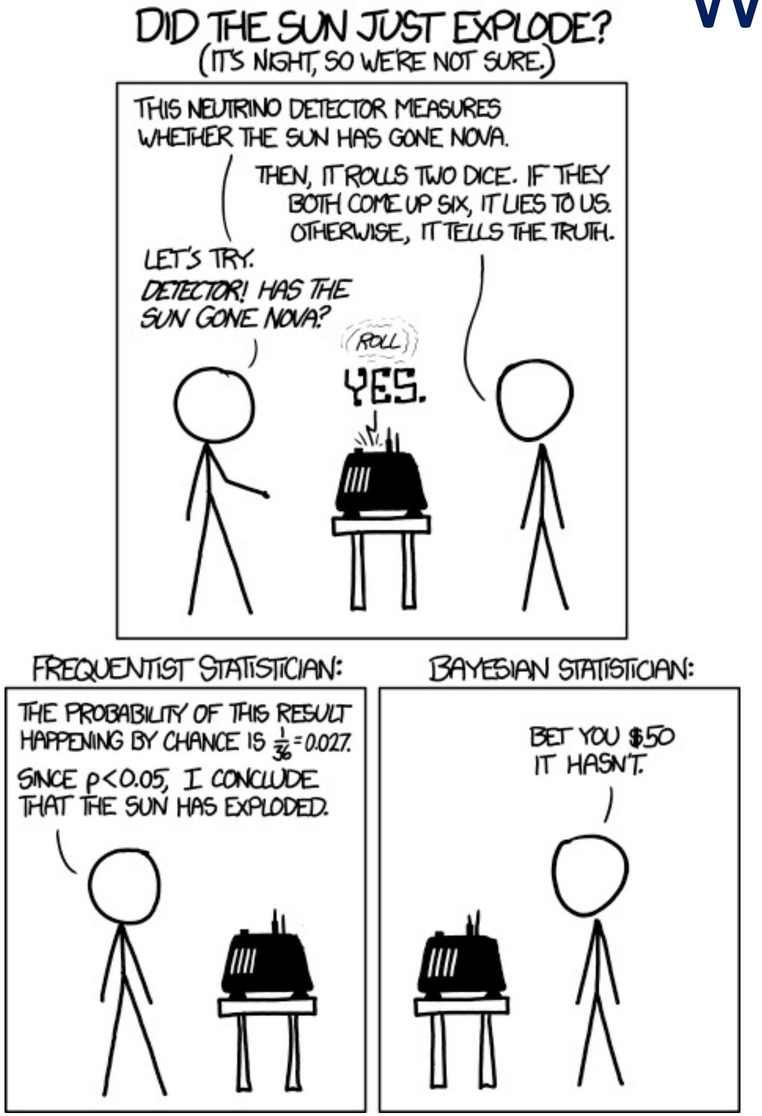
Classical statistical inference

Part 2

Associated notebook:

04-Basic_statistical_inference_frequentists_2/Frequentist_inference_01.ipynb

What is inference?



https://xkcd.com/1132/

Derive **INFORMATION** based on **DATA**

Examples:

- Exoplanet transit $\Rightarrow$ M, d $\Rightarrow$ P(M | d)
- Supernovae distances $\Rightarrow$ Expansion rate H_0

Inference generally implies an underlying *statistical model*: PDF or regression laws with *parameters* θ

Three types of inference:

- Point estimation: “best” θ
- Confidence interval: Confidence around θ
- Hypothesis testing: *data OK w. model ?*

Point estimate $\hat{\theta}$

Example:

θ = mean mag. of a population of galaxies

A: Your data set = subsample of measurements:
A = $\{X_1, X_2, X_3, X_4\}$ where X = mag. (this is a RV)

$$\hat{\theta} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$ ≡ Point estimate of θ

If you do the experiment with another sample ($\Rightarrow$ different *realisation*) you will get another $\hat{\theta}$

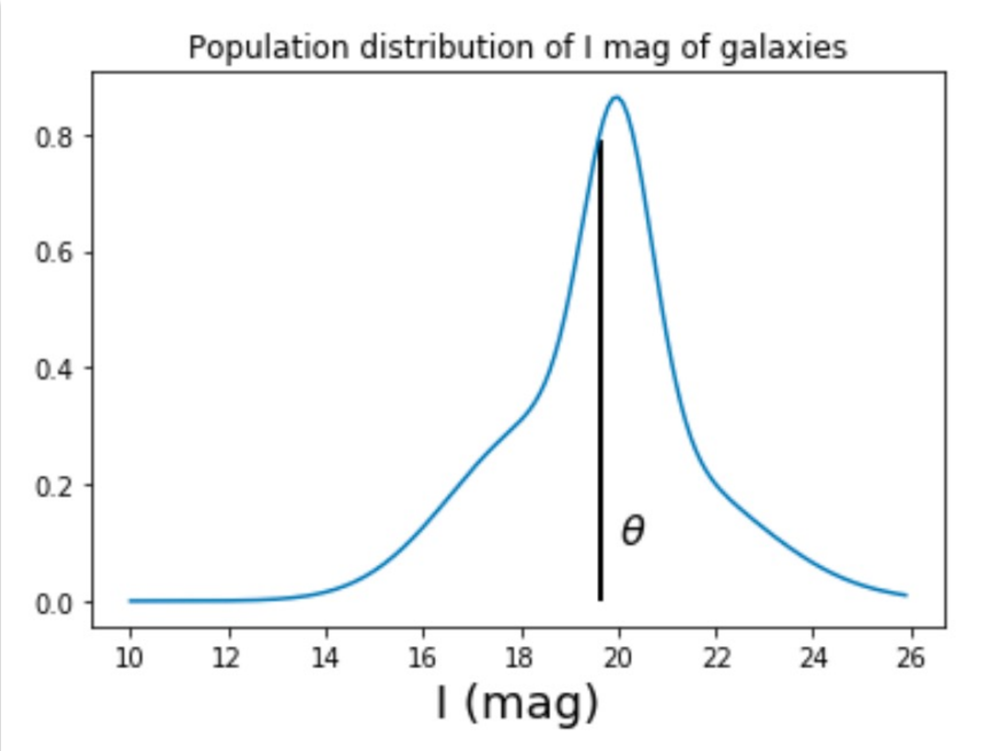
Point estimate $\hat{\theta}$

Generalisation:

$$\hat{\theta} = g(X_1, X_2, X_3, \ldots X_n)$$

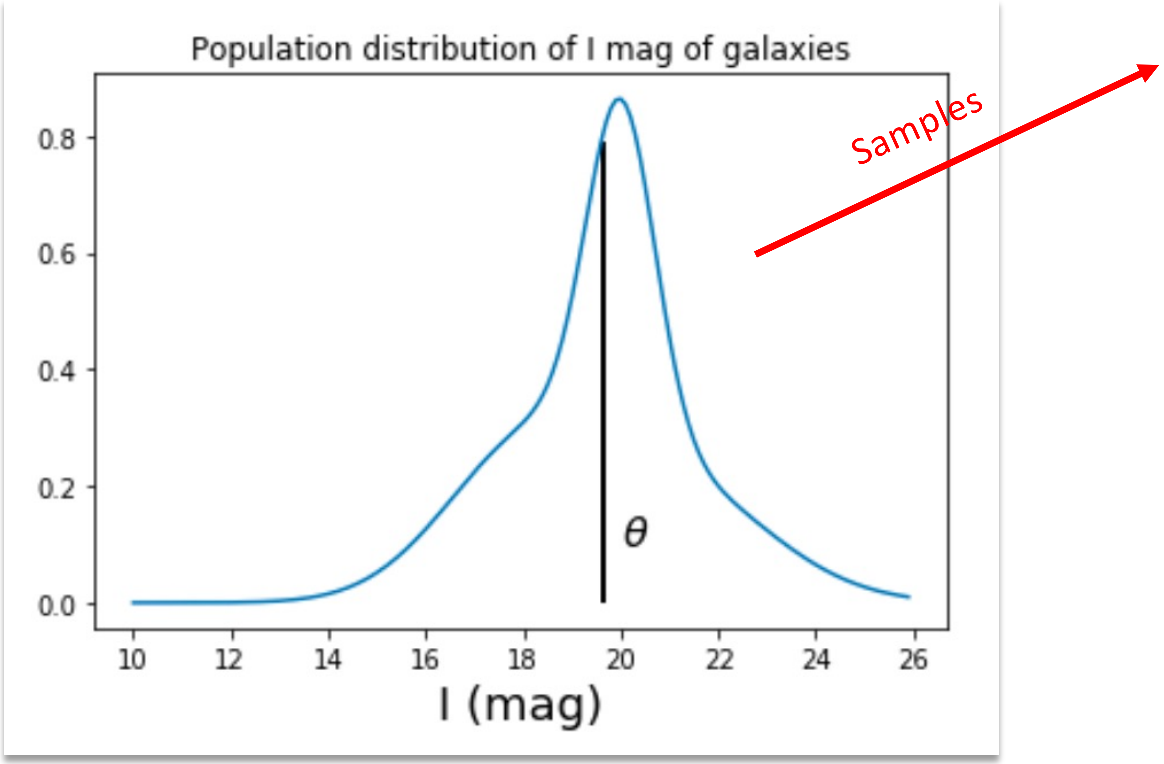
- Point estimate of a param. is a *function* of RV X_1, …
- It is as well a **Random Variable (RV)**
- It can be biased, is characterized by a variance but should ideally be consistent (converges towards θ)
- Distribution of $\hat{\theta}$ is called *sampling distribution*

Point estimate $\hat{\theta}$

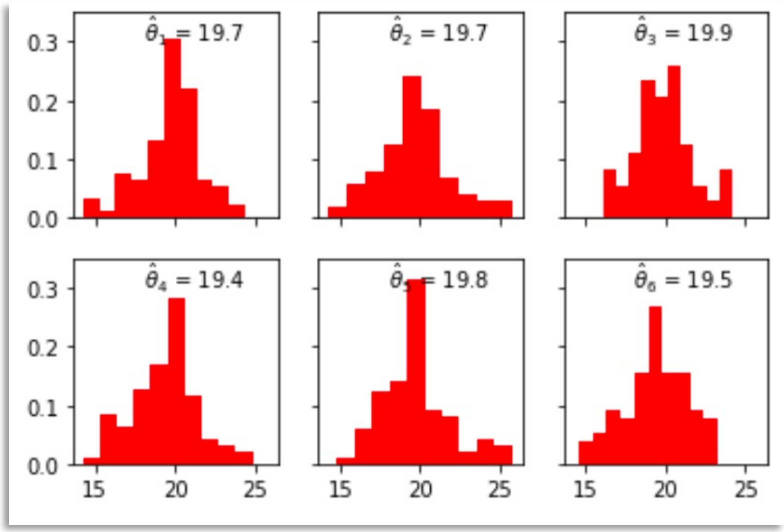


Population mean: $\theta = 19.66$

Point estimate $\hat{\theta}$

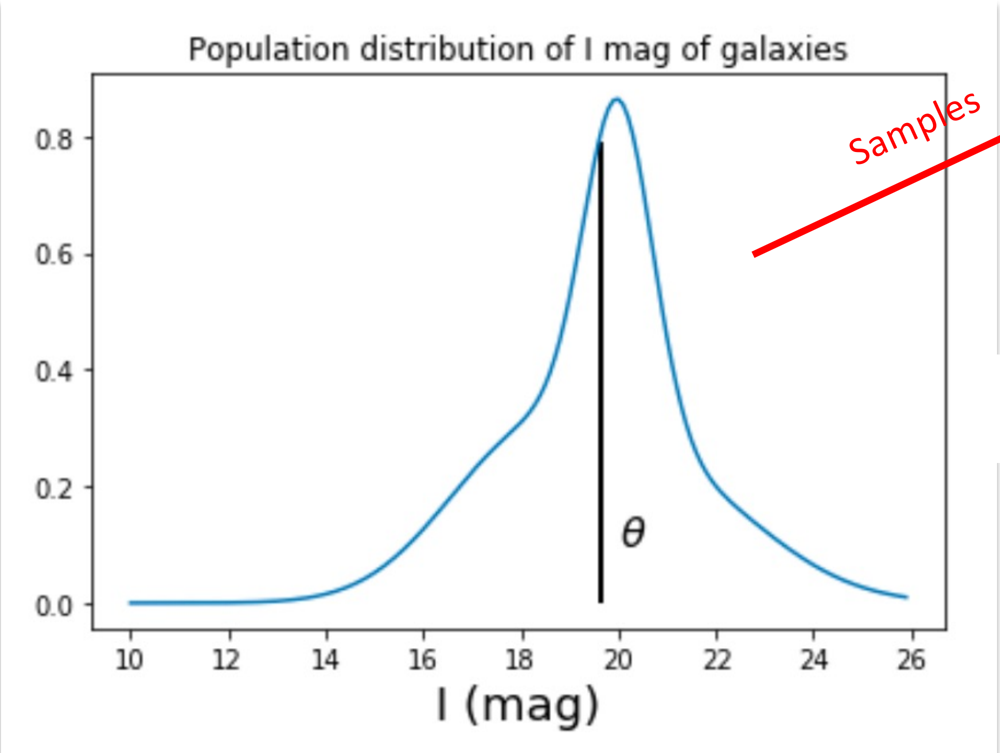


Six different *samples* drawn from the true population



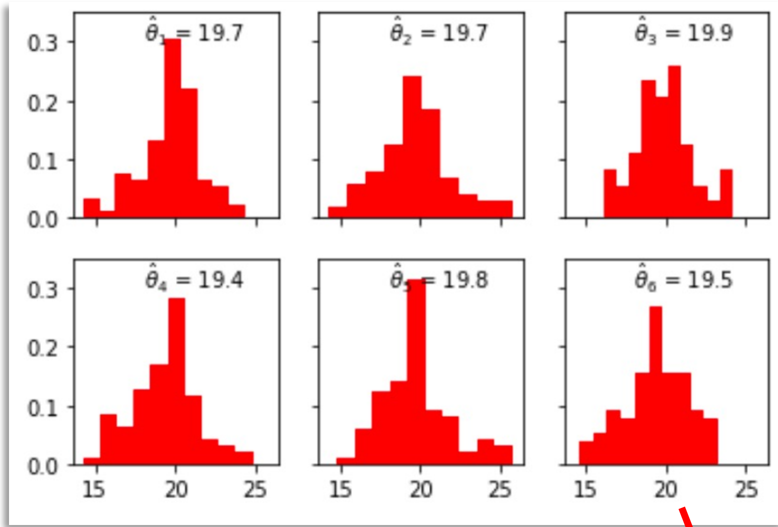
Population mean: $\theta = 19.66$

Point estimate $\hat{\theta}$

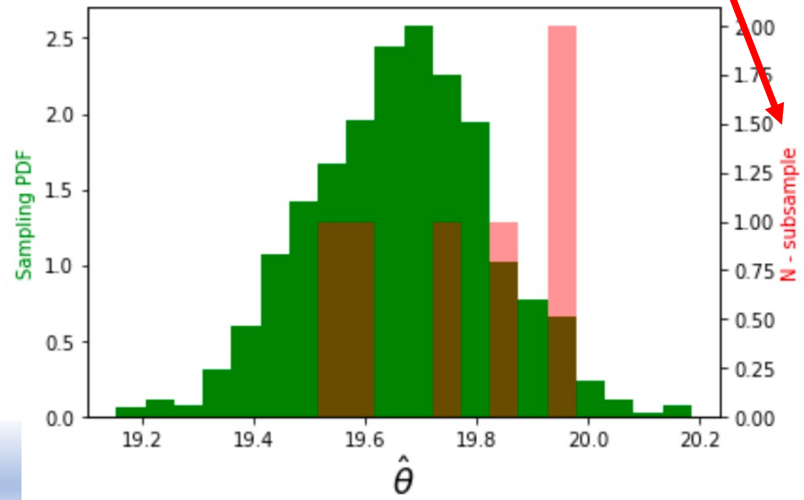


Samples

Six different *samples* drawn from the true population



Distribution of $\{\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k\}$ = sample distribution

