Martingales and the Monkey Problem

Irene Cai

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1 Introduction

In AU2025 term, I worked on topic "Martingales and Monkey Problem" with mentor Hansen Zhang. In this report, I will give a high-level summary of the subject.

2 Definition of a martingale

A martingale is a sequence of random variables where [1] all random variables in the sequence have finite expectations and [2] the conditional expected value of the next observation in the sequence, given all past observations, is equal to the most recent observation. More generally, for sequences that are not martingales by themselves, we might still find some function that generates a martingale from the original (non-martingale) sequence.

3 The Drunkard's Walk

In the "drunkard's walk", the drunkard has a certain initial distance from the cliff. At every time, they step away from the cliff with probability p, or towards with probability $q = 1 - p$. This process continues until either the drunkard falls down the cliff, or they are rescued when they first reach a certain distance from the cliff, assumed to be greater than the initial distance. Solve for the probability of death given the initial distance.

This question is a one-dimensional random walk. When $p = q$, the sequence of random variables representing the drunkard's distance from the cliff at each time is a martingale. This distance is always finite, and given the current distance, the drunkard's next distance is expected to be the same. When $p \neq q$, the distance is not by itself a martingale, but we can apply a specific function to generate a martingale (De Moivre's martingale) with respect to the original sequence.

Representing the distance as a martingale allows us to use the optional stopping theorem. This theorem states that, for certain martingales [1], the expected value of a random variable at a stopping time (in this case, when the drunkard is dead or rescued) is equal to the initial expected value. Expanding the expected value of the sequence at the stopping time allows us to solve for the probability of death with simple algebra. The drunkard's walk *can* be solved without martingale, but the computation is much more convoluted.

The drunkard's walk shows how martingales have properties that make them useful for modeling random processes. For processes that may not have an obvious martingale, it is then attractive to transform them to one.

4 The Monkey Problem and its Solution

A monkey is placed in front of a typewriter, and types capital letters at random. Just before the monkey types a letter at each time, a new gambler arrives and bets \$1 that this letter will be A. If they win, they receive \$26 and bets it all on the next letter being B. If they win again, they bet the $\$26^2$ they have on the next letter, so on and so forth, through the "ABRACADABRA" string. If the gambler lose the bet at any point, the gambler leaves. At time T, the monkey expected to first type "ABRACADABRA". Find T.

The somewhat riddling setup, with the gamblers joining and quitting the game, actually allows this question to be solved as a martingale. Since a gambler enters and bets \$1 at every time, after T times, T dollars are bet. Thus, finding the expected total winnings of all gamblers allow us to solve for the expected value of T. The total winnings of all gamblers is a martingale, because the bets of each individual gamblers is a martingale: their expected bet at any time given their last bet is equal to the last bet.

By the time the monkey typed "ABRACADABRA", there are three gamblers left: the first ("ABRACADABRA"), the eighth ("ABRA"), and the eleventh ("A"). Using the optional stopping theorem, we can solve the expected total winning $\mathrm{E}(26^{11} + 26^4 + 26 - T)$, which gives $\mathrm{E(T)} = 26^{11} + 26^4 + 26$. Assuming a monkey can type 250 keys every minute, this will take nearly 28 million years.

[1] The specific conditions are not relevant to this report because both this and the Monkey problem satisfy them.

5 Conclusion

As thought experiments, the drunkard's walk and the monkey problem reveal how the properties of martingales facilitate mathematical modeling. Therefore, it has applications in modeling real life processes like the stock market, the spread of epidemics through a community, or the population size of species in stable ecosystems.

6 References

[1]Ai, Di. "Martingale and the Monkey Problem". 2011.
[2]Grimmet and Stirkazer. *Probability and Random Processes*. Oxford University Press, Oxford, 2020. Fourth Edition.
[3]Lutsko, Christopher. "Monkeys typing and martingales". 2023.