,\ n \in Z$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha,\ n \in Z$

Solutions of Equations of Form $a\cos\theta + b\sin\theta = c$

We can solve the equation of this type by putting

$a = r\cos\phi$ and $b = r\sin\phi$, provided $\left|\dfrac{c}{\sqrt{a^2+b^2}}\right| \le 1$

PROPERTIES AND SOLUTION OF TRIANGLES

Standard symbols

The following symbols in relation to ΔABC are universally adopted.

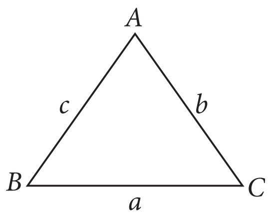
$m\angle BAC = A$

$m\angle ABC = B$

$m\angle BCA = C$

$A + B + C = \pi$

$AB = c,\ BC = a,\ CA = b$



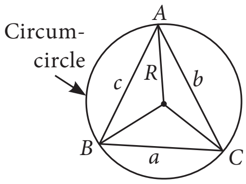
Semi-perimeter of the triangle, $s = \dfrac{a+b+c}{2}$

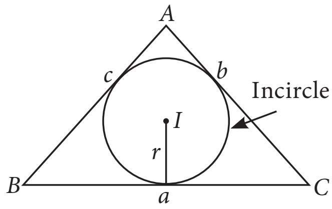
So, $a + b + c = 2s$

The radius of the circumcircle of the triangle, *i.e.*, circumradius = R

The radius of the incircle of the triangle, *i.e.*, inradius = r

Area of the triangle = $\Delta = S$





Sine law

In any triangle ABC, the sides are proportional to the sines of the opposite angles, *i.e.*,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine law

$$\cos A = \frac{b^2+c^2-a^2}{2bc},\ \cos B = \frac{a^2+c^2-b^2}{2ac},$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Projection law

In any ΔABC, $a = b\cos C + c\cos B$, $b = a\cos C + c\cos A$, $c = a\cos B + b\cos A$

Napier's analogy

In any ΔABC,

(i) $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

(ii) $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(iii) $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

Trigonometric Ratios of Half Angles

(i) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}},\ \sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}},$

$\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(ii) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}},\ \cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}},$

$\sin\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

(iii) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}},\ \tan\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}},$

$\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

Area of Triangle

(i) Area of a triangle in terms of sides (Heron's formula)

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

(ii) Area of triangle in terms of one side and sine of three angles.

$$\Delta = \frac{a^2}{2}\cdot\frac{\sin B\sin C}{\sin A} = \frac{b^2}{2}\cdot\frac{\sin A\sin C}{\sin B}$$

$$= \frac{c^2}{2}\cdot\frac{\sin A\sin B}{\sin C}$$

PROBLEMS

1. If $A + B = 45°$, show that $(1 + \tan A)(1 + \tan B) = 2$.
2. If α, β and γ are in A.P., show that
$$\cot\beta = \frac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha}$$
3. Find the maximum and minimum value of $\cos^2\theta - 6\sin\theta\cdot\cos\theta + 3\sin^2\theta + 2$
4. Find the most general solutions for $2^{\sin x} + 2^{\cos x} = 2^{1-1/\sqrt{2}}$.
5. Solve the equation $\sin x + \cos x - 2\sqrt{2}\sin x\cos x = 0$.
6. Solve: $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ in terms of k where k is perimeter of the ΔABC.
7. With usually notations, if in a triangle ABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that:
$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$
8. In a ΔABC, $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show a, b, c are in A.P.
9. A circular wire of radius 7.5 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 120 cm. Find in degrees the angle which is subtended at the centre of the hoop.
10. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced and 72° at the centre, find the length of the rope.

SOLUTIONS

1. We know, $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$$1 = \frac{\tan A + \tan B}{1 - \tan A\tan B} \quad (\because \tan 45° = 1)$$

$\therefore$ $\tan A + \tan B + \tan A\tan B = 1$

or, $1 + \tan A + \tan B + \tan A\tan B = 1 + 1$

$\Rightarrow$ $(1 + \tan A) + \tan B(1 + \tan A) = 2$

$\Rightarrow$ $(1 + \tan A)(1 + \tan B) = 2$

2. Since, α, β and γ are in A.P,

$$\therefore \quad 2\beta = \alpha + \gamma \Rightarrow \cot\beta = \cot\left(\frac{\alpha+\gamma}{2}\right) = \frac{\cos\left(\frac{\alpha+\gamma}{2}\right)}{\sin\left(\frac{\alpha+\gamma}{2}\right)}$$

Multiplying and dividing by $2\sin\left(\frac{\alpha-y}{2}\right)$, we get

$$\cot\beta = \frac{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cdot\sin\left(\frac{\alpha-\gamma}{2}\right)}{2\sin\left(\frac{\alpha+\gamma}{2}\right)\cdot\sin\left(\frac{\alpha-\gamma}{2}\right)} = \frac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha}$$

3. We have, $\cos^2\theta - 6\sin\theta\cdot\cos\theta + 3\sin^2\theta + 2$

$= (1 - \sin^2\theta) - 3\sin 2\theta + 3\sin^2\theta + 2$

$= 2\sin^2\theta - 3\sin 2\theta + 3$

$= (1 - \cos 2\theta) - 3\sin 2\theta + 3$

$= 4 - (\cos 2\theta + 3\sin 2\theta)$ ...(i)

as we have, $-\sqrt{10} \le \cos 2\theta + 3\sin 2\theta \le \sqrt{10}$

$\therefore$ $-\sqrt{10} \le -(\cos 2\theta + 3\sin 2\theta) \le \sqrt{10}$

or, $4 - \sqrt{10} \le 4 - (\cos 2\theta + 3\sin 2\theta) \le 4 + \sqrt{10}$ ...(ii)

From (i) and (ii), we get

$$4 - \sqrt{10} \le \cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2 \le 4 + \sqrt{10}$$

Hence, $4 + \sqrt{10}$ and $4 - \sqrt{10}$ are the maximum and minimum values.

4. As we know, A.M. $\ge$ G.M.

$$\Rightarrow \quad \frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x}\cdot 2^{\cos x}}$$

(equality holds only when $2^{\sin x} = 2^{\cos x}$)

$\therefore$ $2^{\sin x} + 2^{\cos x} \ge 2\cdot\sqrt{2^{\sin x + \cos x}}$ ...(i)

Now, equation (i) admits minimum value when $\sin x + \cos x$ is $(-\sqrt{2})$

$$\left\{\text{using } -\sqrt{a^2+b^2} \le a\cos x + b\sin x \le \sqrt{a^2+b^2}\right\}$$

$\therefore$ $2^{\sin x} + 2^{\cos x} \ge 2\cdot\sqrt{2^{-\sqrt{2}}}$

or $2^{\sin x} + 2^{\cos x} \ge 2\cdot 2^{-\sqrt{2}/2}$

or $2^{\sin x} + 2^{\cos x} \ge 2^{1-1/\sqrt{2}}$

Thus the equation holds only when,

$2^{\sin x} = 2^{\cos x} \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1$

$\Rightarrow \quad x = n\pi + \dfrac{\pi}{4}$ is the general solution.

5. Let $(\sin x + \cos x) = t$ and using the equation

$\sin x \cdot \cos x = \dfrac{t^2 - 1}{2}$, we get

$$t - 2\sqrt{2}\left(\frac{t^2-1}{2}\right) = 0 \quad \Rightarrow \quad \sqrt{2}t^2 - t - \sqrt{2} = 0$$

$\Rightarrow t = \sqrt{2}$ and $-\dfrac{1}{\sqrt{2}}$ $\therefore t_1 = \sqrt{2},\ t_2 = -\dfrac{1}{\sqrt{2}}$ are roots of this quadratic equation, thus the solution of the given equation reduces to the solution of two trigonometric equations.

$$\sin x + \cos x = \sqrt{2} \quad \text{and} \quad \sin x + \cos x = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$$

$$\Rightarrow \quad \sin x \,.\, \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 1$$

Similarly, $\sin x \,.\, \cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{4}\cos x = -\dfrac{1}{2}$

$$\Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = 1 \quad \text{and} \quad \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2}$$

$$\Rightarrow \quad x + \frac{\pi}{4} = (4n+1)\frac{\pi}{2} \quad \text{and} \quad x + \frac{\pi}{4} = n\pi + (-1)^n \,.\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{4} \quad \text{and} \quad x = n\pi + (-1)^n\frac{\pi}{6} - \left(\frac{\pi}{4}\right)$$

6. Here, $b\cos^2\dfrac{C}{2} + c\cos^2\dfrac{B}{2}$

$$= \frac{b}{2}(1 + \cos C) + \frac{c}{2}(1 + \cos B)$$

$$= \frac{b+c}{2} + \frac{1}{2}(b\cos C + c\cos B)$$

[using projection formula]

$$= \frac{b+c}{2} + \frac{1}{2}a = \frac{a+b+c}{2}$$

$\therefore \quad b\cos^2\dfrac{C}{2} + c\cos^2\dfrac{B}{2} = \dfrac{k}{2}$ [where $k = a + b + c$]

7. Let $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13} = k$

$\Rightarrow \quad 2(a + b + c) = 36k \quad$ ...(i)

and $b + c = 11k,\ c + a = 12k,\ a + b = 13k$ ...(ii)

Solving (i) and (ii), we get

and $a = 7k,\ b = 6k,\ c = 5k$

Hence,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36k^2+25k^2-49k^2}{60k^2} = \frac{12}{60} = \frac{1}{5} = \frac{7}{35}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{49k^2+25k^2-36k^2}{70k^2} = \frac{38}{70} = \frac{19}{35}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49k^2+36k^2-25k^2}{84k^2} = \frac{60}{84} = \frac{5}{7} = \frac{25}{35}$$

$$\Rightarrow \quad \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

8. Here $\dfrac{c}{2}(1 + \cos A) + \dfrac{a}{2}(1 + \cos C) = \dfrac{3b}{2}$

$\Rightarrow \quad a + c + (c\cos A + a\cos C) = 3b$

$\Rightarrow \quad a + c + b = 3b \qquad$ (using projection formula)

$\Rightarrow \quad a + c = 2b$

which shows a, b, c are in A.P.

9. Radius of the circular wire is 7.5 cm.

$\therefore$ Length of the circular wire $= 2\pi \times 7.5 = 15\,\pi$ cm

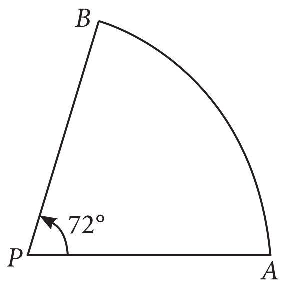
Radius of the hoop = 120 cm

Let θ be the angel subtended by the wire at the centre of the hoop. Then,

$$\theta = \frac{arc}{radius}$$

$$\Rightarrow \theta\left(\frac{15\pi}{120}\right)^c = \left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^o = 22°\,30'$$

10. Let the post be at point P and let PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72°$ and arc $AB = 88$ m. Let r be the length of the rope *i.e.*, $PA = r$ metres.



Here, $\theta = 72° = \left(72 \times \dfrac{\pi}{180}\right)^c = \left(\dfrac{2\pi}{5}\right)^c$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi} = 70 \text{ metres (approx).}$$

CLASS XI | Series 3

ACE YOUR WAY CBSE

Trigonometric Functions and Principle of Mathematical Induction

IMPORTANT FORMULAE

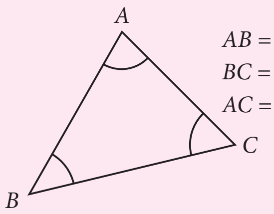
TRIGONOMETRIC FUNCTIONS

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
- $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$
- $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$
- $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$
  $= 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\cos A - \cos B = 2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{B-A}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
- $2\sin A \cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A \sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A \cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A \sin B = \cos(A - B) - \cos(A + B)$
- $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
- $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\cos A \cos 2A \cos 2^2 A \ldots\ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$
- $\sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$
- $\cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2},\ n \in Z$
- $\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$

- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha,\ n \in Z$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in Z$
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$
- $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
- $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
- $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
- $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$

Properties of triangles

- *Sine formula (sine rule)*
  - *The sines of the angles are proportional to the lengths of the opposite sides.*



$AB = c$
$BC = a$
$AC = b$

i.e., $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

- *Cosine formula*
  - $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
  - $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
  - $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$
- *Napolian or Napier's Analogy or Tangent Law*
  - $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$
  - $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$
  - $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$
- *If A, B, C are angles of a triangle ABC, then*
  - $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
  - $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

PRINCIPLE OF MATHEMATICAL INDUCTION

- *The proposition P(n) involving a natural number n is assumed to be true for all natural numbers n if the following two conditions are satisfied :*
- *The proposition P(n) is true for n = 1 i.e., P(1) is true.*
- *P(m) is true $\Rightarrow$ P(m + 1) is true.*
  i.e., P(m + 1) is true whenever P(m) is true.
- *But for many problems, following extended form of the principle of mathematical induction is used.*
- *If P(n) is proposition such that :*
- *P(1), P(2), ..., P(k) are true.*
- *P(m), P(m + 1), ..., P(m + k – 1) are true*
  $\Rightarrow$ P(m + k) is true.
  Then P(n) is true for all natural numbers n.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. In a ΔABC, if $a = \sqrt{3}+1$, $B = 30°$ and $C = 45°$, then find the value of c.
2. In a ΔABC, if $b = \sqrt{3}+1$, $c = \sqrt{3}-1$ and $A = 60°$, then find the value of $\tan\left(\dfrac{B-C}{2}\right)$.
3. If $A + B = \dfrac{\pi}{4}$, then find the value of $(1 + \tan A)(1 + \tan B)$.
4. If $3\tan\theta\tan\varphi = 1$. Prove that $2\cos(\theta+\varphi) = \cos(\theta-\varphi)$
5. Prove that $\cos 55° + \cos 65° + \cos 175° = 0$.

SHORT ANSWER TYPE

6. Assuming the distance of the earth from the moon to be 38400 km and the angle subtended by the moon at the eye of a person on the earth to be 31′ find the diameter of the moon.
7. The angles of a triangle are in A.P. and the ratio of the smallest angle in degrees to the greatest angle in radians is 60 : π. Find the angles of the triangle in degrees and radians.
8. If $m = \tan x + \sin x$ and $n = \tan x - \sin x$, prove that $m^2 - n^2 = 4\sqrt{mn}$
9. Evaluate $\sin\left\{n\pi + (-1)^n \dfrac{\pi}{4}\right\}$; where n is an integer.

10. Prove that $\sin A \cdot \sin(60° - A) \cdot \sin(60° + A) = \frac{1}{4}\sin 3A$.

LONG ANSWER TYPE - I

11. Find the number of solutions of the equation $2\sin^3 x + 2\cos^3 x - 3\sin 2x + 2 = 0$ in $[0, 4\pi]$.

12. In a ΔABC, if $\frac{a}{b^2 - c^2} + \frac{c}{b^2 - a^2} = 0$, then find the value of $\angle B$.

13. If $0 < \theta < \frac{\pi}{2}$, and if $\frac{y+1}{1-y} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$, then find the value of y.

14. Show that $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9 for every natural number n.

15. If α and β be two different roots of the equation $a\cos\theta + b\sin\theta = c$, prove that $\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$.

LONG ANSWER TYPE - II

16. By using law of Mathematical induction, prove that the statement

$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n - 1)\beta)$

equals $\dfrac{\cos\left(\alpha + \frac{(n-1)\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\ \forall n \in N.$

17. If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, prove that :

(i) $\sin^4 A + \sin^4 B = 2\sin^2 A\sin^2 B$

(ii) $\frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1$

18. Show by using the principle of mathematical induction that

$$1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n+1)^2$$

19. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, show that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

20. If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$, prove that

$$\tan\left(\frac{\pi}{4} - \theta\right)\cdot\tan\left(\frac{\pi}{4} - \alpha\right) = m$$

SOLUTIONS

1. We have, $B = 30°$ and $C = 45°$

$\Rightarrow A = 105°$

$\therefore$ By sine rule, $\frac{a}{\sin A} = \frac{c}{\sin C}$

$\Rightarrow \frac{\sqrt{3}+1}{\sin 105°} = \frac{c}{\sin 45°} \Rightarrow c = \frac{(\sqrt{3}+1)}{\sqrt{3}+1} \times \frac{1}{\sqrt{2}} \times 2\sqrt{2} = 2$

2. By Napier's Analogy, we have

$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2} \Rightarrow \tan\left(\frac{B-C}{2}\right) = \frac{2}{2\sqrt{3}}\cot 30° = 1$

3. From given, we have $\tan(A + B) = \tan\frac{\pi}{4}$

$\therefore \tan A + \tan B = 1 - \tan A\tan B$

$\Rightarrow \tan A + \tan B + \tan A\tan B = 1$

$\Rightarrow (1 + \tan A) + \tan B(1 + \tan A) = 1 + 1$

(by adding 1 on both sides)

$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$

4. Given, $3\tan\theta\tan\varphi = 1$ or $\cot\theta\cdot\cot\varphi = 3$

or $\frac{\cos\theta\cos\varphi}{\sin\theta\sin\varphi} = \frac{3}{1}$

By componendo and dividendo, we have

$\frac{\cos\theta\cos\varphi + \sin\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi} = \frac{3+1}{3-1} \Rightarrow \frac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)} = 2$

$\Rightarrow 2\cos(\theta + \varphi) = \cos(\theta - \varphi)$

5. L.H.S. $= \cos 55° + \cos 65° + \cos 175°$

$= 2\cos\frac{55°+65°}{2}\cos\frac{55°-65°}{2} + \cos 175°$

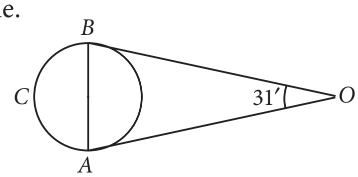
$= 2\cos 60°\cos(-5°) + \cos 175°$

$= 2\times\frac{1}{2}\cos 5° + \cos(180° - 5°) = \cos 5° - \cos 5° = 0$

6. Let AB be the diameter of the moon and O be the observer on earth.

Given, $\angle AOB = 31' = \frac{31}{60}\times\frac{\pi}{180}$ radians.

Since angle subtended by the moon is very small, therefore, its diameter will be approximately equal to the arc of a circle whose centre is the eye of the observer and radius is the distance of the earth from the moon. Also the moon subtends an angle of $31'$ at the centre of this circle.



$\because \theta = \frac{l}{r}$

$\therefore \frac{31}{60}\times\frac{\pi}{180} = \frac{AB}{38400}$

$\Rightarrow AB = \frac{31}{60}\times\frac{22}{7\times180}\times 38400 = 346\frac{26}{63}$ km

7. Let the angles of the triangle be $(\theta - d)°$, $\theta°$, $(\theta + d)°$, where $d > 0$. Then,

$\theta - d + \theta + \theta + d = 180 \Rightarrow 3\theta = 180 \Rightarrow \theta = 60°$

Hence the angles are $(60 - d)°$, $60°$ and $(60 + d)°$

Given, $\dfrac{\text{least angle in degrees}}{\text{greatest angle in radians}} = \dfrac{60}{\pi}$

$$\Rightarrow \frac{(60-d)}{(60+d)\dfrac{\pi}{180}} = \frac{60}{\pi} \qquad \left[\because 1° = \frac{\pi}{180} \text{ radians}\right]$$

$$\Rightarrow \frac{(60-d)}{(60+d)} \cdot \frac{180}{\pi} = \frac{60}{\pi} \Rightarrow \frac{60-d}{60+d} = \frac{1}{3} \Rightarrow d = 30°$$

Thus the angles are 30°, 60°, 90°.

In radians, the angles are

$$\left(30 \times \frac{\pi}{180}\right), \left(60 \times \frac{\pi}{180}\right), \left(90 \times \frac{\pi}{180}\right)$$

i.e., $\dfrac{\pi}{6}, \dfrac{\pi}{3}$ and $\dfrac{\pi}{2}$.

8. L.H.S. $= m^2 - n^2 = (\tan x + \sin x)^2 - (\tan x - \sin x)^2$

$= 4 \tan x \cdot \sin x$

Now, R.H.S. $= 4\sqrt{mn} = 4\sqrt{(\tan x + \sin x)(\tan x - \sin x)}$

$$= 4\sqrt{\frac{\sin^4 x}{\cos^2 x}} = \frac{4 \sin^2 x}{\cos x} = 4 \tan x \cdot \sin x = \text{L.H.S.}$$

9. Since, $\sin(\pi + \theta) = -\sin\theta$

$\therefore \sin(n\pi + \theta) = (-1)^n \sin\theta$

$$\therefore \sin\left\{n\pi + (-1)^n \frac{\pi}{4}\right\} = (-1)^n \sin\left\{(-1)^n \frac{\pi}{4}\right\}$$

$$= (-1)^n(-1)^n \sin\frac{\pi}{4} = (-1)^{2n} \sin\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

10. L.H.S. $= \sin A \cdot \sin(60° - A) \cdot \sin(60° + A)$

$$= \frac{1}{2}\sin A\,[2\sin(60° + A)\cdot\sin(60° - A)]$$

$$= \frac{1}{2}\sin A\,[\cos(60° + A - 60° + A) - \cos(60° + A + 60° - A)]$$

$$= \frac{1}{2}\sin A\,(\cos 2A - \cos 120°)$$

$$= \frac{1}{4}(2\cos 2A \sin A - 2\cos 120° \sin A)$$

$$= \frac{1}{4}\left[\sin(2A + A) - \sin(2A - A) - 2\left(-\frac{1}{2}\right)\sin A\right]$$

$$= \frac{1}{4}(\sin 3A - \sin A + \sin A) = \frac{1}{4}\sin 3A$$

11. We have, $2\sin^3 x + 2\cos^3 x - 3\sin 2x + 2 = 0$

$\Rightarrow \sin^3 x + \cos^3 x + 1 - 3\sin x \cos x = 0$

$$\Rightarrow \sin x + \cos x + 1 = 0 \qquad \left[\begin{array}{l}\because a^3 + b^3 + c^3 - 3abc = 0 \\ \Rightarrow a + b + c = 0\end{array}\right]$$

$$\Rightarrow 2\sin\frac{x}{2}\cos\frac{x}{2} + 2\cos^2\frac{x}{2} = 0$$

$$\Rightarrow 2\cos\frac{x}{2}\left[\cos\frac{x}{2} + \sin\frac{x}{2}\right] = 0$$

$$\Rightarrow \cos\frac{x}{2} = 0 \text{ or } \cos\frac{x}{2} + \sin\frac{x}{2} = 0$$

$$\Rightarrow \cos\frac{x}{2} = 0 \text{ or } \tan\frac{x}{2} = -1$$

Now if, $\cos\dfrac{x}{2} = 0 \Rightarrow x = \pi, 3\pi$

and $\quad \tan\dfrac{x}{2} = -1 \Rightarrow x = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$

Hence, there are 4 solutions.

12. We have, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say)

$$\therefore \frac{a}{b^2 - c^2} + \frac{c}{b^2 - a^2} = 0$$

$$\Rightarrow \frac{k\sin A}{k^2(\sin^2 B - \sin^2 C)} + \frac{k\sin C}{k^2(\sin^2 B - \sin^2 A)} = 0$$

$$\Rightarrow \frac{\sin A}{\sin(B + C)\sin(B - C)} + \frac{\sin C}{\sin(B + A)\sin(B - A)} = 0$$

$$\Rightarrow \frac{1}{\sin(B - C)} + \frac{1}{\sin(B - A)} = 0$$

$\Rightarrow \sin(B - A) + \sin(B - C) = 0$

$\Rightarrow \sin(A - B) = \sin(B - C)$

$\Rightarrow A - B = B - C \Rightarrow A + C = 2B \Rightarrow B = 60°$.

$(\because A + B + C = 180°)$

13. We have, $\dfrac{y+1}{1-y} = \sqrt{\dfrac{1 + \sin\theta}{1 - \sin\theta}}$

$$\Rightarrow \frac{y+1}{1-y} = \sqrt{\frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}} \Rightarrow \frac{1+y}{1-y} = \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$

$$\Rightarrow \frac{1+y}{1-y} = \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} \Rightarrow y = \tan\frac{\theta}{2}$$

14. Let $P(n) = n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9.

Now, $P(1) = 1^3 + 2^3 + 3^3 = 36$, which is divisible by 9

$\therefore \quad P(1)$ is true. ...(i)

Let $P(m)$ be true $\Rightarrow P(m) = m^3 + (m + 1)^3 + (m + 2)^3$ is divisible by 9

$\Rightarrow P(m) = m^3 + (m + 1)^3 + (m + 2)^3 = 9k$, where k is an integer

To prove $P(m+1)$ is true *i.e.*, $P(m+1)$ is divisible by 9

Now, $P(m+1) = (m+1)^3+(m+2)^3+(m+3)^3$

$= (m+1)^3+(m+2)^3+m^3+9m^2+27m+27$

$= [m^3+(m+1)^3+(m+2)^3]+9m^2+27m+27$

$= 9k + 9(m^2+3m+3)$,

which is divisible by 9.

Hence $P(m+1)$ is true whenever $P(m)$ is true. ...(ii)

From (i) and (ii), it follows that $P(n)$ is true for all natural numbers n.

15. Given, α and β are the roots of the equation

$a\cos\theta + b\sin\theta = c$

$\therefore$ $a\cos\alpha + b\sin\alpha = c$ ...(i)

and $a\cos\beta + b\sin\beta = c$ ...(ii)

Subtracting (ii) from (i), we get

$a(\cos\alpha - \cos\beta) + b(\sin\alpha - \sin\beta) = 0$

$\Rightarrow$ $b(\sin\alpha - \sin\beta) = a(\cos\beta - \cos\alpha)$

$$\Rightarrow b\times 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = a\times 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right)=\frac{b}{a}$$

$$\left[\because\ \alpha,\beta \text{ are different, } \therefore\ \sin\left(\frac{\alpha-\beta}{2}\right)\neq 0\right]$$

$$\text{Now, } \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$$

16. Let $P(n) : \cos\alpha + \cos(\alpha+\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Let $n = 1$

$\therefore$ L.H.S. $= P(1) = \cos\alpha$ and

$$\text{R.H.S.} = \frac{\cos(\alpha+0)\cdot\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \cos\alpha$$

$\therefore$ $P(1)$ is true ...(i)

Let $P(k)$ is true, where

$P(k) : \cos\alpha + \cos(\alpha+\beta) + \ldots. + \cos(\alpha+(k-1)\beta)$

$$= \frac{\cos\left(\alpha+\frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Now, we prove that $P(k+1)$ is true whenever $P(k)$ is true.

i.e., $P(k+1) : \cos\alpha + \cos(\alpha+\beta) + \ldots$

$+ \cos(\alpha+(k-1)\beta) + \cos(\alpha+k\beta)$

$$= \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\sin\frac{(k+1)\beta}{2}}{\sin\frac{\beta}{2}}$$

Now, L.H.S. of $P(k+1)$

$= \cos\alpha + \cos(\alpha+\beta) + \ldots$

$+ \cos(\alpha+(k-1)\beta) + \cos(\alpha+k\beta)$

$= P(k) + \cos(\alpha+k\beta)$

$$= \frac{\cos\left(\alpha+\left(\frac{k-1}{2}\right)\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\frac{\beta}{2}} + \cos(\alpha+k\beta)$$

$$= \frac{1}{2}\left[\frac{2\cos(\alpha+\frac{(k-1)}{2}\beta)\sin\frac{k\beta}{2}+2\cos(\alpha+k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{\sin\left(\alpha+k\beta+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}\right]$$

$$= \frac{2\cos\left(\frac{2\alpha+k\beta}{2}\right)\cdot\sin\left(\frac{k+1}{2}\right)\beta}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\cdot\sin\left(\frac{k+1}{2}\right)\beta}{\sin\frac{\beta}{2}} = \text{R.H.S of } P(k+1)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true. ...(ii)

Hence, by (i) and (ii), it proves that $P(n)$ is true for all natural numbers n.

17. Given, $\dfrac{\cos^4 A}{\cos^2 B}+\dfrac{\sin^4 A}{\sin^2 B}=1=\cos^2 A+\sin^2 A$

$$\therefore \frac{\cos^4 A}{\cos^2 B}-\cos^2 A=\sin^2 A-\frac{\sin^4 A}{\sin^2 B}$$

$$\Rightarrow \frac{\cos^2 A(\cos^2 A-\cos^2 B)}{\cos^2 B}=\sin^2 A\frac{(\sin^2 B-\sin^2 A)}{\sin^2 B}$$

$$\Rightarrow \frac{\cos^2 A}{\cos^2 B}(\cos^2 A-\cos^2 B) = \frac{\sin^2 A}{\sin^2 B}[(1-\cos^2 B)-(1-\cos^2 A)]$$

$$\Rightarrow \frac{\cos^2 A}{\cos^2 B}(\cos^2 A-\cos^2 B)=\frac{\sin^2 A}{\sin^2 B}(\cos^2 A-\cos^2 B)$$

$$\Rightarrow \quad (\cos^2 A - \cos^2 B)\left(\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B}\right) = 0$$

If $\cos^2 A - \cos^2 B = 0$, then $\cos^2 A = \cos^2 B$

If $\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} = 0$, then $\cos^2 A \sin^2 B = \sin^2 A \cos^2 B$

$\Rightarrow \cos^2 A\,(1 - \cos^2 B) = (1 - \cos^2 A)\cos^2 B$

$\Rightarrow \cos^2 A - \cos^2 A \cos^2 B = \cos^2 B - \cos^2 A \cos^2 B$

$\Rightarrow \cos^2 A = \cos^2 B$

Thus in both cases, $\cos^2 A = \cos^2 B$

$\therefore \; 1 - \sin^2 A = 1 - \sin^2 B \Rightarrow \sin^2 A = \sin^2 B$

(i) L.H.S. $= \sin^4 A + \sin^4 B$

$= (\sin^2 A - \sin^2 B)^2 + 2\sin^2 A \sin^2 B$

$= 2\sin^2 A \sin^2 B =$ R.H.S. $[\because \; \sin^2 A = \sin^2 B]$

(ii) L.H.S. $= \frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = \frac{\cos^4 B}{\cos^2 B} + \frac{\sin^4 B}{\sin^2 B}$

$[\because \; \cos^2 A = \cos^2 B, \sin^2 A = \sin^2 B]$

$= \cos^2 B + \sin^2 B = 1 =$ R.H.S.

