An Examination of Results for the Matroid Secretary Problem

Spencer Riggins
sriggin2@masonlive.gmu.edu

Abstract

We will be examining the matroid secretary problem, a variation on the classical secretary problem in which the elements chosen must adhere to some combinatorial constraint defined by a matroid. We will elucidate a number of algorithms that work on the matroid secretary algorithm across all matroids. We will then examine how being aware the structure of the matroid allows for construction of better approximation algorithms by examining a number of matroid-specific algorithms. We shall also investigate a number of special cases and related problems to deepen our understanding of the matroid secretary problem.

Introduction

The classical secretary problem can be understood as a manager seeking to hire the best employee possible out of n candidates. The manager interviews each candidate in succession, with the rule that once a candidate is interviewed, the manager must make an online decision to either hire or pass over that candidate. Algorithms for the classical secretary problem seek to offer a competitive ratio for the skill level (weight) of the candidate hired in comparison to the optimal candidate. Dynkin introduced the original formulation of the problem which assumes that an adversary is able to assign weights to the elements, but then those elements arrive in uniform random order, and since then a great deal of research has gone into the problem and some of its variants. [20, 40] A famous result by Lindley showed that, under this model, by interviewing the first n/e candidates as a sample, and then hiring the next candidate that is better than the best sampled candidate, one can pick the optimum candidate with probability $1/e$. [36] It is important to note, however, that Lindley's result, as well as all results presented in this paper, can only provide a $1/e$ competitive result in expectation, and cannot make any true guarantees about the returned selection.

The matroid secretary problem is a relatively new variant that expands the rules of the secretary problem over matroid constraints. The problem was first proposed by Babaioff in 2007 as the following: "The elements of a matroid are presented to an online algorithm in random order. When an element arrives, the algorithm observes its value and must make an irrevocable decision regarding whether or not to accept it. The accepted elements must form an independent set, and the objective is to maximize the combined value of these elements." [2] There are some notable differences between the matroid secretary problem and the classical secretary problem. Notice that in the classical secretary problem we are searching for a way to maximize the probability of selecting the optimal candidate. In the matroid secretary problem we are searching for a competitive algorithm that best approximates the independent set with greatest total weight. Another important stipulation to take notice of is that in the matroid secretary problem we are allowed to hire multiple candidates, as long as those candidates form an independent subset in the matroid.

Background

In order to fully understand the matroid secretary problem, we will first offer a definition for a matroid. A matroid is an algebraic structure $M = (E, I)$ where E is a set of elements, and I is a collection of subsets of E. In order for M to be a matroid, I must satisfy three properties:

1. Non-Emptiness: $\emptyset \in I$
2. Hereditary: $A \subseteq B \; and \; B \in I \implies A \in I$
3. Augmentation: $If \; A, B \in I \; and \; |A| < |B| \; then \; \exists \, x \in A \setminus B \; s.t. \; B \cup \{x\} \in I$

We also take this opportunity to give a few more important definitions regarding matroids, and introduce some notation that will be used throughout the paper:

- Rank: The rank of a matroid is the cardinality of the largest independent set in the matroid. Throughout this paper r will denote matroid rank.
- Basis: A basis in a matroid is some independent subset, such that adding any other element to that subset causes it to become not independent.
- Dual: The dual matroid M^* of matroid M is the matroid in which every independent set is disjoint from some basis in M.
- Minor: A matroid minor M' of matroid M is a matroid obtained by implementing a series of restriction and contraction operations on the elements of M.

With this definition in mind, the matroid secretary algorithm is then given as such: An online mechanism is presented with a matroid $M = (E, I)$ with nonnegative weights assigned to every element in E. The number of elements in E, n, is also given. The elements of E then arrive one by one in an online fashion. When an element arrives, it can either be selected, or not selected, with the restriction that any selected element must maintain independence in the selected set, with the goal of maximizing the total weight of the selected elements. The total weight of the selected independent set is then compared to the independent set in the matroid M with total possible weight to obtain the approximation ratio.

Further specification of the matroid secretary problem can be achieved when considering the method by which weights are assigned to elements, and the way those elements are presented to the online mechanism. Luckily, Soto provided a convenient framework for explaining these methods. [3]

- Full Information Model: Weights of elements are chosen randomly from a known distribution and then presented randomly
- Partial Information Model: Weights of elements are chosen randomly from an unknown distribution and then presented randomly
- Random Assignment Model: An adversary chooses a list of nonnegative weights unknown to the agent which are then assigned to elements randomly
- Zero Information Model: An adversary assigns weights arbitrarily as elements arrive

We will use this framework as a tool for classification of the algorithms to be presented i.e. an algorithm belongs to a model if it provides an approximation for the matroid secretary problem in that model. It is important to take note of a few important results concerning these models. Most importantly one must take note of the cascading nature of the given models. For instance, an algorithm that provides a competitive ratio in the random assignment model will also yield the same competitive ratio for the partial and full information models, and an algorithm that yields a competitive ratio in the zero information model will yield that competitive ratio in all models. A few more formulaic specifications are worth noting for clarity. Gharan and Vondrak, as well as others, also offered classifications for a few other sub-models that cover prior information known about the matroid including whether or not all independent sets in the matroid are known, whether the weights of elements are assigned a numeric weight or only have some comparator function, etc. [2] Some of these variations will be covered in this survey. For the purposes of this paper, though, we will always assume that the weights belong to a numeric distribution and that all independent subsets in the matroid are known to the mechanism at the start of the algorithm, unless otherwise specified.

