Indian Institute of Technology Kanpur

Random Walks and Electrical Networks

A report submitted for

CS648: Randomized Algorithms

Instructor: Prof. Surender Baswana

Contributors:

- Abhinav Garg (210029)
- Avni Maheshwari (210241)
- Shubham Patel (218070709)

September 9, 2025

Contents

1 Basic Theory

1.1 Random Walks

A random walk is a sequence of steps where each step is determined by independent random variables.

For example, In a 1-dimensional random walk, an object starts at position zero on a number line and moves either +1 or -1 with equal probability at each step. Mathematically, this can be expressed as:

$$S_n = S_{n-1} + X_n$$

where S_n is the position after n steps, and X_n represents the step taken at the n^{th} move.

1.2 Electric Circuits

1.2.1 Ohm's Law

Ohm's Law is a fundamental principle in electrical circuits that describes the relationship between voltage (V), current (I), and resistance (R). It states that the current flowing through a conductor is directly proportional to the voltage across it and inversely proportional to its resistance, provided the physical conditions, such as temperature, remain constant. Mathematically, it is expressed as:

$$V = I \cdot R$$

1.2.2 Kirchhoff's Current Law

Kirchhoff's Current Law (KCL) is a fundamental principle in electrical circuit analysis. It states that the total current entering a junction or node in a circuit must equal the total current leaving the node. This Law is based on the conservation of electric charge, ensuring that no charge is lost at the junction. Mathematically, KCL can be expressed as:

$$I_{\text{in}} = I_{\text{out}}$$

1.2.3 Kirchhoff's Voltage Law

Kirchhoff's Voltage Law (KVL) states that the algebraic sum of all voltages in a closed loop is zero. This principle is rooted in the conservation of energy, ensuring that the energy supplied by voltage sources is entirely accounted for by the energy dissipated or stored in circuit components like resistors, capacitors, and inductors. Mathematically, KVL is expressed as:

$$\sum_{k=1}^{n} V_k = 0$$

Here, V_k represents the voltage across each element in the loop. The Law requires consistent traversal direction (clockwise or counterclockwise) to maintain polarity conventions—adding voltage when moving from negative to positive terminals and subtracting when moving from positive to negative terminals.

1.2.4 Superposition

The Superposition Theorem is an essential concept for analyzing linear electrical circuits containing multiple independent sources, such as voltage and current sources. It asserts that the total current or voltage in any branch of the circuit is the algebraic sum of the individual effects produced by each source operating independently.

Conditions for Application:

1. Superposition is applicable only when there are multiple independent sources in the circuit.
2. The circuit must be linear, meaning it should not include non-linear components like diodes or transistors.

1.3 Linear Algebra

1.3.1 Multiplication with a Vector as Weighted Sum of Columns

Multiplying a matrix by a vector can be interpreted as forming a weighted sum of the matrix's columns, where the vector's entries act as the weights.

Mathematical Representation: Let A be an $m \times n$ matrix with columns $a_1, a_2, \ldots, a_n$, and let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be an $n \times 1$ vector. Then the product Ax is:

$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Each entry x_i in x scales the corresponding column a_i, and the results are summed to produce the final vector.

1.3.2 Rank

The rank of a matrix, denoted as $\text{rank}(A)$, is defined as:

- The maximum number of linearly independent rows in the matrix.
- Alternatively, the maximum number of linearly independent columns in the matrix.

For any matrix A, the number of linearly independent rows is always equal to the number of linearly independent columns. This property is known as **rank equality**.

If the rows or columns of a matrix are independent, it means none of them can be written as a linear combination of the others.

1.3.3 Nullspace

The null space (or kernel) of a matrix is the set of all vectors that, when multiplied by the matrix, result in the zero vector. In other words, it consists of all solutions to the equation:

$$Ax = 0$$

where A is an $m \times n$ matrix, x is an $n \times 1$ column vector, and 0 is the zero vector.

The dimension of the null space is called the **nullity** of the matrix.

1.3.4 Relation Between Rank and Nullspace

For an $m \times n$ matrix A:

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

1.3.5 Positive Definiteness on the Reduced Space and Invertibility

Claim: A symmetric matrix M that satisfies

$$v^T M v > 0 \quad \text{for all nonzero } v$$

is **positive definite**; thus, all its eigenvalues are positive, and it is invertible.

Proof (Eigenvalue Argument via the Rayleigh Quotient): Let λ be any eigenvalue of M with corresponding eigenvector x (with $x \neq 0$), so that:

$$Mx = \lambda x$$

Then:

$$x^T Mx = \lambda x^T x$$

Since M is assumed to satisfy $x^T Mx > 0$ for all nonzero x, and since $x^T x > 0$ for $x \neq 0$, it follows that:

$$\lambda = \frac{x^T Mx}{x^T x} > 0$$

Thus, every eigenvalue of M is positive.

1.3.6 Eigenvalues of a Symmetric Matrix

Theorem: If A is a real symmetric matrix ($A = A^T$), then all eigenvalues of A are real.
Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of A, and let $v \in \mathbb{C}^n$ ($v \neq 0$) be the corresponding eigenvector:

$$Av = \lambda v \tag{eq. 3}$$

Take the conjugate transpose of both sides:

$$v^* A^T = \lambda^* v^*,$$

$$\implies v^* A = \lambda^* v^* \tag{eq. 4}$$

where v^* is the conjugate transpose of v, and λ^* is the complex conjugate of λ.

Multiply v^* on the left of the (eq. 3):

$$v^* Av = \lambda^* v^* v.$$

Multiply v on the right of the (eq. 4):

$$v^* Av = \lambda v^* v.$$

Equating the two expressions for $v^* Av$:

$$\lambda v^* v = \lambda^* v^* v.$$

Since $v \neq 0$, we know $v^* v \neq 0$. Divide both sides by $v^* v$ (since $v^* v$ is a scalar):

$$\lambda = \lambda^*.$$

Thus, λ is real.
Conclusion: All eigenvalues of a real symmetric matrix are real.