18. Let $P(n) : 1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n+1)^2$

When $n = 1$, L.H.S. $= 1$ and R.H.S. $= \frac{1}{8}\cdot 3^2 = \frac{9}{8}$

Clearly, $1 < \frac{9}{8}$ $\quad \therefore P(1)$ is true.

Let $P(m)$ be true

$$\Rightarrow \; 1 + 2 + 3 + \ldots + m < \frac{1}{8}(2m+1)^2 \qquad \ldots(1)$$

To prove $P(m + 1)$ is true.

i.e., $1 + 2 + 3 + \ldots + m + m + 1 < \frac{1}{8}(2m+3)^2 = \alpha(\text{say})$ ..(2)

From (1), $1 + 2 + 3 + \ldots + m + m + 1 < \frac{1}{8}(2m+1)^2 + (m + 1) = \beta(\text{say})$ ...(3)

Now, $\alpha - \beta = \frac{1}{8}(2m+3)^2 - \frac{1}{8}(2m+1)^2 - (m+1)$

$= \frac{1}{8}\cdot 4(m+1)\cdot 2 - (m+1) = 0$

$\therefore \; \alpha = \beta$ ...(4)

From (3), $1 + 2 + 3 + \ldots + m + m + 1 < \alpha$ $\quad [\because \; \beta = \alpha]$

or $\quad 1 + 2 + 3 + \ldots + m + m + 1 < \frac{1}{8}(2m+1)^2$

Hence $P(m + 1)$ is true whenever $P(m)$ is true.

So by principle of mathematical induction it follows that $P(n)$ is true for all natural numbers n.

19. We have, $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$

$$= \frac{\frac{n\sin\alpha\cos\alpha}{\cos^2\alpha}}{\frac{1}{\cos^2\alpha} - \frac{n\sin^2\alpha}{\cos^2\alpha}}$$ [Dividing numerator and denominator by $\cos^2\alpha$]

$$= \frac{n\tan\alpha}{\sec^2\alpha - n\tan^2\alpha} = \frac{n\tan\alpha}{1 + \tan^2\alpha - n\tan^2\alpha}$$

$$\tan\beta = \frac{n\tan\alpha}{1 + (1-n)\tan^2\alpha} \qquad \ldots(1)$$

Now, L.H.S. $= \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\cdot\tan\beta}$

$$= \frac{\tan\alpha - \frac{n\tan\alpha}{1 + (1-n)\tan^2\alpha}}{1 + \tan\alpha\,.\,\frac{n\tan\alpha}{1 + (1-n)\tan^2\alpha}} \qquad [\text{From (1)}]$$

$$= \frac{\tan\alpha + (1-n)\tan^3\alpha - n\tan\alpha}{1 + (1-n)\tan^2\alpha + n\tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha + (1-n)\tan^3\alpha}{1 + \tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha(1 + \tan^2\alpha)}{1 + \tan^2\alpha} = (1 - n)\tan\alpha = \text{R.H.S.}$$

20. Given, $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$

$$\Rightarrow \frac{\sin(\theta+\alpha) + \cos(\theta-\alpha)}{\sin(\theta+\alpha) - \cos(\theta-\alpha)} = \frac{1-m+1+m}{1-m-1-m}$$

[By componendo and dividendo]

$$\Rightarrow \frac{\sin(\theta+\alpha) + \sin\left[\frac{\pi}{2} - (\theta-\alpha)\right]}{\sin(\theta+\alpha) - \sin\left[\frac{\pi}{2} - (\theta-\alpha)\right]} = \frac{2}{-2m} = -\frac{1}{m}$$

$$\Rightarrow \frac{\sin\left(\frac{\pi}{4}+\alpha\right).\cos\left(\theta - \frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}+\alpha\right).\sin\left(\theta - \frac{\pi}{4}\right)} = -\frac{1}{m}$$

$$\Rightarrow \tan\left(\frac{\pi}{4}+\alpha\right).\cot\left(\theta - \frac{\pi}{4}\right) = -\frac{1}{m}$$

$$\Rightarrow -\tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\frac{\pi}{4} - \theta\right) = -\frac{1}{m}$$

$$\Rightarrow m = \cot\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \tan\left(\frac{\pi}{4} - \theta\right).\tan\left(\frac{\pi}{4} - \alpha\right)$$

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MPP-3 MONTHLY Practice Problems

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Trigonometric Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The general solution of $e^{-1/\sqrt{2}}(e^{\sin x}+e^{\cos x})=2$ is

(a) $x = m\pi$ (b) $x=\dfrac{(4m+1)\pi}{4}$

(c) $x=\dfrac{(4m+1)\pi}{2}$ (d) None of these

2. If $x \sin a + y \sin 2a + z \sin 3a = \sin 4a$

$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$

$x \sin c + y \sin 2c + z \sin 3c = \sin 4c$

Then, the roots of the equation

$t^3-\left(\dfrac{z}{2}\right)t^2-\left(\dfrac{y+2}{4}\right)t+\left(\dfrac{z-x}{8}\right)=0$, $a, b, c, \neq n\pi$ are

(a) $\sin a, \sin b, \sin c$ (b) $\cos a, \cos b, \cos c$

(c) $\sin 2a, \sin 2b, \sin 2c$ (d) $\cos 2a, \cos 2b, \cos 2c$

3. In ΔABC, $a = 2b$ and $|A - B| = \dfrac{\pi}{3}$, then $\angle C$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$

(c) $\dfrac{\pi}{4}$ (d) none of these

4. If $\dfrac{\sin^2 2x+4\sin^4 x-4\sin^2 x\cos^2 x}{4-\sin^2 2x-4\sin^2 x}=\dfrac{1}{9}$

and $0 < x < \pi$, then the value of x is

(a) $\pi/3$ (b) $\pi/6$

(c) $2\pi/3$ (d) none of these

5. In any triangle ABC,

$a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) =$

(a) $3abc$ (b) $3a + 3b + 3c$

(c) $3 (a - b - c)$ (d) $3a^2 b^2 c^2$

6. Points D, E are taken on the side BC of a ΔABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\dfrac{\sin(x+y)\sin(y+z)}{\sin x \sin z}$ is equal to

(a) 1 (b) 2

(c) 4 (d) none of these

One or More Than One Option(s) Correct Type

7. The equation

$$3^{\sin 2x+2\cos^2 x}+3^{1-\sin 2x+2\sin^2 x}=28$$

is satisfied for the values of x given by

(a) $\cos x = 0$ (b) $\tan x = 0$

(c) $\tan x = -1$ (d) none of these

8. The line joining the in-centre to the circum-centre of a triangle ABC is inclined to the side BC at an angle

(a) $\tan\theta=\left(\dfrac{\cos B+\cos C+1}{\sin C-\sin B}\right)$

(b) $\tan\theta=\left(\dfrac{\cos B+\cos C-1}{\sin C-\sin B}\right)$

(c) $\tan\theta=\left(\dfrac{1-\cos B-\cos C}{\sin B-\sin C}\right)$

(d) none of these

9. The least positive non-integral solution of $\sin \pi(x^2 + x) - \sin \pi x^2 = 0$ is

(a) rational

(b) irrational of the form $\sqrt{p}$

(c) irrational of the form $\dfrac{\sqrt{p}-1}{4}$, where p is an odd integer

(d) irrational of the form $\dfrac{p-1}{4\sqrt{p+4}}$, where p is an odd integer

10. If $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then

(a) $\sin\alpha - \cos\alpha = \sqrt{2}\sin\theta$

(b) $\sin\alpha + \cos\alpha = \sqrt{2}\cos\theta$

(c) $\cos 2\theta = \sin 2\alpha$ (d) $\sin 2\theta + \cos 2\alpha = 0$

11. If $\cos^4 x + \sin^4 x - \sin 2x + (3/4)\sin^2 2x = y$, then

(a) $y = 1$ if $x = 15\pi/2$ (b) $y \neq 0$ for any value of x

(c) $y = 0$ if $x = 15\pi$ (d) $y = 1$ if $\sin 2x = 0$

12. If $A = \sin^2 x + \cos^4 x$, then for all real x

(a) $\dfrac{13}{16} \le A \le 1$ (b) $1 \le A \le 2$

(c) $\dfrac{3}{4} \le A \le \dfrac{13}{16}$ (d) $\dfrac{3}{4} \le A \le 1$

13. $5\sin x - 12\cos x = -13\sin 3x$, if

(a) $\sin(2x - \phi/2) = 0$ (b) $\sin(x + \phi/2) = 0$

(c) $\cos(x + \phi/2) = 0$ (d) $\cos(2x + \phi/2) = 0$

where $\phi = \sin^{-1}\dfrac{12}{13}$.

Comprehension Type

We have $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots +$

$$\cos(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left[\frac{2\alpha + \overline{n-1}\,\beta}{2}\right]$$

and $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots +$

$$\sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\frac{2\alpha + \overline{n-1}\,\beta}{2}\right]\left[\sin\frac{\beta}{2} \neq 0\right]$$

14. $\displaystyle\sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right) = \dots\dots$

(a) $\dfrac{n}{2} + 1$ (b) $\dfrac{n}{2} - 1$

(c) $\dfrac{n}{2}$ (d) none of these

15. $\sqrt{1 + \cos\alpha} + \sqrt{1 + \cos 2\alpha} + \sqrt{1 + \cos 3\alpha} + \dots$ upto n terms $=$

(a) $\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cos\left(\dfrac{n\alpha}{4}\right)$ (b) $\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cos\dfrac{(n+1)\alpha}{4}$

(c) $\sqrt{2}\dfrac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cos\dfrac{(n+1)\alpha}{4}$ (d) none of these

Matrix Match Type

16. Match the following :

	Column I		Column II
P.	$\sin^2 24° - \sin^2 6° =$	1.	2
Q.	$\tan 6° \tan 42° \tan 66° \tan 78° =$	2.	-2
R.	If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x =$	3.	1
S.	If $\cos x + \sec x = -2$, then for a +ve odd integer n, $\cos^n x + \sec^n x$ is	4.	$\frac{\sqrt{5}-1}{8}$

	P	Q	R	S
(a)	4	3	1	2
(b)	3	4	1	2
(c)	1	4	3	2
(d)	2	4	3	1

Integer Answer Type

17. In triangle ABC, $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., then the value of $\cot\frac{A}{2}\cot\frac{C}{2}$ is equal to

18. If $b + c = 3a$, then $\cot\frac{B}{2}\cot\frac{C}{2}$ is equal to

19. The value of $3\tan^6 10° - 27\tan^4 10° + 33\tan^2 10°$ equals

20. The angle θ whose cosine equal to its tangent is given by $\sin\theta = n\sin 18°$. Then n is equal to

◆◆

Keys are published in this issue. Search now! ☺

SELF CHECK

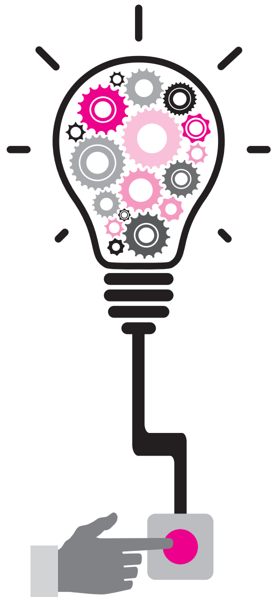
No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

THOUGHT PROVOKING PROBLEMS

BASED ON CBSE



Relations and Functions, Inverse Trigonometric Functions & Matrices and Determinants

1. Show that $f : N \to N$, given by

$$f(x)=\begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even}\end{cases}$$

is both one-one and onto.

2. Let $f, g : R \to R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, for all $x \in R$. Then find fog and gof.

3. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$.

4. Consider the binary operations $* : R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $a \, o \, b = a$ for all, $a, b \in R$. Show that '$*$' is commutative but not associative, 'o' is associative but not commutative.

5. Let $*$ be a binary operation on the set of rational numbers given as $a * b = (2a - b)^2$, $a, b \in Q$. Find $3 * 5$ and $5 * 3$. Is $3 * 5 = 5 * 3$?

6. Let $A = R \times R$ and $*$ be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A.

7. Show that the relation S in the set $A = \{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in Z, |a - b| \text{ is a multiple of } 4\}$ is an equivalence relation. Find the set of all elements related to 1.

8. Show that the relation R defined by $(a, b) \, R \, (c, d) \Rightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation.

9. Let $*$ be a binary operation defined by $a * b = 3a + 4b - 2$. Find $4 * 5$.

10. Let S be the set of all rational numbers except 1 and $*$ be defined on S by $a * b = a + b - ab$, for all $a, b \in S$. Prove that :

(i) $*$ is a binary operation on S.

(ii) $*$ is commutative as well as associative.

11. Prove that

$$\cos^{-1}\left(\frac{\cos x+\cos y}{1+\cos x\cdot\cos y}\right) = 2\tan^{-1}\left(\tan\frac{x}{2}\tan\frac{y}{2}\right).$$

12. Express the equation :

$$\cot^{-1}\frac{y}{\sqrt{1-x^2-y^2}}=2\tan^{-1}\sqrt{\frac{3-4x^2}{4x^2}}-\tan^{-1}\sqrt{\frac{3-4x^2}{x^2}}$$

as a rational integral equation in x and y.

13. If $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$ and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))))$, where $a \in [0, 1]$. Find the relationship between x and y in terms of 'a'.

14. Solve :

$$3\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

15. Solve the equation :

$$\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\pi/2.$$

16. Prove that : $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$

17. Solve for x : $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

18. Find the number of positive integral solutions of the equation :

$$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}.$$

19. Solve for x : $2(\sin^{-1}x)^2 - (\sin^{-1}x) - 6 = 0$

20. If $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$.

21. A trust fund, ₹ 35,000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to orphanage and second bond pays 10% interest per annum which will be given to an N.G.O. (Cancer Aid Society). Using matrix multiplication, determine how to divide ₹ 35,000 among two types of bonds if the trust fund obtains an annual total interest of ₹ 3,200. What are the values reflected in this question?

22. If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then find the values of a and b.

23. Express the matrix $A=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix.

24. Using elementary operations, find the inverse of the matrix $\begin{bmatrix}-1 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$

25. Using properties of determinants, prove that

$$\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}=4a^2b^2c^2.$$

26. Using properties of determinants, prove that

$$\begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix}=4abc.$$

27. Using properties of determinants, prove that

$$\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$$

28. Prove the following, using properties of determinants.

$$\begin{vmatrix}x & y & z\\ x^2 & y^2 & z^2\\ y+z & z+x & x+y\end{vmatrix}$$
$$= (x - y)(y - z)(z - x)(x + y + z).$$

29. Prove that $\begin{vmatrix}yz-x^2 & zx-y^2 & xy-z^2\\ zx-y^2 & xy-z^2 & yz-x^2\\ xy-z^2 & yz-x^2 & zx-y^2\end{vmatrix}$ is divisible by $(x + y + z)$, and hence find the quotient.

30. Find the equation of the line joining $A(1, 3)$ and $B(0, 0)$ using determinants and find the value of k if $D(k, 0)$ is a point such that area of ΔABD is 3 square units.

31. Two schools A and B decided to award prizes to their students for three values, team spirit, truthfulness and tolerance at the rate of ₹ x, ₹ y and ₹ z per student respectively. School A, decided to award a total of ₹ 1,100 for the three values to 3, 1 and 2 students respectively while school B decided to award ₹ 1,400 for the three values to 1, 2 and 3 students respectively. If one prize for all the three values together amount to ₹ 600 then
   (i) Represent the above situation by a matrix equation after forming linear equations.
   (ii) Is it possible to solve the system of equations so obtained using matrices?
   (iii) Which value you prefer to be rewarded most and why?

32. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

33. Two institutions decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate of ₹ x, ₹ y and ₹ z respectively per person. The first institution decided to award respectively 4, 3 and 2 employees with a total prize money of ₹ 37,000 and

the second institution decided to award respectively 5, 3 and 4 employees with a total prize money of ₹ 47,000. If all the three prizes per person together amount to ₹ 12,000 then using matrix method find the value of x, y and z. What values are described in the question?

34. Find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}$.

Hence solve the system of equations
$x + 2y - 3z = -4$; $2x + 3y + 2z = 2$;
$3x - 3y - 4z = 11$.

35. Using matrices, solve the following system of equations :
$2x + y + z = 7$, $x - y - z = -4$, $3x + 2y + z = 10$.

SOLUTIONS

1. Here, $f : N \to N$ s.t.

$$f(x)=\begin{cases}x+1, & \text{if } x \text{ is odd}\\ x-1, & \text{if } x \text{ is even}\end{cases}$$

Let $x, y \in N$ s.t. $f(x) = f(y)$
We shall show that $x = y$
(i) If x and y both are even
$f(x) = f(y) \Rightarrow x - 1 = y - 1 \Rightarrow x = y$
(ii) If x and y both are odd
$f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$
(iii) If x is odd and y is even
$f(x) = f(y) \Rightarrow x + 1 = y - 1 \quad \Rightarrow \quad y - x = 2$ ...(1)
R.H.S. is even but L.H.S. is odd.
$\Rightarrow$ Equation (1) in N is not possible.
$\Rightarrow$ (iii) does not arise.
(iv) If x is even and y is odd, does not arise.
In any case, $f(x) = f(y) \Rightarrow x = y$
$\Rightarrow$ f is one-one
For any $y \in N$ (co-domain), y can be even or odd
When y is odd, $y + 1$ is even, so
$f(y + 1) = (y + 1) - 1 = y$
When y is even, $y - 1$ is odd, so
$f(y - 1) = (y - 1) + 1 = y$
$\Rightarrow$ $f : N \to N$ is onto.
Hence, f is both one-one and onto.

2. Here, $f, g : R \to R$ s.t.

$$f(x)=|x|+x=\begin{cases}2x & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases}$$

and $g(x)=|x|-x=\begin{cases}0 & \text{if } x\ge 0\\ -2x & \text{if } x<0\end{cases}$

$\therefore$ $(fog)(x) = f(g(x)) =\begin{cases}f(0) & \text{if } x\ge 0\\ f(-2x) & \text{if } x<0\end{cases}$

$$=\begin{cases}0 & \text{if } x\ge 0\\ 2(-2x) & \text{if } x<0\end{cases}=\begin{cases}0 & \text{if } x\ge 0\\ -4x & \text{if } x<0\end{cases}$$

and $(gof)(x) = g(f(x))$

$$=\begin{cases}g(2x) & \text{if } x\ge 0\\ g(0) & \text{if } x<0\end{cases}=\begin{cases}0 & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases}=0\ \forall\, x\in R.$$

3. Let $f : N \to S$, $f(x) = 9x^2 + 6x - 5$
Consider, $f(x_1) = f(x_2)$
$\Rightarrow$ $9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$
$\Rightarrow$ $9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$
$\Rightarrow$ $(x_1 - x_2)[9x_1 + 9x_2 + 6] = 0$
$\Rightarrow$ $x_1 = x_2$ $\quad [\because x_1, x_2 \in N]$
$\Rightarrow$ f is one-one.
Since, S is the range of f. $\therefore$ f is onto.
Since, f is one-one and onto.
So, f is invertible.
Let $y \in S$ be arbitrary number.
Consider, $y = f(x) \Rightarrow x = f^{-1}(y)$
$\Rightarrow$ $y = 9x^2 + 6x - 5 \Rightarrow y = (3x + 1)^2 - 6$

$$\Rightarrow \sqrt{y+6}=3x+1 \Rightarrow x=\frac{\sqrt{y+6}-1}{3}$$

Also, $f^{-1}(y)=\dfrac{\sqrt{y+6}-1}{3}$ or $f^{-1}(x)=\dfrac{\sqrt{x+6}-1}{3}$

Now, $f^{-1}(43)=\dfrac{\sqrt{49}-1}{3}=\dfrac{7-1}{3}=2$

and $f^{-1}(163)=\dfrac{\sqrt{169}-1}{3}=\dfrac{13-1}{3}=4$

4. $b * a = |b - a| = |a - b|$ $[\because |-x| = |x|\ \forall\ x \in R]$
$= a * b\ \forall a, b \in R$
$\Rightarrow$ $*$ is commutative on R.
Also, for $a = 2$, $b = 4$, $c = 5$
$(a * b) * c = (2 * 4) * 5 = |2 - 4| * 5$
$= 2 * 5 = |2 - 5| = 3$
and $a * (b * c) = 2 * (4 * 5) = 2 * |4 - 5|$
$= 2 * 1 = |2 - 1| = 1.$
$\therefore$ $(a * b) * c \ne a * (b * c)$
$\Rightarrow$ $*$ is not associative on R.
Also, $(a\ o\ b)\ o\ c = a\ o\ c = a$
and $a\ o\ (b\ o\ c) = a\ o\ b = a$
$\Rightarrow$ $(a\ o\ b)\ o\ c = a\ o\ (b\ o\ c)\ \forall a, b, c \in R$
$\Rightarrow$ o is associative on R.
Now, $a\ o\ b = a$, $b\ o\ a = b$
$\Rightarrow$ $a\ o\ b \ne b\ o\ a$
$\Rightarrow$ o is not commutative on R.

5. We have, $a * b = (2a - b)^2$

$\therefore$ $3 * 5 = (2 \times 3 - 5)^2 = (6 - 5)^2 = 1$

$5 * 3 = (2 \times 5 - 3)^2 = (10 - 3)^2 = 49$

Thus, $3 * 5 \neq 5 * 3$

6. Here $A = R \times R$ and $*$ on A is defined as

$(a, b) * (c, d) = (a + c, b + d)\ \forall (a, b), (c, d) \in R$

Now $(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$

$= (a, b) * (c, d)\ \forall (a, b), (c, d) \in A$

$\Rightarrow$ $*$ is commutative on A.

Again $[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f)$

$= (a+c+e, b+d+f) = (a+(c+e), b+(d+f))$

$= (a, b) * (c + e, d + f)$

$= (a, b) * [(c, d) * (e, f)]\ \forall (a, b), (c, d), (e, f) \in A$

$\Rightarrow$ $*$ is associative on A.

Also $0 \in R$ and $(0, 0) \in A$.

$\therefore$ $\forall (a, b) \in A$, $(a, b) * (0, 0) = (a + 0, b + 0) = (a, b)$

and $(0, 0) * (a, b) = (0 + a, 0 + b) = (a, b)$

$\Rightarrow$ $(0, 0)$ acts as an identity element in A w.r.t. $*$.

7. We have, $A = \{x \in Z : 0 \leq x \leq 12\}$

$\therefore$ $A = \{0, 1, 2, 3, ..., 12\}$

and $S = \{(a, b) : |a - b|$ is a multiple of $4\}$

(i) Reflexive : For any $a \in A$,

$|a - a| = 0$ is a multiple of 4.

Thus, $(a, a) \in S$ $\therefore$ S is reflexive.

(ii) Symmetric : For any $a, b \in A$,

Let $(a, b) \in S \Rightarrow |a - b|$ is a multiple of 4

$\Rightarrow |b - a|$ is a multiple of 4 $\Rightarrow (b, a) \in S$

i.e., $(a, b) \in S \Rightarrow (b, a) \in S$ $\therefore$ S is symmetric.

(iii) Transitive : For any $a, b, c \in A$,

Let $(a, b) \in S$ and $(b, c) \in S$

$\Rightarrow |a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4

$\Rightarrow a - b = \pm 4k_1$ and $b - c = \pm 4k_2$; $k_1, k_2 \in N$

$\Rightarrow (a - b) + (b - c) = \pm 4(k_1 + k_2)$; $k_1, k_2 \in N$

$\Rightarrow a - c = \pm 4(k_1 + k_2)$; $k_1, k_2 \in N$

$\Rightarrow |a - c|$ is a multiple of 4 $\Rightarrow (a, c) \in S$

$\therefore$ S is transitive.

Hence, S is an equivalence relation.

The set of elements related to 1 is $\{5, 9\}$.

8. (i) Reflexive : Let (a, b) be an arbitrary element of $N \times N$

$\Rightarrow a + b = b + a$

$\Rightarrow (a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$

So, R is reflexive.

(ii) Symmetry: Let $(a, b), (c, d) \in N \times N$ such that

$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow b + c = a + d$

$\Rightarrow c + b = d + a \Rightarrow (c, d)\ R\ (a, b)$.

Thus, $(a, b)\ R\ (c, d)$

$\Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$.

So, R is symmetric.

(iii) Transitive: Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

Now, $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ ...(i)

and $(c, d)\ R\ (e, f) \Rightarrow c + f = d + e$ ...(ii)

Adding (i) and (ii), we get

$(a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$.

So, R is transitive.

$\therefore$ R is an equivalence relation.

9. Here $a * b = 3a + 4b - 2$

$\therefore$ $4 * 5 = 3(4) + 4(5) - 2 = 12 + 20 - 2 = 30$

10. We have, $S = Q - \{1\}$

$a * b = a + b - ab\ \forall a, b \in S$

(i) As $a, b \in S \Rightarrow a, b \in Q$ and $a \neq 1, b \neq 1$ ...(1)

$\therefore$ $a + b - ab \in Q$

We check : $a + b - ab \neq 1$

Suppose $a + b - ab = 1 \Rightarrow a + b - ab - 1 = 0$

$\Rightarrow (1 - a)(-1 + b) = 0$

$\Rightarrow$ Either $1 - a = 0$ or $-1 + b = 0$

$\Rightarrow a = 1$ or $b = 1$

This contradicts (1).

$\therefore$ $a + b - ab \neq 1 \Rightarrow a + b - ab \in Q - \{1\} = S$

$\Rightarrow$ $*$ is binary operation on S.

(ii) Let $a, b \in S$

$a * b = a + b - a{\cdot}b = b + a - b{\cdot}a = b * a$, $*$ is commutative in S.

Let $a,b,c \in S$

Then $a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc) = a + b + c - ab - bc - ca + abc$

$= a + b - ab + c - (a + b - ab)\,c = (a * b) * c$

$\therefore$ $*$ is associative

11. Let $\tan\frac{x}{2}\cdot\tan\frac{y}{2} = \tan\theta$ ...(i)

Consider,

$$\frac{\cos x+\cos y}{1+\cos x\cdot\cos y}=\frac{\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}+\dfrac{1-\tan^2(y/2)}{1+\tan^2(y/2)}}{1+\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\cdot\dfrac{1-\tan^2(y/2)}{1+\tan^2(y/2)}}$$

$$=\frac{(1-\tan^2(x/2))\,(1+\tan^2(y/2)) + (1+\tan^2(x/2))\,(1-\tan^2(y/2))}{(1+\tan^2(x/2))\,(1+\tan^2(y/2)) + (1-\tan^2(x/2))\cdot(1-\tan^2(y/2))}$$

$$=\frac{1-\tan^2(x/2)+\tan^2(y/2)-\tan^2(x/2)\tan^2(y/2)+1-\tan^2(y/2)+\tan^2(x/2)-\tan^2(x/2)\tan^2(y/2)}{1+\tan^2(x/2)+\tan^2(y/2)+\tan^2(x/2)\tan^2(y/2)+1-\tan^2(x/2)-\tan^2(y/2)+\tan^2(x/2)\cdot\tan^2(y/2)}$$

$$= \frac{2-2\tan^2(x/2)\tan^2(y/2)}{2+2\tan^2(x/2)\tan^2(y/2)} = \frac{1-\tan^2\theta}{1+\tan^2\theta} \quad \text{\{from (i)\}}$$

$= \cos 2\theta$

$\therefore$ L.H.S. $= \cos^{-1}(\cos 2\theta) = 2\theta$

$$= 2\left\{\tan^{-1}\left(\tan\frac{x}{2}\cdot\tan\frac{y}{2}\right)\right\} = \text{R.H.S.}$$

12. $\cot^{-1}\left(\dfrac{y}{\sqrt{1-x^2-y^2}}\right) = \tan^{-1}\left(\dfrac{\sqrt{1-x^2-y^2}}{y}\right)$

also, $2\tan^{-1}\sqrt{\dfrac{3-4x^2}{4x^2}} = \tan^{-1}\left(\dfrac{4x\sqrt{3-4x^2}}{8x^2-3}\right)$

Hence the given equation is,

$$\tan^{-1}\frac{\sqrt{1-x^2-y^2}}{y} = \tan^{-1}\left(\frac{4x\sqrt{3-4x^2}}{8x^2-3}\right) - \tan^{-1}\sqrt{\frac{3-4x^2}{x^2}}$$

$$= \tan^{-1}\left[\frac{\dfrac{4x\sqrt{3-4x^2}}{8x^2-3} - \sqrt{\dfrac{3-4x^2}{x^2}}}{1+\dfrac{4x\sqrt{3-4x^2}}{8x^2-3}\sqrt{\dfrac{3-4x^2}{x^2}}}\right]$$

$$= \tan^{-1}\left[\frac{(3-4x^2)^{3/2}}{9x-8x^3}\right]$$

$$\therefore \frac{\sqrt{1-x^2-y^2}}{y} = \frac{(3-4x^2)^{3/2}}{9x-8x^3}$$

Squaring and simplifying, we get

$$\frac{1-x^2}{y^2} = 1+\frac{(3-4x^2)^3}{(9x-8x^3)^2} = \frac{27-27x^2}{(9x-8x^3)^2}$$

$$\Rightarrow y^2 = \frac{x^2(9-8x^2)^2}{27}$$

13. Here, $x = \text{cosec}\,(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$

$$\Rightarrow x = \text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$\Rightarrow x = \text{cosec}\left(\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$$

$$\Rightarrow x = \sqrt{3-a^2} \quad \text{...(i)}$$

and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1} a)))))$

$$\Rightarrow y = \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$\Rightarrow y = \sec\left(\cot^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$$

$$\Rightarrow y = \sqrt{3-a^2} \quad \text{...(ii)}$$

From (i) and (ii), $x = y = \sqrt{3-a^2}$

14. Let $\tan^{-1} x = \theta$ for $x \geq 0$

Case I : When $0 \leq x < 1$, then $0 \leq \theta < \pi/4$ and so $0 \leq 2\theta < \pi/2$

$\therefore \sin^{-1}\dfrac{2x}{1+x^2} = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x$

$\cos^{-1}\dfrac{1-x^2}{1+x^2} = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$

$\tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1} x$

Thus,

$$3\left(\sin^{-1}\frac{2x}{1+x^2}\right) - 4\left(\cos^{-1}\frac{1-x^2}{1+x^2}\right) + 2\left(\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{\pi}{3},$$

$\Rightarrow 3(2\tan^{-1}x) - 4(2\tan^{-1} x) + 2(2\tan^{-1} x) = \pi/3$

$\Rightarrow 2\tan^{-1} x = \pi/3 \Rightarrow \tan^{-1}x = \pi/6 \Rightarrow x = 1/\sqrt{3}$

as $0 \leq \dfrac{1}{\sqrt{3}} < 1$, $x = \dfrac{1}{\sqrt{3}}$ is a solution.