Motivation

The matroid secretary problem is of particular interest for study because it naturally represents a number of real world scenarios. In this section, we shall elucidate some of these representations, as well as examine some of the open questions that fall under the matroid secretary problem. The most easy representation to understand is that for uniform matroids, which decomposes simply to the multiple choice secretary problem. Transversal matroids model scenarios in which there is an auction with a number of buyers for which only a specific subsets of a supply of products satisfy their needs. Particularly, transversal matroids perfectly model the unit-demand preference domains previously studied by Mu'alem and Nisan, in which a seller is selling items to a given number of buyers, where the buyers are all willing to pay the same price, but each buyer is only satisfied by elements in a subset of a limited supply of different products. [23] In fact, transversal matroids are able to expand this specification to the case where each buyer has a different price they are willing to pay for each product. Gammoid matroids represent networking scenarios in which traffic from a client must be routed to a server on an edge disjoint path from other clients' routes. Graphic matroids break down to essentially finding a maximum-weight tree in a graph. Then partition and truncated partition matroids represent hiring scenarios in which only a certain number of employees can be hired per department in a company. Obviously, the easy decompositions into other problems makes the matroid secretary problem a very intriguing area of research, and due to these decompositions, a great deal of effort has been put into developing algorithms on specific types of matroid secretary problems. We will attempt to cover many of such algorithms, but it is important to know that a large amount of effort has been put into answering the question posed by Babaioff in his original paper, "Is there a constant competitive algorithm for all matroids in the random assignment model?" [1] As it stands, this question still remains unanswered, but we will cover the current results that tend towards finding an answer to this question, including the current algorithms that work on all matroids, and then later examine some ancillary problems to the matroid secretary problem that provide insight into Babaioff's question.

General Matroid Secretary Algorithms

As mentioned, in this section we will examine matroid secretary algorithms that are able to be used on all matroids. Currently, most research has gone into the zero information model, as well as the random assignment model since those generalize to the other models.

Zero Information Model

We will first examine algorithms that work for all matroids in the zero information model, since this is the area where there has been the most research. Babaioff points out that the expected value of the maximum valued subset in a matroid is $\Omega(\frac{log(n)}{loglog(n)})$, but the expected value for any randomized algorithm for the zero information model when the elements are presented in random order is 2. Thus, randomized algorithms are out of the question for the zero information model matroid secretary problem. However, Babaioff does propose an algorithm in his original paper on the matriod secretary problem which achieves an $O(log(r))$ approximation where r is the rank of the matroid. [1] Essentially, Babaioff's algorithm is a generalization of the secretary algorithm, which gives very little consideration to the matroid constraints on the chosen set. The algorithm works by sampling the first $s = \lceil n/2 \rceil$, without choosing any sampled element. Then, finding a threshold weight by choosing a number j between 0 and $\lceil log(r) \rceil$ and letting the threshold weight be l^{*j} where l^* is the heaviest sampled weight. The returned set is then chosen in a greedy manner by choosing any element whose weight is above the threshold, and maintains independence in the chosen set. Chakraboty and Lachich then gave an $O(\sqrt{log(r)})$ approximation. [14] Their algorithm also begins by sampling the first $\frac{n}{2}$ elements, placing each element into one of $O(log(r))$ weight classes. The algorithm then proceeds one of two ways, depending upon which elements were observed in the sample. The first way to proceed simply selects a weight class and greedily choose any subsequent element in that weight class if that element maintains independence in the chosen set. The second way utilizes the cardinalities of the bases found in the matroids obtained from only the elements in each weight class, and chooses elements based on a comparison of the rank of the heavier elements in the aforementioned bases to the size of the chosen set. This algorithm is quite complicated, but what it hopes to accomplish is preventing elements of heavy weight from being "blocked out" due to the selection of light elements through the utilization of weight classes. The most recent and powerful result is an $O(loglog(r))$ approximation given by Feldman and Zenklusen. [5] This most recent result also utilizes weight classes in much the same way as the second path from Chakraboty and Lachich's result, but with an added restriction that, for any element e to be chosen, there must be a set of lighter elements from the sampled set S' such that $S' \cup \{e\}$ is not an independent set in the matroid. This added constraint essentially allows the algorithm to It is worth noting that, Feldman and Zenklusen's result arose a few months after Lachich gave another $O(loglog(r))$ competitive algorithm in the zero information model, but Lachich's algorithm was quite convoluted, utilizing three different possible branches in logic, similar to the result by Chakraboty and Lachich. However, this algorithm will not be explored in detail due to the proximity in time to Feldman and Zenklusen's result, and the fact that Feldman and Zenklusen's implementation cut down on the hidden constant in the big O approximation quite significantly. [35] As an expansion on these results, Rubinstein also presented an algorithm that operates when weights of elements belong to any downwardly closed set system, not just the positive real numbers or integers, that achieves an $O(log(n))$ competitive ratio. [4]

