First-Order logic (FO)

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FO = First-Order logic

Vocabulary	<u>Relational symbols</u>:	$\Sigma = \{R, S, T, \ldots\}$	(aka <u>signature</u>)
	<u>Variables</u>:	$x, y, \ldots, x_1, x_2, \ldots$	
	<u>Quantifiers</u>:	$\exists, \forall$	
	Boolean connectives:	$\vee, \wedge, \neg, \rightarrow, \leftrightarrow$	

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Syntax $\quad \phi : \quad R(x_1,\dots,x_k) \mid \dots \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \phi \rightarrow \phi \mid \phi \leftrightarrow \phi$

$\exists x\, \phi \mid \forall x\, \phi \mid \dots$

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Semantics

Now a model consists of a universe U^M

+ some mappings $R \mapsto R^M \subseteq U^M \times \dots \times U^M$

$x \mapsto x^M \in U^M$

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Syntax

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+ some mappings $R \mapsto R^M \subseteq U^M \times \ldots \times U^M$

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$M \models \phi_1 \vee \phi_2$ iff $M \models \phi_1$ or $M \models \phi_2$

…

$M \models R(x_1,\ldots,x_k)$ iff $(x_1^M,\ldots,x_k^M) \in R^M$

$M \models \exists x\, \phi$ iff $M[x:=u] \models \phi$ for *some* $u \in U^M$

$M \models \forall x\, \phi$ iff $M[x:=u] \models \phi$ for *every* $u \in U^M$

Examples

Syntax $\phi : \quad R(x_1,\ldots,x_k) \mid \ldots \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \phi \to \phi \mid \phi \leftrightarrow \phi$

$\exists x\ \phi \mid \forall x\ \phi \mid \ldots$

"All humans are mortal. Socrates is human. So Socrates is mortal."

$$\phi(y) = ((\forall x\ A(x) \to B(x))\ \&\ A(y)) \to B(y)$$

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M : $\quad U^M = \{\text{Socrates, Plato, Cyclop, Jupiter}\}$

$\quad A^M = \{\text{Socrates, Plato}\}$

$\quad B^M = \{\text{Socrates, Plato, Cyclop}\}$

$\quad y^M = \text{Socrates}$

Examples

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"There is a node in the graph that is isolated from all other nodes."

$$\phi = \exists x\, \forall y\, \neg(x=y) \to \neg E(x,y)$$

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M : U^M = {nodes of a graph}
E^M = {edges of a graph}

Examples

Syntax $\phi : \quad R(x_1,\ldots,x_k) \;|\; \ldots \;|\; \phi \lor \phi \;|\; \phi \land \phi \;|\; \neg\phi \;|\; \phi \to \phi \;|\; \phi \leftrightarrow \phi$

$\exists x\, \phi \;|\; \forall x\, \phi \;|\; \ldots$

Examples

Syntax ϕ : $R(x_1,\ldots,x_k)$ | … | $\phi \vee \phi$ | $\phi \wedge \phi$ | $\neg\phi$ | $\phi \rightarrow \phi$ | $\phi \leftrightarrow \phi$ |
$\exists x\ \phi$ | $\forall x\ \phi$ | …

"There's a man such that when he runs, everybody runs."

$$\phi = \exists x\ R(x) \rightarrow \forall y\ R(y)$$

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M : $U^M = \{\text{Ben, Han, Leia, Luke}\}$

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Examples

- "R is a function" $\phi = \forall x\, \exists y\; R(x,y) \;\wedge\; \forall z\, R(x,z) \rightarrow y=z$

  in this case, one can use the shorthand
  "R(x)=…" for $\exists y\, R(x,y) \;\wedge\; \forall z\, R(x,z) \rightarrow z=\ldots$

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- "+ admits *zero* and *inverses*" $\phi = \exists x_0\ \ \forall y\ \ x_0+y = y\ \wedge\ \forall y\ \exists z\ \ y+z = x_0$

Exercices

- "f is continuous" $\phi = \forall x\ \forall \varepsilon\ \exists \delta\ \forall y\ \ ||x\text{-}y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$
- "f is uniformly continuous" $\phi = \forall \varepsilon\ \exists \delta\ \forall x\ \forall y\ \ ||x\text{-}y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$

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What is an appropriate signature for the above formulas?

Exercices

- "f is continuous" $\phi = \forall x\ \forall \varepsilon\ \exists \delta\ \forall y\ \ ||x-y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$

- "f is uniformly continuous" $\phi = \forall \varepsilon\ \exists \delta\ \forall x\ \forall y\ \ ||x-y|| < \delta \rightarrow ||f(x) - f(y)|| < \varepsilon$

What is an appropriate <u>signature</u> for the above formulas?

Are the formulas equivalent? Is one a consequence of another? Can you prove it?

(hint: $\exists x\ \forall y\ \alpha \rightarrow \forall y\ \exists x\ \alpha$ *assuming universe is non-empty*)

Exercices

Choose appropriate universes and signatures, and define these properties in FO:

1. "There are infinitely many Prime numbers" $\phi = \ldots$

2. "In the tree, z is the least common ancestor of x and y" $\phi(x,y,z) = \ldots$

3. "Polynomial *p* evaluates to y on x" (for fixed *p*) $\phi_p(x,y) = \ldots$

4. "The graph is strongly connected" $\phi = \ldots$

5. "In the infinite sequence of *a*'s and *b*'s, every *a* is followed by *b*" $\phi = \ldots$

Normal forms

Prenex [+CNF/DNF] as for QBF, i.e. $\phi = Qx_1 \ldots Qx_n \ \alpha(x_1,\ldots,x_n)$

NNF (Negation Normal Form)

$\phi : \quad \exists x\, \phi \ \mid \ \forall x\, \phi \ \mid \ \phi \vee \phi \ \mid \ \phi \wedge \phi \ \mid \ \alpha$

$\alpha : \quad R(x_1,\ldots,x_k) \ \mid \ \neg R(x_1,\ldots,x_k)$

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Lemma Given ϕ ($\leftrightarrow$-free), one can compute in polynomial time an *equivalent* formula ϕ^* in NNF

Proof As for propositional logic, push negations inside:

$$\neg\forall\phi \rightsquigarrow \exists\neg\phi$$
$$\neg\exists\phi \rightsquigarrow \forall\neg\phi$$
$$\neg(\phi_1 \wedge \phi_2) \rightsquigarrow \neg\phi_1 \vee \neg\phi_2$$
$$\neg(\phi_1 \vee \phi_2) \rightsquigarrow \neg\phi_1 \wedge \neg\phi_2$$

Algorithms

Model-checking problem

input: formula ϕ + *finite* model M
output: yes iff $M \models \phi$

Satisfiability problem

input: formula ϕ
output: yes iff $M \models \phi$ for *some* M

(recall: ϕ valid iff $\neg\phi$ is not satisfiable
ϕ, ϕ' equivalent iff $\phi \leftrightarrow \phi'$ is valid)

Algorithms

🎉 **PSPACE** 🎉

Model-checking problem

input: formula ϕ + *finite* model M
output: yes iff $M \vDash \phi$

💀 **UNDECIDABLE** 💀

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Algorithms — model-checking

```
Model-check(φ, M)
    if φ = R(x₁,…,xₖ) then
        if (x₁ᴹ,…,xₖᴹ) ∈ Rᴹ then
            return true
        else
            return false
    else if φ = φ₁ ∨ φ₂ then
        return Model-check(φ₁, M) OR
               Model-check(φ₂, M)
    else if …
    …
    else if φ = ∃x φ' then
        for u ∈ Uᴹ do
            if Model-check(φ', M[x:=u]) then
                return true
        return false
    else if φ = ∀x φ' then
        for u ∈ Uᴹ do
            if NOT Model-check(φ', M[x:=u]) then
                return false
        return true
```

Algorithms — satisfiability

Theorem [Trakhtenbrot '50] Satisfiability of FO is undecidable

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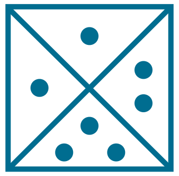
Proof by reduction from Domino (aka Tiling) problem...