1.3.7 Diagonizable Matrices

Theorem: If $B = P^{-1} AP$ for an invertible matrix P, then A and B have the same eigenvalues.
Proof:

The eigenvalues of a matrix A are the roots of its characteristic polynomial:

$$\det(A - \lambda I) = 0.$$

Compute the characteristic polynomial of B:

$$\det(B - \lambda I) = \det(P^{-1} AP - \lambda I).$$

Factor out P^{-1} and P:

$$\det(P^{-1} AP - \lambda I) = \det(P^{-1}(A - \lambda I)P).$$

Using the property $\det(XY) = \det(X)\det(Y)$, we have:

$$\det(P^{-1}(A - \lambda I)P) = \det(P^{-1})\det(A - \lambda I)\det(P).$$

Since $\det(P^{-1}) = \det(P)^{-1}$, we get:

$$\det(P^{-1})\det(P) = 1 \quad \Rightarrow \quad \det(B - \lambda I) = \det(A - \lambda I).$$

Thus, A and B share the same characteristic polynomial and hence the same eigenvalues.

Corresponding Eigenvectors:

Suppose λ is an eigenvalue of A with eigenvector v, so

$$Av = \lambda v.$$

Multiply both sides by P^{-1}:

$$P^{-1}Av = \lambda P^{-1}v.$$

Insert $PP^{-1} = I$ between A and v:

$$P^{-1}A(PP^{-1})v = \lambda P^{-1}v \quad \Rightarrow \quad (P^{-1}AP)(P^{-1}v) = \lambda(P^{-1}v).$$

Let $w = P^{-1}v$. Then:

$$Bw = \lambda w.$$

Since P is invertible and $v \neq 0$, we have $w \neq 0$. Thus, λ is also an eigenvalue of B.

Theorem: If P is diagonalizable (even with repeated eigenvalues), its eigenvectors span the space and are linearly independent.

Proof:

By definition, P is diagonalizable if there exists an invertible matrix V and a diagonal matrix Λ such that:

$$P = V\Lambda V^{-1}.$$

The columns of V are eigenvectors of P, and Λ contains the corresponding eigenvalues along its diagonal.

Since V is invertible, its columns must be linearly independent.

Conclusion: A diagonalizable matrix (even with repeated eigenvalues) has a complete set of linearly independent eigenvectors that span the space.

1.4 Markov Chains

Markov chains are a special type of stochastic process in which the dynamics of the current state are independent of all the previous history of the process just given the previous state.

$$P(S_{t+1}|S_t = a, S_{t-2} = b, ..., S_1 = c) = P(S_{t+1}|S_t = a) \tag{1}$$

So, for the Markov chain, we can define a Transition Probability Matrix P to capture the dynamics. Let n be the total number of possible states. P will be a $n \times n$ matrix with P_{ij} equal to the probability of transitioning from state i to state j, i.e., $P(S_{t+1} = I|S_t = j)$.

The following are properties defined for some kinds of Transition Matrices:

- **Irreducibility:** Every state is reachable from every other state. That is, for all states i, j, there exists $n \in \mathbb{N}$ such that $P^n(i, j) > 0$.
- **Aperiodicity:** The greatest common divisor of return times to any state is 1. Formally, for all states i, the set $\{n \in \mathbb{N} : P^n(i, i) > 0\}$ has greatest common divisor 1.

- **Reversibility:** For the stationary distribution π and for all states i, j,

$$\pi(i)P(i,j) = \pi(j)P(j,i)$$

holds true.

Irreducibility and aperiodicity assure a unique stationary state.

2 Hell–Heaven Problem

2.1 Problem

Suppose a random walker starts walking on a graph, where some nodes are heaven and hell, from a node u. Compute the probability of reaching heaven before hell and prove that it is equal to the voltage of the starting node u when each edge in the graph is replaced by a 1Ω resistor, heaven is connected to a 1V battery, and hell is grounded.

2.2 Solution

Suppose a node v has a degree d_v, then

$$P(\text{reaching heaven before hell from } u) = \frac{1}{d_v} \sum_{v \in N(u)} P(\text{reaching heaven before hell from } v)$$

$$P(\text{reaching heaven before hell from hell}) = 0$$

$$P(\text{reaching heaven before hell from heaven}) = 1$$

Assuming that a 1V battery is connected across heaven and hell with a positive end connected to heaven and hell being grounded and each edge between nodes a and b having a resistance R_{ab},

$$V_u \left(\sum_{v \in N(u)} \frac{1}{R_{uv}} \right) = \sum_{v \in N(u)} \frac{V_v}{R_{uv}}$$

V_u : Voltage at node u.
Assuming all $R_{ab} = 1\Omega$,

$$V_u = \frac{1}{\deg(u)} \sum_{v \in N(u)} V_v$$

Also, we know,

$$V_{\text{heaven}} = 1\text{V}$$

$$V_{\text{hell}} = 0\text{V}$$

Therefore, by equivalence, we can say that

$$P(\text{reaching heaven before hell from } u) = V_u$$

3 Uniqueness, Existence, and Reversibility

We consider an electrical network modeled as an undirected, connected graph $G = (V, E)$, where each edge $(i, j) \in E$ has an associated positive conductance $\rho_{ij} > 0$. The network supports arbitrary current injections or withdrawals I_n at each node n. Our goal is to analyze the behavior of this network using Kirchhoff's Current Law (KCL) and fundamental principles from linear algebra.