Random Assignment Model

The random assignment model was first mentioned by Babaioff in his original 2007 paper, but it was left as an open area for further research. The first true study into this problem was given by Soto in 2013. At this time, Soto presented a $\frac{2e^2}{e-1}$ approximation algorithm for the random assignment model. [3] This algorithm works by decomposing the matroid into its *principal partition*, which is highly related to the theory of *principal sequences*, (uniformly-dense submatroids) covered in detail in the survey by Fujishige, and then running Soto's algorithm for uniformly-dense matroids on those subproblems which will be covered in a later section. [25] The *principal partition* converts the given matroid into a number of uniformly-dense matroids on which the $2e$ competitive algorithm mentioned later in the paper is run. Gharan and Vondrak improved upon Soto's result in 2011, giving an approximation of $\frac{40}{1-e}$. [2] This algorithm works in a different way from Soto's result, as it does not utilize the *principle partition* theory. Instead, Gharan and Vondrak offer a more traditional approach in which the first $\lceil n/2 \rceil$ elements are sampled, and then a threshold weight is chosen equal to the $(\lfloor \frac{r}{4} \rfloor + 1)^{st}$ heaviest sampled weight. The algorithm finishes by greedily choosing any candidate element that is heavier than the threshold weight, and maintains an independent chosen set in a similar fashion to Babaioff's original algorithm from the zero information model. It is important to notice, though, that these results show that constant competitive algorithms do exist for general matroids in the random assignment model.

As it has been shown, the question concerning whether a constant competitive algorithm exists in the zero information model has not been answered, but seemingly that goal is being approached. A constant competitive algorithm for general matroids has even been found in the random assignment model.

Matroid-Specific Algorithms

Since no constant competitive algorithm has yet been found for all matroids, and since many matroids model often-encountered problems, it is important to investigate what approximation ratios can be achieved when the matroid constraints correspond to a specific type of matroid. We will look at algorithms that work for specific matroids in the zero information model, as well as some that work for specific matroids in the random assignment model.

Zero Information Model

The first, and potentially most easily understood matroid-specific algorithm we will overview is the case of uniform matroids. Recall that a uniform matroid is a matroid for which the independent sets are all subsets of $S \subseteq E$ for which S is an independent set if and only if S contains fewer or equal elements to a set threshold r. That is, $|S| \leq r$. Rather trivially, it can be noticed that this problem corresponds exactly to the multiple-choice secretary problem, a problem that had already been studied in detail by Kleinberg, who was able to offer a $(1 + O(1/r^2))$ competitive ratio in expectation. [21] The construction of this algorithm takes a sample of size $\lfloor r/2 \rfloor$, then, in similar fashion to the classical secretary algorithm, chooses any element of weight greater than the heaviest sampled weight until r elements have been chosen or every element has been inspected. Notably, if $r = 1$ then this exactly describes the classical secretary problem and classical secretary algorithm. It is important to notice though, that by the specification of approximation ratio, as r tends towards infinity, the approximation ratio approaches 1. For cases where r is large, Hajiaghayi also offers an algorithm with a constant approximation ratio of 4. [37]

A noticable theme among matroid-specific algorithms will quickly arise. Many matroid-specific algorithms have achieved constant or near constant competitive ratios. Therefore, if the matroid being examined in an implementation of the matroid secretary problem is classified as a type of matroid with a matroid-specific algorithm is associated with it, then the matroid-specific algorithm can be used to achieve a better approximation than the general-matroid algorithms discussed above. Another type matroid that has received a large amount of investigation are laminar matroids. A laminar matroid is a matroid defined over a laminar family of sets F. That is, for any two subsets $B_1, B_2 \in F$, either $B_1 \subseteq B_2$, $B_2 \subseteq B_1$, OR $|B1 \cap B_2| = \emptyset$. In a laminar matroid, each $B_i \in F$ is associated with a capacity c_i. Then a set $S \subseteq E \subseteq U$ is independent if and only if $\forall B \in F$, $|B_i S| \leq c_i$. The first investigation into the matroid-secretary problem over laminar matroids was given by Im and Wang [9], in which a randomized process is used to implement a "kick next" operation where a chosen element of greater weight than some element in an optimal solution of a considered subset of the sample is added with a given probability. This procedure allowed Im and Wang to achieve a competitive ratio of $\frac{3}{16*10^3}$, but this algorithm only works in the random assignment model. Jaillet subsquently improved the approximation ratio to $3\sqrt{3e}$ by sampling 2/3 of the elements and then running the classical secretary problem to find the maximal element of either the even or odd indexed subsets of the sample and the remaining elements, and then returning the union of all elements chosen by those secretary problems. This algorithm improves the competitive ratio, while also working in the zero information model as well as the random assignment model. Ma also offered an algorithm that achieves a 9.6 approximation ratio, which does not beat the $3\sqrt{3e}$ algorithm given by Jaillet, but works on laminar matroids, as well as transversal matroids in the zero information model, a case which will be discussed now. [10]

