Probability and Statistics
Random Variable

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Course requirements

- attendance - maximum 3 absences
- 4 tests during semester - required **minimum 15 points**, maximum 33 points
- final exam test - 5 exercises, maximum 67 points
- total - **minimum 51 points**, maximum 100 points

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- Y... number of smokers in randomly chosen 1000 people from the population of Ostrava city
- Z... number of trials that it takes to roll a die until you get 6

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- Y... number of smokers in randomly chosen 1000 people from the population of Ostrava city
- Z... number of trials that it takes to roll a die until you get 6
- T... waiting time for a train which leaves every 30 minutes
- F... time to failure of an electronic product

Types of Random Variables

Discrete Random Variable (DRV)
... usually express count data (number of something). The number of possible values (outcomes) is countable. You can list all these possible outcomes, although it might take you very very long time.

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Probability distribution

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- An upper-case letter will represent the name of the random variable, e.g. X is number of...
- Its lower-case counterpart will represent the particular value of the random variable.

Probability distribution of DRV

Probability mass function

Let us denote

$$p_i = P(X = x_i) = P(x_i).$$

Then

$$0 \leq p_i \leq 1 \text{ and } \sum_i p_i = 1.$$

Values $P(X = x_i)$ determine the probability mass function and usually, we write these values **to the table**.

Probability distribution of DRV

Probability mass function

$$p_i = P(X = x_i) = P(x_i), \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1$$

Let's consider a random variable X which is defined as number of heads in two coin flips.

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$$P(X = 1) = ?$$
$$P(X = 2) = ?$$

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$$P(X = 0) = 0,5 \cdot 0,5 = 0,25$$
$$P(X = 1) = 0,5 \cdot 0,5 + 0,5 \cdot 0,5 = 0,5$$
$$P(X = 2) = 0,5 \cdot 0,5 = 0,25$$

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x_i	0	1	2
$P(x_i)$	0,25	0,50	0,25

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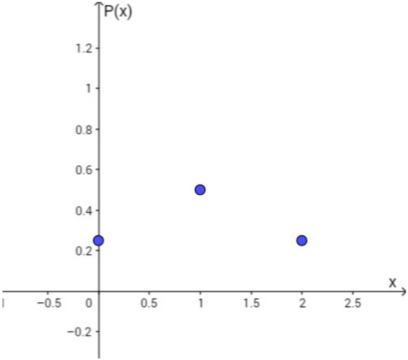

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Important properties

- $0 \leq F(x) \leq 1$
- $F(x)$ is non-decreasing (i.e. increasing or constant)
- $F(x)$ is continuous from the left
- for $x \to -\infty$, $F(x) \to 0$ ("start" in 0)
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Cumulative distribution function can be derived from the probability mass function (and the other way around).

Probability distribution of DRV

Cumulative distribution function

$$F(x) = P(X < x).$$

Let's go back to the random variable X which is defined as number of heads in two coin flips. We know the probability mass function.

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CDF in important points:

$$F(0) = P(X < 0) = 0$$
$$F(1) = P(X < 1) = P(X = 0) = 0,25$$
$$F(2) = P(X < 2) = P(X = 0) + P(X = 1) = 0,25 + 0,5 = 0,75$$

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What about:

$$F(0,5) = ?$$
$$F(1,1) = ?$$
$$F(2,8) = ?$$

Probability distribution of DRV

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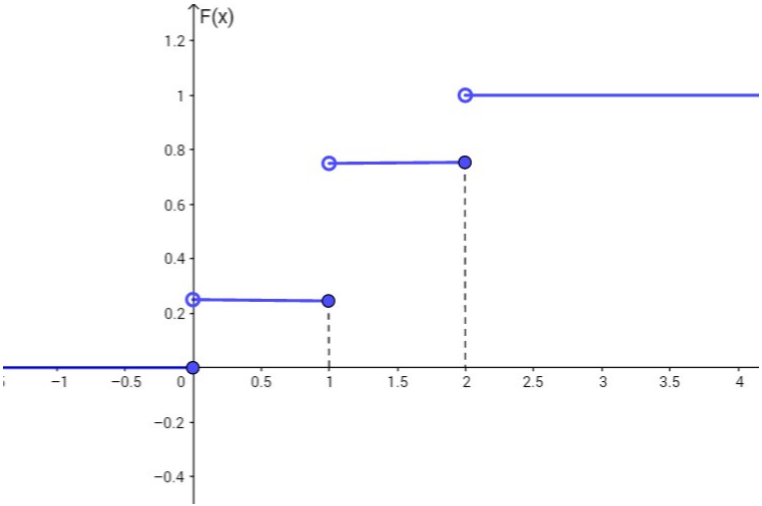
Probability distribution of DRV