Case II : When $x = 1$, $\tan^{-1}\dfrac{2x}{1-x^2}$ is not defined.

$\therefore$ $x = 1$, cannot be a solution.

Case III : If $x > 1$, then $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$ and so $\dfrac{\pi}{2} < 2\theta < \pi$

$\therefore \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \sin^{-1}(\sin 2\theta) = \pi - 2\theta = \pi - 2\tan^{-1} x.$

$\cos^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1} x.$

and $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}(\tan 2\theta) = 2\theta - \pi = 2\tan^{-1} x - \pi$

Thus the given equation becomes,

$$3(\pi - 2\tan^{-1} x) - 4(2\tan^{-1} x) + 2(2\tan^{-1} x - \pi) = \frac{\pi}{3}$$

$\Rightarrow \pi - 10\tan^{-1} x = \dfrac{\pi}{3}$ or $\tan^{-1}x = \dfrac{\pi}{15}$

i.e., $x = \tan\pi/15 < \tan\pi/4 < 1.$

$\therefore$ $x = \tan\pi/15$ is not a solution.

$\therefore$ $x = 1/\sqrt{3}$ is the only solution for the given equation for $x \geq 0$.

15. We have, $\sin(\sin^{-1} 6x) = \sin(-\sin^{-1} 6\sqrt{3}x - \pi/2)$

$\Rightarrow 6x = -\sin(\sin^{-1} 6\sqrt{3}x + \sin^{-1} 1)$

$\Rightarrow 6x = -\sin(\sin^{-1}\sqrt{1-108x^2})$

$$\Rightarrow 6x = -\sqrt{1-108x^2} \quad \text{...(i)}$$

Squaring both sides, we get

$36x^2 = 1 - 108x^2 \Rightarrow 144x^2 = 1$

$\Rightarrow x = 1/12$ and $x = -1/12$

If $x = \dfrac{1}{12}$, we get

$$\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{6}-\frac{\pi}{3}=-\frac{\pi}{2}$$

Thus $x = 1/12$ is the root of given equation. But, when substituting in (i), we get

L.H.S $\Rightarrow 6x = \dfrac{1}{2}$, R.H.S $\Rightarrow -\sqrt{1-108x^2} = -1/2$

Thus L.H.S ≠ R.H.S of equation (i)

Thus $x = -1/12$ is a root of the given equation as it satisfy both given and equation (i).

16. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$=2\left(2\tan^{-1}\frac{1}{5}\right)-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$$

$$=2\left\{\tan^{-1}\frac{2\,(1/5)}{1-(1/5)^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$$

$$=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}$$

$$=\tan^{-1}\left(\frac{2\,(5/12)}{1-(5/12)^2}\right)-\tan^{-1}\left(\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right)$$

$$=\tan^{-1}\left(\frac{120}{119}\right)-\tan^{-1}\left(\frac{1}{239}\right)$$

$$=\tan^{-1}\left(\frac{\frac{120}{119}-\frac{1}{239}}{1+\left(\frac{120}{119}\right)\left(\frac{1}{239}\right)}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

17. We have $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1}x + \cot^{-1} x)^2 - 2\tan^{-1}x\cdot\cot^{-1}x = \dfrac{5\pi^2}{8}$

$\Rightarrow \left(\dfrac{\pi}{2}\right)^2 - 2\tan^{-1}x\cdot(\pi/2 - \tan^{-1}x) = \dfrac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \dfrac{3\pi^2}{8} = 0$

$\Rightarrow \tan^{-1}x = -\pi/4,\ 3\pi/4 \Rightarrow \tan^{-1}x = -\pi/4 \Rightarrow x = -1$

18. Here, $\tan^{-1}x + \cos^{-1}\dfrac{y}{\sqrt{1+y^2}} = \sin^{-1}\dfrac{3}{\sqrt{10}}$.

$\Rightarrow \tan^{-1}x + \tan^{-1}\left(\dfrac{1}{y}\right) = \tan^{-1}(3)$

or $\tan^{-1}\left(\dfrac{1}{y}\right) = \tan^{-1}(3) - \tan^{-1}(x)$

or $\tan^{-1}\left(\dfrac{1}{y}\right) = \tan^{-1}\left(\dfrac{3-x}{1+3x}\right) \Rightarrow y = \dfrac{1+3x}{3-x}$

As x, y are positive integers, $x = 1, 2$ and corresponding $y = 2, 7$.

∴ Solutions are $(x, y) = (1, 2), (2, 7)$ (*i.e.,*) two solutionss.

19. Let $\sin^{-1} x = y$, we get

$2y^2 - y - 6 = 0 \Rightarrow y = 2$ and $y = -1.5$

∴ $\sin^{-1} x = 2$ and $\sin^{-1} x = -1.5$

Since $2 > \pi/2$ and $|-1.5| < \pi/2$, the only solution is $x = \sin(-1.5)$.

20. $A^2 - 5A + 4I$

$$=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}-5\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}+4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}-\begin{bmatrix}10&0&5\\10&5&15\\5&-5&0\end{bmatrix}+\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$=\begin{bmatrix}-1&-1&-3\\-1&-3&-10\\-5&4&2\end{bmatrix}$$

Since, $A^2 - 5A + 4I + X = O \Rightarrow X = -(A^2 - 5A + 4I)$

$$\therefore\ X=\begin{bmatrix}1&1&3\\1&3&10\\5&-4&-2\end{bmatrix}$$

21. Trust fund = ₹ 35,000.

Let ₹ x be invested in the first bond and then ₹ $(35{,}000 - x)$ will be invested in the second bond.

Interest paid on the first bond = 8% = 0.08

Interest paid on the second bond = 10% = 0.10

Total annual interest = ₹ 3,200.

∴ In matrices, $[x \ \ 35{,}000-x]\begin{bmatrix}0.08\\0.10\end{bmatrix}=[3{,}200]$

$\Rightarrow x \times 0.08 + (35{,}000 - x) \times 0.10 = 3{,}200$

$\Rightarrow 8x + 3{,}50{,}000 - 10x = 3{,}20{,}000$

$\Rightarrow 2x = 30{,}000 \Rightarrow x = 15{,}000$

∴ ₹ 15,000 should be invested in the first bond and ₹ 35,000 – ₹ 15,000 = ₹ 20,000 be invested in the second bond.

The values reflected in this question are :

(i) Spirit of investment.

(ii) Giving charity to cancer patients.

(iii) Helping the orphans living in the society.

22. We have, $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$

Consider, $(A + B) = \begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}+\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}=\begin{bmatrix}1+a & 0\\ 2+b & -2\end{bmatrix}$

Now, $(A + B)^2 = \begin{bmatrix}1+a & 0\\ 2+b & -2\end{bmatrix}\begin{bmatrix}1+a & 0\\ 2+b & -2\end{bmatrix}$

$=\begin{bmatrix}(1+a)^2 & 0\\ (2+b)(1+a-2) & 4\end{bmatrix}=\begin{bmatrix}(1+a)^2 & 0\\ (2+b)(a-1) & 4\end{bmatrix}$

Now, $A^2 = \begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

and $B^2 = \begin{bmatrix}a & 1\\ b & -1\end{bmatrix}\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}=\begin{bmatrix}a^2+b & a-1\\ ab-b & b+1\end{bmatrix}$

$\therefore\quad A^2 + B^2 = \begin{bmatrix}a^2+b-1 & a-1\\ ab-b & b\end{bmatrix}$

It is given that $(A + B)^2 = A^2 + B^2$

$\therefore\quad \begin{bmatrix}(1+a)^2 & 0\\ (2+b)(a-1) & 4\end{bmatrix}=\begin{bmatrix}a^2+b-1 & a-1\\ ab-b & b\end{bmatrix}$

By equality of matrices, comparing the corresponding elements, we get

$a - 1 = 0 \Rightarrow a = 1$ and $b = 4$

Also, $(1 + a)^2 = a^2 + b - 1$ and $(2 + b)(a - 1) = ab - b$ satisfied by $a = 1$ and $b = 4$

23. We know that, $A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$

Here, $\frac{1}{2}(A+A')$ is symmetric matrix and $\frac{1}{2}(A-A')$ is skew symmetric matrix.

Now, $A=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}\Rightarrow A'=\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}$

$\therefore\quad \frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}6 & 1 & -5\\ 1 & -4 & -4\\ -5 & -4 & 4\end{bmatrix}$, which is symmetric.

and $\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0 & -5 & -3\\ 5 & 0 & -6\\ 3 & 6 & 0\end{bmatrix}$ which is skew symmetric.

$\therefore\quad A=\frac{1}{2}\begin{bmatrix}6 & 1 & -5\\ 1 & -4 & -4\\ -5 & -4 & 4\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & -5 & -3\\ 5 & 0 & -6\\ 3 & 6 & 0\end{bmatrix}$

$\Rightarrow\quad A=\begin{bmatrix}3 & 1/2 & -5/2\\ 1/2 & -2 & -2\\ -5/2 & -2 & 2\end{bmatrix}+\begin{bmatrix}0 & -5/2 & -3/2\\ 5/2 & 0 & -3\\ 3/2 & 3 & 0\end{bmatrix}$

24. We have $A = IA$

$\Rightarrow\quad \begin{bmatrix}-1 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 3R_1$, we get

$$\begin{bmatrix}-1 & 1 & 2\\ 0 & 3 & 5\\ 0 & 4 & 7\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & -2\\ 0 & 4 & 7\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & -1\\ 3 & 0 & 1\end{bmatrix}A$$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & -2\\ 0 & 4 & 7\end{bmatrix}=\begin{bmatrix}-1 & 1 & -1\\ -2 & 1 & -1\\ 3 & 0 & 1\end{bmatrix}A$$

Applying $R_3 \to R_3 + 4R_2$, we get

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & -2\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}-1 & 1 & -1\\ -2 & 1 & -1\\ -5 & 4 & -3\end{bmatrix}A$$

Applying $R_2 \to R_2 - 2R_3$, we get

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}-1 & 1 & -1\\ 8 & -7 & 5\\ -5 & 4 & -3\end{bmatrix}A$$

Applying $R_1 \to (-1)R_1$, $R_2 \to (-1)R_2$, $R_3 \to (-1)R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & 1\\ -8 & 7 & -5\\ 5 & -4 & 3\end{bmatrix}A$$

Hence, $A^{-1}=\begin{bmatrix}1 & -1 & 1\\ -8 & 7 & -5\\ 5 & -4 & 3\end{bmatrix}$

25. L.H.S. $=\Delta=\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix}$

Taking a, b, c common from C_1, C_2, C_3 respectively, we get

$$\Delta = abc\begin{vmatrix}a & c & a+c\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$$

Applying $C_3 \to C_3 - C_1 - C_2$, we get

$$\Delta = abc\begin{vmatrix}a & c & 0\\ a+b & b & -2b\\ b & b+c & -2b\end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$, we get

$$\Delta = abc\begin{vmatrix} a & c & 0 \\ a & -c & 0 \\ b & b+c & -2b \end{vmatrix}$$

$$= abc(-2b)(-ac-ac) = -2ab^2c(-2ac)$$
$$= 4a^2b^2c^2 = \text{R.H.S.}$$

26. $\text{L.H.S.} = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$, we get

$$\begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Taking -2 common from R_1, we get

$$(-2)\begin{vmatrix} 0 & c & b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $C_2 \to \frac{1}{c}C_2, C_3 \to \frac{1}{b}C_3$, we get

$$(-2)\,bc\begin{vmatrix} 0 & 1 & 1 \\ b & 1+a/c & 1 \\ c & 1 & a/b+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$, we get

$$-2bc\begin{vmatrix} 0 & 0 & 1 \\ b & a/c & 1 \\ c & -a/b & a/b+1 \end{vmatrix}$$

$$= -2bc(-a-a) = 4abc = \text{R.H.S.}$$

27. $\text{L.H.S.} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & cb & c^2+1 \end{vmatrix}$

Applying $R_1 \to \frac{1}{a}R_1,\ R_2 \to \frac{1}{b}R_2,\ R_3 \to \frac{1}{c}R_3$, we get

$$abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1,\ C_2 \to bC_2,\ C_3 \to cC_3$, we get

$$\frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} a^2+b^2+c^2+1 & b^2 & c^2 \\ a^2+b^2+c^2+1 & b^2+1 & c^2 \\ a^2+b^2+c^2+1 & b^2 & c^2+1 \end{vmatrix}$$

Taking $(a^2 + b^2 + c^2 + 1)$ common from C_1, we get

$$(a^2+b^2+c^2+1)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$, we get

$$(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2)(1)(1 - 0)$$
$$= (1 + a^2 + b^2 + c^2) = \text{R.H.S.}$$

28. $\text{L.H.S.} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix}$

Applying $R_3 \to R_3 + R_1$, we get

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

Taking $(x + y + z)$ common from R_3, we get

$$(x+y+z)\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$, we get

$$(x+y+z)\begin{vmatrix} x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \\ 0 & 0 & 1 \end{vmatrix}$$

Taking $(x - y)$ and $(y - z)$ common from C_1 and C_2 respectively, we get

$$(x+y+z)(x-y)(y-z)\begin{vmatrix} 1 & 1 & z \\ x+y & y+z & z^2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x + y + z)(x - y)(y - z)[(y + z) - (x + y)]$$
$$= (x - y)(y - z)(z - x)(x + y + z) = \text{R.H.S.}$$

29. Let $\Delta = \begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} -(x^2+y^2+z^2-xy-yz-zx) & zx-y^2 & xy-z^2 \\ -(x^2+y^2+z^2-xy-yz-zx) & xy-z^2 & yz-x^2 \\ -(x^2+y^2+z^2-xy-yz-zx) & yz-x^2 & zx-y^2 \end{vmatrix}$$

Taking $-(x^2 + y^2 + z^2 - xy - yz - zx)$ common from C_1, we get

$$\Delta = -(x^2 + y^2 + z^2 - xy - yz - zx) \times \begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 1 & xy-z^2 & yz-x^2 \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$$\Delta = -(x^2 + y^2 + z^2 - xy - yz - zx) \times \begin{vmatrix} 0 & (x-y)(x+y+z) & (y-z)(x+y+z) \\ 0 & (x-z)(x+y+z) & (y-x)(x+y+z) \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

Taking $(x + y + z)$ common from R_1 and R_2 both, we get

$$\Delta = -(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)^2 \times \begin{vmatrix} 0 & x-y & y-z \\ 0 & x-z & y-x \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

$\Rightarrow\ \Delta = -(x + y + z)(x^3 + y^3 + z^3 - 3xyz)$
$[(x - y)(y - x) - (x - z)(y - z)]$

$\Rightarrow\ \Delta = -(x + y + z)(x^3 + y^3 + z^3 - 3xyz)$
$(xy + yz + zx - x^2 - y^2 - z^2)$

$\Rightarrow\ \Delta = (x + y + z)(x^3 + y^3 + z^3 - 3xyz)$
$(x^2 + y^2 + z^2 - xy - yz - zx)$

Hence, Δ is divisible by $(x + y + z)$ and quotient is
$(x^3 + y^3 + z^3 - 3xyz)(x^2 + y^2 + z^2 - xy - yz - zx)$

30. Using determinants, the line joining $A(1, 3)$ and $B(0,0)$ is given by $\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1(3x - y) = 0 \Rightarrow y = 3x$

Now, $D(k, 0)$ is a point s.t. area of ΔABD = 3 sq. units

$\therefore\ \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = 3 \Rightarrow -(0 - 3k) = \pm 6 \Rightarrow k = \pm 2$

31. (i) Given, value of prize for team spirit = ₹ x
Value of prize for truthfulness = ₹ y
Value of prize for tolerance = ₹ z
Linear equation for School A is $3x + y + 2z = 1100$
Linear equation for School B is $x + 2y + 3z = 1400$
Linear equation for Prize is $x + y + z = 600$
The corresponding matrix equation is $PX = Q$

where, $P = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $Q = \begin{bmatrix} 1100 \\ 1400 \\ 600 \end{bmatrix}$

(ii) Now $|P| = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -3 + 2 - 2 = -3 \neq 0$

Thus, P^{-1} exists. So, system of equations has unique solution and it is given by $X = P^{-1}Q$

Now, $\text{adj } P = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}$

$$P^{-1} = \frac{1}{|P|} \cdot \text{adj } P = -\frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}$$

Now, $X = P^{-1}Q$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1100 \\ 1400 \\ 600 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -300 \\ -600 \\ -900 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$\Rightarrow\ x = 100, y = 200, z = 300.$

Thus, the above system of equations is solvable.

(iii) The value truthfulness should be rewarded the most because a student who is truthful will be also tolerant and will work with a team spirit in the school.

32. Let the monthly income of Aryan be ₹ $3x$ and that of Babban be ₹ $4x$

Also, let monthly expenditure of Aryan be ₹ $5y$ and that of Babban be ₹ $7y$

According to question,

$3x - 5y = 15000$
$4x - 7y = 15000$

These equations can be written as

$AX = B$

where, $A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = (-21 + 20) = -1 \neq 0$$

Thus, A^{-1} exists. So, system of equations has a unique solution and given by $X = A^{-1}B$

Now, $\text{adj}(A) = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$$

Now, $X = A^{-1}B = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix}$

$\Rightarrow$ $x = 30000$ and $y = 15000$

So, monthly income of Aryan = 3 × 30000 = ₹90000

Monthly income of Babban = 4 × 30000 = ₹120000

From this question we are encouraged to save a part of money every month.

33. According to question, we have

$x + y + z = 12000$

$4x + 3y + 2z = 37000$

$5x + 3y + 4z = 47000$

The system of equations can be written as $AX = B$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 5 & 3 & 4 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 12000 \\ 37000 \\ 47000 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 5 & 3 & 4 \end{vmatrix} = 6 - 6 - 3 = -3 \neq 0$$

$\therefore$ A^{-1} exists. So, system of equations has a unique solution and it is given by $X = A^{-1}B$

$$\text{adj } A = \begin{bmatrix} 6 & -1 & -1 \\ -6 & -1 & 2 \\ -3 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\cdot \text{adj } A = -\frac{1}{3}\begin{bmatrix} 6 & -1 & -1 \\ -6 & -1 & 2 \\ -3 & 2 & -1 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 6 & -1 & -1 \\ -6 & -1 & 2 \\ -3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 12000 \\ 37000 \\ 47000 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -12000 \\ -15000 \\ -9000 \end{bmatrix} = \begin{bmatrix} 4000 \\ 5000 \\ 3000 \end{bmatrix}$$

$\Rightarrow$ $x = 4000$, $y = 5000$; $z = 3000$

The values described in this question are resourcefulness, competence and determination.

34. Here, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = -6 + 28 + 45 = 67 \neq 0$$

$\therefore$ A^{-1} exists.

Now, $\text{adj } A = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\cdot \text{adj } A = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

The given system of equations is

$x + 2y - 3z = -4$

$2x + 3y + 2z = 2$

$3x - 3y - 4z = 11$

The system of equations can be written as $AX = B$

where, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

$\therefore$ A^{-1} exists, so system of equations has a unique solution given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $x = 3$, $y = -2$, $z = 1$.

35. Given equations can be written as $AX = B$

where, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 2 - 4 + 5 = 3 \neq 0$$

$\therefore$ A^{-1} exists.

Now, $\text{adj } A = \begin{bmatrix} 1 & 1 & 0 \\ -4 & -1 & 3 \\ 5 & -1 & -3 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -4 & -1 & 3 \\ 5 & -1 & -3 \end{bmatrix}$$

Since A^{-1} exists. So, system of equations has a unique solution given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -4 & -1 & 3 \\ 5 & -1 & -3 \end{bmatrix}\begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 1$, $y = 2$, $z = 3$.

CLASS XII | Series 3

ACE YOUR WAY CBSE

Matrices and Determinants

Important Formulae

MATRICES

- If A, B and C are any three matrices , then
  - Commutative law : $A + B = B + A$
  - Associative law : $(A + B) + C = A + (B + C)$
    $(AB)C = A(BC)$
  - Distributive law : $A(B + C) = AB + AC$
    $(A + B)C = AC + BC$
  - Existence of identity : $A + O = A = O + A$
    $IA = A = AI$
  - Existence of inverse : $A + (–A) = O = (–A) + A$
- If A and B are comparable matrices and k is a scalar, then $k(A + B) = (kA + kB)$
- If k_1, k_2 are scalars and A is any matrix, then
  - $(k_1 + k_2)A = (k_1A + k_2A)$
  - $k_1(k_2A) = (k_1k_2)A$
- If A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix, then AB exists and it is an $(m \times p)$ matrix.
- $(i, k)^{th}$ element of AB = sum of the products of corresponding elements of i^{th} row of A and k^{th} column of B.
- Properties of Transpose
  - $(A^T)^T = A$
  - $(A+B)^T = A^T+B^T$, where A and B are of same order
  - $(kA)^T = kA^T$, where k be any scalar (real or complex)
  - $(AB)^T = B^T A^T$
  - $(ABC)^T = C^T B^T A^T$
  - A is a symmetric matrix if $A^T = A$.
  - A is a skew symmetric matrix if $A^T = – A$.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- Elementary operations of a matrix are as follows :
  - $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
  - $R_i \to kR_i$ or $C_i \to kC_i$
  - $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$
- If A and B are two square matrices such that $AB = BA = I$, then B is the inverse matrix of A and is denoted by A^{-1}.
- Inverse of a square matrix, if it exists, is unique.

DETERMINANTS

- Determinant of matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(expanding along R_1)

- For square matrix A, $|A|$ satisfy the following properties:
  - $|A^T| = |A|$, where A^T = transpose of A.
  - If we interchange any two rows (or columns), then sign of determinant changes.
  - If any two rows or any two columns are identical or proportional, then value of determinant is zero.
  - If we multiply each element of a row or a column of a determinant by constant k, then value of determinant is multiplied by k.
  - Multiplying a determinant by k means multiply elements of only one row (or one column) by k.

- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can also be expressed as sum of two or more determinants.
- If in each element of a row or a column of a determinant, the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.
- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element a_{ij} of the matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}.
- Cofactor of a_{ij} is given by $A_{ij} = (-1)^{i+j} M_{ij}$
- $\det(AB) = \det A \cdot \det B$, where A and B are matrices of same order.
- $\det(kA) = k^n \det A$, if A is of order $n \times n$
- $\det(A^n) = (\det A)^n$ if $n \in I^+$
- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is cofactor of a_{ij}

- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$ respectively.
- Properties of adjoint
  - $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$
  - $\text{adj }(AB) = (\text{adj } B)\cdot(\text{adj } A)$
  - $|\text{adj } A| = |A|^{n-1}$
  - $|\text{adj }(\text{adj } A)| = |A|^{(n-1)^2}$
  - $|\text{adj } \lambda A| = \lambda^{n-1} \text{ adj } A$
  - $\text{adj }(\text{adj } A) = |A|^{n-2}A$
  - $\text{adj }(A^T) = (\text{adj } A)^T$
  - $\text{adj }(O) = O$, $(\text{adj } I) = I$
  - If A is symmetric, then adj (A) is symmetric.
- If $AB = BA = I$, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ or $B^{-1} = A$ and hence $(A^{-1})^{-1} = A$.
- A square matrix A has inverse if and only if A is non-singular.
- Properties of Inverse
  - $A^{-1} = \dfrac{(\text{adj } A)}{|A|}$, for non-singular square matrix A
  - $AA^{-1} = A^{-1}A = I_n$
  - $(kA)^{-1} = k^{-1}A^{-1}$ if $k \neq 0$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(ABC\ldots Z)^{-1} = Z^{-1}Y^{-1}\ldots B^{-1}A^{-1}$
  - $(A^T)^{-1} = (A^{-1})^T$
  - If $A = \text{diag.}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then
    $A^{-1} = \text{diag.}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})$ when $\lambda_i \neq 0\ \forall\ i$
- A system of equation is consistent or inconsistent according as its solution exists or not.
- If $a_1 x + b_1 y + c_1 z = d_1$
  $a_2 x + b_2 y + c_2 z = d_2$
  $a_3 x + b_3 y + c_3 z = d_3$
  then these equations can be written as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

  - If $|A| \neq 0$, there exists unique solution given by $X = A^{-1}B$.
  - If $|A| = 0$ and $(\text{adj } A)B \neq O$, then there exists no solution.
  - If $|A| = 0$ and $(\text{adj } A)B = O$, then system may or may not be consistent.
- Cramer's Rule : Let there be a system of n simultaneous linear equations in 'n' unknowns given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ and let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$.

Then, $x_1 = \dfrac{D_1}{D}, x_2 = \dfrac{D_2}{D}, \ldots, x_n = \dfrac{D_n}{D}$, provided that $D \neq 0$.

  - If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by
    $x = \dfrac{D_1}{D}, x_2 = \dfrac{D_2}{D}, x_3 = \dfrac{D_3}{D}, \ldots, x_n = \dfrac{D_n}{D}$
  - If $D = 0$ and $D_1 = D_2 = D_3 = \ldots = D_n = 0$, then the given system of equations is consistent, with infinitely many solutions.
  - If $D = 0$ and at least one of the determinants $D_1, D_2, D_3, \ldots, D_n$ is non-zero, then the given system of equations is inconsistent and have no solution.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Construct a 3 × 2 matrix A, whose elements are given by $a_{ij} = \dfrac{(i+2j)^2}{2}$.

2. If a, b, c are in A.P., prove that $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ = 0.

3. Evaluate $\begin{vmatrix} 3 & 7 & 13 \\ -5 & 0 & 0 \\ 0 & 11 & -2 \end{vmatrix}$.

4. If $A = \begin{vmatrix} x & 2 \\ 2 & x \end{vmatrix}$ and $|A^4| = 625$, find the value(s) of x.

5. Find the additive inverse of $A + B$ where A and B are given as $A = \begin{bmatrix} 2 & 5 \\ 9 & 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 3 & -9 \end{bmatrix}$

SHORT ANSWER TYPE

6. Find the values of a and b for which the following hold :

$$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

7. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p.

8. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find the determinant of the matrix $3A^2 - 2B$.

9. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, write A^{-1} in terms of A.

10. If the points (a_1, b_1), (a_2, b_2) and $(a_1 + a_2, b_1 + b_2)$ are collinear then using determinants, show that $a_1b_2 = a_2b_1$.

LONG ANSWER TYPE - I

11. If $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, then calculate AC, BC and $(A + B)C$. Also verify that $(A + B)C = AC + BC$.

12. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, find A^{-1} using elementary row transformations.

13. Using properties of determinants, prove that :

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

14. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ for matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

15. Solve the following system of equations by using matrix method.

$3x + 4y = 7$

$6x + 8y = 14$

LONG ANSWER TYPE - II

16. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$ and $f(x) = x^2 - 5x - 14$, find $f(A)$. Hence obtain A^3.

17. Express the matrix $\begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrix.

18. Using properties of determinants, show that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = a^2 + b^2 + c^2 + 1.$$

19. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for helping others and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, helping others and supervision, suggest one more value which the management of the colony must include for awards.

20. If $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}$, find AB.

Hence solve the system of equations :

$x-y=3,\ 2x+3y+4z=17,\ y+2z=7$

SOLUTIONS

1. $a_{ij}=\dfrac{(i+2j)^2}{2}$; $i=1, 2, 3$ and $j=1, 2$

$$a_{11}=\frac{(1+2\cdot1)^2}{2}=\frac{9}{2},\ a_{12}=\frac{(1+2\cdot2)^2}{2}=\frac{25}{2};$$

$$a_{21}=\frac{(2+2\cdot1)^2}{2}=8,\ a_{22}=\frac{(2+2\cdot2)^2}{2}=18;$$

$$a_{31}=\frac{(3+2\cdot1)^2}{2}=\frac{25}{2},\ a_{32}=\frac{(3+2\cdot2)^2}{2}=\frac{49}{2}.$$

Hence, the required matrix $A=\begin{bmatrix}\frac{9}{2} & \frac{25}{2}\\ 8 & 18\\ \frac{25}{2} & \frac{49}{2}\end{bmatrix}$.

2. Given, a, b, c are in A.P.,

$\therefore\ a+c=2b \Rightarrow a+c-2b=0$ ...(i)

$$\begin{vmatrix}x+1 & x+2 & x+a\\ x+2 & x+3 & x+b\\ x+3 & x+4 & x+c\end{vmatrix}=\begin{vmatrix}0 & 0 & a+c-2b\\ x+2 & x+3 & x+b\\ x+3 & x+4 & x+c\end{vmatrix}$$

(Using $R_1 \to R_1+R_3-2R_2$)

$$=\begin{vmatrix}0 & 0 & 0\\ x+2 & x+3 & x+b\\ x+3 & x+4 & x+c\end{vmatrix}=0 \qquad \text{(Using (i))}$$

3. On expanding along R_2, we get

$$\begin{vmatrix}3 & 7 & 13\\ -5 & 0 & 0\\ 0 & 11 & -2\end{vmatrix}=-(-5)\begin{vmatrix}7 & 13\\ 11 & -2\end{vmatrix}+0\begin{vmatrix}3 & 13\\ 0 & -2\end{vmatrix}-0\begin{vmatrix}3 & 7\\ 0 & 11\end{vmatrix}$$

$=5(-14-143)=-785.$

4. Given, $A=\begin{vmatrix}x & 2\\ 2 & x\end{vmatrix}$ $\therefore\ |A|=x^2-4$

Now, $|A^4|=(|A|)^4=625$

$\Rightarrow\ (x^2-4)^4=625 \Rightarrow (x^2-4)^2=25$

$\Rightarrow\ x^2-4=\pm 5 \Rightarrow x^2-4=5$ or $x^2-4=-5$

$\Rightarrow\ x^2=9$ or $x^2=-1$

$\Rightarrow\ x^2=9$ or $x=3, -3$ ($\because\ x^2\geq 0, \forall x\in R$)

5. Let $C=A+B=\begin{bmatrix}1 & 7\\ 12 & -6\end{bmatrix}$

Now, $(-C)=\begin{bmatrix}-1 & -7\\ -12 & 6\end{bmatrix}$

$\therefore\ (-C)$ is the additive inverse of $A+B$.

6. Given, $\begin{bmatrix}a & b\\ -a & 2b\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}5\\ 4\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}2\cdot a+b\cdot(-1)\\ (-a)\cdot 2+2b\cdot(-1)\end{bmatrix}=\begin{bmatrix}5\\ 4\end{bmatrix} \Rightarrow \begin{bmatrix}2a-b\\ -2a-2b\end{bmatrix}=\begin{bmatrix}5\\ 4\end{bmatrix}$$

$\Rightarrow\ 2a-b=5$ ...(i)

and $-2a-2b=4 \Rightarrow a+b=-2$ ...(ii)

Adding (i) and (ii), we get $3a=3 \Rightarrow a=1$

From (ii), $1+b=-2 \Rightarrow b=-3$

Hence, $a=1, b=-3$.

