FinTech 545 Homework Week 04

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February 23, 2024

1. (a) Classical Brownian Motion:

$$
\begin{aligned}
P_t &= P_{t-1} + r_t \\
E[P_t] &= E[P_{t-1} + r_t] \\
&= E[P_{t-1}] + E[r_t] \\
&= P_{t-1} \\
Var(P_t) &= Var(P_{t-1} + r_t) \\
&= Var(r_t) \\
\sigma_{P_t} &= \sigma
\end{aligned}
$$

   (b) Arithmetic Return System:

$$
\begin{aligned}
P_t &= P_{t-1}(1 + r_t) \\
E[P_t] &= E[P_{t-1}(1 + r_t)] \\
&= P_{t-1}E[1 + r_t] \\
&= P_{t-1} \\
Var(P_t) &= Var(P_{t-1}(1 + r_t)) \\
&= P_{t-1}^2 Var(1 + r_t) \\
&= P_{t-1}^2 \sigma^2 \\
\sigma_{P_t} &= P_{t-1}\sigma
\end{aligned}
$$

(c) Log Return or Geometric Brownian Motion:

$$
\begin{aligned}
P_t &= P_{t-1}e^{r_t} \\
E[P_t] &= E[P_{t-1}e^{r_t}] \\
&= P_{t-1}E[e^{r_t}] \\
&= P_{t-1}M_{r_t}(1) \\
&= P_{t-1}e^0 e^{\frac{1}{2}\sigma^2} \\
&= P_{t-1}e^{\frac{1}{2}\sigma^2} \\
Var(P_t) &= Var(P_{t-1}e^{r_t}) \\
& r_t \text{ follows log-normal distribution } (0, \sigma^2) \\
Var(P_t) &= P_{t-1}^2(e^{\sigma^2} - 1)e^{2\times 0+\sigma^2} \\
&= P_{t-1}^2(e^{\sigma^2} - 1)e^{\sigma^2} \\
\sigma_{P_t} &= P_{t-1}^2\sqrt{(e^{\sigma^2} - 1)e^{\sigma^2}}
\end{aligned}
$$

We set $\mu = 0, \sigma = 0.1$ for r_t with sample size of 100,000. Given $P_{t-1} = 100$. Here is the comparison between theoretical estimate of mean and SD, and simulated results.

i. Classical Browniam Motion

	Mean	Std Dev
Simulated	100.0002	0.0999
Expected	100	0.1
Difference	0.0002	0.0001

Given the large difference between this and other 2 types of return, I suppose the r_t in this question is not the same scale as the others. This r_t should be in **price** unit instead of **percentage** unit. I re-simulated r_t for classical return, using $r_t \sim N(\mu = 0, \sigma^2 = 100)$.

	Mean	Std Dev
Simulated	100.0586	10.0001
Expected	100	10
Difference	0.0586	0.0001

ii. Arithmetic Return System

	Mean	Std Dev
Simulated	100.0203	9.9889
Expected	100	10
Difference	0.0203	0.0111

iii. Geometric Brownian Motion

	Mean	Std Dev
Simulated	100.5205	10.0641
Expected	100.5013	10.0753
Difference	0.0192	0.0112

For each type of return, our simulation results closely align with the expectations.

2. The return given P_t and P_{t-1} under different types of return is:

   (a) Classical Brownian Motion

   $r_t = P_t - P_{t-1}$

   (b) Arithmetic Return

   $r_t = \frac{P_t}{P_{t-1}} - 1$

   (c) Log Return

   $r_t = \ln(\frac{P_t}{P_{t-1}})$

   The VaR of META (on $ basis given most recent price) under these 5 methods are:

```
VaR of META:
    using Normal Distribution:  16.236135
    using EW variance:          8.966972
    using MLE Fitted T:         12.90073
    using Fitted AR(1) Model:   16.187455
    using Historic Simulation:  11.808949
```

   On probability basis:

```
VaR of META:
    using Normal Distribution:  0.054287
    using EW variance:          0.029982
    using MLE Fitted T:         0.043135
    using Fitted AR(1) Model:   0.054124
    using Historic Simulation:  0.039484
```

   The normal distribution and fitted AR(1) model are close to each other. The VaR using EW variance is much smaller than VaR using Normal Distribution, suggesting that recent prices, which are more weighted, do not fluctuate as much as older prices.

   The historical simulation is different from normal and AR(1) model, suggesting that the actual return may not follow a normal distribution. Instead, it is closer to a MLE fitted T distribution.

3. To calculate the VaR of a portfolio on dollar basis, I need to know:

   (1) The total market value of the portfolio

   (2) The weight of each asset inside the portfolio

   (3) The exponentially weighted covariance matrix of the portfolio

   (4) The standard deviation of the entire portfolio's value

   For each asset, we use the most recent price from our price database (last row of DailyPrices table), times the quantity ("Holdings" in portfolio table) of each asset in the portfolio to get the total value of each asset. We aggregate each asset's total value to get (1).

   Then (2) could be derived by using each assets total value divided by (1).

   To calculate (3), we need to calculate the arithmetic return for every asset, using `return_calculate()` function defined in question 2, and calculate the exponentially weighted covariance matrix given $\lambda = 0.94$.

   We can get (4) by combining the weight of each asset and the covariance matrix of the portfolio. More specifically, let $W \in \mathbb{R}^{n\times 1}$ be the weight, $\Sigma \in \mathbb{R}^{n\times n}$ be the return covariance matrix. The standard deviation of the entire portfolio's value is $\sigma = \sqrt{W^T \times \Sigma \times W}$

   We set $\alpha = 0.05$, and get the 5% quantile q of the normal distribution. The VaR is

   $$q \times \sigma \times \text{portfolio total value}$$

   The result given arithmetic return is:

   ```
   Arithmetic Return:
   For Portfolio A, the 95% VaR is: $15426.968
   For Portfolio B, the 95% VaR is: $8082.5724
   For Portfolio C, the 95% VaR is: $18163.2916
   For Portfolio Total, the 95% VaR is: $38941.3757
   ```

   The VaR for portfolio B is smaller than A and C, which could be determined by either a lower portfolio variance, or a smaller portfolio value. The total portfolio has a much larger VaR, largely due to its higher aggregate portfolio value.

Compared to arithmetic return model, we want to choose log return model for a few reasons:

1. Log return is more stable when describing the price fluctuation, especially in short-term. Log return reduces the abnormal fluctuation, and smoothes the price return model.

2. In many cases, we can use normal distribution to approximation log return, which is better fitted with our normal distribution EW covariance method of calculating VaR.

The result given log return is:

```
Log Return:
For Portfolio A, the 95% VaR is: $15433.5151
For Portfolio B, the 95% VaR is: $8089.6162
For Portfolio C, the 95% VaR is: $18081.6128
For Portfolio Total, the 95% VaR is: $38904.8424
```

The VaR for each portfolio is very close to the VaR estimate using arithmetic return system.