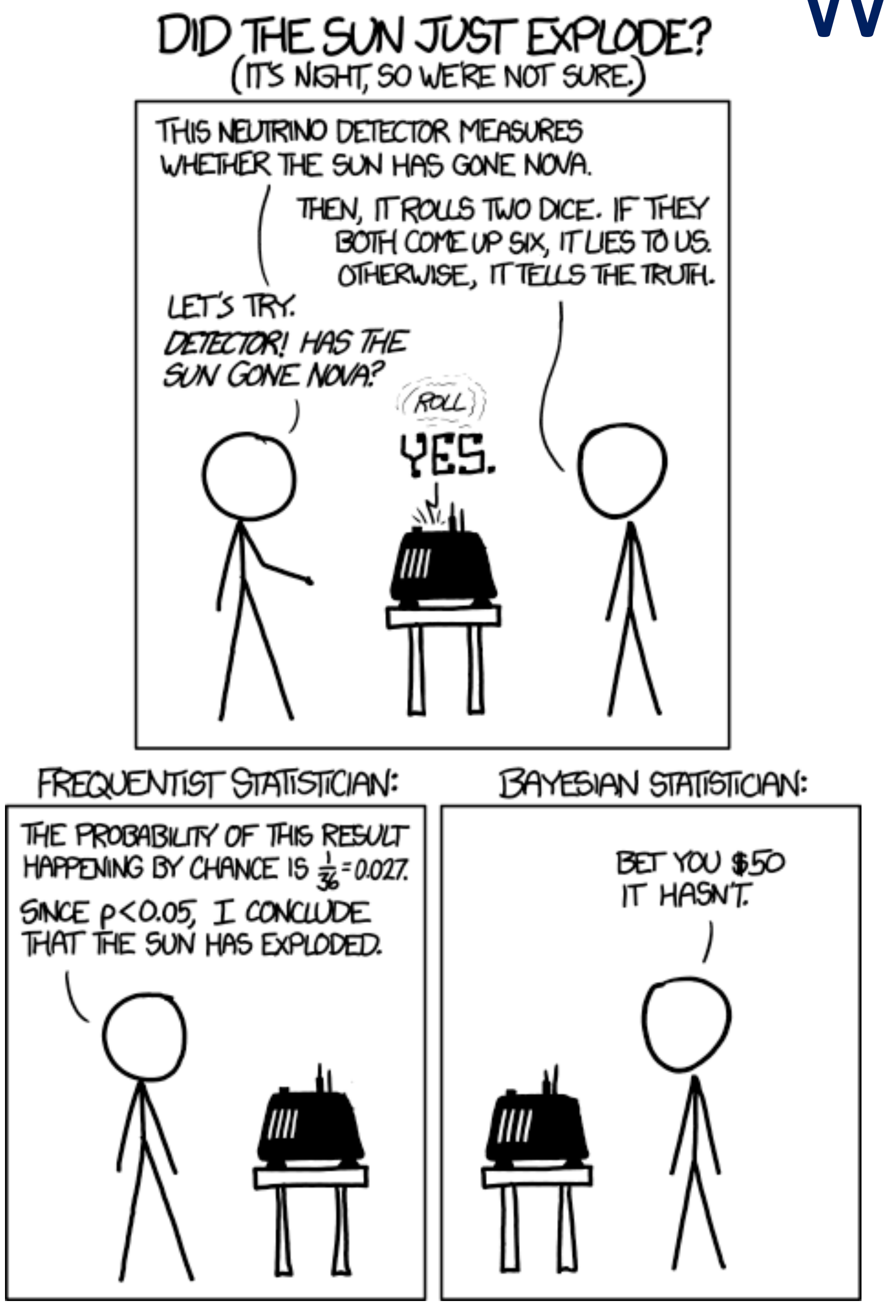
Classical statistical inference

Part 2

Associated notebook:

04-Basic_statistical_inference_frequentists_2/Frequentist_inference_01.ipynb

What is inference?



Derive **INFORMATION** based on **DATA**

Examples:

- Exoplanet transit ⇒ M, d ⇒ P(M | d)
- Supernovae distances ⇒ Expansion rate H_0

Inference generally implies an underlying *statistical model*: PDF or regression laws with *parameters* θ

Three types of inference:

- Point estimation: “best” θ
- Confidence interval: Confidence around θ
- Hypothesis testing: *data OK w. model ?*

https://xkcd.com/1132/

Point estimate $\hat{\theta}$

Example:

θ = mean mag. of a population of galaxies

A: Your data set = subsample of measurements:
A = $\{X_1, X_2, X_3, X_4\}$ where X = mag. (this is a RV)

$$\hat{\theta} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$ ≡ Point estimate of θ

If you do the experiment with another sample (⇒ different *realisation*) you will get another $\hat{\theta}$

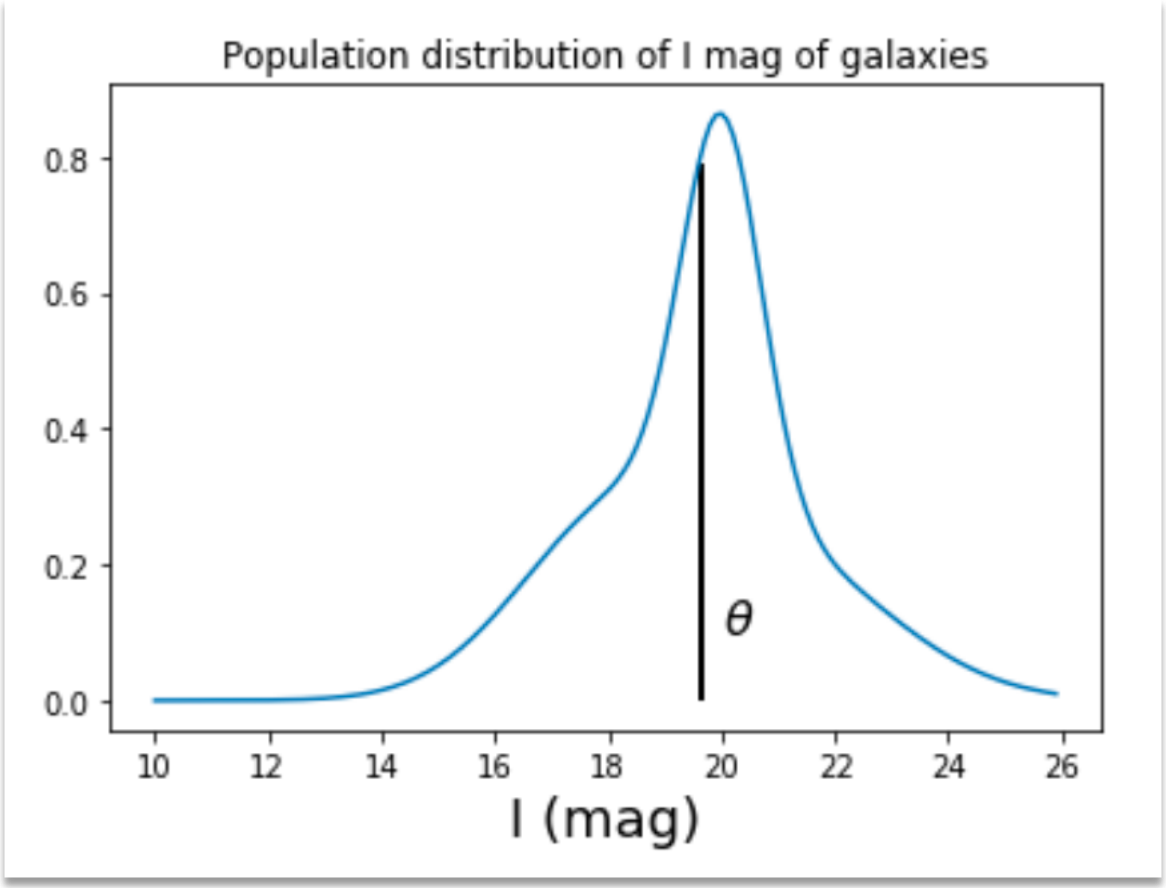
Point estimate $\hat{\theta}$

Generalisation:

$$\hat{\theta} = g(X_1, X_2, X_3, \ldots X_n)$$

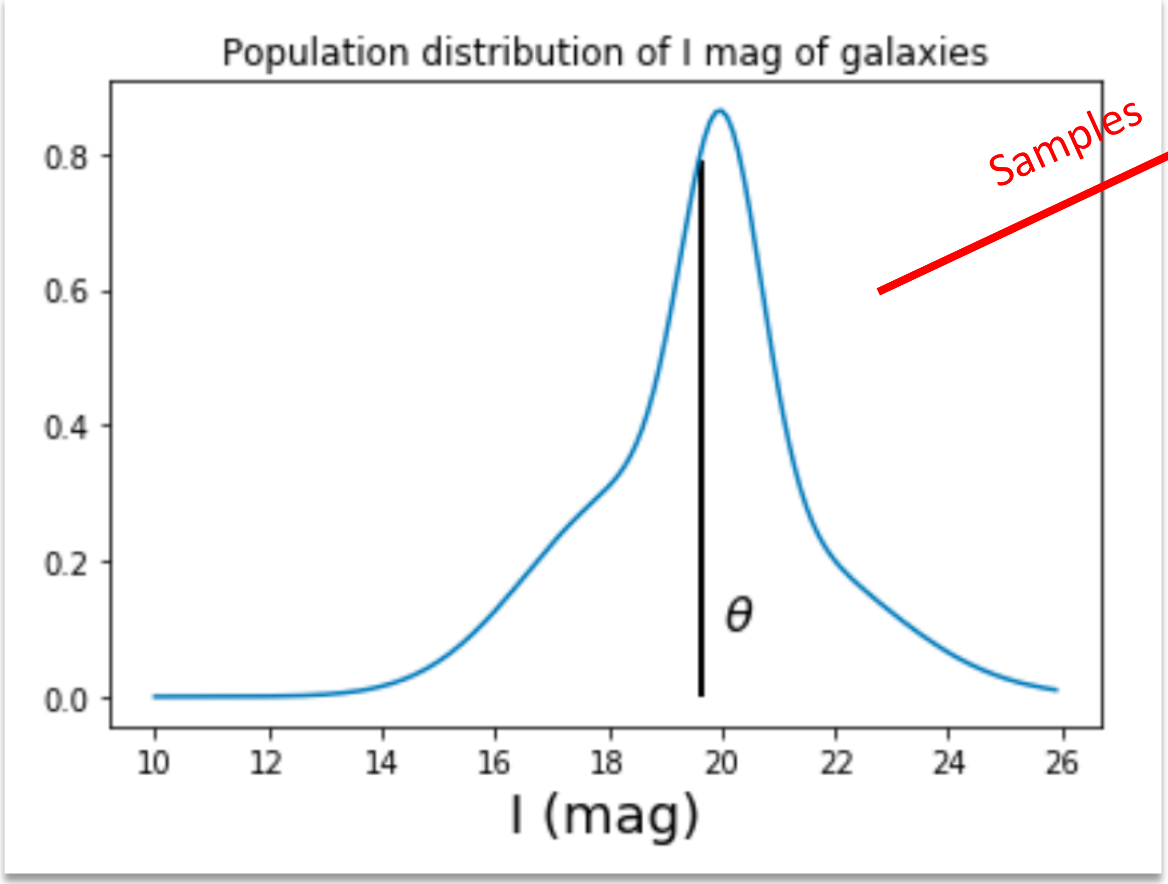
- Point estimate of a param. is a *function* of RV X_1, …
- It is as well a **Random Variable (RV)**
- It can be biased, is characterized by a variance but should ideally be consistent (converges towards θ)
- Distribution of $\hat{\theta}$ is called *sampling distribution*

Point estimate $\hat{\theta}$



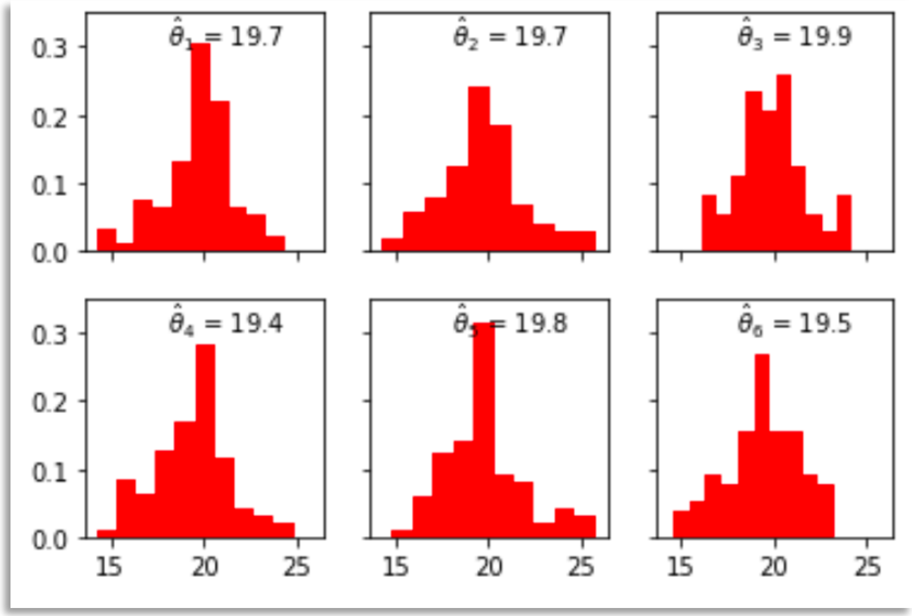
Population mean: $\theta = 19.66$

Point estimate $\hat{\theta}$



Samples

Six different *samples* drawn from the true population



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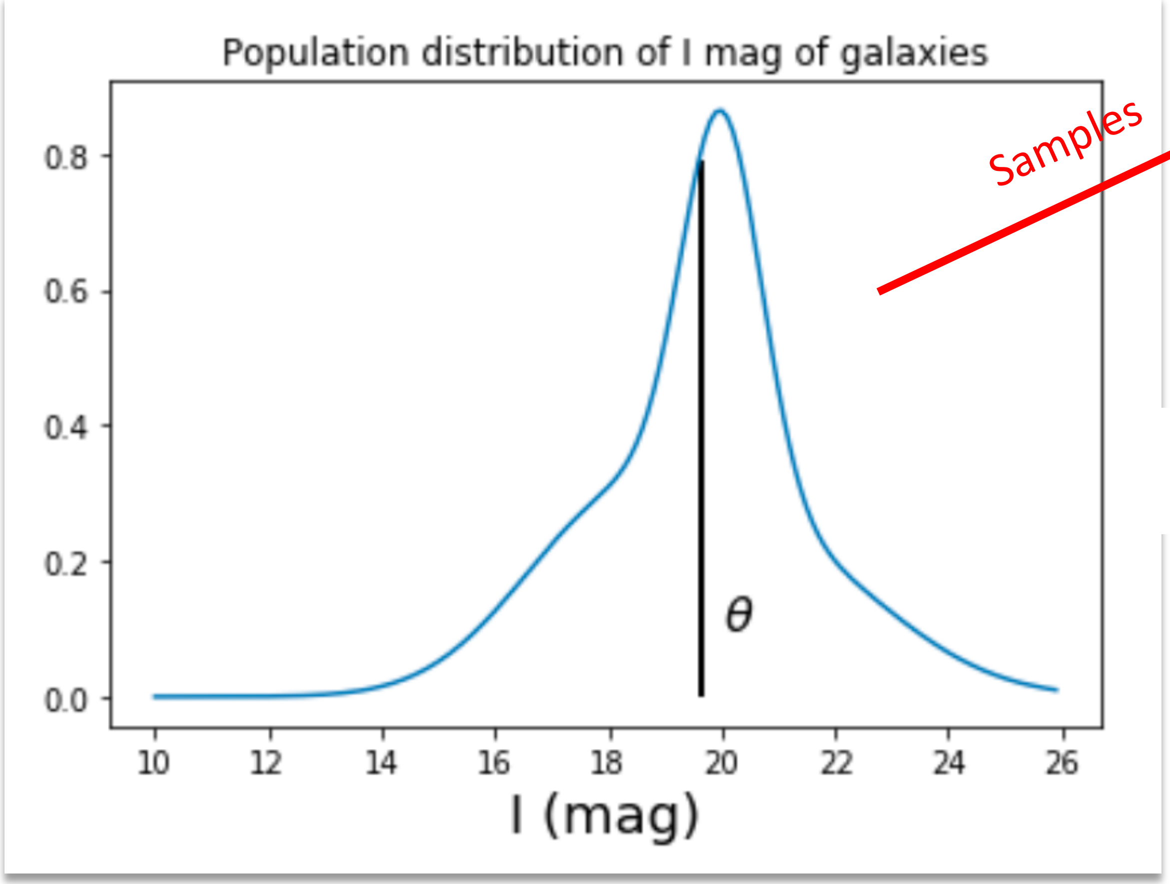

Samples

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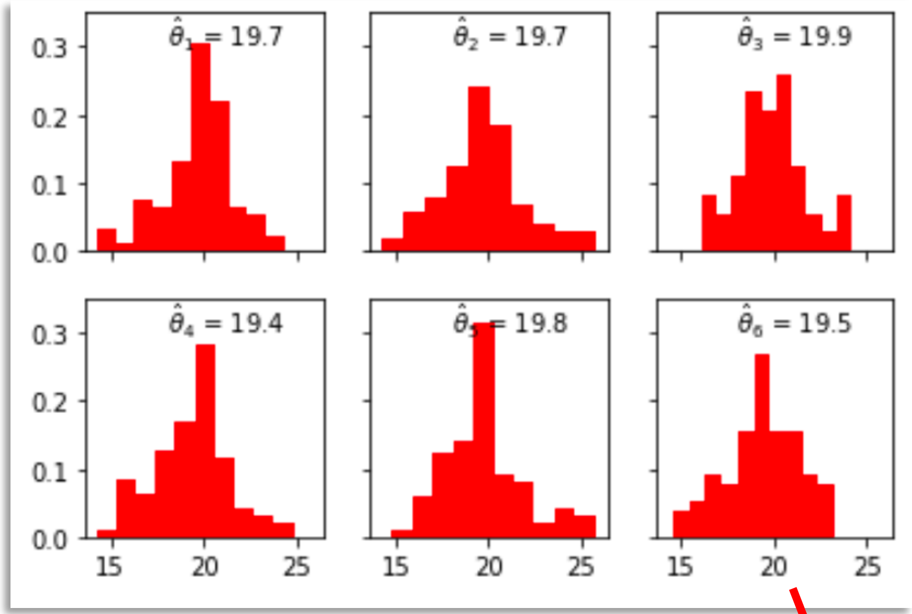
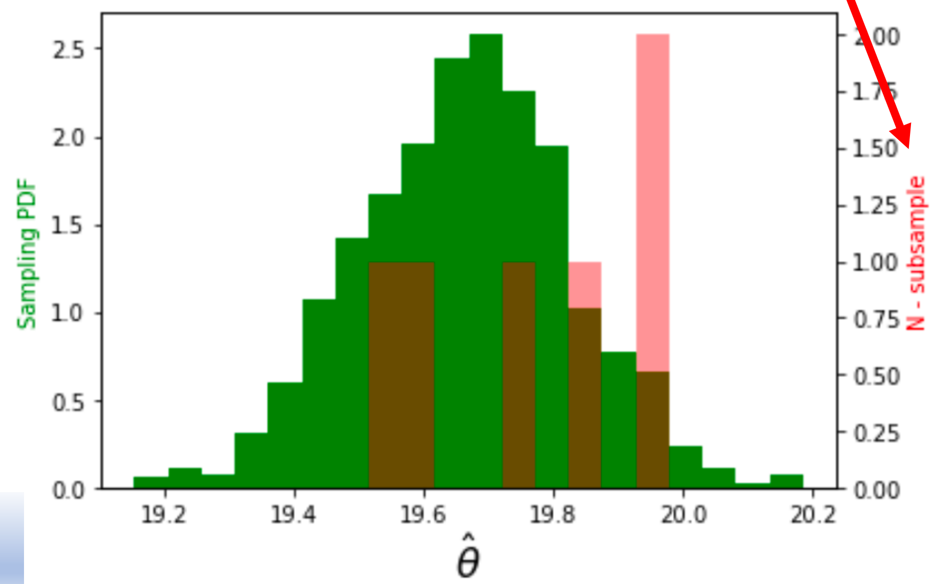