7. $A^2=\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$

$$=\begin{bmatrix}2\cdot2+(-2)(-2) & 2\cdot(-2)+(-2)\cdot 2\\ (-2)\cdot 2+2\cdot(-2) & (-2)(-2)+2\cdot 2\end{bmatrix}=\begin{bmatrix}8 & -8\\ -8 & 8\end{bmatrix}$$

Given $A^2=pA \Rightarrow \begin{bmatrix}8 & -8\\ -8 & 8\end{bmatrix}=p\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}8 & -8\\ -8 & 8\end{bmatrix}=\begin{bmatrix}2p & -2p\\ -2p & 2p\end{bmatrix}$$

$\Rightarrow\ 8=2p,\ -8=-2p \Rightarrow p=4$

8. $A^2=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}4-3 & -2-2\\ 6+6 & -3+4\end{bmatrix}=\begin{bmatrix}1 & -4\\ 12 & 1\end{bmatrix}$

$\therefore\ 3A^2-2B=3\begin{bmatrix}1 & -4\\ 12 & 1\end{bmatrix}-2\begin{bmatrix}0 & 4\\ -1 & 7\end{bmatrix}$

$$=\begin{bmatrix}3 & -12\\ 36 & 3\end{bmatrix}-\begin{bmatrix}0 & 8\\ -2 & 14\end{bmatrix}$$

$$=\begin{bmatrix}3-0 & -12-8\\ 36-(-2) & 3-14\end{bmatrix}=\begin{bmatrix}3 & -20\\ 38 & -11\end{bmatrix}$$

$\Rightarrow\ |3A^2-2B|=\begin{vmatrix}3 & -20\\ 38 & -11\end{vmatrix}=3\cdot(-11)-38\cdot(-20)$

$=-33+760=727$

9. Given, $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$

$\therefore\ adj\ A=\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$

Now, $|A|=\begin{vmatrix}2 & 3\\ 5 & -2\end{vmatrix}=-4-15=-19\neq 0$

$\Rightarrow\ A^{-1}$ exists

$$\therefore \quad A^{-1}=\frac{1}{|A|}adj\,A=-\frac{1}{19}\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$$

$$=\frac{1}{19}\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}=\frac{1}{19}A$$

10. The given points are collinear

$$\therefore \quad \begin{vmatrix}a_1 & b_1 & 1\\ a_2 & b_2 & 1\\ a_1+a_2 & b_1+b_2 & 1\end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix}a_1 & b_1 & 1\\ a_2-a_1 & b_2-b_1 & 0\\ a_2 & b_2 & 0\end{vmatrix}=0$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$\Rightarrow \begin{vmatrix}a_1 & b_1\\ a_2-a_1 & b_2-b_1\end{vmatrix}=0 \qquad \text{[Expanding along } C_3]$$

[Applying $R_2 \to R_1 + R_2$]

$$\Rightarrow \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=0$$

$\Rightarrow$ $a_1b_2 - a_2b_1 = 0$ $\Rightarrow$ $a_1b_2 = a_2b_1$.

11. $$AC=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$$

$$=\begin{bmatrix}0\cdot 2+6\cdot(-2)+7\cdot 3\\ (-6)\cdot 2+0\cdot(-2)+8\cdot 3\\ 7\cdot 2+(-8)\cdot(-2)+0\cdot 3\end{bmatrix}=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}$$

$$BC=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0\cdot 2+1\cdot(-2)+1\cdot 3\\ 1.2+0.(-2)+2\cdot 3\\ 1\cdot 2+2\cdot(-2)+0\cdot 3\end{bmatrix}=\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}$$

$$A+B=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}+\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0+0 & 6+1 & 7+1\\ -6+1 & 0+0 & 8+2\\ 7+1 & -8+2 & 0+0\end{bmatrix}=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}$$

$$\therefore \quad (A+B)\,C=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$$

$$=\begin{bmatrix}0\cdot 2+7\cdot(-2)+8\cdot 3\\ (-5)\cdot 2+0\cdot(-2)+10\cdot 3\\ 8\cdot 2+(-6)(-2)+0\cdot 3\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix} \quad \text{...(i)}$$

$$\text{Now, } AC+BC=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}+\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}=\begin{bmatrix}9+1\\ 12+8\\ 30-2\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix} \text{...(ii)}$$

From (i) and (ii), we get
$(A + B)C = AC + BC$

12. Given, $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 5 & 3\\ 0 & 2 & 1\end{bmatrix}$, then $A = I_3A$

$$\Rightarrow \begin{bmatrix}1 & -1 & 0\\ 2 & 5 & 3\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Using $R_2 \to R_2 - 2R_1$, we get

$$\Rightarrow \begin{bmatrix}1 & -1 & 0\\ 0 & 7 & 3\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Using $R_2 \to R_2 - 3R_3$, we get

$$\Rightarrow \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}A$$

Using $R_3 \to R_3 - 2R_2$, we get

$$\Rightarrow \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & -3\\ 4 & -2 & 7\end{bmatrix}A$$

Using $R_1 \to R_1 + R_2$, we get

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 1 & -3\\ -2 & 1 & -3\\ 4 & -2 & 7\end{bmatrix}A$$

$$\Rightarrow I_3 = BA, \text{ where } B=\begin{bmatrix}-1 & 1 & -3\\ -2 & 1 & -3\\ 4 & -2 & 7\end{bmatrix}$$

$$\text{Hence, } A^{-1}=\begin{bmatrix}-1 & 1 & -3\\ -2 & 1 & -3\\ 4 & -2 & 7\end{bmatrix}$$

13. Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + R_1$, we get

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} a & b & c \\ -b & -c & -a \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$$= -(a+b+c)\begin{vmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$= -(a+b+c)\begin{vmatrix} a-c & b-c & c \\ b-a & c-a & a \\ 0 & 0 & 1 \end{vmatrix}$$

Now expanding along R_3

$= -(a + b + c)((c - a)(a - c) - (b - a)(b - c))$

$= -(a + b + c)(ac - c^2 - a^2 + ac - (b^2 - bc - ab + ac))$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc$ (Using algebraic identities)

14. We have, $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$$\therefore\ AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} -1 & 5 \\ 5 & -14 \end{vmatrix} = 14 - 25 = -11 \neq 0$$

$\Rightarrow\ (AB)^{-1}$ exists.

$$adj\,(AB) = \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$$

$$\therefore\ (AB)^{-1} = \frac{1}{|AB|}adj\,(AB) = \frac{-1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \quad \text{...(i)}$$

Now, $|A| = \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -8 - 3 = -11 \neq 0 \Rightarrow A^{-1}$ exists

$$adj\,A = \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}adj\,A = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$$

$|B| = \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0 \Rightarrow B^{-1}$ exists.

$$adj\,B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix},\quad \therefore\ B^{-1} = \frac{1}{|B|}adj\,B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\text{Now, } B^{-1}A^{-1} = \frac{1}{11}\left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}\right)$$

$$= \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii), we find that $(AB)^{-1} = B^{-1}A^{-1}$.

15. The given system of equations can be written as

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} \ \textit{i.e.}, AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 24 - 24 = 0$

$\Rightarrow$ The given system may or may not be consistent.

$\therefore\ adj\,A = \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$ and

$$(adj\,A)\,B = \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 56 - 56 \\ -42 + 42 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O$$

$\Rightarrow$ The given system is consistent and has infinitely many solutions.

For the solutions of the given system :

Let $y = k$, where k is any number.

$\therefore$ From (i), we get

$$3x + 4k = 7 \text{ and } 6x + 8k = 14 \Rightarrow x = \frac{1}{3}(7 - 4k) \text{ and}$$

$$x = \frac{1}{6}(14 - 8k) = \frac{1}{3}(7 - 4k)$$

Hence, the solutions of the given system is

$x = \frac{1}{3}(7 - 4k)$, $y = k$, where k is any number

16. We have, $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$$\therefore\ A^2 = AA = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3 + (-5)\cdot(-4) & 3\cdot(-5) + (-5)\cdot 2 \\ -4\cdot 3 + 2\cdot(-4) & -4\cdot(-5) + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$$

Given $f(x) = x^2 - 5x - 14$

$\Rightarrow\ f(A) = A^2 - 5A - 14I_2$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 29-15-14 & -25-(-25)-0 \\ -20-(-20)-0 & 24-10-14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\Rightarrow \; A^2 = 5A + 14I_2$ ...(i)

$\therefore \; A^3 = AA^2 = A(5A + 14I_2)$ (Using (i))

$= A(5A) + A(14I_2) = 5AA + 14(AI_2)$

$= 5A^2 + 14A = 5(5A + 14I_2) + 14A$ (Using (i))

$$= 39A + 70I_2 = 39\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} + 70\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 117 & -195 \\ -156 & 78 \end{bmatrix} + \begin{bmatrix} 70 & 0 \\ 0 & 70 \end{bmatrix} = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$$

17. Let $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & -2 \\ -1 & 7 & 1 \end{bmatrix}$

Therefore, $A + A^T = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & -2 \\ -1 & 7 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 5 & 0 \\ 5 & 10 & 5 \\ 0 & 5 & 2 \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & \frac{5}{2} & 0 \\ \frac{5}{2} & 5 & \frac{5}{2} \\ 0 & \frac{5}{2} & 1 \end{bmatrix}$

Now $P^T = \left[\frac{1}{2}(A + A^T)\right]^T$

$$= \begin{bmatrix} 4 & \frac{5}{2} & 0 \\ \frac{5}{2} & 5 & \frac{5}{2} \\ 0 & \frac{5}{2} & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & \frac{5}{2} & 0 \\ \frac{5}{2} & 5 & \frac{5}{2} \\ 0 & \frac{5}{2} & 1 \end{bmatrix} = P$$

Hence, P is symmetric.

Now, $A - A^T = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & -2 \\ -1 & 7 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4-4 & 2-3 & -1-1 \\ 3-2 & 5-5 & 7+2 \\ 1+1 & -2-7 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 9 \\ 2 & -9 & 0 \end{bmatrix}$$

Let $Q = \frac{A - A^T}{2} = \frac{1}{2}\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 9 \\ 2 & -9 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & \frac{9}{2} \\ 1 & -\frac{9}{2} & 0 \end{bmatrix}$

$$Q^T = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & \frac{9}{2} \\ 1 & -\frac{9}{2} & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -\frac{9}{2} \\ -1 & \frac{9}{2} & 0 \end{bmatrix} = -Q$$

Hence Q is skew symmetric.

Now $\left(\frac{A + A^T}{2}\right) + \left(\frac{A - A^T}{2}\right) = P + Q$

$$= \begin{bmatrix} 4 & \frac{5}{2} & 0 \\ \frac{5}{2} & 5 & \frac{5}{2} \\ 0 & \frac{5}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & \frac{9}{2} \\ 1 & -\frac{9}{2} & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix} = A$$

Hence, $A = \left(\frac{A + A^T}{2}\right) + \left(\frac{A - A^T}{2}\right)$

= symmetric + skew symmetric matrix

18. Using $C_1 \to aC_1$, $C_2 \to bC_2$ and $C_3 \to cC_3$, we get

$$\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^3+a & ab^2 & ac^2 \\ ba^2 & b^3+b & bc^2 \\ ca^2 & cb^2 & c^3+c \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} a^2+b^2+c^2+1 & b^2 & c^2 \\ a^2+b^2+c^2+1 & b^2+1 & c^2 \\ a^2+b^2+c^2+1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (a^2+b^2+c^2+1)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$= (a^2+b^2+c^2+1)\begin{vmatrix}1 & b^2 & c^2\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix}$$

$= (a^2 + b^2 + c^2 + 1) \cdot 1 \cdot (1 - 0)$

$= a^2 + b^2 + c^2 + 1.$

19. According to given conditions, we have

$x + y + z = 12,\ 2x + 3(y + z) = 33,\ x + z = 2y$

i.e., $x + y + z = 12, \quad 2x + 3y + 3z = 33,$

$x - 2y + z = 0.$

The given system of equations can be written as $AX = B$

where $A = \begin{bmatrix}1 & 1 & 1\\2 & 3 & 3\\1 & -2 & 1\end{bmatrix}, X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B = \begin{bmatrix}12\\33\\0\end{bmatrix}$

Now, $|A| = \begin{vmatrix}1 & 1 & 1\\2 & 3 & 3\\1 & -2 & 1\end{vmatrix}$

$= 1\ (3 + 6) - 1\ (2 - 3) + 1\ (-4 - 3)$

$= 9 + 1 - 7 = 3 \neq 0$

$\Rightarrow$ A^{-1} exists and so the given system has a unique solution $X = A^{-1} B$.

Now, $adj\ A = \begin{bmatrix}9 & -3 & 0\\1 & 0 & -1\\-7 & 3 & 1\end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{1}{|A|}adj\ A = \dfrac{1}{3}\begin{bmatrix}9 & -3 & 0\\1 & 0 & -1\\-7 & 3 & 1\end{bmatrix}$

$\therefore \quad X = A^{-1}B = \dfrac{1}{3}\begin{bmatrix}9 & -3 & 0\\1 & 0 & -1\\-7 & 3 & 1\end{bmatrix}\begin{bmatrix}12\\33\\0\end{bmatrix}$

$= \dfrac{1}{3}\begin{bmatrix}108-99+0\\12+0+0\\-84+99+0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \dfrac{1}{3}\begin{bmatrix}9\\12\\15\end{bmatrix} = \begin{bmatrix}3\\4\\5\end{bmatrix} \Rightarrow x = 3, y = 4, z = 5.$

$\therefore$ The number of awardees for honesty = 3, for helping others = 4 and supervising the work = 5. Apart from the values honesty, helping others and supervision, the management of the colony may include the value vigilance because if some members are vigilant they can save the residents of the colony from crimes and mishaps.

20. $AB = \begin{bmatrix}1 & -1 & 0\\2 & 3 & 4\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}2 & 2 & -4\\-4 & 2 & -4\\2 & -1 & 5\end{bmatrix}$

$= \begin{bmatrix}2+4+0 & 2-2+0 & -4+4+0\\4-12+8 & 4+6-4 & -8-12+20\\0-4+4 & 0+2-2 & 0-4+10\end{bmatrix}$

$= \begin{bmatrix}6 & 0 & 0\\0 & 6 & 0\\0 & 0 & 6\end{bmatrix} = 6\,I_3$

$\Rightarrow \quad A\left(\dfrac{1}{6}B\right) = I_3 \Rightarrow A^{-1} = \dfrac{1}{6}B$ (By def. of inverse)

$\Rightarrow \quad A^{-1} = \dfrac{1}{6}\begin{bmatrix}2 & 2 & -4\\-4 & 2 & -4\\2 & -1 & 5\end{bmatrix}$

The given system of equations is

$x - y + 0z = 3$

$2x + 3y + 4z = 17$

$0x + y + 2z = 7$

This system can be written as

$\begin{bmatrix}1 & -1 & 0\\2 & 3 & 4\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\17\\7\end{bmatrix}$

i.e., $AX = C$ where $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $C = \begin{bmatrix}3\\17\\7\end{bmatrix}$

As A^{-1} exists, the given system has a unique solution $X = A^{-1}\,C$

$\Rightarrow \quad X = \dfrac{1}{6}\begin{bmatrix}2 & 2 & -4\\-4 & 2 & -4\\2 & -1 & 5\end{bmatrix}\begin{bmatrix}3\\17\\7\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \dfrac{1}{6}\begin{bmatrix}6+34-28\\-12+34-28\\6-17+35\end{bmatrix} = \dfrac{1}{6}\begin{bmatrix}12\\-6\\24\end{bmatrix} = \begin{bmatrix}2\\-1\\4\end{bmatrix}$

$\Rightarrow \quad x = 2, y = -1, z = 4$

Hence, the solution of the given system of equations is $x = 2, y = -1, z = 4$.

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MPP-3 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Matrices and Determinants

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The rank of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is equal to

(a) 4 (b) 3 (c) 5 (d) 1

2. If $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$= k\,(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then k is equal to

(a) 1 (b) -1 (c) $\alpha\beta$ (d) $\alpha\beta\gamma$

3. For a fixed positive integer n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix},$$

then $\left[\dfrac{D}{(n!)^3} - 4\right]$ is divisible by

(a) $3n$ (b) n^2

(c) n (d) none of these

4. If n is a positive integer, then

$$\begin{vmatrix} {}^{n+2}C_n & {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} \\ {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} \\ {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} & {}^{n+6}C_{n+4} \end{vmatrix} =$$

(a) 3 (b) -1 (c) -5 (d) -9

5. If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = O$, then for any positive integer p, $A^{p+1} = B^k\,[B + (p + 1)\,C]$, k is

(a) $p + 1$ (b) p (c) $p - 1$ (d) $p + 2$

6. If $A + B + C = \pi$, then

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$$ is equal to

(a) 1 (b) 0 (c) -1 (d) 2

One or More Than One Option(s) Correct Type

7. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then

(a) $A^2 - 4A - 5I_3 = O$ (b) $A^{-1} = \frac{1}{5}(A - 4I_3)$

(c) A^3 is not invertible (d) A^2 is invertible

8. The determinant

$$\Delta = \begin{vmatrix} b & c & b\lambda + c \\ c & d & c\lambda + d \\ b\lambda + c & c\lambda + d & a\lambda^3 + 3c\lambda \end{vmatrix}$$ is equal to

zero, if

(a) b, c, d are in A.P. (b) b, c, d, are in G.P.

(c) b, c, d are in H.P.

(d) λ is a root of $ax^3 - bx^2 + cx - d = 0$

9. Suppose $a_1, a_2, \ldots\ldots$ real numbers, with $a_1 \neq 0$. If $a_1, a_2, a_3, \ldots.$ are in A.P. then

(a) $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{bmatrix}$ is singular

(b) the system of equations $a_1x + a_2y + a_3z = 0$, $a_4x + a_5y + a_6z = 0$, $a_7x + a_8y + a_9z = 0$ has finite number of solutions

(c) $B = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$ is non singular

(d) none of these

10. Let A, B and C be 2×2 matrices with entries from the set of real numbers. Define * as follows : $A * B = 1/2\,(AB + BA)$, then

(a) $A * B = B * A$ (b) $A * A = A^2$

(c) $A * (B + C) = A * B + A * C$

(d) $A * I = A$

11. For all values of λ, the rank of the matrix

$$A=\begin{bmatrix}1 & 4 & 5\\ \lambda & 8 & 8\lambda-6\\ 1+\lambda^2 & 8\lambda+4 & 2\lambda+21\end{bmatrix}$$

(a) for $\lambda = 2$, $\rho(A) = 1$ (b) for $\lambda = -1$, $\rho(A) = 2$
(c) for $\lambda \neq 2, -1$, $\rho(A) = 3$ (d) none of these

12. If D_1 and D_2 are two 3×3 diagonal matrices, then
(a) D_1D_2 is a diagonal matrix
(b) $D_1^2 + D_2^2$ is a diagonal matrix
(c) $D_1D_2 = D_2D_1$
(d) D_1^n is a diagonal matrix $\forall n \in N$

13. The value of θ for which the system of linear equations in x, y, z given as $(\sin 3\theta)\,x - y + z = 0$, $(\cos 2\theta)\,x + 4y + 3z = 0$, $2x + 7y + 7z = 0$, has a non trivial solution
(a) $\dfrac{m\pi}{2}$ (b) $m\pi$
(c) $n\pi + (-1)^n \dfrac{\pi}{6}$ (d) none of these

Comprehension Type

If $A=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, evaluate three roots $(\lambda_1, \lambda_2, \lambda_3)$ of λ from the equation $|A - \lambda I| = 0$. Construct a non-null invertible matrix X such that $AX = \lambda X$ where λ = diagonal $(\lambda_1, \lambda_2, \lambda_3)$.

14. The three roots of λ are
(a) 1, 0, 2 (b) −1, 1, 3 (c) 2, −2, 3 (d) 5, 1, −3

15. Matrix A satisfies
(a) $A^2 - (\lambda_1 + \lambda_2)\,A + \lambda_1\lambda_2 I = 0$
(b) $A^2 - (\lambda_1 + \lambda_3)\,A + \lambda_1\lambda_3 I = 0$
(c) $A^2 - (\lambda_2 + \lambda_3)\,A + \lambda_2\lambda_3 I = 0$
(d) none of these

Matrix Match Type

16. Match the following :

	Column I		Column II
P.	If $\Delta(x)=\begin{vmatrix}x^2-5x+3 & 2x-5 & 3\\ 3x^2+x+4 & 6x+1 & 9\\ 7x^2-6x+9 & 14x-6 & 21\end{vmatrix}$ $= ax^3 + bx^2 + cx + d$, then	1.	$3a + 4b + 5c + d = 141$
Q.	If $\Delta(x)=\begin{vmatrix}2x^3-3x^2 & 5x+7 & 2\\ 4x^3-7x & 3x+2 & 1\\ 7x^3-8x^2 & x-1 & 3\end{vmatrix}$ $= a + bx + cx^2 + dx^3 + ex^4$, then	2.	$a + 2b + 3c + 5d = 156$
R.	If $\Delta(x)=\begin{vmatrix}x-1 & 5x & 7\\ x^2-1 & x-1 & 8\\ 2x & 3x & 0\end{vmatrix}$ $= ax^3 + bx^2 + cx + d$, then	3.	$c + d = 119$
		4.	$3a + 2b + 5c + 5d = 187$

	P	Q	R
(a)	1	3	4
(b)	1	4	2
(c)	4	3	1
(d)	2	3	4

Integer Answer Type

17. If $\begin{bmatrix}\cos\dfrac{2\pi}{3} & -\sin\dfrac{2\pi}{3}\\ \sin\dfrac{2\pi}{3} & \cos\dfrac{2\pi}{3}\end{bmatrix}^k = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ then the least value of $k(k \neq 0)$ is ______.

18. If $D_p = \begin{vmatrix}p & 15 & 8\\ p^2 & 35 & 9\\ p^3 & 25 & 10\end{vmatrix}$, then the value of $\sqrt[5]{\left(-\dfrac{1}{100}\right)\displaystyle\sum_{p=1}^{5} D_p - 37}$ must be ______.

19. If $\begin{vmatrix}a & a^2 & 1+a^3\\ b & b^2 & 1+b^3\\ c & c^2 & 1+c^3\end{vmatrix}=0$ and $\vec{A}=(1, a, a^2)$, $\vec{B}=(1, b, b^2)$ and $\vec{C}=(1, c, c^2)$ are non-coplanar, then the value of $(-abc)$ is ______.

20. If the system of equations $3x - 2y + z = 0$, $\lambda x - 14y + 15z = 0$, $x + 2y + 3z = 0$ have a non trivial solution, then the difference of digits of λ is ______.

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 175

JEE MAIN

1. The mirror image of the curve given by $\arg\left(\frac{z-3}{z-i}\right)=\frac{\pi}{6}$ in the real axis is given by

(a) $\arg\left(\frac{z+3}{z+i}\right)=\frac{\pi}{6}$ (b) $\arg\left(\frac{z-3}{z+i}\right)=\frac{\pi}{6}$

(c) $\arg\left(\frac{z+i}{z+3}\right)=\frac{\pi}{6}$ (d) $\arg\left(\frac{z+i}{z-3}\right)=\frac{\pi}{6}$

2. In a triangle ABC with incentre I, if $AI + AB = BC$, then $B + 3C =$

(a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{6}$ (c) π (d) $\frac{7\pi}{6}$

3. Let $y = f(x)$ be the solution of the equation $\frac{dy}{dx}=xy+x^3y^3,\ y(0)=1.$ The area bounded by the curve $y = f(x)$ and the lines, $y = 0$, $x = \pm 1$ is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

4. The vertices of a triangle are $A(-8, 5)$, $B(-15, -19)$, $C(1, -7)$. The bisector of $\angle A$ is $ax + 2y + c = 0$ where $c - a =$

(a) 67 (b) 68 (c) 71 (d) 73

5. The product of real values of x such that $(\log_x 2)^2-(\log_2 x)^2=\log_{2x}\left(\frac{2}{x}\right)$ is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $2^{\sqrt{2}}$ (d) None of these

JEE ADVANCED

6. If the sum of the digits of the number $N = 2000^{11} - 2011$ is S, then the sum of the digits of S is

(a) 10 (b) 13 (c) 15 (d) 16

COMPREHENSION

The curve $y = f(x)$ passes through the point (0, 1) and the curve $y=g(x)=\int_{-\infty}^{x} f(t)\,dt$ passes through the point $\left(0,\frac{1}{2}\right)$. The tangents drawn to the curves at the point with equal abscissae intersect on the x-axis. Then

7. $\lim_{x\to 0}\frac{(f(x))^2-1}{x}=$

(a) 1 (b) 2 (c) 3 (d) 4

8. The area bounded by the curve $y = g(x)$ and the lines $x = 0$, $x = 1$, $y = x$ is

(a) $\frac{e^2-1}{2}$ (b) $\frac{e^2-2}{3}$ (c) $\frac{e^2-3}{3}$ (d) $\frac{e^2-3}{4}$

INTEGER TYPE

9. Two circles pass through the points (0, –1) and (0, 1) and touch the line $y = mx + c$. If the circles intersect orthogonally, then $c^2 - m^2$ is

MATRIX MATCH

10. The equation $(1+\lambda)x^2 - 2\lambda xy + (\lambda - 2)y^2 - 4x + 3 = 0$ represents

List I		List II	
(P)	a pair of lines if $\lambda =$	1.	-4
(Q)	a parabola if $\lambda =$	2.	-2
(R)	an ellipse if $\lambda =$	3.	$\frac{2}{7}$
(S)	a hyperbola if $\lambda =$	4.	$\frac{1}{2}$

	P	Q	R	S
(a)	3	2	1	4
(b)	1	2	4	3
(c)	3	2	4	1
(d)	2	3	1	4

See Solution Set of Maths Musing 174 on page no. 84

JEE ADVANCED

SOLVED PAPER

PAPER-1

SECTION 1 (Maximum Marks : 28)

- *This section contains SEVEN questions.*
- *Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four options is(are) correct.*
- *For each question, darken the bubble(s) corresponding to the correct option in the ORS.*
- *For each question, marks will be awarded in one of the following categories :*

  Full Marks : +4 If only the bubble corresponding to all the correct option(s) is (are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding to each correct option, provided NO incorrect option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –2 In all other cases.
- *For example, if [a], [c] and [d] are all the correct options for a question, darkening all these three will get +4 marks; darkening only [a] and [d] will get +2 marks; and darkening [a] and [b] will get –2 marks, as a wrong option is also darkened.*

1. Let X and Y be two events such that

$$P(X)=\frac{1}{3}, P(X/Y)=\frac{1}{2} \text{ and } P(Y/X)=\frac{2}{5}. \text{ Then}$$

(a) $P(X\cap Y)=\frac{1}{5}$ (b) $P(X'/Y)=\frac{1}{2}$

(c) $P(Y)=\frac{4}{15}$ (d) $P(X\cup Y)=\frac{2}{5}$

2. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right)=y$, then which of the following is (are) possible value (s) of x?

(a) $1+\sqrt{1+y^2}$ (b) $-1+\sqrt{1-y^2}$

(c) $1-\sqrt{1+y^2}$ (d) $-1-\sqrt{1-y^2}$

3. Let $f : R \to (0, 1)$ be a continuous function. Then , which of the following function(s) has (have) the value zero at some point in the interval (0, 1)?

(a) $x^9 - f(x)$ (b) $f(x)+\int_0^{\frac{\pi}{2}} f(t)\sin t\, dt$

(c) $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\, dt$ (d) $e^x-\int_0^{x} f(t)\sin t\, dt$

4. Let $[x]$ be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?

(a) $x = 2$ (b) $x = 1$

(c) $x = -1$ (d) $x = 0$

5. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$ and midpoint (h, k), then which of the following is (are) possible value(s) of p, h and k?

(a) $p = -2, h = 2, k = -4$

(b) $p = 2, h = 3, k = -4$

(c) $p = -1, h = 1, k = -3$

(d) $p = 5, h = 4, k = -3$

6. Which of the following is (are) NOT the square of a 3×3 matrix with real entries?

(a) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (b) $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$

(c) $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$ (d) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$

7. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right angled triangle?

(a) $a, 4, 2$ (b) $a, 4, 1$ (c) $2a, 8, 1$ (d) $2a, 4, 1$

SECTION 2 (Maximum Marks : 15)

- ***This section contains FIVE questions.***
- ***The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive.***
- ***For each question, darken the bubble corresponding to the correct option in the ORS.***
- ***For each question, marks will be awarded in one of the following categories :***

  Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

  Zero Marks : 0 In all other cases.

8. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side?

9. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points?

10. Let $f : R \to R$ be a differentiable function such that

$$f(0), = 0, f\left(\frac{\pi}{2}\right) = 3 \text{ and } f'(0) = 1.$$

If $g(x) = \int_x^{\pi/2} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]dt$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$

11. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$

12. Words of length 10 are formed using the letters $A, B, C, D, E, F, G, H, I, J$. Let x be the number of such words where no letter is repeated; and let y be the number such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$

SECTION 3 (Maximum Marks : 18)

- ***This section contains SIX questions of matching type.***
- ***This section contains TWO tables (each having 3 columns and 4 rows).***
- ***Based on each table, there are THREE questions.***
- ***Each question has FOUR options [a], [b], [c], and [d]. ONLY ONE of these four options is correct.***
- ***For each questions, darken the bubble corresponding to the correct option in the ORS.***
- ***For each question, marks will be awarded in one of the following categories :***

  Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

  Zero Marks : 0 If none of the bubbles is darkened

  Negative Marks : –1 In all other cases

Answer Q. 13, Q. 14 and Q. 15 by appropriately matching the information given in the three columns of the following table.

Columns 1, 2, 3 contain conics, equations of tangents to the conics and points of contact, respectively.		
Column 1	**Column 2**	**Column 3**
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

13. The tangents to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination?

(a) (II) (iv) (R) (b) (IV) (iii) (S)
(c) (II) (iii) (R) (d) (IV) (iv) (S)

14. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1, 1), then which of the following options is the only CORRECT combination for obtaining its equation?