A transversal matroid is a matroid that can be represented by a bipartite graph, where the elements E are the elements on the left side of the graph. The elements in each independent subset $A \subseteq I$ correspond to a perfect matching in the bipartite graph, where the elements in A are all the elements that are matched on the left side of the graph. The first algorithm for transversal matroids was given by Babaioff in his original paper, but this method assumed that the degree of the left side of the graph d is bounded, offering a $4d$ approximation. [1] The $4d$ approximation for transversal matroids was subsequently improved to a 16 competitive ratio by Dimitrov and Plaxton. [8] Dimitrov and Plaxton's result works by sampling the first $\lceil n/2 \rceil$ elements and examining the possible matchings in the sample. Then, when presented with a candidate e from the left side of the graph, if a possible matched element of e on the right side of the graph is not matched by some element in the sample, then e is greedily chosen and added to the returned set of elements. Notice that while this algorithm does take into account the possible matchings in the bipartite graph, it does not consider the weights of the selected edges. That 16 competitive ratio was subsequently improved to an 8 competitive ratio by Korula and Pál. [6] This algorithm works by selecting edges instead of left elements, which in turn returns a selection of left elements incident to those edges. This algorithm finds the greatest possible weight edge connected to a matched element from the sample, and then only chooses edges that are associated with a greater weight than the left element's largest incident weight from the sample matchings.

Constant competitive algorithms have also been found for certain classes of linear matroids. Linear matroids are defined as follows, let F be some field, $A \in F^{m \times n}$ is an $m \times n$ matrix over F. Then $S = \{1, 2, ..., n\}$ is the index of the columns of A. $I \subseteq S$ if and only if the columns indexed by I are linearly independent. We then define a linear matroid as k-sparse if for every column in A, that column contains at least k non-zero entries. Soto presents an algorithm for k-sparse linear matroids that works by decomposing the original matroid M into n independent problems, one per column in A, for which the original secretary problem from Lindley can be run, thus offering a ke approximation ratio for k-sparse linear matroid secretary problems. [3] [36]

Regular matroids are matroids that can be represented over all fields. Dinitz and Kortsarz gave a $9e$ competitive algorithm for matroid secretary problems over such matroids. [7] This algorithm works similarly to Soto's algorithms for linear matroids by decomposing the regular matroid into its basic matroids, the theory of which is described in detail by previous work from Seymour. [27] Dinitz and Kortsarz also note that this algorithm can also be extended to broader classes of matroids, including $max - flow\ min - cut$ matroids, which are any binary matroids for which no minor of the matroid is isomorphic to the dual of the Fano matroid. A deeper investigation of $max - flow\ min - cut$ matroids are also investigated by Seymour. [28]

Korula and Pál offer another $2e$ competitive algorithm for graphic matroids. [6] Graphic matroids are matroids which can be represented by an undirected weighted graph, where the elements of a matroid are represented by the edges in the graph, and the independent sets are all subsets of edges in the graph that do not form a cycle. The algorithm works as follows: Assign an arbitrary ordering to the nodes in the graph. Then create two directed graphs, one in which edges go from the higher to lower nodes, and one in which the edges go from lower to higher nodes. Each node is assigned to only one of the digraphs with probability $1/2$. As edges are presented to the algorithm in an online fashion a separate secretary problem is run for each of the nodes, where the elements are the edges leaving the node in its assigned digraph. The edges selected by the algorithm are the union of the edges selected by the secretary problems from each node. Analysis of the $2e$ approximation ratio is quite simple for this algorithm because the 2 arises from splitting the graph into two digraphs, and the e arises from running the classical secretary algorithm on each node.

Soto gives another algorithm for the related class of co-graphic matroids, matroids which are the dual of a graphic matroid. A co-graphic matroid has independent sets which are all edges such that there exists some forest in the graph such that adding any of the edges from the independent set to the forest forms a cycle. Soto's algorithm relies heavily on Edmond's matroid partitioning theory, and works in the following way: [33] Partition the tree T that covers the 3-edge-connected graph G_3 into three subtrees T_1, T_2, and T_3. Then since the three trees are bases in the co-graphic matroid constructed from G_3, greedily selecting all elements from one of the trees T_i will yield an approximation ratio of 3. Soto's algorithm expands upon this result to yield a $3e$ approximation ratio for a graph G of any connectedness. [3]

The final result by Soto that will be discussed gives an e approximation for uniformly-dense matroids. A uniformly-dense matroid is a matroid M in which $\forall A \subseteq E$, $max_A \frac{|A|}{r(A)}$, where $r(A)$ is the rank of the matroid obtained by only considering the elements of A under the constraints given by M. This specification can be most easily visualized for the case of graphic matroids. That is, there are no areas of the graph where a large number of edges are incident to a small number of nodes. The algorithm works by sampling the first $\lceil n/2 \rceil$ elements and maintaining an set of the r heaviest elements T. Then as elements are presented, if the candidate element is heavier than the lightest element in T, it replaces the light element in T and that element is chosen. [3] Soto also gives mention that the same co-graphic matroid algorithm described above can be used to achieve a competitive ratio that decreases with the overall density of the matroid.