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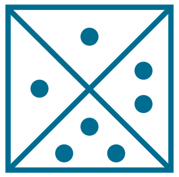


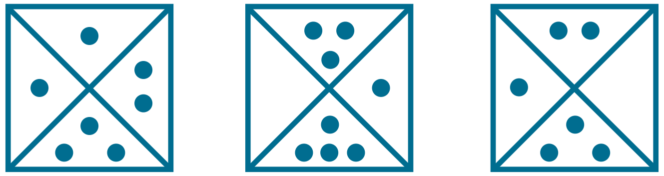
Reduction from P to P': Algorithm A that solves P by using an oracle that returns solutions to P'

(think of "P easier than P'")

e.g. many-one reduction: for all x P(x) iff P'(A(x))

The (undecidable) Domino problem

Domino

Input: 4-sided dominos: 

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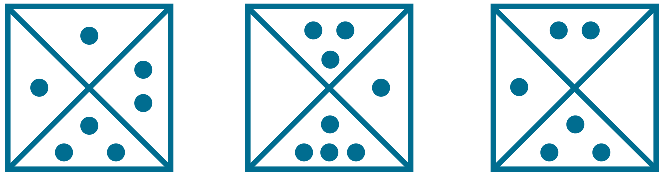
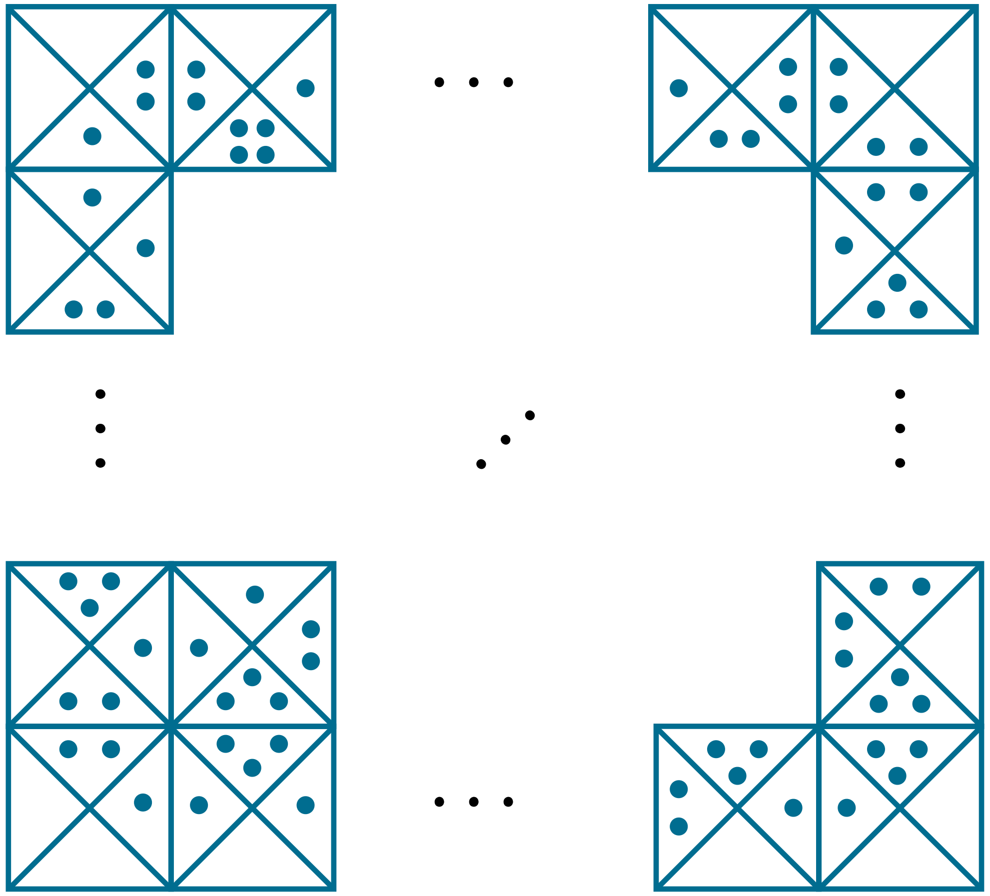
Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)



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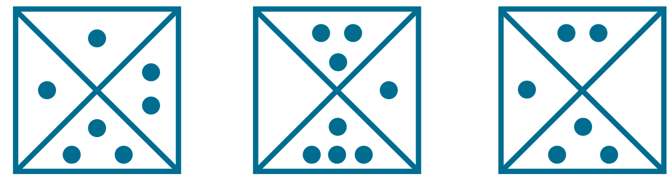
Input: 4-sided dominos: 

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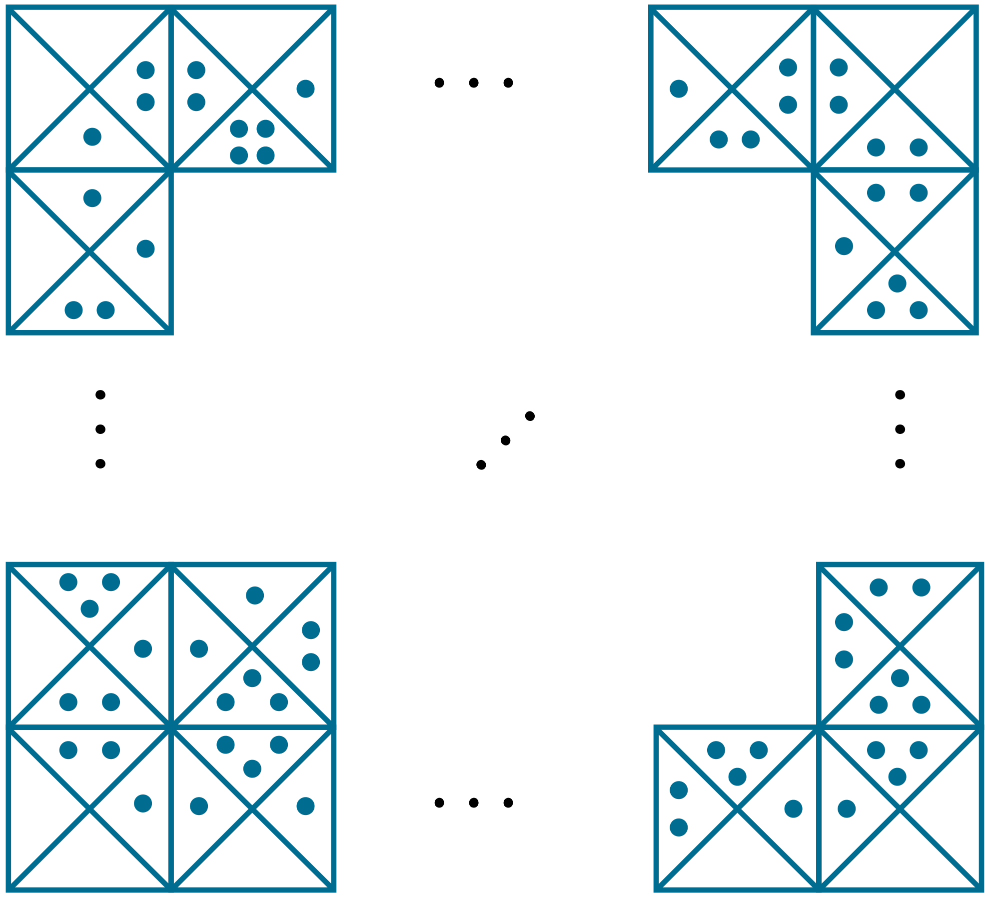
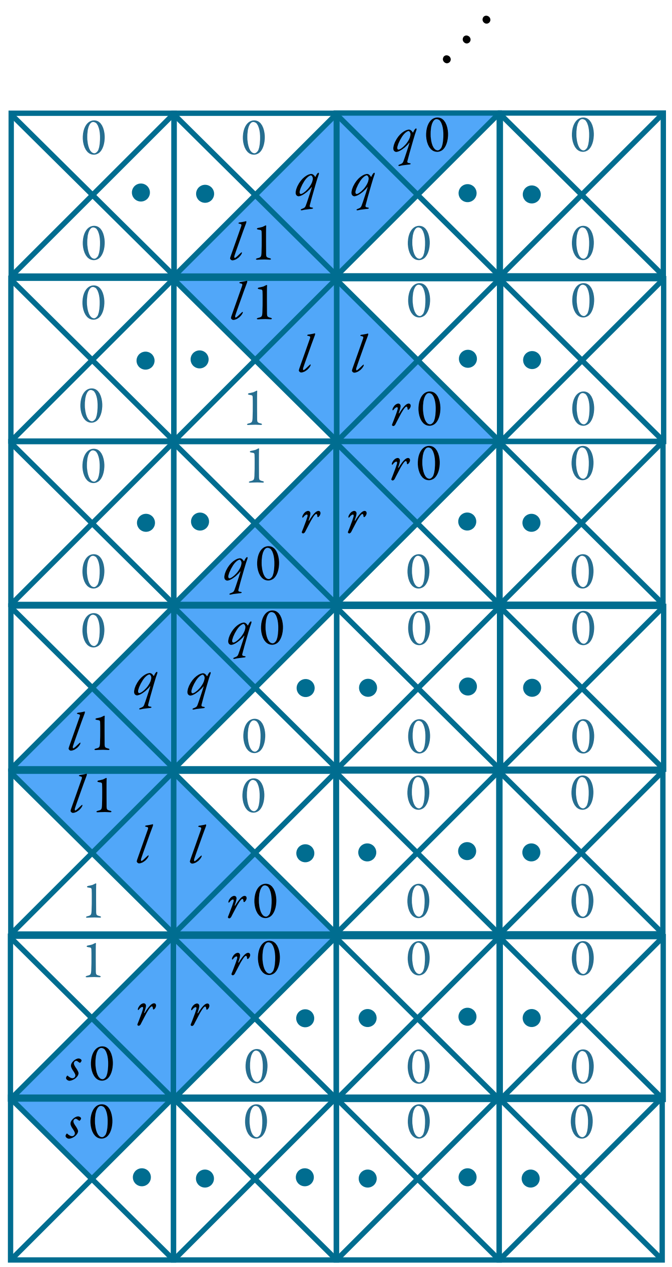


Rules: sides must match,
you can't rotate the dominos, but you can 'clone' them.