(a) (I) (i) (P) (b) (III) (i) (P)
(c) (II) (ii) (Q) (d) (I) (ii) (Q)

15. If a tangent to a suitable conic (Column I) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(a) (III) (i) (P) (b) (I) (ii) (Q)
(c) (II) (iv) (R) (d) (III) (ii) (Q)

Answer Q.16, Q.17 and Q. 18 by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x\log_e x$, $x \in (0, \infty)$

- *Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.*
- *Column 2 contains information about the limiting behaviour of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.*
- *Column 3 contains information about increasing/ decreasing nature of $f(x)$ and $f'(x)$.*

	Column 1	Column 2	Column 3
(I)	$f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim\limits_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II)	$f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim\limits_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III)	$f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim\limits_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0, 1)
(IV)	$f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim\limits_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

16. Which of the following options is the only CORRECT combination?

(a) (II) (ii) (Q) (b) (III) (iii) (R)
(c) (IV) (iv) (S) (d) (I) (i) (P)

17. Which of the following options is the only CORRECT combination?

(a) (III) (iv) (P) (b) (II) (iii) (S)
(c) (I) (ii) (R) (d) (IV)(i) (S)

18. Which of the following options is the only INCORRECT combination?

(a) (I) (iii) (P) (b) (II) (iii) (P)
(c) (III) (i) (R) (d) (II) (iv) (Q)

PAPER-2

SECTION 1 (Maximum Marks : 21)

- *This section contains SEVEN questions.*
- *Each question has FOUR options (a), (b), (c) and (d). ONLY ONE of these four options is correct.*
- *For each question, darken the bubble corresponding to the correct option in the ORS.*
- *For each question, marks will be awarded in one of the following categories :*

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.
Zero Marks : 0 If none of the bubbles is darkened.
Negative Marks : –1 In all other cases.

1. Let S {1, 2, 3, ..., 9}. For k = 1, 2, ..., 5, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$

(a) 125 (b) 252
(c) 210 (d) 126

2. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} + \overrightarrow{OR}\cdot\overrightarrow{OS} = \overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS}$$
$$= \overrightarrow{OQ}\cdot\overrightarrow{OR} + \overrightarrow{OP}\cdot\overrightarrow{OS}$$

Then the triangle PQR has S as its

(a) orthocentre (b) centroid
(c) circumcentre (d) incentre

3. If $f : R \to R$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}, f(1)=1$, then

(a) $f'(1) \le 0$ (b) $\frac{1}{2} < f'(1) \le 1$
(c) $0 < f'(1) \le \frac{1}{2}$ (d) $f'(1) > 1$

4. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is

(a) $14x + 2y - 15z = 1$ (b) $14x - 2y + 15z = 27$
(c) $14x + 2y + 15z = 31$ (d) $-14x + 2y + 15z = 3$

5. How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries M^TM is 5?

(a) 162 (b) 135 (c) 126 (d) 198

6. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = (\sqrt{4+\sqrt{9+\sqrt{x}}})^{-1}dx, x > 0$$

and $y(0)=\sqrt{7}$, then $y(256)=$

(a) 9 (b) 3 (c) 80 (d) 16

7. Three randomly chosen non-negative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is

(a) $\frac{5}{11}$ (b) $\frac{1}{2}$ (c) $\frac{36}{55}$ (d) $\frac{6}{11}$

SECTION 2 (Maximum Marks : 28)

- ***This section contains SEVEN questions.***
- ***Each question has FOUR options [a], [b], [c] and [d]. ONE OR MORE THAN ONE of these four options is (are) correct.***
- ***For each question, darken the bubble(s) corresponding to the correct option(s) in the ORS.***
- ***For each question, marks will be awarded in one of the following categories :***

Full Marks : +4 If only the bubble(s) corresponding to the all the correct option(s) is (are) darkened.
Partial Marks : +1 For darkening a bubble corresponding to each correct option, provided NO incorrect option is darkened.
Zero Marks : 0 If none of the bubbles is darkened.
Negative Marks : –2 In all other cases.

- ***For example, if [a], [c] and [d] are all the correct options for a question darkening all these three will get +4 marks; darkening only [a] and [d] will get +2 marks; and darkening [a] and [b] will get –2 marks, as a wrong option is also darkened.***

8. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then

(a) $\frac{1}{2} < \alpha < 1$ (b) $0 < \alpha \le \frac{1}{2}$
(c) $\alpha^4 + 4\alpha^2 - 1 = 0$ (d) $2\alpha^4 - 4\alpha^2 + 1 = 0$

9. If $g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1}(t)dt$, then

(a) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (b) $g'\left(\frac{\pi}{2}\right) = -2\pi$
(c) $g'\left(-\frac{\pi}{2}\right) = -2\pi$ (d) $g'\left(-\frac{\pi}{2}\right) = 2\pi$

10. Let $f(x) = \frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x \ne 1$. Then

(a) $\lim_{x \to 1^-} f(x)$ does not exist
(b) $\lim_{x \to 1^+} f(x)$ does not exist
(c) $\lim_{x \to 1^+} f(x) = 0$ (d) $\lim_{x \to 1^-} f(x) = 0$

11. If $f : R \to R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$ and $f(0) = 1$, then

(a) $f'(x) < e^{2x}$ in $(0, \infty)$ (b) $f(x) > e^{2x}$ in $(0, \infty)$
(c) $f(x)$ is increasing in $(0, \infty)$
(d) $f(x)$ is decreasing in $(0, \infty)$

12. Let α and β be non-zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha \cos\beta = 1$. Then which of the following is/are true?

MPP-3 CLASS XI ANSWER KEY

1.	(b)	**2.**	(b)	**3.**	(b)	**4.**	(b)	**5.**	(a)
6.	(c)	**7.**	(a, c)	**8.**	(b, c)	**9.**	(c, d)	**10.**	(a,b,c,d)
11.	(a,b,d)	**12.**	(d)	**13.**	(a, c)	**14.**	(b)	**15.**	(c)
16.	(a)	**17.**	(3)	**18.**	(2)	**19.**	(1)	**20.**	(2)

(a) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(b) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(c) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

(d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

13. If $f(x)=\begin{vmatrix}\cos 2x & \cos 2x & \sin 2x\\ -\cos x & \cos x & -\sin x\\ \sin x & \sin x & \cos x\end{vmatrix}$ then

(a) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$
(b) $f(x)$ attains its minimum at $x = 0$
(c) $f(x)$ attains its maximum at $x = 0$
(d) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

14. If $I=\sum_{k=1}^{98}\int_{k}^{k+1}\frac{k+1}{x(x+1)}dx$, then

(a) $I<\frac{49}{50}$ (b) $I > \log_e 99$
(c) $I < \log_e 99$ (d) $I>\frac{49}{50}$

SECTION 3 (Maximum Marks : 12)

- *This section contains TWO paragraphs.*
- *Based on each paragraph, there are TWO questions.*
- *Each question has FOUR options [a], [b], [c] and [d]. ONLY ONE of these four options is correct.*
- *For each question, darken the bubble corresponding to the correct option in the ORS.*
- *For each question, marks will be awarded in one of the following categories :*
  Full marks : +3 If only the bubble corresponding to the correct option is darkened.
  Zero Marks : 0 In all other cases.

Paragraph I

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}$ and $\overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}$ and $\overrightarrow{PQ}$, respectively, of a triangle PQR.

15. $|\overrightarrow{OX}\times\overrightarrow{OY}|=$

(a) $\sin(P + R)$ (b) $\sin 2R$
(c) $\sin(P + Q)$ (d) $\sin(Q + R)$

16. If the triangle PQR varies, then the minimum value of $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ is

(a) $-\frac{3}{2}$ (b) $\frac{5}{3}$ (c) $-\frac{5}{3}$ (d) $\frac{3}{2}$

Paragraph II

Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0, 1, 2, ...$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a+b\sqrt{5}=0$, then $a = 0 = b$.

17. $a_{12} =$

(a) $2a_{11} + a_{10}$ (b) $a_{11} - a_{10}$
(c) $a_{11} + 2a_{10}$ (d) $a_{11} + a_{10}$

18. If $a_4 = 28$, then $p + 2q =$

(a) 21 (b) 14
(c) 7 (d) 12

ANSWER KEYS

PAPER 1

1. (b,c)	**2.** (b,d)	**3.** (a,c)	**4.** (a,b,c)	**5.** (b)
6. (b,d)	**7.** (a,b,c)	**8.** (6)	**9.** (2)	**10.** (2)
11. (1)	**12.** (5)	**13.** (a)	**14.** (d)	**15.** (a)
16. (a)	**17.** (b)	**18.** (c)		

PAPER 2

1. (d)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (d)
6. (b)	**7.** (d)	**8.** (a,d)		
9. (No choice is correct)				
10. (b,d)	**11.** (b,c)	**12.** (a,b)	**13.** (c,d)	**14.** (c,d)
15. (c)	**16.** (a)	**17.** (d)	**18.** (d)	

For detailed solutions please refer
MTG JEE Advanced Explorer

OLYMPIAD CORNER

1. Let $a_n = \dfrac{n^2+1}{\sqrt{n^4+4}}$ for $n = 1, 2, 3, \ldots$

and let b_n be the product $a_1\, a_2 \ldots a_n$. Prove that

$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}}$$

and deduce that $\dfrac{1}{(n+1)^3} < \dfrac{b_n}{\sqrt{2}} - \dfrac{n}{n+1} < \dfrac{1}{n^3}$

for all positive integers n.

2. Find the largest constant k such that

$$\frac{kabc}{a+b+c} \le (a+b)^2 + (a+b+4c)^2$$

for all $a, b, c, > 0$.

3. In ΔABC, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC, AB, respectively. Determine the angles $\angle A$, $\angle B$, $\angle C$, if

$$\angle BDE = 24°, \angle CED = 18°.$$

4. In a group of nine mathematicians each speaks at most three languages and any two of them speak at least one common language. Show that at least five of them share a common language.

5. Find the exact value of

$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\ldots\cos\left(\frac{16\pi}{17}\right).$$

SOLUTIONS

1. Observe that, $n^4 + 4 = n^4 + 4n^2 + 4 - 4n^2$

$$= (n^2 + 2)^2 - 4n^2$$
$$= (n^2 + 2n + 2)\,(n^2 - 2n + 2)$$

Also $(k - 1)^2 + 2\,(k - 1) + 2 = k^2 + 1$

and $(k + 1)^2 - 2\,(k + 1) + 2 = k^2 + 1$

These observations imply that in the product $a_1 a_2 \ldots a_{k-1}\; a_k\; a_{k+1} \ldots a_n$, the numerator of a_k, $1 + k^2$, is cancelled by $\sqrt{(k-1)^2+2(k-1)+2}$ in the denominator of a_{k-1} and $\sqrt{(k+1)^2+2(k+1)+2}$ in the denominator of a_{k+1}.

Therefore, $b_n = \dfrac{\sqrt{2}\sqrt{1+n^2}}{\sqrt{2+2n+n^2}}$,

since all the numerator terms cancel except the numerator of the first and last terms.

To obtain the inequalities, we consider

$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{1+n^2}}{\sqrt{n^2+2n+2}}$$

and $$\frac{b_n}{\sqrt{2}} - \frac{n}{n+1} = \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}} - \frac{n}{n+1}$$

$$= \frac{(n+1)\sqrt{n^2+1} - n\sqrt{n^2+2n+2}}{(n+1)\sqrt{n^2+2n+2}} \qquad \ldots\text{(i)}$$

Let r represent the expression in (i). Then r can be written as :

$r =$

$$\frac{2n+1}{(n+1)\sqrt{n^2+2n+2}\,\left((n+1)\sqrt{n^2+1}+n\sqrt{n^2+2n+2}\right)}$$

$$= \frac{2n+1}{n(n+1)(n^2+2n+2)+(n+1)^2\sqrt{n^2+2n+2}\,\sqrt{n^2+1}}.$$

Now,

$$r < \frac{2(n+1)}{(n+1)[n^3+(n+1)n^2]} < \frac{2(n+1)}{(n+1)2n^3} = \frac{1}{n^3}.$$

To obtain the other inequality :

This is equivalent to

$$(2n + 1)(n + 1)^3 > (n + 1)[n(n^2 + 2n + 2) + (n + 1)\sqrt{n^2+1}\sqrt{n^2+2n+2}]$$

or $$(2n + 1)(n + 1)^2 > n(n^2 + 2n + 2) + (n + 1)\sqrt{n^2+1}\sqrt{n^2+2n+2}.$$

This is equivalent to

$$n^3+3n^2+2n+1>(n+1)\sqrt{n^2+1}\sqrt{n^2+2n+2}$$

or $$\frac{n^2}{n+1}+n^2+n+1>\sqrt{n^2+1}\sqrt{n^2+2n+2}.$$

If $n = 1$, $7/2 > \sqrt{10}$ is true.

If $n \geq 2$, $n^2/(n + 1) > 1$, then

$$n^2+n+1+\frac{n^2}{n+1}>n^2+n+2.$$

If we can show that $n^2+n+2>\sqrt{n^2+1}\sqrt{n^2+2n+2}$ for $n \geq 2$, we will be done. $(n^2 + n + 2) > \sqrt{n^2+1}\sqrt{n^2+2n+2}$ iff $(n^2 + n + 2)^2 > (n^2 + 1)(n^2 + 2n + 2)$ iff $2(n^2 + n + 1) > 0$, $n \geq 2$.

This is clearly true, hence the inequalities are true.

2. By the A.M.-G.M. inequality,

$$\begin{aligned}(a + b)^2 + (a + b + 4c)^2 &= (a + b)^2 + (a + 2c + b + 2c)^2\\ &\geq (2\sqrt{ab})^2+(2\sqrt{2ac}+2\sqrt{2bc})^2\\ &= 4ab + 8ac + 8bc + 16c\sqrt{ab}.\end{aligned}$$

Therefore,

$$\frac{(a+b)^2+(a+b+4c)^2}{abc}\cdot(a+b+c)$$

$$\geq\frac{4ab+8ac+8bc+16c\sqrt{ab}}{abc}\cdot(a+b+c)$$

$$=\left(\frac{4}{c}+\frac{8}{b}+\frac{8}{a}+\frac{16}{\sqrt{ab}}\right)(a+b+c)$$

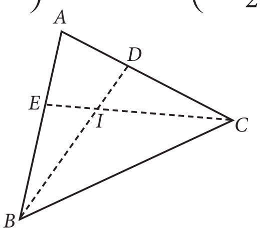
$$=8\left(\frac{1}{2c}+\frac{1}{b}+\frac{1}{a}+\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{ab}}\right)\left(\frac{a}{2}+\frac{a}{2}+\frac{b}{2}+\frac{b}{2}+c\right)$$

$$\geq 8\left(5\sqrt[5]{\frac{1}{2a^2b^2c}}\right)\left(5\sqrt[5]{\frac{a^2b^2c}{2^4}}\right)=100,$$

again by the A.M.-G.M. Inequality. Hence the largest constant k is 100. For $k = 100$, equality holds if and only if $a = b = 2c > 0$.

3. Let I be the incenter of triangle ABC. Let D' be the projection onto AC of I, and let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$, as usual. Now $\angle ADB =\frac{1}{2}\beta+\gamma$. So,

$$ID=\frac{r}{\sin\left(\frac{1}{2}\beta+\gamma\right)} \text{ and } IE=\frac{r}{\sin\left(\beta+\frac{1}{2}\gamma\right)} \qquad \text{...(1)}$$



Applying the law of sines to triangle IDE we have

$$\frac{\sin\left(\beta+\frac{1}{2}\gamma\right)}{\sin\left(\frac{1}{2}\beta+\gamma\right)}=\frac{\sin 18^\circ}{\sin 24^\circ}. \qquad \text{...(2)}$$

As $\angle BDE = 24°$ and $\angle CED = 18°$,

we have $\angle DIE = 138° = 90^\circ+\frac{1}{2}\alpha \Rightarrow \alpha=96^\circ$.

Thus, $\beta + \gamma = 84°$ so $\gamma = 84° - \beta$ and $\frac{1}{2}\gamma=42^\circ-\frac{1}{2}\beta.$ ...(3)

Now using (3) in (2) we get

$$\frac{\sin\left(42^\circ+\frac{1}{2}\beta\right)}{\sin\left(84^\circ-\frac{1}{2}\beta\right)}=\frac{\sin 18^\circ}{\sin 24^\circ}.$$

Expanding and doing some calculations gives

$$\tan\frac{1}{2}\beta=\frac{\sin 18^\circ\sin 84^\circ-\sin 24^\circ\sin 42^\circ}{\sin 24^\circ\cos 42^\circ+\sin 18^\circ\cos 84^\circ},$$

and this equals $\tan 6°$

{To see this we have

$$\frac{\sin 18^\circ\sin 84^\circ-\sin 24^\circ\sin 42^\circ}{\sin 24^\circ\cos 42^\circ+\sin 18^\circ\cos 84^\circ}=\frac{\sin 6^\circ}{\cos 6^\circ}$$

just in case

$$\sin 18^\circ \sin^2 84^\circ - \sin 24^\circ \sin 42^\circ \cos 6^\circ$$
$$= \sin 6^\circ \sin 24^\circ \cos 42^\circ + \sin 18^\circ \sin^2 6^\circ,$$

$\sin 18° [\cos^2 6° - \sin^2 6°]$

$$= \sin 24^\circ [\sin 6^\circ \cos 42^\circ + \cos 6^\circ \sin 42^\circ],$$
$$\sin 18^\circ \cos 12^\circ = \sin 24^\circ \sin 48^\circ,$$
$$\sin 18^\circ = 2 \sin 12^\circ \sin 48^\circ,$$
$$\sin 18^\circ = \cos 36^\circ - \cos 60^\circ,$$
$$\sin 18^\circ = -2 \sin^2 18^\circ + \frac{1}{2}.$$

giving the equivalent condition

$$2\sin^2 18^\circ+\sin 18^\circ-\frac{1}{2}=0.$$

This is the same as $\sin 18° = -\frac{1}{4}(-1+\sqrt{5})$, which is true.}

So, $\frac{1}{2}\beta = 6°, \beta = 12°$ and $\alpha = 96°, \beta = 12°, \gamma = 72°$.

4. If any one of them speaks less than three languages then by the pigeon hole principle one of the languages he (or she) speaks should be spoken by four others and we are done. Suppose now that each speaks three languages. Three cases arise:

(i) Two mathematicians have all the three languages in common. Here again we are through since one of the three languages should be spoken by at least three of the remaining seven mathematicians.

(ii) Some two mathematicians have two languages in common, say, M_1 and M_2 have L_1 and L_2 in common. Let L_3 and L_4 be the third languages of M_1 and M_2 respectively. If there is one more mathematician having L_1 and L_2 as two of his languages then we are done as follows: Suppose M_3 speaks (L_1, L_2, L_5). Of the remaining six mathematicians if there are more than two who speak L_1 or L_2 then we are through; if not, there are four mathematicians who do not speak L_1 or L_2. But then these four are forced to speak L_3, L_4, L_5 to be able to converse with M_1, M_2, M_3 and so there will be five mathematicians who speak L_3, L_4 and L_5. So suppose only two have L_1 and L_2 common. Consider the pair (L_3, L_4). If some three mathematicians speak both these languages we are done as before; if not, there are five who do not speak L_3 and L_4 simultaneously. But then these five should speak either L_1 or L_2 to be able to converse with both M_1 and M_2 which in turn implies that either L_1 or L_2 is spoken by at least five mathematicians.

(iii) Any two of the nine mathematicians have exactly one language in common. Let M_1 speak (L_1, L_2, L_3). The remaining eight have to speak either L_1 or L_2 or L_3 and so one of these three languages has to be spoken by at least three more mathematicians, say L_1 is spoken by M_2, M_3, M_4 also. We will show that L_1 is spoken by all the nine mathematicians. Suppose not; say M_5 has L_2 in common with M_1. Now M_5 has to speak to each of M_2, M_3, M_4 in a different language other than L_2 (since any two mathematicians have only one common language) which is not possible as no mathematician speaks more than three languages and we are done.

5. Let $I = \cos\frac{2\pi}{17}\cos\frac{4\pi}{17}\cos\frac{6\pi}{17}\ldots\cos\frac{16\pi}{17}$;

then,

$$I\sin\frac{2\pi}{17} = \frac{1}{2}\sin\frac{4\pi}{17}\cos\frac{4\pi}{17}\cos\frac{6\pi}{17}\ldots\cos\frac{16\pi}{17}$$

$$= \frac{1}{4}\sin\frac{8\pi}{17}\cos\frac{8\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}\cos\frac{16\pi}{17}$$

$$= \frac{1}{8}\sin\frac{16\pi}{17}\cos\frac{16\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}$$

$$= -\frac{1}{16}\sin\frac{2\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}$$

Thus,

$$I\sin\frac{6\pi}{17} = -\frac{1}{16}\sin\frac{6\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}$$

$$= -\frac{1}{32}\sin\frac{12\pi}{17}\cos\frac{12\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17}$$

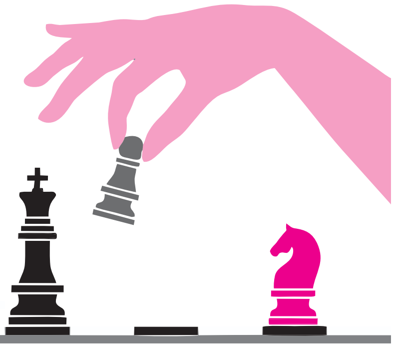
$$= \frac{1}{64}\sin\frac{10\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17}$$

$$= -\frac{1}{128}\sin\frac{14\pi}{17}\cos\frac{14\pi}{17}$$

$$= -\frac{1}{256}\sin\frac{28\pi}{17} = \frac{1}{256}\sin\frac{6\pi}{17}.$$

Therefore, $I = \frac{1}{256}$.

Challenging PROBLEMS ON CALCULUS



1. Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{[P(x)]}{P([x])}$, where $P(x)$ is a polynomial with positive coefficients, ([·] denotes greatest integer function)
(a) 0 (b) –1
(c) 1 (d) does not exist

2. The number of non-differentiable points in $(0, 2\pi)$ of the function $f(x)=\lim\limits_{m\to\infty}\sqrt[2m]{\cos^{2m}x+\sin^{2m}x}$, $x\in R$ is/are
(a) 0 (b) 1 (c) 2 (d) 4

3. If f is twice continuously differentiable on R such that $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -1$, then for $a \in R$, $\lim\limits_{x\to\infty}\left(f\left(\dfrac{a}{\sqrt{x}}\right)\right)^x =$
(a) e^{-a^2} (b) $e^{-a^2/2}$ (c) a^2 (d) e^{a^2}

4. For a positive integer n, a local extremum value of the function $f(x)=\left(1+x+\dfrac{x^2}{2!}+....+\dfrac{x^n}{n!}\right)e^{-x}$ is
(a) 0 (b) 1 (c) e^{-1} (d) e

5. Compute $\lim\limits_{m\to\infty}\left[2\sin^2\dfrac{m^5}{m+1}+\cos^2\dfrac{m^5}{m+1}\right]^{1/m}$
(a) 0 (b) 1 (c) e (d) e^2

6. Evaluate $\lim\limits_{n\to\infty}(n+1+n\cos n)^{\frac{1}{2n+n\sin n}}$
(a) 0 (b) 1 (c) e (d) e^2

7. Evaluate the limit $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)\left(1+\dfrac{2}{n^2}\right)....\left(1+\dfrac{n}{n^2}\right)$
(a) 1 (b) e (c) e^2 (d) $\sqrt{e}$

8. If $f(x)$ is an increasing function from $R \to R$ such that $f''(x) > 0$, $f(x) \neq 0$ and f^{-1} exists then $\dfrac{d^2(f^{-1}(x))}{dx^2}$ is
(a) = 0 (b) < 0
(c) > 0 (d) depends on $f(x)$

9. Consider the function $f(x)=\dfrac{x^2-21x+1}{x^2+1}$, then for real x and y the maximum value of $|f(x) - f(y)|$ is
(a) 20 (b) 21 (c) 40 (d) 42

10. Evaluate the limit $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{k(n-k)!+(k+1)}{(k+1)!\,(n-k)!}$
(a) 0 (b) 1 (c) e (d) $e - 1$

11. For real numbers x, y, z we have $[x]-y=2[y]-z=3[z]-x=\dfrac{5}{21}$, ([.] denotes greatest integer function), then $x + y + z$ has the value
(a) $\dfrac{41}{7}$ (b) $\dfrac{44}{7}$ (c) $-\dfrac{41}{7}$ (d) $-\dfrac{44}{7}$

12. Let $f(x)$ be a real valued function defined on positive reals such that
(i) $f(x) < f(y)$ if $x < y$ and
(ii) $f\left(\dfrac{2xy}{x+y}\right)=\dfrac{f(x)+f(y)}{2}$ for all x, y. Then
(a) $f(x)$ is positive for all x
(b) $f(x)$ is negative for all x
(c) $f(x)$ is negative for exactly one value of x
(d) $f(x)$ is negative for some value of x.

13. The number of real solutions to the system of equations
$\sqrt{x^2-2x+6}\cdot\log_3(6-y)=x,$
$\sqrt{y^2-2y+6}\cdot\log_3(6-z)=y,$ and
$\sqrt{z^2-2z+6}\cdot\log_3(6-x)=z$ is/are
(a) 0 (b) 1 (c) 2 (d) 4

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14. A function $f: R \to R$ such that for all $x \in R$,

$$f\left(x-\frac{b}{a}\right)+2x \le \frac{a}{b}x^2+\frac{2b}{a} \le f\left(x+\frac{b}{a}\right)-2x$$

where a, b be fixed positive real numbers then minimum value of $f(1)$ is

(a) 1 (b) 1/2 (c) 2 (d) 3/2

15. If p, q are relatively prime positive integers then

$$\left\{\frac{p}{q}\right\}+\left\{\frac{2p}{q}\right\}+\ldots+\left\{\frac{(q-1)p}{q}\right\}= \_\_\_\_,$$ $\{.\}$ denotes the fractional part.

(a) q (b) $q-1$

(c) $\frac{q-1}{2}$ (d) $\frac{(p-1)(q-1)}{2}$

16. Let $m = 144^{\sin^2 x} + 144^{\cos^2 x}$, then the number of possible integral values of m are

(a) 24 (b) 120 (c) 122 (d) 145

17. The number of values of n for which all solutions of the equation $x^3 - 3x + n = 0$ are integers is/are

(a) 0 (b) 1 (c) 2 (d) 3

18. For $n \in N$, let t_n be defined by $\left(1+\frac{1}{n}\right)^{n+t_n} = e$, then $\lim\limits_{n\to\infty} t_n =$

(a) 0 (b) 1/2 (c) 1 (d) –1

19. There exist a function $f: N \to Z$ such that

(i) $f(5) = 21$ and

(ii) $f(xy) = f(x) + f(y) + kf(\gcd(x, y))$ for all $x, y \in N$, then number of possible integral values of k are

(a) 0 (b) 1 (c) 2 (d) 3

20. Given that the polynomial $f(x) = x^n + a_1x^{n-1} + .. + a_n$ with integral coefficient is equal to 5 for four distint integers a, b, c, d. The number of integral k such that $f(k) = 7$ is/are

(a) 0 (b) 1 (c) 2 (d) 4

SOLUTIONS

1. (c): Since $P(x)$ is a polynomial with positive coefficients for $x > 1$.

We have, $\frac{P(x)-1}{P(x)} \le \frac{[P(x)]}{P[(x)]} \le \frac{P(x)}{P(x-1)}$

So, by Sandwich Rule, $\lim\limits_{x\to\infty}\frac{[P(x)]}{P([x])}=1$

2. (c): Notice that $f(x) = \max.\{|\sin x|, |\cos x|\}$.
Clearly, f is continuous on R and non-differentiable at $x = \pi/4$ and $x = 3\pi/4$ in $(0, \pi)$.

3. (b) : Let $L=\lim\limits_{x\to\infty}\left[f\left(\frac{a}{\sqrt{x}}\right)\right]^x$.

Taking log on both sides, we get

$$\log L=\lim_{x\to\infty} x\log f\left(\frac{a}{\sqrt{x}}\right)=\lim_{t\to 0^+}\frac{\log(f(a\sqrt{t}))}{t},$$

where $x=\frac{1}{t}$

$$=\lim_{t\to 0}\frac{af'(a\sqrt{t})}{2\sqrt{t}\cdot f(a\sqrt{t})} \quad \text{(using L Hospital's rule)}$$

$$=\lim_{t\to 0}\frac{a^2f''(a\sqrt{t})}{2f(a\sqrt{t})+2a\sqrt{t}\,f'(a\sqrt{t})}=\frac{-a^2}{2}$$

Hence, $L=e^{-a^2/2}$

4. (b) : $f'(x)=\frac{-x^n}{n!}e^{-x},\ f'(x)=0 \Rightarrow x=0$

If n is even, $f'(x) < 0$ for $x \ne 0$. So no extrema.
If n is odd, $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$.
So, $f(0) = 1$ is a (global) max. value of f.

5. (b) : Since, $1\le\left[2\sin^2\frac{m^5}{m+1}+\cos^2\frac{m^5}{m+1}\right]^{1/m}\le\sqrt[m]{2}$

it follows that $\lim\limits_{m\to\infty}\left[2\sin^2\frac{m^5}{m+1}+\cos^2\frac{m^5}{m+1}\right]^{1/m}=1$

(By sandwich rule)

6. (b) : We have,

$$1<(1+n+n\cos n)^{\frac{1}{2n+n\sin n}}<(1+2n)^{\frac{1}{2n+n\sin n}}$$

and $1<(1+2n)^{\frac{1}{2n+n\sin n}}<(1+2n)^{1/n}$

So, by Sandwich rule, we have

$$\lim_{n\to\infty}(1+n+n\cos n)^{\frac{1}{2n+n\sin n}}=1$$

7. (d) : Let $L=\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\ldots\left(1+\frac{n}{n^2}\right)$

Taking log on both sides, we have

$$\log L=\lim_{n\to\infty}\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2}{n^2}\right)+\ldots$$

$$+\log\left(1+\frac{n}{n^2}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \log\left(1+\frac{k}{n^2}\right)$$

Now, using expansion of $\log(1 + x)$ series, we have

$$x - \frac{x^2}{2} < \log(1+x) < x,\ x > 0$$

So, $\frac{k}{n^2} - \frac{k^2}{2n^4} < \log\left(1+\frac{k}{n^2}\right) < \frac{k}{n^2}$

Now, summing these $k = 1, 2, ..., n$ inequalities and putting $n \to \infty$, we have

$$\log L = \frac{1}{2} \text{ i.e. } L = e^{1/2} = \sqrt{e}$$

8. (b) : f is an increasing function. So, $f'(x) > 0$ and $f''(x) > 0$ is given.