Random Assignment Model

Kordecki also offers another randomized algorithm for graphic matroids that assumes the weights of the graph were distributed by a random process, and leans heavily on the work of Janson on randomized graphs. [24, 29] This algorithm is important to note because as the numbers of elements becomes small, the approximation ratio of this algorithm tends towards 1, potentially beating the constant approximation rate given by Soto's algorithm for the general random assignment model.

Kesselheim then offers an algorithm for transversal matroids in the random assignment model that beats Korula and Pál's $2e$ competitive algorithm. Kesselheim's result operates by taking a sample set, and then greedily choosing any element that belongs to the optimal solution of the matroid secretary problem that results only from considering the previously observed elements and the candidate element. This algorithm cuts Korula and Pál's $2e$ competitive ratio down to e. [18]

Special Case Algorithms

The problem classifications given by Soto cover a large amount of possible matroid secretary problem specifications in a very concise way, but they are not by any means exhaustive. At the core, though, the models define the rules of the matroid secretary problem. In this section we will examine some other variants of the matroid secretary problem that are defined by less easily categorized rules.

Non-Uniform Arrival Model

Under all previously mentioned models, the order that elements are presented are assumed to either be random, or determined by an adversary to the secretary mechanism. Kesselheim and Tönnis examine a case in which element arrival order is instead determined by some submodular function. [17] Under this model, Kesselheim and Tönnis were able to construct a $(1+O(k^{\frac{1}{3}})+\epsilon)$ competitive approximation for the uniform matroid, multiple-choice secretary problem. In their analysis, k is a chosen number less than or equal to p, the number of distinct matroid elements that can be assumed to be δ-close to the real distribution function, and ϵ is another small value associated with a theorem concerned with determining the degree to which the matroid elements adhere to the given distribution function. Regardless, it can be noted that the value of ϵ can tend towards zero under the correct assumptions. Notably, for some scenarios this algorithm beats the $(1 + O(1/r^2))$ competitive ratio offered by Kleinberg's algorithm in the zero information model. Kesselheim and Tönnis also prove that a variation of Korula and Pál's transversal matroid algorithm exists for the non-uniform arrival model that offers an approximation ratio of $\Omega(\frac{\delta^2}{(k+1)!}\frac{n}{ln\,n})$, beating the zero information model algorithm only in cases where the number of elements in the matroid are very small, and it can be assumed that the element arrival order holds very strictly to the distribution function. [16]

Free Order Model

Another special case concerning element arrival order that has received particular interest is the free order model, which relaxes the specification of the problem to allow the mechanism to choose the order in which elements are presented. Note that this problem is still of importance because the mechanism does not know the weights of the elements. Jaillet presents an algorithm for the free order model matroid secretary problem. [11] Despite the relaxed constraint on the order of elements presented, this algorithm still starts, as many secretary algorithms do, by taking a random sampling of the first $\lfloor n/2 \rfloor$ elements. The sampled elements are then arranged in order of weight $A_i = \{a_1, a_2, ..., a_i\}$, lowest to highest. The mechanism then uses its ability to choose the order of presented elements to present all elements in $span(A_i) \setminus span(A_{i-1})$, and chooses the candidate element if that element maintains independence in the chosen set, and if the weight of the presented element is greater than the weight of a_i. After this process is completed, if any elements have not yet been presented, then they are simply chosen if they maintain independence in the chosen set. This algorithm achieves a 9 approximation in the free order model. As a note, other inquiries have been made into the relaxation of element arrival orders, including Vardi, who recently studied a version of the uniform matroid secretary problem where each element arrives twice, in which a 0.727 approximation can be achieved. [39]

Related Problems

We will now examine some other classes of matroid problems in order to gain insight into the given results in the matroid secretary problem.

Truncation Matroid Secretary Problems

Truncation matroids can be constructed from any matroid M by a simple operation. For any matroid with rank r, define a threshold k, and remove any independent subsets in I that are of greater rank than the threshold k. It is important to notice that uniform matroids, discussed earlier, can be obtained by performing this truncation procedure on any matroid where the independent sets equal to all subsets of E. The results of Karger's work on random sampling for matroid optimization problems is expanded upon by Babaioff to prove that for any matroid secretary algorithm, if there exists an algorithm that offers a d competitive ratio, then by a simple reduction, a $max(13d, 400)$ competitive ratio algorithm can also be constructed. [12, 33] Thus, any matroid-specific algorithm that works for a particular class of matroid mentioned above, can also yield a constant competitive algorithm for any truncated version of that matroid.