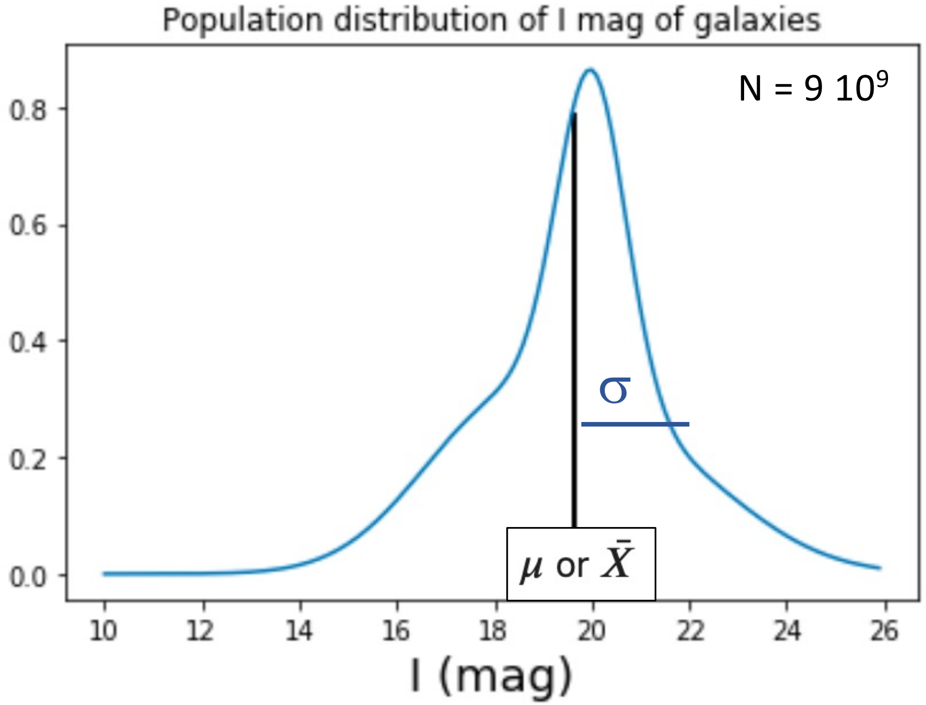


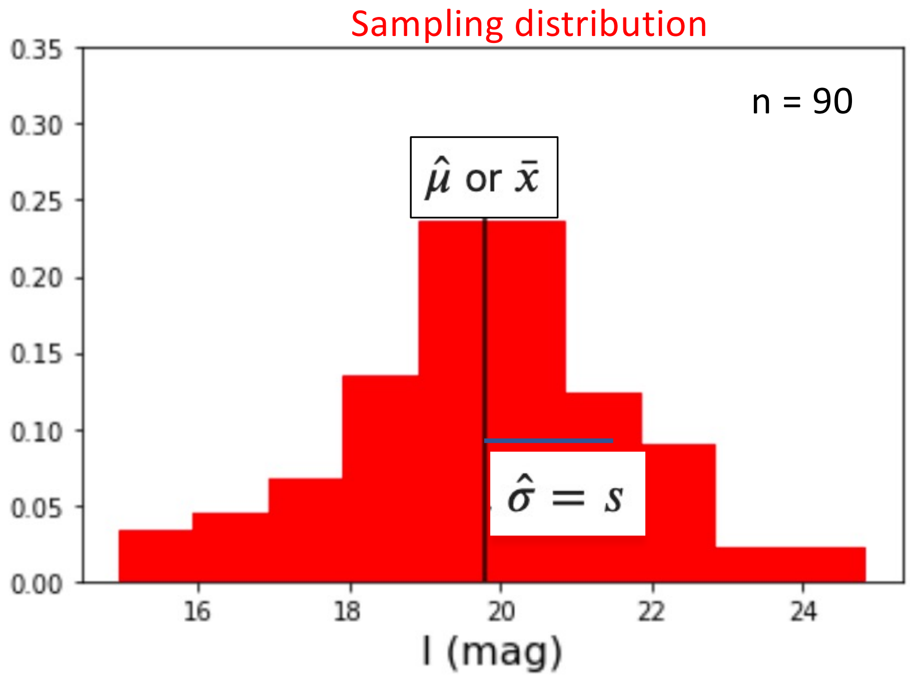
Population mean: $\theta = 19.66$

Go to: Sect. II.1 of the notebook

Summary statistics

Name	Population Statistics	Sample Statistics
size	N	n
mean	$\mu = \bar{X} = \frac{\sum_i X_i}{N}$	$\hat{\mu} = \bar{x} = \frac{\sum_i x_i}{n}$
Variance	$\sigma^2 = \frac{\sum_i (X_i - \bar{X})^2}{N}$	$s^2 = \frac{\sum_i (x_i - \bar{x})^2}{n-1}$
Standard deviation	$\sigma = \sqrt{\sigma^2}$	$\hat{\sigma} = s = \sqrt{s^2}$





Summary (sample) statistics: *standard error*

Standard error (stde) ≠ Standard deviation (std)

Name	Formula
Standard error on the mean	$stde(\bar{x}) = \frac{s}{\sqrt{n}}$
Standard error on the sample stdev	$stde(s) = s/\sqrt{2(n-1)}$
Standard error on proportions	$stde(\hat{p}) = \frac{\sqrt{p\,(1-p)}}{\sqrt{n}}$
Standard error on p^{th} percentile (h_p = PDF at p^{th} percentile)	$stde(q^p) = \frac{1}{h_p} \frac{\sqrt{p\,(1-p)}}{\sqrt{n}}$

Central limit theorem

When independent **random variables** are added, their *sum* tends towards a normal distribution (if n >>)

- This is true even if the original RV are not normally distributed
- Sampling dist. of mean tends (for large *n*) towards a Normal distribution
- Sampling dist. of variance (for large *n)* does NOT tend towards a Normal distribution

Go to: Sect. II.1.1. of the notebook

Distribution of sample quantities

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$

- Sample distribution of $\hat{\mu} \sim N(\mu, \sigma/\sqrt{n}) \iff Z = \dfrac{\mu - \hat{\mu}}{(\sigma/\sqrt{n})} \sim N(0,1)$
- Sample distribution of $t = \dfrac{\mu - \hat{\mu}}{(s/\sqrt{n})} \sim t(n-1)$ s is derived from the sample dist.