Let $g(x) = f^{-1}(x)$ then $f(g(x)) = x$

So, $g'(x) = \frac{1}{f'(g(x))} \Rightarrow g'(x) > 0$

Again differentiating,

$$g''(x) = \frac{-1}{(f'(g(x)))^2} \times f''(g(x)) \times g'(x)$$

Hence, $g''(x) < 0$

9. (b) : $f'(x) = \frac{21(x^2-1)}{(x^2+1)^2}$

So, $f(x)$ increases for $x \in (-\infty, -1] \cup [1, \infty)$ and decreases for $(-1, 1)$. So, $f_{max} = f(-1)$ and $f_{min} = f(1)$

So, for any x, y

$$|f(x) - f(y)| \le |f(-1) - f(1)| \le 21$$

10. (b) : Given summation

$$= \sum_{k=0}^{n} \frac{k}{(k+1)!} + \sum_{k=0}^{n} \frac{1}{k!(n-k)!}$$

$$= \sum_{k=0}^{n} \left[\frac{1}{k!} - \frac{1}{(k+1)!}\right] + \frac{1}{n!} \sum_{k=0}^{n} {}^nC_k$$

$$= 1 - \frac{1}{(n+1)!} + \frac{2^n}{n!}$$

So, required limit = $1 - 0 + 0 = 1$ as $n \to \infty$

11. (b) : From given equation, we have

$$x = 3[z] - \frac{5}{21} \Rightarrow [x] = 3[z] - 1$$

Similarly from the other relations, we have

$[y] = [x] - 1$, $[z] = 2[y] - 1$

Solving these three equations, we get

$[x] = 2$, $[y] = 1$, $[z] = 1$

So, $x = 3(1) - \frac{5}{21}$ etc.

Hence, $x + y + z = \frac{44}{7}$

12. (d) : Given f is an increasing function. So either $\lim_{x\to 0^+} f(x) = -\infty$ or $\lim_{x\to 0^+} f(x) = a$ for some real number a.

Let us assume that $\lim_{x\to 0^+} f(x) = a$

Now, in the given relation we fix x and $y \to 0^+$ then since $\lim_{y\to 0^+} \frac{2xy}{x+y} = 0$, we have

$$a = \frac{f(x)+a}{2} \text{ i.e. } f(x) = a = \text{constant function}$$

Contradiction.

Hence, we should have $\lim_{x\to 0^+} f(x) = -\infty$

$\Rightarrow f(x) < 0$ for some positive x.

13. (b) : $(x, y, z) \in (-\infty, 6)$. We rewrite the equations given as

$$y = 6 - 3^{\frac{x}{\sqrt{x^2-2x+6}}}.$$ Similarly for the others.

Consider a function

$$f(t) = 6 - 3^{\frac{t}{\sqrt{t^2-2t+6}}},\ t \in (-\infty, 6)$$

Now, $f'(t) < 0$ in $t \in (-\infty, 6)$. So, f is strictly decreasing

$\Rightarrow f(f(f(t)))$ is also decreasing.

The given relation gives $f(x) = y$, $f(y) = z$, $f(z) = x$

$\Rightarrow f(f(f(x))) = x$. Hence, $f(x) = x$

Now, $y = f(x)$ is a decreasing function and $y = x$ is an increasing function.

So, $f(x) = x$ has only one solution $x = 3$.

$x = y = z = 3$

14. (c): Putting $y = x - \frac{b}{a}$ in the L.H.S. of the inequality, we have $f(y) \le \frac{a}{b}y^2 + \frac{b}{a}$ and while putting $y = x + \frac{b}{a}$ in the R.H.S of the inequality given, we have

$$f(y) \ge \frac{a}{b}y^2 + \frac{b}{a}$$

Hence, $f(x) = \frac{a}{b}x^2 + \frac{b}{a}$.

15. (c): If $a = bc + r$, $r \to$ remainder then $\left\{\frac{a}{b}\right\} = \frac{r}{b}$

Let $r_1, r_2, ..., r_{q-1}$ be the remainders of the numerators in the given question. Since p, q are relatively prime, these remainders represent a permutation of the numbers 1, 2, 3, ..., $(q-1)$. Hence, the given expression becomes

$$\frac{r_1}{q}+\frac{r_2}{q}+\ldots+\frac{r_{q-1}}{q}=\frac{1}{q}+\frac{2}{q}+\ldots.+\frac{q-1}{q}=\frac{q-1}{2}$$

16. (c): Consider the function $f(t) = 144^t + 144^{1-t}$,

Where $t = \sin^2 x$, $t \in [0, 1]$

$\therefore \quad f'(t) = \log 144 \cdot [144^t - 144^{1-t}]$

$\Rightarrow \quad f'(t) = 0$ for $t = 1/2$

So, f is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right)$

So, $f_{\min} = f\left(\frac{1}{2}\right) = 24$ and $f_{\max} = f(0) = f(1) = 145$

So every integer between 24 and 145 is possible.

17. (c): If the given equation has integer root α then

$n = -\alpha^3 + 3\alpha \quad \Rightarrow \quad n$ is an integer too.

Let $f(x) = x^3 - 3x$, $f'(x) = 3(x-1)(x+1)$

So, f is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $(-1, 1)$ and $f(-1) = 2$, $f(1) = -2$.

So $f(x) = -n$ has three real roots iff $|n| \le 2$

$\Rightarrow \quad$ Possible $n = \{-2, -1, 0, 1, 2\}$

Moreover, one of the integer solutions has to be $\{1, 0, -1\}$.

So, $n \in \{f(1), f(0), f(-1)\} = \{-2, 0, 2\}$

Direct checking gives, valid values of n are 2, -2

18. (b) : Given, $\left(1+\frac{1}{n}\right)^{n+t_n} = e,$

Taking log on both sides, we have

$$t_n = \frac{1}{\log\left(1+\frac{1}{n}\right)} - n, \text{ let } 1+\frac{1}{n} = y;\ n \to \infty,\ y \to 1$$

So, $\lim\limits_{n\to\infty} t_n = \lim\limits_{y\to 1}\left[\frac{1}{\log y} - \frac{1}{y-1}\right]$

$= \lim\limits_{y\to 1} \frac{y-1-\log y}{(y-1)\log y} \quad \left(\frac{0}{0} \text{ form}\right)$

$= \frac{1}{2}$ (using L Hospital's rule twice)

19. (c): Putting $x = y$, we have $f(x^2) = (k+2) f(x)$

So, $f(x^4) = (k+2) f(x^2) = (k+2)(k+2) f(x)$

$= (k+2)^2 f(x)$ ... (1)

Putting $y = x^3$, we have

$$f(x^4) = f(x) + f(x^3) + kf(x) = (k+1)f(x) + f(x^3)$$
$$= (k+1)f(x) + f(x) + f(x^2) + kf(x)$$
$$= (2k+2)f(x) + f(x^2)$$
$$= (2k+2)f(x) + (k+2)f(x) = (3k+4)f(x) \quad \text{... (2)}$$

Divide (1) and (2), we have,

$$(k+2)^2 f(x) = (3k+4)f(x) \Rightarrow k = 0, -1$$

20. (a) : Let $f(x) = (x-a)(x-b)(x-c)(x-d)\, g(x) + 5$ where $g(x)$ is a polynomial with integral coefficient of $(n-4)^{\text{th}}$ degree. If $f(k) = 7$ then

$(k-a)(k-b)(k-c)(k-d)\, g(k) = 2$ ... (i)

Now, 2 has only 4 distinct integral factors (2, 1, -1, -2). Since $(k-a)$, $(k-b)$, $(k-c)$, $(k-d)$ are distinct the above product (i) can't be 2.

Hence, no such k exists.

JEE WORK OUTS

Time : 1 hr. **Marks :** 60

PAPER-1

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct ans. 3 marks and wrong ans., no negative mark]

1. The function $f(x) = \begin{cases} |x-3|, x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, x < 1 \end{cases}$ is

(a) continuous at $x = 1$
(b) derivable at $x = 1$
(c) not derivable at $x = 1$
(d) not derivable at $x = 3$

2. The solutions of the quadratic equation $(3|x| - 3)^2 = |x| + 7$, which belongs to the domain of definition of the function $y = \sqrt{(x-4)x}$ is/are

(a) $\frac{1}{9}$ (b) 2
(c) –2 (d) $-\frac{1}{9}$

3. Which of the following functions (is) are injective maps ?

(a) $f(x) = |x + 1|, x \in [-1, \infty)$
(b) $g(x) = x + \frac{1}{x}, x \in (0, \infty)$
(c) $h(x) = x^2 + 4x - 5, x \in (0, \infty)$
(d) $k(x) = e^{-x}, x \in [0, \infty)$

4. Let $y = g(x) = \frac{x+2}{x-1}$, then

(a) $g(1) = 3$ (b) $x = g(y)$
(c) y increases with x for $x > 1$
(d) g is a rational function of x

5. For all $x \in (0, 1)$

(a) $e^x < 1 + x$ (b) $\log_e(1 + x) < x$
(c) $\sin x > x$ (d) $\log_e x > x$

6. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) f(y)$ for all $x, y \in R$ and $f(x) \neq 0$ for any x in R. Let function be differentiable at $x = 0$ and $f'(0) = 2$. Then

(a) $f(x)$ is continuous at $x = 0$
(b) $f(x)$ is not differentiable at infinitely many points
(c) $f(x) = e^{2x}$ (d) None of these

7. The function $f(x) = 1 + |\sin x|$ is

(a) continuous everywhere
(b) not differentiable at an infinite number of points
(c) differentiable nowhere
(d) not differentiable at $x = 0$

8. Let $[x]$ denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is

(a) continuous at $x = 0$
(b) continuous in $(-1, 0)$
(c) differentiable at $x = 1$
(d) differentiable in $(-1, 1)$

9. If $f(x) = 1 + 2\sin x + 2\cos^2 x, 0 \leq x \leq \pi/2$, then $f(x)$ is

(a) greatest at $x = \pi/6$
(b) least at $x = \pi/6$
(c) increasing in $\left(0, \frac{\pi}{6}\right)$ and decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$
(d) least at $x = 0$ and $x = \pi/2$

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10. If $f(x)=\sin^3 x+\lambda\sin^2 x, -\frac{\pi}{2}<x<\frac{\pi}{2}$ then

(a) $f(x)$ has a point of inflection if $\lambda = 0$
(b) $f(x)$ has exactly one point of maximum and exactly one point of minimum if $|\lambda| < 3/2$
(c) $f(x)$ has exactly one point of maximum and exactly one point of minimum if $\lambda \in \left(-\frac{3}{2},0\right)\cup\left(0,\frac{3}{2}\right)$
(d) all of the above

INTEGER TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct ans. 3 marks and wrong ans., no negative mark]

11. $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$, where [.] denotes the greatest integer function is

12. Consider the function $f(x) = x - |x - x^2|, -1 \le x \le 2$. Find the number of points of non-differentiability of $f(x)$ for $x \in [-1, 2]$.

13. $\lim_{x\to 0}\frac{\cos^2(1-\cos^2(1-\cos^2(.....-\cos^2(x)))...)}{\sin\left\{\pi\left(\frac{\sqrt{x+4}-2}{x}\right)\right\}}=\frac{k}{\pi}$

where $k =$

14. The left hand limit of $f(x)=\left\{\frac{|x|^3}{3}-\left[\frac{x}{3}\right]^3\right\}$, where $[x]$ denotes the greatest integer less than or equal to x, is

15. The derivative of $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$ is at $x=\frac{\pi}{3}$ is $k - 5$, if $k =$

16. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

17. If $f(x + y) = f(x) \cdot f(y)$ for all x and y and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ is equal to

18. A food company produces x quality A and y quality B items per day, where $y(5 - x) = 10 (4 - x)$, $0 \le x \le 4$. If the profit on each quality A item is twice the profit on quality B item, then the most profitable number of quality A items per day to manufacture is.

19. If $f(x)=\begin{cases}2-|x^2+5x+6|, & x\neq -2\\ a^2+1, & x=-2\end{cases}$ then the minimum value of $|a|$ so that $f(x)$ has maxima at $x = -2$ is

20. From a fixed point A on the circumference of a circle of radius r, the perpendicular AY is let fall on the tangent at P. The maximum area of the triangle APY is $\frac{3\sqrt{3}}{k}r^2$, for $k =$

PAPER-2

Time : 1 hr. **Marks :** 60

SINGLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices, (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. [Correct ans. 3 marks & wrong ans.-1]

1. $\lim_{x\to 1}\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}=$

(a) $n(n + 1)$ (b) $\frac{n(n+1)}{2}$
(c) $n + 1$ (d) $\frac{3n}{2}$

2. $\lim_{x\to 0}\frac{\tan^3\sqrt{x}\ \ln(1+3x)}{\tan^{-1}\sqrt{x}^2 e^5\sqrt[3]{x}-1}=$

(a) $\frac{\sqrt{3}}{5}$ (b) 1 (c) $\frac{3}{5}$ (d) 0

3. Value of $\lim_{n\to\infty}\left\{\prod_{i=1}^{n}\cos\left(\frac{x}{2^i}\right)\right\}$ is

(a) $\sin x$ (b) $\sin x/x$ (c) $x/\sin x$ (d) 1

4. $\lim_{x\to 0} 1^{\operatorname{cosec}^2 x}+2^{\operatorname{cosec}^2 x}+.....+n^{\operatorname{cosec}^2 x\sin^2 x}$ is

(a) n (b) n^2 (c) 0 (d) 1

5. If $y=\sin x$, then $\frac{d^2}{dy^2}(\cos^7 x)$ is equal to

(a) $35 \cos^3 x - 42 \cos^5 x$ (b) $35 \cos^3 x + 42 \cos^5 x$
(c) $42 \cos^3 x - 35 \cos^5 x$ (d) $-35 \cos^3 x - 42 \cos^5 x$

6. If $5f(x)+3f\left(\frac{1}{x}\right)=x+2$ and $y = x\, f(x)$, then $\left.\frac{dy}{dx}\right|_{x=1}$ is equal to

(a) 14 (b) –1 (c) 1 (d) 7/8

7. If $f(x)=\dfrac{x}{\sin x}$ and $g(x)=\dfrac{x}{\tan x}$ where $0 < x \le 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions

(b) both $f(x)$ and $g(x)$ are decreasing functions

(c) $f(x)$ is an increasing function

(d) None of these

8. A square piece of tin of side 18 cm. is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. The maximum possible volume of the box is given by (in cm^2)

(a) 420 (b) 338

(c) 432 (d) None of these

9. The function $f(x)=\dfrac{1}{1+x\tan x}$ has

(a) one point of minimum in the interval $(0, \pi/2)$

(b) one point of maximum in the interval $(0, \pi/2)$

(c) no point of maximum, no point of minimum in the interval $(0, \pi/2)$

(d) two points of maxima in the interval $(0, \pi/2)$

10. $\lim\limits_{h\to 0}\dfrac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$

(a) does not exist (b) is equal to –3/2

(c) is equal to 3/2 (d) is equal to 3

PARAGRAPH TYPE

This section contains 2 paragraph. Based upon each of the paragraphs 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [Correct ans. 3 marks and wrong ans.-1]

Paragraph 1

In calculus the derivative of any function $y = f(x)$ is defined as

$$Df(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Now instead of this usual definition of derivative $Df(x)$, define a new kind of derivative $D^*f(x)$, which can be calculated by the formula

$$D^*f(x)=\lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}; \text{ where } f^2(x)=(f(x))^2$$

11. If $f(x)=\dfrac{x}{\ln x}$; then $D^*f(x)$ is

(a) $\dfrac{\ln x-1}{(\ln x)^2}$ (b) $\dfrac{2x(\ln x-1)}{(\ln x)}$

(c) $\dfrac{2x(\ln x-1)}{(\ln x)^2}$ (d) $\dfrac{2x(\ln x-1)}{(\ln x)^3}$

12. If function $g(x) = x^x$, then $D^*g(x)\,|_{x=1}$ is

(a) 1 (b) $2e^e$

(c) 2 (d) not defined

Paragraph 2

Given a function 'g' which has a derivative $g'(x)$ for every real 'x' and which satisfy $g'(0) = 2$ and $g(x + y) = e^y g(x) + e^x g(y)$ for all x and y.

13. The function $g(x)$ is

(a) $x(2 + xe^x)$ (b) $x(e^x + 1)$

(c) $2x \cdot e^x$ (d) $x + \ln(x + 1)$

14. The range of the function $g(x)$ is

(a) R (b) $\left[-\dfrac{2}{e},\infty\right)$

(c) $\left[-\dfrac{1}{e},\infty\right)$ (d) $[0, \infty)$

Paragraph 3

Consider a function defined in $[-2, 2]$ is

$$f(x)\begin{cases}\{x\}, & -2\le x<-1\\ |\operatorname{sgn}(x)|, & -1\le x\le 1\\ \{-x\}, & 1<x\le 2\end{cases}$$

where $\{x\}$ denotes the fractional part function.

15. The function $f(x)$ is continuous at $x =$

(a) –2 (b) –3/2

(c) –1 (d) All of these

16. The total number of points of discontinuity of $f(x)$ in $x \in [2, 2]$ are

(a) 1 (b) 3

(c) 4 (d) infinite

MATCHING LIST TYPE

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which one is correct. [Correct ans. 3 marks and wrong ans.-1]

17. Math the following:

	Column I		Column II
P.	Period of the function $f(x) = \lvert\sin(\cos x)\rvert + \cos(\sin x)$	1.	$\pi/2$
Q.	Period of the function $f(x) = [\sin(4x)] + \lvert\cos 4x\rvert$ ([.] denotes the greatest integer function)	2.	can't be determined
R.	If the function $f: R \to R$ be such that $f(x) = \pi x - [\pi x]$, where [.] denotes the greatest function, then the period of the function $f^{-1}(x)$ is	3.	2π
S.	Period of the function $f(x) = \min\{\sin x, \lvert x\rvert\}$ is	4.	π

	P	Q	R	S
(a)	1	2	3	4
(b)	4	1	2	3
(c)	1	2	4	3
(d)	1	4	2	3

18. If $L = \lim\limits_{x\to 0} \dfrac{a\sin x - bx + cx^2 + x^3}{2x^2 \log(1+x) - 2x^3 + x^4}$ and is finite, then match the following

	Column I		Column II
P.	$a + b$	1.	0
Q.	c	2.	6
R.	20 L	3.	12
S.	a	4.	1.5

	P	Q	R	S
(a)	1	2	3	4
(b)	2	4	1	3
(c)	3	1	4	2
(d)	1	4	2	3

19. Math the following.

	Column I		Column II
P.	The ratio of altitude to the radius of the cylinder of maximum value that can be inscribed in a given sphere is	1.	$1/\sqrt{2}$
Q.	The ratio of radius to the altitude of the cone of the greatest volume which can be inscribed in a given sphere is	2.	$\sqrt{2}$
R.	The cone circumscribing of sphere of radius 'r' has the minimum volume if its semi vertical angle is θ is such that $33 \sin \theta =$	3.	32/3
S.	The greatest value of x^3y^4 if $2x + 3y = 7$ and $x \ge 0$, $y \ge 0$ is	4.	11

	P	Q	R	S
(a)	1	2	4	3
(b)	1	4	2	3
(c)	1	2	3	4
(d)	2	1	4	3

20. Match the following:

	Column I		Column II
P.	Range of $\sqrt{[\sin 2x] - [\cos 2x]}$ is	1.	{1, 2, 3}
Q.	Domain of $\sqrt{{}^{x^2+4^x}C_{2x^2+3}}$	2.	{1}
R.	Range of $\sqrt{\log(\cos(\sin x))}$ is	3.	{0, 1}
S.	Range of $[\lvert\sin x\rvert + \lvert\cos x\rvert]$ is	4.	{0, 1}

	P	Q	R	S
(a)	3	1	4	2
(b)	2	1	4	3
(c)	1	2	4	3
(d)	4	1	2	3

ANSWER KEY

PAPER-1

1. (a, b, d) **2.** (c, d) **3.** (a, c, d)
4. (b, d) **5.** (b) **6.** (a, c) **7.** (a, b, d)
8. (a, b, d) **9.** (a, c, d) **10.** (a, c)
11. (0) **12.** (2) **13.** (4) **14.** (9) **15.** (4)
16. (1) **17.** (6) **18.** (3) **19.** (1) **20.** (8)

PAPER-2

1. (b) **2.** (c) **3.** (b) **4.** (a) **5.** (a)
6. (d) **7.** (c) **8.** (c) **9.** (b) **10.** (d)
11. (d) **12.** (c) **13.** (c) **14.** (b) **15.** (d)
16. (b) **17.** (b) **18.** (c) **19.** (d) **20.** (a)

For detailed solution to the Sample Paper, visit our website www. vidyalankar.org

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The readers' & comments and suggestions regarding the problems and solutions offered are always welcome.

1. Let $X = \{x : x = n^3 + 2n + 1, n \in R\}$ and $Y = \{x : x = 3n^2 + 7, n \in R\}$ then $X \cap Y$ is a subset of

(a) $\{x : x = 3n + 5, n \in N\}$
(b) $\{x : x = n^2 + n + 1, n \in N\}$
(c) $\{x : x = 7n - 1, n \in N\}$
(d) $\{x : x = 3n^2 + 5, n \in N\}$

2. If $z \neq 0$ is a complex number such that $\text{Arg}(z) = \pi/4$, then

(a) $\text{Re}(z^2) = 0$ (b) $\text{Im}(z^2) = 0$
(c) $\text{Re}(z^2) = \text{Im}(z^2)$ (d) $\text{Re}(z) = 0$

3. In a triangle PQR, $\angle R = \pi/2$. If $\tan(P/2)$ and $\tan(Q/2)$ are the roots of the equation $ax^2 + bx + c = 0$ where $a \neq 0$, then

(a) $a + b = c$ (b) $b + c = a$
(c) $a + c = b$ (d) $b = c$

4. Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$

then value of $\int_{\pi/4}^{\pi/2} f(x)\,dx$ is

(a) 0 (b) $\pi/48$
(c) $-\dfrac{\pi}{2} - \dfrac{\pi}{15\sqrt{2}}$ (d) $1 - \dfrac{1}{\sqrt{2}} - \dfrac{\pi}{8} - \dfrac{1}{2}\log 2$

5. The determinant of a skew symmetric matrix of order 3 is

(a) 1 (b) 0 (c) -1 (d) 2

6. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\dfrac{a_{10} - 2a_8}{2a_9}$ is

(a) 1 (b) 2 (c) 3 (d) 4

7. If 3[rd], 6[th] and last term of HP is $\dfrac{1}{3}, \dfrac{1}{5}$ and $\dfrac{3}{203}$ respectively. Find number of terms.

(a) 50 (b) 70 (c) 85 (d) 100

8. Consider the A.P. $a_1, a_2, ..., a_n, ...$ and the G.P. $b_1, b_2, ..., b_n, ...$ such that $a_1 = b_1 = 1; a_9 = b_9$ and $\sum_{r=1}^{9} a_r = 369$ then

(a) $b_6 = 27$ (b) $b_7 = 27$
(c) $b_8 = 81$ (d) $b_9 = 18$

9. The coefficient of $a^8 b^4 c^9 d^9$ in $\{ab(c + d) + cd(a + b)\}^{10}$ is

(a) $\dfrac{(10)!}{8!4!9!9!}$ (b) $10!$
(c) 2520 (d) $5!$

10. If $(10)^9 + 2(11)^1 (10)^8 + 3(11)^2 (10)^7 + ... + 10(11)^9 = k(10)^9$, then k is equal to

(a) 100 (b) 110 (c) $\dfrac{121}{10}$ (d) $\dfrac{441}{100}$

SOLUTIONS

1. (c) : $x \in X \cap Y \Rightarrow n^3 + 2n + 1 = 3n^2 + 7$
$\Rightarrow n^3 - 3n^2 + 2n - 6 = 0 \Rightarrow (n - 3)(n^2 + 2) = 0$
$\Rightarrow n = 3$ as $n \in R \Rightarrow x = 3 \times 3^2 + 7 = 34$
In (a) and (b), $x \neq 34$ for any $n \in N$ and in (c), $x = 34$ for $n = 5$.
Hence $X \cap Y$ is a subset of (c).

2. (a) : As $\arg(z) = \pi/4$, we can write

$z = r\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$, where $r = |z|$

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$$\Rightarrow \quad z^2 = r^2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

(Using De Moivre's Theorem)

$$= r^2(0+i) = ir^2 \Rightarrow \text{Re}(z^2) = 0$$

3. (a) : We have

$$\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = \frac{-b}{a} \text{ and } \tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right) = \frac{c}{a}$$

Now, $P + Q = \dfrac{\pi}{2} \Rightarrow \tan\left(\dfrac{P}{2} + \dfrac{Q}{2}\right) = \tan\left(\dfrac{\pi}{4}\right)$

$$\Rightarrow \frac{\tan(P/2) + \tan(Q/2)}{1 - \tan(P/2)\tan(Q/2)} = 1 \Rightarrow \frac{-b/a}{1 - c/a} = \frac{-b}{a-c} = 1$$

$\Rightarrow a - c = -b \Rightarrow c = a + b$

4. (d) : Applying $R_2 \to R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ -\sin^2 x & 0 & 0 \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= \sin^2 x\left[\cos x \operatorname{cosec}^2 x - \cos^2 x(\sec^2 x + \cot x \operatorname{cosec}^2 x)\right]$$

$$= \cos x - \sin^2 x - \frac{\cos^3 x}{\sin x}$$

$$= \cos x - \frac{1}{2}(1 - \cos 2x) - \left(\frac{1}{\sin x} - \sin x\right)\cos x$$

$$\text{Thus } \int_{\pi/4}^{\pi/2} f(x)dx = \int_{\pi/4}^{\pi/2} \cos x dx - \frac{1}{2}\int_{\pi/4}^{\pi/2} dx$$

$$+ \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos 2x dx - \int_{\pi/4}^{\pi/2}\left(\frac{1}{\sin x} - \sin x\right)\cos x dx$$

$$= 1 - \frac{1}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{2}\log 2$$

5. (b) : $|A^T| = |-A| = -|A| \Rightarrow |A| = 0$

6. (c) : The roots of $x^2 - 6x - 2 = 0$ are α and β

So $\alpha + \beta = 6$ and $\alpha\beta = -2$

Then $a_n - 2a_{n-2} = \alpha^n - \beta^n - 2(\alpha^{n-2} - \beta^{n-2})$

$= (\alpha^n - 2\alpha^{n-2}) - (\beta^n - 2\beta^{n-2})$

$= \alpha^{n-2}(\alpha^2 - 2) - \beta^{n-2}(\beta^2 - 2)$ ...(1)

Since α and β are roots of $x^2 - 6x - 2 = 0$

So, $\alpha^2 - 6\alpha - 2 = 0$ or $\alpha^2 - 2 = 6\alpha$

and $\beta^2 - 6\beta - 2 = 0$ or $\beta^2 - 2 = 6\beta$

Putting $\alpha^2 - 2 = 6\alpha$ and $\beta^2 - 2 = 6\beta$ in (1), we get

$$a_n - 2a_{n-2} = \alpha^{n-2}\cdot 6\alpha - \beta^{n-2}\cdot 6\beta$$

$$= 6\left(\alpha^{n-1} - \beta^{n-1}\right) = 6a_{n-1}$$

Hence, $a_n - 2a_{n-2} = 6a_{n-1}$

For $n = 10$, we have $a_{10} - 2a_8 = 6a_9$ or $\dfrac{a_{10} - 2a_8}{2a_9} = 3$

7. (d) : As 3rd, 6th and last term of H.P. are $\dfrac{1}{3}, \dfrac{1}{5}$ and $\dfrac{3}{203}$ respectively.

So, 3rd, 6th & last term of A.P. are $3, 5$ & $\dfrac{203}{3}$ resp.

Let a and d be first term and common difference of A.P.

$T_3 = a + 2d = 3$ ...(1) $T_6 = a + 5d = 5$ ...(2)

$T_n = a + (n-1)d = \dfrac{203}{3}$ ...(3)

On, solving (1) and (2), $d = \dfrac{2}{3}$ and $a = \dfrac{5}{3}$

Putting value of a and d in (3), we get

$n = 100$ $\therefore$ Number of terms is 100.

8. (b) : $a_1 = b_1 = 1; a_9 = 1 + 8d = b_9 = 1\cdot R^8$

$$\text{Now } \sum_{r=1}^{9} a_r = \frac{9}{2}(1 + a_9) = \frac{9}{2}\left(1 + R^8\right) = 369 \Rightarrow R = \sqrt{3}$$

$$\therefore b_7 = 1\cdot R^6 = 1(\sqrt{3})^6 = 27$$

9. (c) : $(abc + abd + acd + bcd)^{10}$

$$= a^{10}b^{10}c^{10}d^{10}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{10}$$

$\therefore$ coefficient of $a^8b^4c^9d^9$

$= \text{coefficient of } a^{-2}b^{-6}c^{-1}d^{-1}$ in

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{10} = \frac{10!}{2!6!1!1!} = \frac{10\times 9\times 8\times 7}{2} = 2520$$

10. (a) : Let $S = (10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 9(11)^8(10) + 10(11)^9$ ...(1)

$\left(\text{A.G.P. with } r = \dfrac{11}{10}\right)$

$$\Rightarrow \frac{11S}{10} = (11)^1(10)^8 + 2(11)^2(10)^7 + \ldots + 9(11)^9 + (11)^{10} \quad \ldots(2)$$

Subtracting (2) from (1), we get

$$-\frac{S}{10} = (10)^9 + (11)^1(10)^8 + (11)^2(10)^7 + \ldots + (11)^9 - (11)^{10}$$

$$= \frac{(10)^9\left[\left(\frac{11}{10}\right)^{10} - 1\right]}{\frac{11}{10} - 1} - (11)^{10}\left(\because S_n = \frac{a(r^n - 1)}{r - 1}, r > 1\right)$$

This gives $S = (10)^{11}$ and hence $k = 100$.

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JEE Main 2018

MOCK TEST PAPER

Series-1

Time: 1 hr 15 min.

The entire syllabus of Mathematics of JEE MAIN is being divided in to eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

Unit No.	Topic	Syllabus In Details
UNIT NO.1	Sets, Relations & Functions	Sets and their representation; Union, intersection and compliment of sets and their algebraic properties; Power set. Ordered pairs, Cartesian product of sets. Definition of relation. Domain and range of a relation. Types of relations, equivalence relations. Function as a special kind of relation from one set to another. Domain, co-domain and range of a function. One-one, into and onto functions, Composition of functions. Real valued function of a real variable, domain and range of these functions, constant, identity, polynomial, rational, modulus, Signum and greatest integer functions with their graphs, sum, difference, product and quotients of functions.
	Trigonometry	Measurement of Trigonometric Angles. Definition of Trigonometric functions, Associated angles
	Co-ordinate Geometry-2D	Cartesian coordinates, distance between two points, section formulae, shift of origin. Locus problems.

