Markov Chains and Electrical Networks
Relaxation Times

May 2025

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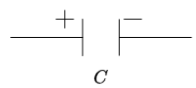
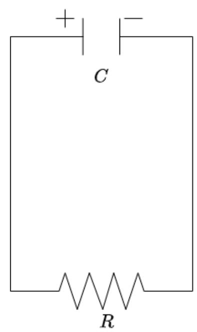
1 Background - Electrical Networks

1.1 Capacitor



Charge = Capacitance × Voltage - $Q = CV$

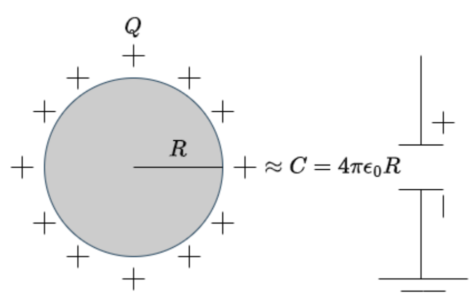
$$Q = Q_0 e^{-\frac{t}{RC}}$$



Current input to capacitor $= I_{in} = \frac{dQ}{dt} = C\frac{dV}{dt}$
For a CR-circuit, charge decays as $Q = Q_0 e^{-\frac{t}{\tau}}, \tau = CR$

Relaxation Time τ is defined as (CR in the above case) the time for a charge on the capacitor to decay to 37% ($\frac{1}{e}$) of the current charge.

1.2 Metallic Sphere as Capacitor



A metallic sphere with radius R, charge Q has potential $V = \frac{Q}{4\pi\epsilon_0 R}$
It can also be viewed as a capacitor with capacitance $4\pi\epsilon_0 R$.

2 Background - Markov Chains

2.1 Markov Chain

Consists of n states.
Matrix $M \in \mathcal{R}^{n \times n}$ such that $M_{ij} = P(s_{t+1} = i | s_t = j)$
$1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_n \geq -1$ are eigen values of M
Define $\lambda_2' = max_{i \neq 1} |\lambda_i|$

Spectral Gap is defined as $1 - \lambda_2$

Absolute Spectral Gap is defined as $1 - \lambda_2'$

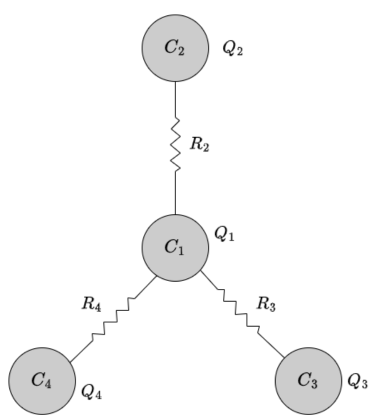
Relaxation Time τ is defined as $\frac{1}{1-\lambda_2'}$

3 Result

The relaxation time of a Markov chain, which can also be represented as a random walk on a graph, is the same as the relaxation time of a C-R network on the graph.

This result holds for Markov chains satisfying theorem 1, have $\lambda_2 = \lambda_2'$ and have a stationary state. The relation for cases where $\lambda_2 \neq \lambda_2'$ is discussed later.

4 Capacitors on a graph



Consider the above configuration of the capacitors with different charges. Let us calculate what equations govern the change of potential at C_1.

$$I_{in} = \frac{dQ_1}{dt} = C_1 \frac{dV_1}{dt}$$

$$\frac{V_2 - V_1}{R_2} + \frac{V_3 - V_1}{R_3} + \frac{V_4 - V_1}{R_4} = C_1 \frac{dV_1}{dt}$$
$$\frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4} - V_1 \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) = C_1 \frac{dV_1}{dt} \tag{1}$$

5 Normal random walk

For simplicity, let us begin with a simple random walk where, at each step, the walker selects one of the neighboring vertex (not the same as itself) with equal probability.

