Games in Logic

Goal: check which properties / languages are *definable* in a logic (e.g. FO)

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Examples

- Is the property "Universe has even cardinality" definable in FO(E) ?
- Is the class of "Strongly connected graphs" definable in FO(E) ?
- Is the language $L=(AA)^*$ definable in FO(≤,A,B) ?

Warmup — The evaluation game

Goal: check whether $M \models \phi$

```
Model-check(φ, M)
    if φ = R(x₁,...,xₖ) then
        if (x₁ᴹ,...,xₖᴹ) ∈ Rᴹ then
            return true
        else
            return false
    else if φ = φ₁ ∨ φ₂ then
        return Model-check(φ₁, M) OR
                 Model-check(φ₂, M)
    else if ...
    ...
    else if φ = ∃x φ' then
        for u ∈ Uᴹ do
            if Model-check(φ', M[x:=u]) then
                return true
        return false
    else if φ = ∀x φ' then
        for u ∈ Uᴹ do
            if NOT Model-check(φ', M[x:=u]) then
                return false
        return true
```

Warmup — The evaluation game

Goal: check whether $M \vDash \phi$

Construct a <u>two-player game</u> $G_{\phi,M}$ whose <u>winner</u> determines whether $M \vDash \phi$

```
Model-check(φ, M)
    if φ = R(x1,...,xk) then
        if (x1^M,...,xk^M) ∈ R^M then
            return true
        else
            return false
    else if φ = φ1 v φ2 then
        return Model-check(φ1, M) OR
               Model-check(φ2, M)
    else if ...
    ...
    else if φ = ∃x φ' then
        for u ∈ U^M do
            if Model-check(φ', M[x:=u]) then
                return true
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Players: Eve, Adam

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Warmup — The evaluation game

Goal: check whether $M \models \phi$

Players: Eve, Adam

Arena: subformulas α of ϕ

+ binding $\lambda : \mathrm{FreeVars}(\alpha) \to U^M$

Construct a two-player game $G_{\phi,M}$ whose winner determines whether $M \models \phi$

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Recall: negations pushed inside

$\neg\forall\phi \rightsquigarrow \exists\neg\phi \qquad \neg\exists\phi \rightsquigarrow \forall\neg\phi$

$\neg(\phi \wedge \psi) \rightsquigarrow \neg\phi \vee \neg\psi \qquad \ldots$

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At each position (α, λ) of the arena

- if $\alpha = R(x_1,\ldots,x_k)$ then game ends, Eve wins if $(\lambda(x_1),\ldots,\lambda(x_k)) \in R^M$, otherwise Adam wins
- if $\alpha = \neg R(x_1,\ldots,x_k)$ then game ends, Adam wins if $(\lambda(x_1),\ldots,\lambda(x_k)) \in R^M$, otherwise Eve wins

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- if $\alpha = \alpha_1 \vee \alpha_2$ then Eve can choose $\alpha' \in \{\alpha_1, \alpha_2\}$, game continues at position (α', λ)

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- if $\alpha = \exists x\, \alpha'(x)$ then Eve can choose any element $u \in U^M$ to be bound to x, game continues at position (α', λ') where $\lambda' = \lambda[x := u]$

Warmup — The evaluation game