Student distribution

- Sample distribution of $S = \dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1).$

Chi square distribution

Classical statistical inference: *confidence interval*

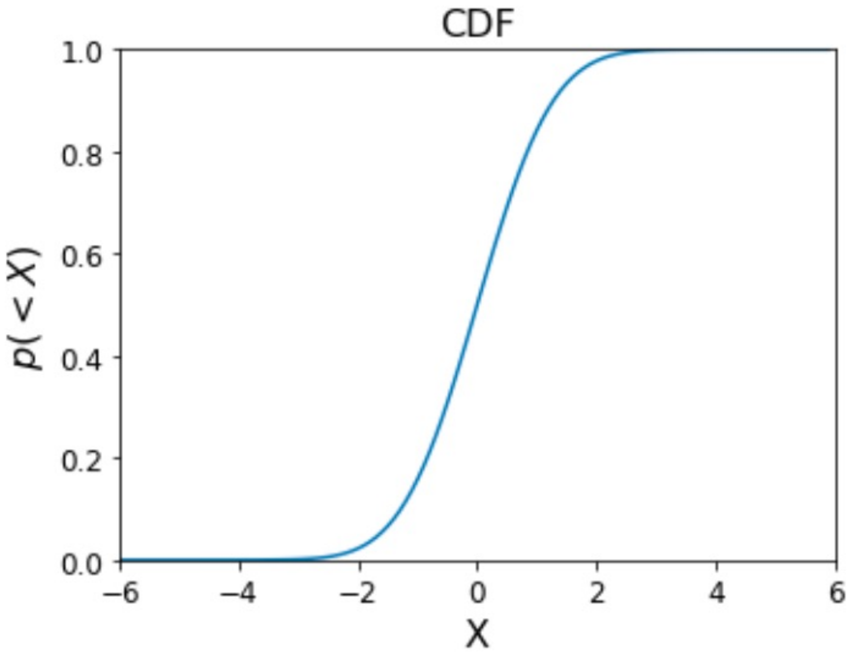
Part 3

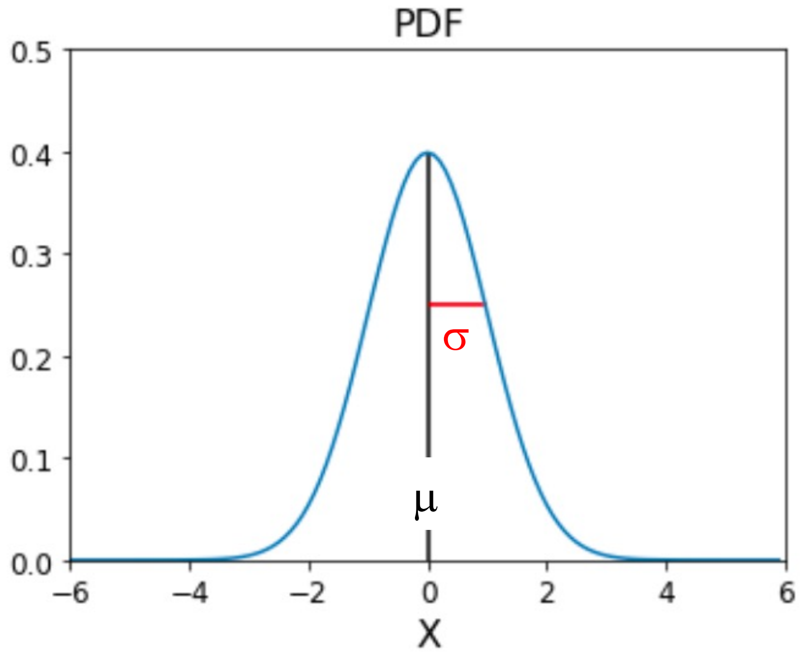
Associated notebook:

[04-Basic_statistical_inference_frequentists_2/Frequentist_inference_02.ipynb]

Confidence interval on a RV (descriptive stat)

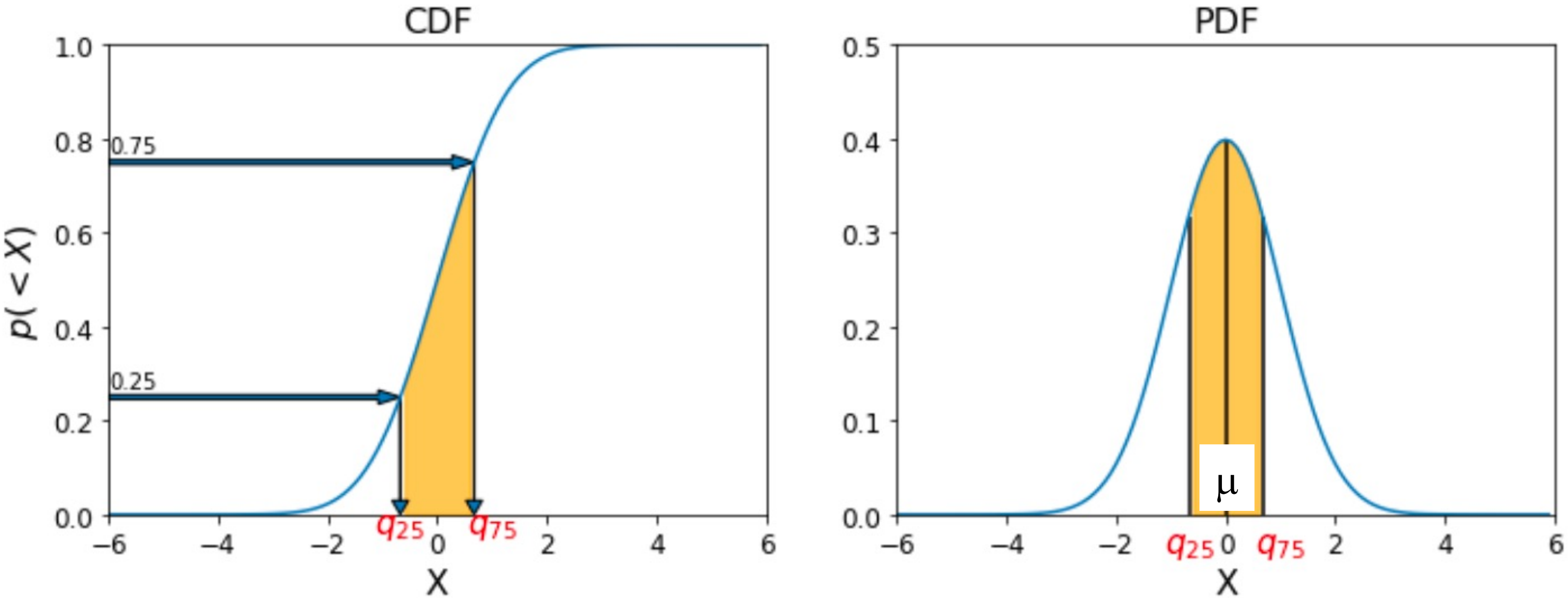
If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Confidence interval on a RV (descriptive stat)