5.1 Setting resistances

Take $R_2, R_3, R_4 = 1\Omega$ since in the random walk it is equally likely to go from 1 to 2,3,4. Our equations become:

$$V_2 + V_3 + V_4 - 3V_1 = C_1 \frac{dV_1}{dt} \tag{2}$$

5.2 Setting capacitance

If we set $C_1 = \deg(1) = 3$, then the equations become:

$$V_2 + V_3 + V_4 - 3V_1 = 3\frac{dV_1}{dt}$$
$$\frac{V_2 + V_3 + V_4}{3} - V_1 = \frac{dV_1}{dt}$$
$$\frac{V_2 + V_3 + V_4}{deg(1)} - V_1 = \frac{dV_1}{dt} \tag{3}$$

5.3 Eigen-Voltage at start

Let us deal with the case that the initial voltage distribution is an eigenvector for the random walk's M^T. Let at the start:

$$V^0 = [V_1^0, V_2^0, ..., V_n^0]^T = \alpha \cdot e_i$$

where α is a constant and e_i is eigenvector of M^T of the random walk with eigen-value λ_i. Since e_i is eigenvector, hence:

$$M^T e_i = \lambda_i e_i$$
$$M^T V^0 = \lambda_i V^0$$

Focusing on the computation of 1^{st} element of RHS, i.e. V_1^0:

$$\frac{1}{deg(1)} \cdot V_2^0 + \frac{1}{deg(1)} \cdot V_3^0 + \frac{1}{deg(1)} \cdot V_4^0 = \lambda_i V_1^0$$

$$\frac{V_2^0 + V_3^0 + V_4^0}{deg(1)} = \lambda_i V_1^0 \tag{4}$$

Substituting Eqn 4 into Eqn 3, at the start the following would hold:

$$\lambda_i V_1 - V_1 = \frac{dV_1}{dt}$$
$$(\lambda_i - 1)V_1 = \frac{dV_1}{dt}$$
$$-(1 - \lambda_i)V_1 = \frac{dV_1}{dt} \tag{5}$$

At the start, equations similar to Eqn 5 would hold for all the nodes. If Eqn 5 is true throughout the duration, then the solution would be:

$$V_1 = V_1^0 e^{-(1-\lambda_i)t} \tag{6}$$

Eqn 5 would indeed hold throughout the duration, not only at the start. This is because, at the start, the exponential decay holds for all the nodes. Hence, in dt time, all of the voltages decrease by a fraction of $\lambda_i dt$. Therefore, Eqn 4 still holds since voltages have decreased by the same fraction. Hence, Eqn 5 throughout making Eqn 6 as the solution.

Hence, here the relaxation time would be:

$$\tau = \frac{1}{1 - \lambda_i} \tag{7}$$

5.4 Generic voltage at start

Inspired from [Eqn 22, paper], for all nodes, Eqn 3 can also be written in the following form:

$$M^T V - V = \frac{dV}{dt}$$
$$(M^T - I)V = \frac{dV}{dt} \tag{8}$$

where the eigenvalues of $M^T - I$ are $(1 - \lambda_i)$ and eigenvectors are same as M^T.

Following Theorem 1 is from Pg 7, Math Course to solve $Ax = \frac{dx}{dt}$:

Theorem 1 *Suppose A is an $n \times n$ matrix of real constants. If A has n real linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct), then the vector functions $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ defined by*

$$\mathbf{x}_k(t) = e^{\lambda_k t}\mathbf{v}_k, \quad \text{for } k = 1, 2, \ldots, n$$

are a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$ on any interval. The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

Using this to solve Eqn 8. Since matrix M^T satisfies the criteria, hence $M^T - I$ also satisfies the criteria. Define $a_i = c_i e_{i1}$ The voltage solution would be:

$$\begin{aligned} V_1 &= a_1 e^{(1-\lambda_1)t} + a_2 e^{(1-\lambda_2)t} + ... + a_n e^{(1-\lambda_n)t} \\ V_1 &= a_1 e^{(1-1)t} + a_2 e^{(1-\lambda_2)t} + ... + a_n e^{(1-\lambda_n)t} \\ V_1 &= a_1 + a_2 e^{(1-\lambda_2)t} + ... + a_n e^{(1-\lambda_n)t} \end{aligned} \tag{9}$$

According to Eqn 9, $1 - \lambda_2$ is the dominant term in governing the voltage. This is because $\lambda_2 \geq \lambda_3 ...$, hence $-(1 - \lambda_2) \geq -(1 - \lambda_3) ....$

For this case, if we define the **effective** relaxation time, it would be:

$$\tau \approx \frac{1}{1 - \lambda_2} \tag{10}$$

Interesting side note
Analyzing Theorem 1 and Eqn 9. At t=0, we need the voltage to be the same as the initial voltage, so c_i is adjusted as that. Now, if we observe the second equation in the theorem carefully, we can note that each of the decays discussed for eigen-voltage at the start is super-posed to produce the final solution. This can be used to extract the eigenvectors.