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- if $\alpha = \exists x\, \alpha'(x)$ then Eve can choose any element $u \in U^M$ to be bound to x, game continues at position (α', λ') where $\lambda' = \lambda[x:=u]$
- if $\alpha = \forall x\, \alpha'(x)$ then Adam can choose any element $u \in U^M$ to be bound to x, game continues at position (α', λ') where $\lambda' = \lambda[x:=u]$

Warmup — The evaluation game

Goal: check whether $M \models \phi$

Players: Eve, Adam

Lemma
$M \models \phi$ iff Eve has a <u>strategy</u> to win $G_{\phi,M}$

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Definability vs elementary equivalence vs n-equivalence

Notation P : property (i.e. set of models), M : model, ϕ : FO formula

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intuitively,

no formula can distinguish M *from* M'

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Lemma If there are M,M' such that
$M \in P$, $M' \notin P$, and M,M' elementary equivalent
then P is *not* definable in FO

Definability vs elementary equivalence vs n-equivalence

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Definability vs elementary equivalence vs n-equivalence

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2) M,M' <u>elementary equivalent</u> if for every ϕ $\quad M \models \phi$ iff $M' \models \phi$

3) ϕ has <u>quantifier rank</u> n if it has at most n nested quantifiers

Example $\phi = \forall x \forall y\, (\neg E(x,y) \vee (\exists z\, E(x,z)) \vee (\exists t\, E(t,y))\,)$
has quantifier rank 3 (q.r. can be $\ll$ # quantifiers)

Lemma If there are M,M' such that
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then P is *not* definable in FO

Definability vs elementary equivalence vs n-equivalence

1)	P defined by ϕ	if	for every M	$M \in P$ iff $M \models \phi$
2)	M,M' elementary equivalent	if	for every ϕ	$M \models \phi$ iff $M' \models \phi$
3)	ϕ has quantifier rank n	if	it has at most n nested quantifiers	
4)	M,M' are n-equivalent	if	for every ϕ with q.r. n	$M \models \phi$ iff $M' \models \phi$

Lemma If there are M,M' such that
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Definability vs elementary equivalence vs n-equivalence

1) P defined by ϕ if for every M: $M \in P$ iff $M \models \phi$

2) M,M' elementary equivalent if for every ϕ: $M \models \phi$ iff $M' \models \phi$

3) ϕ has quantifier rank n if it has at most n nested quantifiers

4) M,M' are n-equivalent if for every ϕ with q.r. n: $M \models \phi$ iff $M' \models \phi$

Lemma If for every n there are M,M' such that $M \in P$, $M' \notin P$, and M,M' n-equivalent
then P is *not* definable in FO

Definability vs elementary equivalence vs n-equivalence

1)	P defined by ϕ	if	for every M	$M \in P$ iff $M \models \phi$
2)	M,M' elementary equivalent	if	for every ϕ	$M \models \phi$ iff $M' \models \phi$
3)	ϕ has quantifier rank n	if	it has at most n nested quantifiers	
4)	M,M' are n-equivalent	if	for every ϕ with q.r. n	$M \models \phi$ iff $M' \models \phi$

Lemma If for every n there are M,M' such that
$M \in P$, $M' \notin P$, and M,M' n-equivalent
then P is *not* definable in FO