The (undecidable) Domino problem

Domino - Why is it undecidable?

It can encode *halting* computations of Turing machines:



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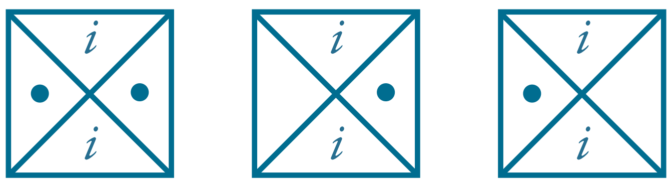

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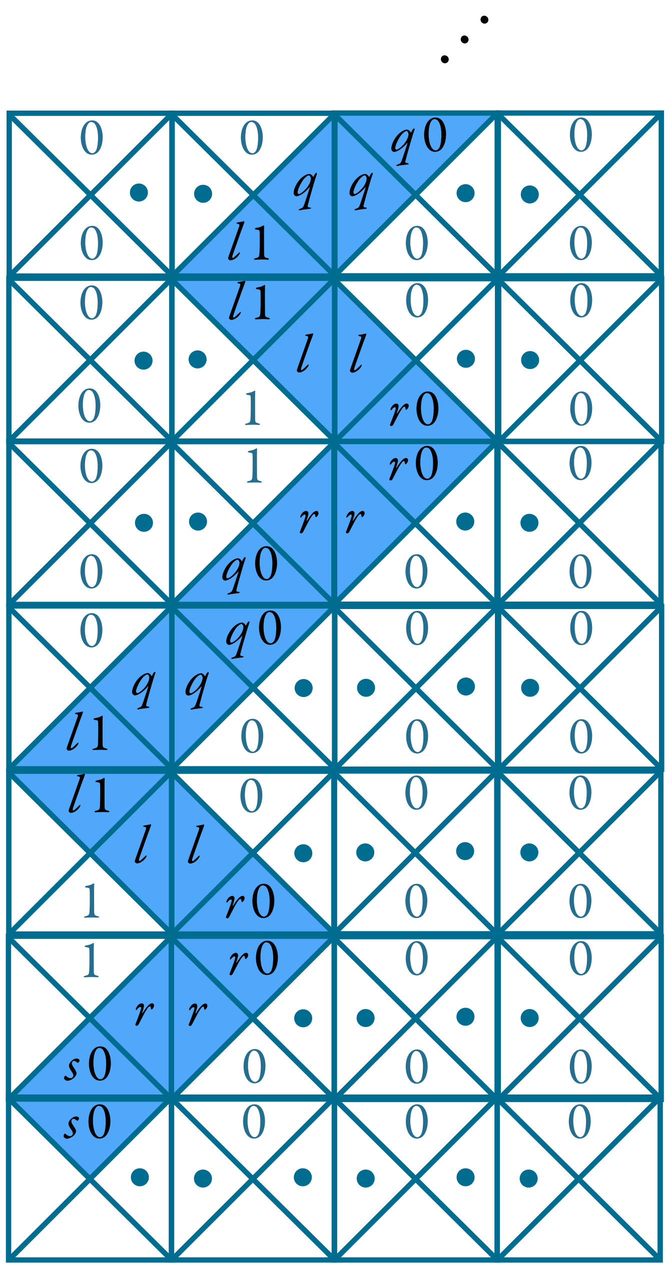

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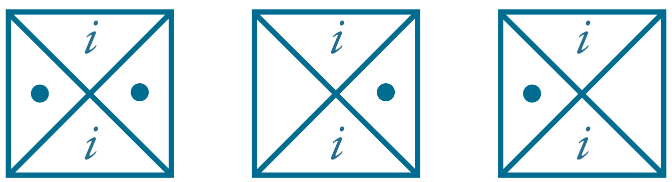

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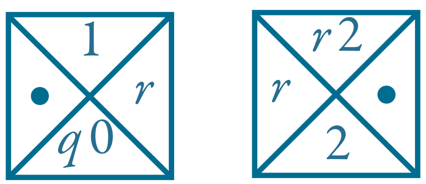

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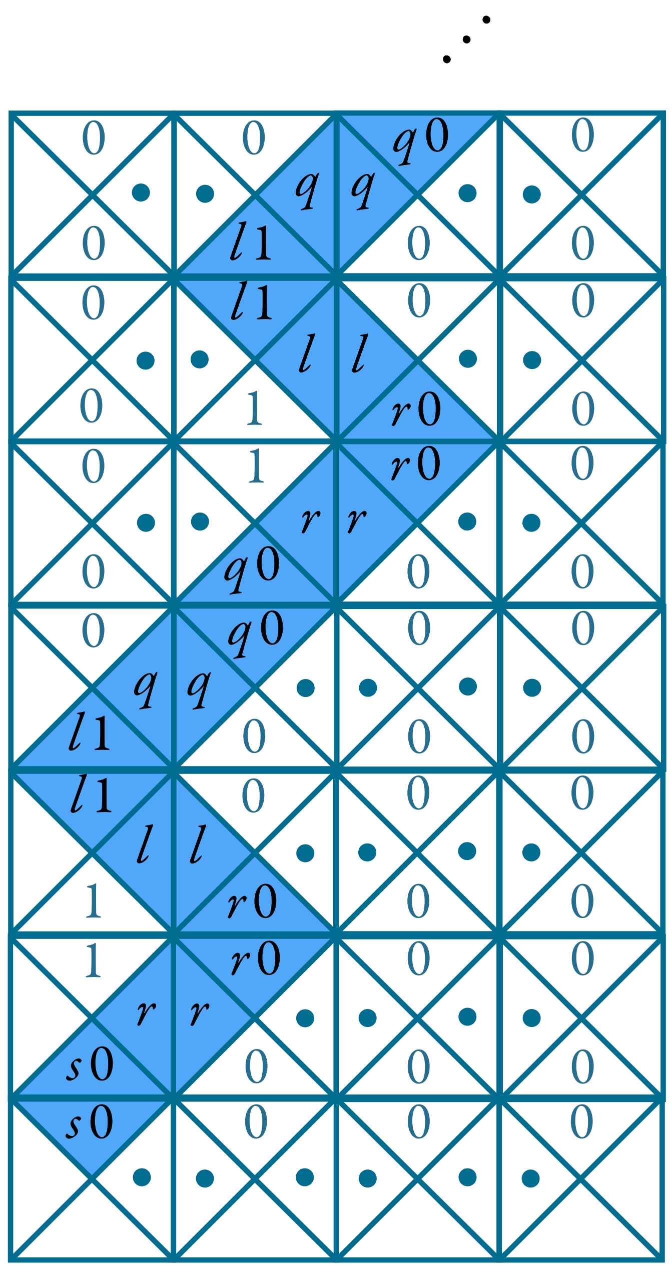

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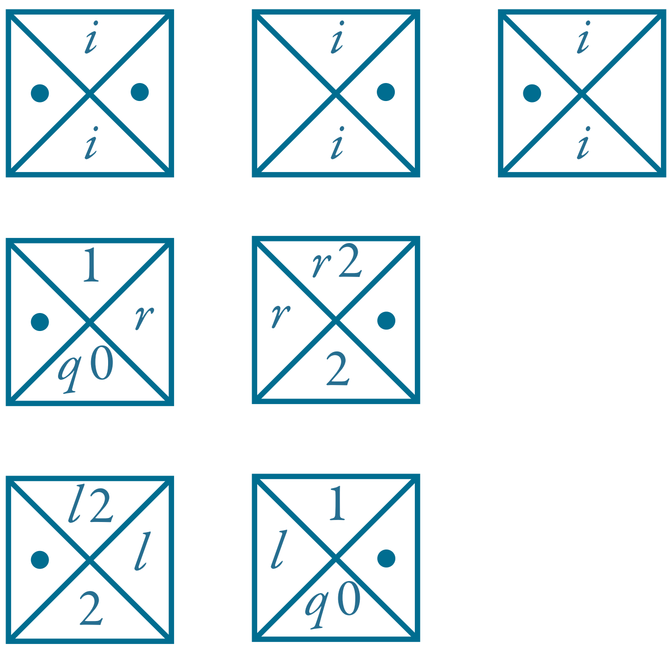