Cumulative distribution function of the random variable X which is defined as the number of heads in two coin flips

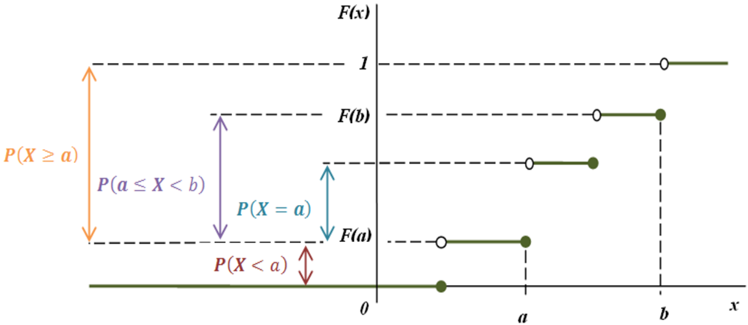


Probability distribution of DRV

Cumulative distribution function of the random variable X which is defined as the number of heads in two coin flips

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 0,25, & 0 < x \leq 1 \\ 0,75, & 1 < x \leq 2 \\ 1, & 2 < x \end{cases}$$

Relationship between CDF and PMF



- $P(X \leq a) = P(X < a) + P(X = a)$
- $P(X > a) = 1 - P(X \leq a)$
- $P(a \leq X < b) = P(X < b) - P(X < a)$

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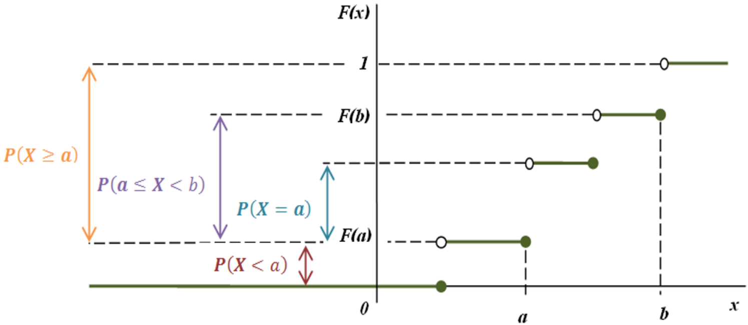

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Usually, the CDF is denoted with a graph or a formula.

Numerical characteristics of DRV

Expected value
...is the weighted average of all possible values that the random variable can take on. The weights correspond to the probabilities.

$$E(X) = \sum_i x_i P(x_i)$$

Variance and Standard deviation ...are measures expressing how far a set of numbers is spread out (how far the numbers are from the expected value).

$$var(X) = E(X^2) - [E(X)]^2$$
$$sd(X) = \sqrt{var(X)}$$

1. An experiment consists in injecting a virus to three rats and checking if they survive or not. It is known that the probability of surviving is 0,8 for the first rat, 0,7 for the second and 0,6 for the third. Let's consider a random variable S describing the number of surviving rats.
   a) Find the probability mass function of the random variable S.
   b) Find the cumulative distribution function of the random variable S.
   c) Calculate the expected value and the standard deviation.
   d) Calculate $P(X \leq 1)$, $P(X \geq 2)$ and $P(X = 1,5)$.

2. The cumulative distribution function of random variable Z is:

$$F(z) = \begin{cases} 0, & z \leq -2 \\ 0,2, & -2 < z \leq 0 \\ 0,52, & 0 < z \leq 1 \\ 0,93, & 1 < z \leq 4 \\ 1, & 4 < z \end{cases}$$

Draw this function and derive the probability mass function.

Probability distribution of CRV

Probability density function

... function $f(x)$ which is non-zero and the total area under the curve is 1, i.e.

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Moreover, the probability density function $f(x)$ is derivative of the cumulative distribution function $F(x)$, i.e.

$$F'(x) = f(x), \text{ or } F(x) = \int_{-\infty}^{x} f(t)dt$$

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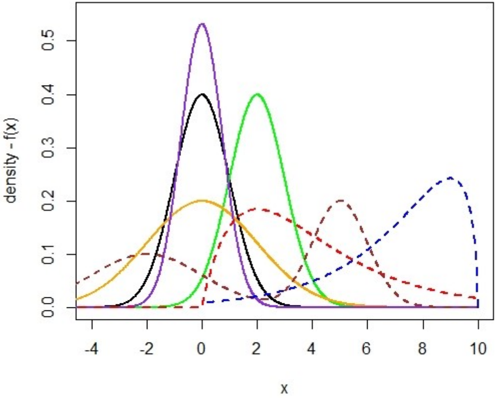
Considering the probability density function for DRV doesn't make sense! However, the probability mass function for CRV does exist but it is a zero function!