New goal: check whether
M,M' are n-equivalent

Construct a new game $G_{M,M'}$
whose winner determines whether
M,M' are n-equivalent

Ehrenfeucht-Fraïssé games

Duplicator **Spoiler**

Ehrenfeucht-Fraïssé games

M, M' are
n-equivalent!

Duplicator **Spoiler**

Ehrenfeucht-Fraïssé games

Duplicator

Spoiler

Ehrenfeucht-Fraïssé games

Duplicator

Spoiler

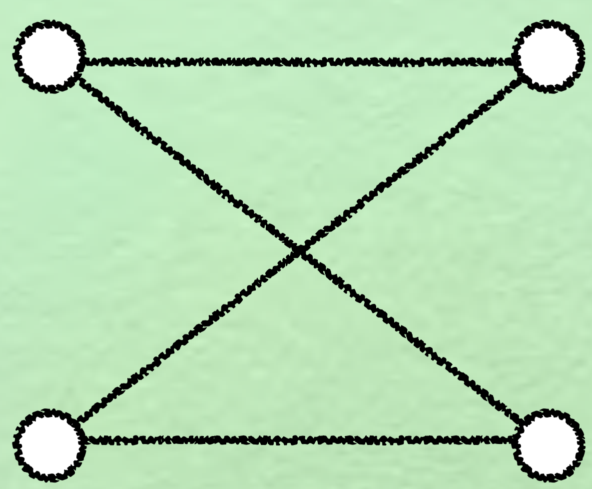
Play for n <u>rounds</u> on the arena whose <u>positions</u> are tuples

$$(u_1,\dots,u_i, v_1,\dots,v_i) \in U^M \times \dots \times U^M \times U^{M'} \times \dots \times U^{M'}$$

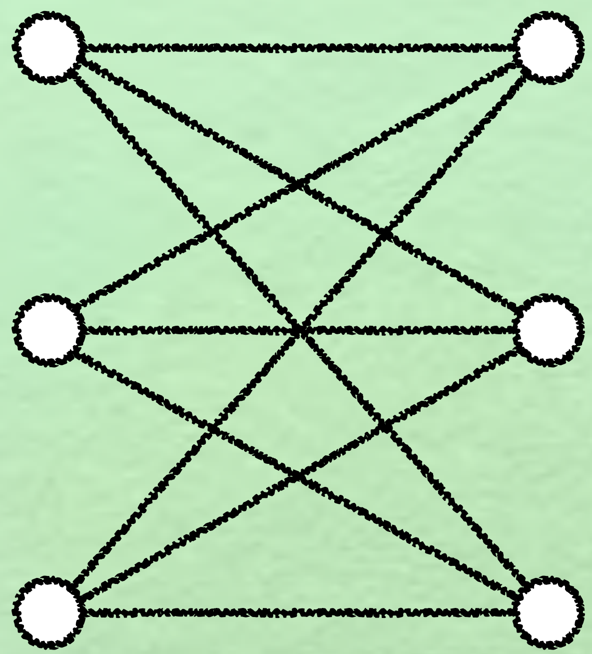
At each round i

- **Spoiler** chooses an element u_i from U^M (or v_i from $U^{M'}$)
- **Duplicator** responds with an element v_i from $U^{M'}$ (resp. u_i from U^M)
- **Duplicator** <u>survives</u> if $M\,|\,\{u_1,\dots,u_i\}$ and $M'\,|\,\{v_1,\dots,v_i\}$ are <u>isomorphic</u>

Example How many rounds can Duplicator survive ?



M



M’

Ehrenfeucht-Fraïssé games

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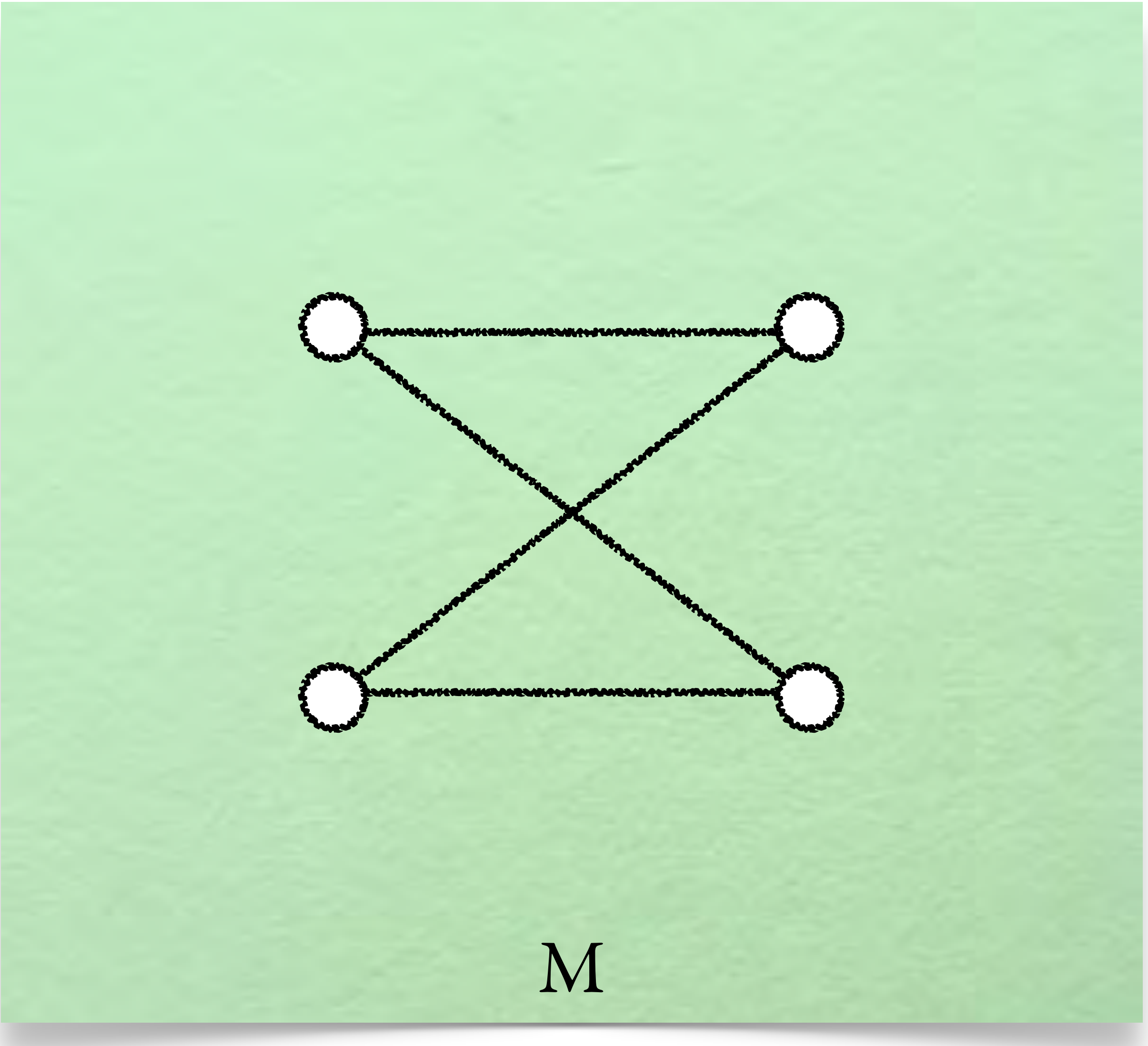

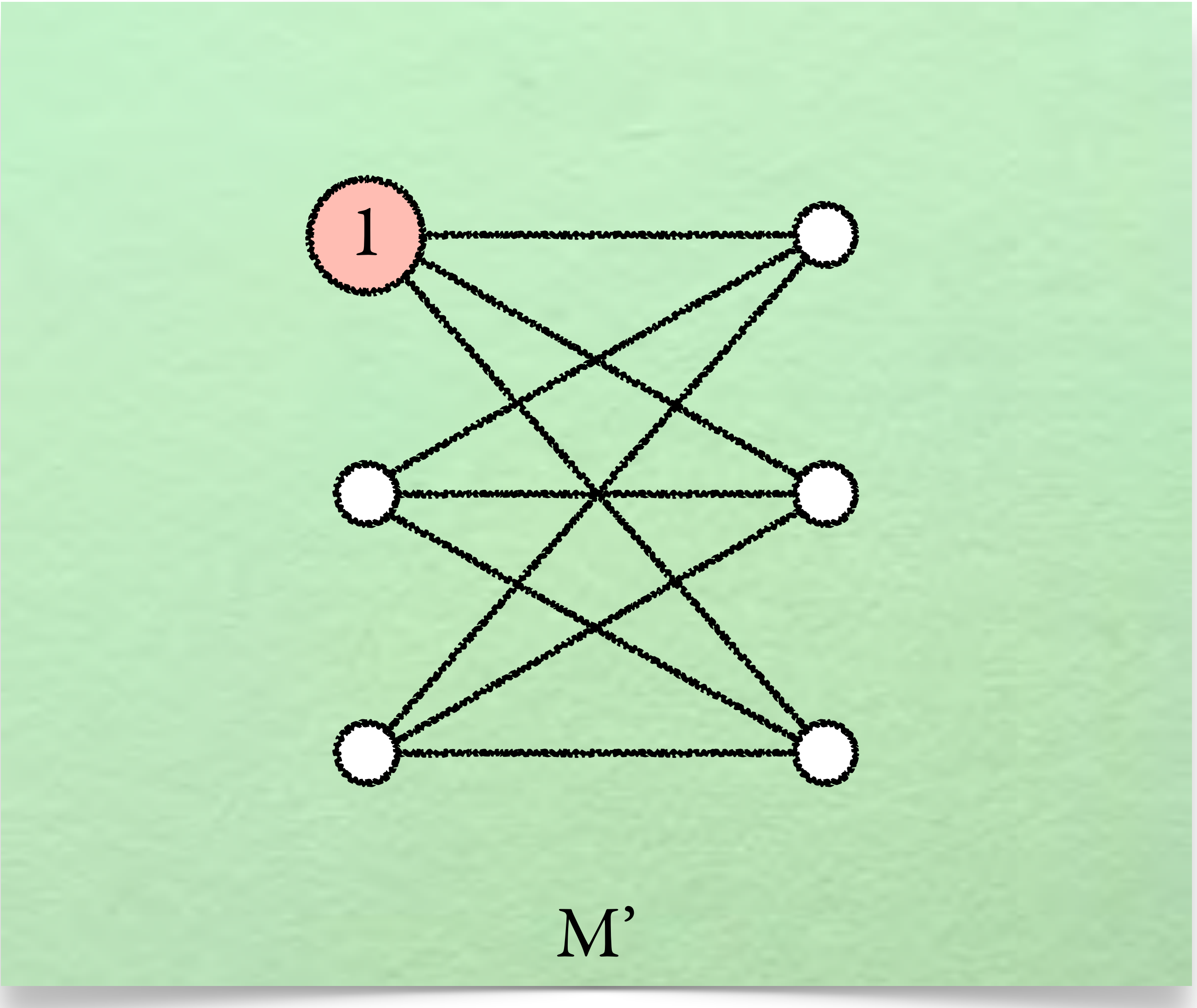

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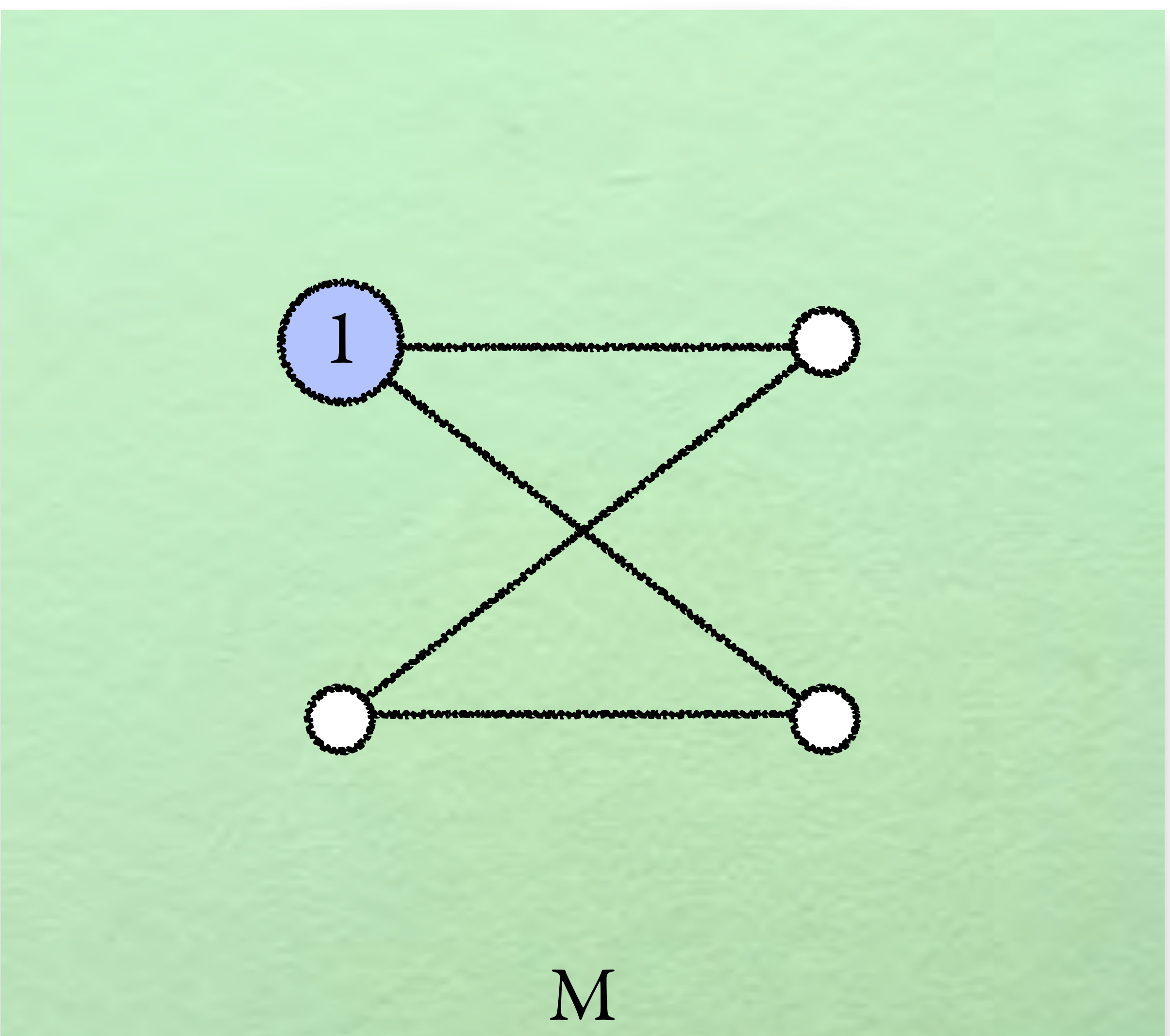

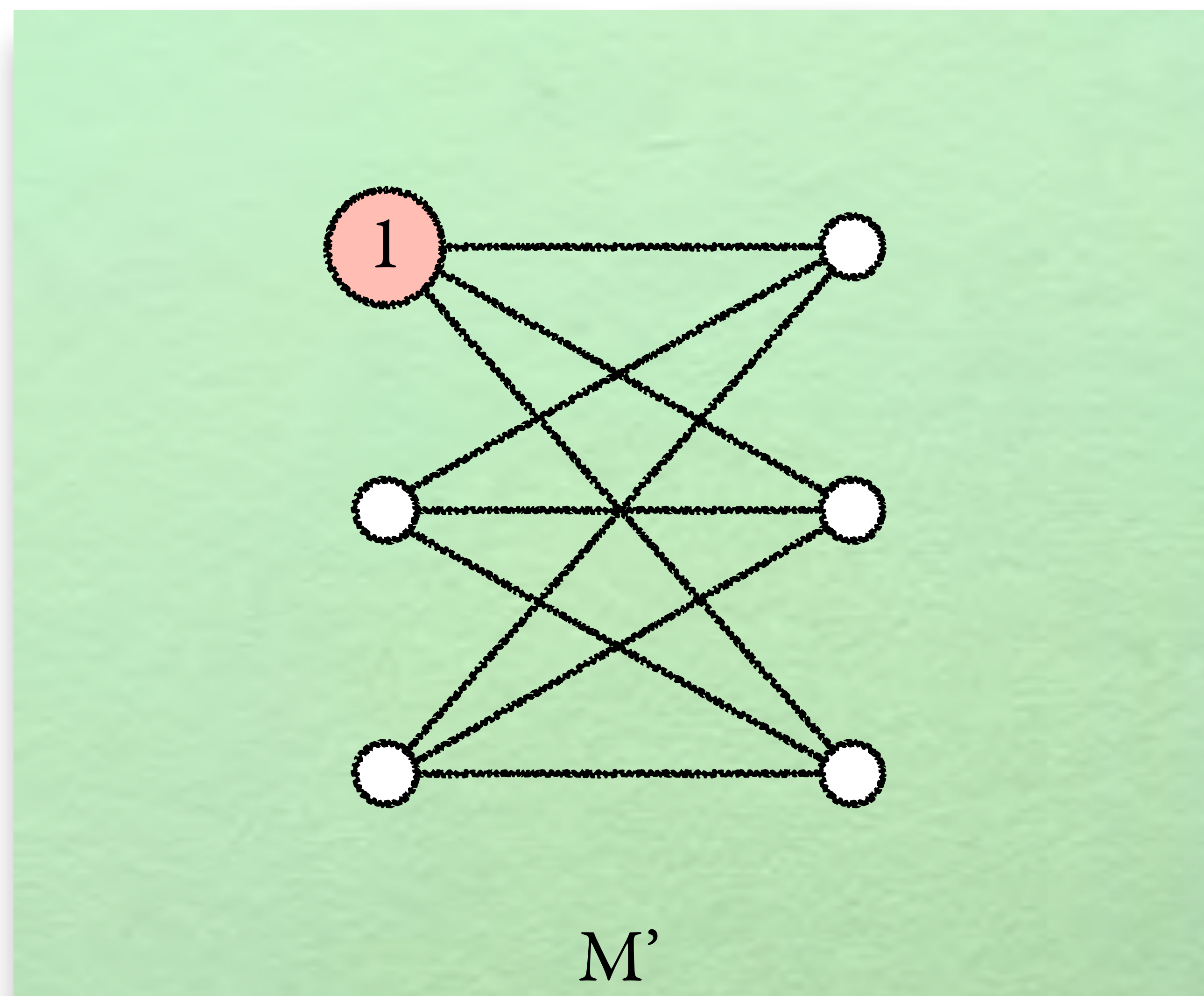

Ehrenfeucht-Fraïssé games

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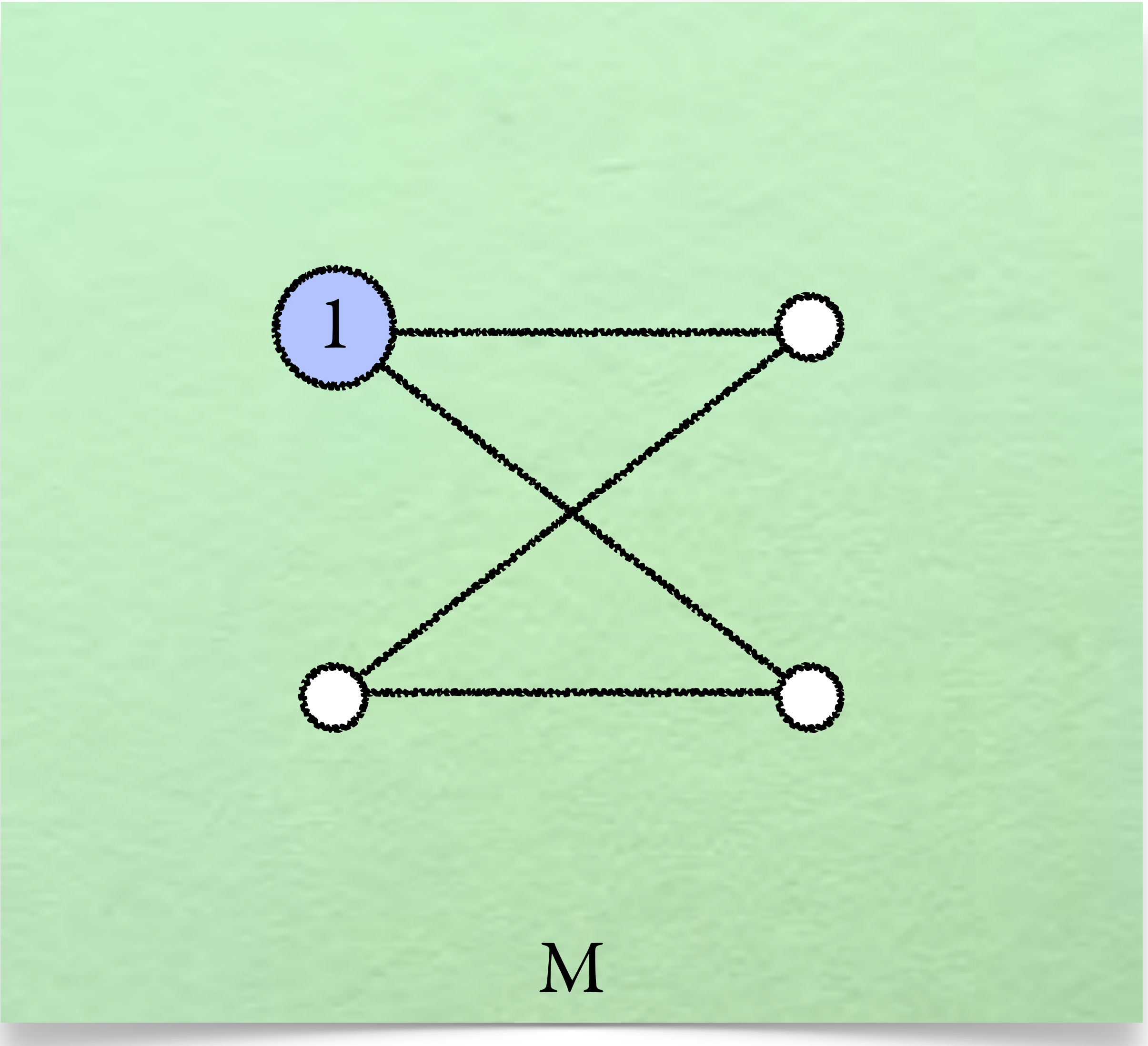

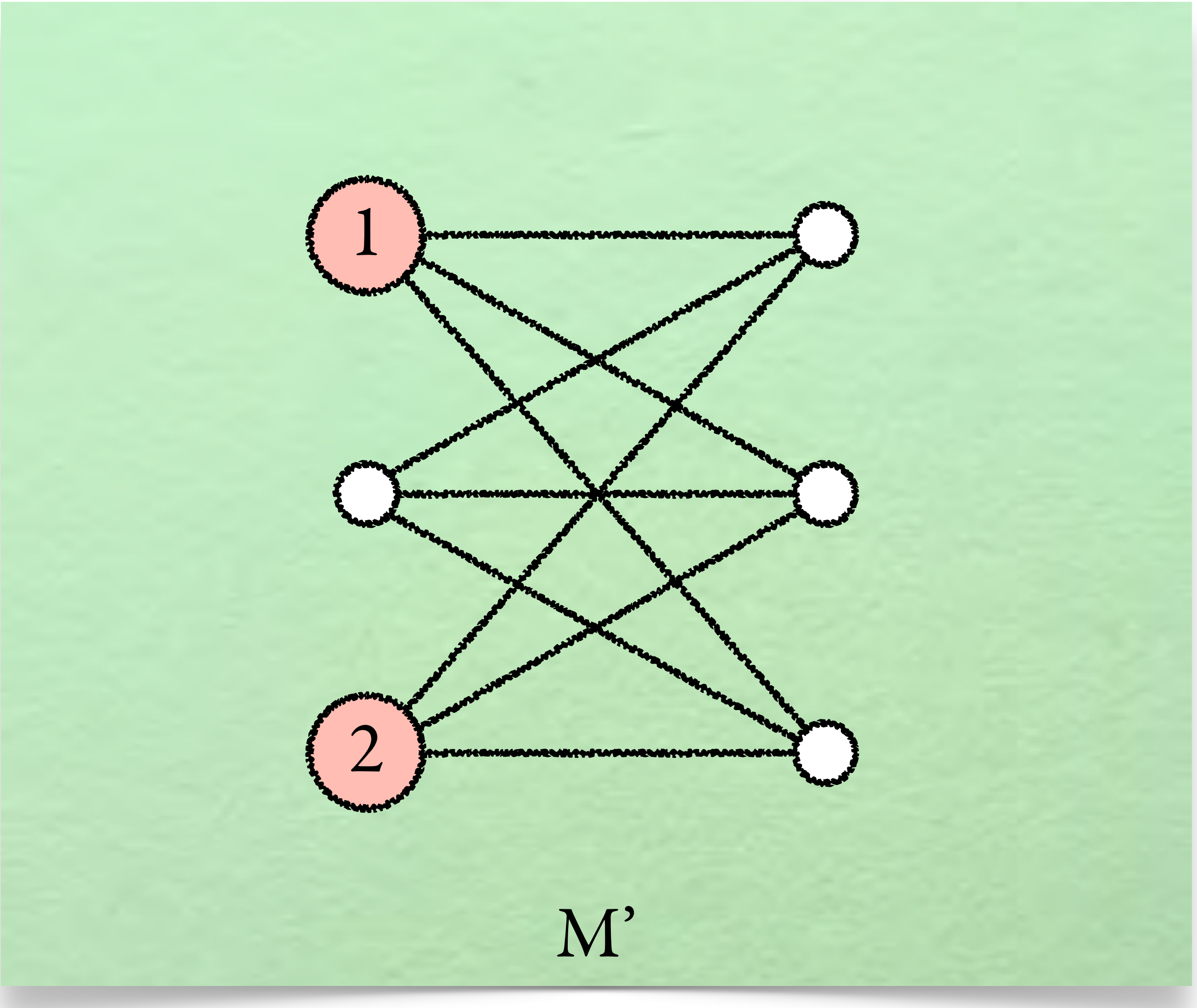

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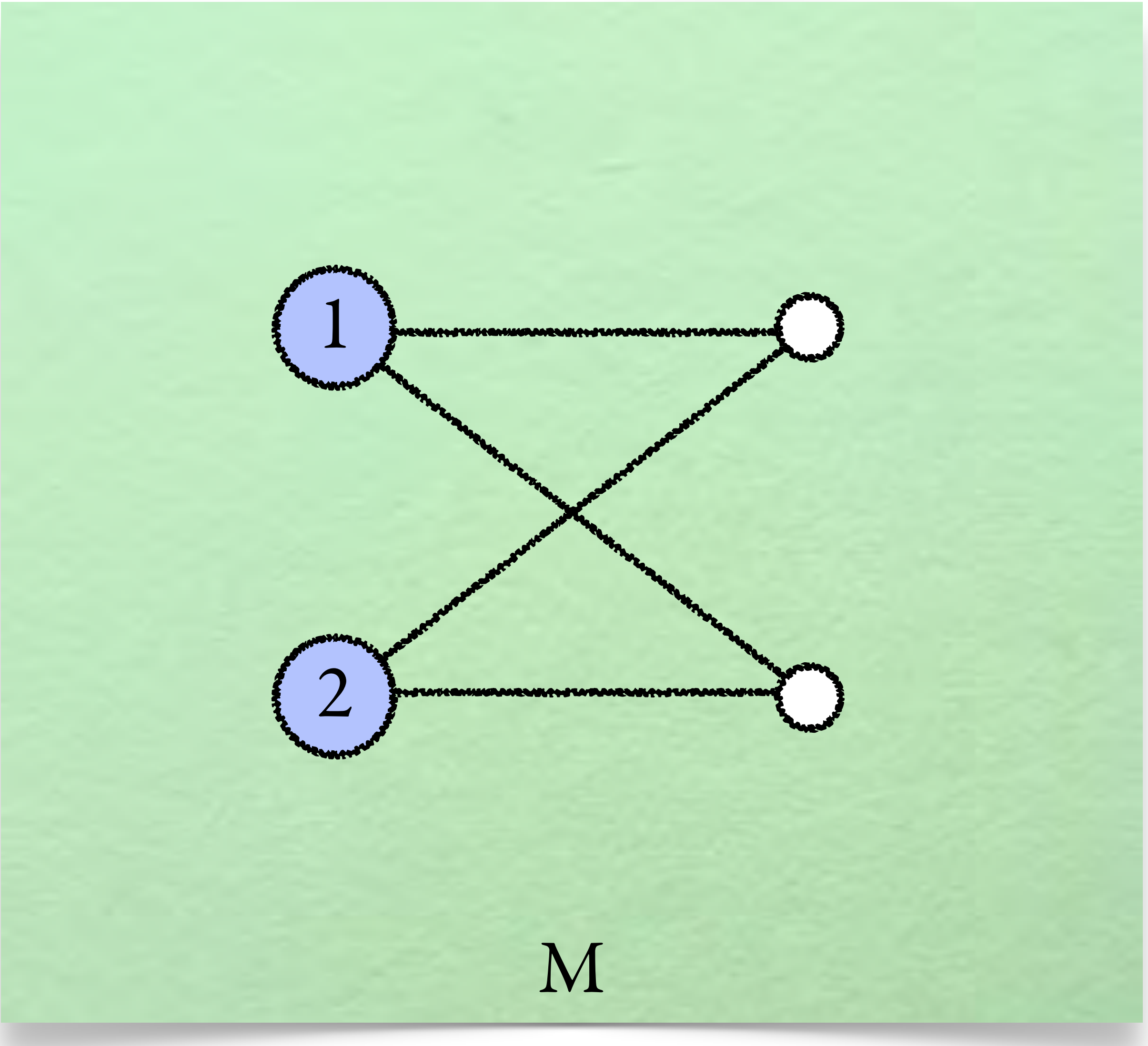

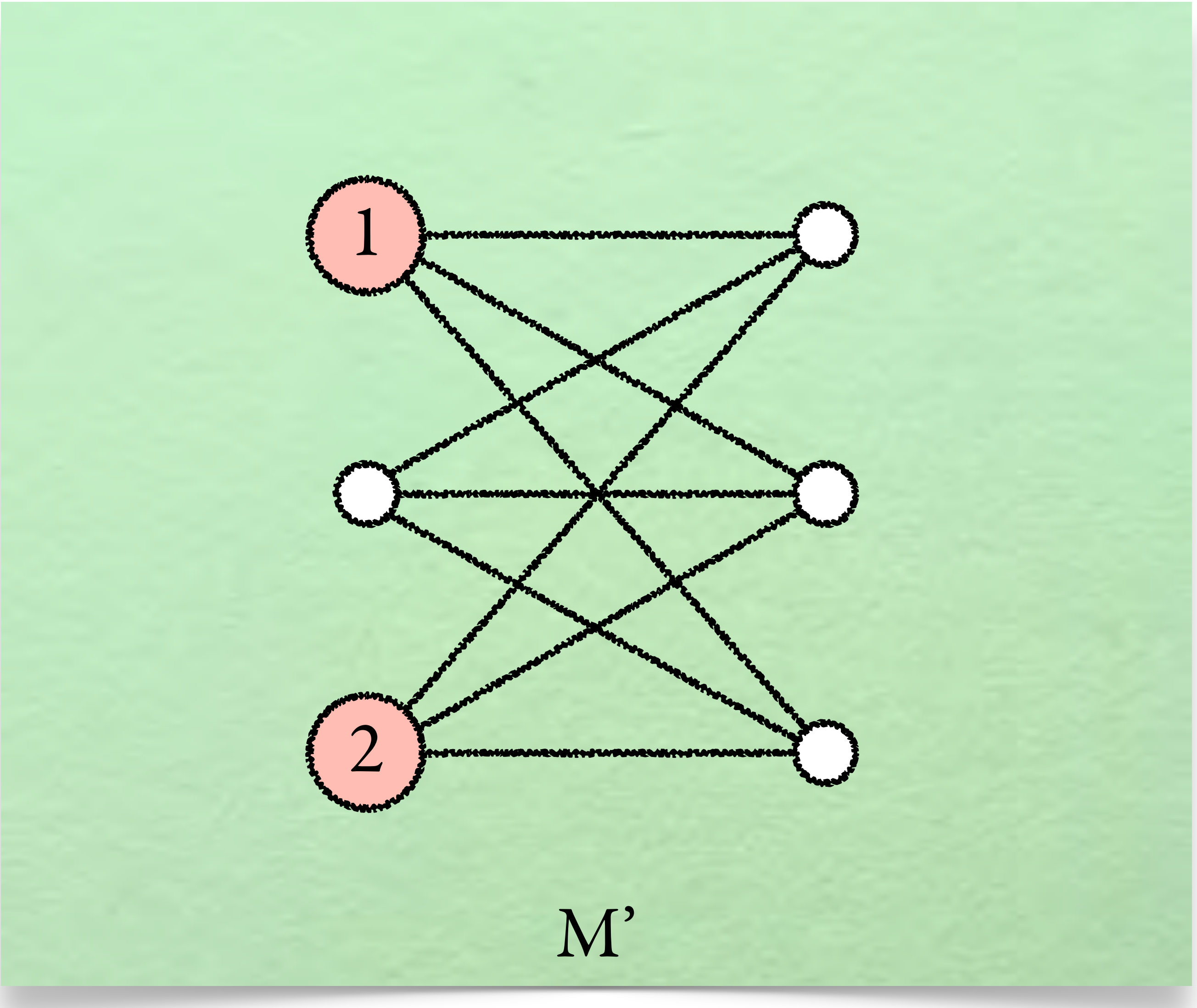

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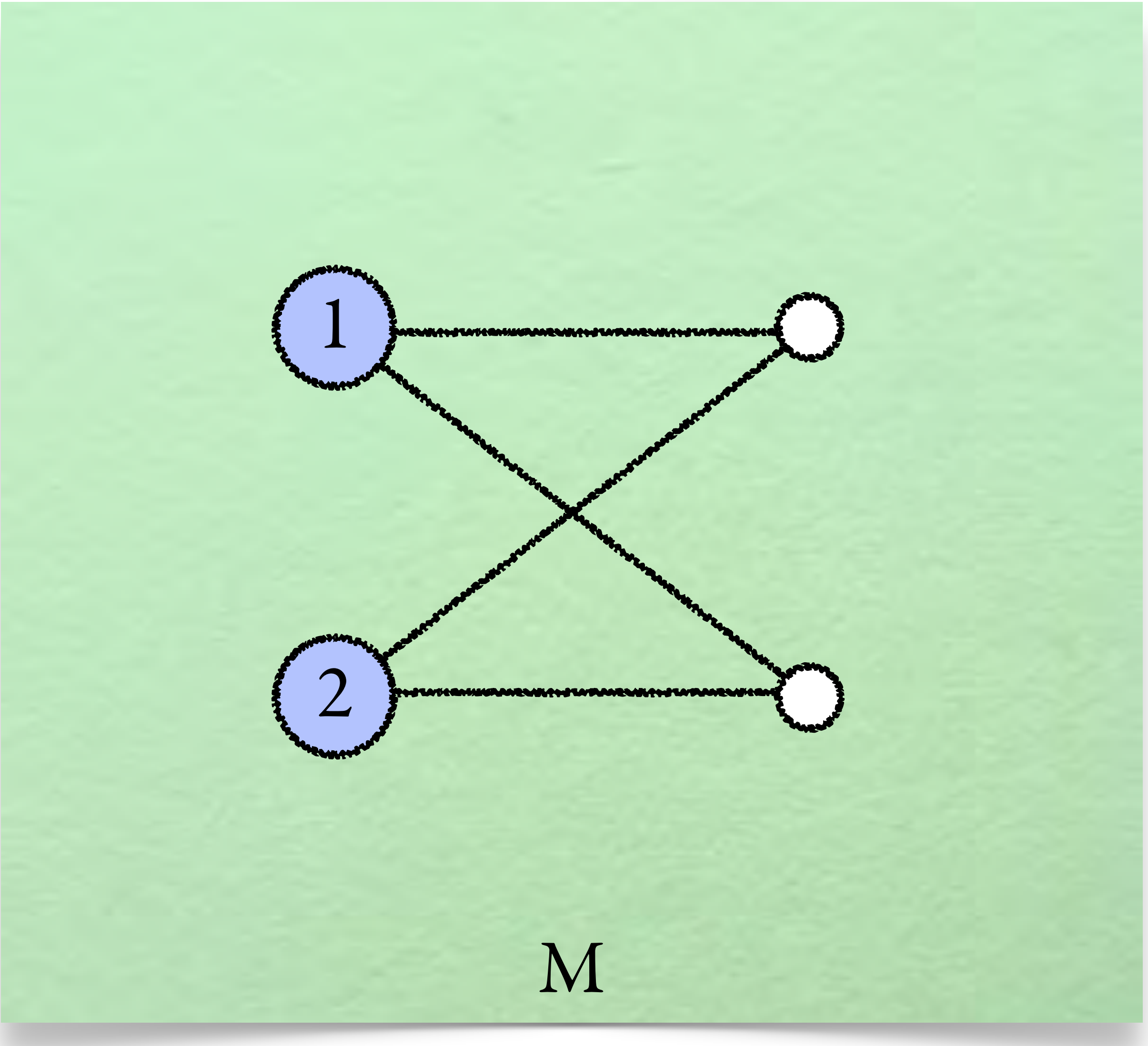

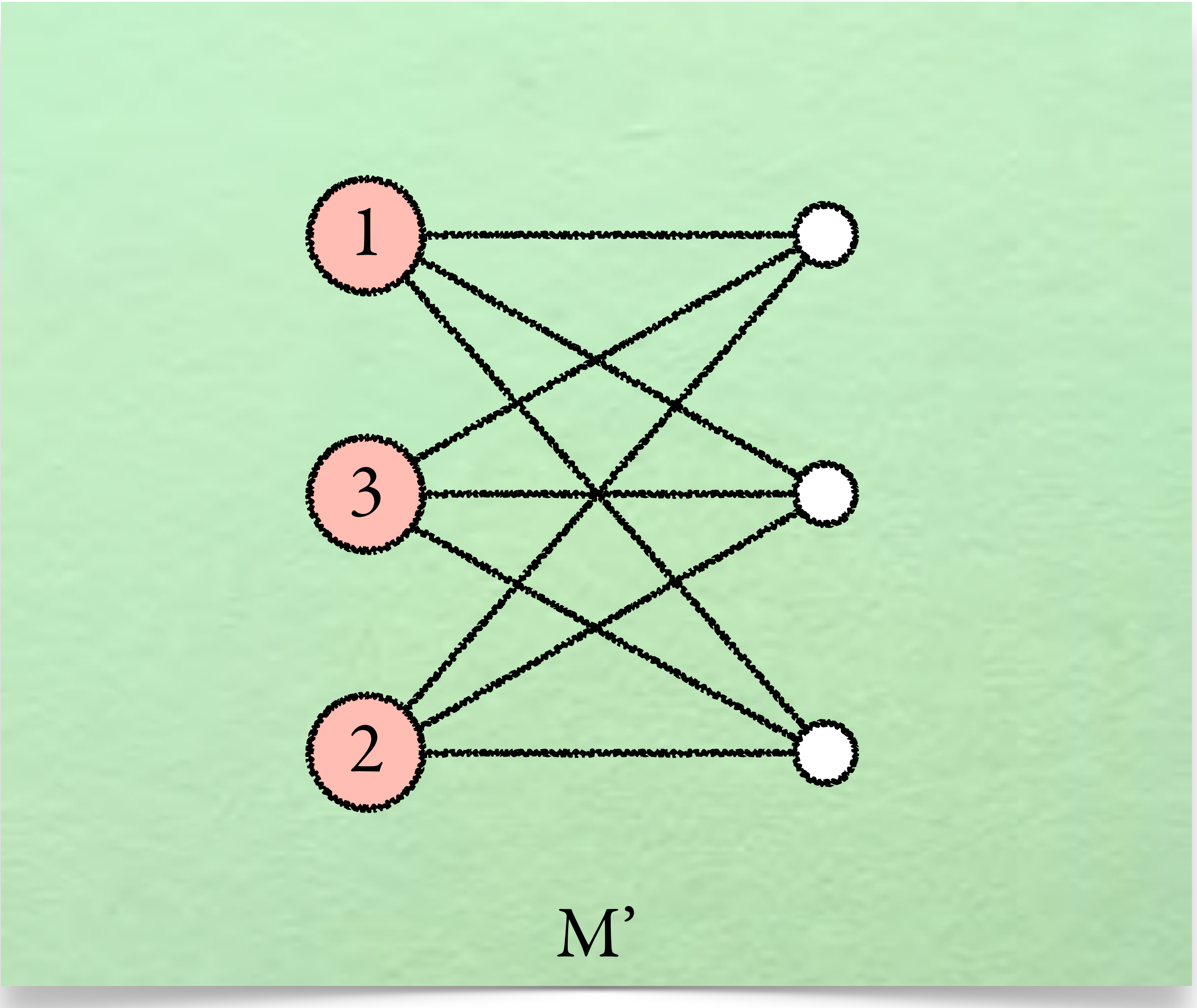

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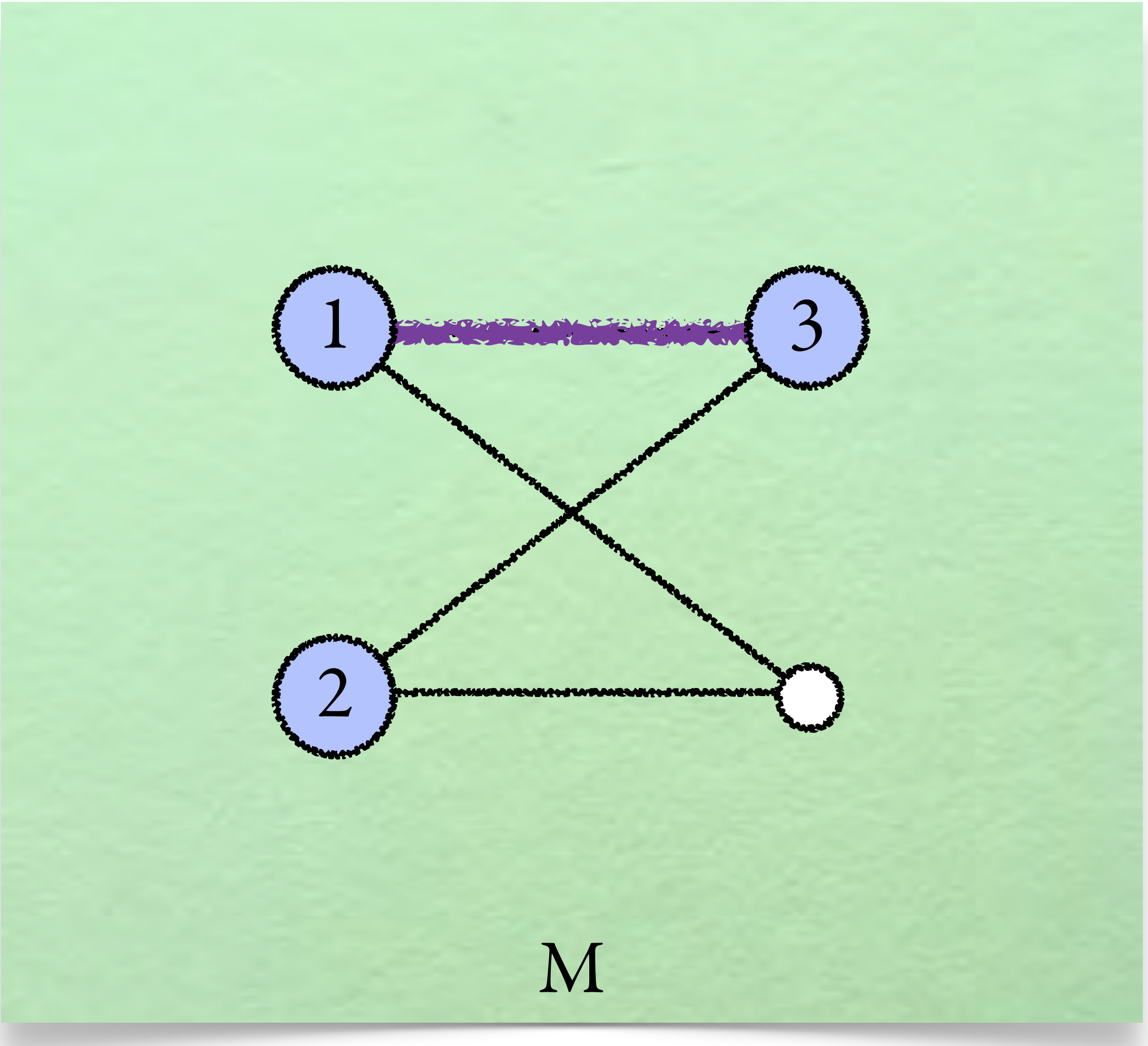

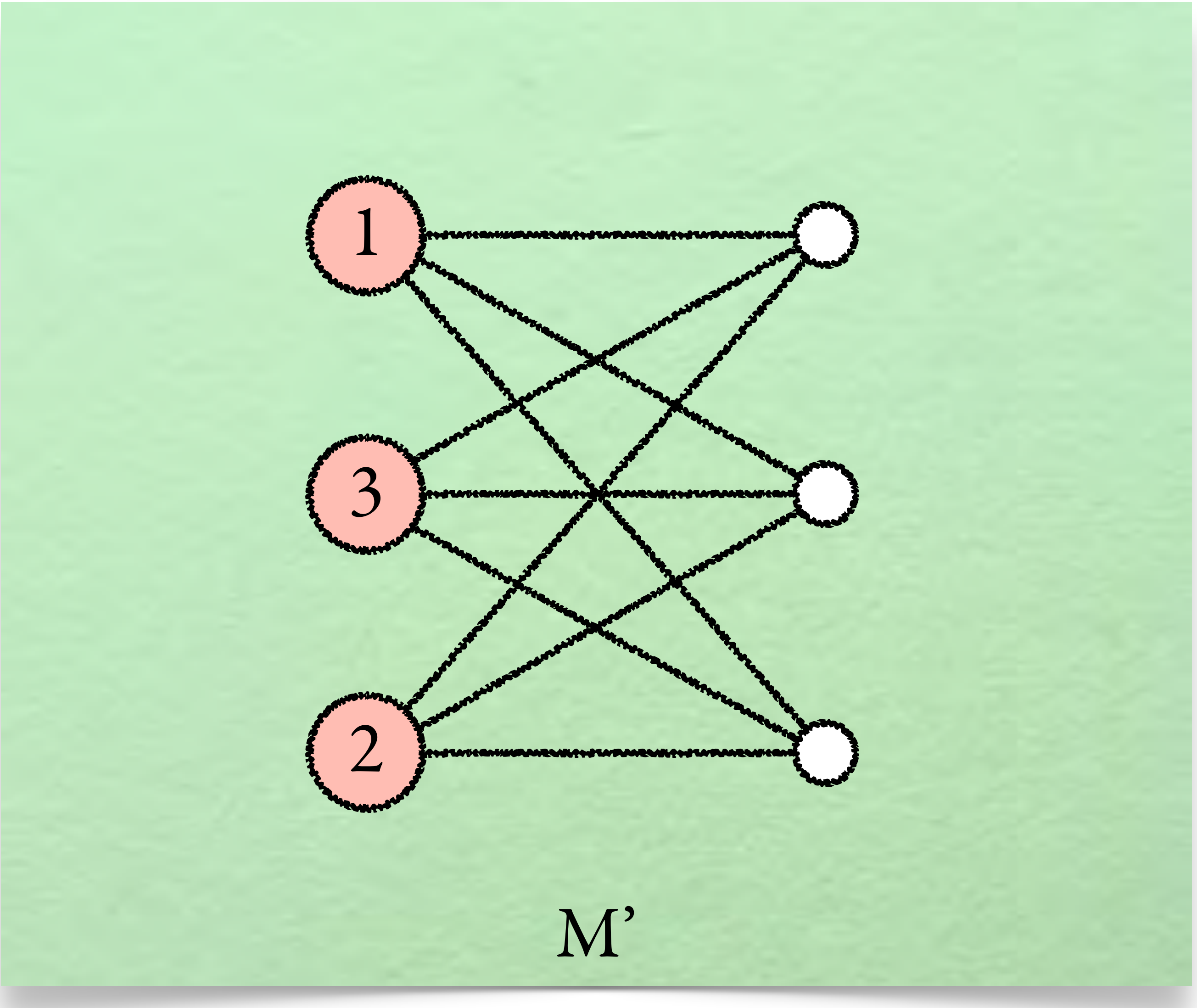

Ehrenfeucht-Fraïssé games

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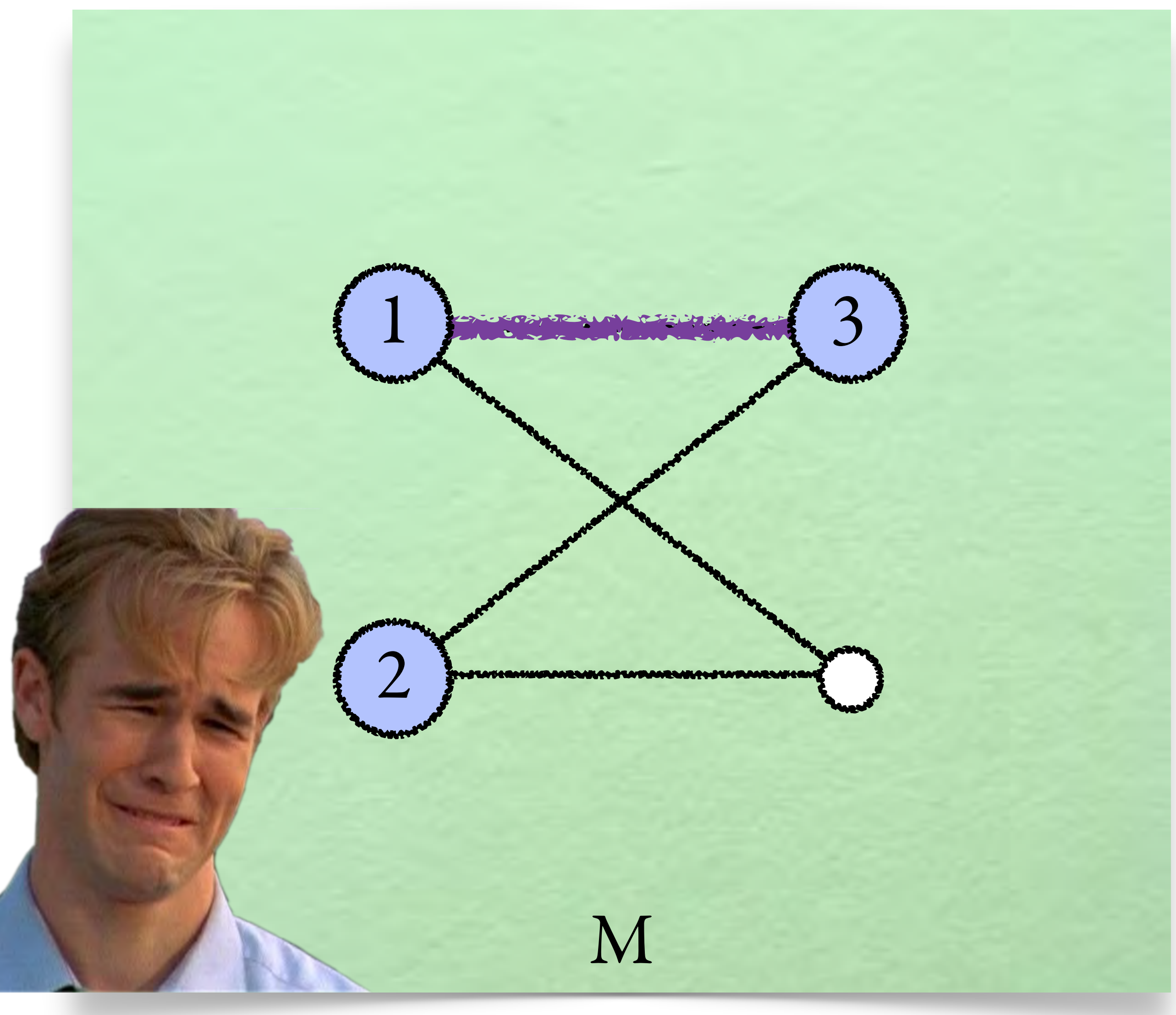

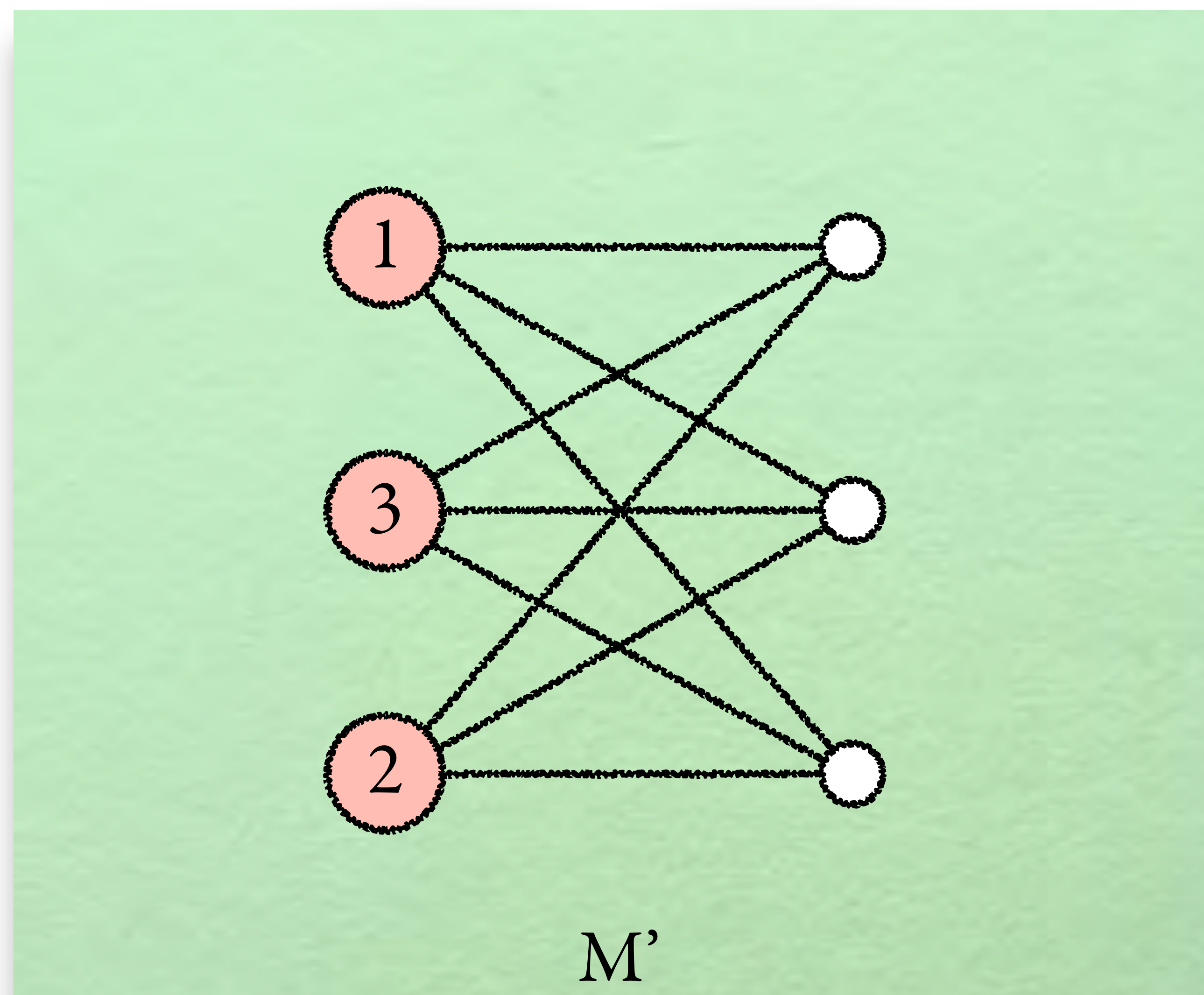

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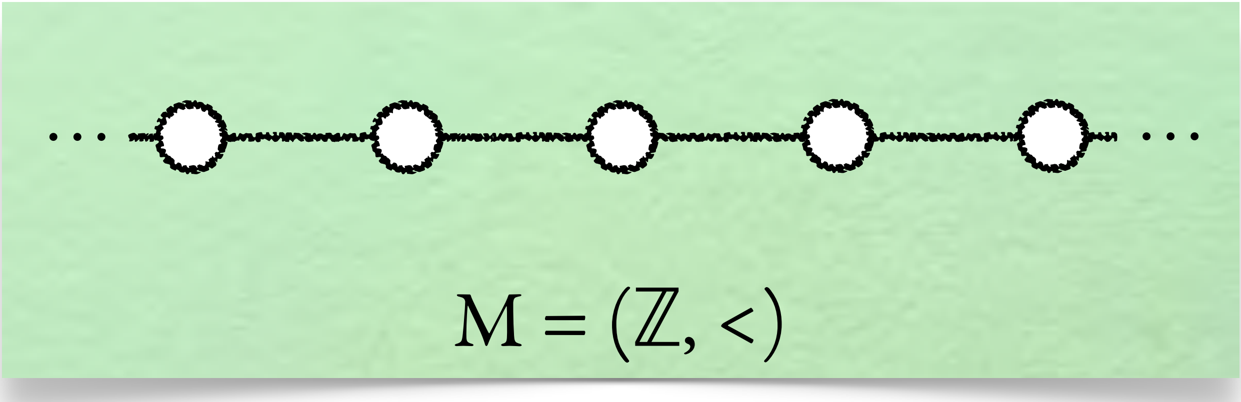

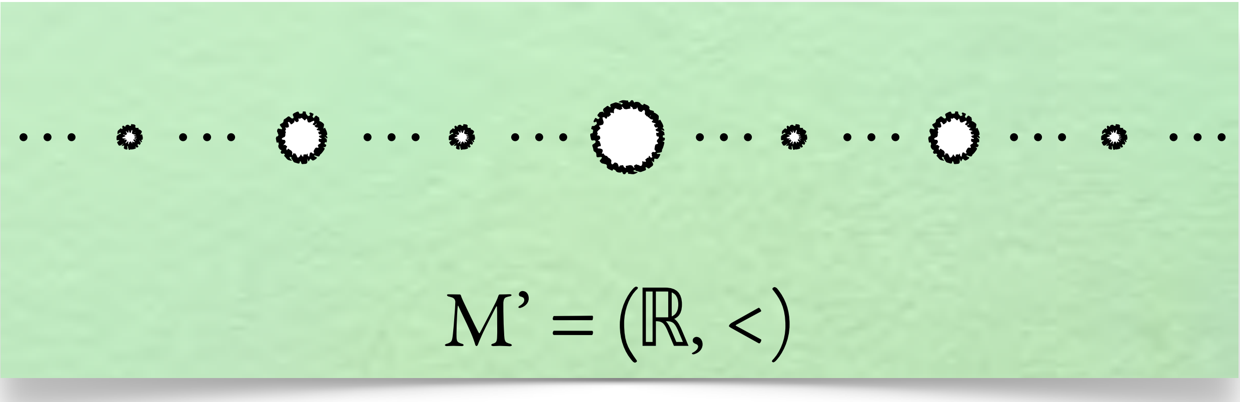

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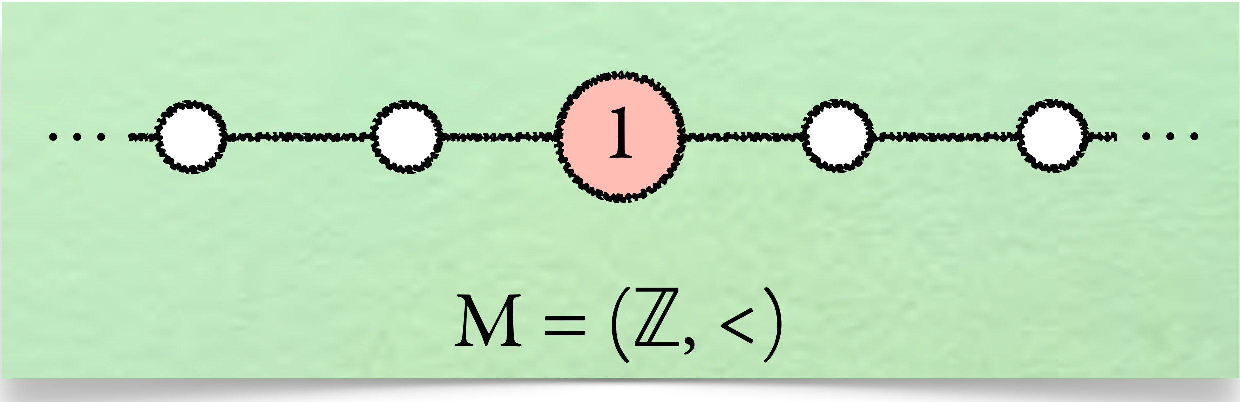

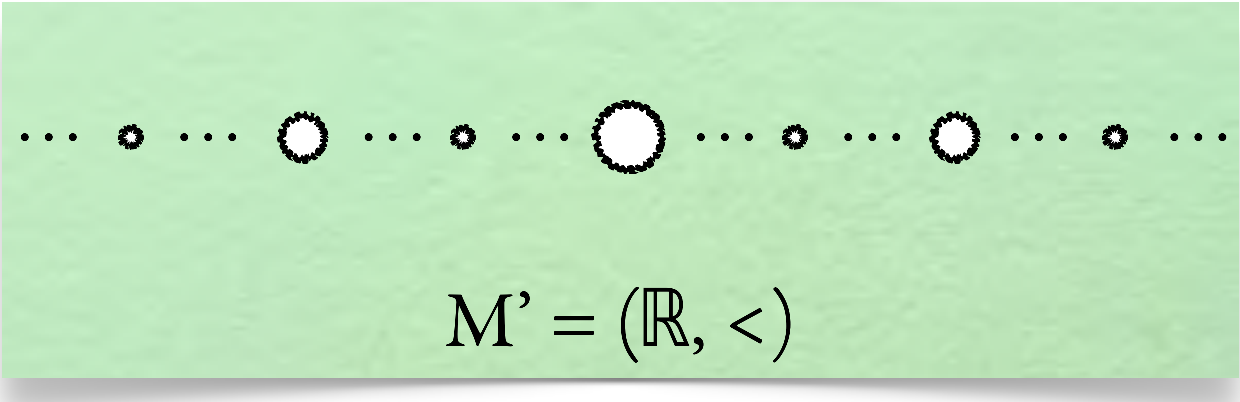

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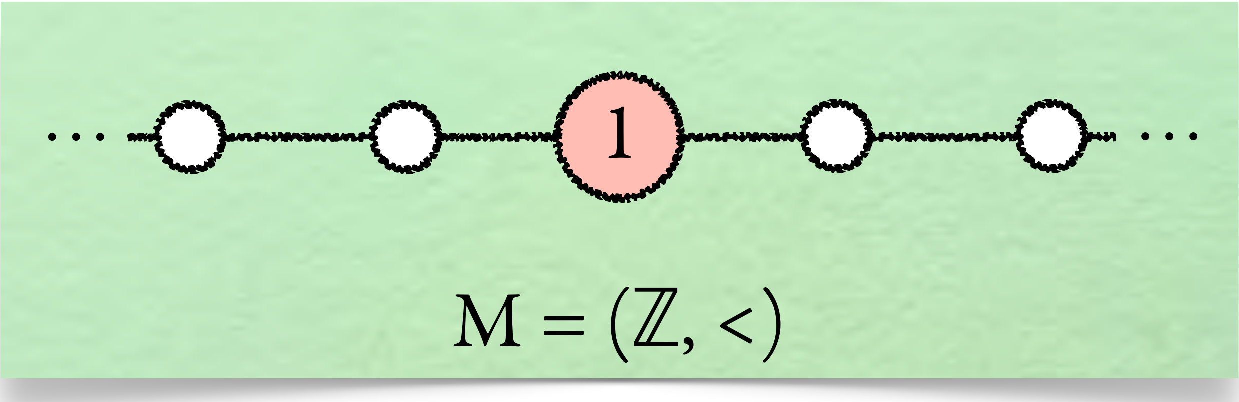

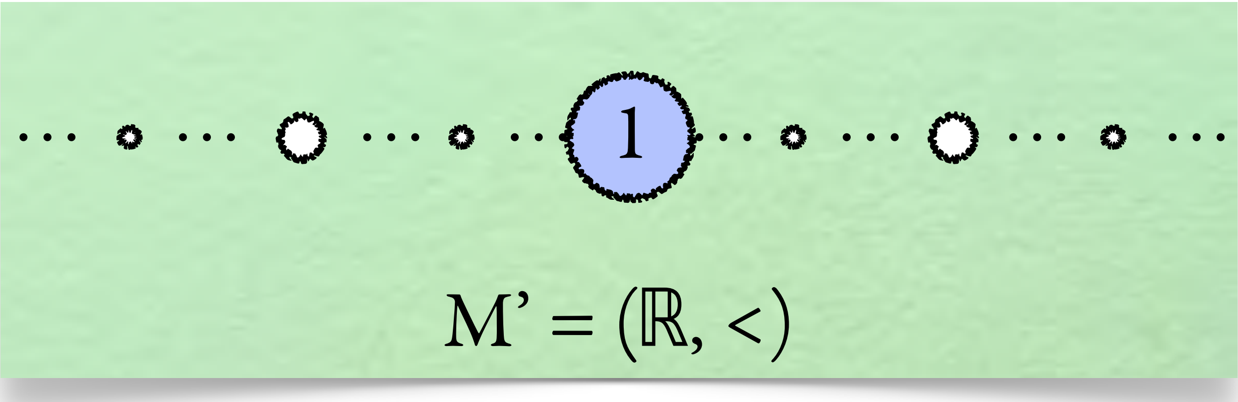

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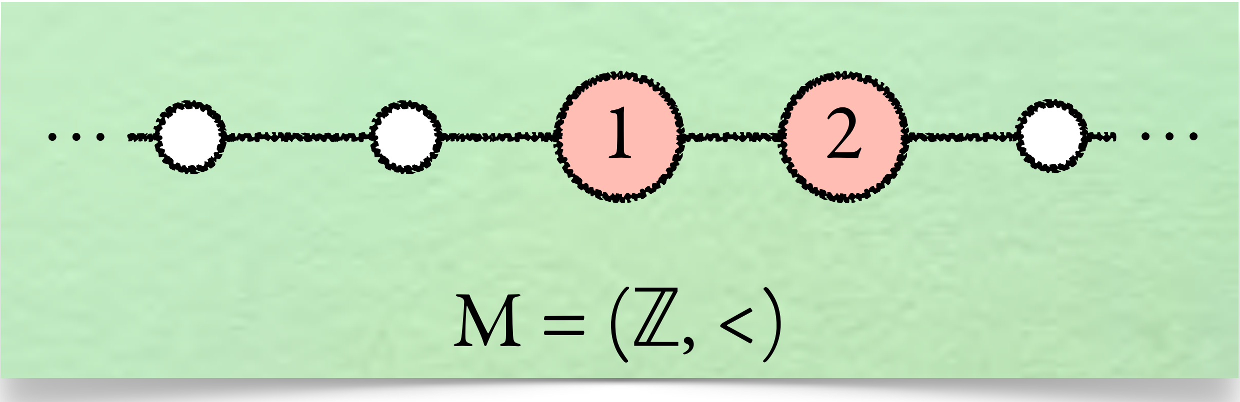

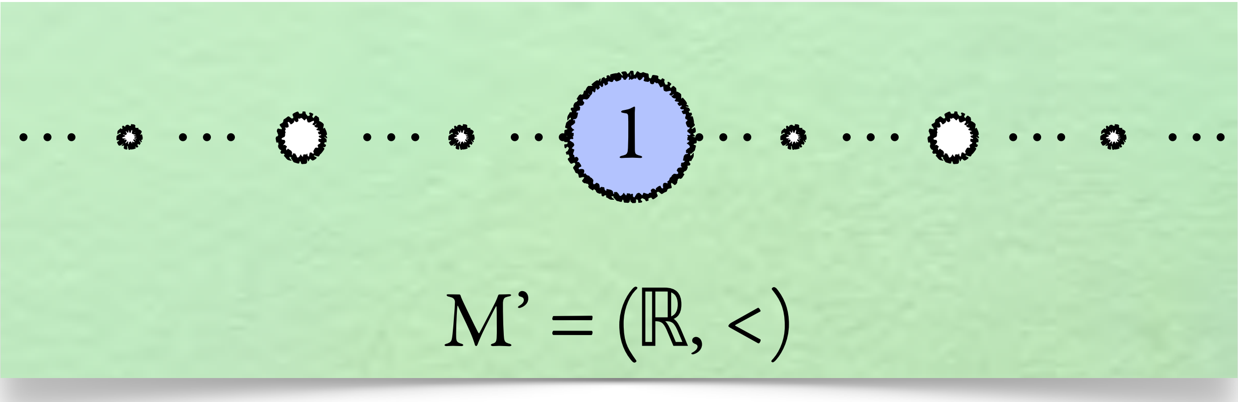

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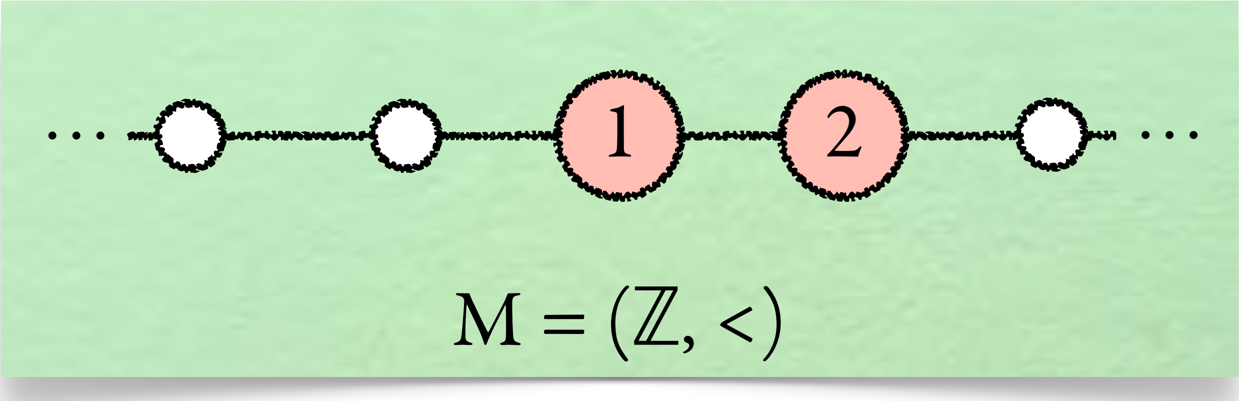

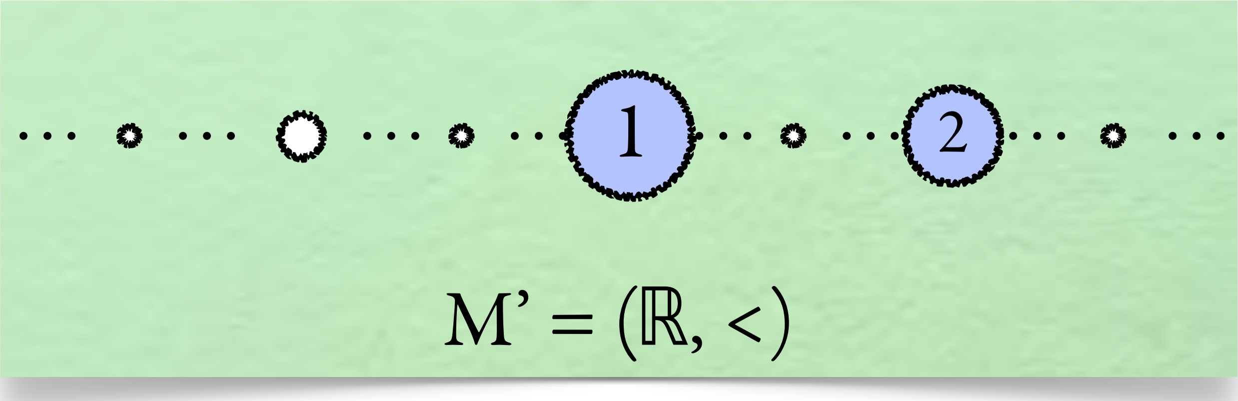

Ehrenfeucht-Fraïssé games

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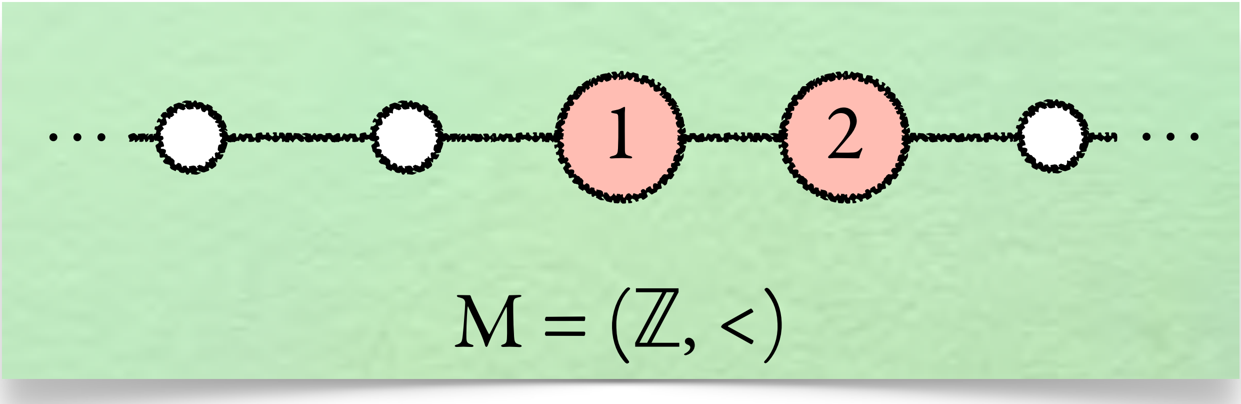

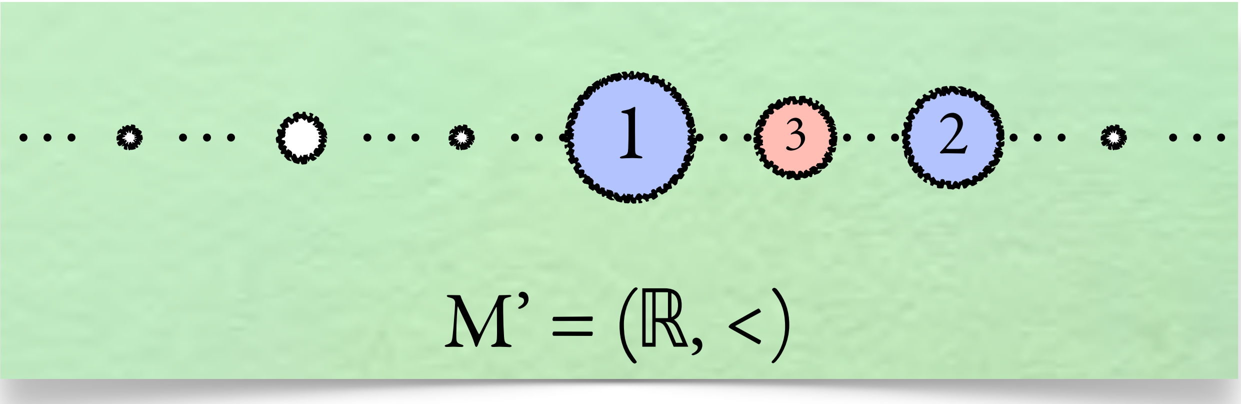

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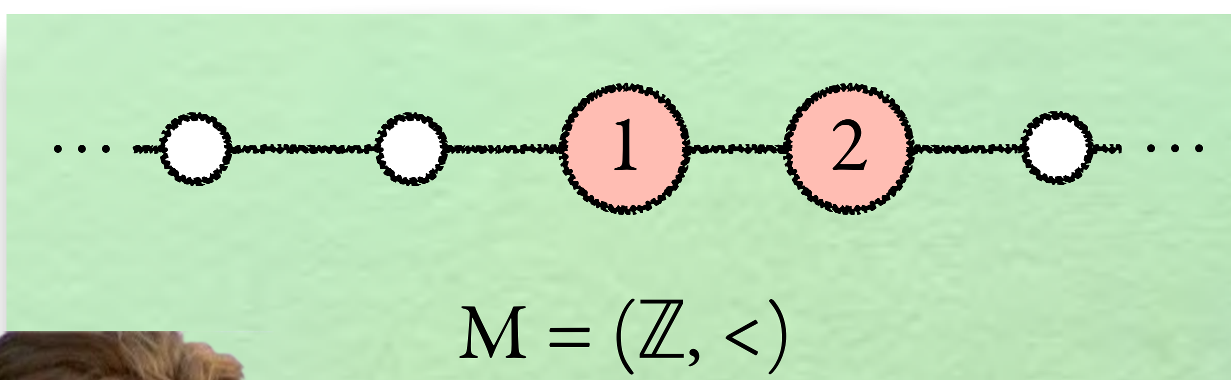

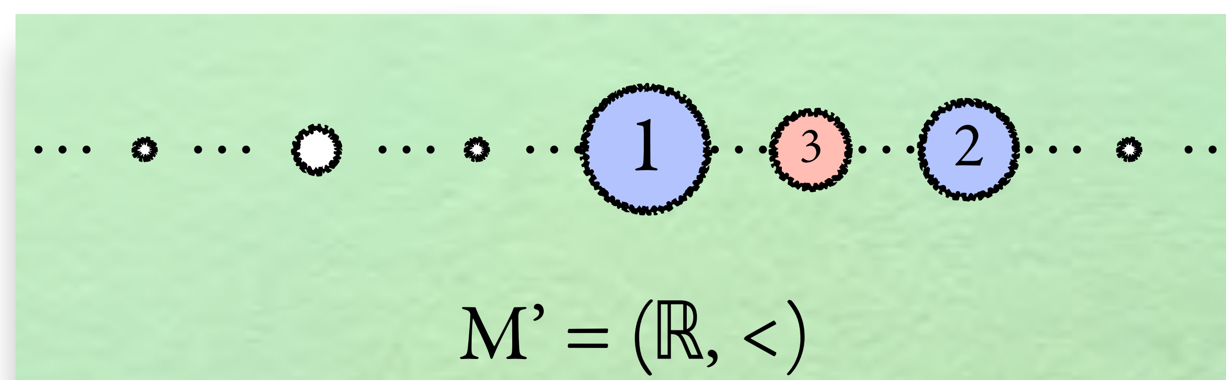

Ehrenfeucht-Fraïssé games

On non-isomorphic *finite* models, Spoiler always wins, eventually… Why?