3.1 Kirchhoff's Current Law in the cicuit

Let v_n be the voltage at node n, and suppose $N(n)$ is the set of neighbors of node n. The current flowing from node n to its neighbor i is given by Ohm's Law:

$$I_{n\to i} = \rho_{ni}(v_n - v_i).$$

By Kirchhoff's Current Law (KCL), the total current flowing out of node n must equal the net current I_n injected at node n. Hence:

$$\sum_{i\in N(n)} \rho_{ni}(v_n - v_i) = I_n.$$

This can be rearranged as:

$$\left(\sum_{i\in N(n)} \rho_{ni}\right) v_n - \sum_{i\in N(n)} \rho_{ni} v_i = I_n.$$

Let us define a matrix $A \in \mathbb{R}^{N\times N}$ associated with the electrical network G & conductances ρ_{ij} as follows:

- For all i, define the diagonal entries:
$$A_{ii} = \sum_{j\in N(i)} \rho_{ij}.$$
- For all $i \neq j$, define the off-diagonal entries:
$$A_{ij} = \begin{cases} -\rho_{ij}, & \text{if } (i,j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Let $v \in \mathbb{R}^N$ be the vector of node voltages, and $I \in \mathbb{R}^N$ be the vector of current injections. Then, the KCL equations for the entire network are written compactly as:

$$Av = I.$$

3.2 Zero Net Current

A key property of the matrix A is that every row (and likewise every column) sums to zero. Let $\mathbf{1} \in \mathbb{R}^N$ denote the all-ones vector. Then:

$$\mathbf{1}^T A = 0$$

Consider:

$$\sum_n I_n = \mathbf{1}^T I$$

$$\implies \sum_n I_n = \mathbf{1}^T A v$$

$$\implies \sum_n I_n = (\mathbf{1}^T A)v = \mathbf{0}^T v$$

$$\implies \sum_n I_n = 0.$$

Thus, the total current entering the circuit equals the total current leaving it; that is, in a valid electrical network, the net current flow is zero.

3.3 Existence, Uniqueness, and Reversibility

Claim: For a connected electrical network,

$$\ker(A) = \{c\mathbf{1} \mid c \in \mathbb{R}\},$$

i.e., the nullspace of A consists exactly of the constant vectors.

Proof:

For each node i,

$$(A\mathbf{1})_i = A_{ii} \cdot 1 + \sum_{j \neq i} A_{ij} \cdot 1 = \left(\sum_{j:(i,j)\in E} \rho_{ij} \right) - \sum_{j:(i,j)\in E} \rho_{ij} = 0.$$

$$\implies A\mathbf{1} = 0$$

Hence, $\mathbf{1} \in \ker(A)$, and so any constant multiple $c\mathbf{1}$ is also in $\ker(A)$.

Suppose $\mathbf{x} \in \ker(A)$, i.e.,

$$A\mathbf{x} = 0$$

$$\implies \mathbf{x}^T A\mathbf{x} = 0 \qquad \text{(eq. 1)}$$

Now let us look at $\mathbf{x}^T A\mathbf{x}$

$$\mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{N} x_i (A\mathbf{x})_i = \sum_{i=1}^{N} x_i \left(A_{ii} x_i + \sum_{j \neq i} A_{ij} x_j \right).$$

Substitute the definitions:

$$\mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{N} x_i \left(\left(\sum_{j \in N(i)} \rho_{ij} \right) x_i - \sum_{j \in N(i)} \rho_{ij} x_j \right).$$

This equals

$$\sum_{i=1}^{N} \sum_{j \in N(i)} \rho_{ij} (x_i^2 - x_i x_j).$$

Since each edge (i, j) appears twice (once in the sum for i and once for j), we can rewrite the expression as

$$\mathbf{x}^T A\mathbf{x} = \frac{1}{2} \sum_{(i,j)\in E} \rho_{ij} \left[x_i^2 - 2x_i x_j + x_j^2 \right] = \sum_{(i,j)\in E} \rho_{ij} (x_i - x_j)^2.$$

$$\implies \mathbf{x}^T A\mathbf{x} = \sum_{(i,j)\in E} \rho_{ij} (x_i - x_j)^2 \qquad \text{(eq. 2)}$$

From (eq. 1) and (eq. 2):

$$A\mathbf{x} = 0 \implies \mathbf{x}^T A\mathbf{x} = 0$$

$$\implies \sum_{(i,j)\in E} \rho_{ij} (x_i - x_j)^2 = 0$$

Therefore, in order for $A\mathbf{x} = 0$, it is necessary that

$$\sum_{(i,j)\in E} \rho_{ij} (x_i - x_j)^2 = 0.$$

Since each $\rho_{ij} > 0$ and every squared difference is non-negative, this sum can only vanish if $(x_i - x_j)^2 = 0$ for every edge $(i, j) \in E$. That is, $x_i = x_j$ for all connected pairs. Because the graph is connected, this

implies all entries of $\mathbf{x}$ are equal, so $\mathbf{x} = c\mathbf{1}$ for some constant c.

Thus, the nullspace of A is exactly the set of constant vectors $c\mathbf{1}$. This gives us that Nullity$(A) = 1$ which implies that $\mathbf{Rank}(\mathbf{A}) = \mathbf{N} - \mathbf{1}$.

Now, to eliminate the singularity of A, fix a reference node by setting $v_1 = 0$. Every voltage vector v is now uniquely represented by a vector $v' \in \mathbb{R}^{N-1}$ for the remaining nodes. In other words, we consider the subspace

$$\left\{v \in \mathbb{R}^N \mid v_1 = 0\right\}.$$

In this subspace, the only constant vector is the zero vector. Consequently, if v' is nonzero, then the extended vector $v = (0, v')$ is not constant, and there exists at least one edge (i, j) for which $v_i \neq v_j$. Hence,

$$v^T A v = \sum_{(i,j)\in E} \rho_{ij}(v_i - v_j)^2 > 0.$$

This shows that the quadratic form restricted to the subspace (which is represented by A') is strictly positive.