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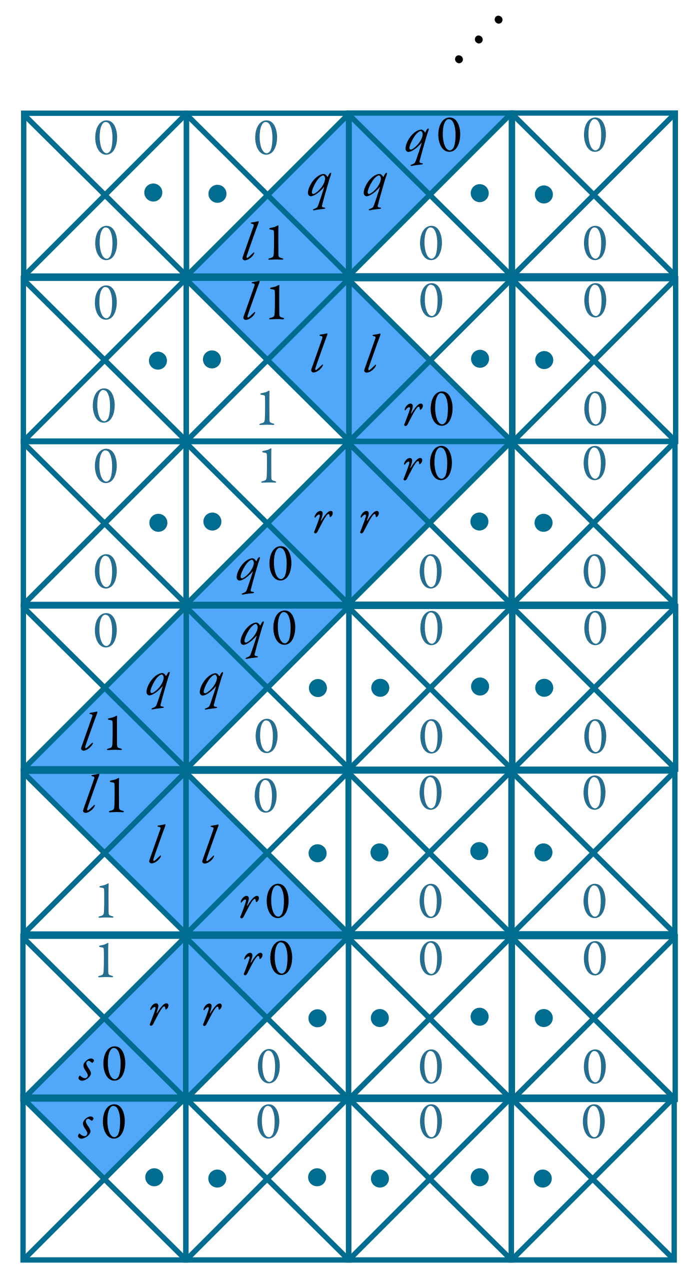

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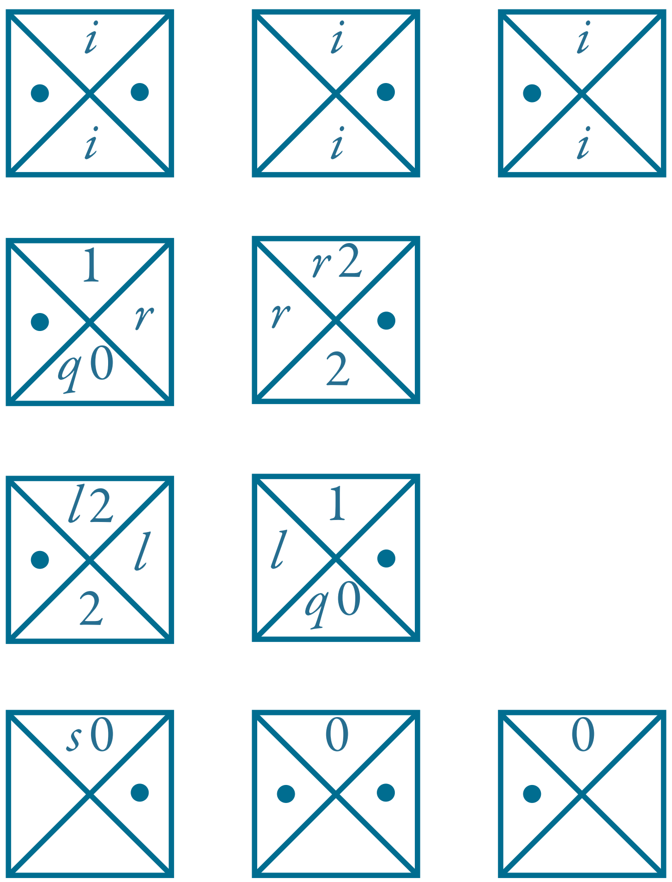

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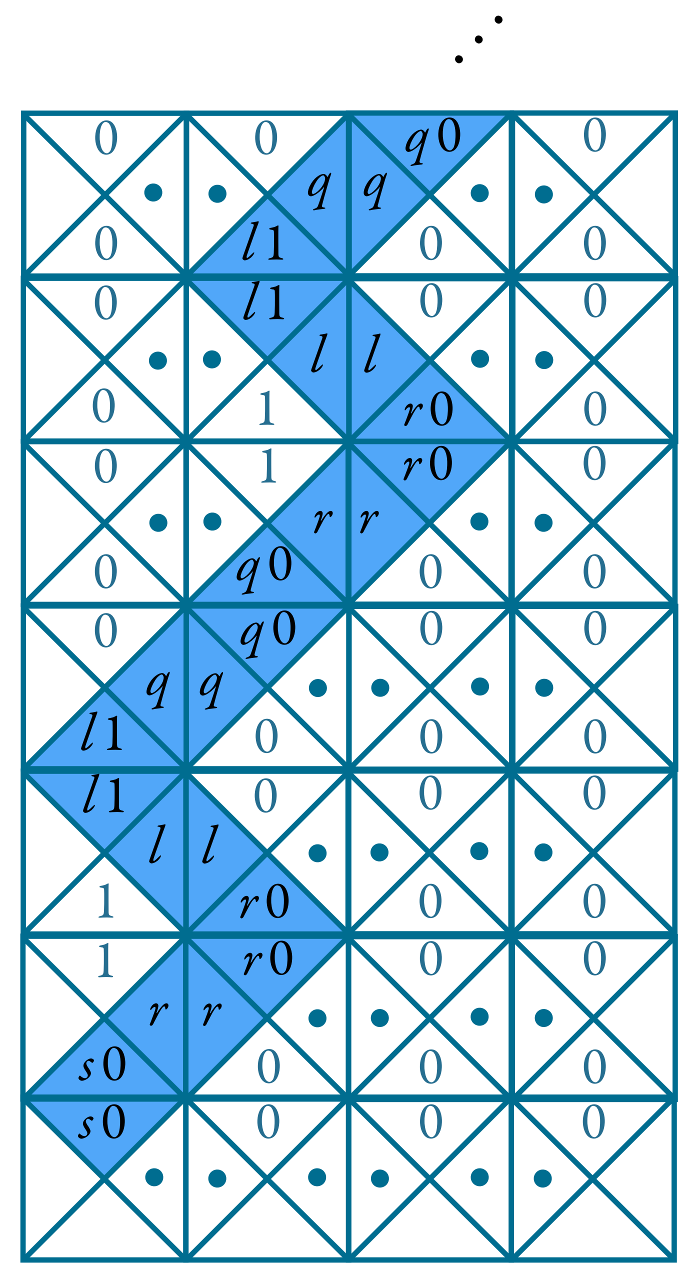

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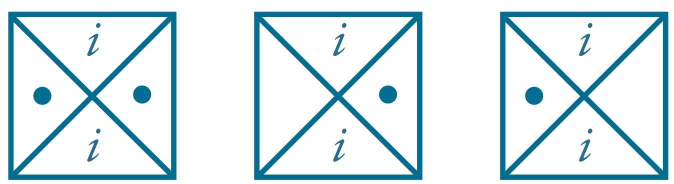

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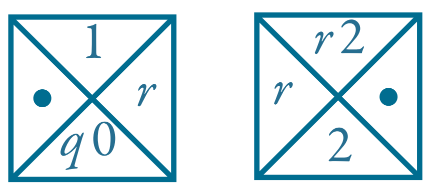

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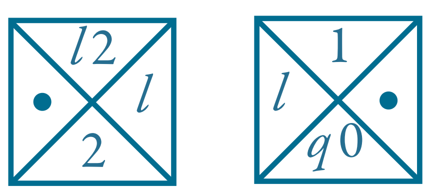

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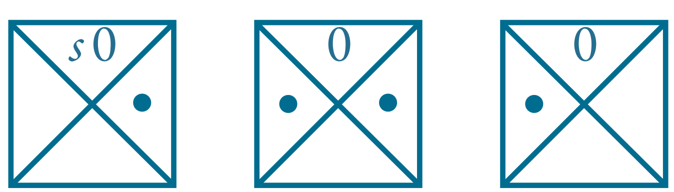