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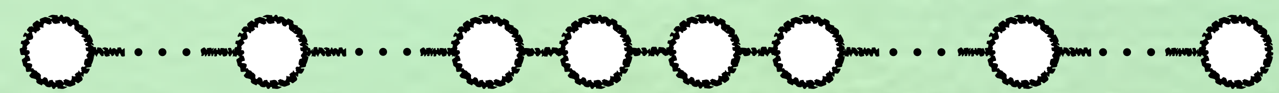

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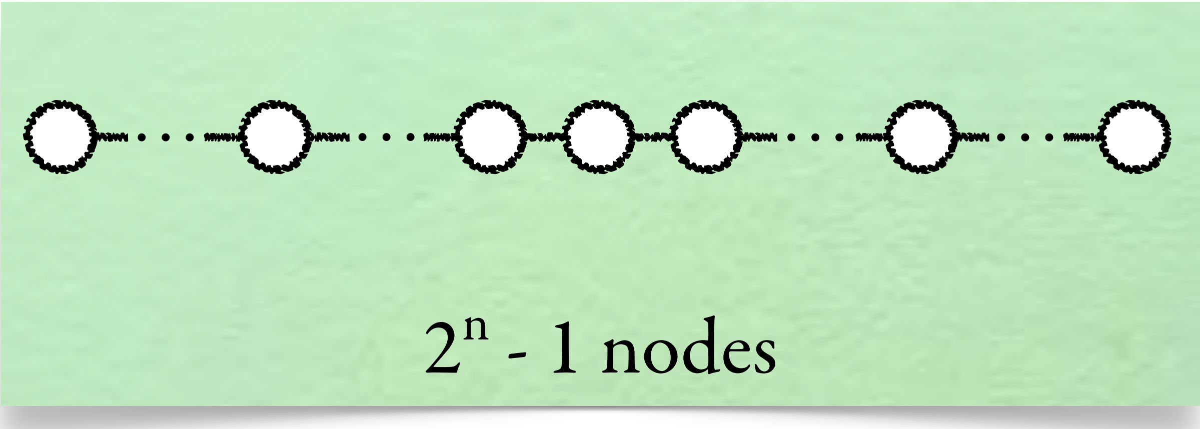

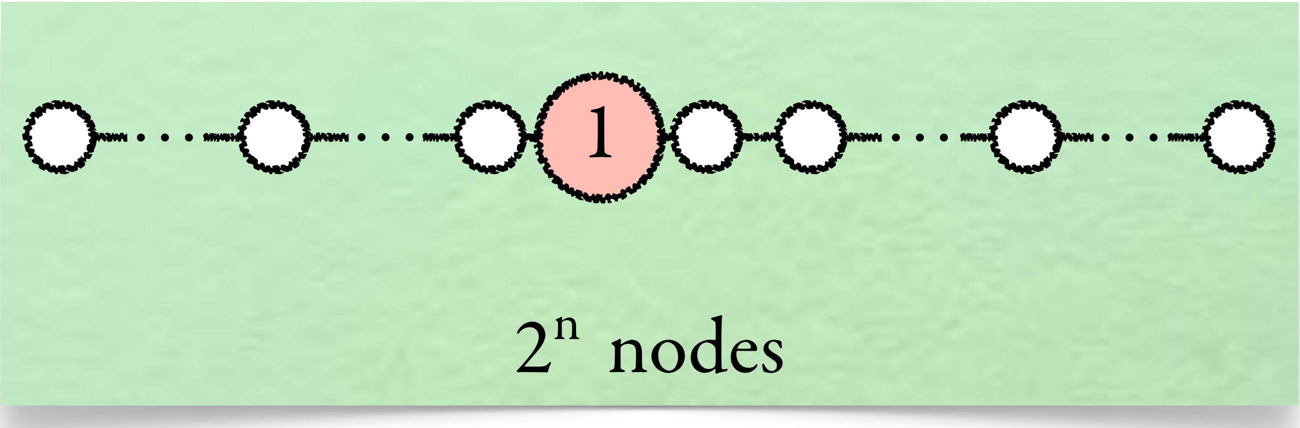

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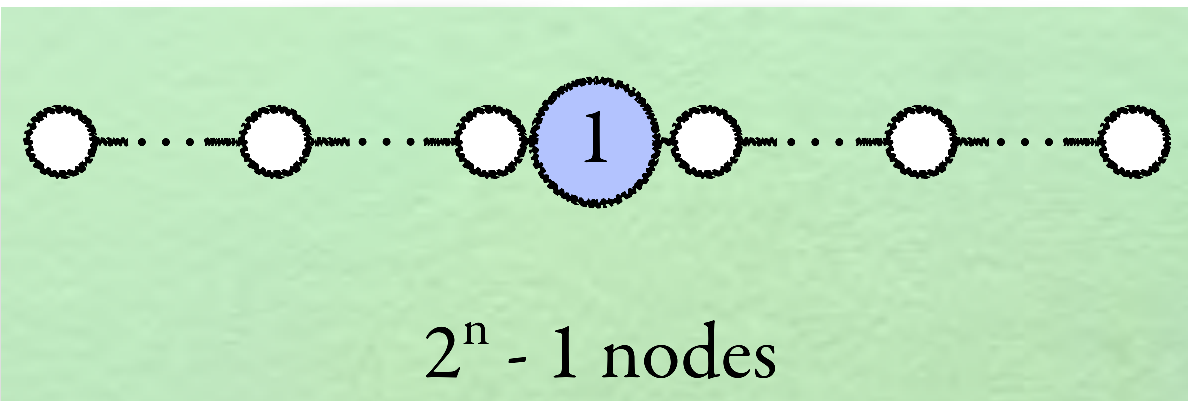

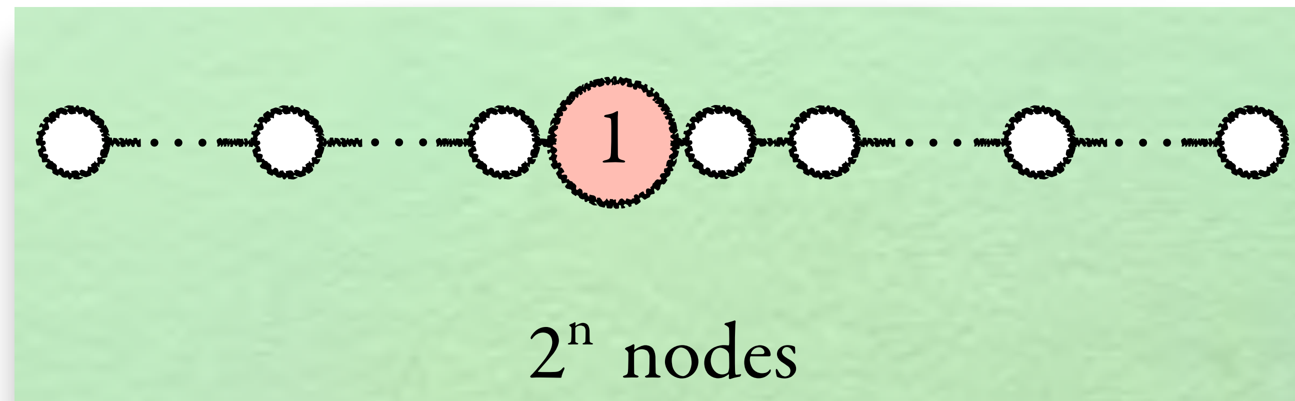

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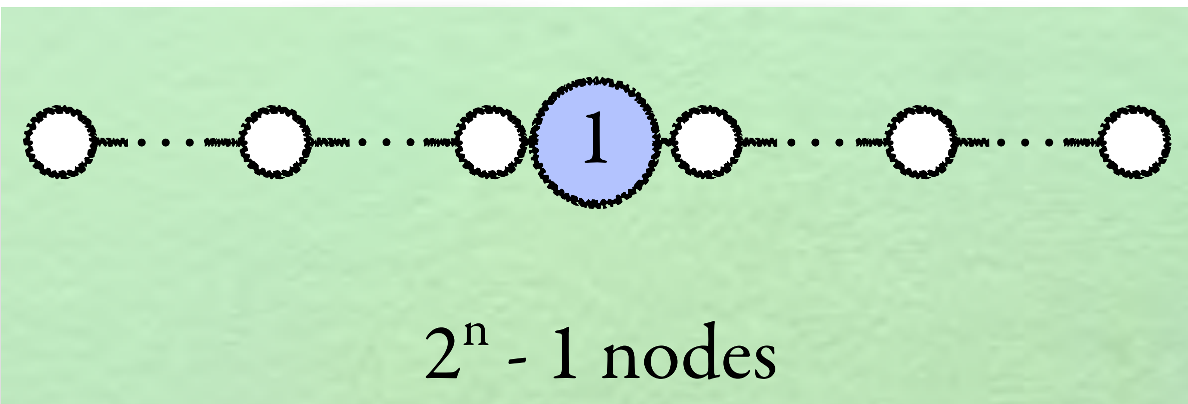

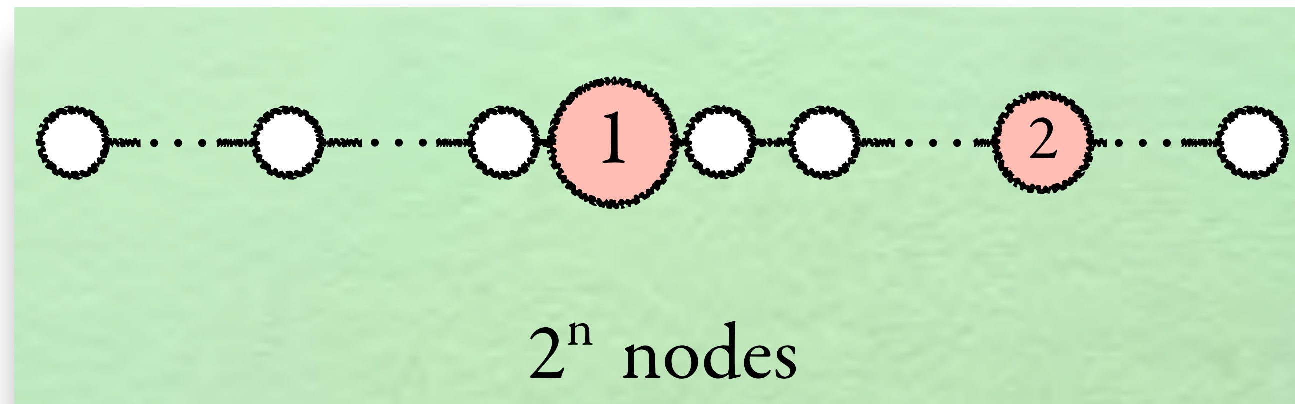

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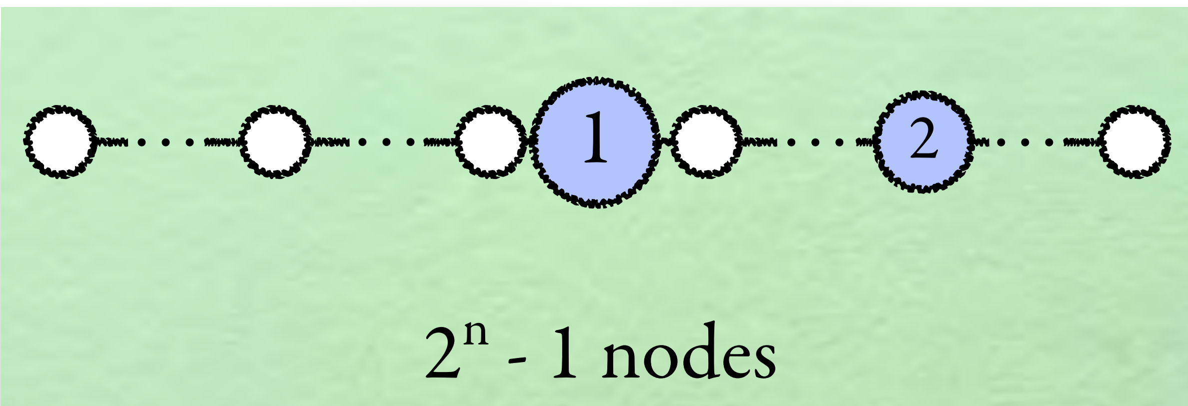

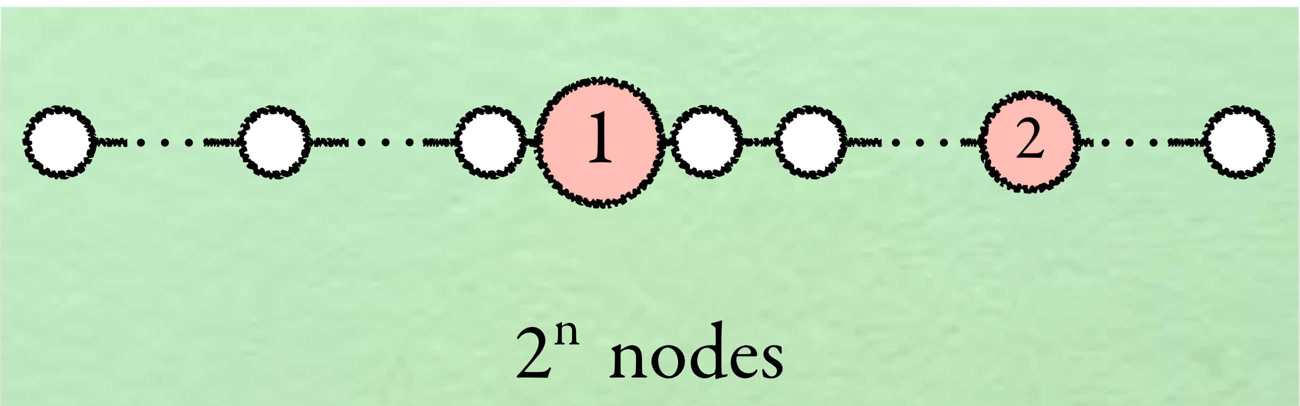

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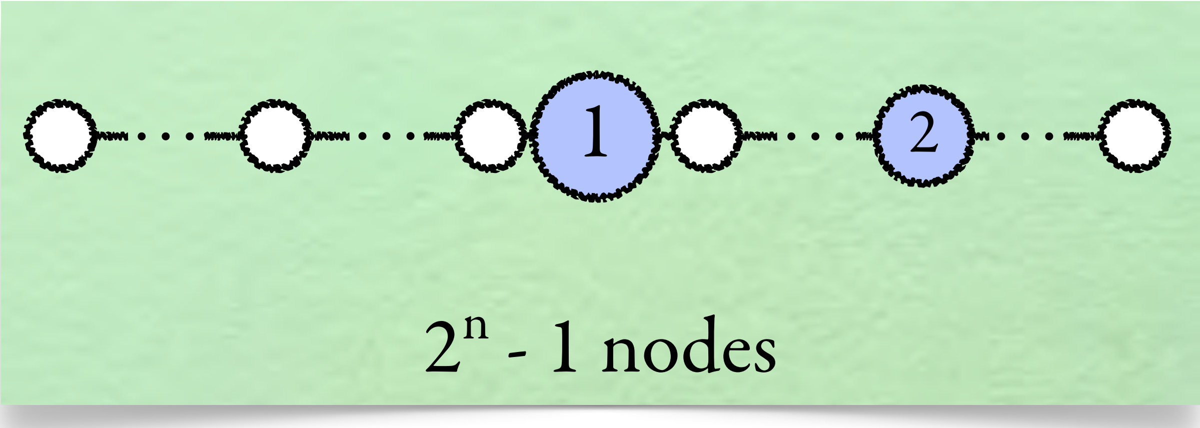

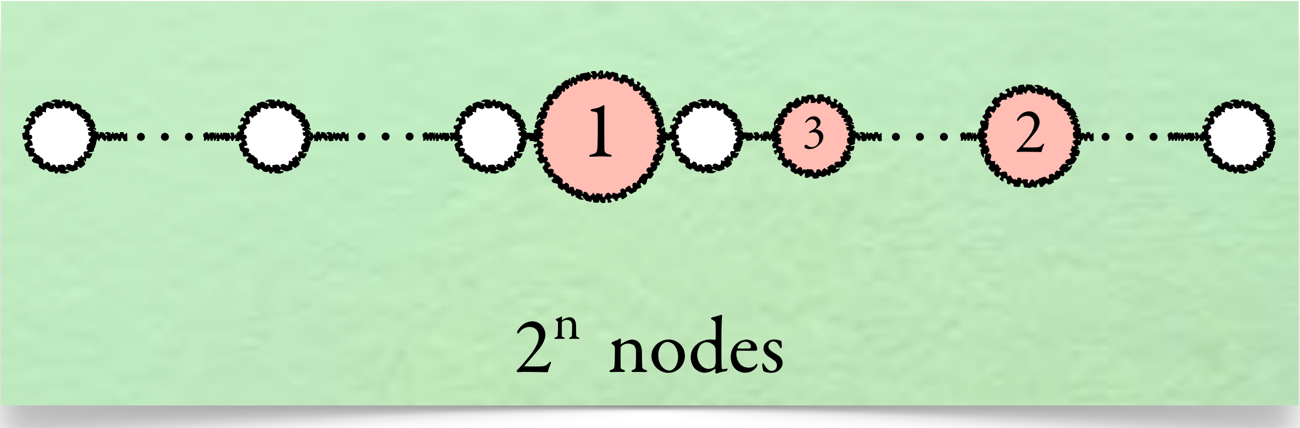

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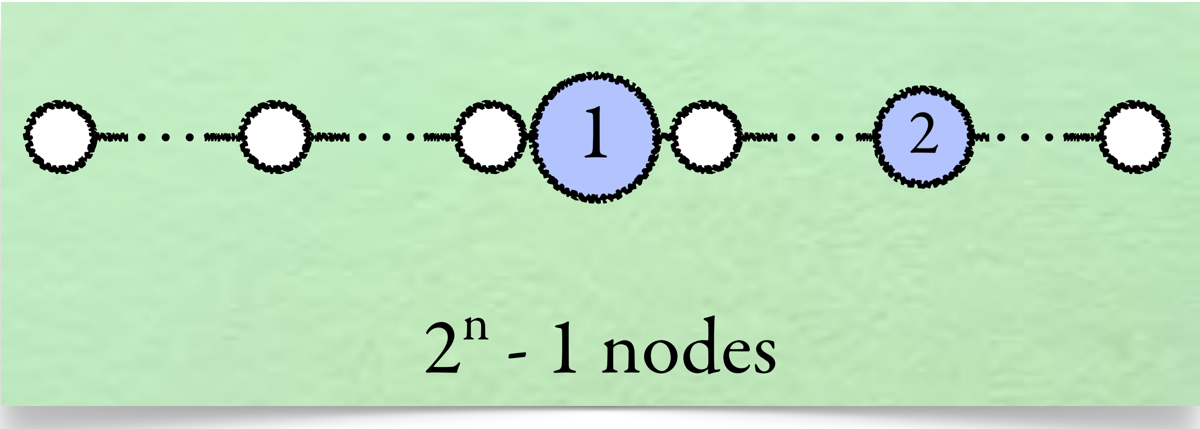

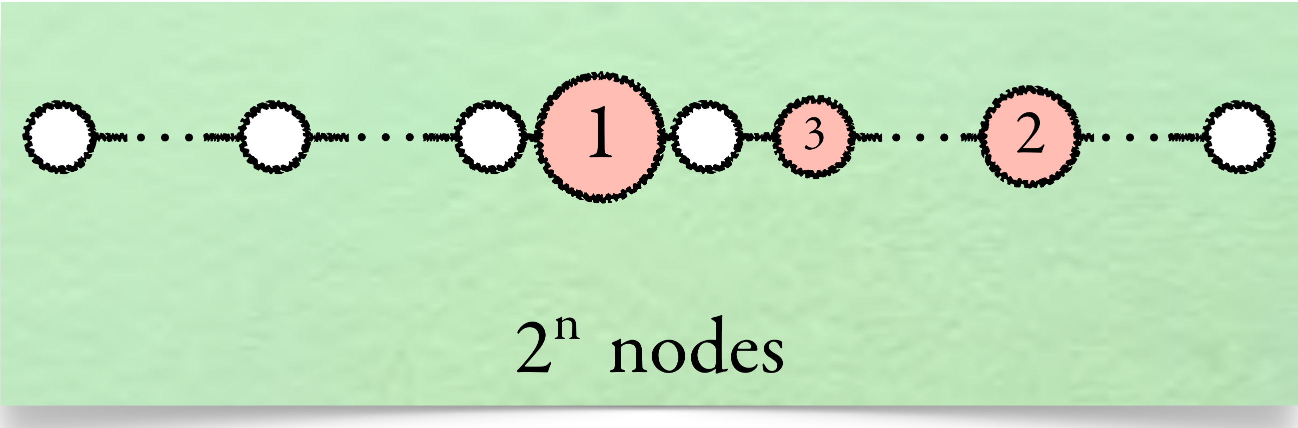

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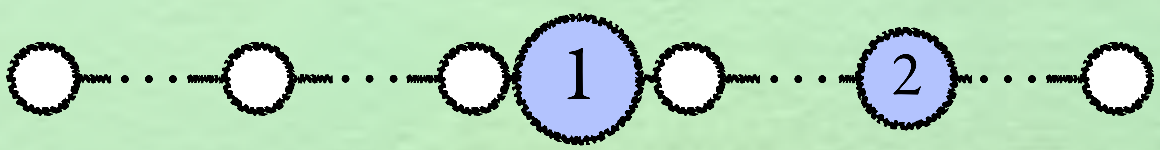

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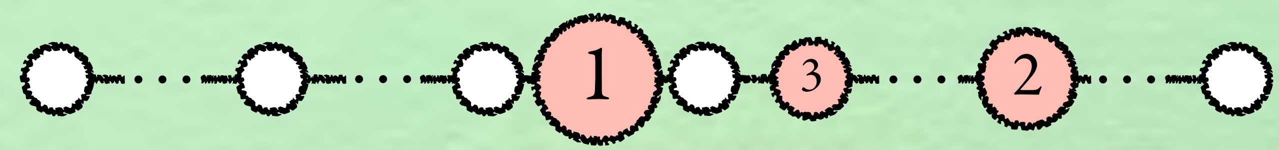

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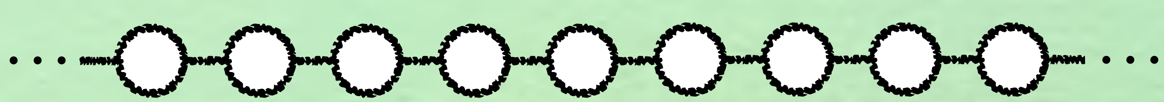

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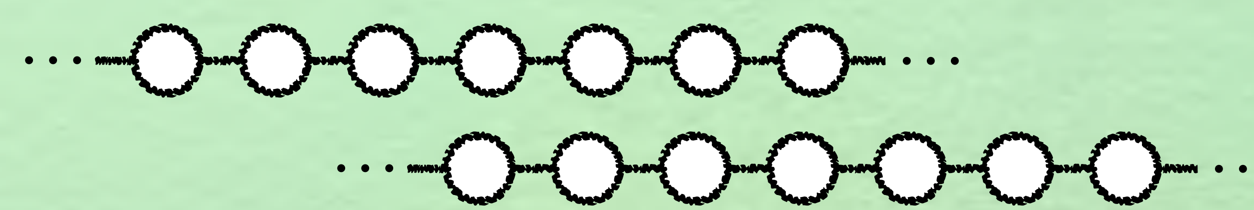

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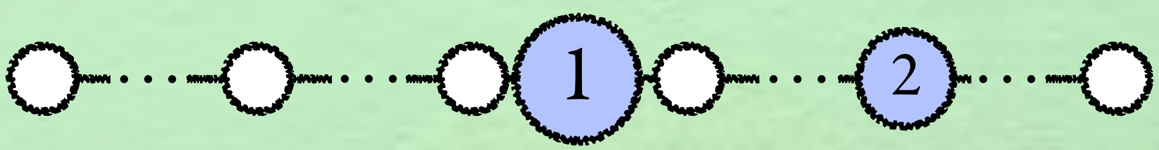

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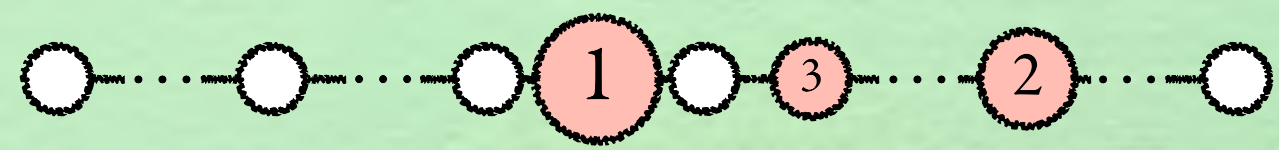

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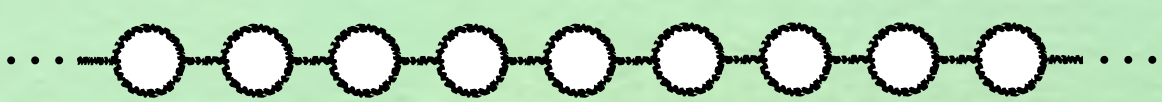

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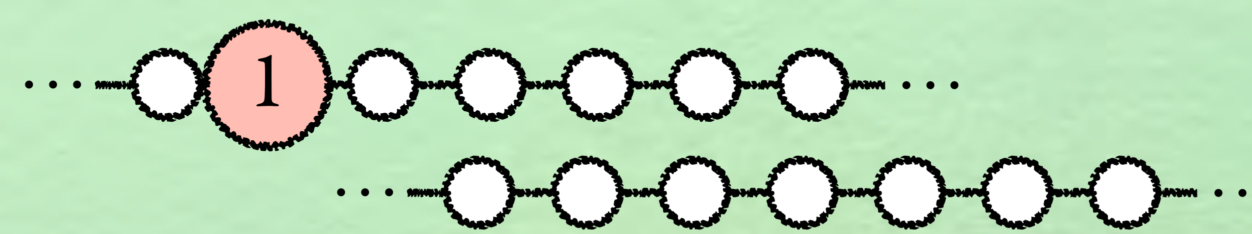

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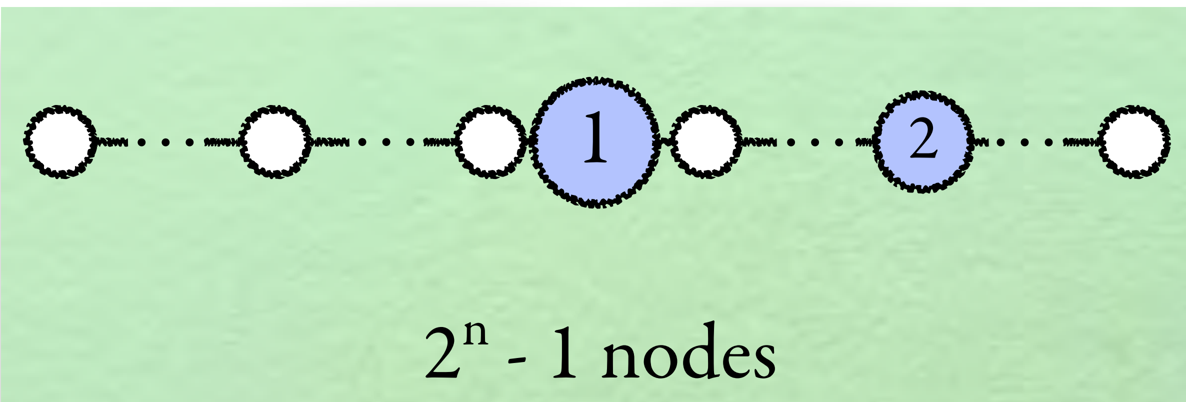

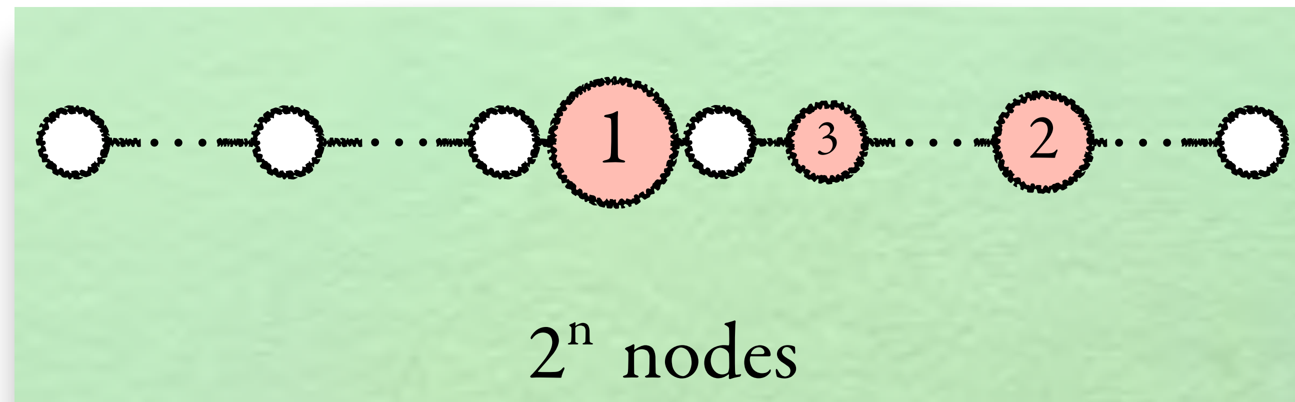

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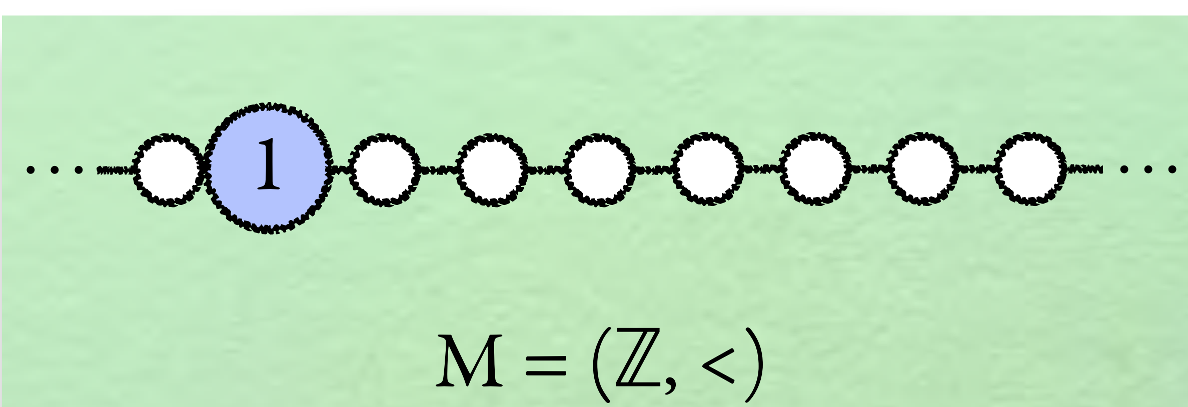

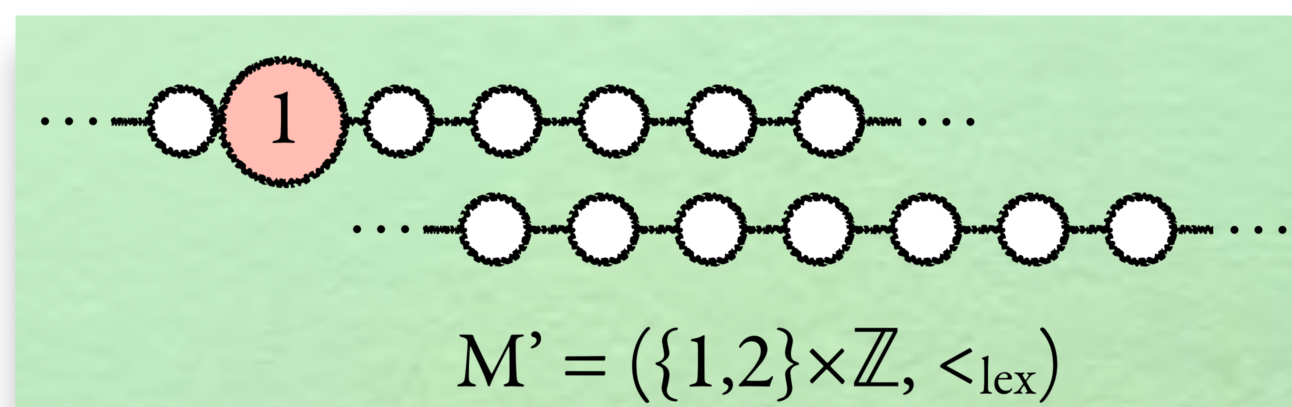

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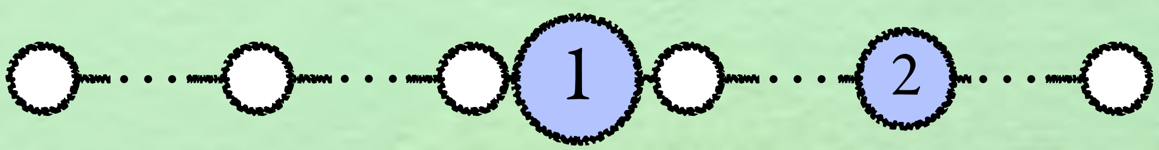

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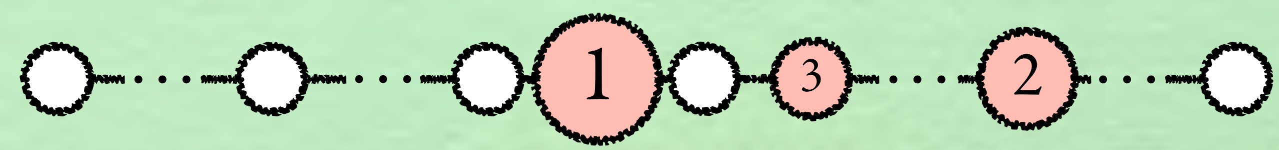

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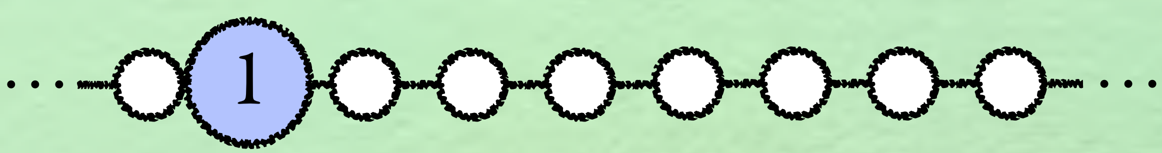

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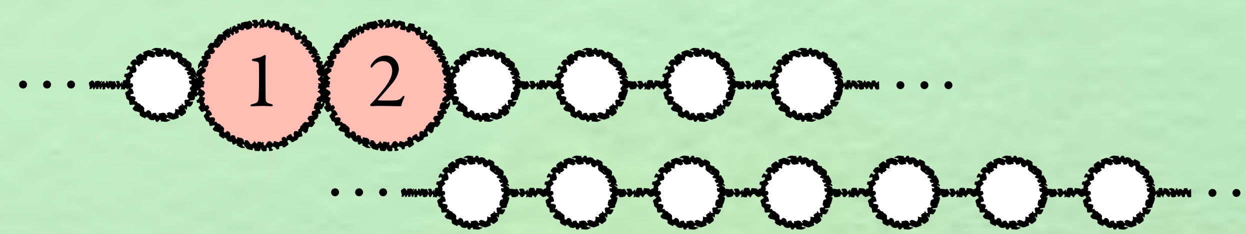

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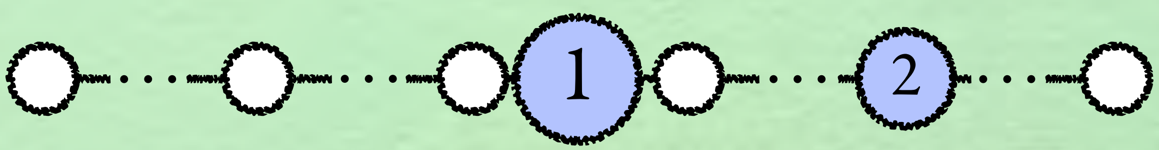

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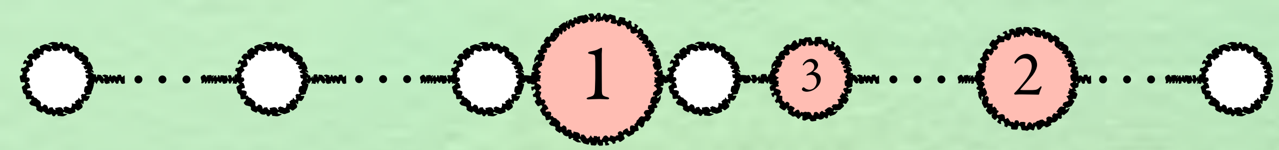

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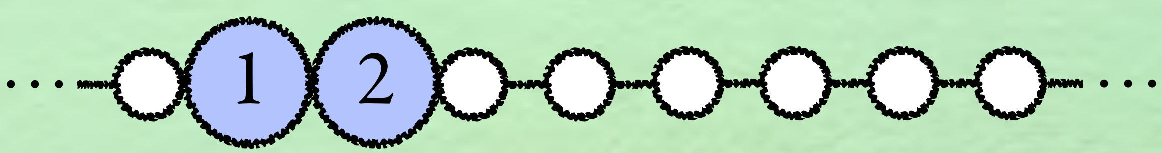

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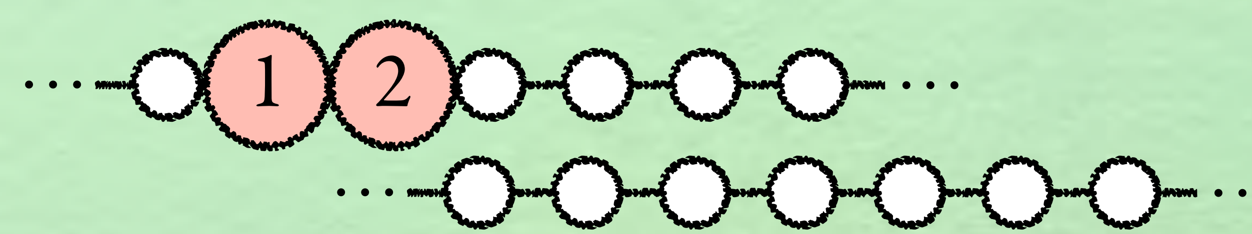

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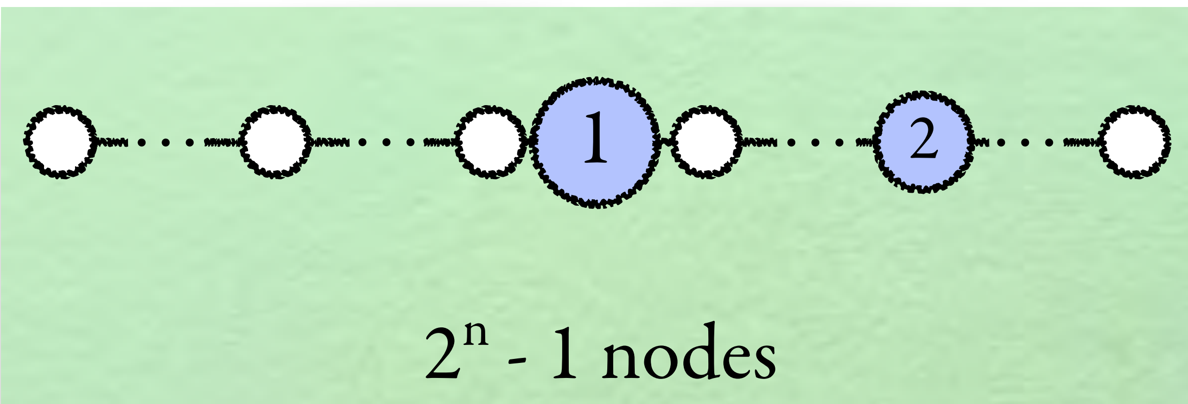

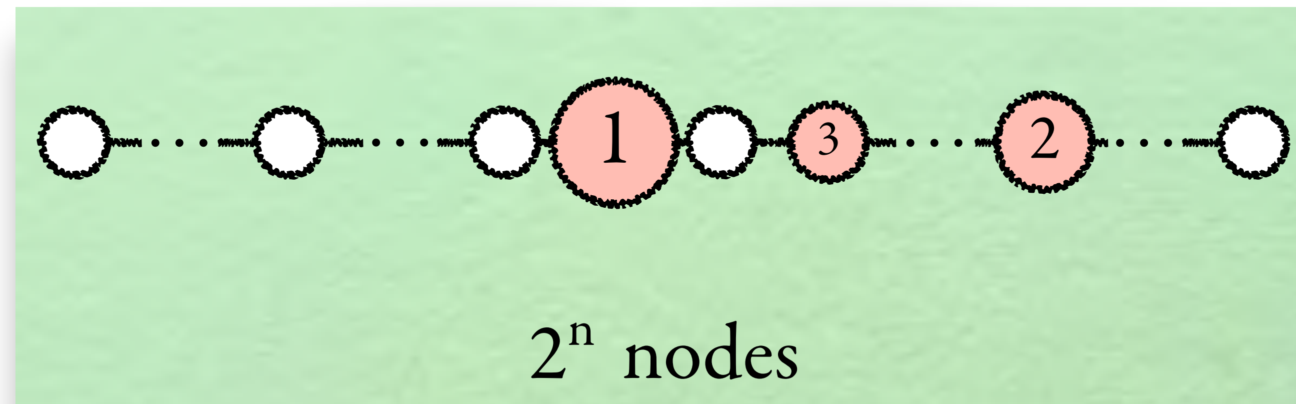

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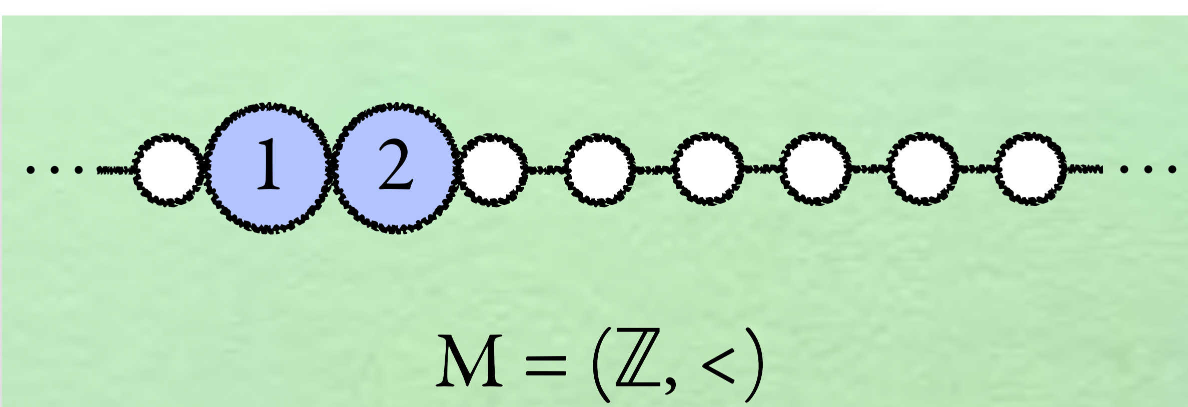

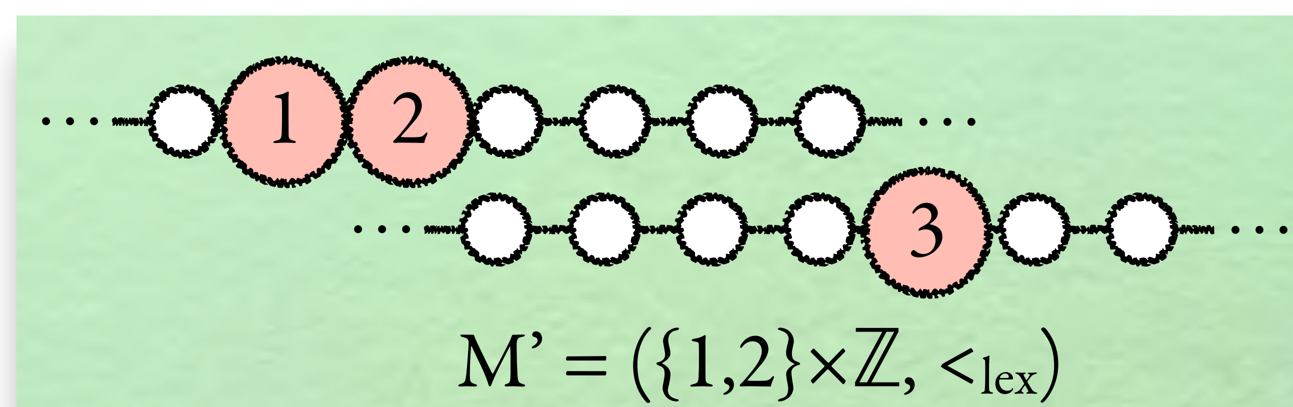

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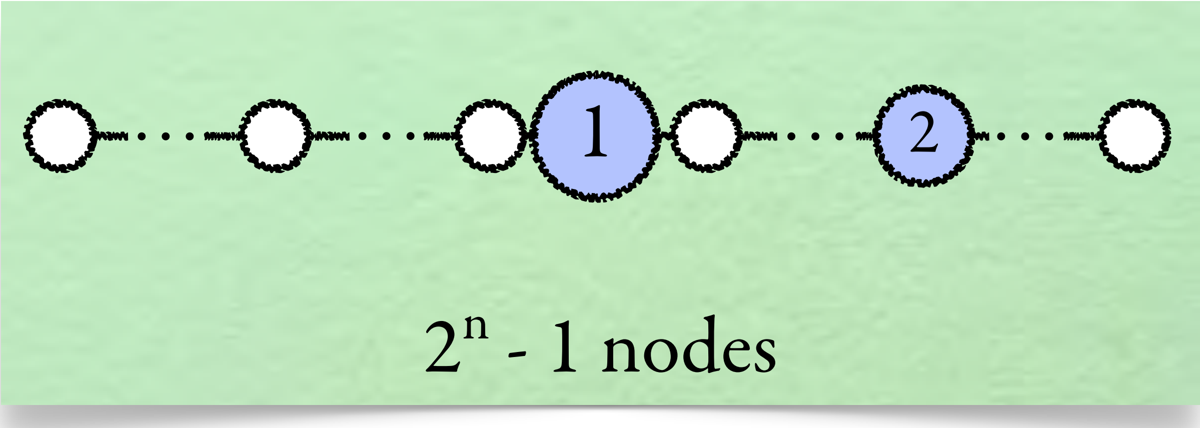

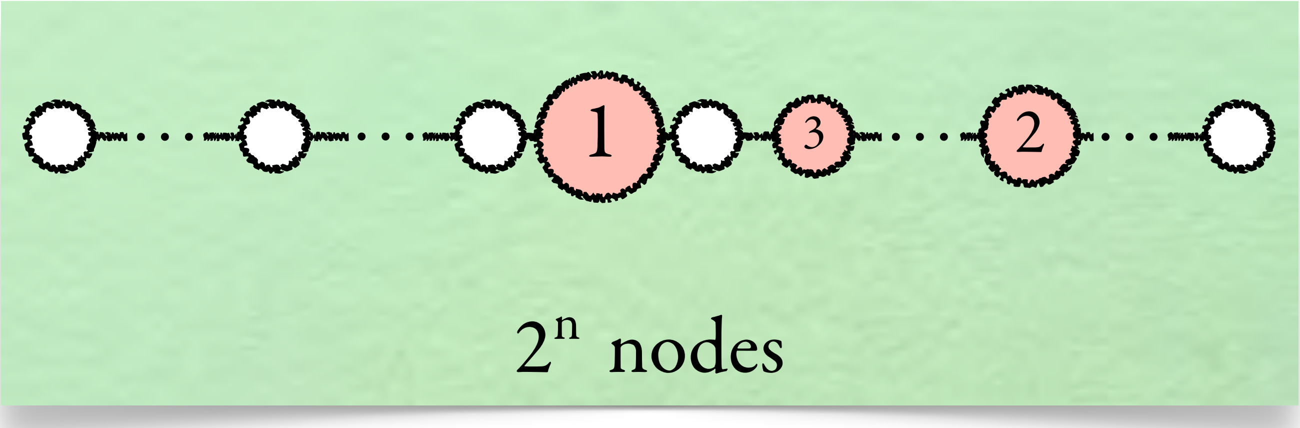

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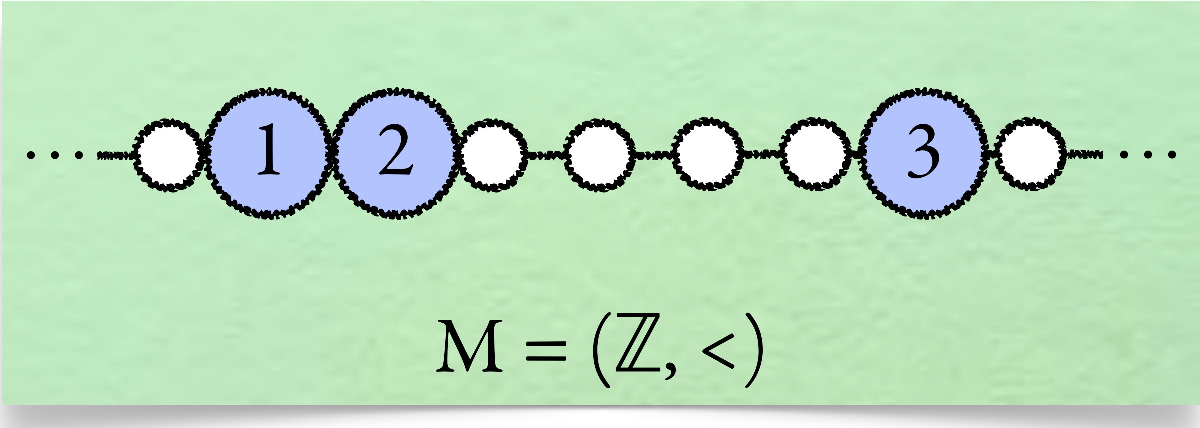

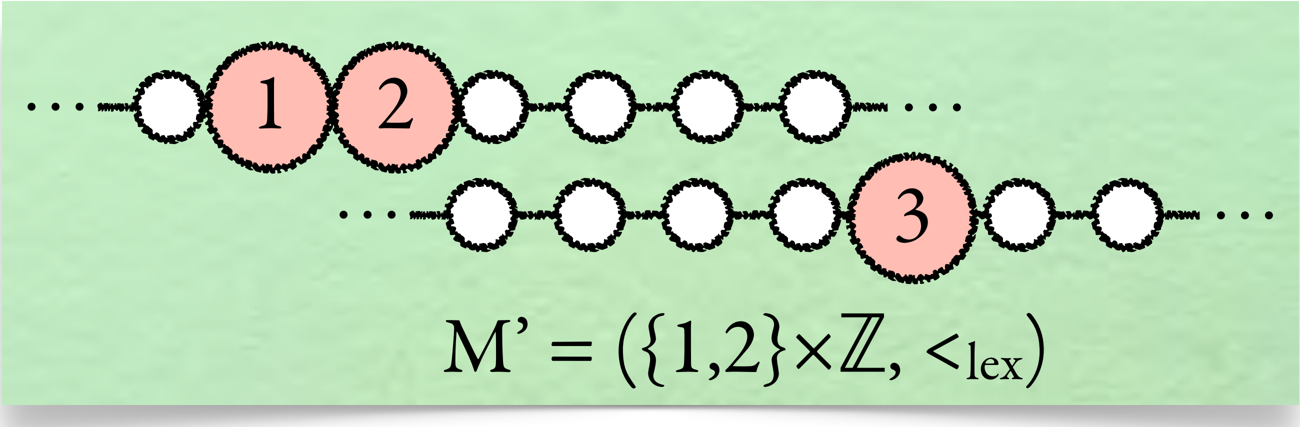

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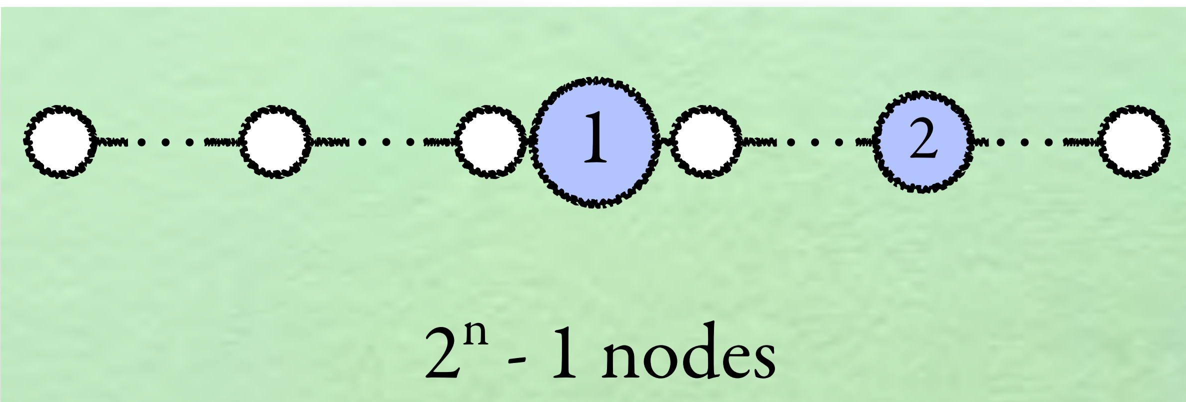

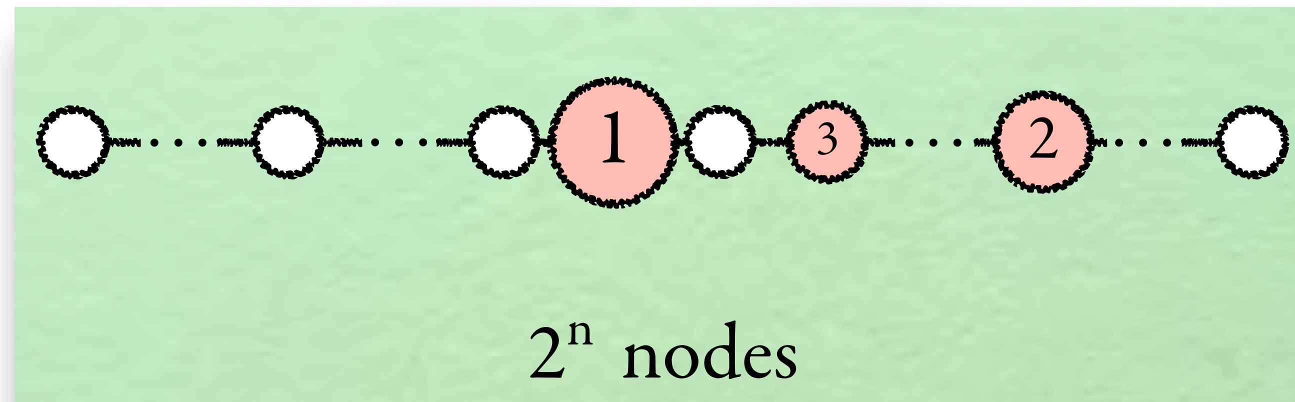

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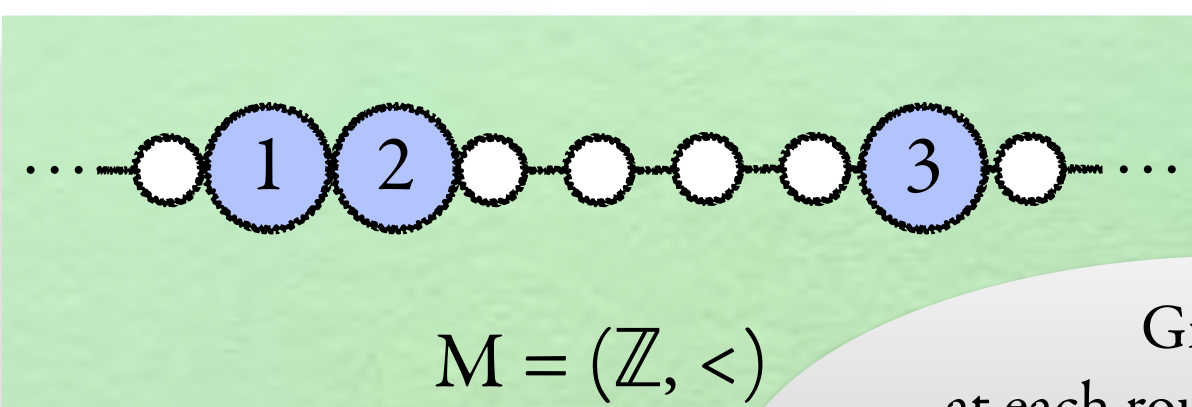

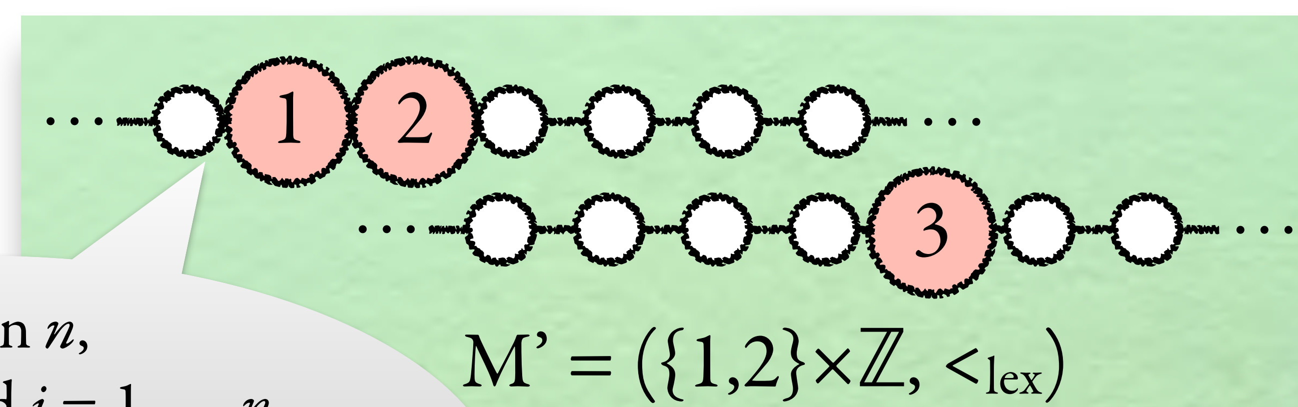

Given n,
at each round $i = 1, ..., n$,
pairs of marked nodes in M and M'
must be either at *equal distance*
or *at distance* $\geq 2^{n-i}$

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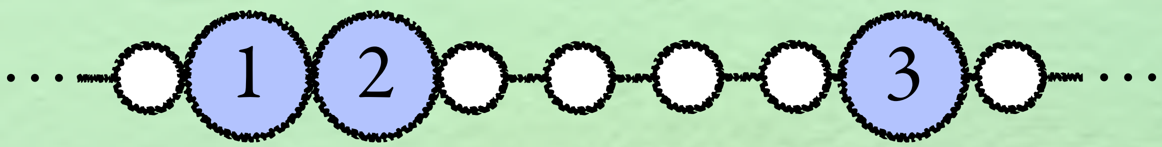

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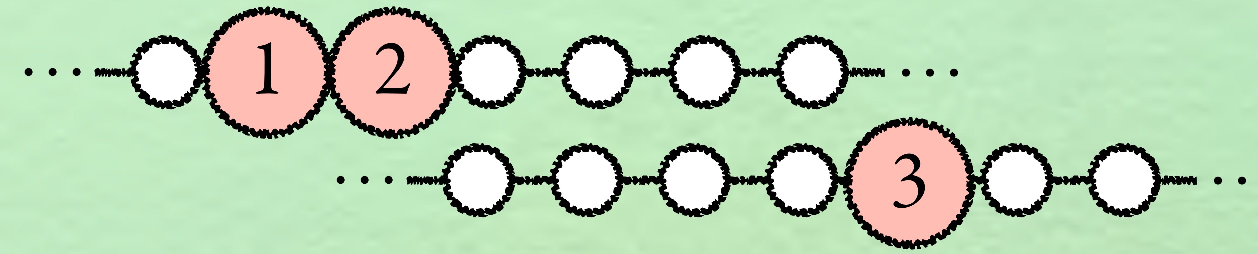

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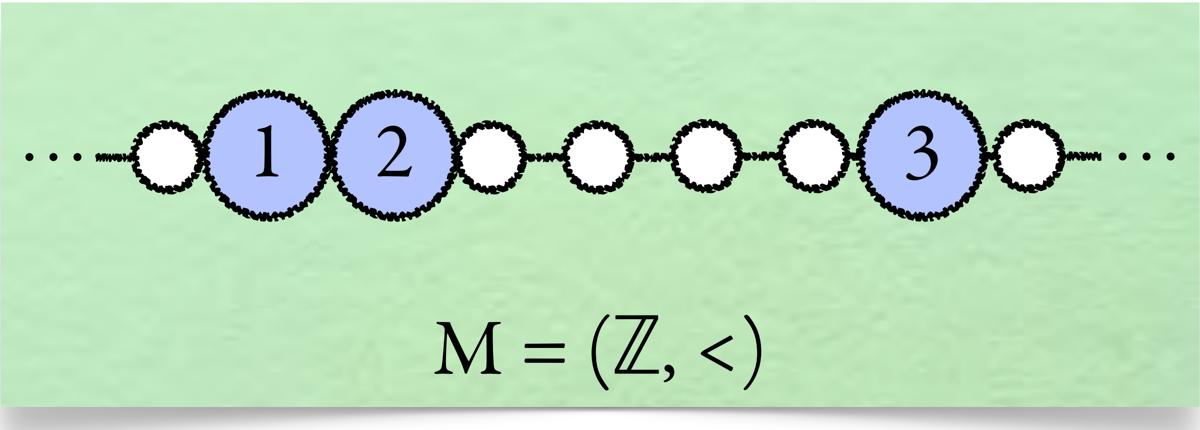