$$\forall x \in \mathbb{R} : P(X = x) = 0$$

Probability distribution of CRV

Probability density function



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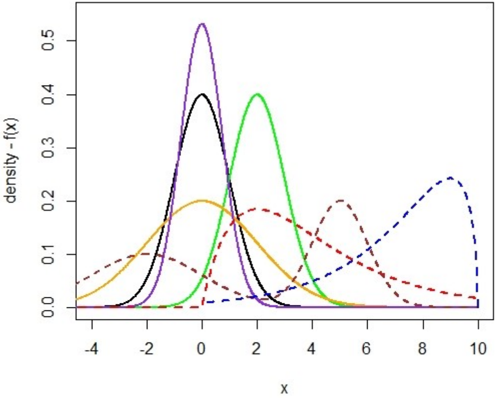
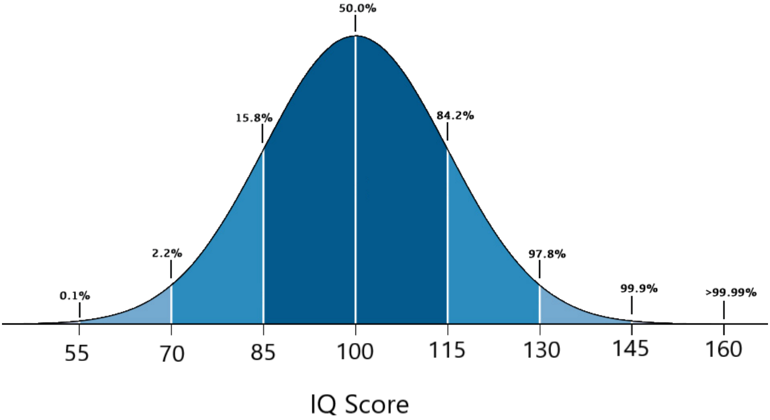


One of the possible shapes of the probability density function is "the bell curve". In reality, this "bell curve" shape is desirable thanks to its favourable properties.

Probability distribution of CRV

Probability density function



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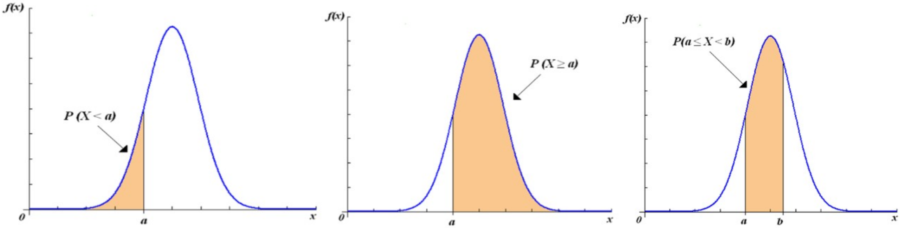

Numerical characteristics and important properties for CRV

- $P(X < a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$
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Considering **CRV** strict inequalities ($>$, $<$) are interchangeable with their "non-strict" counterparts ($\geq$, $\leq$). This is not possible for **DRV**!!!

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- $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$
- formulas for $var(X)$ and $sd(X)$ correspond to those defined in DRV

Numerical characteristics for linear transformation of RV

Let's assume

$$Y = a + bX.$$

Then

$$E(Y) = E(a + bX) = a + bE(X)$$
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These laws are valid for **both CRV and DRV**.

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$$\begin{aligned} E(Y) &= E(a + bX) = a + bE(X) \\ var(Y) &= var(a + bX) = b^2 var(X) \end{aligned}$$

These laws are valid for **both CRV and DRV**.

There are no such laws for non-linear transformations, e.g.

$$Y = X^2, V = \sqrt{Y}, Z = e^W$$

3. Let's go back to the exercise with the experiment with rats. The probability mass function of the random variable X describing the number of surviving rats was

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$P(x_i)$	0,024	0,188	0,452	0,336

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$$Y = 100X$$

Calculate the probability mass function, the cumulative distribution function, the expected value and the variance for the random variable Y. Use the known information about the random variable X.

5. Continuous random variable W is defined with following cumulative distribution function.

$$F(w) = \begin{cases} 0, & w \in (-\infty, 0) \\ w^2, & w \in \langle 0, 1 \rangle \\ 1, & w \in (1, +\infty). \end{cases}$$

Find the probability density function, the expected value and $P(0,2 < W < 0,6)$.

Discrete Probability Distribution

… coming next week …