Subadditive Objective Function

Typically, the objective function in the matroid secretary problem is submodular. In particular, the objective function is almost always addition of positive integer or real weights, but work has also been done on matroid secretary problems with subadditive objective functions. That is, adding an element to the chosen function does not have a natural additive effect. For clarity, a subadditive function is over a domain A where $\forall x, y \in A\ f(x+y) \leq f(x) + f(y)$. For example, square root is a subadditive function. This obviously adds many new possibilities to explore in relation to the matroid secretary problem, because the subadditive property must be taken into account when choosing elements. Rubinstein, as well as others, have explored the uniform matroid, multiple-choice secretary problem, and achieved an $O(log(n)log^2(r))$ approximation, but further work has yet to be done on this variant of the problem. [22]

Submodular Objective Functions

As mentioned, most research into secretary problems examines the case where total set weight is determined by normal addition, but Kesselheim and Tönnis also give results for when the objective function is not just integer or real number addition, but any submodular function. The original work for this algorithm on general matroids by Bateni gave an $O(log^2(r))$ competitive ratio. [31] This result was then vastly improved by Calinescu with an $(e+1)$ competitive algorithm for general matroids with submodular objective functions, which improves upon and extends a $(1+2^r)$ approximation given by Babaioff [26, 38] Kesselheim and Tönnis then presented results stating that if an algorithm yields a d approximation under any submodular function, (e.g. the matroid-specific zero information model algorithms for uniform and transversal matroids), then a $\frac{d}{e}(1-\frac{\sqrt{r-1}}{(r+1)\sqrt{2\pi}})$ approximation algorithm can be constructed for all submodular uniform matroid secretary problems, and a $d/4$ approximation algorithm can be constructed for all submodular transversal matroid secretary problems. Work on other matroid classes, as well as general matroid secretary algorithms with submodular objective functions has not been explored specifically. [16] However, Feldman and Zenklusen did prove a more general result stating that for any matroid, a d competitive algorithm in the zero information model can be used to construct a $24d(3d+1)$ algorithm for the submodular function over the same matroid.[15]

Ordinal Matroid Secretary Problem

In most research on the matroid secretary problem, it is assumed that the weights of elements are fully revealed, as elements are presented. In the ordinal matroid secretary problem, however, the mechanism does not see the true weights of elements, but is only supplied with a function that returns which of the weights is larger. Seemingly, this would put great strain on matroid secretary algorithms, but many of the given algorithms do not actually consider the true value of elements, and instead only compare values anyway. Soto cemented this notion in 2017, by designing a number of constant competitive algorithms for specific matroid classes in the ordinal matroid secretary problem. [30]

Matroid Prophet Inequality Problem

The matroid prophet inequality problem is one closely related to the matroid secretary problem, which generalizes the prophet inequality problem over matroid constraints. That is, each element is assigned a hidden weight from a known distribution, and as elements are presented in an online fashion the mechanism must choose whether to select or pass over that element, with the weight of the element being revealed only after the selection has been made. The goal, just as in the matroid secretary problem, is to select a subset of maximal weight. It turns out, though, that low approximation rate algorithms for the matroid prophet inequality problem are much easier to construct than those for the matroid secretary problem. Kleinberg and Weinberg in their 2012 paper, by utilizing a few powerful theorems about matroid bases present an algorithm that achieves a 2 approximation ratio for the matroid prophet inequality problem. This matches Krengel and Sucheston's 2 approximation ratio for the classical prophet inequality problem, analogous to the classical secretary problem, which has been proven optimal. [19, 32]

Knapsack Secretary Problem

Another secretary variant that has gained the interest of some of the frontrunners in matroid secretary problem research is the knapsack secretary problem. The knapsack secretary problem is a multiple choice secretary problem under a knapsack constraint rather than a matroid constraint. Specifically, each element is assigned a value and weight, and the mechanism must select a maximal value set such that the total weight of the selected set is below a given knapsack size. Babaioff presented a constant $10e$ approximation for the algorithm when weights, are variable, and an e approximation ratio when all elements have the same weight. Importantly, these algorithms operate under the random assignment model, since knapsack problems can generally fail when presented with arbitrarily bad inputs. The constant-weight algorithm is extremely similar to Soto's uniformly-dense matroid algorithm. This algorithm takes a sample of elements, maintains a set T of elements of size k, where k is the total number of elements that can be selected (calculated from the weight of elements and the knapsack size), and when an element arrives if it has value greater than any element in T, it is chosen and replaces the lowest valued element in T. The variable-weight algorithm utilizes a branching logic depending on whether an unusually large element is observed in the sample. If a large element is observed, a threshold weight and value are calculated in a randomized way by normalizing the total knapsack size to 1, and elements are chosen if their value is above the calculated value threshold, and price below the calculated price threshold. If no large element is observed, then the weight normalization procedure is done and the constant weight algorithm is run with the added restriction that if an element is chosen with weight greater than $1/k$, then it is automatically rejected. [13]

