# Vector Calculus

GATE EC

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# Vector Calculus

## Basics of Vectors

- $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad |\vec{V}| = \sqrt{x^2 + y^2 + z^2}$
- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{u}$
- Del operator $(\nabla) = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ for Cartesian system.
- Gradient of a function, grad f $= \nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$
- $\nabla f$ indicates normal to the surface f(x, y, z) = C
- Divergence, div F $= \nabla \cdot F = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$ Where, F = a$\hat{\imath}$ + b$\hat{\jmath}$ + c$\hat{k}$
- Curl, $\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a & b & c \end{vmatrix} = \hat{\imath}\left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right) + \hat{k}\left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)$

## Standard formulae

1. div grad f $= \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2. curl grad f = $\nabla \times \nabla f = 0$
3. div curl f = $\nabla \cdot \nabla \times f = 0$
4. curl curl f = $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f$
5. grad div f = $\nabla(\nabla \cdot f) = \nabla \times (\nabla \times f) + \nabla^2 f$

# Line Integrals

## Green's Theorem

If P(x, y) and Q(x, y) are continuous and differentiable in a region bounded by closed curve C, then:

$$\int_C (Pdx + Qdy) = \int \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \cdot dy$$

# Surface Integrals

## Stoke's theorem

It relates line integral and surface integrals

$$\oint_C F \cdot dr = \int \int_S (\nabla \times F) \cdot N \, ds$$

Where N is unit vector normal to the surface.

# Volume Integrals

## Gauss Divergence theorem

It relates surface integral and volume integral

$$\int_S \int F \cdot ds = \int_V \int \int div \, F \, dv$$

## Points to Remember

Circulation along every closed surface is zero for irrotational field

| Coordinate system | | |
|---|---|---|
| Cartesian<br>(x, y, z) | Cylindrical<br>($\rho$, $\phi$, Z) | Spherical<br>(r, $\theta$, $\phi$) |
| $-\infty < x < \infty$ | $0 \leq \rho < \infty$ | $0 \leq r < \infty$ |
| $-\infty < y < \infty$ | $0 \leq \phi \leq \pi$ | $0 \leq \theta \leq \pi$ |
| $-\infty < z < \infty$ | $-\infty < z < \infty$ | $0 \leq \phi \leq 2\pi$ |

| System | U | V | W | $h_1$ | $h_2$ | $h_3$ |
|---|---|---|---|---|---|---|
| Cartesian | X | Y | Z | 1 | 1 | 1 |
| Cylindrical | $\rho$ | $\phi$ | Z | 1 | $\rho$ | 1 |
| Spherical | r | $\theta$ | $\phi$ | 1 | r | r sin $\theta$ |

## Relation between direction of Cartesian and spherical system

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

## Relation between Cartesian and cylindrical system

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

## Del Operator

$$\boxed{\vec{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial u}\widehat{a_u} + \frac{1}{h_2}\frac{\partial}{\partial v}\widehat{a_v} + \frac{1}{h_3}\frac{\partial}{\partial w}\widehat{a_w}}$$

Various operations:

I) $\vec{\nabla} \cdot V =$ gradient $= \sum \frac{1}{h_1}\frac{\partial}{\partial u} V \hat{a}_u$ where V is scalar Quantity.

II) $\vec{\nabla} \cdot \vec{A} = divergence = \frac{1}{h_1 h_2 h_3} \sum \frac{\partial}{\partial u} h_2 h_3 A_u$

III) $\vec{\nabla} \times \vec{A} = curl = \frac{1}{h_1 h_2 h_3} \left\| \begin{bmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 A_u & h_2 A_v & h_3 h_w \end{bmatrix} \right\|$

IV) $\nabla^2$A = Laplacian operator $= \frac{1}{h_1 h_2 h_3} \sum \frac{\partial}{\partial u} \frac{h_2 h_3}{h_1} \frac{\partial}{\partial u} A_u$

## Points to Remember

- $\vec{\nabla} \cdot \vec{A} = 0$, A field is solenoidal
- $\vec{\nabla} \times \vec{A} = 0$,A field is ir-rotational or conservative in nature
- III) $\nabla^2 A = 0$ A is harmonic in nature