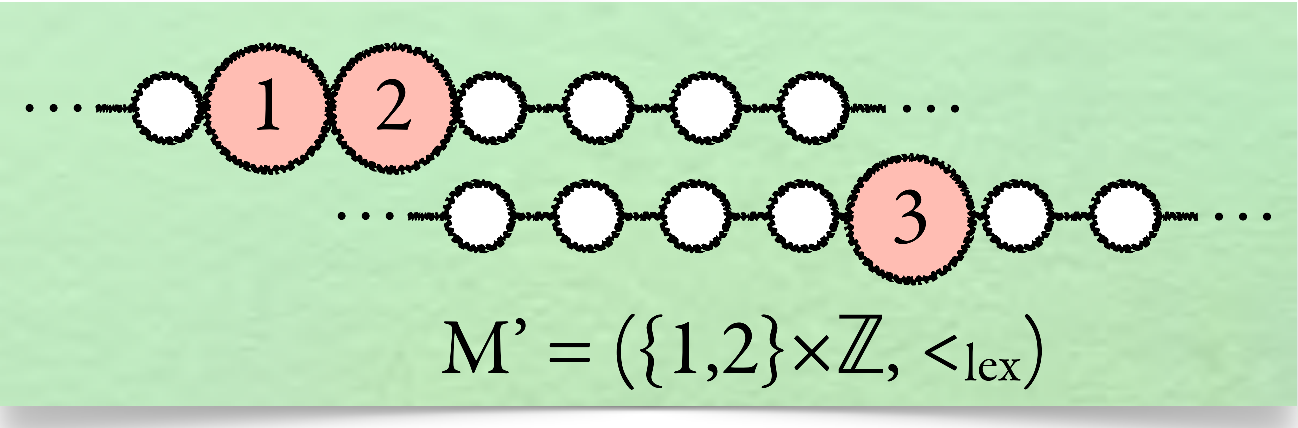

Lemma If for every n there are M,M' such that
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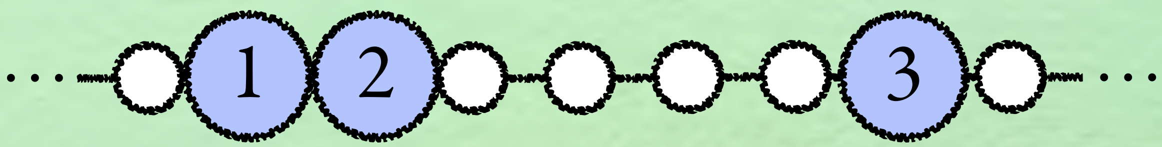

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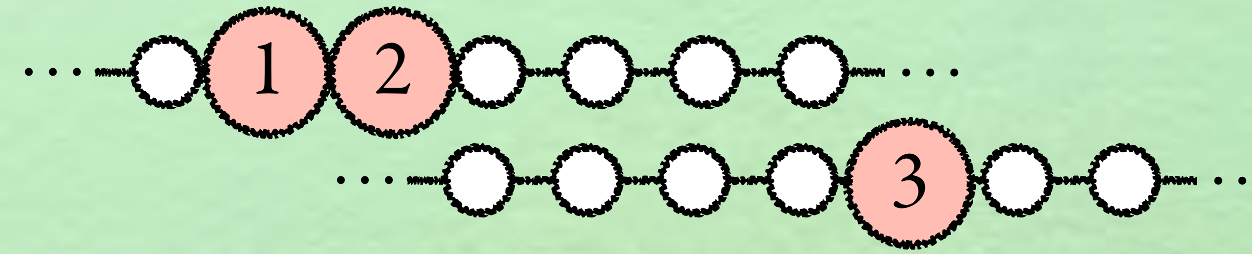

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Lemma If for every n there are M,M' such that $M \in P$, $M' \notin P$, and M,M' n-equivalent then P is *not* definable in FO

Theorem M,M' n-equivalent iff Duplicator survives n rounds in $G_{M,M'}$
[Fraïssé '50, Ehrenfeucht '60]

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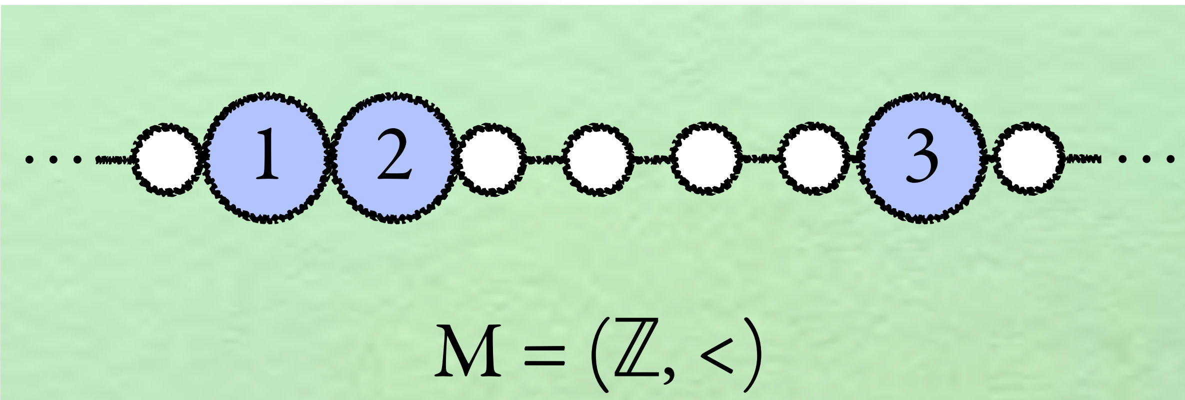

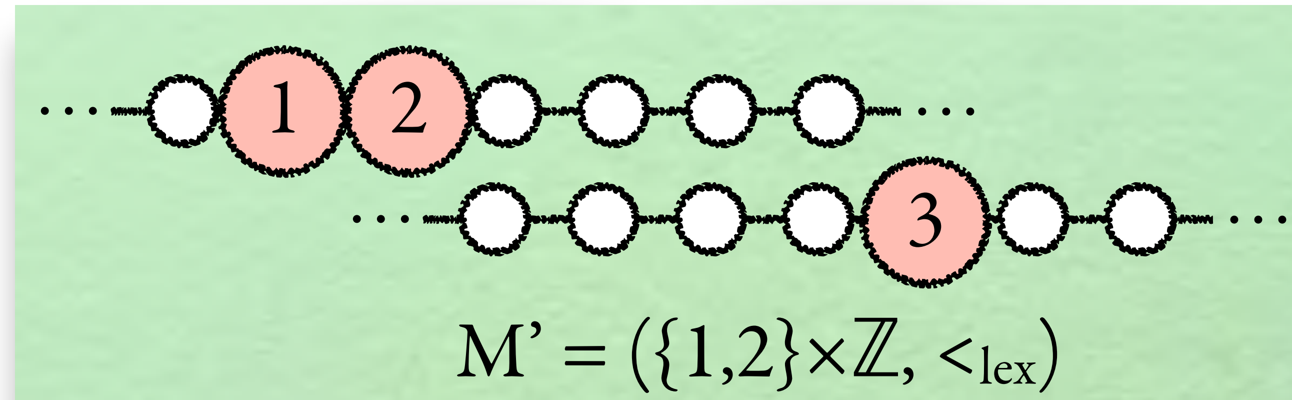

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In particular, P = {discrete orders} is *not* definable in FO,
since $\mathbb{Z} \in P$ and $\{1,2\}\times\mathbb{Z} \notin P$

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Example P={connected graphs}. Given n, find M$\in$P, M'$\notin$P where Duplicator survives n rounds

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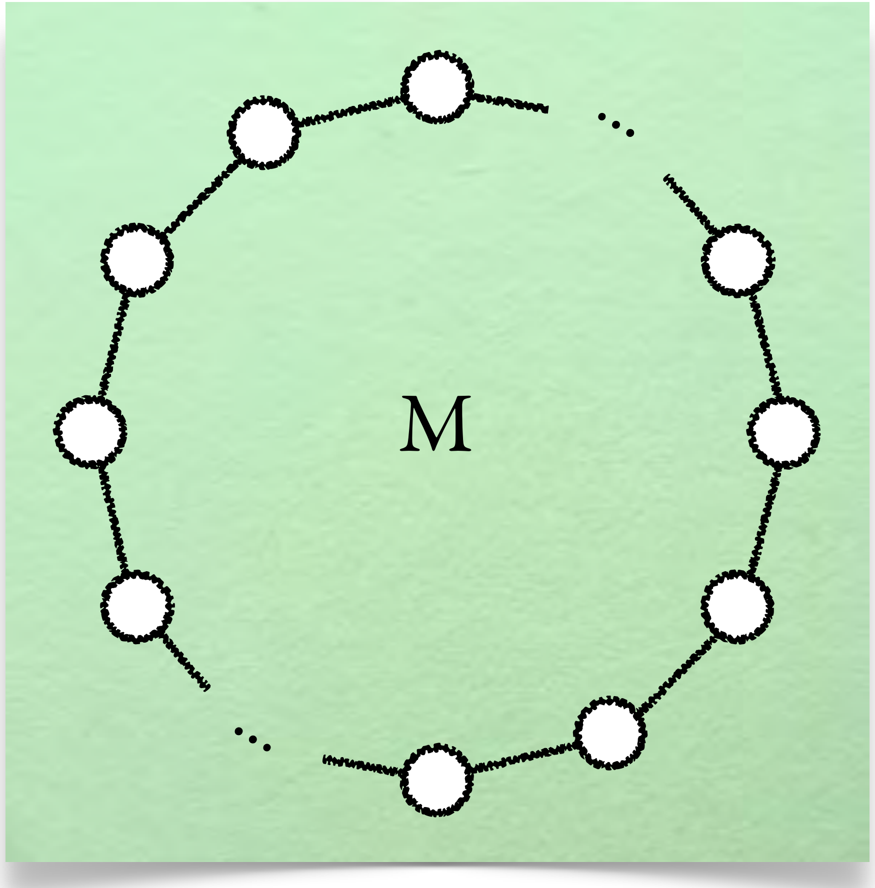

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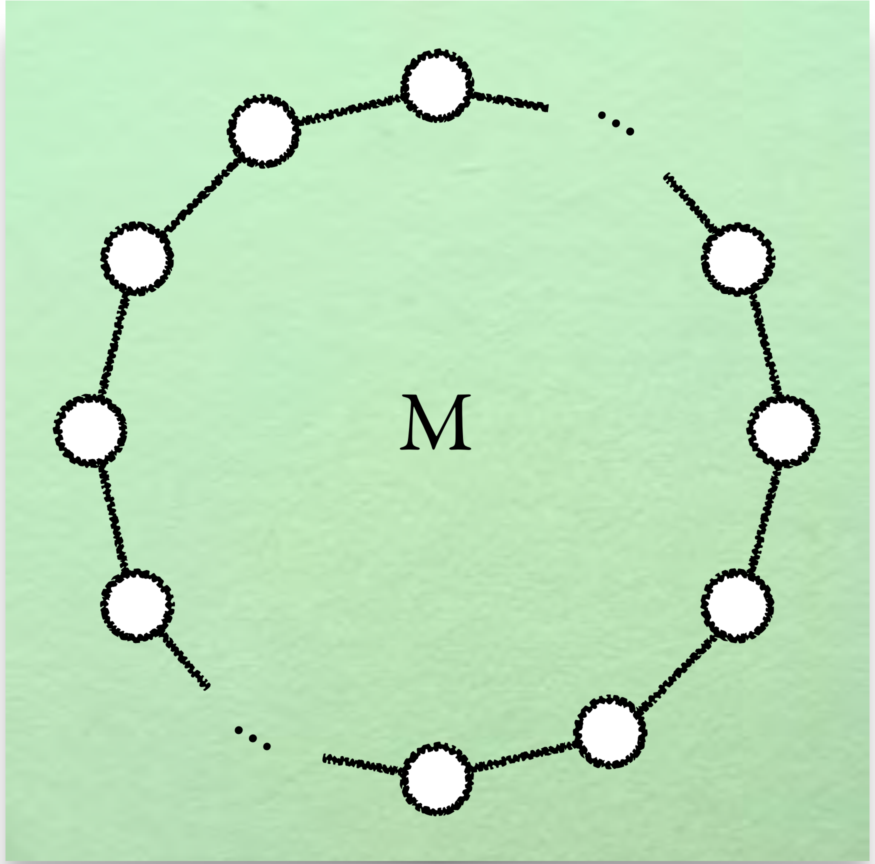

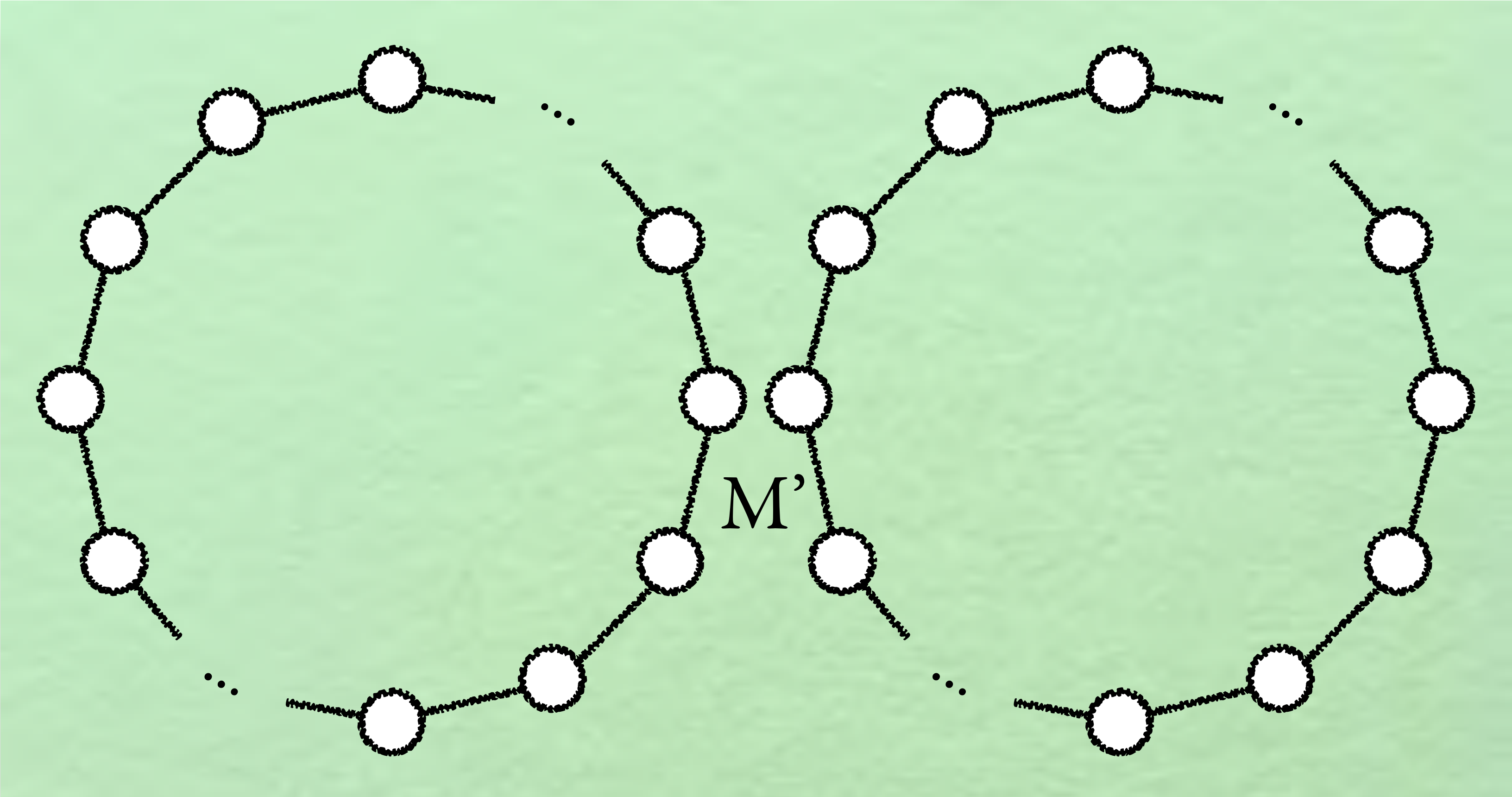

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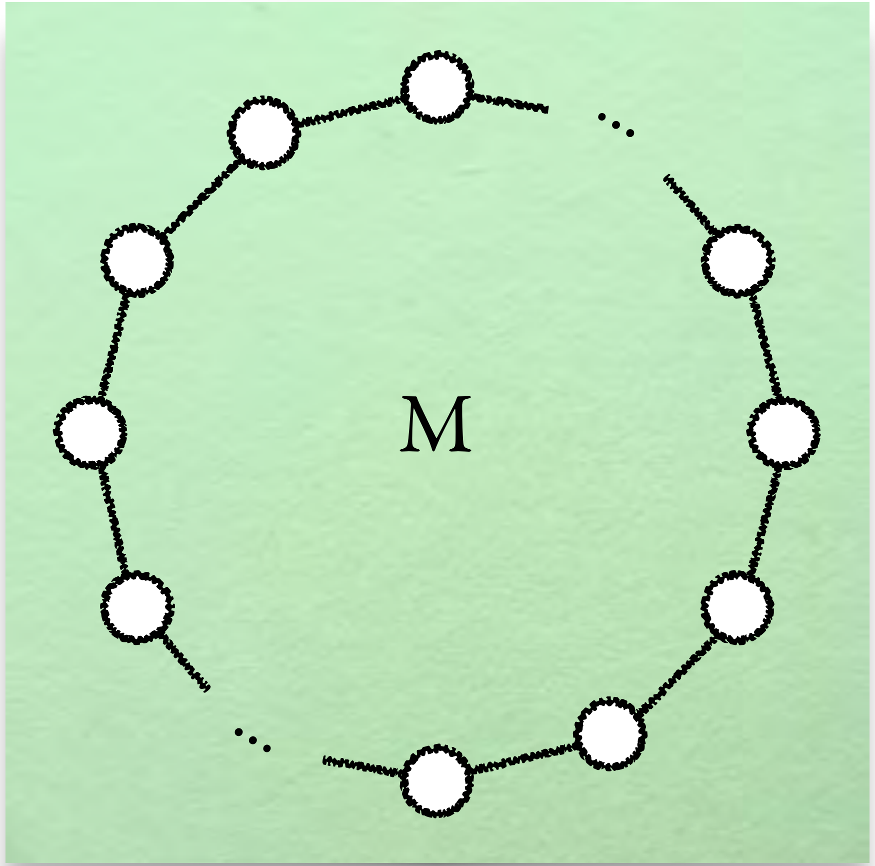

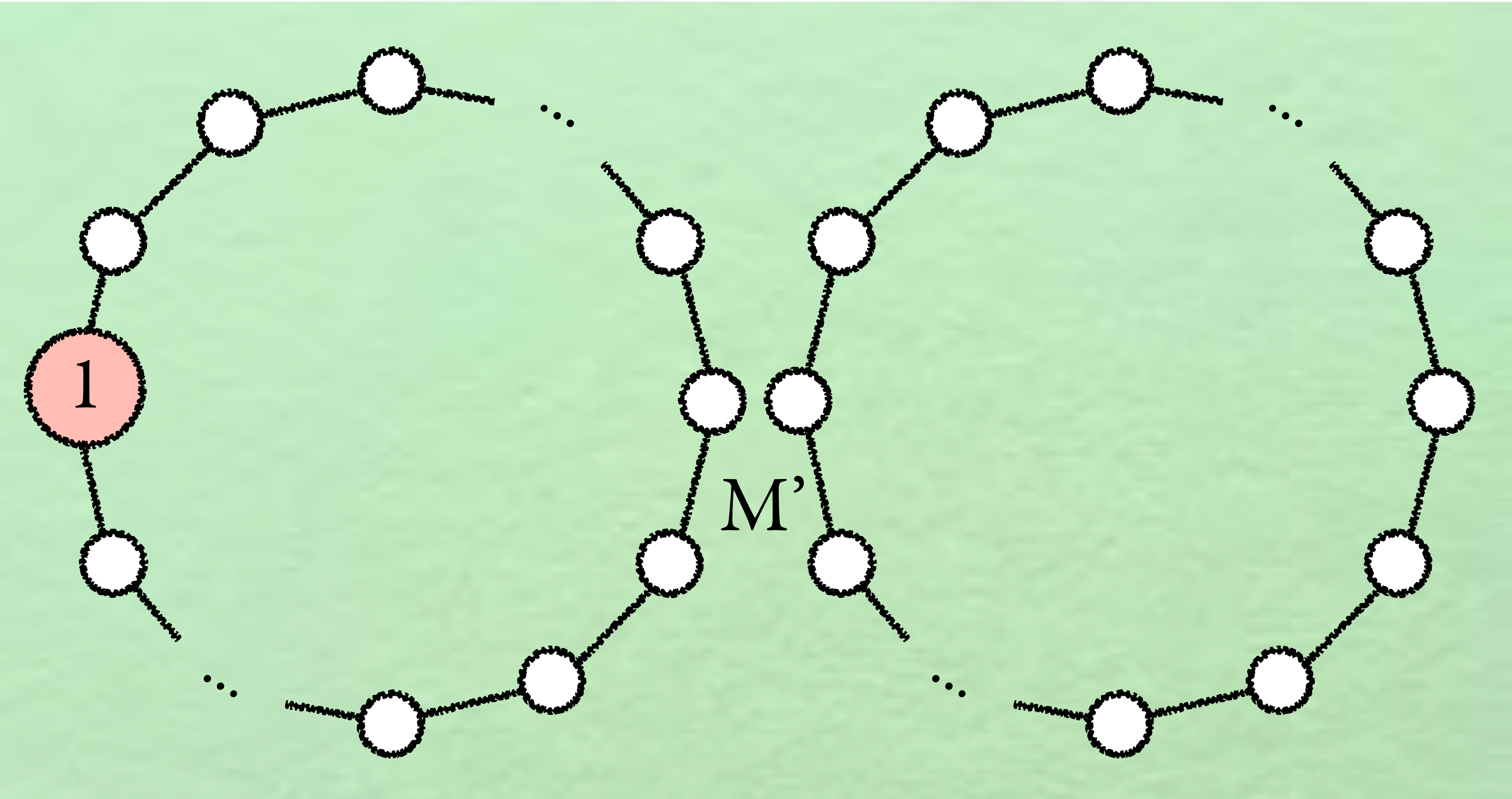

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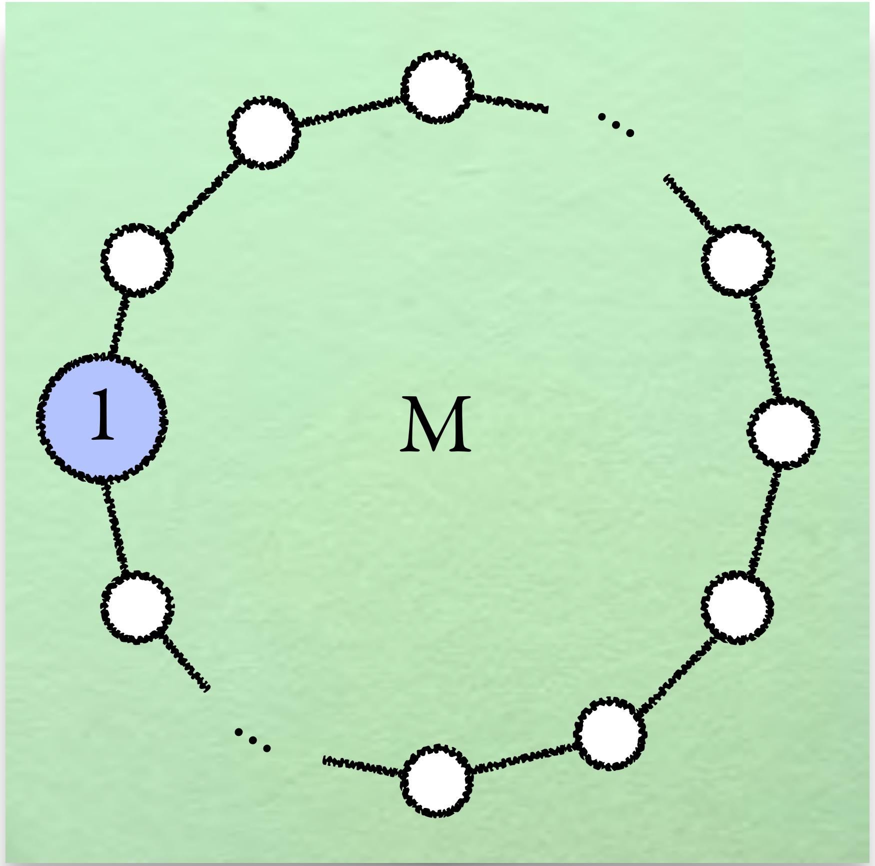

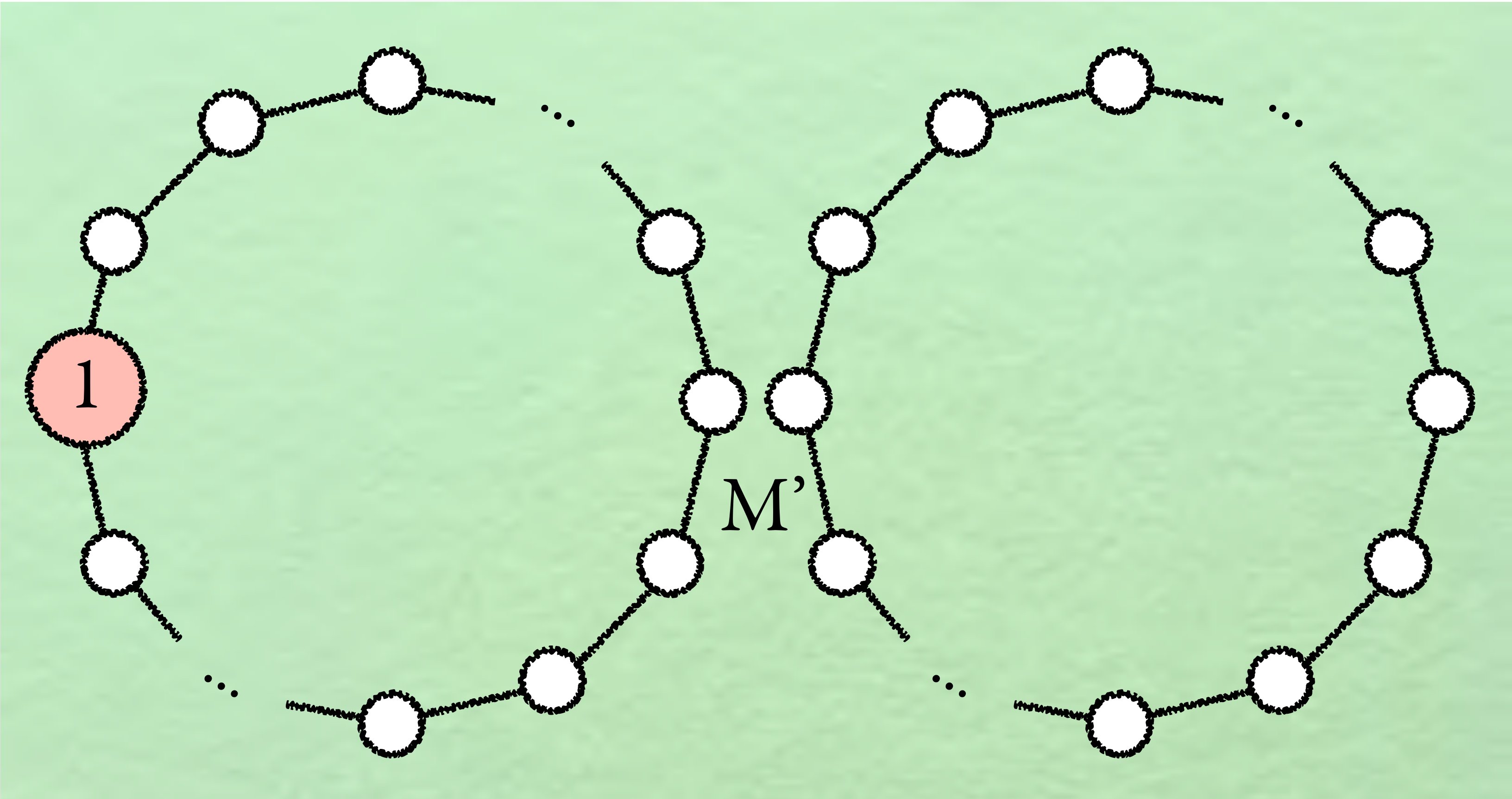

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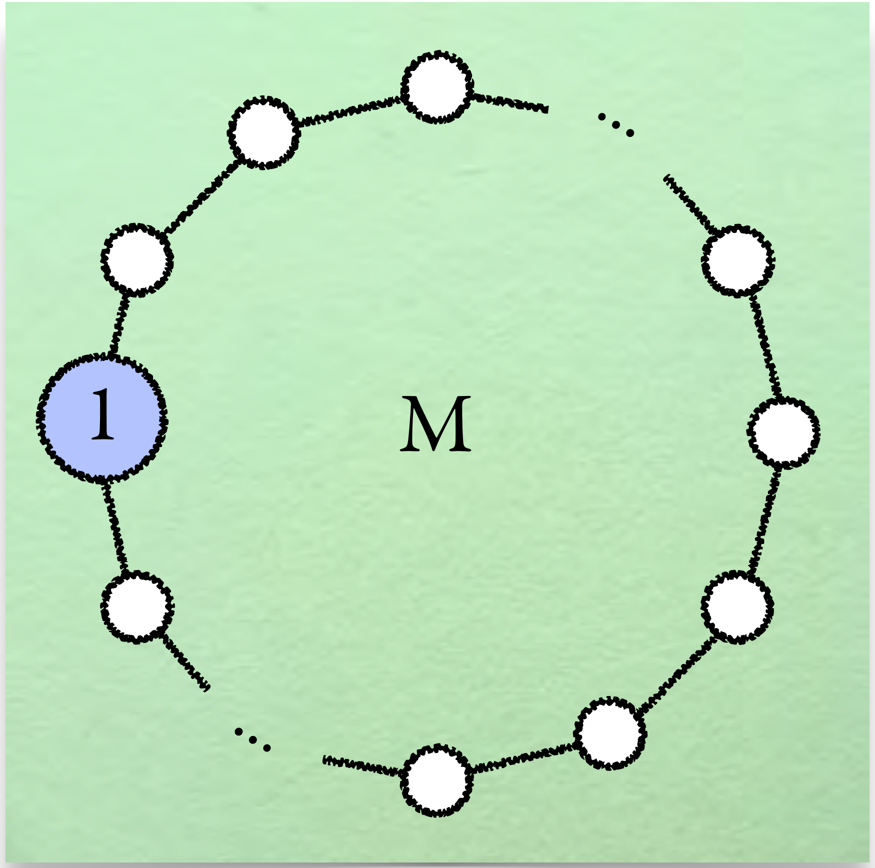

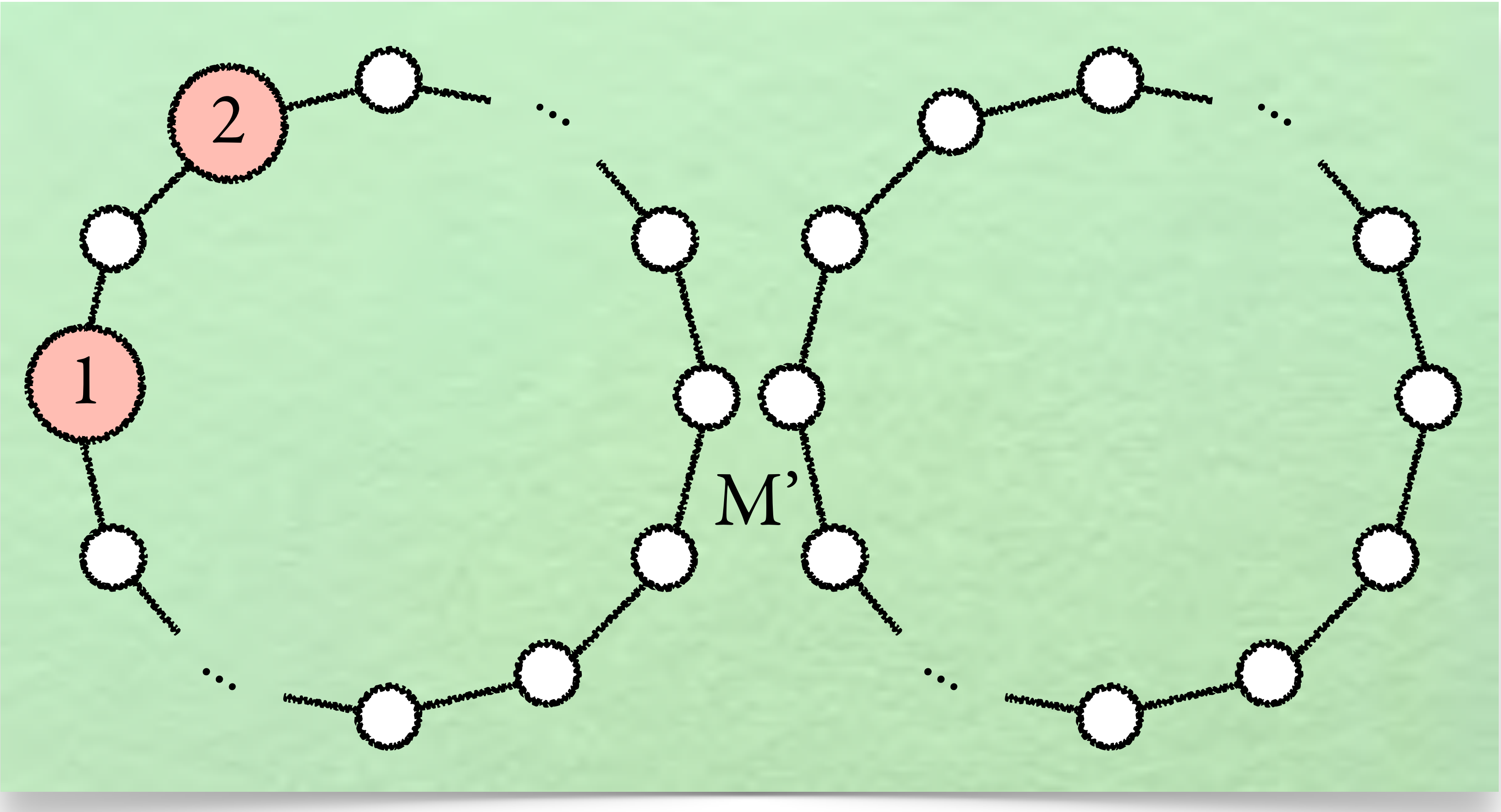

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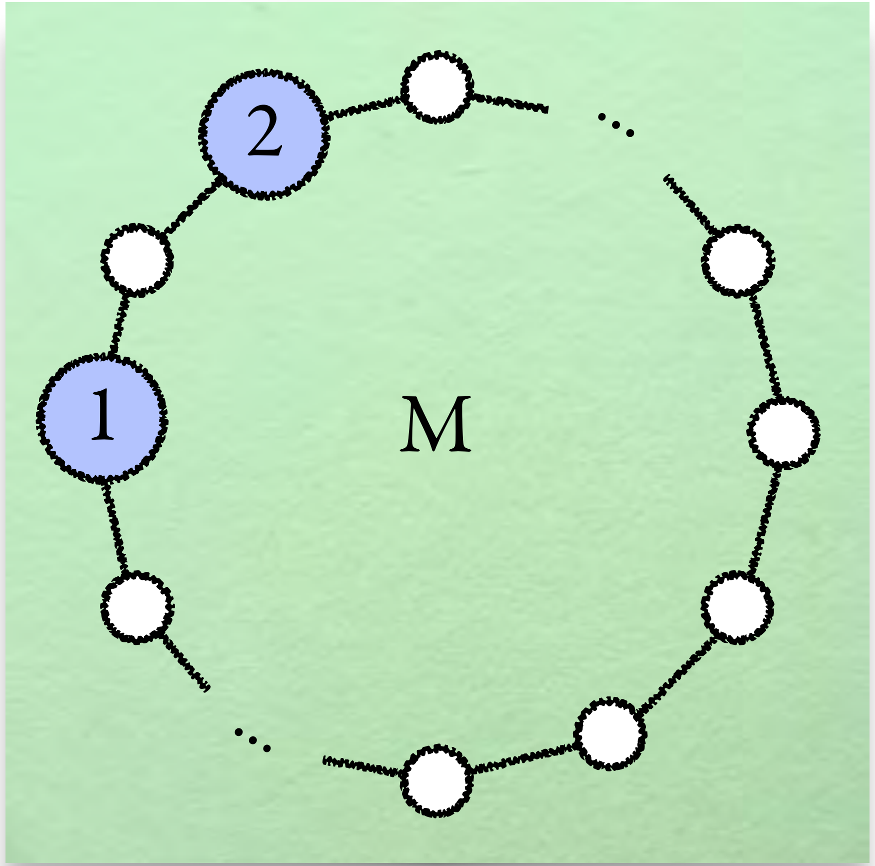

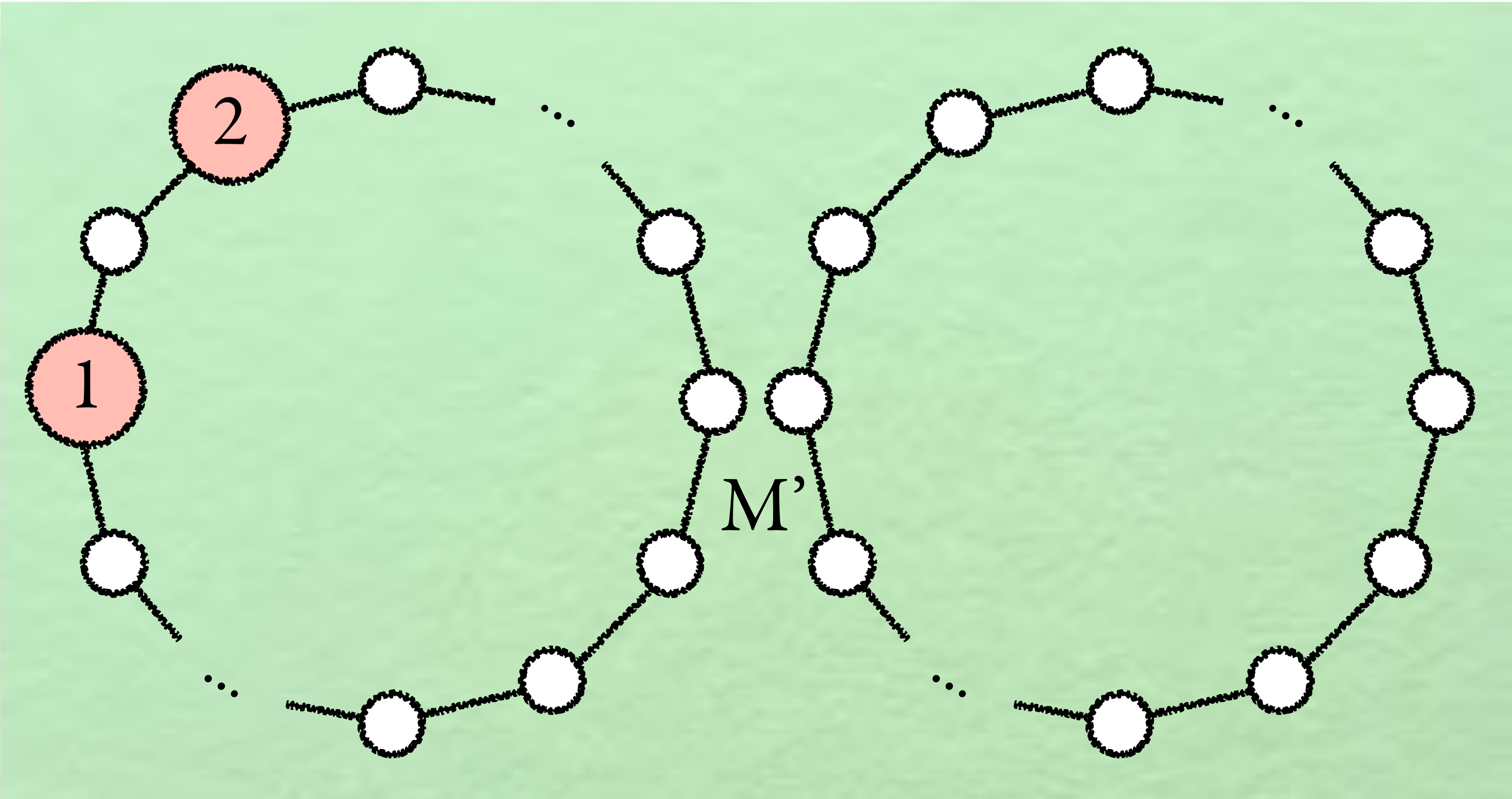

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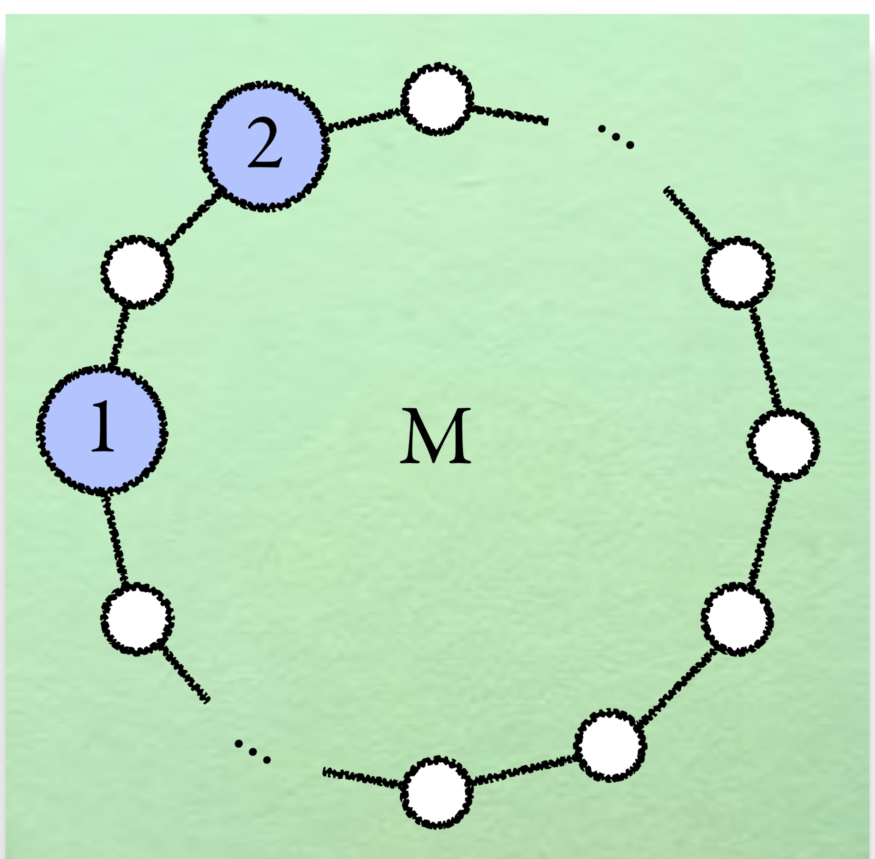

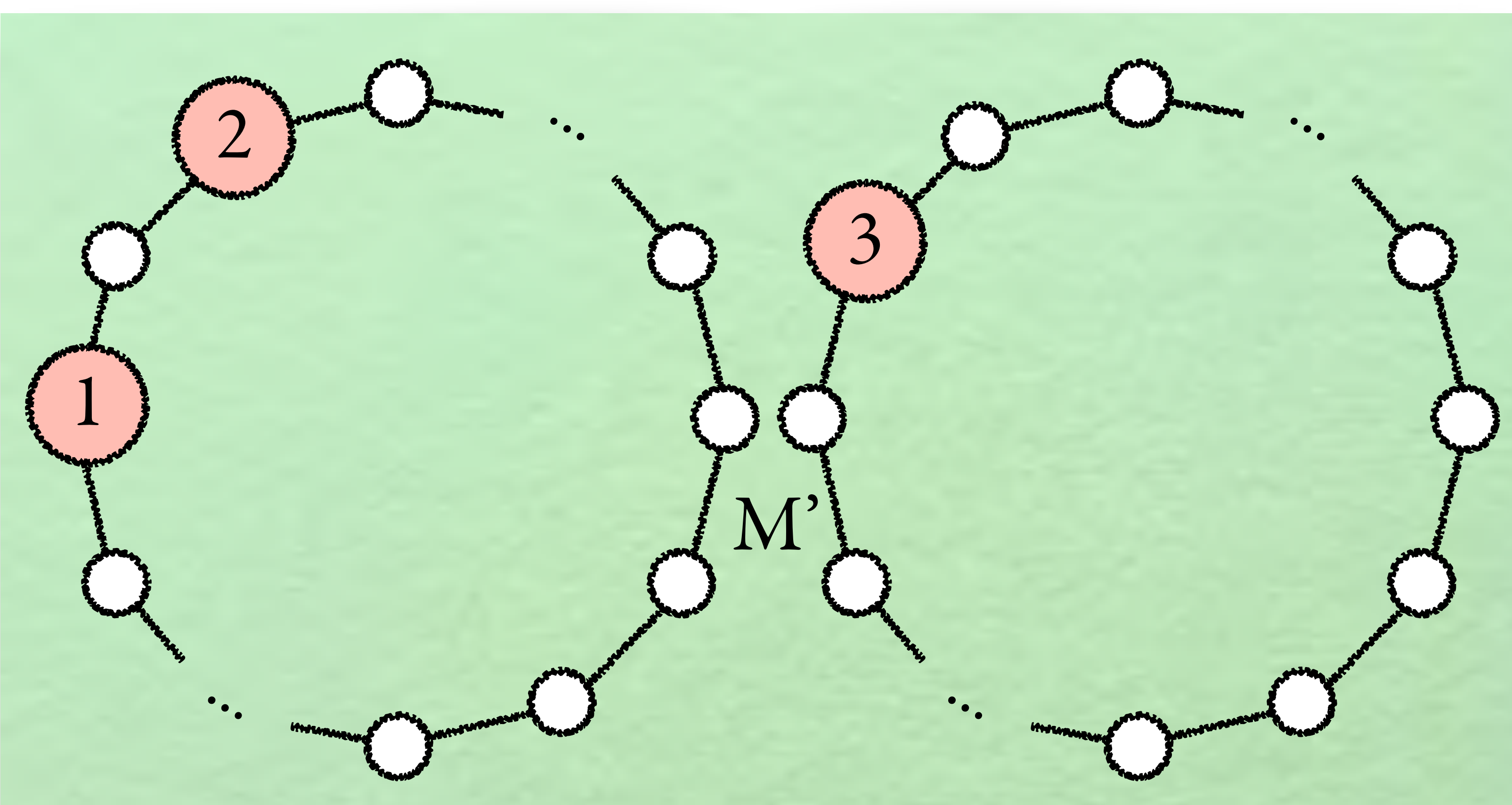

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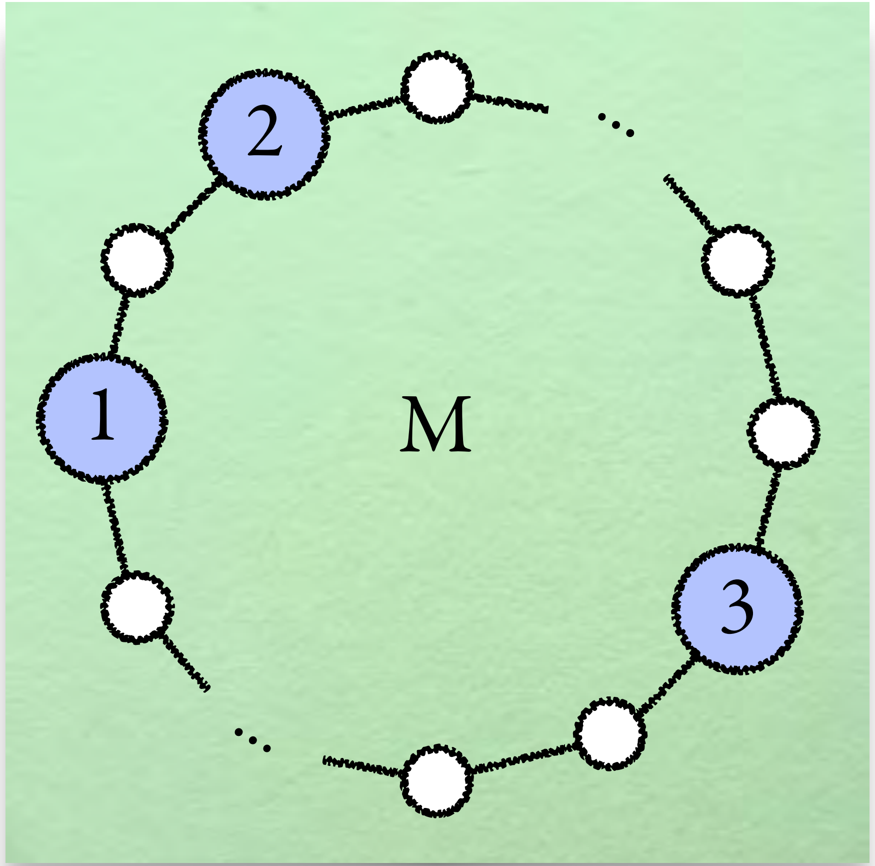

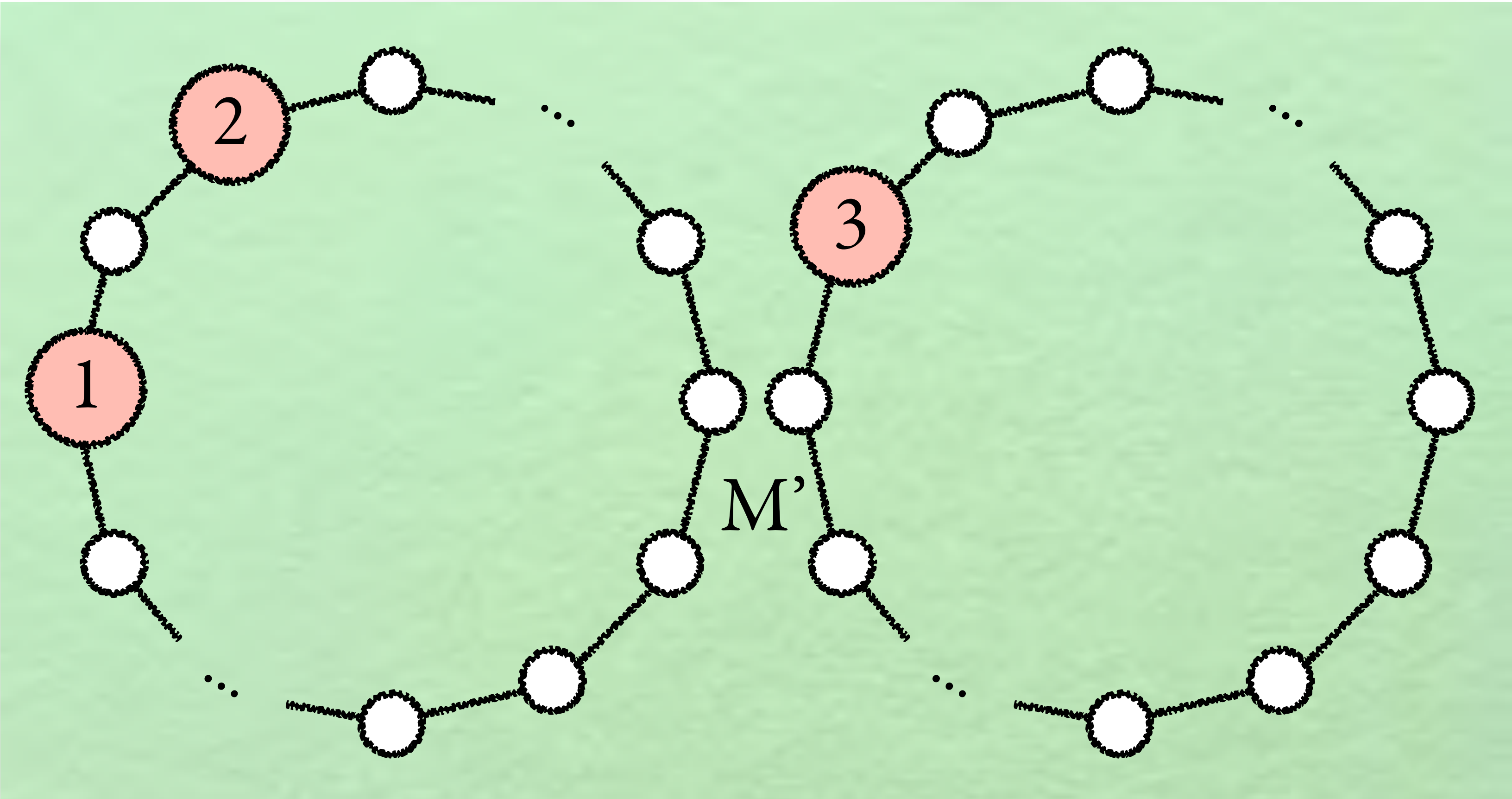

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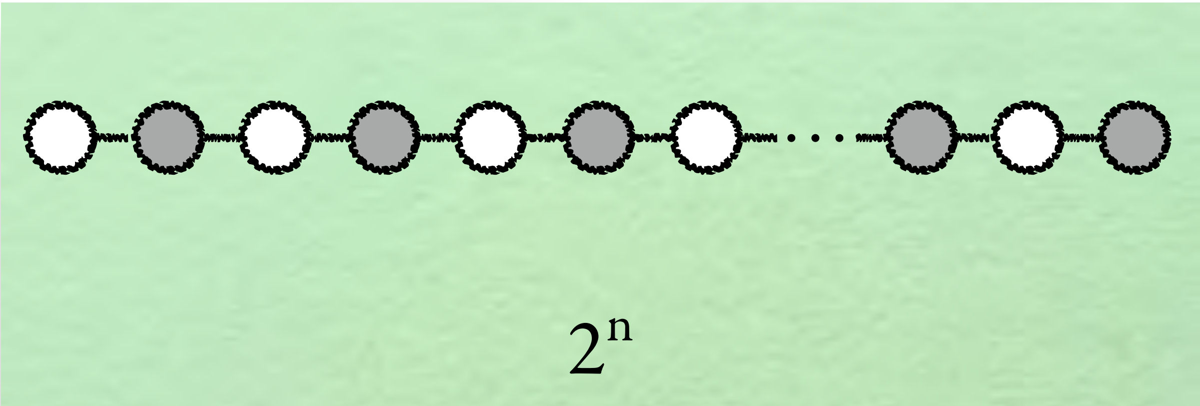

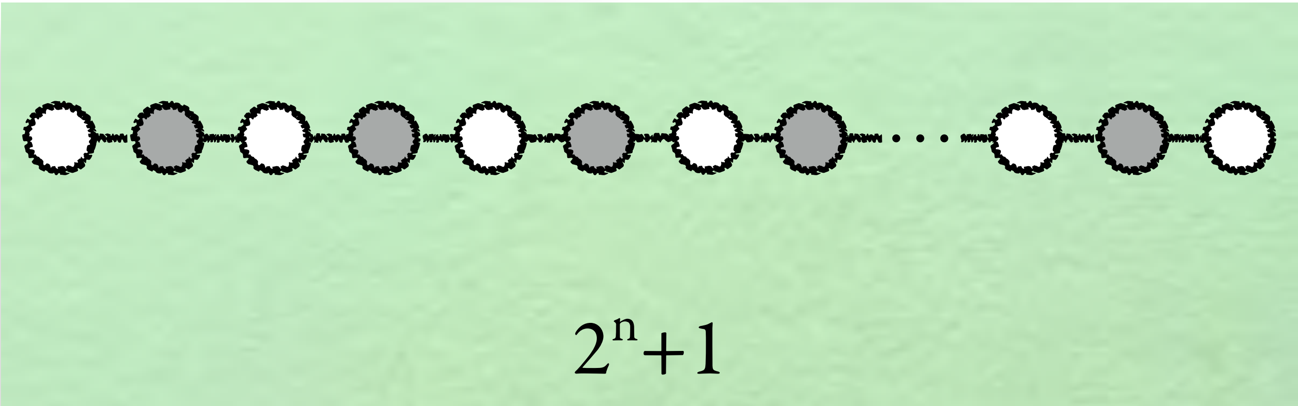

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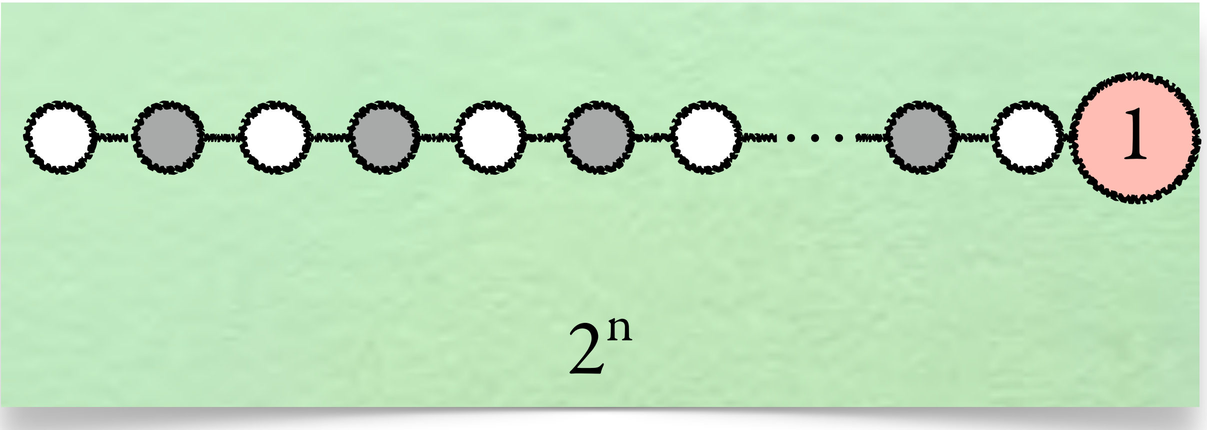

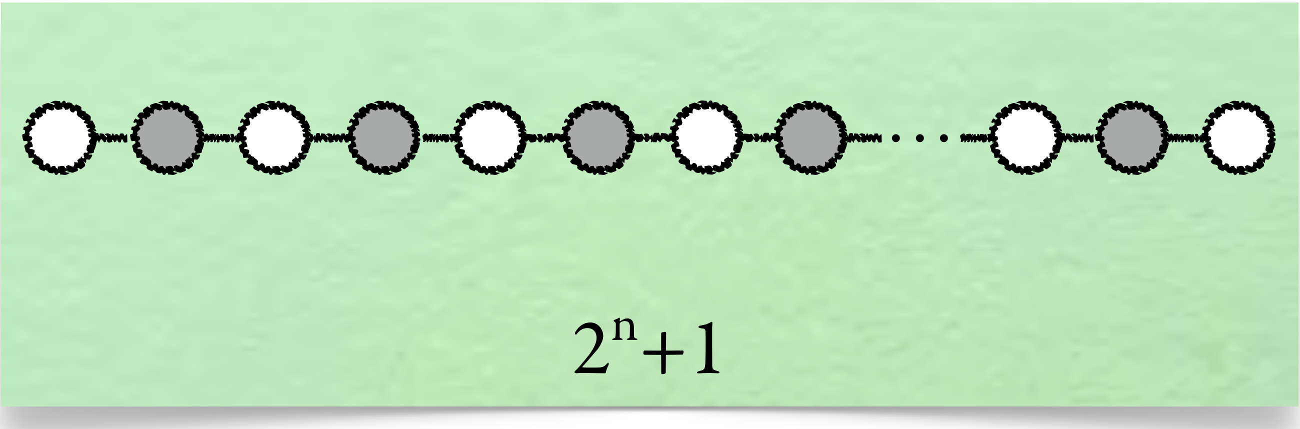

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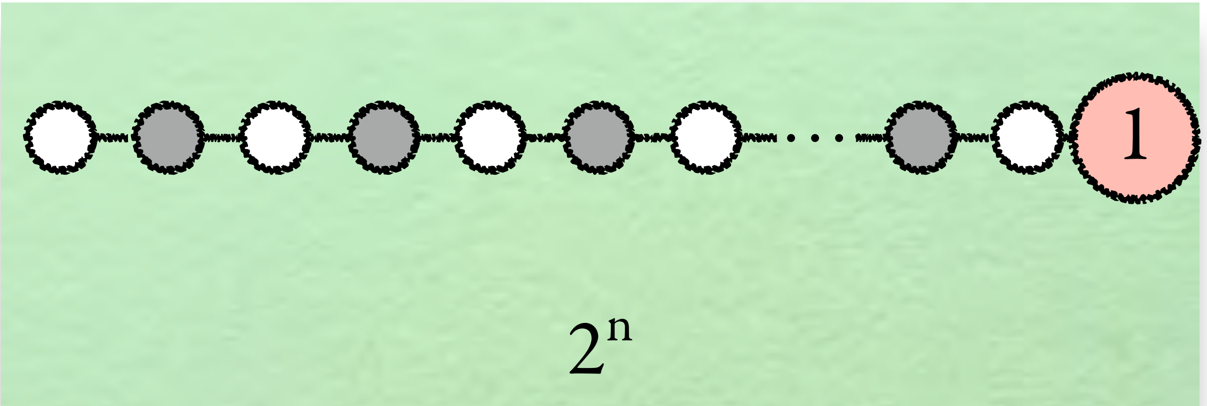

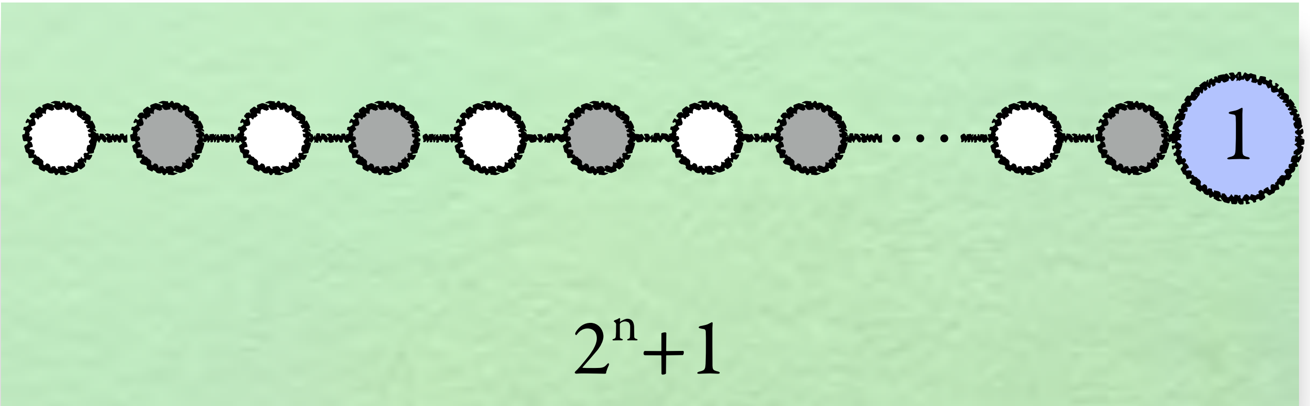

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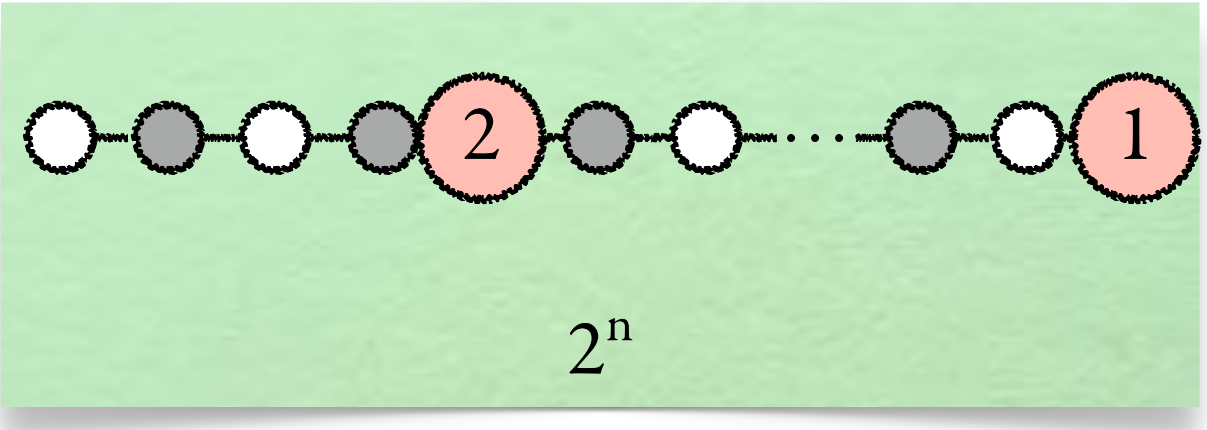

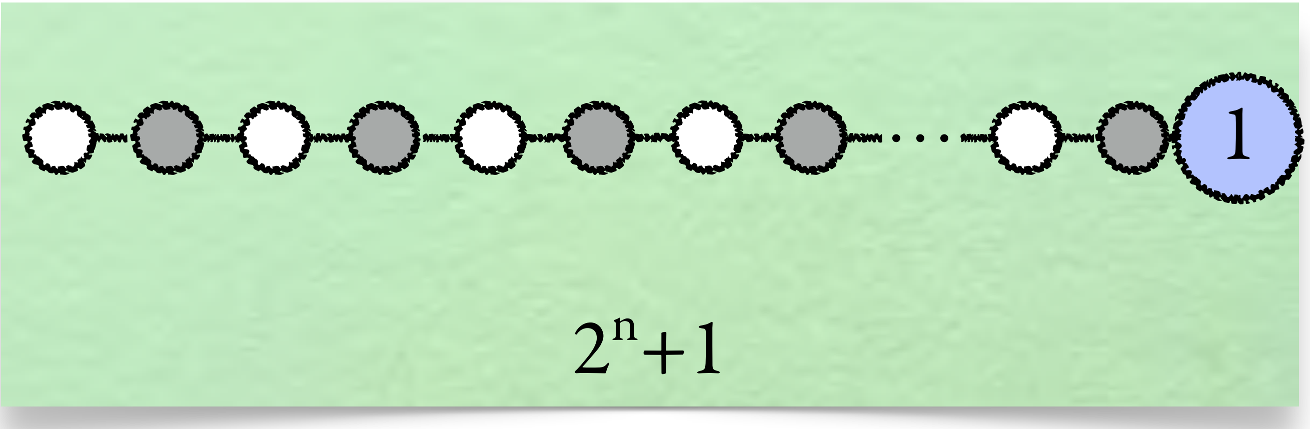

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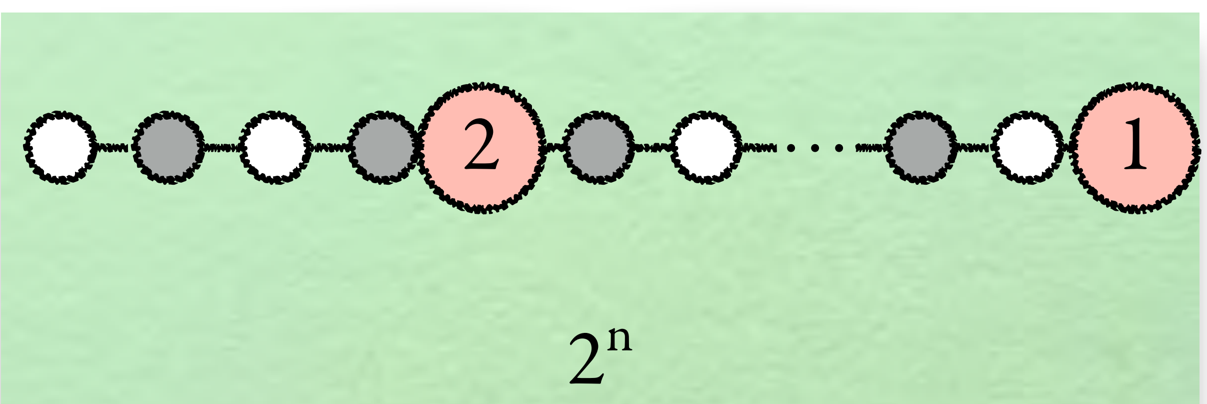

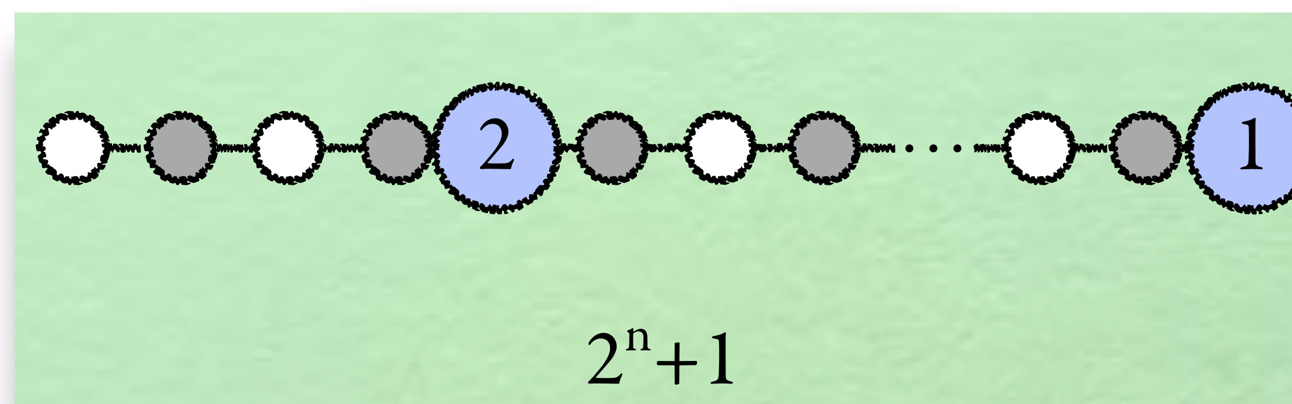

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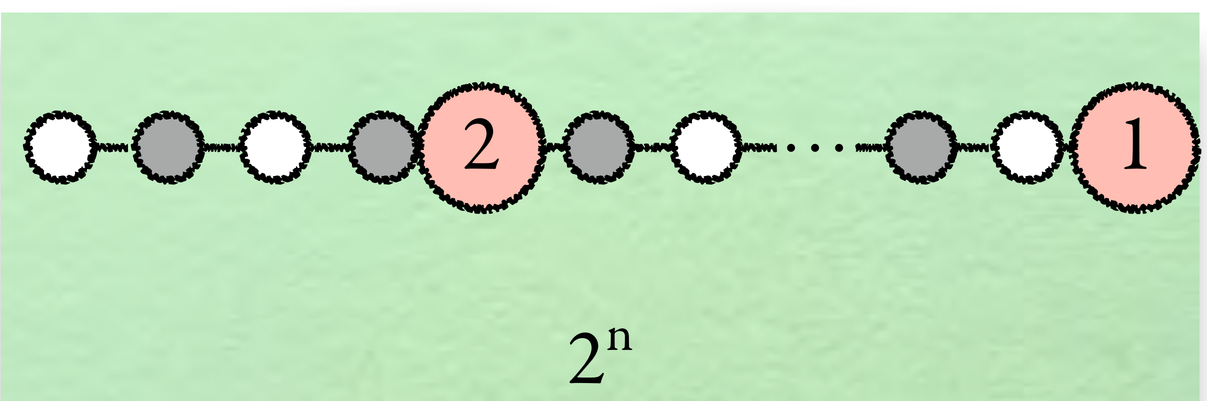

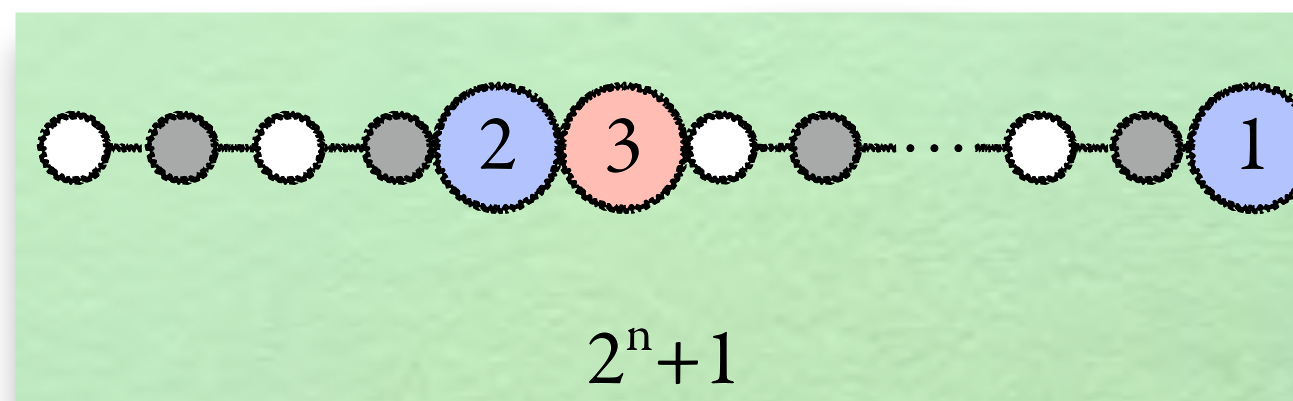

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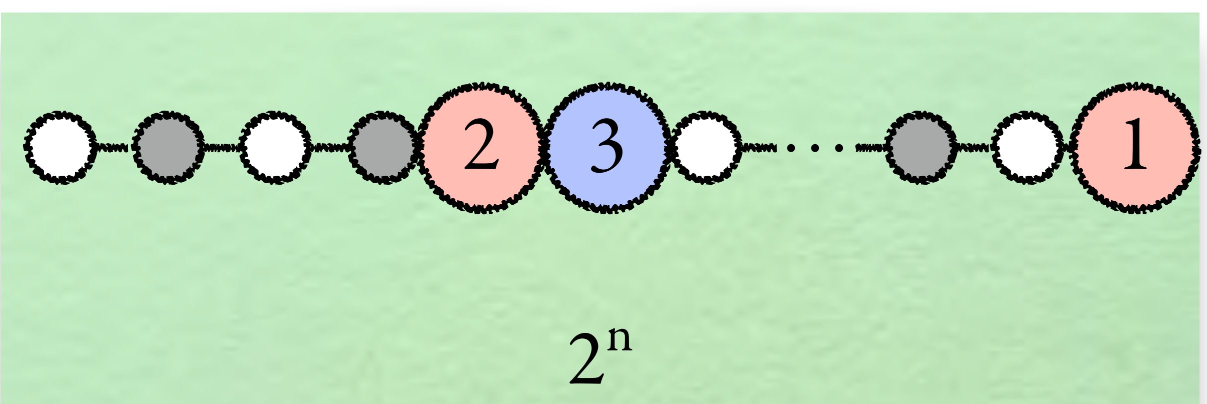

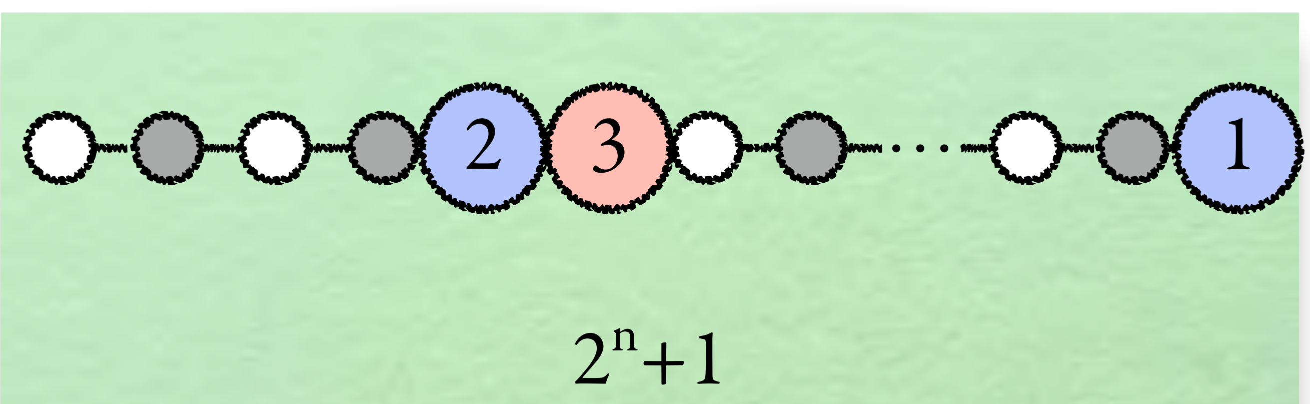

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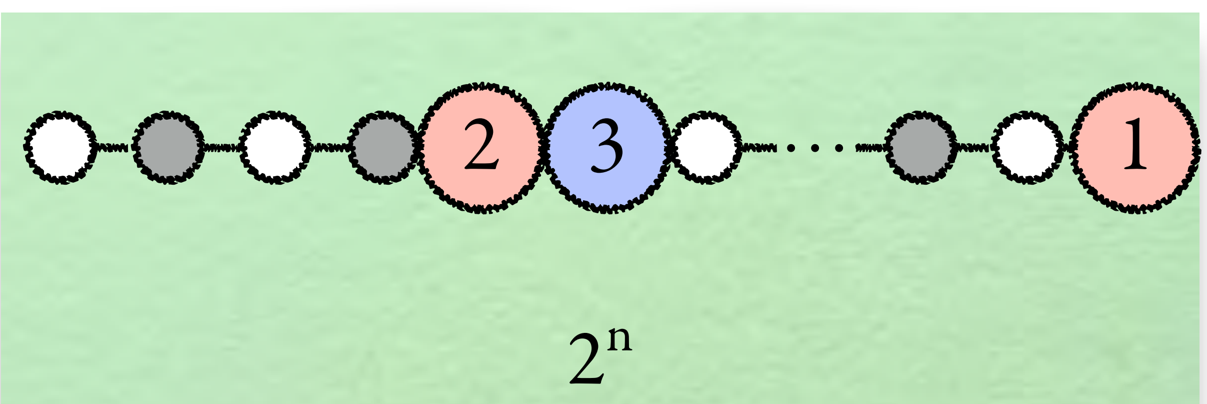

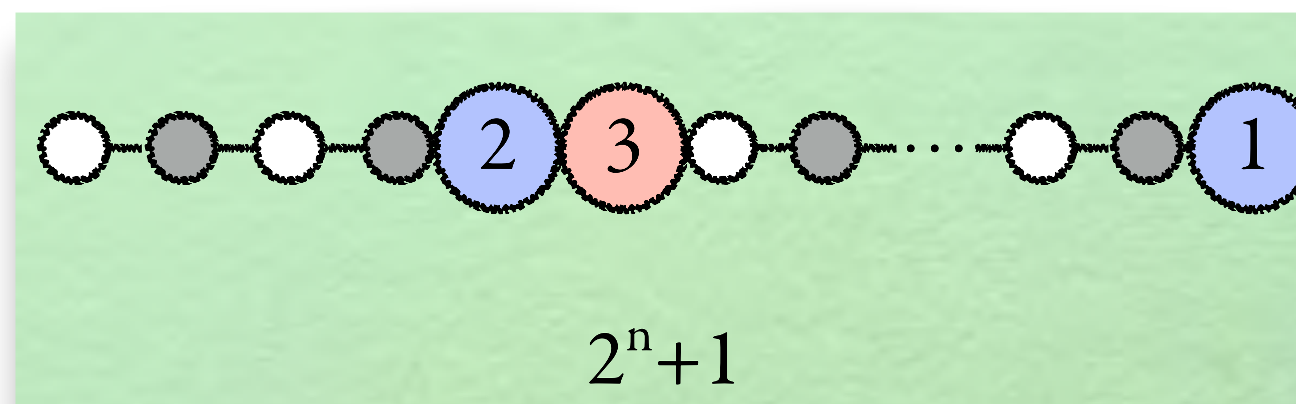

Rule of thumb If Spoiler plays "close" to previous pebbles,
then Duplicator responds *isomorphically within the corresponding neighbourhoods*
otherwise Duplicator plays "far" but has freedom of choice

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé games

Several properties can be proved to be *not* definable in FO:

- connectivity
- parity (i.e. even / odd)
- 2-colorability
- finiteness
- acyclicity

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Your turn now!

...

Ehrenfeucht-Fraïssé games — soundness

Theorem M,M' *n*-equivalent iff Duplicator survives *n* rounds in $G_{M,M'}$
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Proof (if direction — from Duplicator's strategy to *n*-equivalence)

Consider ϕ in NNF and with quantifier rank n