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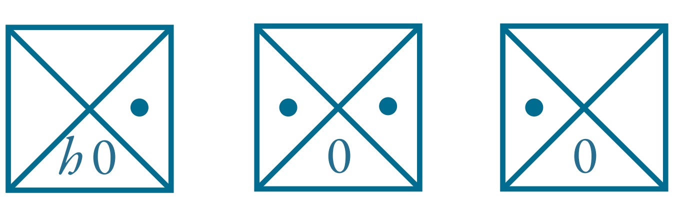
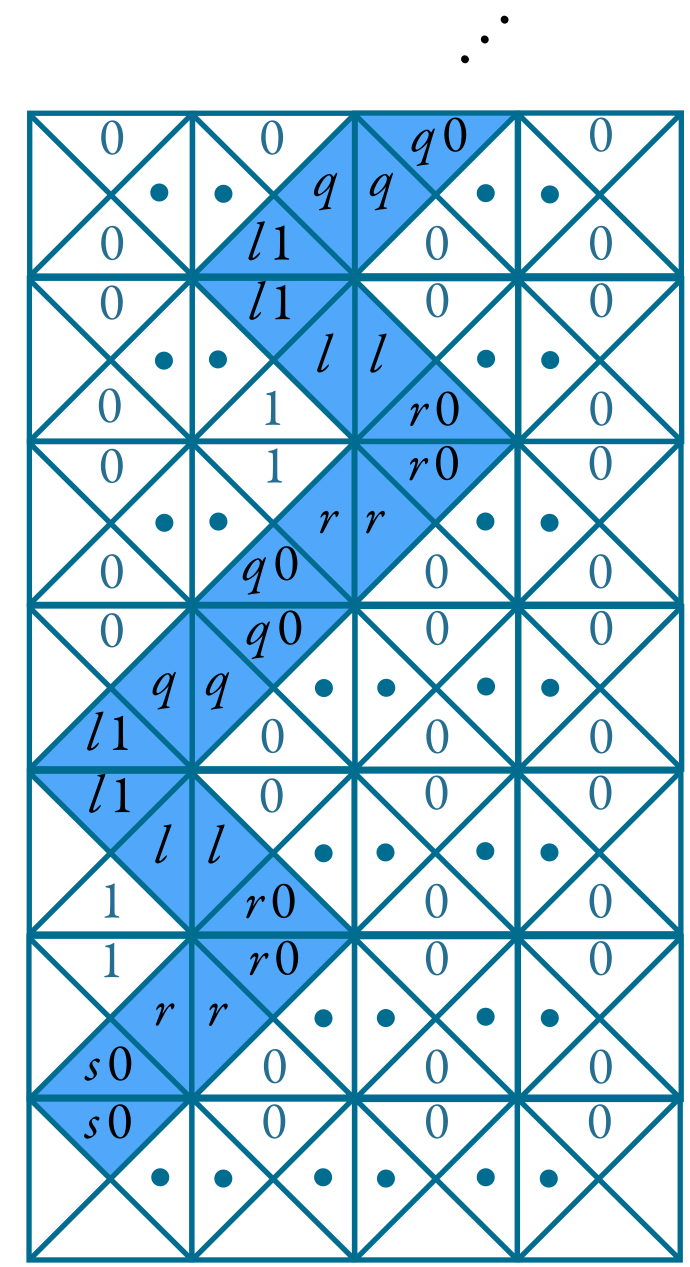


(halting configuration)

. . .



Domino reduces to Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…

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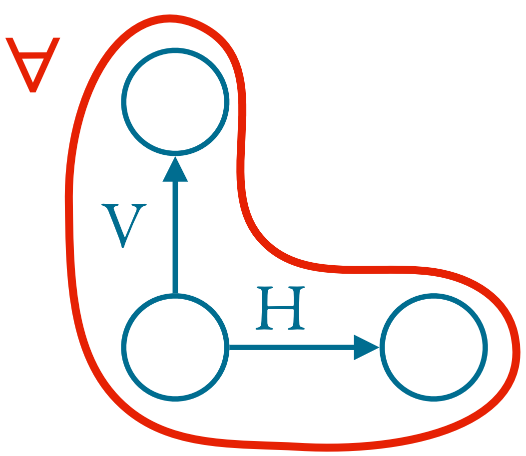

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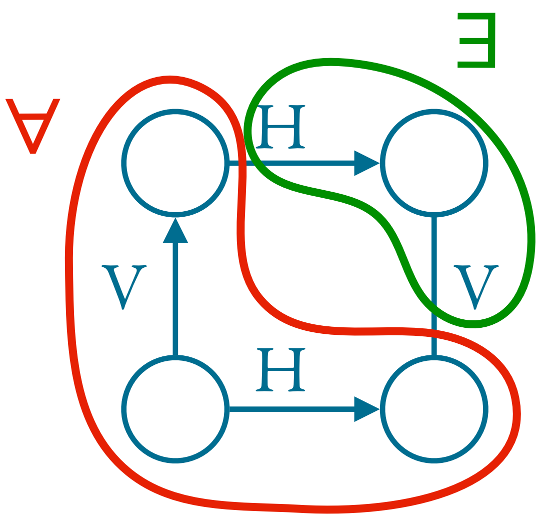

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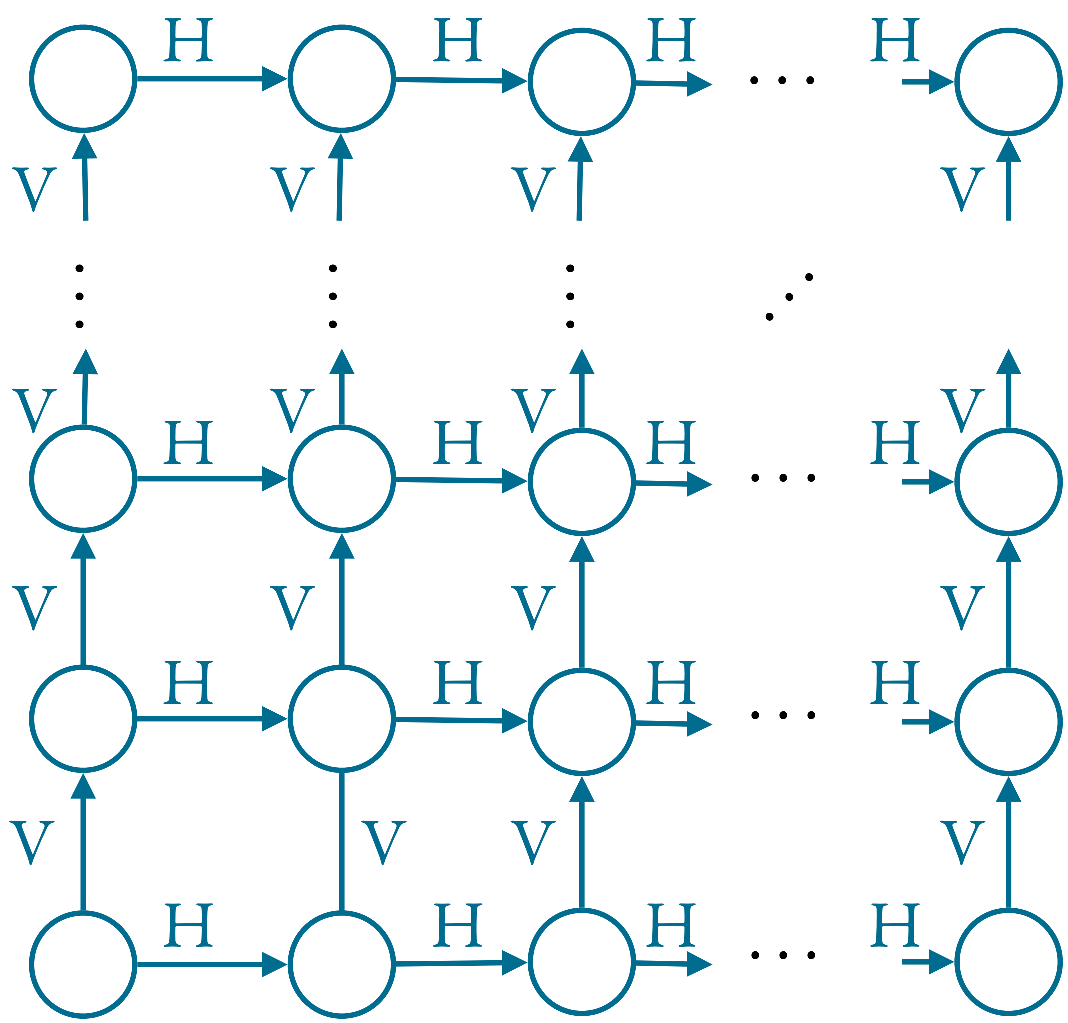