Distribution of $\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k\}$ = sample distribution

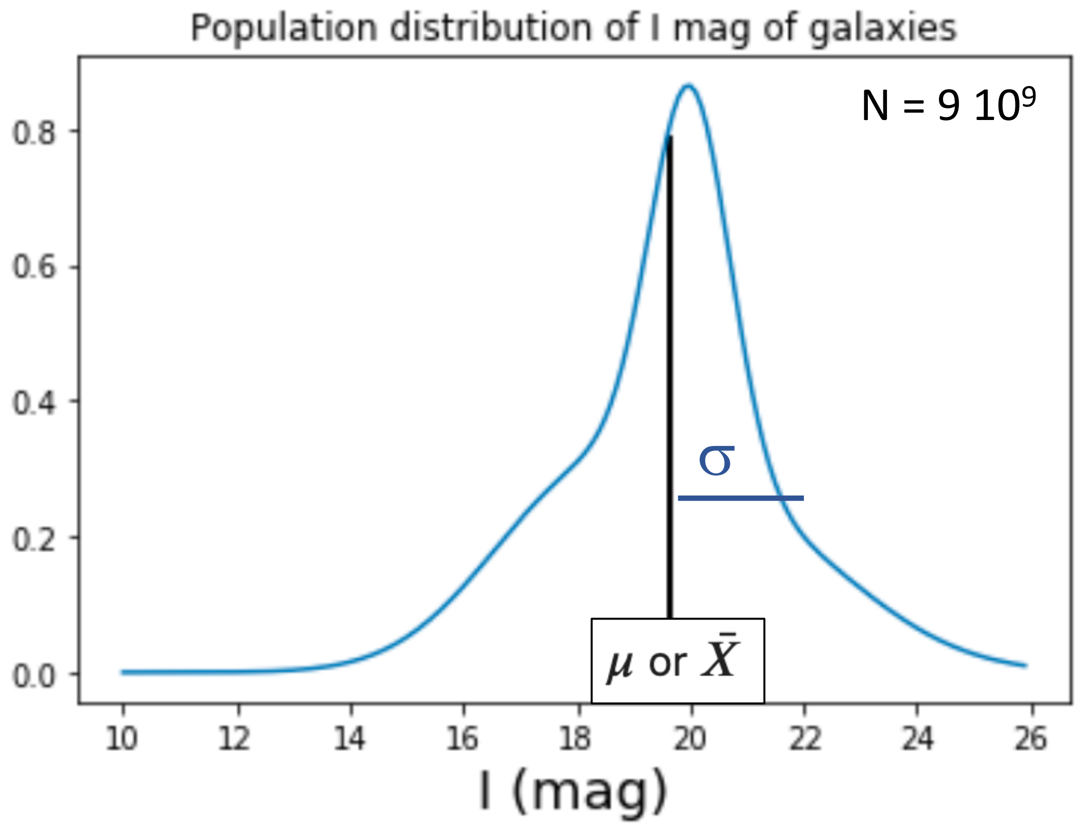


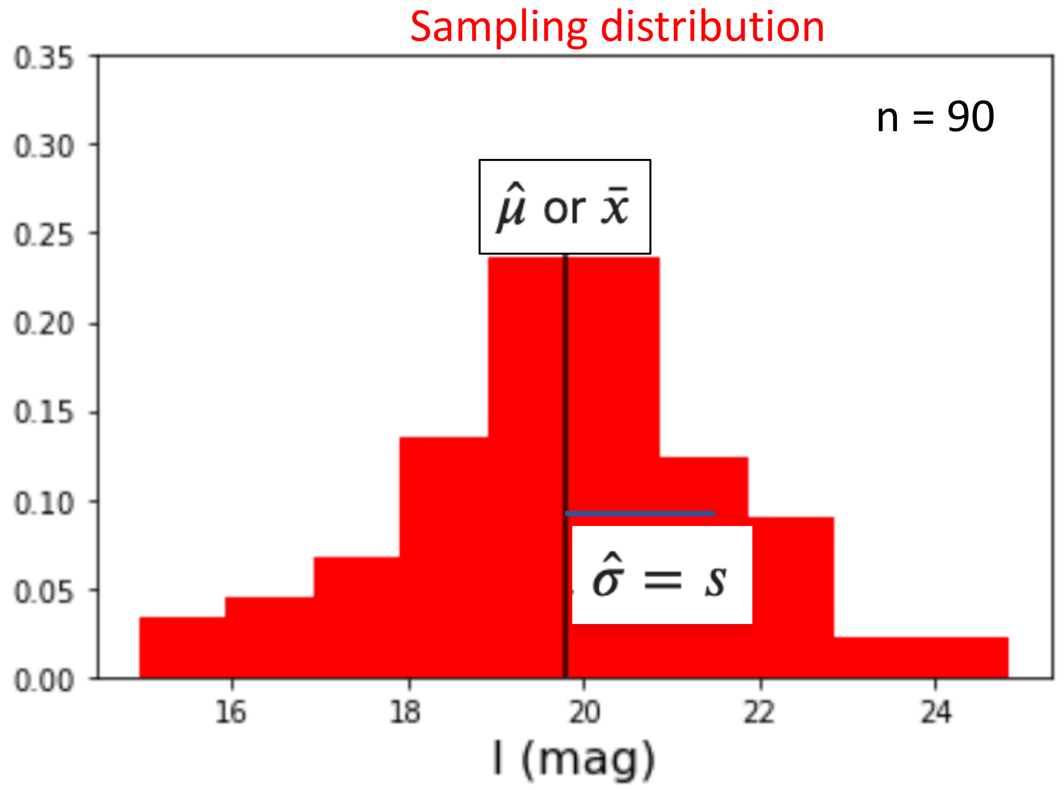
Population mean: $\theta = 19.66$

Go to: Sect. II.1 of the notebook

Summary statistics

Name	Population Statistics	Sample Statistics
size	N	n
mean	$\mu = \bar{X} = \frac{\sum_i X_i}{N}$	$\hat{\mu} = \bar{x} = \frac{\sum_i x_i}{n}$
Variance	$\sigma^2 = \frac{\sum_i (X_i - \bar{X})^2}{N}$	$s^2 = \frac{\sum_i (x_i - \bar{x})^2}{n-1}$
Standard deviation	$\sigma = \sqrt{\sigma^2}$	$\hat{\sigma} = s = \sqrt{s^2}$





Summary (sample) statistics: *standard error*

Standard error (stde) ≠ Standard deviation (std)

Name	Formula
Standard error on the mean	$stde(\bar{x}) = \frac{s}{\sqrt{n}}$
Standard error on the stdev	$stde(s) = s/\sqrt{2(n-1)}$
Standard error on proportions	$stde(\hat{p}) = \frac{\sqrt{p\,(1-p)}}{\sqrt{n}}$

Central limit theorem

When independent **random variables** are added, their *sum* tends towards a normal distribution (if n >>)

- This is true even if the original RV are not normally distributed
- Sampling dist. of mean tends (for large *n*) towards a Normal distribution
- Sampling dist. of variance (for large *n)* does NOT tend towards a Normal distribution

Go to: Sect. II.1.1. of the notebook

Distribution of estimators

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$

- Sample distribution of $\hat{\mu} \sim N(\mu, \sigma/\sqrt{n}) \quad \Longleftrightarrow \quad Z = \dfrac{\bar{X} - \hat{\mu}}{(\sigma/\sqrt{n})} \sim N(0, 1)$
- Sample distribution of $t = \dfrac{\bar{X} - \hat{\mu}}{(s/\sqrt{n})} \sim t(n-1)$ s is derived from the sample dist.