A symmetric matrix is positive definite if, for every nonzero vector x, the quadratic form $x^T M x > 0$. We have just shown that if $x \in \mathbb{R}^{N-1}$ (representing the non-reference voltages) is nonzero, then

$$x^T A' x > 0.$$

Because A' is symmetric (inherited from A) and positive definite on $\mathbb{R}^{N-1}$, it follows from standard linear algebra that:

- All eigenvalues of A' are positive.
- A' is nonsingular (invertible).

A matrix with all positive eigenvalues cannot have a zero eigenvalue, so the determinant is nonzero, and the matrix is invertible.

Existence: The linear system $Av = I$ has a solution if and only if $I \in \text{im}(A)$. From earlier, $\text{im}(A) = \{x \in \mathbb{R}^N \mid \mathbf{1}^T x = 0\}$, which holds for all valid current configurations by conservation of current. Hence, a solution exists.

Uniqueness: The nullspace of A consists of constant vectors, implying that solutions to $Av = I$ are unique up to an additive constant. By fixing one voltage (e.g., $v_1 = 0$), this ambiguity is removed. Then, the reduced system $A'v' = I'$ has a unique solution.

Reversibility: If the current vector is negated, i.e., $I \mapsto -I$, the solution to $Av = -I$ is simply $v \mapsto -v$, since $A(-v) = -Av$. Thus, reversing the current injections reverses the voltage configuration as well.

4 Hitting Time and Commute Time

4.1 Hitting Time

The Hitting Time H_{uv} is the expected number of steps of the walk that starts from u and ends at v. Therefore, by definition, H_{uv} on a general graph can be written as:

$$H_{uv} = 1 + \frac{1}{\deg(u)} \sum_{y \in N(u)} H_{yv}$$

Taking inspiration from the Heaven-Hell Problem, we replace each edge with a 1Ω resistor and then apply KCL on the graph:

$$V_u = \frac{1}{\deg(u)} \sum_{y \in N(u)} V_y$$

To compensate for the constant term in the equation for H_{uv}, we connect each node with a current source of deg(the node) A, i.e., a node x is connected with a current source of $\deg(x)$ A. The node v, along with the current source mentioned earlier, is also connected to a current source of value $-2m$ A to maintain the conservation of current and is grounded.

Therefore, the equation of voltage at node u is:

$$V_u = \frac{1}{\deg(u)} \left[\deg(u) + \sum_{y \in N(u)} V_y \right] = 1 + \frac{1}{\deg(u)} \sum_{y \in N(u)} V_y$$

Therefore, by equivalence, we can say that:

$$V_u \text{ (for the circuit mentioned above) } = H_{uv}$$

4.2 Commute Time

The Commute time C_{uv} is the expected number of steps of the walk that starts and ends at u and visits v at least once during the walk. Therefore, by definition:

$$C_{uv} = H_{uv} + H_{vu}$$

We know $H_{uv} = V_u$ when each edge is replaced by a 1Ω resistor, and each node is connected with a current source of deg(the node) A, i.e., a node x is connected with a current source of $\deg(x)$ A. The node v, along with the current source mentioned earlier, is also connected to a current source of value $-2m$ A to maintain the conservation of current and is grounded.

Similarly, for H_{vu}, we replace each edge with a 1Ω resistor and connect each node with a current source of deg(the node) A, i.e., a node x is connected with a current source of $\deg(x)$ A. The node u, along with the current source mentioned earlier, is also connected to a current source of value $-2m$ A to maintain the conservation of current and is grounded. We get $H_{vu} = V_v$.

Now, we reverse the directions of current at each node for the second graph and then superimpose both the graphs to get the circuit where $2m$ A current enters at u and leaves at v.

By the properties of superposition, we know that:

$$V_u = H_{uv}$$
$$V_v = -H_{vu}$$

Since no current is entering or leaving the intermediate nodes, we can reduce the circuit to a simpler circuit where the nodes u and v are connected by a resistance R_{uv} and a current $I = 2m$ A enters at node u and leaves at v.

R_{uv}: The equivalent resistance between node u and v

On applying Ohm's Law, we get:

$$V_u - V_v = IR_{uv}$$

$$H_{uv} - (-H_{vu}) = 2m \cdot R_{uv}$$

$$H_{uv} + H_{vu} = 2m \cdot R_{uv}$$

$$C_{uv} = 2m \cdot R_{uv}$$

Therefore, the commute time is equal to:

$$\boxed{C_{uv} = 2 \times (\text{Total number of edges}) \times R_{uv}}$$

where R_{uv} is the equivalent resistance between u and v when each edge of the graph is replaced by 1Ω.

5 Mixing of Markov Chains

5.1 Bound on Eigenvalues of Transition Probability Matrix

Let P be the transition matrix of a finite Markov chain defined as:

1. All entries $P_{ij} \geq 0$.
2. Each row sums to 1: $\sum_j P_{ij} = 1$ for all i.

Let λ be an eigenvalue of P with eigenvector $\mathbf{v} \neq \mathbf{0}$, such that:

$$P\mathbf{v} = \lambda\mathbf{v}.$$

Let v_k be the component of $\mathbf{v}$ with the maximum absolute value, i.e.,

$$|v_k| = \max_i |v_i|$$

Consider the k-th component of $P\mathbf{v}$:

$$(P\mathbf{v})_k = \sum_j P_{kj} v_j.$$

By the eigen equation, this equals λv_k:

$$\lambda v_k = \sum_j P_{kj} v_j.$$

$$|\lambda| \cdot |v_k| = \left| \sum_j P_{kj} v_j \right| \leq \sum_j P_{kj} |v_j|.$$

(Since we apply the triangle inequality.)