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$

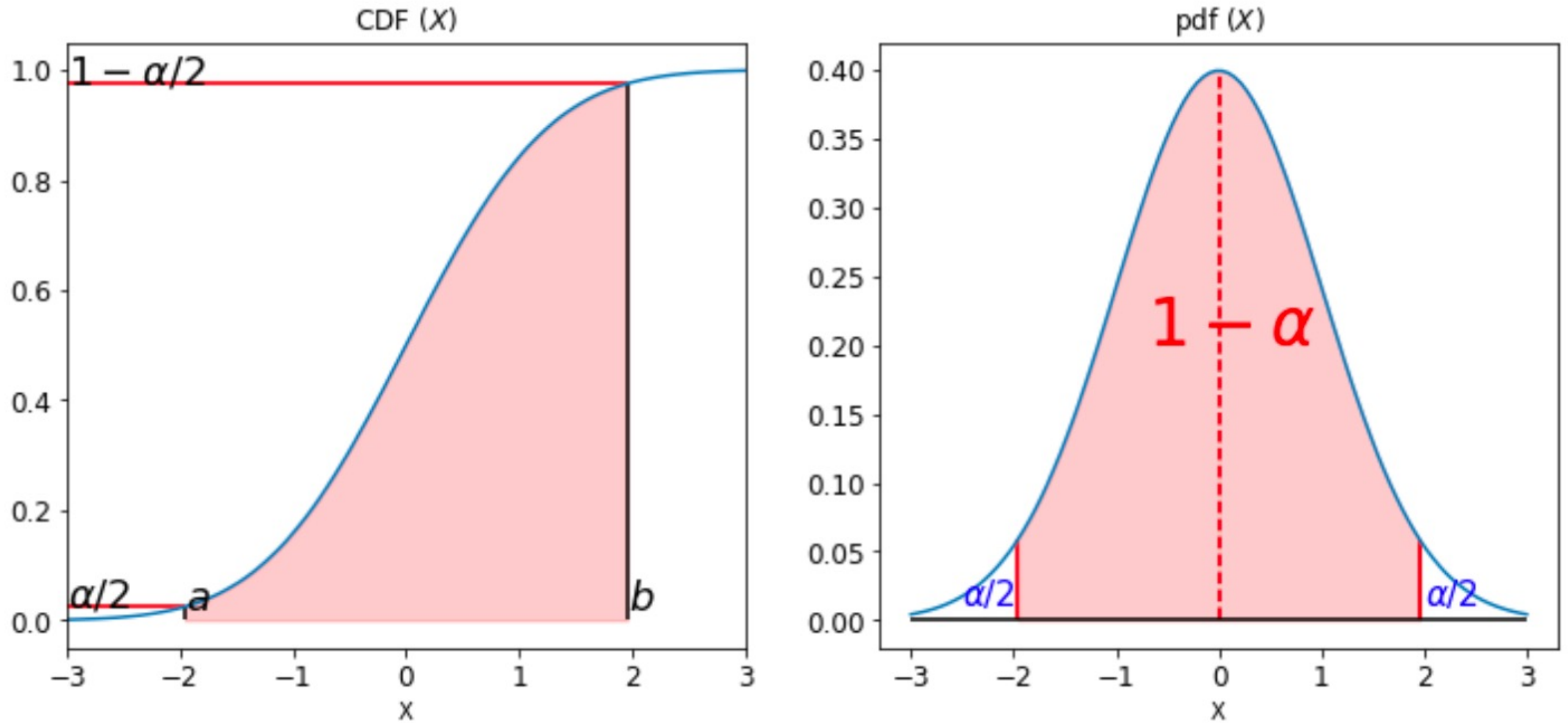


Interquartile IQ = $[q_{25}, q_{75}] \equiv 50\ \%$ CI around μ

$1 - \alpha$ CI $\equiv [a, b]$ such that $p(a \leq X \leq b) = (1 - \alpha)$

Confidence interval on a RV (descriptive stat)

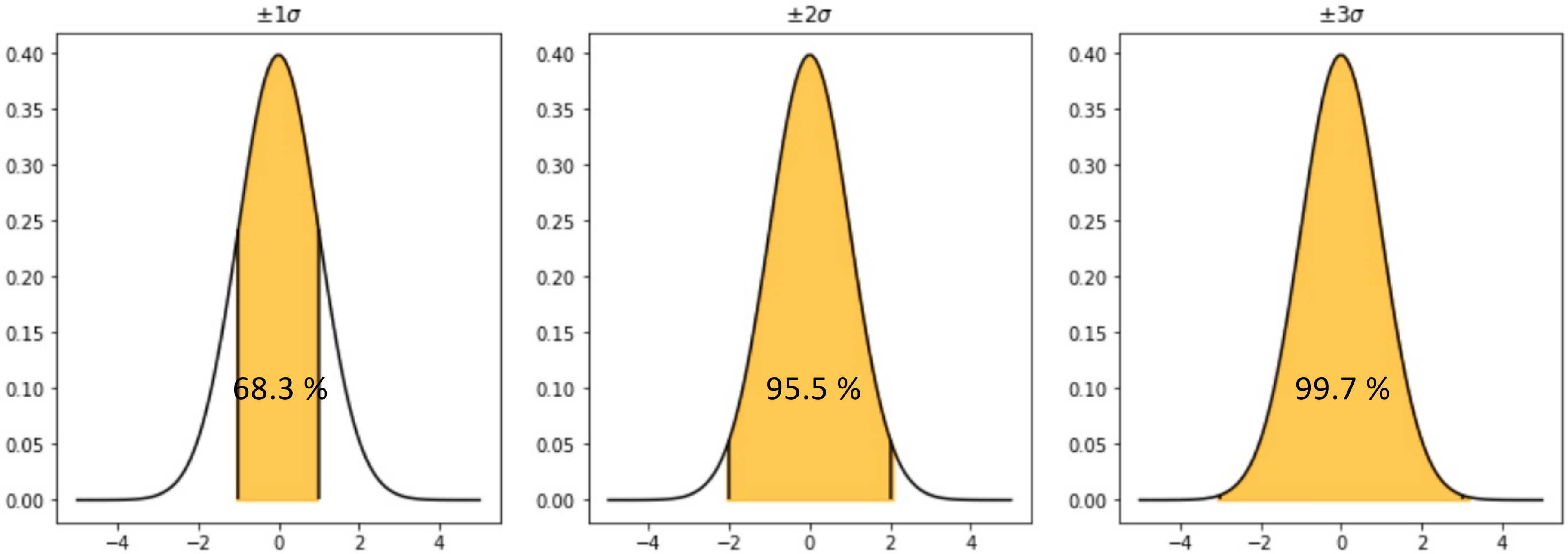
If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$



$1-\alpha$ CI $\equiv [a, b]$ such that $p(a \leq X \leq b) = (1-\alpha)$

Confidence interval on a RV (descriptive stat)

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$



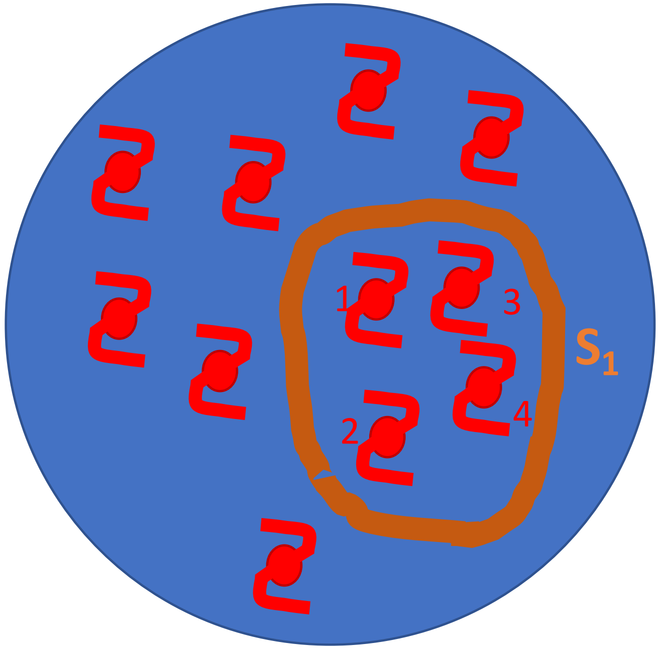
$[\mu - 1\ \sigma, \mu + 1\ \sigma] \equiv 68.3\ \%$

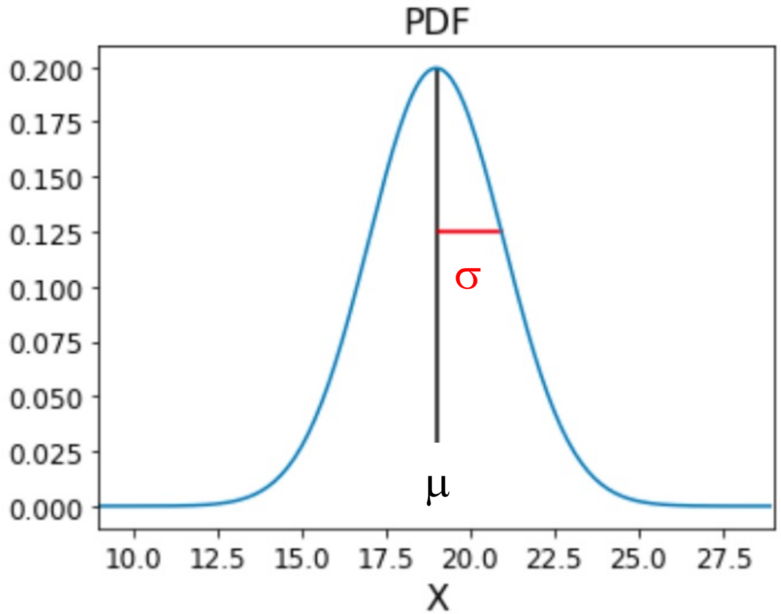
$[\mu - 2\ \sigma, \mu + 2\ \sigma] \equiv 95.5\ \%$