Student distribution

- Sample distribution of $S = \dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1).$

Chi square distribution

Classical statistical inference: *confidence interval*

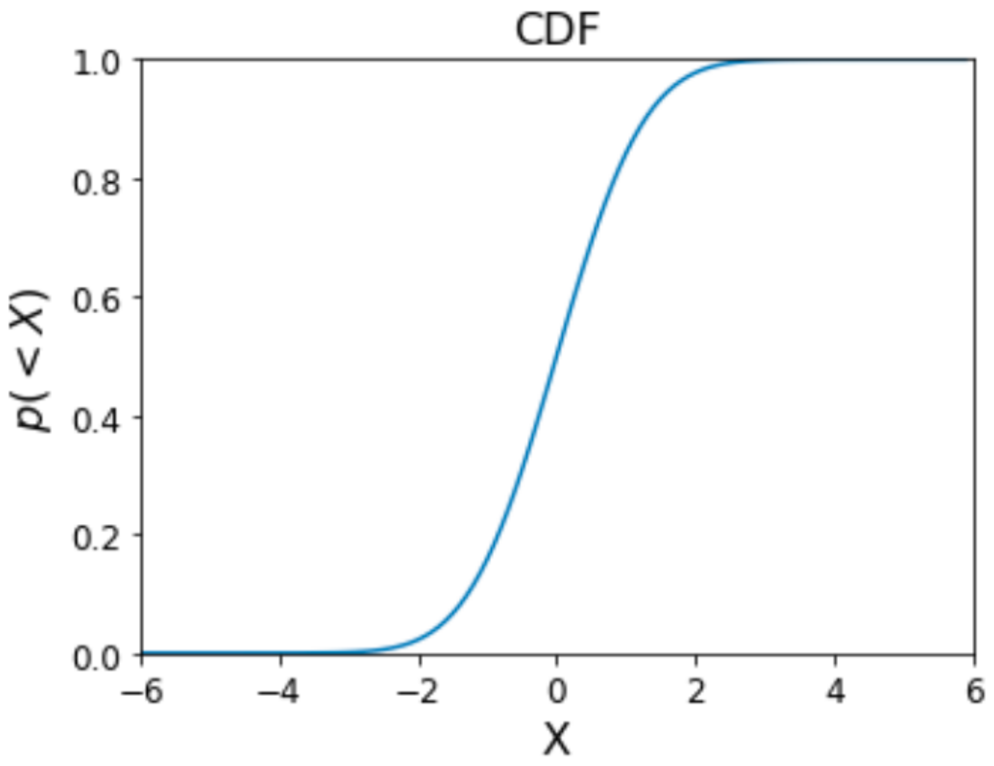
Part 3

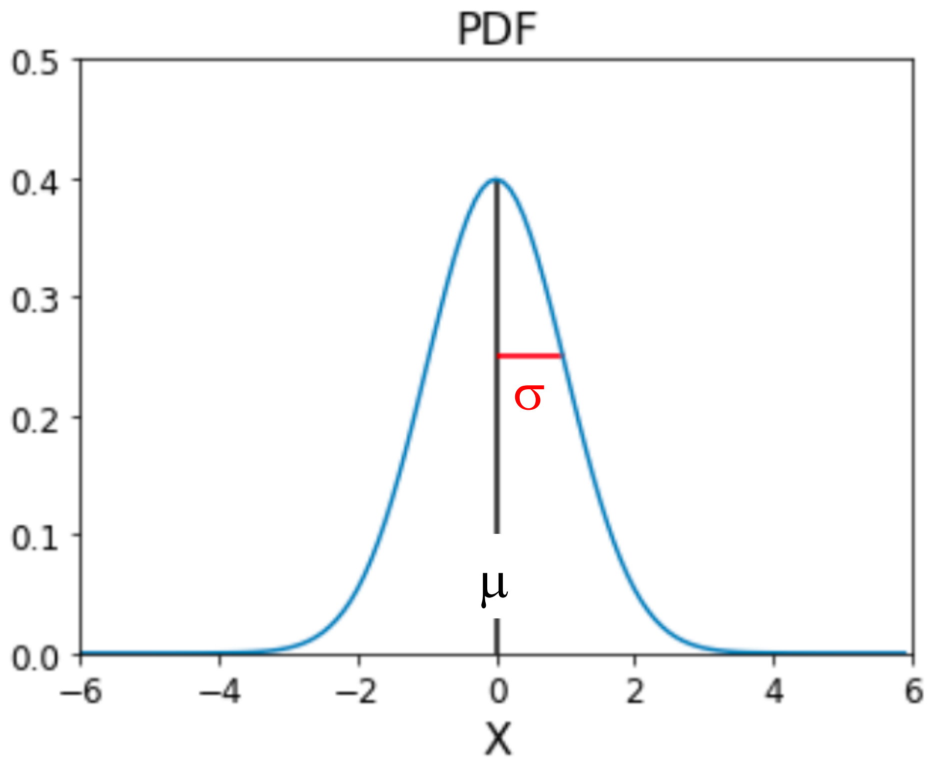
Associated notebook:

04-Basic_statistical_inference_frequentists_2/Frequentist_inference_02.ipynb

Confidence interval on a RV (descriptive stat)

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





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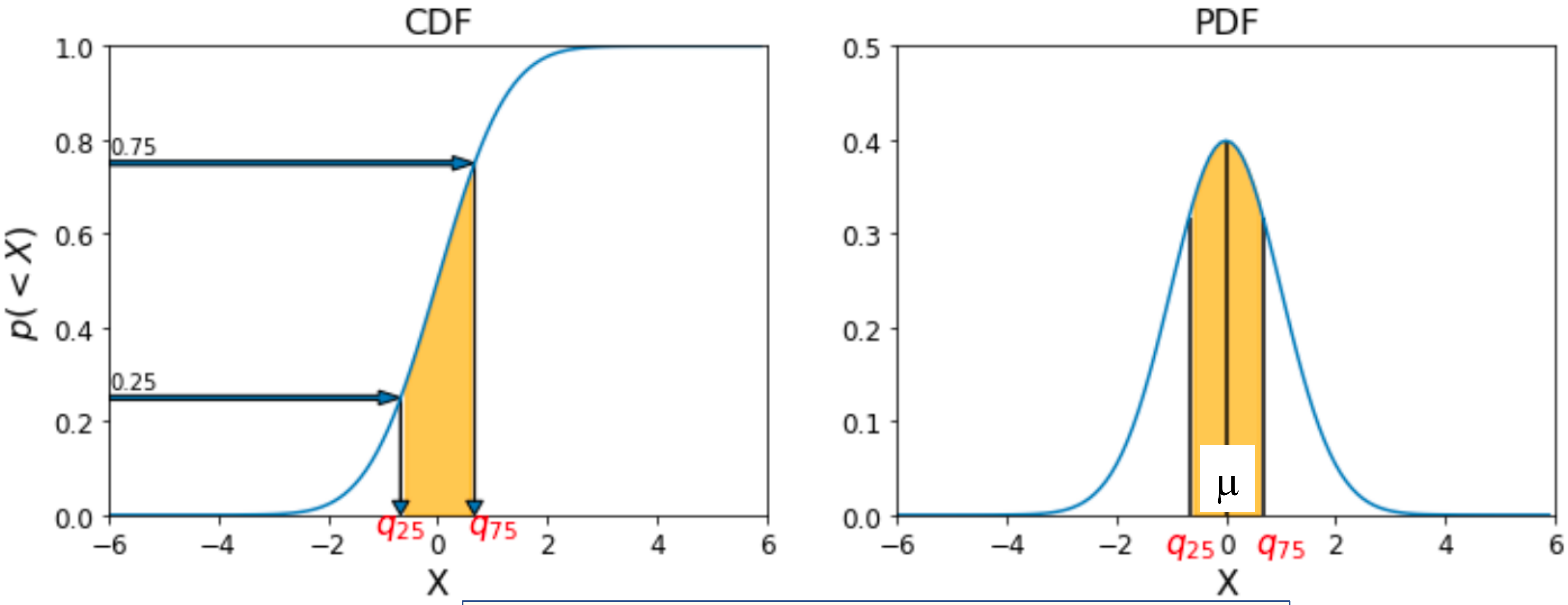
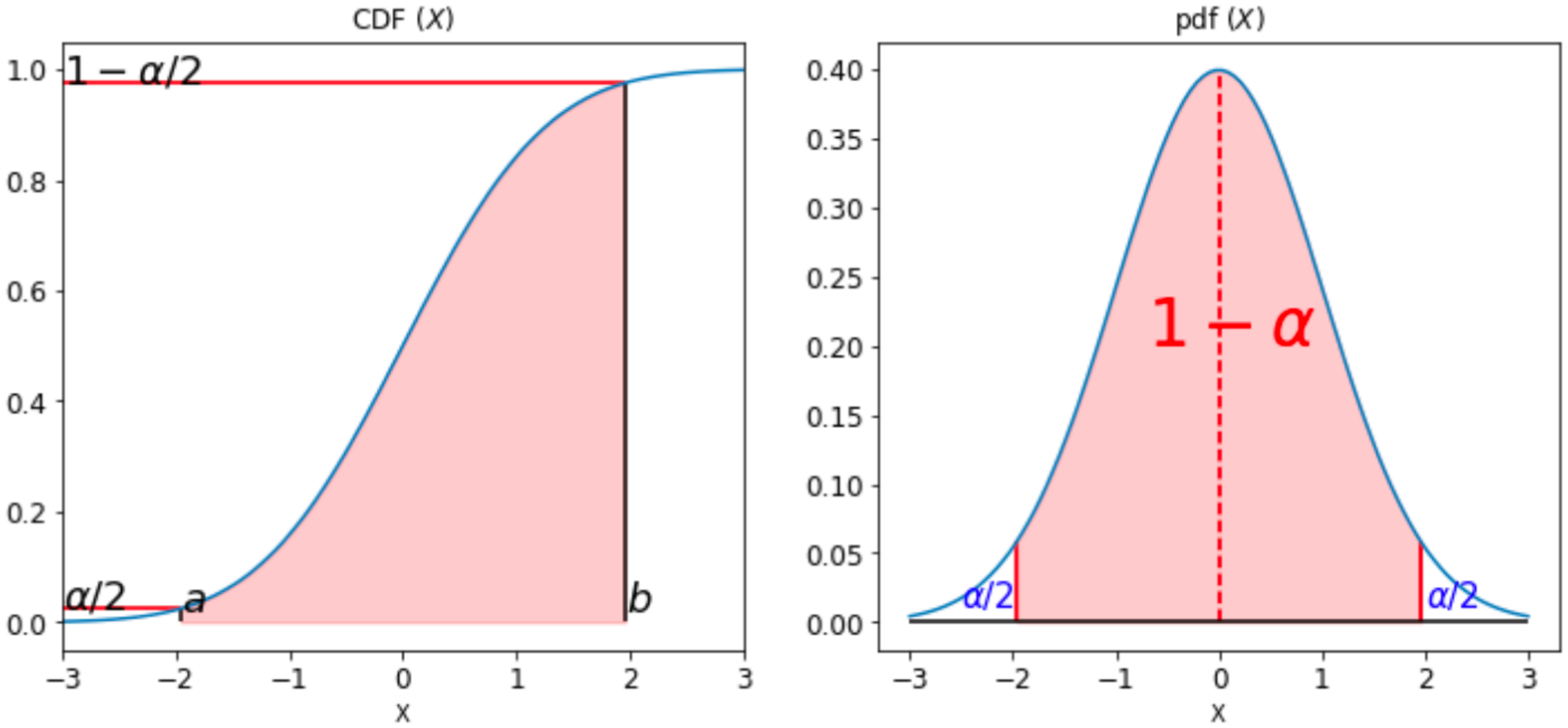