Since $|v_j| \leq |v_k|$ for all j, we have:

$$\sum_j P_{kj} |v_j| \leq \sum_j P_{kj} |v_k| = |v_k| \sum_j P_{kj}.$$

Using the fact that $\sum_j P_{kj} = 1$, we get:

$$|\lambda| \cdot |v_k| \leq |v_k|.$$

Since $\mathbf{v} \neq \mathbf{0}$, we know that $|v_k| > 0$. Dividing both sides by $|v_k|$, we obtain:

$$|\lambda| \leq 1.$$

Thus, all eigenvalues λ of P satisfy $|\lambda| \leq 1$.

5.2 Span of Eigenvectors of reversible Markov chains

Let P be a reversible Markov chain. The stationary distribution π is unique and strictly positive; that is, $\pi(i) > 0$ for all i.

Define the diagonal matrix

$$D = \text{diag}(\sqrt{\pi(1)}, \ldots, \sqrt{\pi(n)}),$$

and let

$$Q = DPD^{-1}.$$

Then the entries of Q are given by

$$Q_{i,j} = (DPD^{-1})_{i,j} = \sqrt{\pi(i)} \cdot p_{i,j} \cdot \frac{1}{\sqrt{\pi(j)}}.$$

Now assume the detailed balance condition:

$$(*) \quad \pi(i)p_{i,j} = \pi(j)p_{j,i}.$$

Then we compute:

$$\begin{aligned} Q_{j,i} &= \sqrt{\pi(j)} \cdot p_{j,i} \cdot \frac{1}{\sqrt{\pi(i)}} \\ &= \frac{1}{\sqrt{\pi(j)}} \cdot \pi(j)p_{j,i} \cdot \frac{1}{\sqrt{\pi(i)}} \\ &= \frac{1}{\sqrt{\pi(j)}} \cdot \pi(i)p_{i,j} \cdot \frac{1}{\sqrt{\pi(i)}} \quad \text{(by (*))} \\ &= \sqrt{\pi(i)} \cdot p_{i,j} \cdot \frac{1}{\sqrt{\pi(j)}} = Q_{i,j}. \end{aligned}$$

Therefore, Q is symmetric.

Since Q is symmetric, all its eigenvalues are real, and Q is diagonalizable. Moreover, since Q is similar to P (i.e., $P = D^{-1}QD$), it follows that P is also diagonalizable and has real eigenvalues.

5.3 Mixing Time

Hence, the eigenvectors of P (and consequently of P^T) are linearly independent. Let there be n states in our Markov chain. Thus, any vector $v \in \mathbb{R}^n$ can be expressed as a linear combination of the eigenvectors of P^T, i.e.,

$$v_0 = \sum_{i=1}^{n} \alpha_i \epsilon_i,$$

where ϵ_i denotes the i-th eigenvector of P.

Without loss of generality, let ϵ_1 be the eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

Now, let v_j denote the probability distribution after j steps. Then,

$$v_j^T P = v_{j+1}^T.$$

Taking the transpose of both sides yields:

$$P^T v_j = v_{j+1}.$$

Therefore, it follows that:

$$v_j = (P^T)^j v_0.$$

Since v_0 can be expressed as a linear combination of the eigenvectors of P^T, we have:

$$v_j = (P^T)^j \left(\sum_{i=1}^{n} \alpha_i \epsilon_i \right) = \sum_{i=1}^{n} \alpha_i (P^T)^j \epsilon_i = \sum_{i=1}^{n} \alpha_i \lambda_i^j \epsilon_i.$$

Thus,

$$v_j = \alpha_1 \lambda_1^j \epsilon_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i.$$

Since $\lambda_1 = 1$, we obtain:

$$v_j = \alpha_1 \epsilon_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i. \tag{6}$$

Note that P^T is sum-preserving; that is, $\mathbf{1}^T v = \mathbf{1}^T (P^T v)$ for any $v \in \mathbb{R}^n$. Therefore,

$$\mathbf{1}^T v_j = \mathbf{1}^T \left(\alpha_1 \epsilon_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i \right) = \mathbf{1}^T (\alpha_1 \epsilon_1) + \mathbf{1}^T \left(\sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i \right).$$

As noted earlier, $|\lambda_i| \leq 1$ for all i, and since we have separated the term with $\lambda_1 = 1$, the remaining λ_i's satisfy $|\lambda_i| < 1$. Moreover, $\lambda_i = -1$ is not possible, as it would contradict the stationary state condition.

For sufficiently large j, we observe that

$$\mathbf{1}^T \left(\sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i \right) \longrightarrow 0.$$

And since the sum is preserved, we have:

$$\mathbf{1}^T \left(\sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i \right) = 0 \quad \forall j.$$

It follows that:

$$\mathbf{1}^T v_j = \mathbf{1}^T (\alpha_1 \epsilon_1) = \alpha_1.$$

Since v_j must be a valid probability vector, we have:

$$\mathbf{1}^T v_j = 1 \quad \Rightarrow \quad \alpha_1 = 1.$$

From equation (6), we then have:

$$v_j = \epsilon_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i,$$

which implies:

$$v_j - \epsilon_1 = \sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i,$$

and hence:

$$|v_j - \epsilon_1| = \left| \sum_{i=2}^{n} \alpha_i \lambda_i^j \epsilon_i \right|.$$

By bounding by the largest $|\lambda_i|$, we obtain:

$$|v_j - \epsilon_1| \leq \sum_{i=2}^{n} |\alpha_i \lambda_i^j \epsilon_i| \leq |\lambda_2|^j . C,$$

as all ϵ_i are normalized and $\sum \alpha_i = C$ and therefore:

$$|v_j - \epsilon_1| \leq |\lambda_2|^j . C$$

6 Army of Random Walkers

6.1 Motivation

Is the connection between the random walks and electrical networks merely due to them having similar equations? Is there a deeper connection between these two phenomena causing this equivalence? To try to explore this direction more, we begin our journey.