6 Generic Random Walks

6.1 Self-Transition Probability

To incorporate the self-transition probability, we will appropriately modify the capacitance. Consider a random walk, where the probability of transitioning to any of the neighbors is $\frac{1}{deg(1)+1}$ and is also the probability of staying in the same state.

Since the probability of transitioning to different neighbors is equal, set the resistances to be equal:

$$V_2 + V_3 + V_4 - 3V_1 = C_1 \frac{dV_1}{dt}$$

What should be value of C_1?
Set $C_1 = deg(1) + 1$:

$$\begin{aligned} V_2 + V_3 + V_4 - 3V_1 &= 4 \frac{dV_1}{dt} \\ \frac{V_2 + V_3 + V_4}{4} - \frac{3}{4} V_1 &= \frac{dV_1}{dt} \\ \frac{V_1 + V_2 + V_3 + V_4}{4} - V_1 &= \frac{dV_1}{dt} \\ \frac{V_1 + V_2 + V_3 + V_4}{deg(1) + 1} - V_1 &= \frac{dV_1}{dt} \end{aligned}$$

Now, we can proceed with an analysis similar to the case of normal random walks.

6.2 Un-equal transition probabilities

To assign different probabilities to different neighbors of 1, adjust the resistances R_2, R_3, R_4 appropriately. Lower resistance, higher probability.

To assign self-transition probability, adjust C_1 with respect to the resistances. Higher capacitance, higher probability. For correctness ensure capacitance is at least $\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$.

6.3 Overcoming overloading of resistances

Till now, we discussed how to set the resistances between a given node and its neighbors to satisfy the equation at a given node. But what was overlooked was that when setting R_{xy}, what if a different value is needed wrt node x and wrt node y? To tackle this, diodes can be used to ensure that both the resistances do not interfere with each other. So, each edge will be replaced with two pairs of resistance diodes, one used while setting each node.

6.4 If $\lambda_2 \neq \lambda_2'$

Then compute the matrix $N = M^2$. For this matrix, $\lambda^N = (\lambda^M)^2$. Hence, all eigenvalues are positive, and the eigenvalue with a larger value corresponds to a larger magnitude in M. Use this to find $\lambda_2^{M'}$.

7 Experimentation

We perform a simple experiment to compare the relaxation time of a Markov chain with the corresponding capacitor network. We use the graph (a) below for our experimentation. The Markov chain is a simple random walk on a graph, with $\lambda_2 = \lambda_2'$, to ensure that relaxation times are the same.

The eigenvalues for this random walk are:

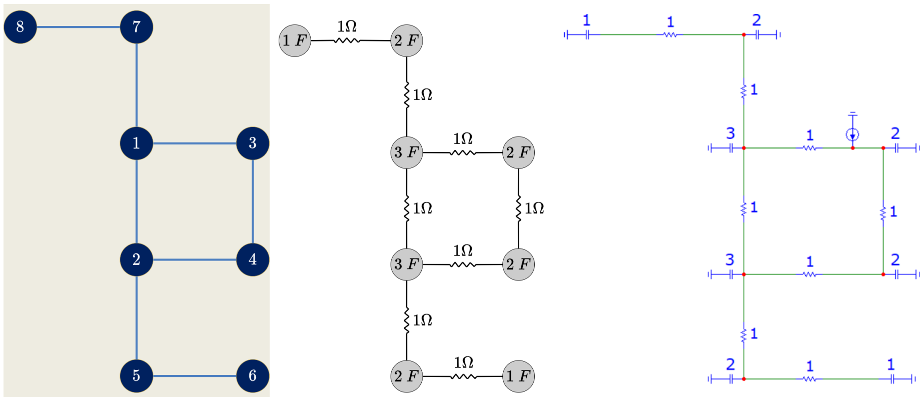
$$1, 0.795, 0.629, 0, 0, -0.629, -0.795, -1$$