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Need to prove that $M' \vDash \phi$

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Eve binds x to $u \in U^M$

evaluation game $G_{M,\phi}$

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EF game $G_{M,M'}$

Spoiler places pebble u

Eve binds x to u ∈ U^M

evaluation game $G_{M,\phi}$

?

Eve

evaluation game $G_{M',\phi}$

Ehrenfeucht-Fraïssé games

Theorem M,M' *n*-equivalent iff Duplicator survives *n* rounds in $G_{M,M'}$

[Fraïssé '50, Ehrenfeucht '60]

Proof (if direction — from Duplicator's strategy to *n*-equivalence)

True wins the *evaluation game* $G_{M,\phi}$

Consider ϕ in NNF and with quantifier rank n

Suppose $M \vDash \phi$ and Duplicator survives *n* rounds in $G_{M,M'}$

Need to prove that $M' \vDash \phi$

True wins the *evaluation game* $G_{M',\phi}$

4 cases based on subformula:

- disjunction
- conjunction
- existential quantification
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EF game $G_{M,M'}$

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Duplicator responds with v

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Suppose M,M' are *n*-equivalent (i.e. *for every* ϕ of q.r. *n*, $M \vDash \phi$ iff $M' \vDash \phi$)
Need to construct a strategy for Duplicator ...

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ϕ needs to be *strong enough to cover all cases for Duplicator*

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We need a HUGE ϕ !

Hintikka formulas

Level-n Hintikka formula of M = strongest formula (up to logical equivalence) of quantifier rank n that holds on M

$$\phi^n_M$$

≈ "FO theory of M relativised to q.r. n"

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$$\phi_M^n$$

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Constructed inductively on n:

$$\phi_M^0 = \bigwedge_{\substack{\alpha \text{ atomic} \\ M \models \alpha}} \alpha \;\wedge\; \bigwedge_{\substack{\alpha \text{ atomic} \\ M \not\models \alpha}} \neg\alpha$$

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Ehrenfeucht-Fraïssé games — completeness

We give a strategy for Duplicator

under the hypothesis M,M' *n*-equivalent

$$\phi^{n}_{M} = \bigwedge_{\substack{u \in U^M \\ M_u = M[x:=u]}} \exists x\, \phi^{n-1}_{M_u} \quad \wedge \quad \forall x \bigvee_{\substack{u \in U^M \\ M_u = M[x:=u]}} \phi^{n-1}_{M_u}$$

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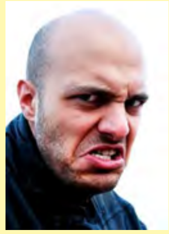
EF game $G_{M,M'}$



Spoiler places pebble $u \in U^M$

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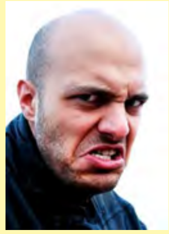
evaluation game $G_{M,\phi}$

Adam picks left conjunct $\wedge\ \exists x \ldots$
then picks conjunct $\exists x\, \phi^{n-1}_{M_u}$

EF game $G_{M,M'}$



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Facts:

- $M \models \phi^{n}_{M}$

  and Eve has a *standard* winning strategy in $G_{M,\phi}$

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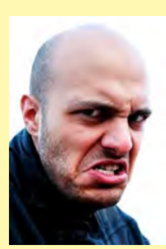

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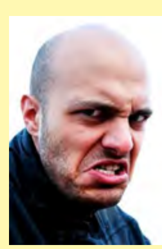
evaluation game $G_{M,\phi}$

Adam picks left conjunct $\bigwedge \exists x \ldots$
then picks conjunct $\exists x\, \phi^{n-1}_{M_u}$

Eve can safely bind x to u
and continue the game

EF game $G_{M,M'}$



Spoiler places pebble $u \in U^M$

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We give a strategy for Duplicator

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Facts:

- $M \vDash \phi^n_M$

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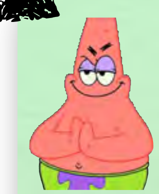
evaluation game $G_{M,\phi}$

Adam picks left conjunct $\bigwedge \exists x \dots$
then picks conjunct $\exists x \, \phi^{n-1}_{M_u}$

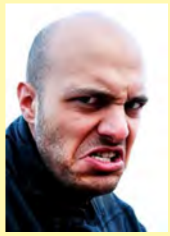
Eve can safely bind x to u
and continue the game

evaluation game $G_{M',\phi}$



Adam picks same conjunct $\exists x \, \phi^{n-1}_{M_u}$

EF game $G_{M,M'}$