Interquartile IQ = $[q_{25}, q_{75}] \equiv 50\,\%$ CI around μ

$1-\alpha$ CI $\equiv [a, b]$ such that $p(a \leq X \leq b) = (1-\alpha)$

Confidence interval on a RV (descriptive stat)

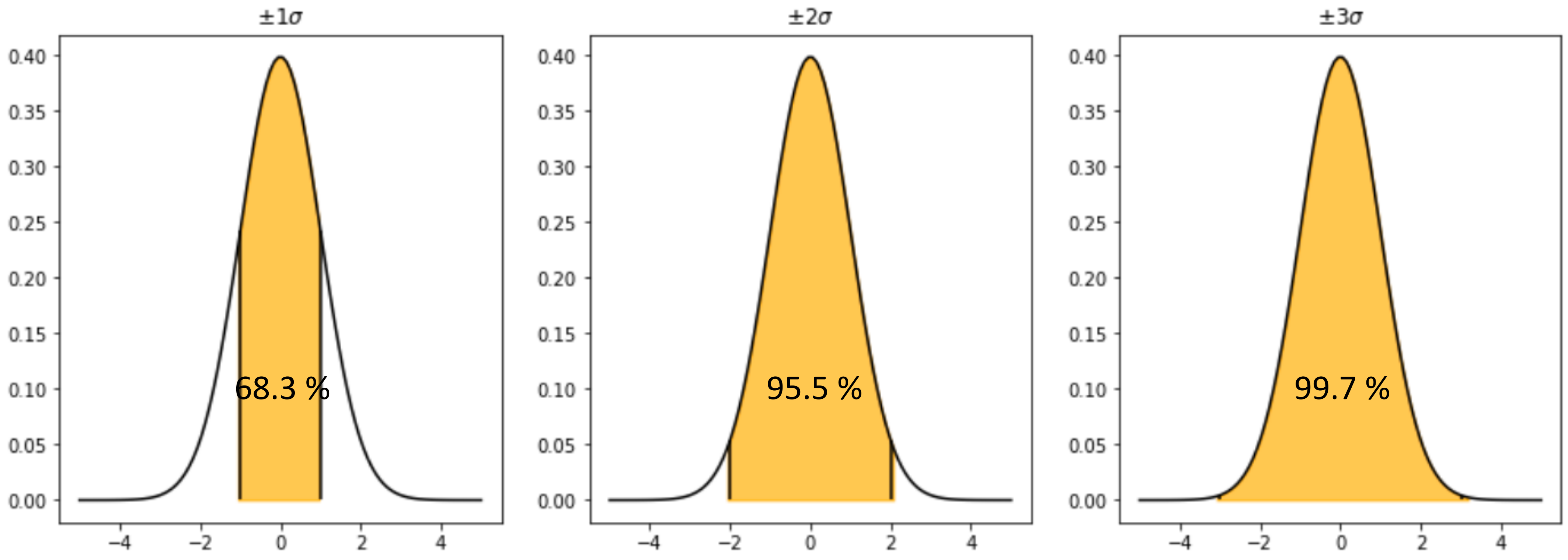
If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$



$1 - \alpha$ CI ≡ [a, b] such that $p(a \leq X \leq b) = (1 - \alpha)$

Confidence interval on a RV (descriptive stat)

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$



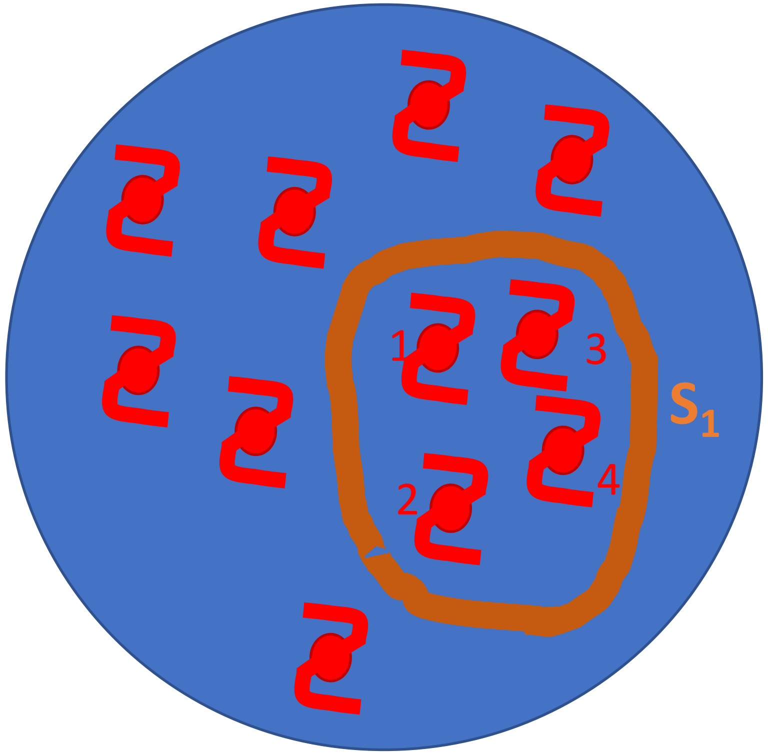
$[\mu - 1\ \sigma, \mu + 1\ \sigma] \equiv 68.3\ \%$

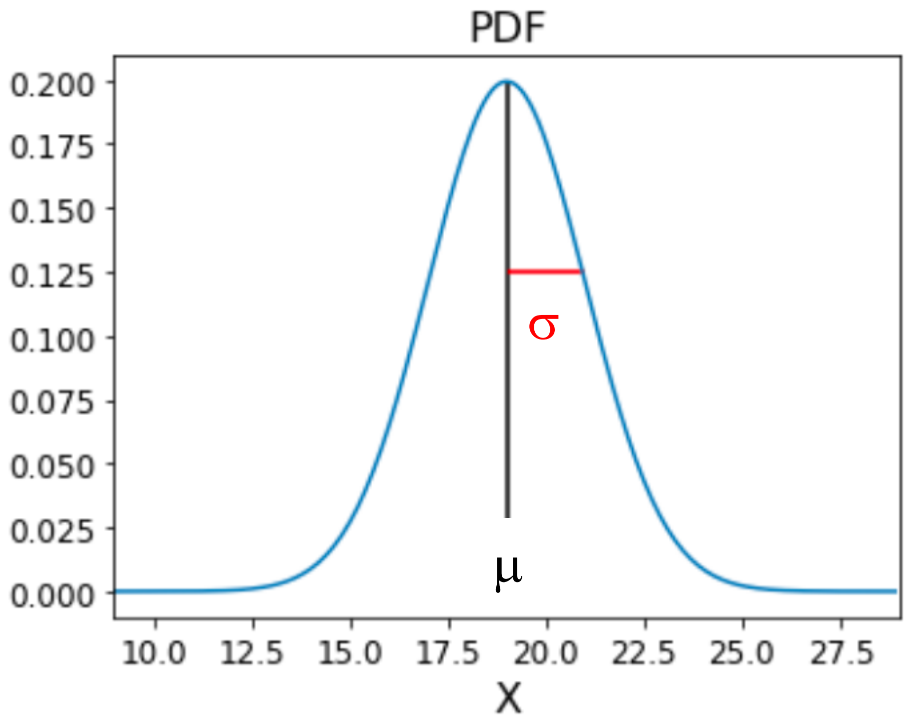
$[\mu - 2\ \sigma, \mu + 2\ \sigma] \equiv 95.5\ \%$