Hence the relaxation time is:

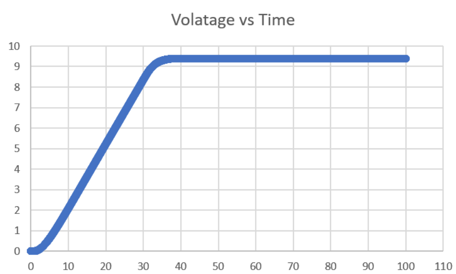
$$\frac{1}{1-\lambda_2'} = \frac{1}{1-0.795} = 4.89$$

Since it is a simple random walk, the resistances would be 1 Ω, and the capacitance would be the same as the degree of the vertex. Which gives the following capacitor network (b).

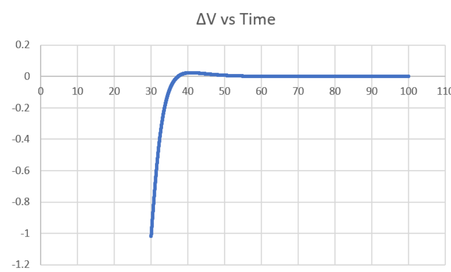
Then, we create this circuit in the simulation, as shown in (c). A small current source is added (at node 3), which emits a pulse of current for 30 seconds. This modifies the charge distribution on the capacitors. Then, we measure how the charge distributes with time to reach equilibrium, from which we measure relaxation time.



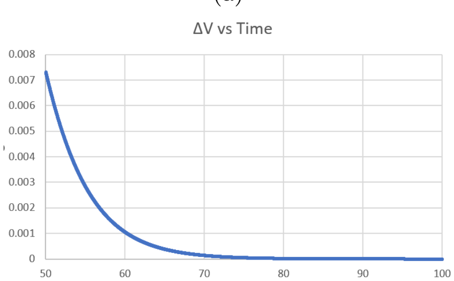
(a) Markov Chain (b) Capacitor Netwok (c) Circuit Simulation



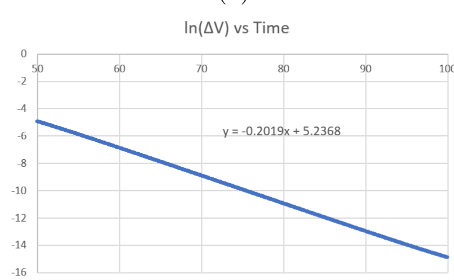
(a)



(b)



(c)



(d)

Following (a) is the plot for voltage at one of the nodes. We can see that for the first 30s, it increases due to pulse and then decays and stabilizes. Figure (b) shows how far the current voltage is from the stable-state voltage. As we can see, it gets closer to 0 as time progresses. Let us focus from the 50s onwards once things are stable. In Figure (c), we can observe that there is an exponential decay. Let us take the natural logarithm on the y-axis and determine the slope as in (d). The slope is found to be -0.2019, and hence, the relaxation time is $\mathbf{4.95} \approx \mathbf{4.89}$, which is within 1.3% of error.

8 Some Related Work / References

Pg 20, Frank Kelly shows that **Capacitor networks and Continuous Reversible Markov chains are equivalent.** The differences between presented and related work are:

- Discrete vs continuous Markov chains
- 'Almost same relaxation time' vs. 'equivalent progress of chain.'
- Diodes help to extend to non-reversible MC vs no need of diode in reversible

Other Related Works or References:

- Electrical Network for Determining the Eigenvalues and Eigenvectors of Real Symmetric Matrix
  Uses a circuit of Capacitors and Inductors.
- Power grid analysis using random walks
  The equation for representing capacitors in electric networks
  (Based on Nodal Analysis)
- Denumerable Markov Chains - Book
  Overall equivalence between electricity and random walks
  Capacitors with some sets
- Math Course
  Capacitor, Electrical Network, and Eigenvalues
- Pg 7, Math Course
  Solution to $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$