Martingales & the Monkey Problem

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What is a martingale?

Martingale

- Etymology: a type of gambling strategy
- S_n is always finite
- The conditional expected value of the next observation, given all the past observations, is equal to the most recent observation.
- → (General case) If a sequence X_n is not a martingale in itself, we could still find some function φ such that $\{S_n = \varphi(X_n) : n \geq 1\}$ is a martingale.

(3) Definition. A sequence $\{S_n : n \geq 1\}$ is a **martingale** with respect to the sequence $\{X_n : n \geq 1\}$ if, for all $n \geq 1$:

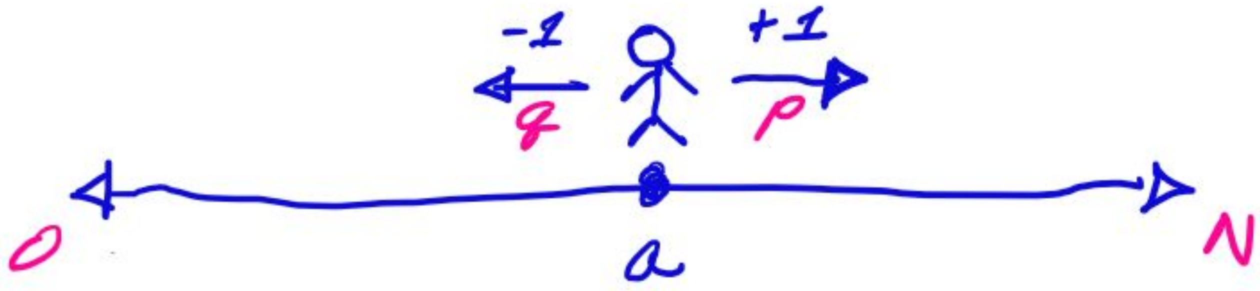
(a) $\mathbb{E}|S_n| < \infty$,

(b) $\mathbb{E}(S_{n+1} \mid X_1, X_2, \dots, X_n) = S_n$.

Example: Random Walk

Random Walk

- There is a drunkard near the edge of the cliff, with initial distance **A**.
- The drunkard takes random steps away from the cliff (+1) with probability **p**, or towards the cliff (-1) with probability **q = 1 - p**.
- The drunkard's position at time n: $\mathbf{S_n}$
- At position **0**, the drunkard falls down the cliff. At **N**, the drunkard is rescued.
- Find: probability that the drunkard falls down the cliff.



Random Walk - Solutions

- **For $p = q = \frac{1}{2}$, $P = 1 - A/N$**

  For $p \neq q$, $P = ((q/p)^A - (q/p)^N)/(1 - (q/p)^N)$
- Method 1: without martingales
  - more intuitive, *but* more complicated to compute
  - differential equations were involved

Random Walk - Solutions

- Method 2: with martingales
- De Moivre's martingale: set $Y_n = (q/p)^{Sn}$ -> Y_n is a martingale wrt S_n (which is not a martingale when $p \neq q$).
- **T** = the number of steps taken before the drunkard is dead or rescued (stopping time)
- Optional stopping theorem
    - *(under certain conditions)* $E(Y_n) = (q/p)^A$ for all n; therefore, $E(Y_T) = (q/p)^A$ as well.
- -> $E(Y_T) = (q/p)^0\, p_A + (q/p)^N\, (1 - p_A)$ $\quad$ $p_{,A} = P(\text{falling down the cliff} \mid S_0 = A)$

  -> $(q/p)^A = p_A + (q/p)^N\, (1 - p_A)$

  -> solve for $p_A = ((q/p)^A - (q/p)^N)/(1 - (q/p)^N)$

The Monkey Problem

The Monkey Problem

- A monkey is placed in front of a typewriter and types a capital letter at random.
- Just before the monkey types a letter at time n = 1, 2, …, a new gambler arrives and bets that the n^{th} letter will be A.
- If he wins, he receives \$26 and bets it all on the $(n+1)^{th}$ letter being B. If he wins again, he bets the $\$26^2$ he has that the $(n+2)^{th}$ sequence, so on and so forth, through the “**ABRACADABRA**” sequence.
- At any point, if he loses the bet, he leaves.
- **T** is the first time by which the monkey types **ABRACADABRA.** Find **E(T)**.

Solution to the Monkey Problem

- Recall Set-up: at every time n, a new gambler enters and bets \$1. After T times, T dollars are bet. Finding the expected value of total winnings of all gamblers -> allows us to solve for E(T)
- 3 gamblers by the time of ABRACADABRA: 1st, 8th ("ABRA"), 11th ("A")
- Expected total winning: $26^{11} + 26^{4} + 26$ **- T** (why is this a martingale?)

Solution to the Monkey Problem

- optional stopping theorem (again)
- -> $E(X_T) = E(X_0) = 0$

  $E(X_T) = E(26^{11} + 26^4 + 26 - T) = 0$

  $26^{11} + 26^4 + 26 - E(T) = 0$

  -> $\mathbf{E(T) = 26^{11} + 26^4 + 26 = 3{,}670{,}344{,}487{,}444{,}778}$
- Assuming a monkey can press 250 keys in a minute, this will take

  14,681,377,949,779 minutes = 27,932,606 years (Lutsko, 2023).

Applications

Application of Martingales

- Martingales have properties that make modeling other random processes easier.
- Predicting other random processes
  - The stock market
  - Epidemics
  - Ecosystem - the population of a species in a stable ecosystem

Thank you for listening :)

Works Cited

- Ai, Di. “Martingale and the Monkey Problem”. 2011
- Grimmet and Stirkazer. *Probability and Random Processes*. Oxford University Press, Oxford, 2020. Fourth Edition.
- Lutsko, Christopher. “Monkeys typing and martingales”. 2023