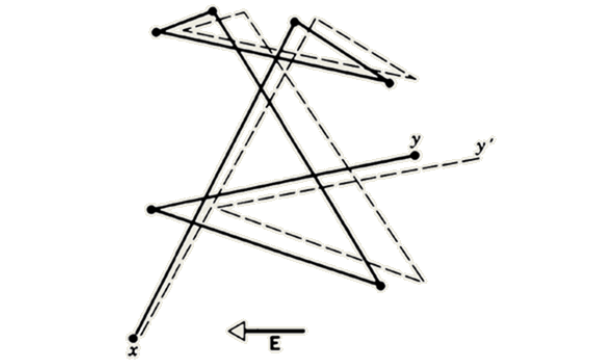

Figure 7 The solid line segments show an electron moving from x to y, making six collisions en route. The dashed lines show what its path *might* have been in the presence of an applied electric field **E**. Note the gradual but steady drift in the direction of $-\mathbf{E}$. (Actually, the dashed lines should be slightly curved to represent the parabolic paths followed by the electrons between collisions.)

Figure 1: Concept of Drift Velocity.

Diving deeper into electrical networks. What is current? Current is essentially due to the flow of electrons. To be more precise, the electrons can be considered somewhat similar to a random walker. They move in a straight direction until they collide with the metallic lattice and then resume their motion in a random direction. Due to an electric field, they are accelerated opposite to the direction of the field until they again collide with the lattice. This leads them to have an effective drift velocity that is much smaller than their traveling velocity.

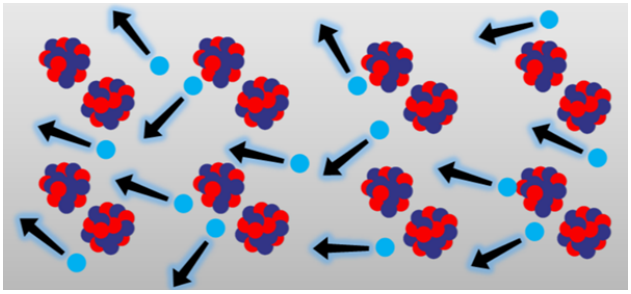

Figure 2: Motion of electrons.

Ok, but what is the use of all this? We observe similar dynamics at other places where we have sort of multiple random movements at molecular scale leading to equations highly similar to electric networks.

For example, consider the relation between the pressure and velocity of fluids. In the fluid, we have multiple molecules with kinetic energy moving randomly. The overall velocity of the fluid is proportional to the pressure difference analogous to the current, which is proportional to the voltage difference, and the resistance of the pipe analogous to resistors.

$$\Delta P = Q * \frac{8\mu L}{\pi R^4} \tag{2}$$

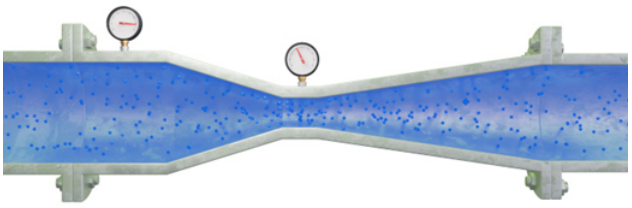

Figure 3: Pressure difference and Fluid Transfer Rate.

Additionally, consider the case of heat transfer and temperature difference. The temperature of a solid is determined by how energetic are the random vibrations of the molecules in the solid. Here, the rate of heat transfer is proportional to the temperature difference and inversely to the heat resistivity of the material.

$$\Delta T = \dot{Q} * \frac{L}{\kappa \pi R^2} \tag{3}$$

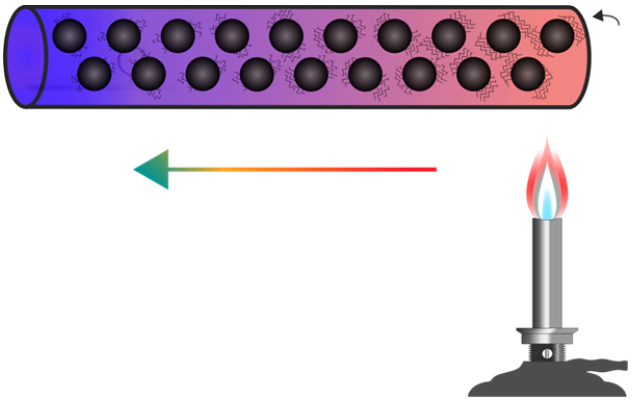

Figure 4: Temperature Difference and Heat Transfer Rate in Solids.

In the above cases, we can see that some sort of random motion is undertaken by an enormous number of microscopic particles. Also, the observation is the macroscopic effect of dynamics being governed by equations analogous to electric circuits. We also note that this correlation does not directly imply causation. With the discussed overall motivation, we tried to come up with a deeper connection and were able to take a small step in that direction by showing equivalence between an army of random walkers (similar to multiple electrons) on a graph with electrical networks.

6.2 Problem

Given a connected graph with n vertices and m edges, there are N random walkers on the graph. At each time step, a random walker chooses a neighboring vertex and goes to it. After a long time, calculate the expected distribution, i.e., the expected number of random walkers on each vertex. A variant of the same problem would be as follows. Additionally, there are certain vertices where, at every time step, a given number of random walkers are spawned (i.e., newly created), and at certain vertices, the walkers are killed (i.e., removed).