1. Which is the simplified representation of $(A' \cap B' \cap C) \cup (B \cap C) \cup (C \cap A)$ Where A, B, C are subsets of universal set S?
(a) A (b) B
(c) C (d) $S \cap (A \cup B \cup C)$

2. For any three sets A_1, A_2, A_3, let $B_1 = A_1$, $B_2 = A_2 - A_1$ and $B_3 = A_3 - (A_1 \cup A_2)$ then
(a) $A_1 \cup A_2 \cup A_3 \supset B_1 \cup B_2 \cup B_3$
(b) $A_1 \cup A_2 \cup A_3 = B_1 \cup B_2 \cup B_3$
(c) $A_1 \cup A_2 \cup A_3 \subset B_1 \cup B_2 \cup B_3$
(d) none of these

3. If $A = \{a, b\}, B = \{c, d\}, C = \{d, e\}$, then $\{(a,c),(a,d),(a,e),(b,c),(b,d),(b,e)\} =$
(s) $A \cap (B \cup C)$ (b) $A \cup (B \cap C)$
(c) $A \times (B \cup C)$ (d) $A \times (B \cap C)$

4. The relation R defined in $A = \{1, 2, 3\}$ by aRb, if $|a^2 - b^2| \leq 5$. Which of the following is false?
(a) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$
(b) $R^{-1} = R$
(c) Domain of $R = \{1, 2, 3\}$
(d) Range of $R = \{5\}$

5. Let n be a fixed positive integer. Define a relation R in the set Z of integers by aRb if and only if $a - b$ divides n. The relation R is
(a) reflexive (b) symmetric
(c) transitive
(d) an equivalence relation.

6. Let R be a relation defined on the set of integers and it is given by $(x, y) \in R \Leftrightarrow |x - y| \leq 1$. Then R is
(a) reflexive and transitive
(b) reflexive and symmetric
(c) an equivalent relation
(d) symmetric and transitive

7. The range of the function $f(x) = \sin[x], -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ where $[x]$ denotes the greatest integer is :
(a) $\{0\}$ (b) $\{0, -1\}$
(c) $\{0, \pm \sin 1\}$ (d) $\{0, -\sin 1\}$

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8. Let $f(x)=\frac{\sqrt{\sin x}}{1+\sqrt[3]{\sin x}}$ Then $D(f)$ contains:
(a) $(0, \pi)$ (b) $(0, 2\pi)$
(c) $(2\pi, 4\pi)$ (d) $(4\pi, 6\pi)$

9. Domain of the function $f(x) = \sin^{-1}(2x^2 + 3x + 1)$ is :
(a) $(-1, 1)$ (b) $(-\infty, \infty)$
(c) $\left[-\frac{3}{2}, 0\right]$ (d) $\left(-\infty, -\frac{1}{2}\right) \cup (2, \infty)$

10. Let $f:(4,6) \to (6,8)$ be a function defined by $f(x)=x+\left[\frac{x}{2}\right]$ (where $[\cdot]$ denotes the greatest integer function), then $f^{-1}(x)$ is equal to
(a) $x-\left[\frac{x}{2}\right]$ (b) $-x-2$
(c) $x - 2$ (d) $\frac{1}{x+\left[\frac{\pi}{6}\right]}$

11. If the real valued function $f(x)=\frac{a^x-1}{x^n(a^x+1)}$ is even, then n is equals to
(a) 2 (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) 3

12. Let $g(x) = 1 + x - [x]$, $[x]$ is the greatest integer not greater than x.
If $f(x)=\begin{cases}-1, & x<0 \\ 0, & x=0 \\ 1, & x>0\end{cases}$, then for all x, $f(g(x))$ equals
(a) x (b) 1 (c) $f(x)$ (d) $g(x)$

13. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is
(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1+\sqrt{1+4\log_2 x})$
(c) $\frac{1}{2}(1-\sqrt{1+4\log_2 x})$
(d) none of these.

14. Let $f(x) = (x + 1)^2 - 1 (x \geq -1)$ then the set $\{x : f^{-1}(x) = f(x)\}$ contains:
(a) $\left\{0, -1, \frac{-3+i\sqrt{3}}{2}, \frac{-3-i\sqrt{3}}{2}\right\}$
(b) $\{0, 11\}$ (c) $\{0, -1\}$
(d) $\{x : x(x - 1) = 0\}$

15. If R_1 and R_2 are symmetric relations (not disjoint) on a set A, then the relation $R_1 \cap R_2$ is
(a) reflexive (b) symmetric
(c) transitive (d) none of these.

16. If $f(x)=\begin{cases}x^3+1, & x<0 \\ x^2+1, & x\geq 0\end{cases}$, $g(x)=\begin{cases}(x-1)^{\frac{1}{3}}, & x<1 \\ (x-1)^{\frac{1}{2}}, & x\geq 1\end{cases}$
then $(gof)(x)$ =
(a) $x, \forall x \in R$ (b) $x-1, \forall x \in R$
(c) $x+1, \forall x \in R$ (d) none of these

17. Let, then $f(x)=-1+|x-1|,\ -1\leq x\leq 3$ and $g(x)=2-|x+1|, -2\leq x\leq 2$, then $(fog)(x)$ =
(a) $\begin{cases}x+1, & -2\leq x\leq 0 \\ x-1, & 0<x<2\end{cases}$ (b) $\begin{cases}x-1, & -2\leq x\leq 0 \\ x+1, & 0<x<2\end{cases}$
(c) $\begin{cases}-x-1, & -2\leq x\leq 0 \\ x-1, & 0<x<2\end{cases}$ (d) none of these

18. $f : R \times R \to R$ such that $f(x + iy) = \sqrt{x^2+y^2}$ then f is
(a) many-one into (b) one-one and onto
(c) many-one and onto (d) one-one and into.

19. Let U be a universal set and $A \cup B \cup C = U$. Then $((A - B) \cup (B - C) \cup ((C - A))' =$
(a) $A\cup B\cup C$ (b) $A\cap B\cap C$
(c) $A\cup(B\cap C)$ (d) $A\cap(B\cup C)$

20. The period of the function $f(x)=|\cos 4x|+|\sin 4x|$ is
(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

21. If $\theta=\frac{\pi}{4n}$, Where n is positive integer, then the value of $\tan\theta\tan 2\theta\tan 3\theta.....\tan(2n-1)\theta=$
(a) 1 (b) $\tan\theta$ (c) $\cot\theta$ (d) 0

22. If a and b are positive the minimum value of $\frac{a^2}{\cos^2 x}+\frac{b^2}{\sin^2 x}$ is
(a) $a + b$ (b) $a^2 + b^2$
(c) $(a + b)^2$ (d) $(a - b)^2$

23. If $1+\sin x+\sin^2 x+...\infty=4+2\sqrt{3}, 0<x<\pi$ then x equal to
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ or $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

24. If $x=y\cos\frac{2\pi}{3}=z\cos\frac{4\pi}{3}$, then $xy+yz+zx=$
(a) -1 (b) 0 (c) 1 (d) 2

25. In the interval $\left[\frac{5\pi}{4}, \frac{3\pi}{2}\right]$, the value of

$\sqrt{\frac{1}{\cos^2\alpha} - \frac{2}{\cot\alpha}} =$

(a) $1-\tan\alpha$ (b) $\tan\alpha - 1$
(c) $\cot\alpha - 1$ (d) $1-\cot\alpha$

26. If the orthocentre and centroid of a triangle are (–3, 5) and (3, 3) then the circumcentre is
(a) (6, 2) (b) (0, 8)
(c) (6, – 2) (d) (0, 4).

27. Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2$ will not contain term in y and the constant term, is

(a) $\left(\frac{3}{4}, 2\right)$ (b) $\left(\frac{3}{4}, -2\right)$
(c) $\left(-\frac{3}{4}, -2\right)$ (d) $\left(-\frac{3}{4}, 2\right)$

28. If in a ΔABC (whose circumcentre is origin), $a \le \sin A$, then for any point (x, y) inside the circum circle of triangle ABC :

(a) $|xy| \le \frac{1}{8}$ (b) $|xy| > \frac{1}{8}$
(c) $\frac{1}{8} < |xy| < \frac{1}{2}$ (d) none of these

29. The coordinates of B and C of a triangle ABC are (–6, 0) and (6, 0). If $\angle BAC = 90°$, the locus of the centroid of the triangle is
(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 4$
(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 36$

30. Area of the triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2) where a, x_1, x_2 are in G.P. with common ratio r and b, y_1, y_2 are in G.P. with common ratio s, is

(a) $\frac{ab}{2}(r-1)(s-1)(s-r)$

(b) $\frac{1}{2}ab(r+1)(s+1)(s-r)$

(c) $\frac{1}{2}ab(r+1)(s+1)(r-s)$

(d) $\frac{1}{2}ab(r+1)(s+1)(r-s)$

SOLUTIONS

1. (c): $(A' \cap B' \cap C) \cup (B \cap C) \cup (C \cap A)$

$= \left[(A \cup B)' \cap C\right] \cup \left[(B \cup A) \cap C\right]$

$= \left[C \cap (A \cup B)'\right] \cup \left[C \cap (A \cup B)\right]$

$= C \cap \left[(A \cup B)' \cup (A \cup B)\right] = C \cap U = C$

2. (b) : $B_1 \cup B_2 = A_1 \cup (A_2 - A_1) = A_1 \cup (A_2 \cap A_1')$

$= (A_1 \cup A_2) \cap (A_1 \cup A_1') = (A_1 \cup A_2) \cap U = A_1 \cup A_2$

$B_1 \cup B_2 \cup B_3 = (B_1 \cup B_2) \cup B_3 = (A_1 \cup A_2) \cup [A_3 - (A_1 \cup A_2)]$

$= (A_1 \cup A_2) \cup \left[A_3 \cap (A_1 \cup A_2)'\right]$

$= \left[(A_1 \cup A_2) \cup A_3\right] \cap \left[(A_1 \cup A_2) \cup \left((A_1 \cup A_2)'\right)\right]$

$= [A_1 \cup A_2 \cup A_3] \cap U = A_1 \cup A_2 \cup A_3$

3. (c) : Since the elements of the given set are ordered pairs ∴ it has been derived from a cartesian product. Again since the number of elements of set A is 2 and the number of elements of the given set is 6 ∴ the number of elements of the set with which the cartesian product of set A produces the given set is 6 ÷ 2 = 3.
∴ the set will be $B \cup C = \{c, d\} \cup \{d, e\} = \{c, d, e\}$
∴ the given set will be equal to $A \times (B \cup C)$.

4. (d) : Given, $A = \{1, 2, 3\}$ and $R = \{(a, b) : |a^2 - b^2| \le 5\}$
Now, $|1^2 - 1^2| = 0 < 5$ ∴ $(1, 1) \in R$ *i.e.* $1R1$
$|2^2 - 2^2| = 0 < 5$ ∴ $(2, 2) \in R$ *i.e.*, $2R2$
$|3^2 - 3^2| = 0 < 5$ ∴ $(3, 3) \in R$ *i.e.*, $3R3$
$1^2 - 2^2| = 3 < 5$ ∴ $(1, 2) \in R$ *i.e.*, $1R2$
$|1^2 - 3^2| = 8 > 5$ ∴ $(1, 3) \notin R$ *i.e.*, $1 \not R\, 3$
$|2^2 - 3^2| = 5$ ∴ $(2, 3) \in R$ *i.e.*, $2R3$
Similarly $(2, 1) \in R$ *i.e.*, $2R1$, $(3, 2) \in R$ *i.e.*, $3R2$
Thus $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3) (2, 1), (3, 2)\}$
Now $R^{-1} = \{ (y, x) : (x, y) \in R\}$
$= \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (1, 2), (2, 3)\} = R$
Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$ and range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$

5. (d) : The given relation may be written in set-builder form as
$R = \{(a, b) : a - b$ divides n and $a, b \in Z\}$
As $a - a = 0$ and 0 divides n ∴ $(a, a) \in R$
∴ R is reflexive.
Let $a, b \in Z$ such that $(a, b) \in R$
Then $(a, b) \in R \Rightarrow a - b$ divides n.
$a - b = nk$ for some integer $k \Rightarrow b - a = n(-k)$
∴ $(a, b) \in R \Rightarrow (b, a) \in R$
∴ R is symmetric.
Now, $(a,b), (b,c) \in R$

Now, $a - b = nc_1$ and $b - c = nc_2$ for some integers c_1 and c_2.

$\therefore \quad (a - b) + (b - c) = n(c_1 + c_2)$

$\Rightarrow \quad a - c = nk$ where $k = c_1 + c_2$, an integer.

$\Rightarrow \quad (a, c) \in R.$

$\therefore \quad (a,b), (b,c) \in R \Rightarrow (a,c) \in R$

$\therefore$ R is transitive and hence R is an equivalence relation.

6. (b) : The given relation may be written in set-builder form as $R = \{(x, y) : |x - y| \le 1 \text{ and } x, y \in Z\}$

As $x - x = 0 \Rightarrow |x - x| = 0 \le 1$ (true)

$\therefore \quad (x, x) \in R$ therefore R is reflexive.

Let $x, y \in Z$ such that $(x, y) \in R$, then

$(x, y) \in R \Rightarrow |x - y| \le 1 \Rightarrow |y - x| \le 1 \Rightarrow (y, x) \in R$

Therefore, R is symmetric.

Let $x, y, z \in Z$ such that $(x, y), (y, z) \in R$, then $|x - y| \le 1$ and $|y - z| \le 1$

$\not\Rightarrow |x - z| \le 1$

Thus $(x, y), (y, z) \in R \not\Rightarrow (x, z) \in R$

Therefore, R is not transitive.

7. (d) : The given function is $f(x) = \sin[x]$

Clearly $-\frac{\pi}{4} \le x < 0 \Rightarrow [x] = -1$

and $0 \le x < \frac{\pi}{4} \Rightarrow [x] = 0$

8. (a) : The given function is $f(x) = \dfrac{\sqrt{\sin x}}{1 + \sqrt[3]{\sin x}}$

$f(x)$ is defined when $\sin x > 0$ and $1 + \sqrt[3]{\sin x} \ne 0$

Clearly $x \in (0, \pi)$, thus the required domain is $(0, \pi)$

9. (c) : The given function $f(x) = \sin^{-1}(2x^2 + 3x + 1)$

$f(x)$ will be defined when $-1 \le 2x^2 + 3x + 1 \le 1$

$$\Rightarrow -\frac{1}{2} \le x^2 + \frac{3}{2}x + \frac{1}{2} \le \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} \le (x)^2 + 2x\frac{3}{4} + \left(\frac{3}{4}\right)^2 + \frac{1}{2} - \frac{9}{16} \le \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} \le \left(x + \frac{3}{4}\right)^2 - \frac{1}{16} \le \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} + \frac{1}{16} \le \left(x + \frac{3}{4}\right)^2 \le \frac{1}{2} + \frac{1}{16}$$

$$\Rightarrow -\frac{7}{16} \le \left(x + \frac{3}{4}\right)^2 \le \frac{9}{16} \Rightarrow 0 \le \left(x + \frac{3}{4}\right)^2 \le \frac{9}{16}$$

$$\Rightarrow -\frac{3}{4} \le \left(x + \frac{3}{4}\right) \le \frac{3}{4} \quad \left[\because 0 \le x^2 \le a^2 \Rightarrow -a \le x \le a\right]$$

$$\Rightarrow -\frac{3}{2} \le x \le 0$$

10. (c): Here, we have $4 < x < 6 \Rightarrow 2 < \frac{x}{2} < 3 \ \therefore \left[\frac{x}{2}\right] = 2$

So, $f(x) = x + \left[\frac{x}{2}\right] = x + 2, x \in (4, 6)$

Let $y \in (6, 8)$ such that $f(x) = y$. Now $f(x) = y$ gives

$y = x + 2 \Rightarrow x = y - 2$

$\therefore f^{-1}(y) = y - 2 \Rightarrow f^{-1}(x) = x - 2$

11. (d) : Given $f(x) = \dfrac{a^x - 1}{x^n(a^x + 1)}$ is even.

$$\therefore \quad f(x) = f(-x) \Rightarrow \frac{a^x - 1}{x^n(a^x + 1)}$$

$$= \frac{a^{-x} - 1}{(-x)^n(a^{-x} + 1)} = \frac{1 - a^x}{(-x)^n(1 + a^x)} \frac{-(a^x - 1)}{(-1)^n x^n(a^x + 1)}$$

$$\Rightarrow 1 = \frac{-1}{(-1)^n}$$

Clearly the above relation is true for $n = 3$

12. (b) : The given functions are

$$g(x) = 1 + x - [x] \text{ and } f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

We see that $g(x) = 1 + x - [x] = 1 + (x - [x])$

$= 1 + \{x\} \ge 1 \ (\because \ 0 \le \{x\} < 1)$

$\therefore \quad f(g(x)) = 1$ for all x

13. (b) : Here $f(x) = 2^{x(x-1)} = y$ (say)

$\Rightarrow \log_2 y = x(x-1)\log_2 2 \Rightarrow x(x-1) - \log_2 y = 0$

$$\Rightarrow x^2 - x - \log_2 y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$

$$\therefore \quad x = \frac{1 + \sqrt{1 + 4\log_2 y}}{2} \quad (\because \ x > 0)$$

$$f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$$

14. (c): Given that $f(x) = (x + 1)^2 - 1$

Let $y = f(x) \Rightarrow (x + 1)^2 - 1 = y$

$\Rightarrow x = \sqrt{y + 1} - 1 \ (x \ge -1).$

$\Rightarrow f^{-1}(y)=\sqrt{y+1}-1 \Rightarrow f^{-1}(x)=\sqrt{x+1}-1$

Now $f(x) = f^{-1}(x)$ gives

$$\Rightarrow (x+1)^2-\sqrt{x+1}=0 \Rightarrow \sqrt{x+1}\left\{(x+1)^{\frac{3}{2}}-1\right\}=0$$

$$\Rightarrow x+1=0 \text{ or } (x+1)^{\frac{3}{2}}=1 \Rightarrow x=-1 \text{ or } x=0$$

Thus $\{x : f^{-1}(x) = f(x)\} = \{-1, 0\}$

15. (b) : Let $a, b \in A$ such that $(a, b) \in R_1 \cap R_2$ then $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow$ aR_1b and aR_2b

$\Rightarrow$ bR_1a and bR_2a [$\because$ R_1 and R_2 are symmetric]

$\Rightarrow$ $(b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow$ $(b, a) \in R_1 \cap R_2$

Hence $R_1 \cap R_2$ is symmetric

16. (a) : Here, $f(x)=\begin{cases} x^3+1, & x<0 \\ x^2+1, & x\ge 0 \end{cases}$

and $=\begin{cases} (x-1)^{\frac{1}{3}}, & x<1 \\ (x-1)^{\frac{1}{2}}, & x\ge 1 \end{cases}$

When $x < 0$, then $(gof)(x)=g(f(x))=g(x^3+1)$

$$=\left\{\left(x^3+1\right)-1\right\}^{\frac{1}{3}} \; [\because \; x<0 \therefore x^3+1<1]=\left(x^3\right)^{\frac{1}{3}}=x$$

When $x \ge 0$, then $(gof)(x)=g(f(x))$

$$=g(x^2+1)=\left\{\left(x^2+1\right)-1\right\}^{\frac{1}{2}} \; [\because x\ge 0 \therefore x^3+1\ge 1]$$

$$=\left(x^2\right)^{\frac{1}{2}}=|x|=x \quad [\because \; x \ge 0]$$

Therefore $(gof)(x) = x$ for all $x \in R$

17. (d): $(fog)(x)=\begin{cases} f(x+3), & 1\le x+3\le 2 \\ f(-x+1), & -1\le -x+2\le 2 \end{cases}$

$$=\begin{cases} f(x+3), & 1\le x+3\le 2 \\ f(-x+1), & -1\le -x+1\le 1 \\ f(-x+1), & 1\le -x+1\le 2 \end{cases} = \begin{cases} x+1, & -2\le x\le -1 \\ -x-1, & -1\le x\le 0 \\ x-1, & 0\le x\le 2 \end{cases}$$

18. (a) : Since $f(x + iy) = f(x - iy)$ $\therefore$ f is many one.

Also Range $\in R^+$ and Co-domain $\in R$

$\therefore$ range $\subset$ co domain $\Rightarrow f$ is into

Hence f is many one into.

19. (b) : $(A-B)\cup(B-C)\cup(C-A)$

$=(A\cup B\cup C)-(A\cap B\cap C)$

$=(A\cap B\cap C)' \; [\because A\cup B\cup C=U]$

$\therefore \left[(A-B)\cup(B-C)\cup(C-A)\right]^c$

$=\left[(A\cap B\cap C)'\right]' = A\cap B\cap C.$

20. (d) : Here $f(x) = |\cos 4x| + |\sin 4x|$

$$\therefore f\left(x+\frac{\pi}{8}\right)=\left|\cos 4\left(x+\frac{\pi}{8}\right)\right|+\left|\sin 4\left(x+\frac{\pi}{8}\right)\right|$$

$=|\sin 4x|+|\cos 4x|= f(x).$

$\because f\left(x+\frac{\pi}{8}\right)= f(x)$, therefore the period of the given function is $\frac{\pi}{8}$.

21. (a) : $\tan\theta\tan 2\theta\tan 3\theta.....\tan(2n-1)\theta$

$$\left[\theta=\frac{\pi}{4n} \therefore 2n\theta=\frac{\pi}{2} \text{ and } \tan(2n-1)\theta=\tan(2n\theta-\theta)\right.$$

$$\left.=\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta\right]$$

$= \tan\theta\tan((2n - 1)\theta)\tan 2\theta\tan((2n - 2)\theta)....\tan n\theta\cot n\theta$

$= (\tan\theta\cot\theta)(\tan 2\theta\cot 2\theta)....(\tan n\theta\cot n\theta) = 1 \cdot 1 \cdot 1.... \cdot 1 = 1$

22. (c) : We have, $\frac{a^2}{\cos^2 x}+\frac{b^2}{\sin^2 x}$

$= a^2\sec^2x + b^2\operatorname{cosec}^2x$

$= a^2(1 + \tan^2x) + b^2(1 + \cot^2x)$

$= a^2 + b^2 + (a\tan x)^2 + (b\cot x)^2$

$= a^2 + b^2 + (a\tan x - b\cot x)^2 + 2ab\tan x\cot x$

$\ge a^2 + b^2 + (a\tan x - b\cot x)^2$

$$\therefore \frac{a^2}{\cos^2 x}+\frac{b^2}{\sin^2 x}\ge (a+b)^2$$

23. (d): $1+\sin x+\sin^2 x+...\infty=4+2\sqrt{3}, 0<x<\pi$

$$\Rightarrow \frac{1}{1-\sin x}=4+2\sqrt{3} \Rightarrow 1-\sin x=\frac{1}{4+2\sqrt{3}}$$

$$=\frac{4-2\sqrt{3}}{4}=\frac{2-\sqrt{3}}{2}$$

$$\Rightarrow \sin x=1-\frac{2-\sqrt{3}}{2}=\frac{\sqrt{3}}{2} \quad \Rightarrow x=\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

24. (b): $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3} = \frac{1}{k}\text{(say)}$

$$\therefore \frac{1}{x} = k, \frac{1}{y} = k\cos\frac{2\pi}{3} = -\frac{k}{2}, \text{ and } \frac{1}{z} = k\cos\frac{4\pi}{3} = \frac{-k}{2}$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = k - \frac{k}{2} - \frac{k}{2} = 0 \Rightarrow xy + yz + zx = 0$$

25. (b) : $\sqrt{\frac{1}{\cos^2\alpha} - \frac{2}{\cot\alpha}} = \sqrt{\sec^2\alpha - 2\tan\alpha}$

$$= \sqrt{1+\tan^2\alpha - 2\tan\alpha} = \sqrt{(1-\tan\alpha)^2}$$

$$= |1-\tan\alpha| = \tan\alpha - 1$$

$$\left[\because \tan\alpha > 1 \text{ when } \alpha \in \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right]\right]$$

26. (a) : From geometry , we know that the centroid divides the line segment joining orthocentre and circumcentre in the ratio 2:1.

$$\therefore 3 = \frac{2\times x + 1\times(-3)}{2+1}$$

$$\Rightarrow x = 6 \text{ and } 3 = \frac{2\times y + 1\times 5}{2+1} \Rightarrow y = 2$$

So, the required circumcentre is (6, 2) .

27. (b) : Let us shift the origin to a point(h, k) such that $x = x' + h$ and $y = y' + k$

The given equation of the curve is

$y^2 + 4y + 8x - 2 = 0$ ......(1)

Now by translation (1) becomes

$(y' + k)^2 + 4(y' + k) + 8(x' + h) - 2 = 0$

$\Rightarrow\ y'^2 + 2(k + 2)y' + 8x' + k^2 + 4k + 8h - 2= 0$

Now in order to vanish the y-term and constant term, we make

$k + 2 = 0$ and $k^2 + 4k + 8h - 2 = 0$

$\Rightarrow\ k = -2$ and $h = \frac{3}{4}$

Therefore, the origin must be shifted to $\left(\frac{3}{4}, -2\right)$

28. (a) : Given $a \le \sin A \Rightarrow \frac{a}{\sin A} \le 1$

$$\Rightarrow 2R \le 1 \Rightarrow R \le \frac{1}{2}\ \left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$$

Thus for any point (x, y) inside the circumcircle , we have $x^2 + y^2 \le \frac{1}{4}$

Again we have

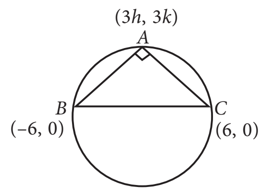
$$\frac{x^2+y^2}{2} \ge \sqrt{x^2y^2} \ge |xy|\ [\because\ \text{A.M.} \ge \text{G.M.}] \therefore |xy| \le \frac{1}{8}$$

29. (b) : Let (h, k) be the co-ordinates of the centroid. Let the co-ordinates of A be (α, β).

$$\therefore\ h = \frac{\alpha + 6 - 6}{3} \Rightarrow \alpha = 3h$$

$$\text{and } k = \frac{\beta + 0 + 0}{3} \Rightarrow \beta = 3k$$

Thus $A(\alpha,\beta) = A(3h,3k)$. Now $AB^2 + AC^2 = BC^2$ gives



$(3h + 6)^2 + (3k)^2 + (3h - 6)^2 + (3k)^2 = (6 + 6)^2$

$\Rightarrow\ 9h^2 + 36 + 9k^2 + 9h^2 + 36 + 9k^2 = 144$

$\Rightarrow\ 18h^2 + 18k^2 = 72$

$\Rightarrow\ h^2 + k^2 = 4$

$\therefore$ The locus of the centroid is $x^2 + y^2 = 4$

30. (a) : We have, $x_1 = ar$, $x_2 = ar^2$, $y_1 = bs$, $y_2 = bs^2$

The area of the triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}ab\begin{vmatrix} 1 & 1 & 1 \\ r & s & 1 \\ r^2 & s^2 & 1 \end{vmatrix} = \frac{1}{2}ab\begin{vmatrix} 1 & 1 & 1 \\ r-1 & s-1 & 0 \\ r^2-1 & s^2-1 & 0 \end{vmatrix}$$

(Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$)

$$= \frac{1}{2}ab(r-1)(s-1)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ r+1 & s+1 & 0 \end{vmatrix}$$

(Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$)

and taking $(r - 1)$, $(s - 1)$ common from C_1 and C_2 respectively

$$= \frac{1}{2}ab(r-1)(s-1)(s-r)$$

◈◈

PRACTICE PAPER 2018

Useful for JEE (MAIN & ADVANCED), BITSAT and other Engineering Exams

1. Which of the following is negative?
   (a) $\cos(\tan^{-1}(\tan 4))$ (b) $\sin(\cot^{-1}(\cot 4))$
   (c) $\tan(\cos^{-1}(\cos 5))$ (d) $\cot(\sin^{-1}(\sin 4))$

2. If in a right angled triangle ABC, $4 \sin A \cos B - 1 = 0$ and $\tan A$ is real then A, B, C are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) none of these

3. If α and β are roots of the equation $ax^2 + bx + c = 0$ then roots of the equation $a(2x + 1)^2 - b(2x + 1)(3 - x) + c(3 - x)^2 = 0$ are
   (a) $\frac{2\alpha+1}{\alpha-3}, \frac{2\beta+1}{\beta-3}$ (b) $\frac{3\alpha+1}{\alpha-2}, \frac{3\beta+1}{\beta-2}$
   (c) $\frac{2\alpha-1}{\alpha-2}, \frac{2\beta+1}{\beta-2}$ (d) none of these

4. Sum of series $\sum_{r=1}^{n}(r^2+1)\,r!$ is
   (a) $(n + 1)!$ (b) $(n + 2)! - 1$
   (c) $n(n + 1)!$ (d) none of these

5. If the sides a, b, c of a triangle ABC are in A.P. then $\frac{b}{c}$ belongs to
   (a) $(0, 2/3)$ (b) $(1, 2)$
   (c) $(2/3, 2)$ (d) $(2/3, 7/3)$

6. The sum of coefficients of the last eight terms in the expansion of $(1 + x)^{16}$ is equal to
   (a) 2^{15} (b) 2^{14}
   (c) $2^{15}-\frac{1}{2}\frac{(16)!}{(8!)^2}$ (d) none of these

7. If $(1 + x)^5 = a_0 + a_1x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5x^5$. Then the value of $(a_0 - a_2 + a_4)^2 + (a_1 - a_3 + a_5)^2$ is equal to
   (a) 243 (b) 32
   (c) 1 (d) 2^{10}

8. If all the words formed from the letters of the word "HORROR" are arranged in the opposite order as they are in a dictionary, then the rank of the word "HORROR" is
   (a) 57 (b) 58 (c) 56 (d) 59

9. Let $ax + by + c = 0$ be a variable straight line, where a, b and c are 1st, 3rd and 7th terms of an increasing A.P. respectively. Then the variable straight line always passes through a fixed point which lies on
   (a) $y^2 = 4x$ (b) $x^2 + y^2 = 5$
   (c) $3x + 4y = 9$ (d) $x^2 + y^2 = 13$