$[\mu - 3\ \sigma, \mu + 3\ \sigma] \equiv 99.7\ \%$

Confidence interval from an estimator

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$

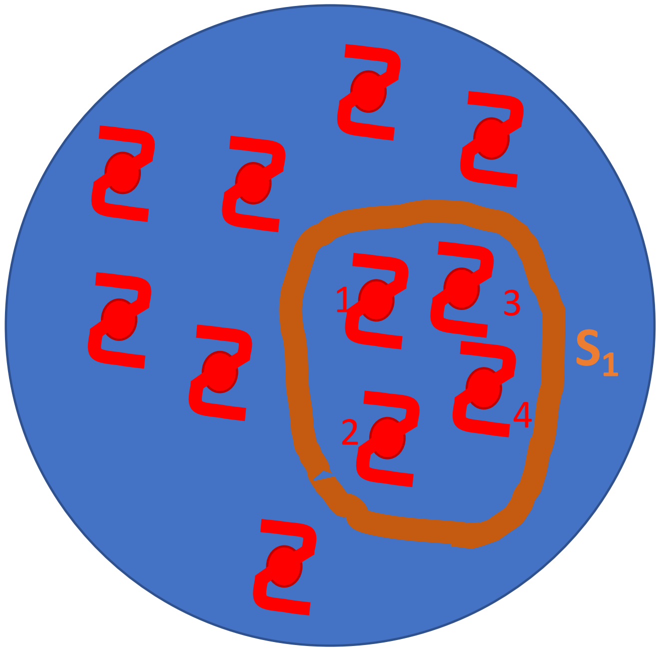


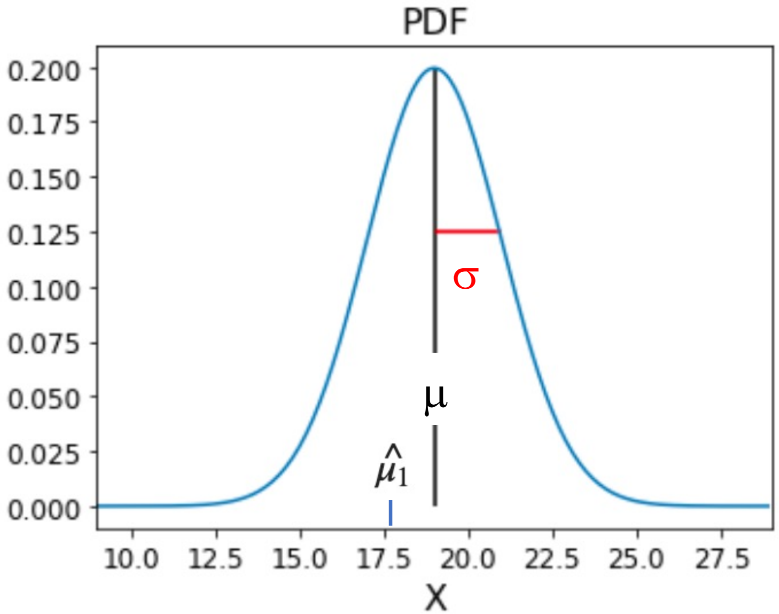


$S_1 = \{x_1, x_2, x_3, x_4\}$

Confidence interval from an estimator

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$



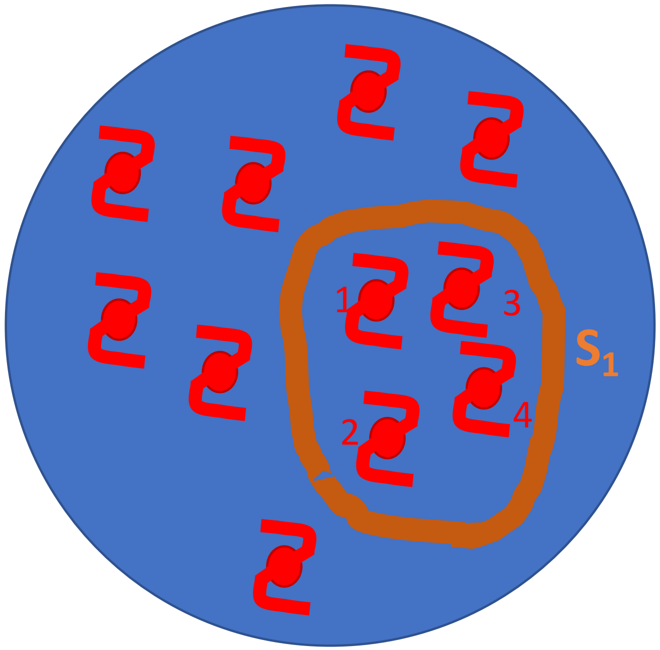


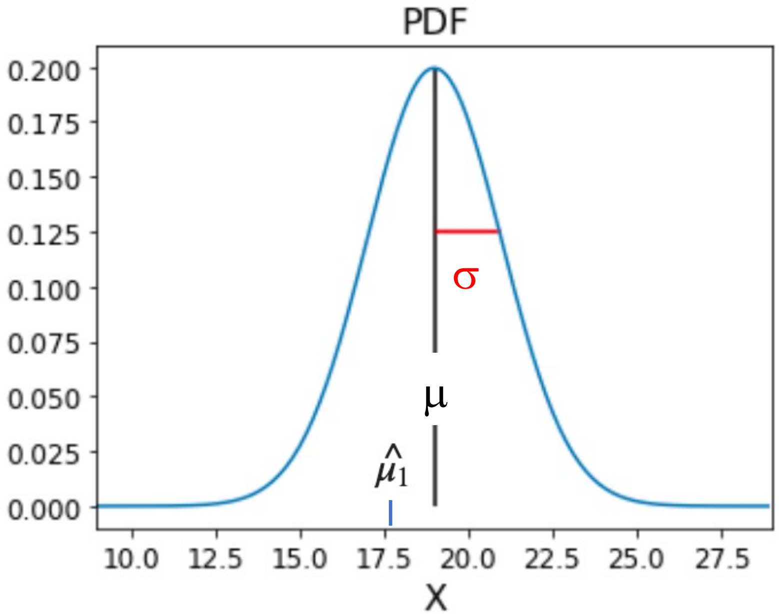
$S_1 = \{x_1, x_2, x_3, x_4\}$

$$S_1 = \{19.45, 16.20, 16.43, 19.10\}$$
$$\hat{\mu}_1 = 17.8$$

Confidence interval from an estimator

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





$S_1 = \{x_1, x_2, x_3, x_4\}$

$$S_1 = \{19.45, 16.20, 16.43, 19.10\}$$
$$\hat{\mu}_1 = 17.8$$

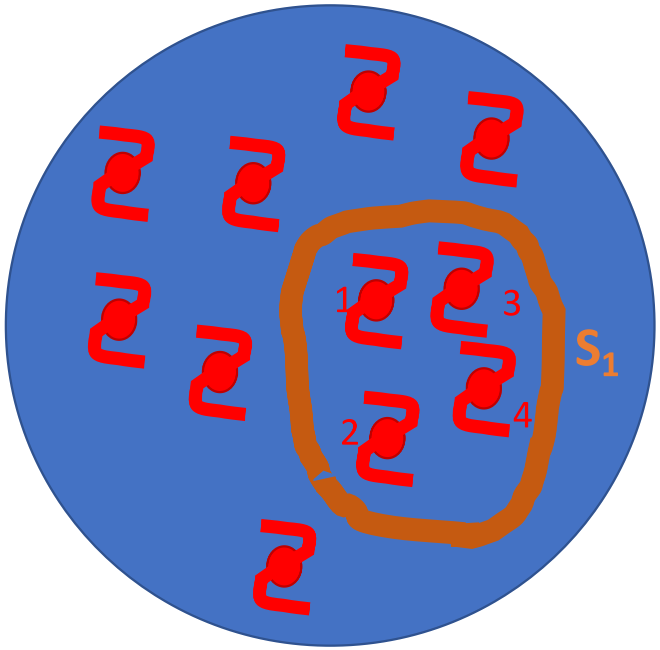
95.5 % CI on sample mean?

