Optimization Methods (CS1.404)
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Affine Set

A set $C \subseteq \mathbb{R}^d$ is said to be an affine set if for any two distinct points, the line passing through these points also lied in the set C. Thus, if $\mathbf{x}_1, \mathbf{x}_2 \in C$, then $\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in C$, $\forall \theta \in \mathbb{R}$.

- C is an affine set if and only if it contains every affine combination of its points.
- For example, solution of a linear equation is an affine set.

Convex Set

Definition

A set $\mathcal{X} \subseteq \mathbb{R}^d$ is called convex, if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in \mathcal{X}, \ \forall \lambda \in [0, 1]$.

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- Note that $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2,\ \forall \lambda \in [0,1]$ represents the line segment joining $\mathbf{x}_1, \mathbf{x}_2$. For $\mathcal{X}$ to be a convex set, this line segment has to lie inside the set $\mathcal{X}$.

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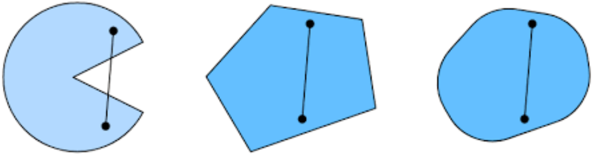

Convex Combination

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Definition: Convex Combination

Given a finite number of points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ in a real vector space, a convex combination of these points is a point of the form $\lambda_0\mathbf{x}_0 + \lambda_1\mathbf{x}_1 + \ldots + \lambda_n\mathbf{x}_n$ where $\lambda_i \geq 0,\ i = 0, 1, \ldots, n$ and $\sum_{i=0}^{n} \lambda_i = 1$.

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As a particular example, every convex combination of two points lies on the line segment between the points.

Intersection of Convex Sets

Result

Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k \subseteq \mathbb{R}^d$ be convex sets. Then $\cap_{i=1}^k \mathcal{X}_i$ is also a convex set.

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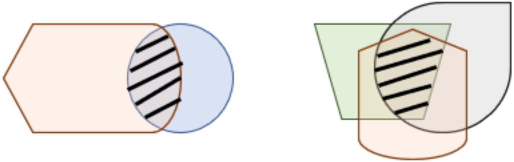


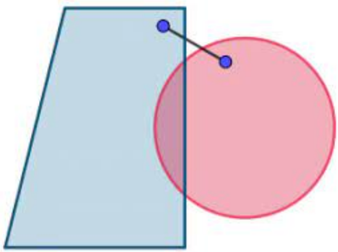
Result

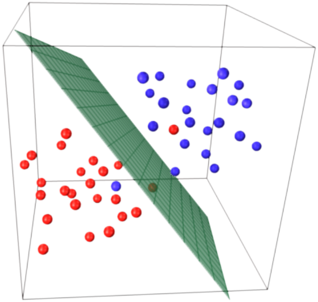
Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k \subset \mathbb{R}^d$ be convex sets. Then $\cup_{i=1}^{k} \mathcal{X}_i$ may not be a convex set.

Union of Convex Sets

Result

Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k \subset \mathbb{R}^d$ be convex sets. Then $\cup_{i=1}^k \mathcal{X}_i$ may not be a convex set.

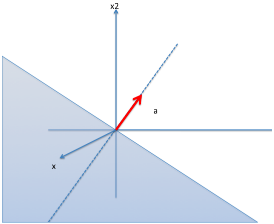




A hyperplane in $\mathbb{R}^d$ is a set of the form $\{\mathbf{x} \mid \mathbf{w}^T\mathbf{x} = b\}$ where $\mathbf{w} \in \mathbb{R}^d$ is normal to the hyperplane and $b \in \mathbb{R}$ is offset parameter.

Hyperplane is convex set also.

Halfspaces



- A half-space is either of the two parts into which a hyperplane divides.
- A half-space may be specified by a linear inequality, derived from the linear equation that specifies the defining hyperplane.
- A strict linear inequality specifies an open half-space:
  $w_1x_1 + w_2x_2 + \ldots + w_dx_d > b$
- A non-strict inequality specifies a closed half-space:
  $w_1x_1 + w_2x_2 + \ldots + w_dx_d \geq b$
- Here, one assumes that not all of the real numbers a1, a2, ..., an are zero.

Closed half-spaces $\{\mathbf{x} \in \mathbb{R}^d \mid w_1x_1 + w_2x_2 + \ldots + w_dx_d \geq b\}$ and $\{\mathbf{x} \in \mathbb{R}^d \mid w_1x_1 + w_2x_2 + \ldots + w_dx_d \leq b\}$ are convex sets.

Theorem

Let $\mathcal{X} \subset \mathbb{R}^n$ be a nonempty compact (closed and bounded) set and $f : \mathcal{X} \to \mathbb{R}$ be a continuous function on $\mathcal{X}$. Then, f attains a minimum and a maximum on $\mathcal{X}$.

- Weierstrass Theorem is not a necessary condition.

Closest Point Theorem

Theorem

let $S \subset \mathbb{R}^n$ be a nonempty, closed convex set and $\mathbf{y} \notin S$. Then there exists a unique point $\mathbf{x}_0 \in S$ with minimum distance from $\mathbf{y}$. Further $\mathbf{x}_0$ is the minimum distance point if and only if $(\mathbf{y} - \mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \leq 0, \forall \mathbf{x} \in S$.