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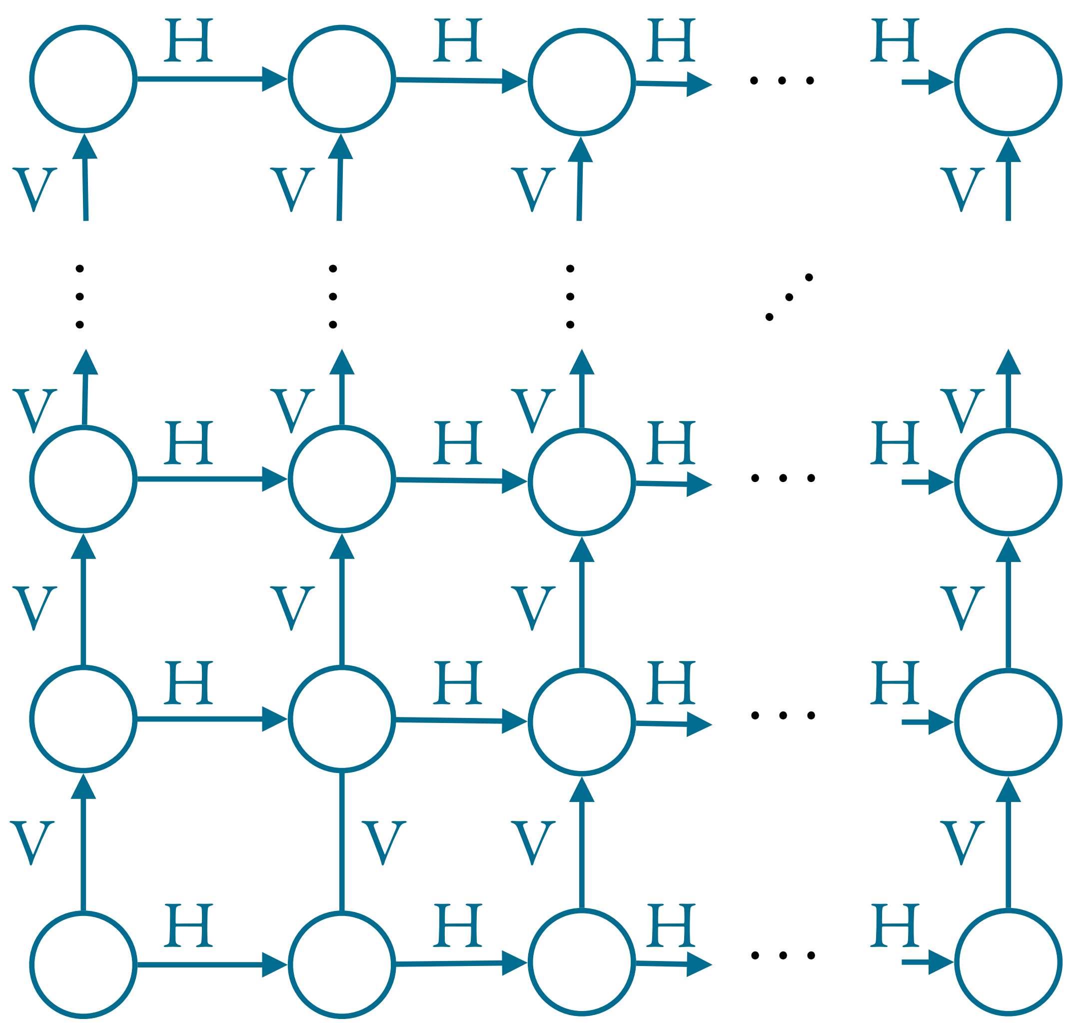

2. **Assign one domino to each node:**

a unary relation

$$\mathbf{D}_{\boxtimes}(\mathbf{x})$$

for each domino $\boxtimes$

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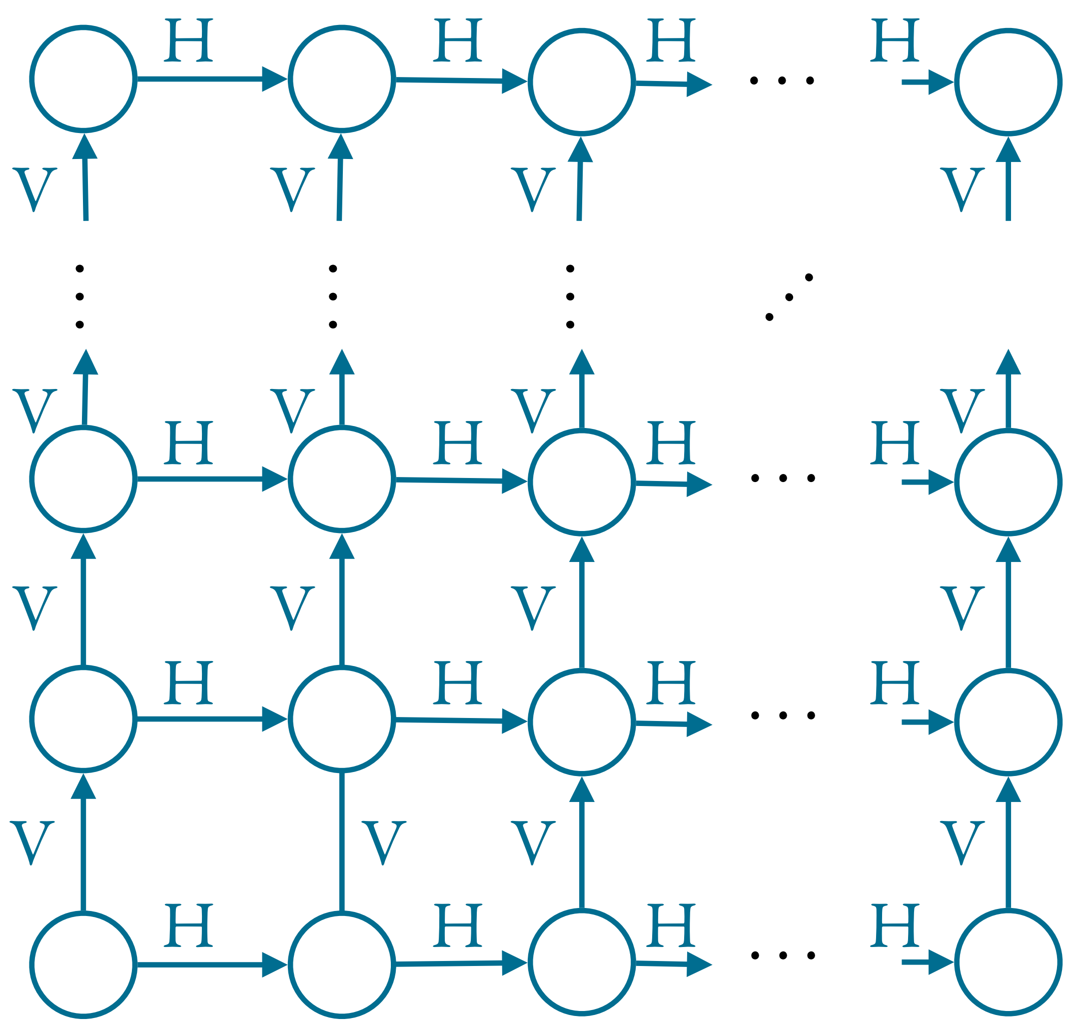