Conclusion

Throughout this paper, we have examined many results for a number of variants of the matroid secretary problem over a series of different matroids. We have explored algorithms that operate generally on any matroid, as well as algorithms that operate on specific types of matroids. We have also provided motivation for studying the matroid secretary problem, highlighting how matroid secretary problems can often break down into other algorithmic problems, and showed that many matroid secretary problems easily model real-life situations. We have shown that constant time competitive algorithms exist for a large number of specific matroids, including uniform matroids, transversal matroids, and graphic matriods. We have also examined the current state of the art algorithms for solving the matroid secretary problem on general matroids. We examined some specific rule constrictions, and relaxations, and showed how they affect the achievable approximation ratios of matroid secretary problems. We finally examined some closely related problems to the matroid secretary problem, in hopes to gain a wider perspective on some of the presented results.

Future Work

A number of matroid specific algorithms were presented in this paper, many of which do achieve a constant approximation ratio in the matroid secretary problem. However, there still exist many classes of matroids for which constant competitive algorithms do not exist. Most notably, no constant competitive algorithm exists for gammoid matroids, which easily model traffic in a network. Finding algorithms with approximation ratios for other classes of matroids that beat the approximation ratios given by the general matroid secretary algorithms is an obvious area for improvement in the field. The main area of concern, though, still remains with Babaioff's open question posed along with the original presentation of the matroid secretary problem: "Does a constant competitive ratio algorithm exist for general matriods in the zero information model?" Some clear barriers do stand in the way of finding the answer to this question. Chiefly among those barriers is a theorem proven by Babaioff stating that certain types of algorithms are incapable of achieving a constant competitive ratio for the matroid secretary problem, including any algorithm that begins by taking a sample of elements. [33] This is discouraging, as nearly all of the algorithms explored so far employ this strategy. However, some results do point towards a positive answer to the question. For instance, a constant competitive algorithm was found for the random assignment model. Also, constant competitive time algorithms have been found for other optimization problems including the knapsack secretary problem, and the extremely closely related matroid prophet inequality problem. This result is the proverbial holy grail of the matroid secretary problem, and it will be compelling to see what progress is made towards it in the coming years.

References

[1] Moshe Babaioff, Nicole Immorlica, and Robert Kleinberg. 2007a. Matroids, secretary problems, and online mechanisms. In Proc. 18th ACM Symp. on Discrete Algorithms. 434-443.

[2] Shayan Oveis Gharan and Jan Vondrák. 2011. On variants of the matroid secretary problem. In Proc. 19th European Symp.

[3] José A. Soto. 2013. Matroid secretary problem in the random-assignment model. SIAM J. Comput. 42, 1 (2013), 178-211.

[4] Aviad Rubinstein. 2016. Beyond matroids: Secretary problem and prophet inequality with general constraints. In Proc. 48th ACM Symp. on Theory of Computing. 324-332.

[5] Moran Feldman, Ola Svensson, and Rico Zenklusen. 2015. A simple O(loglog(rank))-competitive algorithm for the matroid secretary problem. In Proc. 26th ACM Symp. on Discrete Algorithms. 1189-1201.

[6] Nitish Korula and Martin Pál. 2009. Algorithms for secretary problems on graphs and hypergraphs. In Proc. 36th Intl. Colloq. on Automata, Languages and Programming. 508-520.

[7] Michael Dinitz and Guy Kortsarz. 2014. Matroid secretary for regular and decomposable matroids. SIAM J. Comput. 43, 5 (2014), 1807-1830.

[8] Nedialko B. Dimitrov and C. Greg Plaxton. 2012. Competitive weighted matching in transversal matroids. Algorithmica 62, 1-2 (2012), 333-348.

[9] Sungjin Im and Yajun Wang. 2011. Secretary problems: Laminar matroid and interval scheduling. In Proc. 22nd ACM Symp. on Discrete Algorithms. 1265-1274.

[10] Tengyu Ma, Bo Tang, and Yajun Wang. 2016. The simulated greedy algorithm for several submodular matroid secretary problems. Theory Comput. Syst. 58, 4 (2016), 681-706.

[11] Patrick Jaillet, José A. Soto, and Rico Zenklusen. 2013. Advances on matroid secretary problems: Free order model and laminar case. In Integer Programming and Combinatorial Optimization - Proc.16th International Conference, IPCO 2013, Valparaso, Chile, March 18-20, 2013. 254-265.

[12] David Karger. 1998. Random sampling and greedy sparsification for matroid optimization problems. Mathematical Programming 82 (1998), 41-81.

[13] Moshe Babaioff, Nicole Immorlica, David Kempe, and Robert Kleinberg. 2007b. A knapsack secretary problem with applications. In APPROX-RANDOM. 16-28.

