Optimization Methods (CS1.404)
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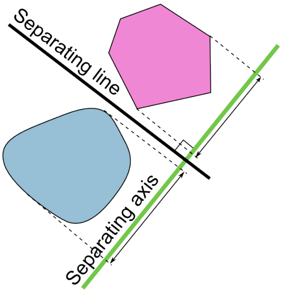
Separating Hyperplane



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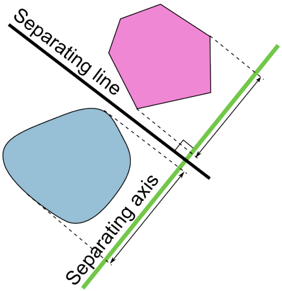
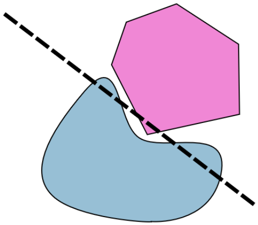




Separating Hyperplane Theorem

Theorem

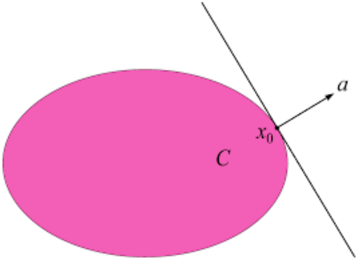
Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two disjoint, non-empty closed convex subsets of $\mathbb{R}^d$ (one of them being bounded), then there exists a non-zero vector $\mathbf{w} \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $\mathbf{w}^T\mathbf{x} \geq b, \ \forall \mathbf{x} \in \mathcal{X}_1$ and $\mathbf{w}^T\mathbf{x} \leq b, \ \forall \mathbf{x} \in \mathcal{X}_2$. Thus, $\mathbf{w}^T\mathbf{x} = b$ separates $\mathcal{X}_1$ and $\mathcal{X}_2$.

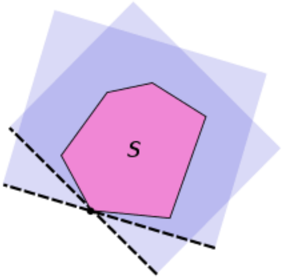
- If $\mathbf{w}^T\mathbf{x} > b, \ \forall \mathbf{x} \in \mathcal{X}_1$ and $\mathbf{w}^T\mathbf{x} < b, \ \forall \mathbf{x} \in \mathcal{X}_2$, the the hyperplane $\mathbf{w}^T\mathbf{x} = b$ is said to strictly separate $\mathcal{X}_1$ and $\mathcal{X}_2$.
- The hyperplane is said to strongly separate $\mathcal{X}_1$ and $\mathcal{X}_2$ if $\mathbf{w}^T\mathbf{x} \geq b + \epsilon, \ \forall \mathbf{x} \in \mathcal{X}_1$ and $\mathbf{w}^T\mathbf{x} \leq b, \ \forall \mathbf{x} \in \mathcal{X}_2$ for some $\epsilon > 0$. If $\mathcal{X}_1 \cap \mathcal{X}_2 = \phi$, then strong separation happens.

Supporting Hyperplane Theorem

Theorem

If C is convex, then there exists a supporting hyperplane at every boundary point of C. Let $\mathbf{x}_0$ be a boundary point of set C, then there exists $\mathbf{a} \neq \mathbf{0}$ such that $\{\mathbf{x} \,:\, \mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_0\}$ is a supporting hyperplane to C and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_0,\ \forall \mathbf{x} \in C$.





Convex Hull

Definition

Let $\mathcal{X} \subset \mathbb{R}^n$ be a set. The **convex hull** of $\mathcal{X}$ is the intersection of all convex sets containing it.

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Lemma

The convex hull of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is the set of all convex combinations of these vectors.

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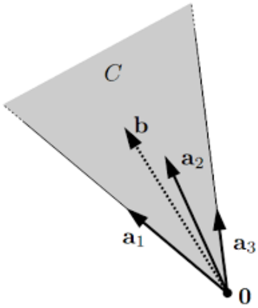
- The convex hull of finite number of vectors is a convex set. **Homework: Prove this statement.**

Convex Cone

Definition

Given vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, the convex cone generated by these vectors is the set of all non-negative linear combinations of $\mathbf{a}_i$'s. That is,

$$\text{Convex-Cone}(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \{t_1\mathbf{a}_1 \ + \ \ldots \ + \ t_n\mathbf{a}_n \mid t_1, \ldots, t_n \geq 0\}$$



Farkas Lemma

Lemma

Let $A \in \mathbb{R}^{m \times n}$ and let $C = \{A\mathbf{x} \mid \mathbf{x} \geq \mathbf{0}\}$. Note that C is a closed convex cone. Then, exactly, one of the two systems has a solution:

1. $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
2. $A^T\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} < 0$