$[\mu - 3\ \sigma, \mu + 3\ \sigma] \equiv 99.7\ \%$

Confidence interval on from an estimator

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$

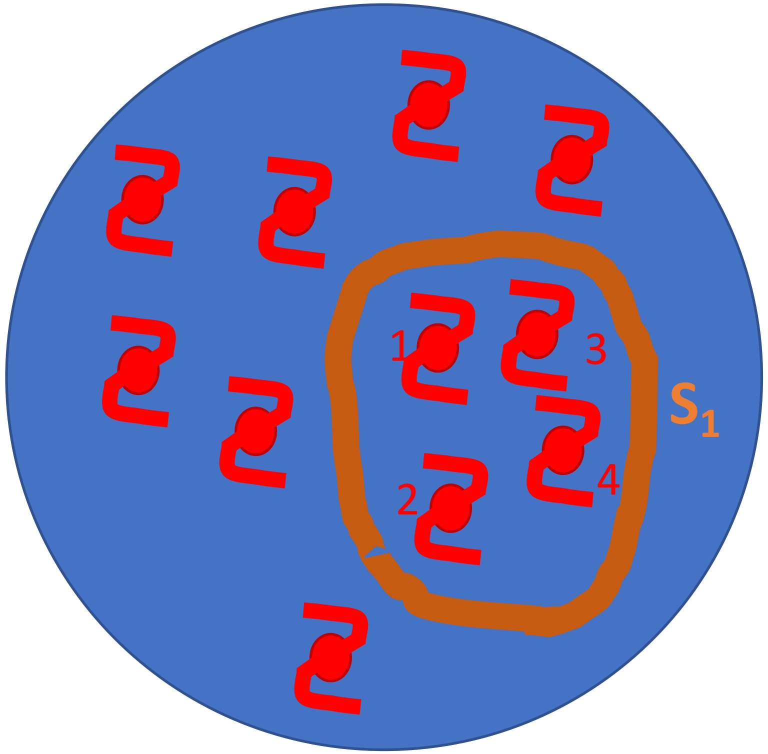


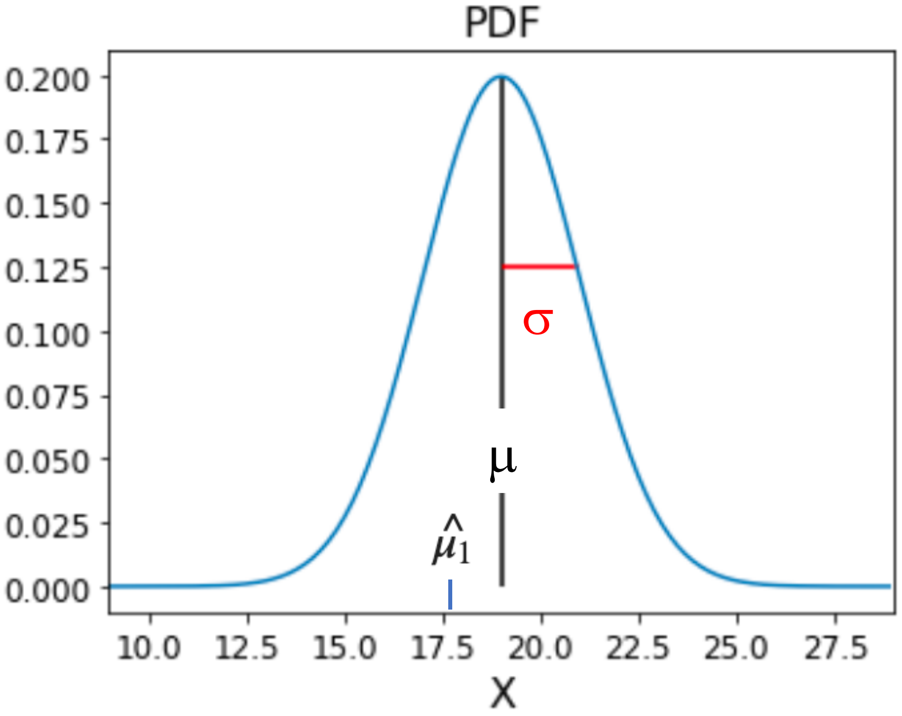


$S_1 = \{x_1, x_2, x_3, x_4\}$

Confidence interval from an estimator

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





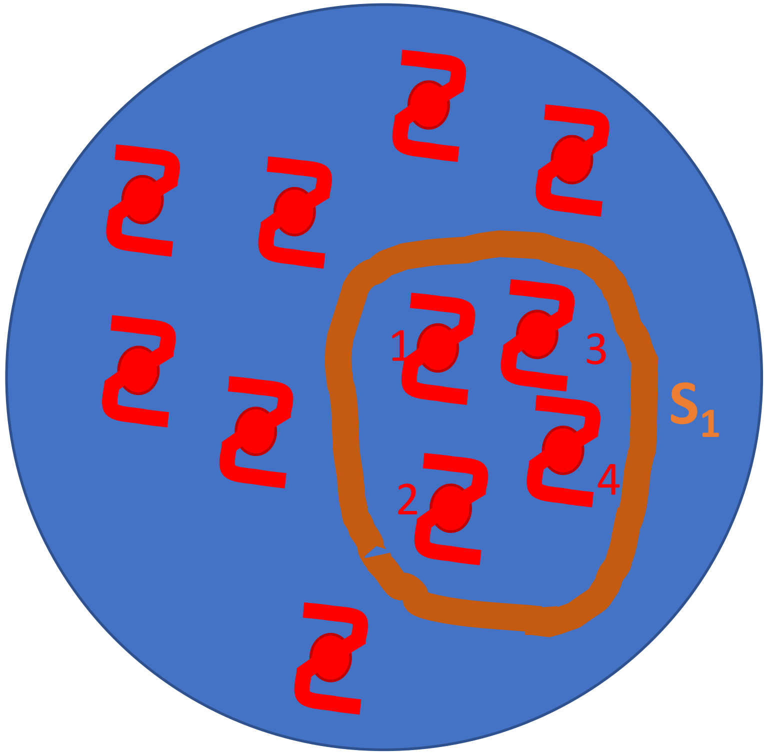
$S_1 = \{x_1, x_2, x_3, x_4\}$

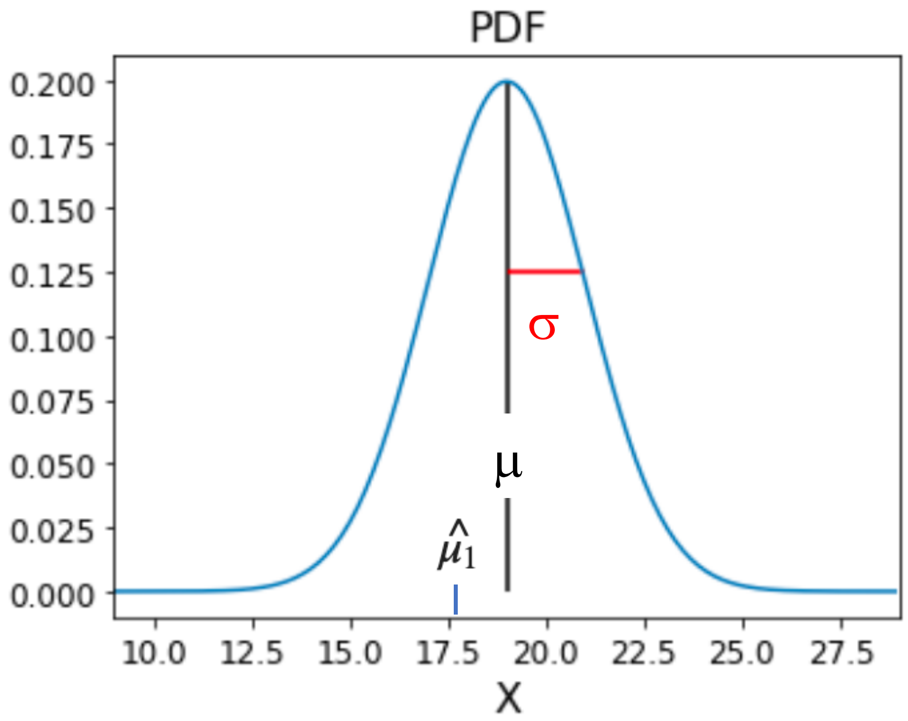
$$S_1 = \{19.45, 16.20, 16.43, 19.10\}$$

$$\hat{\mu}_1 = 17.8$$

Confidence interval from an estimator

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





$S_1 = \{x_1, x_2, x_3, x_4\}$

$$S_1 = \{19.45, 16.20, 16.43, 19.10\}$$

$$\hat{\mu}_1 = 17.8$$

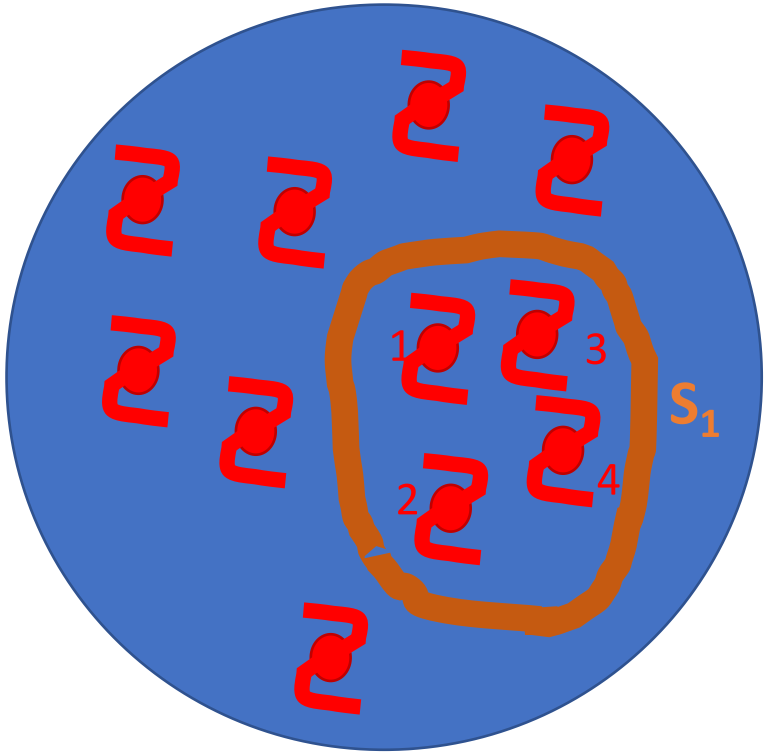
95.5 % CI on $\hat{\mu}$?