[14] Sourav Chakraborty and Oded Lachish. 2012. Improved competitive ratio for the matroid secretary problem. In Proc. 23rd ACM Symp. on Discrete Algorithms. 17021712.

[15] Moran Feldman and Rico Zenklusen. 2015. The submodular secretary problem goes linear. In Proc. 56th IEEE Symp. on Foundations of Computer Science. 486-505.

[16] Thomas Kesselheim and Andreas Tönnis. 2016. Submodular Secretary Problems: Cardinality, Matching, and Linear Constraints. Retrieved from http://arxiv.org/abs/1607.08805 CoRR abs/1607.08805. on Discrete Algorithms. 1265-1274.

[17] Thomas Kesselheim, Robert D. Kleinberg, and Rad Niazadeh. 2015. Secretary problems with non-uniform arrival order. In Proc. 47th ACM Symp. on Theory of Computing. 879-888.

[18] Thomas Kesselheim, Klaus Radke, Andreas Tnnis, and Berthold Vcking. 2013. An optimal online algorithm for weighted bipartite matching and extensions to combinatorial auctions. In Proc. 11th European Symp. on Algorithms. 589-600.

[19] Robert D. Kleinberg and Seth Matthew Weinberg. 2012. Matroid prophet inequalities. In Proc. 44th ACM Symp. on Theory of Computing. 123-136.

[20] Eugene B. Dynkin. 1963. The optimum choice of the instant for stopping a Markov process. Sov. Math. Dokl. 4 (1963), 627-629.

[21] Robert D. Kleinberg. 2005. A multiple-choice secretary problem with applications to online auctions. In Proc. 16th ACM Symp. on Discrete Algorithms. 630-631.

[22] Aviad Rubinstein and Sahil Singla. 2017. Combinatorial prophet inequalities. In Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA17). 1671-1687.

[23] Ahuva Mu'alem and Noam Nisan. 2002. Truthful approximation mechanisms for restricted combinatorial auctions. In Proc. 17th AAAI Conf. on Artificial Intelligence. 379-384.

[24] S. Janson, T. uczak, and A. Ruci nski. Random Graphs. Wiley, New York, 2000.

[25] Satoru Fujishige. Theory of principal partitions revisited. In Research Trends in Combinatorial Optimization, pages 127-162, 2009.

[26] Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrak. Maximizing a submodular set function subject to a matroid constraint (extended abstract). In IPCO, pages 182-196, 2007.

[27] P. D. Seymour. Matroid minors. In R. L. Graham, M. Grotschel, and L. Lovasz, editors, Handbook of combinatorics (vol. 1), pages 527550. MIT Press, Cambridge, MA, USA, 1995.

[28] P. D. Seymour, The matroids with the max-flow min-cut property, J. Combin. Theory Ser. B 23 (1977), 189-222.

[29] Kordecki, Wojciech, Secretary Problem: Graphs, Matroids and Greedoids, 2018. https://arxiv.org/abs/1801.00814.

[30] J. A. Soto, A. Turkieltaub, and V. Verdugo. Strong algorithms for the ordinal matroid secretary problem. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018, pages 715-734, 2018.

[31] Mohammad Hossein Bateni, Mohammad Taghi Hajiaghayi, and Morteza Zadimoghaddam. Submodular secretary problem and extensions. ACM Trans. Algorithms, 9(4):32, 2013.

[32] Ulrich Krengel, and Louis Sucheston. Prophet Compared to Gambler: An Inequality for Transforms of Processes. Ann. Probab. Volume 15, Number 4 (1987), 1593-1599.

[33] Moshe Babaioff, Nicole Immorlica, David Kempe, and Robert Kleinberg. 2018. Matroid Secretary Problems. J. ACM 65, 6, Article 35 (November 2018), 26 pages.

[34] J. Edmonds. Minimum partition of a matroid into independent subsets. Journal of research of the National Bureau of Standards. Section B, 69:6772, 1965.

[35] O. Lachish. O(log log rank) competitive-ratio for the matroid secretary problem, 2014. http://arxiv.org/abs/1403.7343.

[36] Lindley, D.V., "The Use of Prior Probability Distributions in Statistical Inference and Decision," Proceedings of the Fourth Berkeley Symposium in Mathematical Statistics and Probability 1 (1969), 453-468

[37] Hajiaghayi, M. T., Kleinberg, R., and Parkes, D. C. Adaptive Limited-Supply Online Auctions. In Proc. ACM Conf. on Electronic Commerce (2004), pp. 7180.

[38] Moshe Babaioff, Ron Lavi, and Elan Pavlov. Mechanism design for single-value domains. In Proceedings of the 20th National Conference on Artificial Intelligence (AAAI 2005), pages 241247, 2005

[39] S. Vardi. The returning secretary. In 32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany, pages 716-729, 2015.

[40] Peter R. Freeman. 1983. The secretary problem and its extensions: A review. Internat. Statist. Rev. 51, 2 (1983), 189-206.