2. **Assign one domino to each node:**

a unary relation

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for each domino $\boxtimes$

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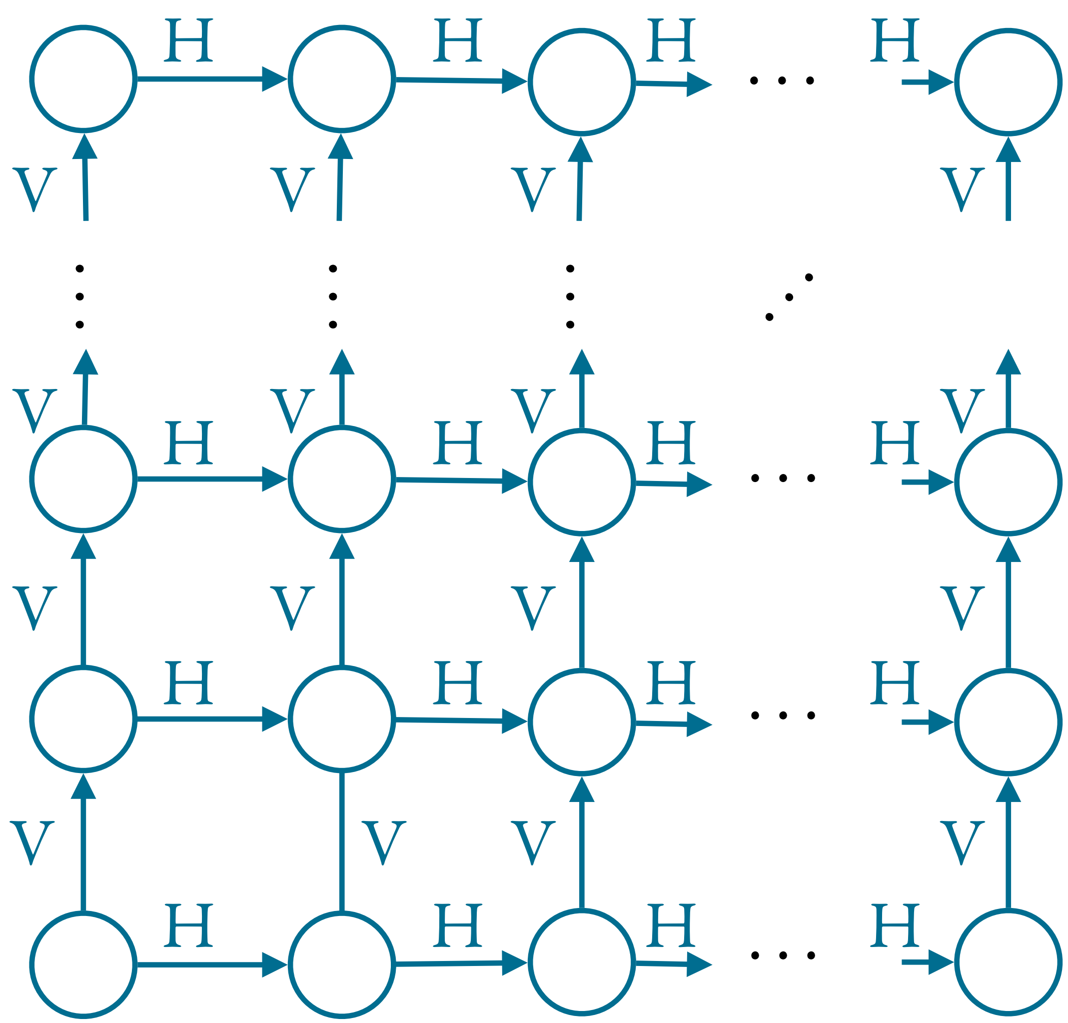

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4. **Borders are white.**

Recap + quiz

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- <u>Satisfiability</u> for FO (does $M \models \phi$ for some M?) is **undecidable**

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What about

- Validity for FO? (Problem def.: does $M \vDash \phi$ for every M?)
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Can you recall the complexity of analogous problems for

- Propositional logic?
- QBF?

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Example

$FO[\mathbb{N},<] = \{ \exists x\, (x=x),\ \forall x \exists y\, x<y,\ \exists y\, \forall x\, \neg(x<y),\ \forall x \forall y\, x=y \vee x<y \vee y<x,\ \ldots \}$

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(notation abuse: relation = is often present, but not explicitly listed
any symbol R is often identified with its relation R^M)

Some fancy FO theories

(Skip)

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How do I compare them?

Logical reductions

<u>Reduction</u> from P to P':

Algorithm A that solves P by using an oracle that returns solutions to P'

e.g. for all x P(x) iff P'(A(x))

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FO interpretation of M into M': a mapping $\alpha : R \mapsto \alpha_R$ such that

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- interpretation of $M = (\mathbb{N}, \leq)$ into $M' = (\mathbb{N}, +)$

$$\alpha_{\leq}(x, y) = \exists z \; y = x + z$$

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  $\approx (\mathbb{Q}, \leq)$

$$\alpha_{\leq_{\text{inorder}}}(x, y) = \exists x', y', z \ \ (x=z\cdot 0\cdot x' \wedge y=z\cdot 1\cdot y') \vee (x=y\cdot 0\cdot x') \vee (y=x\cdot 1\cdot x')$$

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In fact, an FO interpretation of M into M' is more complex (and powerful)

- <u>definitions of relations</u>: $\alpha_R(\bar{x})$ such that $R^M = \{ \bar{u} \mid M'[\bar{x} := \bar{u}] \vDash \alpha_R(\bar{x}) \}$

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- <u>quotient</u>: $\alpha_=(\bar{x},\bar{y})$ such that $M[\ldots] \vDash (\bar{x}=\bar{y})$ iff $M'[\ldots] \vDash \alpha_=(\bar{x},\bar{y})$

  (e.g. to interpret $(\mathbb{Q},+,\cdot)$ into $(\mathbb{Z},+,\cdot)$)

Logical reductions

Given M' and an FO interpretation $\alpha = (\alpha_U, \alpha_=, \alpha_R, \alpha_S, \ldots)$
the interpreted model is $\alpha(M') = (U^M, R^M, S^M, \ldots)$ where

- $U^M = \{\, [\bar{u}]_{\approx} \mid M'[\bar{x} := \bar{u}] \vDash \alpha_U(\bar{x}) \,\}$
- $\bar{u} \approx \bar{v}$ iff $M'[\bar{x} := \bar{u}, \bar{y} := \bar{v}] \vDash \alpha_=(\bar{x},\bar{y})$
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Theorem If $\alpha = (\alpha_U, \alpha_=, \alpha_R, \alpha_S, \ldots)$ is an FO interpretation of M into M' then FO[M] *reduces to* FO[M'], namely, there is an algorithm A_α

for all ϕ $\quad M \vDash \phi$ iff $M' \vDash A_\alpha(\phi)$

Some fancy FO theories

$\mathrm{FO}[\mathbb{N}, +, \cdot\,]$	$=$	Peano arithmetic
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Proof by reduction from undecidable Hilbert's 10th problem... [Matiyasevic '70]

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Given a polynomial $p(x,y,z,...)$
tell whether $p(x,y,z,...) = 0$ for *some integers* x, y, z

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$$(\mathbb{Z},+,\cdot, x,y,z,\ldots,t) \vDash \phi_p \quad \text{iff} \quad p(x,y,z)=t$$

2. Interpret $(\mathbb{Z},+,\cdot,0)$ into $(\mathbb{N},+,\cdot)$

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Programs verification

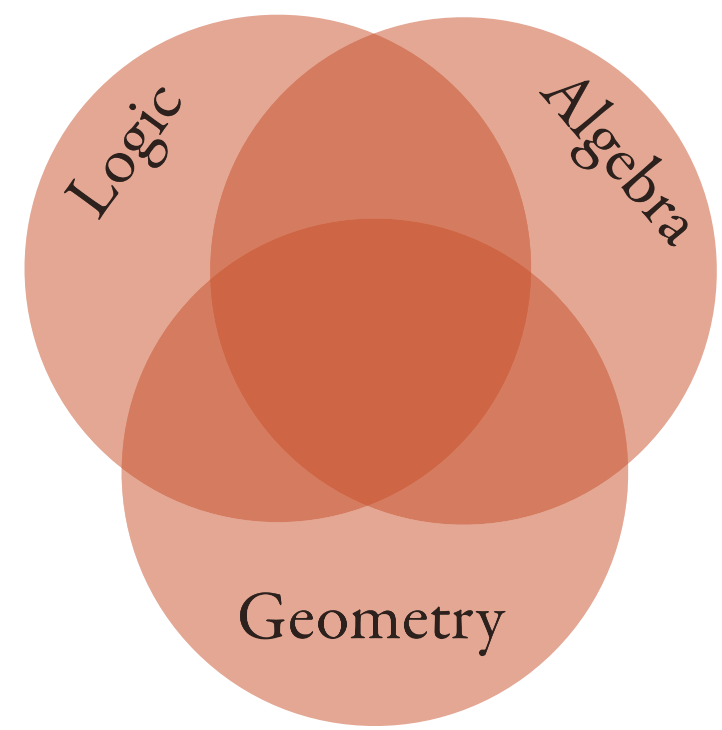
Continuous & discrete dynamical systems

Computer graphics

Robotics

Coding theory & Cryptography

Grammars & Transducers



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Every FO formula ϕ over $(Z,+,0,1,\leq,|)$ can be effectively transformed into an equivalent quantifier-free formula ϕ^*

Proof idea

Show how to remove an innermost quantifier Qz from $\phi = \ldots Qz\ \alpha(\ldots, z)$

Assume:

- $Qz = \exists z$ (if not, treat $\forall z$ as $\neg\exists z \neg$)
- α is $\vee$-free (if not, commute $\exists$ and $\vee$)

Example

temporarily assume formulas are over the reals or the rationals...

$$\exists z\ \alpha(x,y,z) = \exists z\ (2x + 4y - 3z \leq 7) \wedge (3x - y + 2z \leq -4)$$

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$$6 \mid 4x + 8y - 14 + m \ \wedge\ 4x + 8y - 14 + m \leq -9x + 3y - 12$$

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$\mathrm{FO}[\mathbb{N}, +, \cdot\,] = $ Peano arithmetic — 💀 **UNDECIDABLE** 💀 (reduction from H's 10th)

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FO[{0,1}, =] — The FO theory of Boolean algebra

Lemma Given any QBF ϕ without free variables,
one can construct an FO formula ϕ^* such that

$$\models \phi \quad \text{iff} \quad (\{0,1\}, =) \models \phi^*$$

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Corollary FO[{0,1}, =] encodes the set of valid QBF formulas

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FO[V_R, E_R] — The FO theory of the "random" graph

A different perspective and a coarser view on expressiveness...