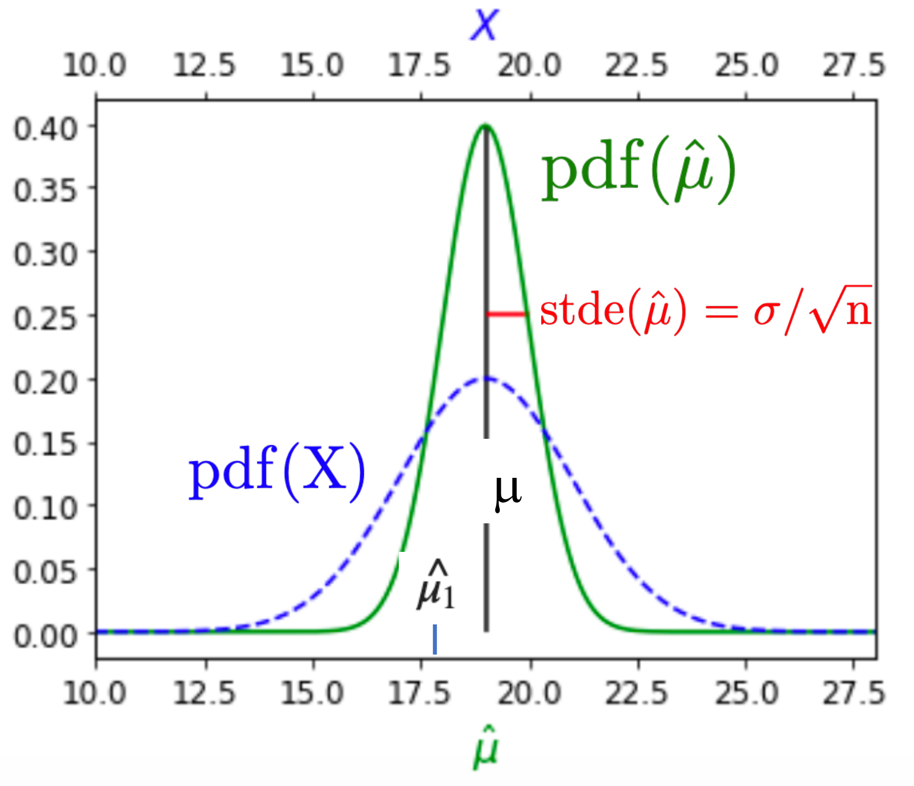
We need to look at the distribution of the estimator $\hat{\mu}$

Confidence interval on the mean

95.5 % CI on $\hat{\mu}$?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

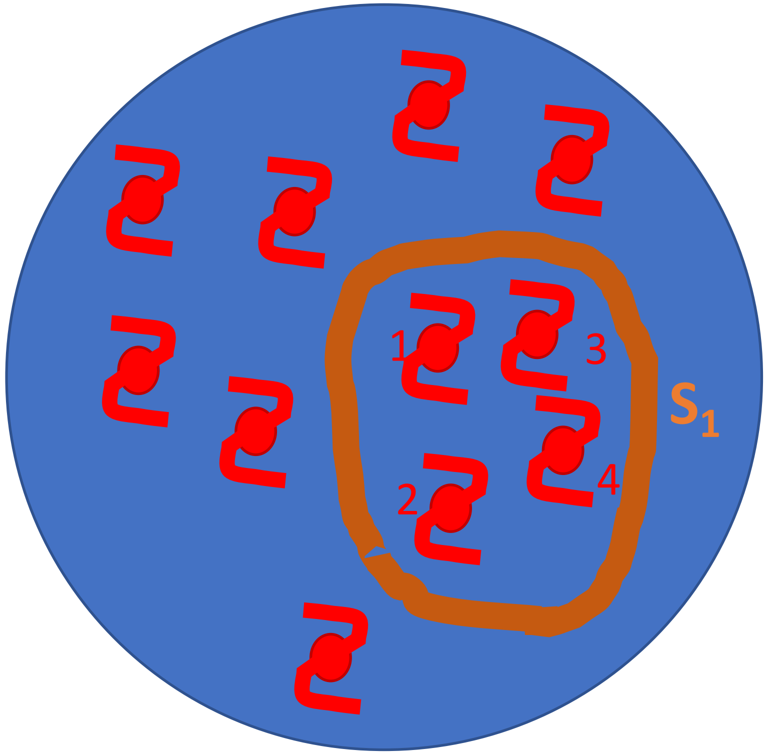
$$\boxed{\hat{\mu} \sim N(\mu, \sigma/\sqrt{n})}$$

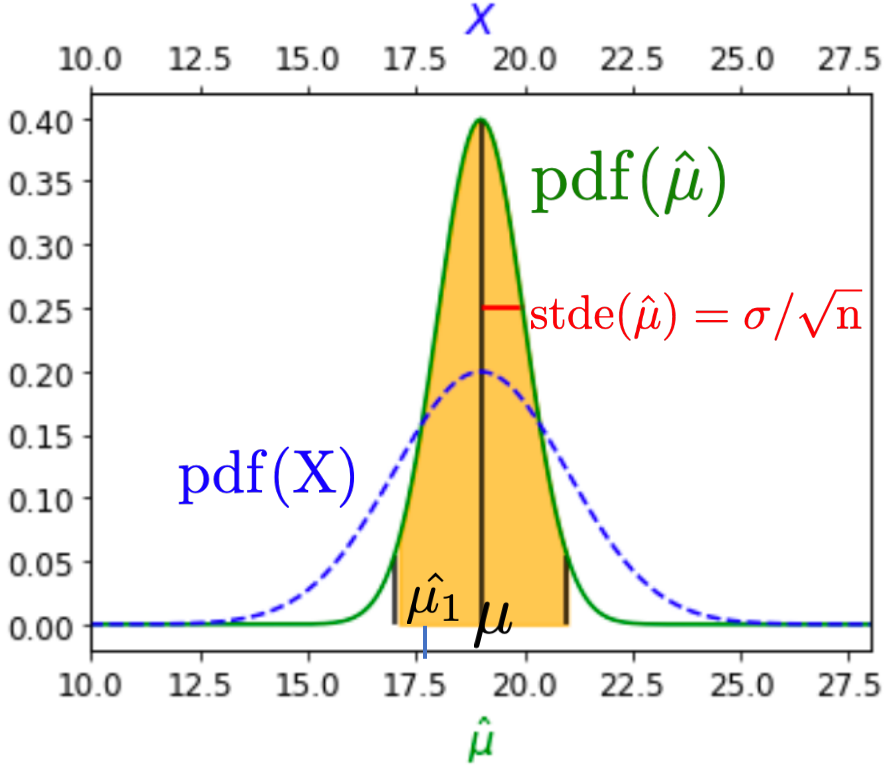
$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

Confidence interval on the mean

95.5 % CI on $\hat{\mu}$?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

$$\boxed{\hat{\mu} \sim N(\mu, \sigma/\sqrt{n})}$$

$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

$$p(\hat{\mu}_1 \in [\mu \pm 2\,\mathrm{stde}]) = 0.955$$

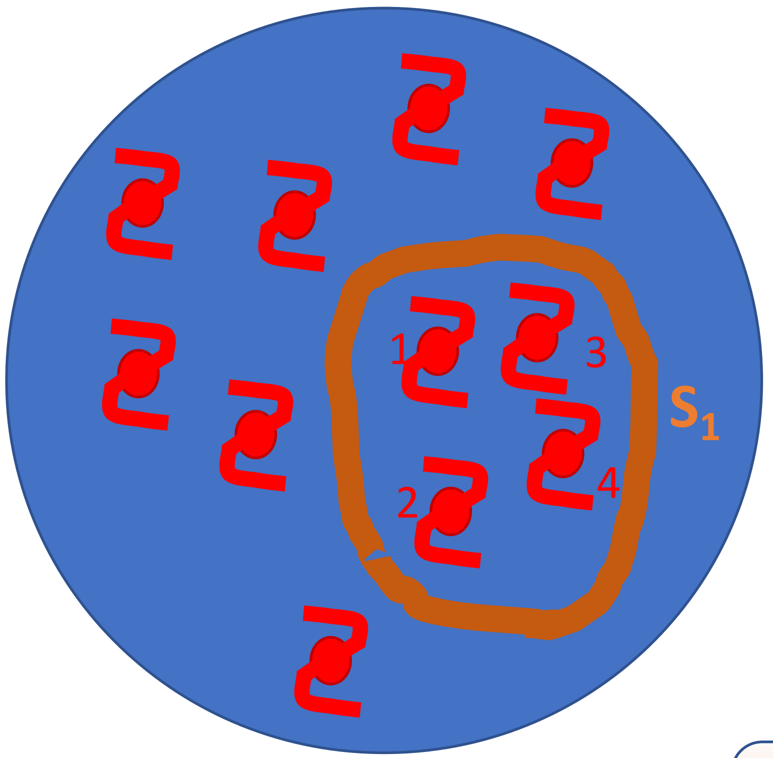
$$p(\mu \in [\hat{\mu}_1 \pm 2\,\mathrm{stde}]) = 0.955$$