Spoiler places pebble $u \in U^M$

Ehrenfeucht-Fraïssé games — completeness

We give a strategy for Duplicator

under the hypothesis M,M' *n*-equivalent

$$\phi^{n}_{M} \;=\; \bigwedge_{\substack{u \in U^M \\ M_u = M[x:=u]}} \exists x\, \phi^{n-1}_{M_u} \;\wedge\; \forall x \bigvee_{\substack{u \in U^M \\ M_u = M[x:=u]}} \phi^{n-1}_{M_u}$$

Facts:

- $M \vDash \phi^{n}_{M}$

  and Eve has a *standard* winning strategy in $G_{M,\phi}$

- $M' \vDash \phi^{n}_{M}$

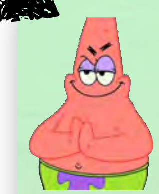
  so Eve has also a winning strategy in $G_{M',\phi}$

evaluation game $G_{M,\phi}$

Adam picks left conjunct $\bigwedge \exists x \ldots$ then picks conjunct $\exists x\, \phi^{n-1}_{M_u}$

Eve can safely bind x to u and continue the game

evaluation game $G_{M',\phi}$



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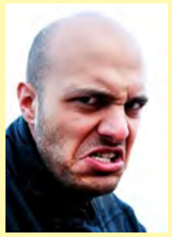

Spoiler places pebble $u \in U^M$

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We give a strategy for Duplicator

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$$\phi^n_M \;=\; \bigwedge_{\substack{u \in U^M \\ M_u = M[x:=u]}} \exists x\, \phi^{n-1}_{M_u} \;\wedge\; \forall x \bigvee_{\substack{u \in U^M \\ M_u = M[x:=u]}} \phi^{n-1}_{M_u}$$

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Adam picks left conjunct $\bigwedge \exists x$ ...
then picks conjunct $\exists x\, \phi^{n-1}_{M_u}$

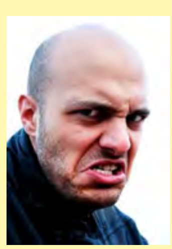
Eve can safely bind x to u
and continue the game

evaluation game $G_{M',\phi}$

Adam picks same conjunct $\exists x\, \phi^{n-1}_{M_u}$

Eve binds x to *some* element v
and continue the game

EF game $G_{M,M'}$



Spoiler places pebble $u \in U^M$

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We give a strategy for Duplicator

under the hypothesis M,M' *n*-equivalent

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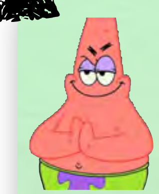

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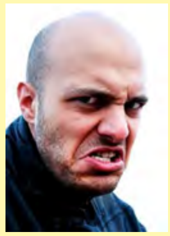

Spoiler places pebble $u \in U^M$

Duplicator responds with $v \in U^{M'}$

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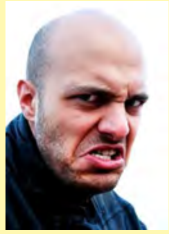

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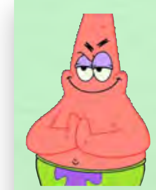

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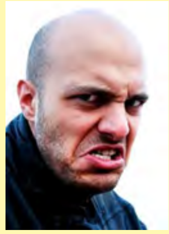
EF game $G_{M,M'}$



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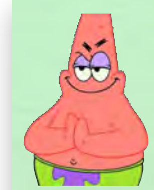

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Eve picks *some* disjunct $\phi^{n-1}_{M_u}$ and continue the game

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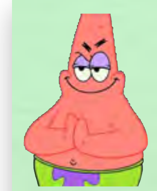

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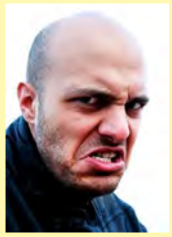
Duplicator responds with $u \in U^M$



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Ehrenfeucht-Fraïssé games — completeness

We give a strategy for Duplicator

under the hypothesis M,M' *n*-equivalent

$$\phi^{n}_{M} = \bigwedge_{\substack{u \in U^M \\ M_u = M[x:=u]}} \exists x\, \phi^{n-1}_{M_u} \;\wedge\; \forall x \bigvee_{\substack{u \in U^M \\ M_u = M[x:=u]}} \phi^{n-1}_{M_u}$$

Facts:

- $M \models \phi^n_M$

  and Eve has a *standard* winning strategy in $G_{M,\phi}$

- $M' \models \phi^n_M$

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evaluation game $G_{M,\phi}$

Adam picks same conjunct $\forall x \bigvee \ldots$ then binds x to u

evaluation game $G_{M',\phi}$

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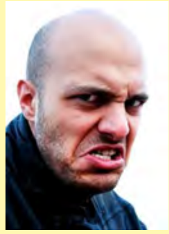

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Ehrenfeucht-Fraïssé games — a few more things

Theorem M,M' *n*-equivalent iff Duplicator survives *n* rounds in $G_{M,M'}$

iff ϕ^n_M and $\phi^n_{M'}$ are logically equivalent

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Theorem M,M' *n*-equivalent iff Duplicator survives *n* rounds in $G_{M,M'}$