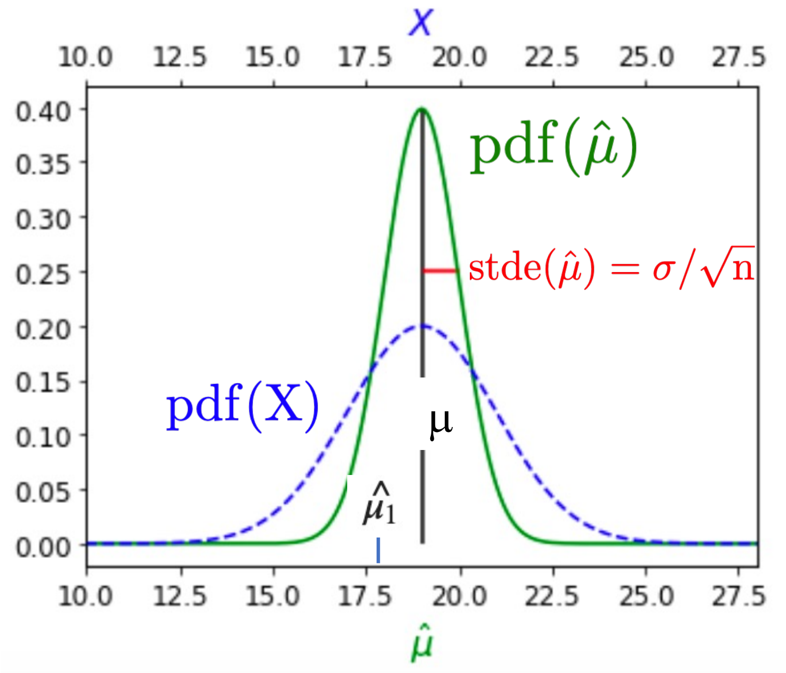
We need to look at the distribution of the sample mean

Confidence interval on the mean

95.5 % CI on $\hat{\mu}$?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

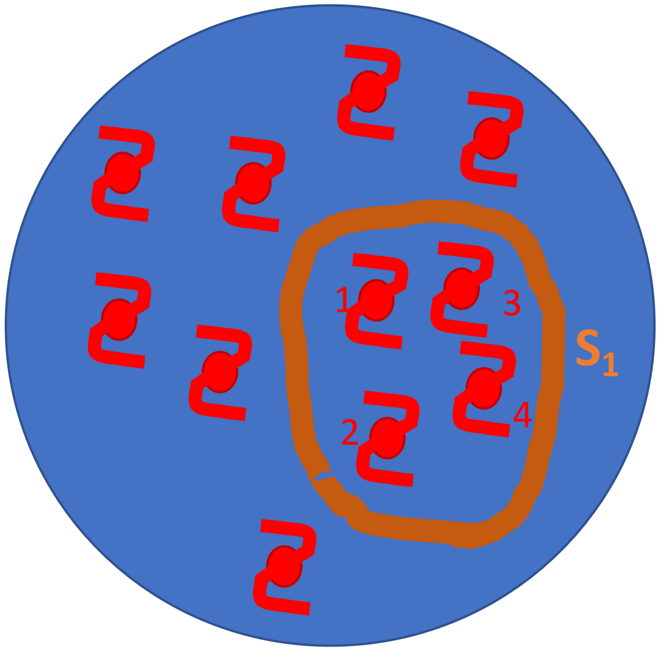
$$\boxed{\hat{\mu} \sim N(\mu, \sigma/\sqrt{n})}$$

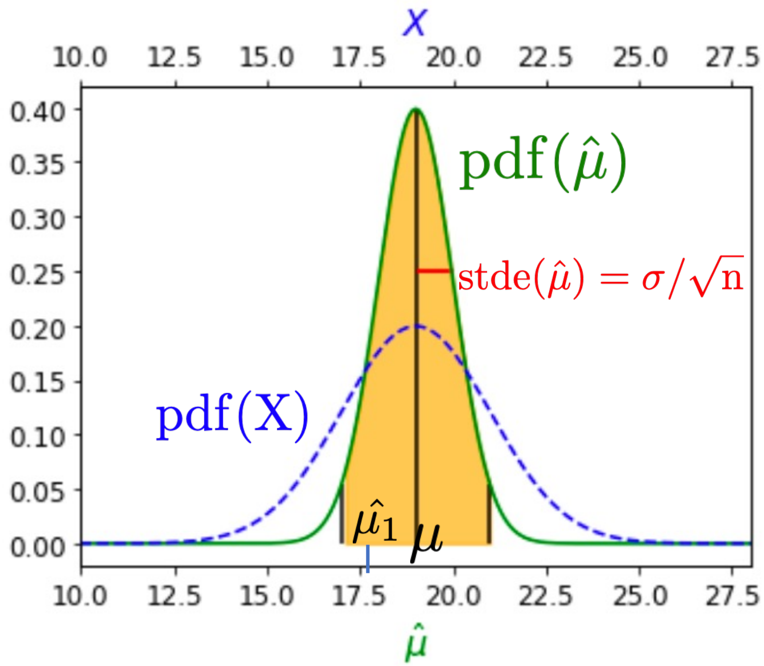
$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

Confidence interval on the mean

95.5 % CI on $\hat{\mu}$?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

$$\boxed{\hat{\mu} \sim N(\mu, \sigma/\sqrt{n})}$$

$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

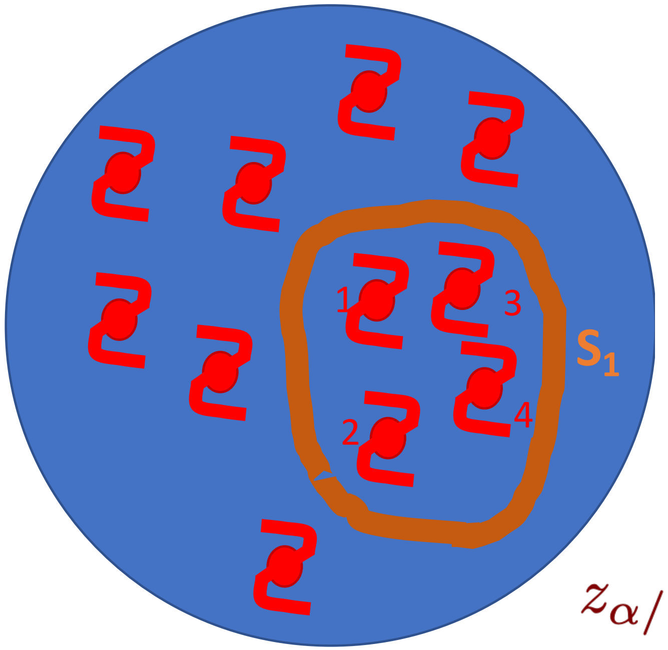
$$p(\hat{\mu}_1 \in [\mu \pm 2\,\mathrm{stde}]) = 0.955$$