10. Three equal circles each of radius r touch one another. The radius of the circle touching all the three given circles internally is
    (a) $(2+\sqrt{3})\,r$ (b) $\frac{(2+\sqrt{3})}{\sqrt{3}}\,r$
    (c) $\frac{(2-\sqrt{3})}{\sqrt{3}}\,r$ (d) $(2-\sqrt{3})\,r$

11. The equation of the tangent to the parabola $y = (x - 3)^2$ parallel to the chord joining the points $(3, 0)$ and $(4, 1)$ is
    (a) $2x - 2y + 6 = 0$ (b) $2y - 2x + 6 = 0$
    (c) $4y - 4x + 13 = 0$ (d) $4x + 4y = 13$

12. The number of rational points on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is
    (a) ∞ (b) 4 (c) 0 (d) 2

13. The angle between the tangents from $(-2, -1)$ to the hyperbola $2x^2 - 3y^2 = 6$ is
    (a) $\tan^{-1}(2)$ (b) $\pi/3$
    (c) $\tan^{-1}(1/2)$ (d) $\pi/6$

14. A function $F(x)$ satisfies the functional equation $x^2 F(x) + F(1 - x) = 2x - x^4$ for all real x. $F(x)$ must be
    (a) x^2 (b) $1 - x^2$
    (c) $1 + x^2$ (d) $x^2 + x + 1$

15. $\lim_{x\to 0} \frac{(2^{\sin x}-1)[\ln(1+\sin 2x)]}{x\tan^{-1}x}$ is equal to

(a) ln2 (b) 2 ln2 (c) $(\ln 2)^2$ (d) 0

16. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at

(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

17. If $y=\frac{1}{x}$, then the value of $\frac{dy}{\sqrt{1+y^4}}+\frac{dx}{\sqrt{1+x^4}}+3$ is equal to

(a) 0 (b) 3 (c) 4 (d) –3

18. The points of contact of the vertical tangents to the curve whose parametric equation is given as $x = 2 - 3\sin\theta, y = 3 + 2\cos\theta$ (where θ is a parameter) are

(a) (2, 5), (2, 1) (b) (–1, 3), (5, 3)
(c) (2, 5), (5, 3) (d) (–1, 3), (2, 1)

19. If $f(x)$ and $g(x)=f(x)\sqrt{1-2(f(x))^2}$ are monotonically increasing, then $\forall\ x \in R$

(a) $|f(x)| \le 1$ (b) $|f(x)| < \frac{2}{3}$
(c) $|f(x)| < \frac{1}{2}$ (d) $|f(x)| < \frac{1}{\sqrt{2}}$

20. Let $f(x)=\begin{cases} 2x^2+2/x^2 & ; \quad 0<|x|\le 2 \\ 1 & ; \quad x=0 \end{cases}$

Then $f(x)$ has

(a) least value 4 but no greatest value
(b) greatest value 4
(c) neither greatest nor least value
(d) least value 1 but no greatest value

21. If $f\left(\frac{3x-4}{3x+4}\right)=x+2$ then $\int f(x)dx$ is equal to

(a) $e^{x+2}\ln\left|\frac{3x-4}{3x+4}\right|+c$

(b) $-\frac{8}{3}\ln|(1-x)|+\frac{2}{3}x+c$

(c) $\frac{8}{3}\ln|x-1|+\frac{x}{3}+c$

(d) none of these

22. If $I_1=\int_0^{\pi/2}\frac{x}{\sin x}dx$ and $I_2=\int_0^1\frac{\tan^{-1}x}{x}dx$, then $\frac{I_1}{I_2}=$

(a) 1 (b) 1/2 (c) 2 (d) $\pi/2$

23. If A_n is the area bounded by $y = x$ and $y = x^n$, $n \in N$, then $A_2 . A_3 .... A_n =$

(a) $\frac{1}{n(n+1)}$ (b) $\frac{1}{2^n n(n+1)}$

(c) $\frac{1}{2^{n-1}n(n+1)}$ (d) $\frac{1}{2^{n-2}n(n+1)}$

24. The degree of the differential equation whose general solution is given by

$$y=(c_1+c_2)\cos(x+c_3)-c_4e^{x+c_5}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is

(a) 5 (b) 4 (c) 1 (d) 2

25. If $|z^2 - 1| = |z|^2 + 1$ then z lies on a

(a) circle (b) parabola
(c) ellipse (d) none of these

26. A can hit a target 4 times in 5 shots, B three times in 4 shots and C twice in 3 shots. They fire a target if exactly two of them hit the target then the chance that it is C who has missed is

(a) 6/13 (b) 1/5
(c) 4/5 (d) 4/15

27. In a quadrilateral $ABCD$, let

$$\Delta=\begin{vmatrix}\cos A & \sin A & \cos(A+D)\\ \cos B & \sin B & \cos(B+D)\\ \cos C & \sin C & \cos(C+D)\end{vmatrix}, \text{ then } \Delta \text{ is}$$

(a) independent of A and B only
(b) independent of B and C only
(c) independent of A, B and C only
(d) independent of A, B, C and D all

28. If $A=\begin{bmatrix}0 & -1 & 2\\ 1 & 0 & 3\\ -2 & -3 & 0\end{bmatrix}$, then $A + 2A^T$ equals

(a) A (b) $-A^T$ (c) A^T (d) $2A^2$

29. A vector of magnitude 3, bisecting the angle between the vectors $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$ and making an obtuse angle with $\vec{b}$ is

(a) $\frac{3\hat{i}-\hat{i}}{\sqrt{6}}$ (b) $\frac{\hat{i}+3\hat{j}-2\hat{k}}{\sqrt{14}}$

(c) $\frac{3(\hat{i}+3\hat{j}-2\hat{k})}{\sqrt{14}}$ (d) $\frac{3\hat{i}-\hat{j}}{\sqrt{10}}$

30. If plane $ax - by + cz = d$ contains the line $\frac{x-a}{a} = \frac{y-2d}{b} = \frac{z-c}{c}$, then $\frac{b}{d} =$

(a) 1 (b) 2 (c) 0 (d) –3

SOLUTIONS

1. **(d) :** (a) $\cos(\tan^{-1}(\tan 4)) = \cos(\tan^{-1}\tan(4-\pi))$
$= \cos(4-\pi) = -\cos 4 > 0$
(b) $\sin(\cot^{-1}(\cot 4)) = \sin(\cot^{-1}\cot(4-\pi))$
$= \sin(4-\pi) = -\sin 4 > 0$
(c) $\tan(\cos^{-1}(\cos 5)) = \tan(\cos^{-1}\cos(2\pi-5))$
$= \tan(2\pi - 5) = -\tan 5 > 0$
(d) $\cot(\sin^{-1}(\sin 4)) = \cot(\sin^{-1}\sin(\pi-4))$
$= \cot(\pi - 4) = -\cot 4 < 0$

2. **(a) :** Since $4\sin A\cos B = 1$, so A and B cannot be 90º (as if $B = 90º$, then $\cos B = 0$ and if $A = 90°$, $\tan A$ is not defined)
$\therefore$ $C = 90º$ and $B = 90° - A$
$\therefore$ $4\sin A\cos(90° - A) = 1$
$\Rightarrow \sin^2 A = \frac{1}{4} \Rightarrow \sin A = \frac{1}{2} \Rightarrow A = \frac{\pi}{6}$
$\Rightarrow B = \frac{\pi}{3}$
So angle $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ are in A.P.

3. **(b) :** $a\frac{(2x+1)^2}{(x-3)^2} + b\frac{(2x+1)}{(x-3)} + c = 0$
$\Rightarrow \frac{2x+1}{x-3} = \alpha$ or $\frac{2x+1}{x-3} = \beta$
$\Rightarrow 2x + 1 = \alpha x - 3\alpha \Rightarrow x(\alpha - 2) = 1 + 3\alpha$
$\Rightarrow x = \frac{1+3\alpha}{\alpha-2}, \frac{1+3\beta}{\beta-2}$

4. **(c) :** $T_r = (r^2 + 1 + r - r)\, r!$
$\Rightarrow T_r = r(r+1)! - (r-1)\,r!$
$S_n = n(n+1)!$

5. **(c) :** $a = 2b - c$
$a + b > c \Rightarrow 2b - c + b > c \Rightarrow \frac{b}{c} > \frac{2}{3}$
Also, $b + c > a \Rightarrow b + c > 2b - c \Rightarrow \frac{b}{c} < 2$
Again $c + a > b \Rightarrow 2b > b \Rightarrow b > 0$
$\therefore \frac{b}{c} \in \left(\frac{2}{3}, 2\right)$

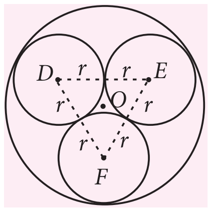
6. **(c) :** Sum of the coefficients of last eight terms
$= {}^{16}C_9 + {}^{16}C_{10} + \ldots + {}^{16}C_{16} = \frac{2^{16} - {}^{16}C_8}{2}$

7. **(b) :** Put $x = i$
$(1+i)^5 = (a_0 - a_2 + a_4) + i(a_1 - a_3 + a_5)$
$\Rightarrow |1+i|^5 = |(a_0 - a_2 + a_4) + i(a_1 - a_3 + a_5)|$
$\Rightarrow (a_0 - a_2 + a_4)^2 + (a_1 - a_3 + a_5)^2 = 2^5 = 32$

8. **(b) :** RRROOH
R □□□□□ $\frac{5!}{2!2!} = 30$
O □□□□□ $\frac{5!}{3!} = 20$
HR □□□□ $\frac{4!}{2!2!} = 6$
HORRRO – 1 HORROR – 1
Total = 30 + 20 + 6 + 1 + 1 = 58

9. **(d) :** Let the common difference of A.P. is d then $b = a + 2d$ and $c = a + 6d$, so variable straight line will be
$ax + (a + 2d)y + a + 6d = 0$
$\Rightarrow a(x + y + 1) + d(2y + 6) = 0$
which always passes through (2, –3).

10. **(b) :** ΔDEF is equilateral with side $2r$. If radius of circumcircle DEF is R_1, then



Area of $\Delta DEF = \frac{\sqrt{3}}{4}(2r)^2 = \sqrt{3}r^2$
$\sqrt{3}r^2 = \frac{2r \cdot 2r \cdot 2r}{4R_1} \Rightarrow R_1 = \frac{2r}{\sqrt{3}}$
$\therefore$ Radius of the circle touching all the three given circles $= r + R_1$
$= r + \frac{2r}{\sqrt{3}} = \frac{(2+\sqrt{3})r}{\sqrt{3}}$

MPP-3 CLASS XII ANSWER KEY

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(b)	**5.**	(b)
6.	(b)	**7.**	(a,b,d)	**8.**	(b,d)	**9.**	(a,c)		
10.	(a,b,c,d)			**11.**	(a,b,c)	**12.**	(a,b,c,d)		
13.	(b,c)	**14.**	(b)	**15.**	(d)	**16.**	(b)	**17.**	(3)
18.	(3)	**19.**	(1)	**20.**	(7)				

11. (c) : $y' = 2(x-3) = 1$ gives the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ and the required tangent is $y - \frac{1}{4} = 1\left(x - \frac{7}{2}\right)$

or $4y - 4x + 13 = 0$.

12. (a) : $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Let any point on ellipse be $(3\cos\theta, 2\sin\theta)$

Since $\sin\theta$ and $\cos\theta$ can be rational for infinite many values of $\theta \in [0, 2\pi]$.

13. (c) : $\frac{x^2}{3} - \frac{y^2}{2} = 1$; $y + 1 = m(x+2)$

or $y = mx + (2m-1)$ touches the hyperbola

$\because$ $c^2 = a^2m^2 - b^2$ $\therefore$ $(2m-1)^2 = 3m^2 - 2$

$\Rightarrow$ $m^2 - 4m + 3 = 0 \Rightarrow m = 1$ and 3

$\therefore$ $\tan\theta = \left|\frac{3-1}{1+3\times 1}\right| = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$

14. (b) : We have, $x^2F(x) + F(1-x) = 2x - x^4$

Replacing x by $(1-x)$ gives ...(i)

$(1-x)^2 F(1-x) + F(x) = 2(1-x) - (1-x)^4$ ...(ii)

Eliminating $F(1-x)$ from (i) and (ii), we get

$F(x) = 1 - x^2$

15. (b) : $\lim_{x\to 0} \frac{(2^{\sin x} - 1)[\ln(1+\sin 2x)]}{x^2 \frac{\tan^{-1} x}{x}}$

$= \lim_{x\to 0} \frac{2^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x} \times \frac{\ln(1+\sin 2x)}{\sin 2x} \times \frac{\sin 2x}{2x} \times 2$

$= 2\ln 2$

16. (d) : Note that $f(x) = 0$ for each integral value of x.

Also, if $0 \le x < 1$, then $0 \le x^2 < 1$

$\therefore$ $[x] = 0$ and $[x^2] = 0 \Rightarrow f(x) = 0$ for $0 \le x < 1$.

Next, if $1 \le x < \sqrt{2}$, then

$1 \le x^2 < 2 \Rightarrow [x] = 1$ and $[x^2] = 1$

Thus, $f(x) = [x]^2 - [x^2] = 0$ if $1 \le x < \sqrt{2}$

It follows that $f(x) = 0$, if $0 \le x < \sqrt{2}$

This shows that $f(x)$ must be continuous at $x = 1$.

However, at points x other than integers and not lying between 0 and $\sqrt{2}$, $f(x) \ne 0$.

Thus, f is discontinuous at all integers except 1.

17. (b) : $y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} \Rightarrow x^2dy + dx = 0$

$\Rightarrow \frac{x^2}{\sqrt{1+x^4}}dy + \frac{dx}{\sqrt{1+x^4}} = 0$

$\Rightarrow \frac{dy}{\sqrt{\frac{1}{x^4}+1}} + \frac{dx}{\sqrt{1+x^4}} = 0 \Rightarrow \frac{dy}{\sqrt{1+y^4}} + \frac{dx}{\sqrt{1+x^4}} = 0$

$\Rightarrow \frac{dy}{\sqrt{1+y^4}} + \frac{dx}{\sqrt{1+x^4}} + 3 = 3$

18. (b) : For vertical tangents $\frac{dx}{d\theta} = 0$ so, we have

$-3\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Corresponding to these values of θ, we have

$x = 2 - 3\sin\frac{\pi}{2} = -1,\ y = 3 + 2\cos\frac{\pi}{2} = 3;$

$x = 2 - 3\sin\frac{3\pi}{2} = 2 + 3 = 5,\ y = 3 + 2\cos\frac{3\pi}{2} = 3$

Thus the required points are $(-1, 3)$, $(5, 3)$.

19. (c) : $g'(x) = \frac{[1 - 4(f(x))^2]f'(x)}{\sqrt{1 - 2(f(x))^2}}$

Now, as $f(x)$ and $g(x)$ are monotonically increasing, $f'(x) > 0$ and $g'(x) > 0 \Rightarrow |f(x)| < \frac{1}{2}$

20. (d) : For $x \to 0$

$2x^2 + \frac{2}{x^2} \to \infty$ also $2\left(x^2 + \frac{1}{x^2}\right) \ge 4$

21. (b) : Put $\frac{3x-4}{3x+4} = t$

$\Rightarrow$ $3x - 4 = 3xt + 4t \Rightarrow x = \frac{4t+4}{3(1-t)}$

$f(t) = \frac{4t+4}{3(1-t)} + 2$

$\Rightarrow$ $f(x) = \frac{4x+4}{3(1-x)} + 2 = \frac{4(x-1)+8}{3(1-x)} + 2$

$\Rightarrow$ $f(x) = 2 - \frac{4}{3} - \frac{8}{3(x-1)} = \frac{2}{3} - \frac{8}{3(x-1)}$

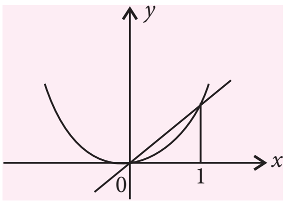
$\therefore$ $\int f(x)dx = \frac{2}{3}x - \frac{8}{3}\ln|x-1| + c$

22. (c) : $I_2 = \int_0^1 \frac{\tan^{-1} x}{x} dx, x = \tan\theta$

$$\Rightarrow I_2 = \int_0^{\pi/4} \frac{2\theta}{\sin 2\theta} d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x} dx = \frac{1}{2}.I_1$$

$$\Rightarrow \frac{I_1}{I_2} = 2$$

23. (d) :



$$A_n = \int_0^1 (x - x^n)\, dx = \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)}$$

Thus $A_2 \cdot A_3 \cdot A_4 .....A_n = \frac{1}{2^{n-1}}\left(\frac{1}{3}.\frac{2}{4}.\frac{3}{5}...\frac{n-1}{n+1}\right)$

$$= \frac{1}{2^{n-2} \cdot n(n+1)}$$

24. (c) : We can write $y = A\cos(x + B) - Ce^x$

where $A = c_1 + c_2$, $B = c_3$ and $C = c_4 e^{c_5}$

$$\frac{dy}{dx} = -A\sin(x+B) - Ce^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -A\cos(x+B) - Ce^x \Rightarrow \frac{d^2y}{dx^2} + y = -2Ce^x$$

$$\Rightarrow \frac{d^3y}{dx^3} + \frac{dy}{dx} = -2Ce^x = \frac{d^2y}{dx^2} + y$$

$$\Rightarrow \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

Which is a differential equation of degree 1

25. (d) : On putting $z = x + iy$ the equation is same as $|x^2 - y^2 + 2ixy - 1| = x^2 + y^2 + 1$

$\Rightarrow \quad (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow \quad x = 0$

$\Rightarrow \quad z$ lies on imaginary axis, so (a), (b), (c) are ruled out.

26. (a) : Let A represents the event 'A hits the target', B represents the event 'B hits the target', C represents the event 'C hits the target' and E be the event that exactly two of A, B and C hit the target.

Then $P(A) = \frac{4}{5}$, $P(B) = \frac{3}{4}$ and $P(C) = \frac{2}{3}$

$\therefore \quad P(C^c/E)$

$$= \frac{P(A)P(B)P(C^c)}{P(A)P(B)P(C^c) + P(A)P(B^c)P(C) + P(A^c)P(B)P(C)}$$

$$= \frac{6}{13}$$

27. (d) : Applying $C_3 \to C_3 - C_1 \cos D + C_2 \sin D$, we get $\Delta = 0$, hence Δ is independent of A, B, C, D all.

28. (c) : $A^T = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}, 2A^T = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -6 \\ 4 & 6 & 0 \end{bmatrix}$

$$2A^T + A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = A^T$$

Alternate solution :

$A^T = -A$ $\quad$ ($\because$ A is skew symmetric)

So $2A^T + A = A^T + A - A = A^T$.

29. (c) : A vector bisecting the angle between $\vec{a}$ and $\vec{b}$ is

$\frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|}$; in the case $\frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}} \pm \frac{\hat{i}-2\hat{j}+\hat{k}}{\sqrt{6}}$

i.e., $\frac{3\hat{i}-\hat{j}}{\sqrt{6}}$ or $\frac{\hat{i}+3\hat{j}-2\hat{k}}{\sqrt{6}}$

A vector of magnitude 3 along these vectors is

$\frac{3(3\hat{i}-\hat{j})}{\sqrt{10}}$ or $\frac{3(\hat{i}+3\hat{j}-2\hat{k})}{\sqrt{14}}$

Now, $\frac{3}{\sqrt{14}}(\hat{i}+3\hat{j}-2\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k})$ is negative and hence

$\frac{3}{\sqrt{14}}(\hat{i}+3\hat{j}-2\hat{k})$ makes an obtuse angle with $\vec{b}$

30. (b) : Given plane contains the line

$\Rightarrow \quad a^2 - b^2 + c^2 = 0$ ... (i)

and $a^2 - 2bd + c^2 = 0$ ...(ii)

By using (i) and (ii) we get $b/d = 2$

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Solution Sender of Maths Musing

SET-174

1. N. Jayanthi (Hyderabad)
2. V. Damodhar Reddy (Telangana)
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MATHS MUSING

SOLUTION SET-174

1. **(b) :** We have $a_n = a_{n-1} - a_{n-2} + a_{n-3} - a_{n-4}$ for $n \geq 5$

$\therefore\ a_5 = a_4 - a_3 + a_2 - a_1 = 23 - 20 + 75 - 11 = 67$

$a_6 = a_5 - a_4 + a_3 - a_2 = -a_1$

$a_7 = a_6 - a_5 + a_4 - a_3 = -a_2$

$a_8 = -a_3,\ a_9 = -a_4,\ a_{10} = -a_5$

$a_{11} = a_1,\ a_{12} = a_2,\ a_{13} = a_3,\ a_{14} = a_4,\ a_{15} = a_5$

Hence, $a_{31} - a_{53} + a_{75} = a_1 - a_3 + a_5 = 11 - 20 + 67 = 58.$

2. **(b) :** $xyz = 1,\ x + \dfrac{1}{z} = 5 \Rightarrow x + xy = 5$

But $y + \dfrac{1}{x} = 29$

$$\therefore\ \frac{5-x}{x} = 29 - \frac{1}{x} = \frac{29x-1}{x} \Rightarrow x = \frac{1}{5}$$

Now, $y = 29 - \dfrac{1}{x} \Rightarrow y = 29 - 5 = 24$

Also, $\dfrac{1}{z} = 5 - x = 5 - \dfrac{1}{5} = \dfrac{24}{5} \Rightarrow z = \dfrac{5}{24}$

$$\therefore\ z + \frac{1}{y} = \frac{5}{24} + \frac{1}{24} = \frac{1}{4}$$

3. **(d) :** $\Delta\left(\dfrac{1}{r^2} + \Sigma\dfrac{1}{r_1^2}\right) = \dfrac{1}{\Delta}\left(s^2 + \Sigma(s-a)^2\right)$

$$= \frac{1}{\Delta}(a^2+b^2+c^2) = 4\left(\frac{a^2+b^2+c^2}{4\Delta}\right)$$

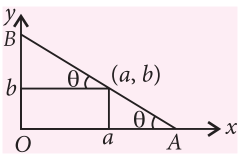
$= 4(\cot A + \cot B + \cot C)$

4. **(c) :** $OA = a + b\cot\theta$

$OB = b + a\tan\theta$

$(a, b) = (4, 1)$

$\Rightarrow OA + OB = 5 + 4\tan\theta + \cot\theta$



$\geq 5 + 2\sqrt{4\tan\theta\cdot\cot\theta} = 9$, by A.M. and G.M. inequality.

5. **(a) :** $\begin{vmatrix} a & y & z \\ x & b & z \\ x & y & c \end{vmatrix} = 0$

Using $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} a & y & z \\ x-a & b-y & 0 \\ x-a & 0 & c-z \end{vmatrix} = 0$$

Using $C_1 \to \dfrac{C_1}{x-a},\ C_2 \to \dfrac{C_2}{y-b},\ C_3 \to \dfrac{C_3}{z-c}$, we get

$$\begin{vmatrix} \frac{a}{x-a} & \frac{y}{y-b} & \frac{z}{z-c} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0 \Rightarrow \frac{a}{x-a} + \frac{y}{y-b} + \frac{z}{z-c} = 0$$

$$\Rightarrow\ \Sigma\frac{a}{x-a} = -2 \text{ and } \Sigma\frac{x}{x-a} = 1\ \therefore\ \Sigma\frac{x+a}{x-a} = -1$$

6. **(b, c) :** Let $a < b < c$ be selected numbers, $a + b + c = 30$

$a = 0,\ (b, c) = (1, 29), (2, 28), ...(14, 16)$, 14 pairs

$a = 1,\ (b, c) = (2, 27), (3, 26), ...(14, 15)$, 13 pairs

$a = 2,\ (b, c) = (3, 25), (4, 24), ...(13, 15)$, 11 pairs

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a = 9,\ (b, c) = (10, 11)$, one pair

The total number of ways $= 1 + 2 + 3 + ... + 15$

$-(3 + 6 + 9 + 12 + 15) = 75 = 3 \times 5^2$

7. **(c) :** $[\sin^{-1}x] = \begin{cases} 0 \text{ in } [0, \sin 1) \\ 1 \text{ in } [\sin 1, 1] \end{cases}$

$[\cos^{-1}x] = \begin{cases} 1 \text{ in } [0, \cos 1] \\ 0 \text{ in } (\cos 1, 1] \end{cases}$

$$\therefore\ f(x) = [\sin^{-1}x] + [\cos^{-1}x] = \begin{cases} 1 \text{ in } [0, \cos 1] \\ 0 \text{ in } (\cos 1, \sin 1) \\ 1 \text{ in } [\sin 1, 1] \end{cases}$$

$f(x)$ is not differentiable at $x = \cos 1$ and $\sin 1$.

8. **(a) :** $f(x) = 0$ in $(\cos 1, \sin 1)$

The length of the interval $= \sin 1 - \cos 1$

$$= \sqrt{2}\left(\frac{\sin 1}{\sqrt{2}} - \frac{\cos 1}{\sqrt{2}}\right) = \sqrt{2}\sin\left(1 - \frac{\pi}{4}\right).$$

9. **(d) :** The girls can sit in 1, 2; 2, 3 ; . . . . ; 11, 12 = 22 ways. If one boy sits between them, they sit in 1, 3; 2, 4; . . . . . .; 10, 12 = 20 ways

If two boys sit between them, they sit in 1, 4; 2, 5; .....; 9 12 = 18 ways

The desired number is $12\,! - 60\cdot 10\,! = \dfrac{6}{11}\cdot 12\,!$

10. **(d): P.** $xy = \dfrac{1}{4},\ x = \dfrac{t}{2},\ y = \dfrac{1}{2t}$

Tangent $\dfrac{x}{2t} + \dfrac{t}{2}y = \dfrac{1}{2} \Rightarrow OA = t,\ OB = \dfrac{1}{t}$

$\therefore\ OA\cdot OB = 1$

Q. Tangent at $(\cos\theta, \sin\theta)$ is $x\cos\theta + y\sin\theta = 1$

$OA = \sec\theta,\ OB = \operatorname{cosec}\theta \Rightarrow \dfrac{1}{OA^2} + \dfrac{1}{OB^2} = 1$

R. Tangent at $(\cos^4\theta, \sin^4\theta)$ is

$x\sin^2\theta + y\cos^2\theta = \sin^2\theta\cos^2\theta$

$\therefore\ OA = \cos^2\theta,\ OB = \sin^2\theta,\ OA + OB = 1$

S. Tangent at $(\cos^3\theta, \sin^3\theta)$ is

$x\sin\theta + y\cos\theta = \sin\theta\cos\theta$

$OA = \cos\theta,\ OB = \sin\theta,\ AB = 1$

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YOU ASK WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. If $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}/2$ and $\vec{A}, \vec{B}, \vec{C}$ are unit vectors then find the angle between $\vec{A}$ and $\vec{C}$.

(Prashant Aditya, Patna)

Ans. Given, $\vec{A}\times(\vec{B}\times\vec{C})=\dfrac{\vec{B}}{2}$

Also, $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}=\dfrac{\vec{B}}{2}$

On comparing coefficients, we have

$$(\vec{A}\cdot\vec{C})\vec{B}=\frac{\vec{B}}{2} \quad\text{and}\quad (\vec{A}\cdot\vec{B})\vec{C}=0$$

$$\Rightarrow\ \vec{A}\cdot\vec{C}=\frac{1}{2}\ \Rightarrow\ |\vec{A}||\vec{C}||\cos\theta|=\frac{1}{2}\ \Rightarrow\ \theta=\frac{\pi}{3}$$

2. Let $a_1, a_2, \ldots$ be positive numbers in G.P.. For each n, let A_n, G_n, H_n be the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \ldots, a_n$. Express the geometric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n, H_1, H_2, \ldots, H_n$.

(Kushagra Pandey, Hyderabad)

Ans. Let the G.P. be $a_1, a_1r, a_1r^2, \ldots, a_1r^{n-1}$

$$A_n=\frac{a_1+a_2+\ldots+a_n}{n}=\frac{a_1}{n}\ (1+r+r^2+\ldots+r^{n-1})$$

$$=\frac{a_1}{n}\left(\frac{1-r^n}{1-r}\right)$$

$$G_n=(a_1a_2\ldots a_n)^{\frac{1}{n}}=\left(a_1^n r^{1+2+\ldots+(n-1)}\right)^{\frac{1}{n}}=a_1r^{(n-1)/2}$$

$$H_n=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}=\frac{n}{\frac{1}{a_1}\left(1+\frac{1}{r}+\ldots+\frac{1}{r^{n-1}}\right)}$$

$$=na_1\cdot r^{n-1}\left(\frac{1-r}{1-r^n}\right)$$

$$\therefore\quad A_nH_n=a_1^2r^{(n-1)}=G_n^2 \qquad \ldots(1)$$

The G.M. of $G_1, G_2, \ldots, G_n$ is $(G_1G_2\ldots G_n)^{\frac{1}{n}}$

$$=(A_1H_1\cdot A_2H_2\ldots A_nH_n)^{\frac{1}{2n}}$$

$$=(A_1A_2\ldots A_n\times H_1H_2\ldots H_n)^{1/2n}.$$

3. Let $\Delta_r=\begin{vmatrix} r-1 & n & 6\\ (r-1)^2 & 2n^2 & 4n-2\\ (r-1)^3 & 3n^3 & 3n^2-3n\end{vmatrix}$.

Find the sum $\sum\limits_{r=1}^{n}\Delta_r$. *(Vidushi Varshney, Delhi)*

Ans. $$\sum_{r=1}^{n}\Delta_r=\begin{vmatrix}\sum\limits_{r=1}^{n}(r-1) & n & 6\\ \sum\limits_{r=1}^{n}(r-1)^2 & 2n^2 & 4n-2\\ \sum\limits_{r=1}^{n}(r-1)^3 & 3n^3 & 3n^2-3n\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{n(n-1)}{2} & n & 6\\ \dfrac{(n-1)n(2n-1)}{6} & 2n^2 & 2(2n-1)\\ \dfrac{(n-1)^2n^2}{4} & 3n^3 & 3n(n-1)\end{vmatrix}$$

$$=\frac{1}{48}\begin{vmatrix}(n-1)n & 2n & 12\\ (n-1)n(2n-1) & 12n^2 & 12(2n-1)\\ (n-1)^2n^2 & 12n^3 & 12n(n-1)\end{vmatrix}$$

By taking factors $(n-1)\,n$ from C_1, $2n$ from C_2 and 12 from C_3, we get

$$\frac{1}{2}(n-1)n^2\begin{vmatrix}1 & 1 & 1\\ 2n-1 & 6n & 2n-1\\ (n-1)n & 6n^2 & n(n-1)\end{vmatrix}=0$$

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