Spring 2025 DRP: Martingales

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In this DRP, we explored the theory of martingales, following our study of measure-theoretic probability last quarter. The aim was to understand conditional expectation, formalize martingales, and study their convergence properties and applications.

1 Conditional Expectation

We began with the definition of conditional expectation $E[X \mid \mathcal{F}]$ as the unique $\mathcal{F}$-measurable random variable satisfying $\int_A E[X \mid \mathcal{F}]dP = \int_A X dP$ for all $A \in \mathcal{F}$. For $X \in L^2$, this coincides with the L^2-orthogonal projection onto the subspace of $\mathcal{F}$-measurable functions. Key properties include linearity, the tower property $E[E[X \mid \mathcal{G}] \mid \mathcal{F}] = E[X \mid \mathcal{F}]$ for $\mathcal{F} \subseteq \mathcal{G}$, and the 'taking out what is known' rule: if Y is $\mathcal{F}$-measurable and bounded (or $XY \in L^1$), then $E[XY \mid \mathcal{F}] = YE[X \mid \mathcal{F}]$.

2 Martingales and Almost Sure Convergence

A sequence $(M_n, \mathcal{F}_n)$ is a martingale if M_n is integrable, $\mathcal{F}_n$-measurable, and $E[M_{n+1} \mid \mathcal{F}_n] = M_n$. Submartingales and supermartingales relax this equality to inequalities.

We studied the Martingale Convergence Theorem which states that a martingale with uniformly bounded L^1-norm converges almost surely (and in L^1 if uniformly integrable). There was also a nice construction of the Radon–Nikodym derivatives using martingale techniques.

3 Doob's Inequality and L^p Convergence

Doob's maximal inequality provides for a nonnegative submartingale (X_n) and $p > 1$,

$$\| \sup_{k \leq n} X_k \|_p \leq \frac{p}{p-1} \|X_n\|_p.$$

Consequently, L^p-bounded martingales (for $p > 1$) converge almost surely and in L^p. This inequality is crucial in proving convergence results and in bounding stopping times.

4 Square Integrable Martingales

Martingales in L^2 enjoy orthogonal decompositions. The Doob decomposition splits any submartingale process into a martingale and an increasing predictable process. We also saw the Pythagorean theorem of martingale differences:

$$E[M_n^2] = E[M_0^2] + \sum_{k=1}^{n} E[(M_k - M_{k-1})^2].$$

5 Uniform Integrability and L^1 Convergence

Uniform integrability is the necessary and sufficient condition for L^1-convergence of martingales: a martingale M_n converges in L^1 to M if and only if it is uniformly integrable. In this case, it also converges almost surely. We studied sufficient conditions for uniform integrability, such as boundedness in L^p for some $p > 1$.

6 Optional Stopping Theorems

We proved the Optional Stopping Theorem under various hypotheses on the stopping time T (e.g., bounded T, integrable bound on increments, or L^1-boundedness of $M_{T\wedge n}$). Formally,

$$E[M_T] = E[M_0]$$

for suitable (M_n) and T. Applying optional stopping to simple symmetric random walks gives us classical results such as gambler's ruin probabilities and expected stopping times.

7 Concentration Inequalities: Azuma–Hoeffding

To quantify how tightly a martingale remains concentrated around its starting value, we can use the Azuma–Hoeffding inequality. Let $(M_n, \mathcal{F}_n)$ be a martingale whose increments satisfy the almost sure bounds

$$|M_k - M_{k-1}| \leq c_k, \quad k = 1, 2, \ldots, n.$$

Then for any $t > 0$,

$$P\big(|M_n - M_0| \geq t\big) \leq 2\exp\Big(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\Big).$$

This inequality gives us concentration bounds for martingales with bounded increments and underlies applications in sums of bounded independent random variables to McDiarmid's bounded-differences method and many other concentration results.

Conclusion

First and foremost, I want to thank Nila and Leon for being such amazing mentors! I learned so much from you guys these past two quarters and had a lot of fun along the way. This quarter, we studied how martingale theory captures the idea of a "fair" stochastic process through conditional expectation, and how Doob's inequalities combined with uniform integrability guarantee convergence in both almost-sure and L^p senses. Tracing the history of the development of martingales and stochastic processes was also interesting, which we formed our presentation around. Applications to random walks through optional stopping, and concentration of measure through Azuma-Hoeffding inequality helped us see how useful martingale techniques can be. Building on these foundations, we are excited to study more probability in the coming quarters.