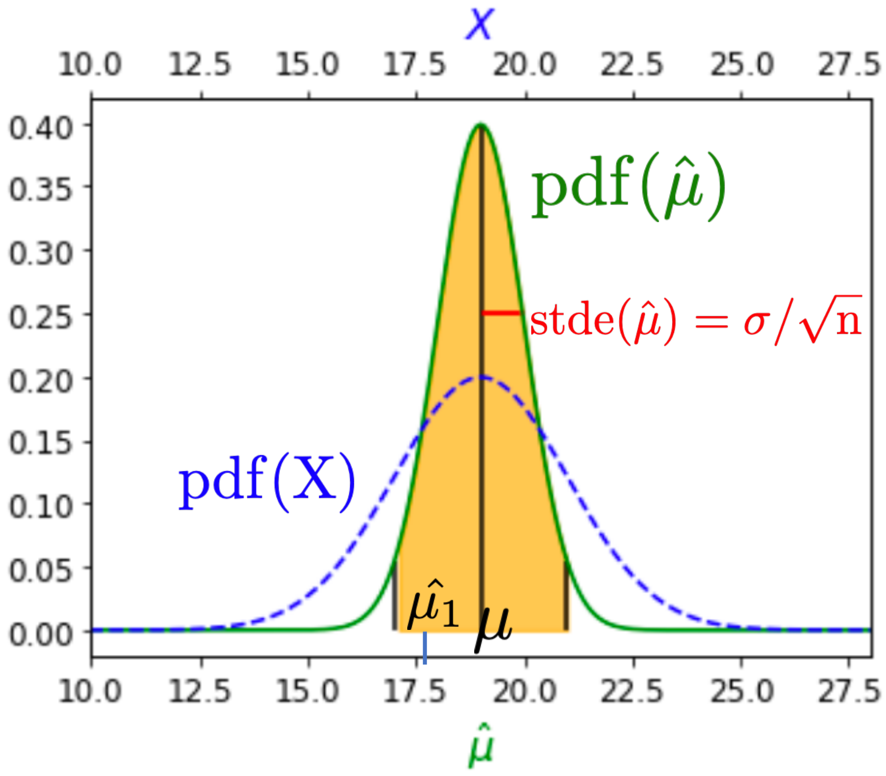
Confidence interval on the mean

95.5 % CI on μ ?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$

Case 1: σ is known





$$\boxed{\hat{\mu} \sim N(\mu, \sigma/\sqrt{n})}$$

$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

$$\mathrm{CI} \equiv [\hat{\mu} \pm \mathrm{m} \times \mathrm{stde}]$$

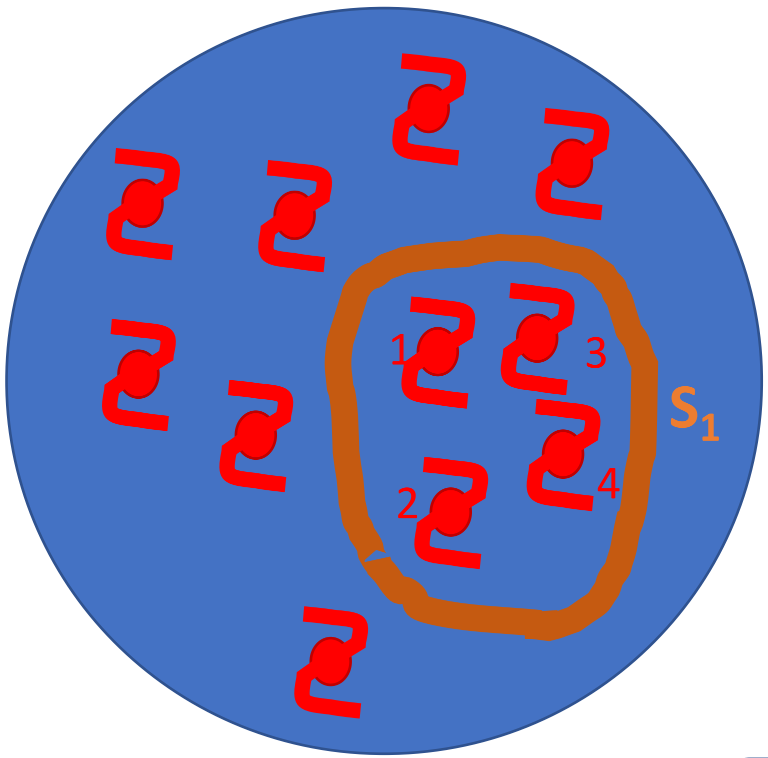
m depends on the distribution of the normalised estimator:

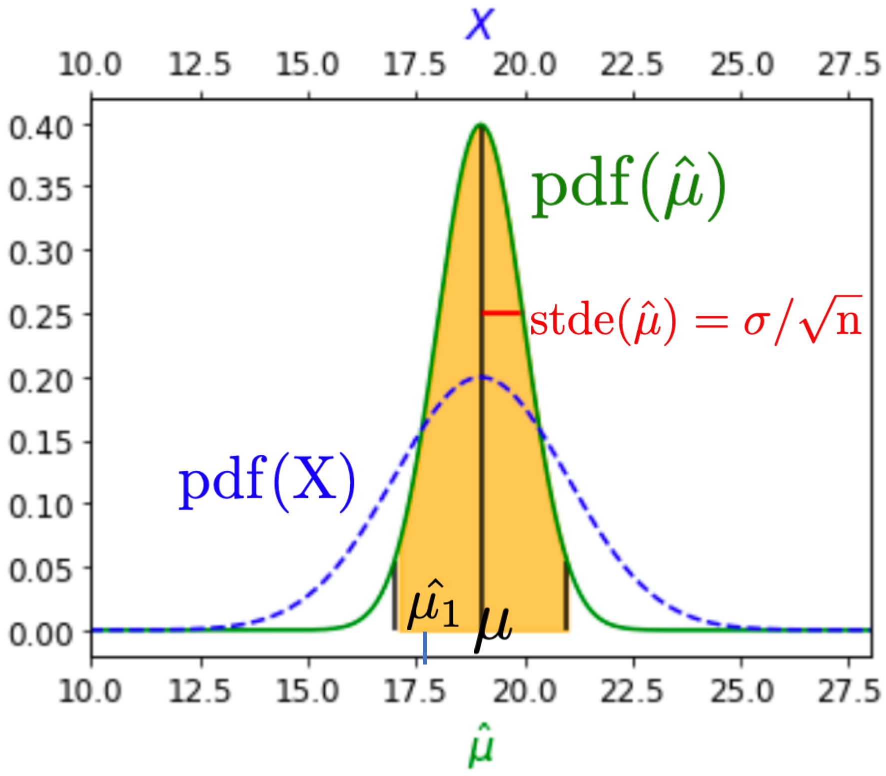
$$\frac{\mu - \hat{\mu}}{\mathrm{stde}}$$

Confidence interval on the mean

95.5 % CI on μ ?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

$$Z = \frac{\mu - \hat{\mu}}{\sigma/\sqrt{\mathrm{n}}} \sim N(0,1)$$

Case 2: σ is unknown

$$\mathrm{stde}(\hat{\mu}) = \mathrm{s}/\sqrt{\mathrm{n}}$$

$$t = \frac{\mu - \hat{\mu}}{\mathrm{s}/\sqrt{\mathrm{n}}} \sim t(n-1)$$

$$\mathrm{CI} \equiv [\hat{\mu} \pm \mathrm{m} \times \mathrm{stde}]$$

Go to: Student_vs_Gauss.ipynb

Confidence interval

“Generic” strategy:

- Look at the PDF of the estimator of interest (the latter is often an output of Bayesian analysis) or of a normalised estimator of known distribution.

- Define region around your estimator (or a normalised estimator for which you know the distribution) that encloses **(1 - α) X 100 %** of the area under the PDF.

- For CI around the mean:

$$\mathrm{CI}_\alpha = [\bar{\mathrm{x}} - \mathrm{q}_{\alpha/2}\,\mathrm{stde}, \bar{\mathrm{x}} + \mathrm{q}_{\alpha/2}\,\mathrm{stde}]$$

$$q_{\alpha/2} = CDF^{-1}(1-\alpha/2)$$

For CI on other statistics (e.g. variance, difference between 2 means, proportions) : see Frequentist_inference_03.ipynb
(Material in Frequentist_inference_03.ipynb is supplementary material)

Go to: Sect. II.2. of the notebooks