6.3 Solution

Dynamics of Multiple Random Walkers

Let us denote the expected number of walkers on a vertex u as ρ_u. At a given time step, all the walkers on u would go to different neighbors of u. The walkers incoming on u would be from its neighbors. The probability that a given walker on a neighboring vertex v comes to u is $\frac{1}{d_v}$. Hence the expected number of walkers coming from v to u would be $\sum \frac{\rho_v}{d_v}$. Hence, the expected number of walkers on vertex u will be $\sum \frac{\rho_v}{d_v}$. Here, we realize that the dynamics governing electric networks and an army of random walkers are not the same.

$$\rho_u = \sum \frac{\rho_v}{d_v} \tag{4}$$

$$V_u = \sum \frac{V_v}{d_u} \tag{5}$$

The Challenge

In the case of electric networks, the voltage of a given node u is the sum of the voltages of its neighbors inversely weighed by the degree of u. But in the case of an army, the density at a given node u is the sum of the densities of its neighbors inversely weighed by the degree of the neighbor. This proves that ρ_u and V_u can not be considered equivalent.

What should we do at this point of our journey? We began very ambitiously with the aim of trying to delve deeper into the understanding behind the equivalence of these phenomena. Also, we found that both the density and voltages can be expressed in a similar form as the weighted average of their neighbors. But sadly, they are governed by quite different dynamics. Should we leave and discard all the efforts?

Tackling the Challenge

It turns out the answer is no! All that is required is a bit of creativity to utilize the results we have. Coincidentally, it turns out that after the lecture on hashing in which the $O(s^2)$ method was modified to reach an $O(s)$ method, we were able to retry and solve it. The problem has nearly been solved; it just requires a bit of tweaking. Advice to the reader is to try sufficiently before proceeding beyond else on seeing the solution, one might feel that by trying a bit more, even I could have tackled the challenge :)

The idea is to define a new variable $\psi_u = \frac{\rho_u}{d_u}$. Below is how the equations are updated to reflect the equivalence. For the variant of the problem, utilizing current sources would lead to the solution.

$$\psi_u = \sum \frac{\psi_v}{d_u} \tag{6}$$

Intersting Interpretation

Based on this, we can see that the current in an edge of the circuit is proportional to the number of random walkers crossing the edge. This is because $I = \frac{\Delta V}{1}$ and the number of walkers crossing an edge (u,v) is $\frac{\rho_u}{d_u} - \frac{\rho_v}{d_v} = \psi_u - \psi_v = \Delta\psi \approx \Delta V$. This hints that considering the current and electrons as an army of random walkers could be a path to ponder over.

6.4 By-product: Stationary Distribution

Finally, as a by-product of the result, we can compute the stationary state of the transition matrix quite elegantly using the above result. The electric network to be considered would be with a similar structure as the graph and with 1Ω resistors instead of the edges. Here, there would be no voltage or current sources. In such an electric network, the solution would be to have the voltages of all the nodes to be equal. Hence, ψ_u is equal for all, leading to density being proportional to the degree of the node. This density is proportional to the probability of a random walker being on that node. This is because these are independent random walkers. For an independent random walker, the stationary distribution dictates the probability after a sufficiently long time. By linearity of expectation, these two can be shown to be the same. Hence, for the stationary state, p_u is proportional to d_u, and the vector can be computed by normalizing this value.

7 Experimentation

7.1 Hell-Heaven Problem

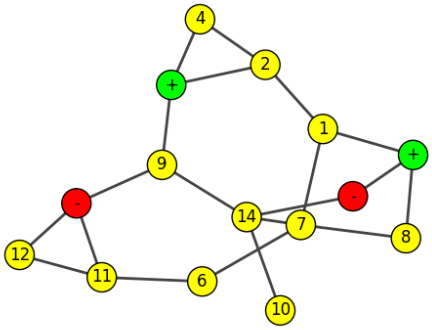

Figure 5: Hell-Heaven Graph Experimentation

We empirically verify the result of the Hell-Heaven problem by performing experimentation on the graph shown in Figure 5. For each node in the graph, we run 10,000 random walks starting at that node and count the number of times it successfully reaches a Heaven node before a Hell node. We compare it with the number predicted by the electric network voltages. The results are highlighted in Table 1. We find that they are close to each other, helping empirically verify the claim.

Index	Empirical	Predicted	Error (%)
1	8417	8409	0.10
2	9330	9363	0.35
4	9677	9681	0.04
6	3682	3664	0.49
7	5868	5863	0.09
8	7990	7931	0.74
9	4386	4482	2.14
10	3361	3448	2.52
11	1568	1465	7.03
12	702	732	4.10
14	3411	3448	1.07

Table 1: Number of Successes for 10,000 trials.

7.2 Commute Time

For the experimentation related to commute time, we utilize the graph shown in Figure 6. We perform 1,000,000 random walks starting at node u and record the commute time taken for each random walk.

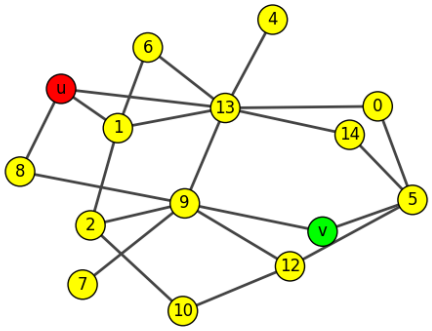

Figure 6: Graph for the Experimentation of Commute Time.

We first observe the distribution of the commute time as shown in Figure 7. As can be seen, the minimum commute time is equal to twice the shortest distance between u and v, as expected.