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So,

1. ϕ^n_M can be used as a representant of the *n*-equivalence class of M
2. For every ϕ' of q.r. n, $\phi' \in FO[M]$ iff ϕ' is a logical consequence of ϕ^n_M

Another use of Ehrenfeucht-Fraïssé games — 0/1 Law

Theorem (0/1 Law)
[Glebskii et al. '69, Fagin '76]

Every FO formula ϕ is
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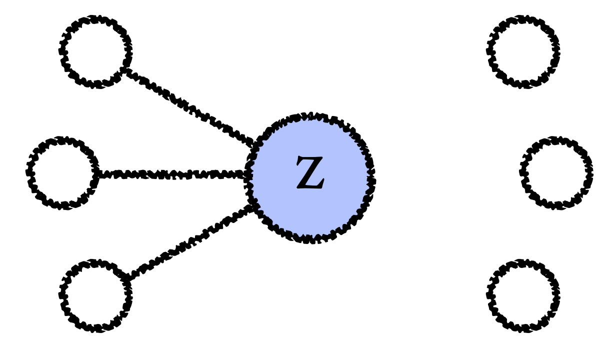
Let n = quantifier rank of ϕ

$$\delta_n = \forall x_1, \ldots, x_n\ \forall y_1, \ldots, y_n\ \exists z\ \bigwedge_{i,j} x_i \neq y_j \wedge E(x_i, z) \wedge \neg E(y_j, z)$$

(<u>extension formula</u>)



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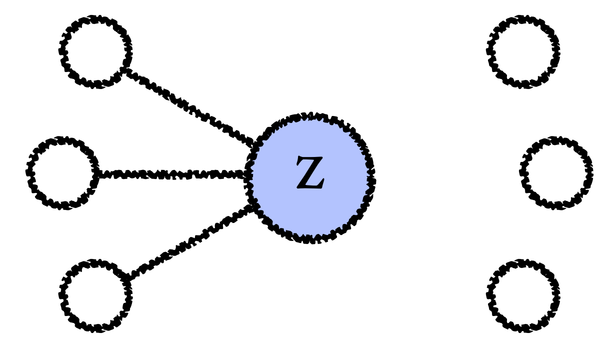

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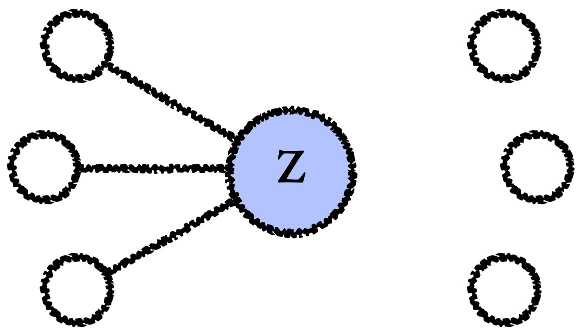

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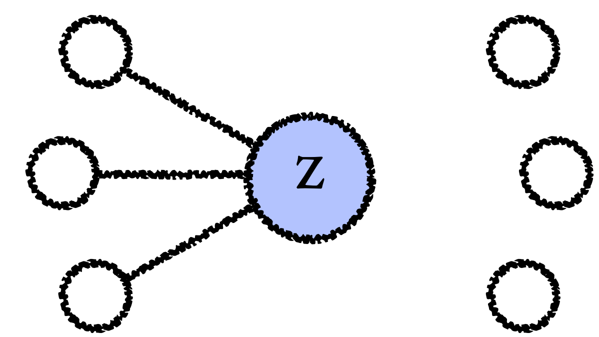

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2 cases

a) There is M $M \vDash \delta_n \wedge \phi$ $\Rightarrow$ (by Fact 1) for every M' if $M' \vDash \delta_n$ then $M' \vDash \phi$

Thus, $P_\infty[\delta_n] \leq P_\infty[\phi]$

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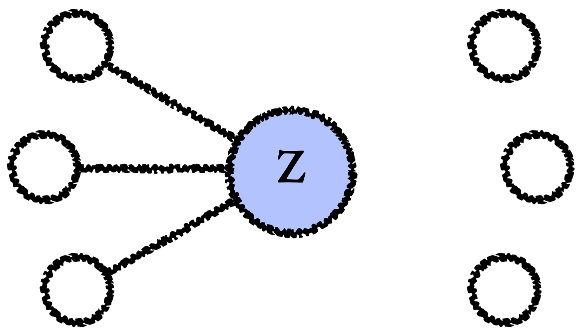

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b) There is no $M \vDash \delta_n \wedge \phi$ $\Rightarrow$ (by Fact 2) there is $M \vDash \delta_n$

$\Rightarrow$ $M \vDash \delta_n \wedge \neg\phi$ $\Rightarrow$ (by case a) $P_\infty[\neg\phi] = 1$

Yes another use of games: synthesis (evaluation games in disguise!)

A. Church

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" Given a *requirement* which a *circuit* is to satisfy,
we may suppose the requirement expressed in
some suitable logistic system which is an
extension of restricted recursive arithmetic.
The *synthesis problem* is then to find
recursion equivalences representing
a circuit that satisfies the given requirement
(or alternatively, to determine that there is no such circuit). "

A. Church

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Recall again the plain reachability problem encoded in QBF

k bits

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
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$\exists \bar{p}_1 \ldots \exists \bar{p}_n \ \phi_{path}(\bar{p}_1,\ldots,\bar{p}_n)$

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$$\exists \bar{p}_1 \ldots \exists \bar{p}_n \;\; \phi_{path}(\bar{p}_1,\ldots,\bar{p}_n)$$

Now, suppose first 2 bits ($\bar{q}_i$) are controlled by Environment

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If so, can we synthesise a strategy for Circuit?

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$$\forall \bar{q}_1 \exists \bar{p}_1 \ldots \forall \bar{q}_n \exists \bar{p}_n \ \ \phi_{path}(\bar{q}_1,\bar{p}_1,\ldots,\bar{q}_n,\bar{p}_n)$$

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Things to remember

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- EF games are a powerful tool (sound & complete) to study <u>definability in FO</u>

  technique: 1) given property P and $n \in \mathbb{N}$
  2) find two models $M \in P$, $M' \notin P$ (which may depend on n !)
  3) show that Duplicator has strategy to survive n rounds in $G_{M,M'}$

- EF games can also be easily adapted to other logics and problems

What next?

More models: infinite words, infinite trees

More power: MSO = Monadic Second-order logic

More tools: automata

An appetiser — FO logic over words

Fix $\Sigma = \{A, B, C, \ldots\}$ set of unary relational symbols

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So,

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- $\phi = \exists x\, \forall y\ (y \leq x \rightarrow A(y)) \wedge (y > x \rightarrow B(y))$ defines $L_\phi = A^*B^*$

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- Can you define in FO the language $L = A^*\, B\, A^*$? And $L = (AA)^*$?