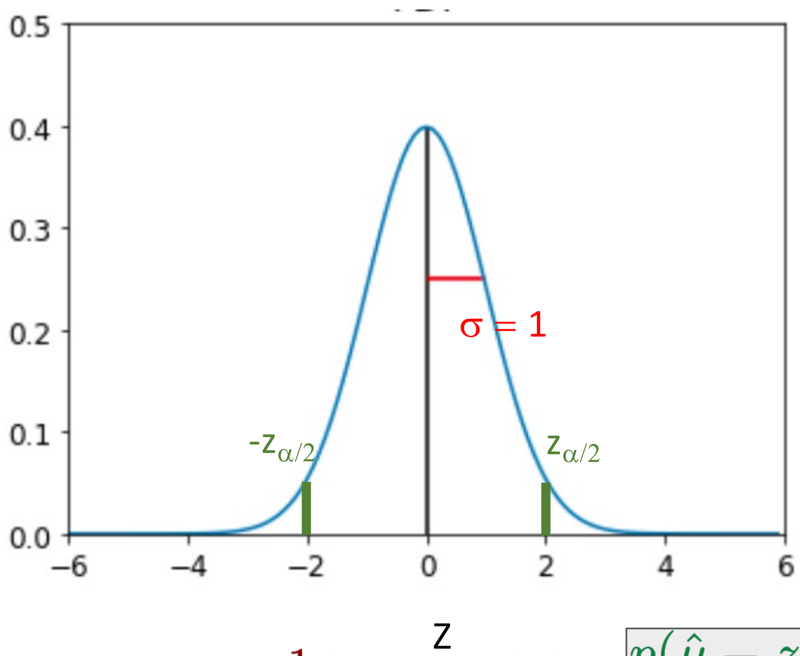
$$p(\mu \in [\hat{\mu}_1 \pm 2\,\mathrm{stde}]) = 0.955$$

Confidence interval on the mean

95.5 % CI on μ ?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

$$Z = \frac{\mu - \hat{\mu}}{\sigma/\sqrt{\mathrm{n}}} \sim N(0,1)$$

$p\,(-z_{\alpha/2} < Z < z_{\alpha/2}) = \alpha * 100\ \%$

$p\,(-z_{\alpha/2} * \mathrm{stde} < \mu - \hat{\mu} < z_{\alpha/2} * \mathrm{stde})$

$= \quad \alpha * 100\ \%$

$$z_{\alpha/2} = CDF^{-1}(1 - \alpha/2)$$

$$p(\hat{\mu} - z_{\alpha/2}\,\mathrm{stde} < \mu < \hat{\mu} + z_{\alpha/2}\,\mathrm{stde}) = \alpha \times 100\%$$

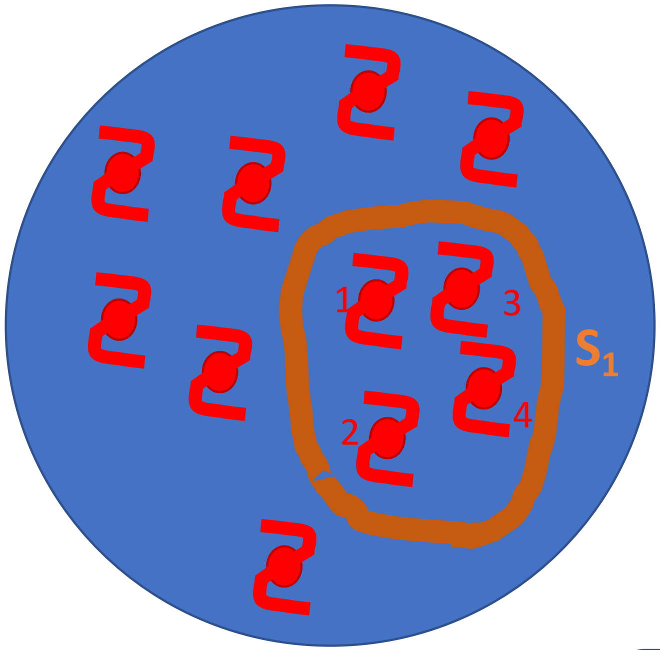
$$\mathrm{CI} \equiv [\hat{\mu} \pm \mathrm{m} \times \mathrm{stde}]$$

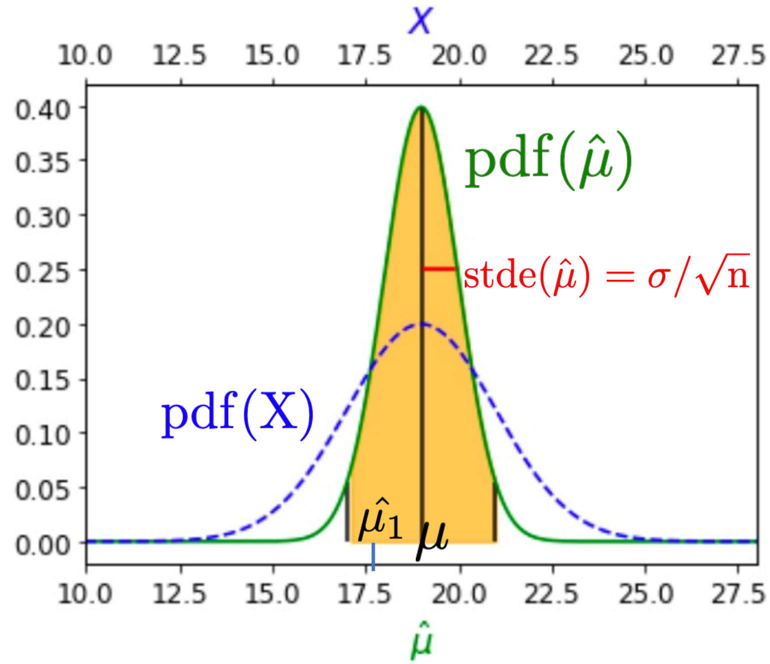
For 95.5 % CI, $z_{\alpha/2}$ = m = 2

Confidence interval on the mean

95.5 % CI on μ ?

If RV $\{X_i\}$ whose population is distributed as $N(\mu, \sigma)$





Case 1: σ is known

$$\mathrm{stde}(\hat{\mu}) = \sigma/\sqrt{\mathrm{n}}$$

$$Z = \frac{\mu - \hat{\mu}}{\sigma/\sqrt{\mathrm{n}}} \sim N(0,1)$$

Case 2: σ is **un**known

$$\mathrm{stde}(\hat{\mu}) = \mathrm{s}/\sqrt{\mathrm{n}}$$

$$t = \frac{\mu - \hat{\mu}}{\mathrm{s}/\sqrt{\mathrm{n}}} \sim t(n-1)$$

$$\mathrm{CI} \equiv [\hat{\mu} \pm \mathrm{m} \times \mathrm{stde}]$$

Go to: Student_vs_Gauss.ipynb

Confidence interval

“Generic” strategy:

- Look at the PDF of a normalised/scaled estimator of known distribution.

- Define region around your (normalised) estimator (of know the distribution) that encloses **(1 - α) X 100 %** of the area under the PDF.

- For CI around the mean:

$$\mathrm{CI}_\alpha = [\,\mu - \mathrm{q}_{\alpha/2}\,\mathrm{stde}, \mu + \mathrm{q}_{\alpha/2}\,\mathrm{stde}]$$

$$q_{\alpha/2} = CDF^{-1}(1-\alpha/2)$$

For CI on other statistics (e.g. variance, difference between 2 means, proportions) : see Frequentist_inference_03.ipynb
(Material in Frequentist_inference_03.ipynb is supplementary material)

Go to: Sect. II.2. of the notebooks