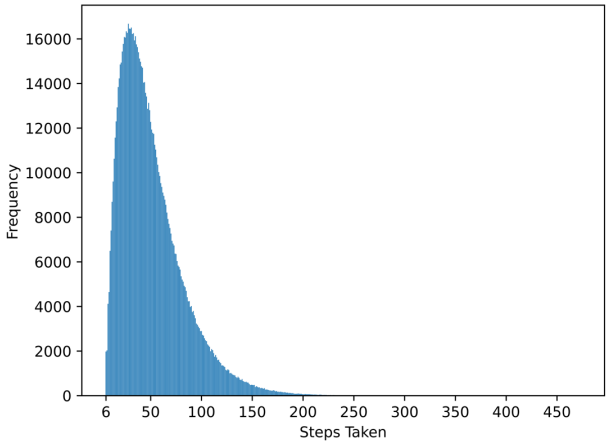

Figure 7: Frequency vs. Steps Taken to Complete the Walk.

During the analysis of Commute Time, a part that can be further researched is how tight the probability of the actual commute time being close to the expected commute time is. To empirically derive this, we first begin by observing the cumulative distribution, in which for a given commute time C_{uv} value, we plot the number of walks that have taken greater than equal to C_{uv}. As can be seen in Figure 8, the probability of deviation appears to be inverse-exponential with the amount of deviation.

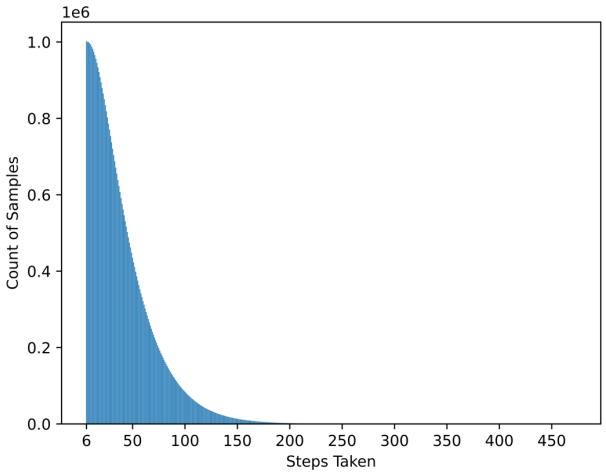

Figure 8: Cumulative distribution of Count of Walks vs Steps.

To verify this hypothesis, we take the logarithm on the y-axis and plot it as Figure 9. To our immense surprise, the data nicely fits a straight line, especially beyond the expected value $E[C_{uv}]$. Thus, this indicates that empirically:

$$P(C_{uv} - E[C_u v] \geq t) \approx e^{\frac{-t}{\tau}} \tag{7}$$

where τ is a problem-dependent constant.

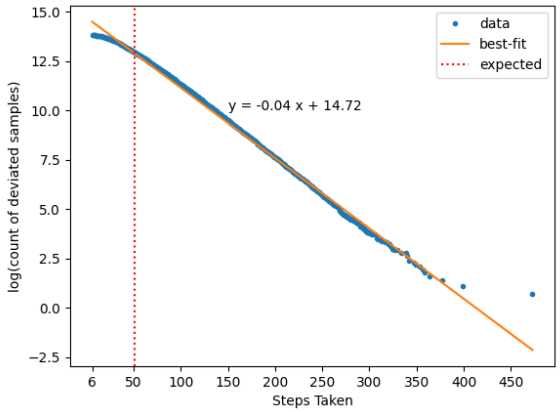

Figure 9: Empirically determining bounds on deviation.

8 Capacitor bounding the deviation - Future Work

We have been able to prove the equivalence between multiple random walkers and electrical circuits. This sort of implies that electrons can be considered similar to random walkers. A possible direction to build on this idea is to utilize the idea of capacitors to get a bound on the probability of deviation in commute time, as shown in Figure 10. This could be because, in terms of random walkers, the capacitor could be considered similar to a battery whose charge decreases when more random walkers arrive. This could help us establish a relationship between the two. This could be an exciting future direction to explore.

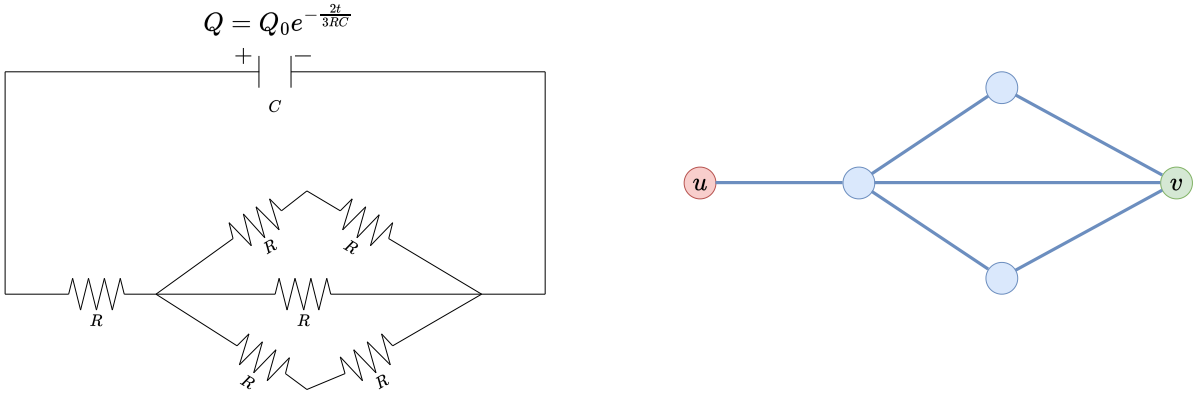

Figure 10: Possible Utility of Capacitor for bounding deviation of probability.