What percentage of finite graphs verify a given FO sentence?

Probability of a formula

$P_n[\phi] \quad = \quad$ probability that ϕ holds on a <u>random</u> finite graph with n nodes

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Example For ϕ = "the graph is complete",

we have $P_n[\phi] = \dfrac{1}{2^{n(n-1)}}$

and hence $P_\infty[\phi] = 0$

Probability of a formula

Theorem (0/1 Law)
[Glebskii et al. '69, Fagin '76]

Every FO formula ϕ is
either <u>almost surely true</u> $(P_\infty[\phi] = 1)$
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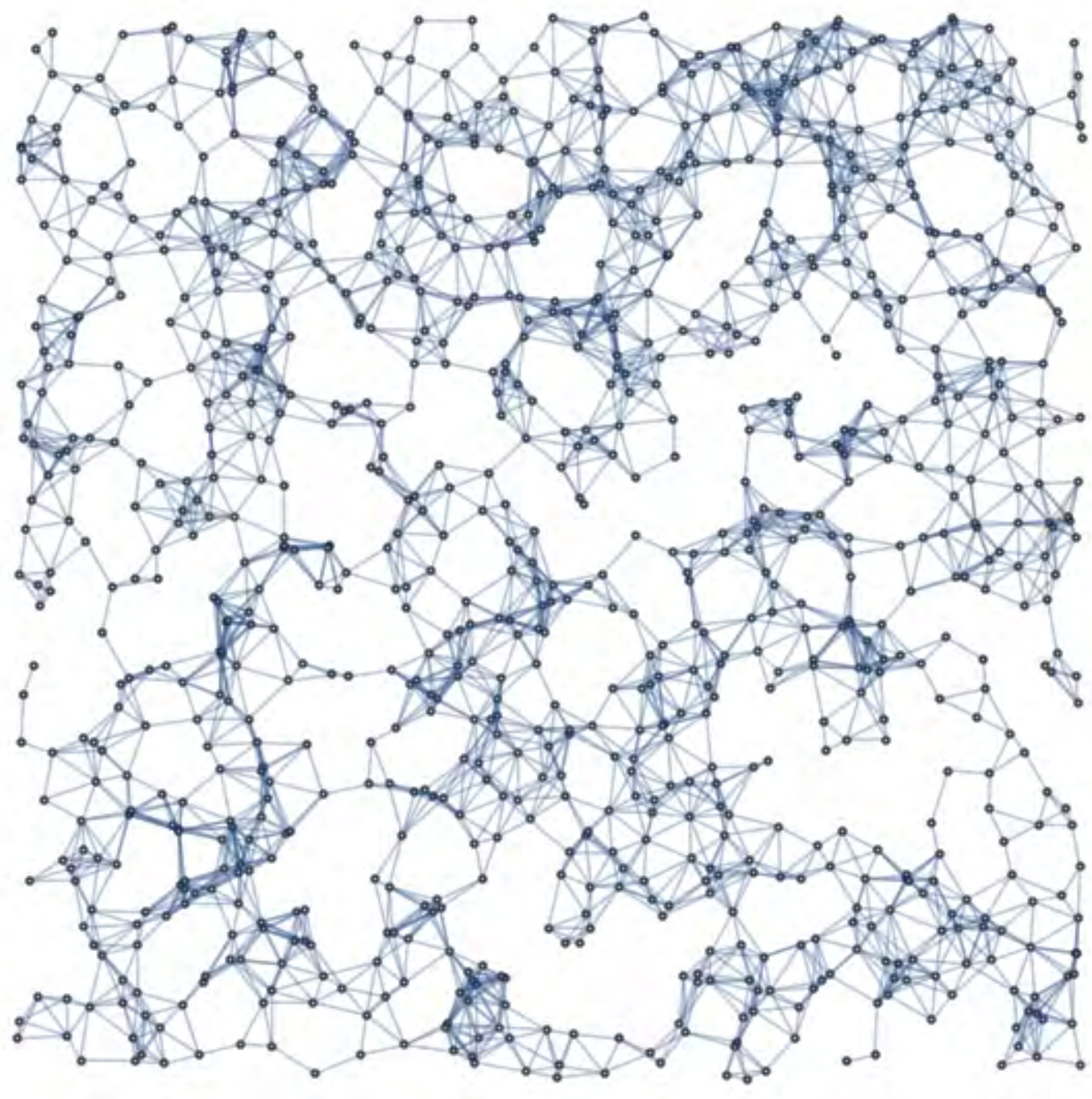
- ϕ = "even number of nodes" $P_\infty[\phi]$ not even defined
- ϕ = "more edges than nodes" $P_\infty[\phi] = 1$ (yet not FO-definable...)

Your turn!

The “random” infinite graph

Every FO formula ϕ is either almost surely true or almost surely false,
and this depends on whether $(V_R, E_R) \vDash \phi$

The “random” graph
(V_R, E_R)



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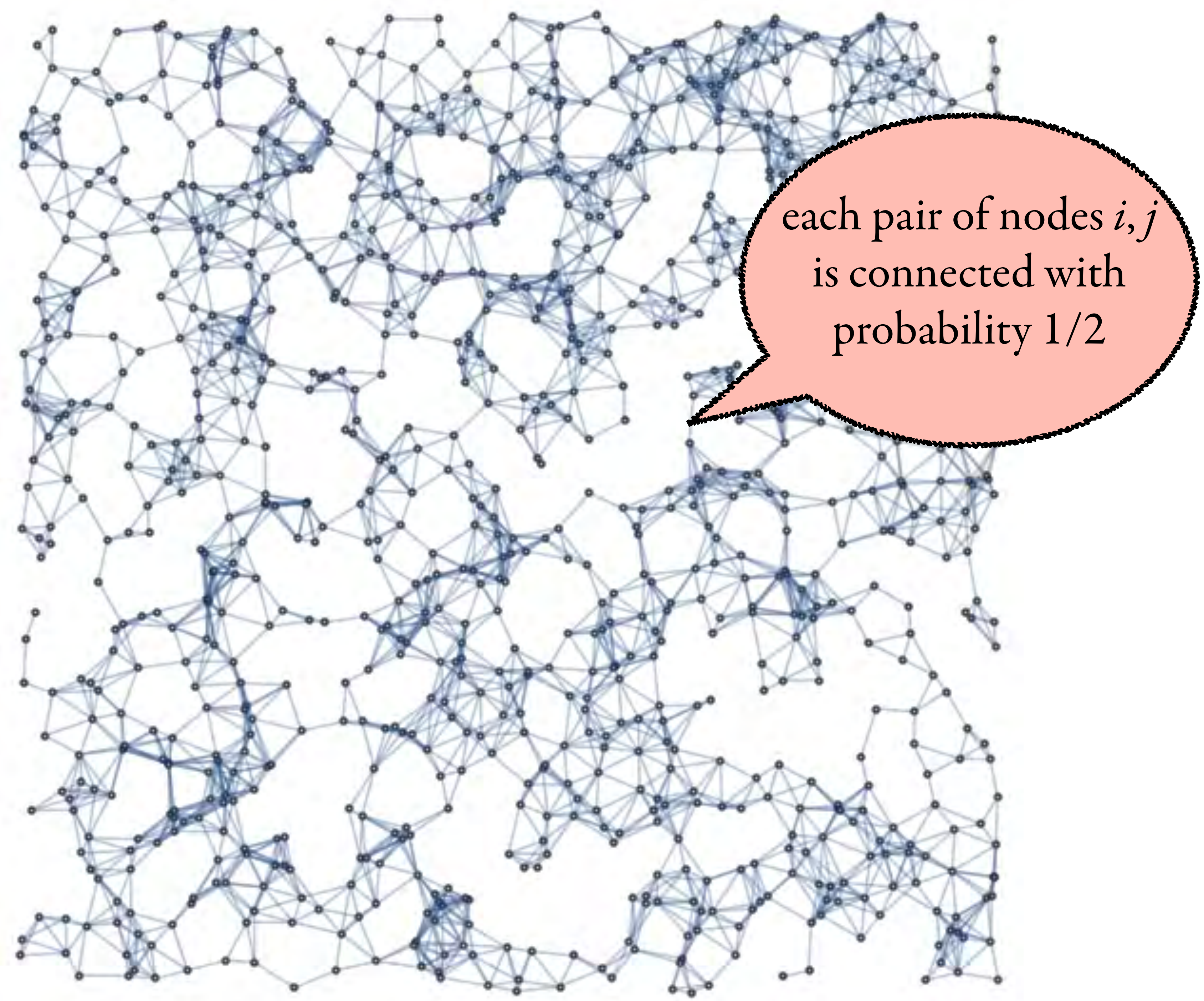

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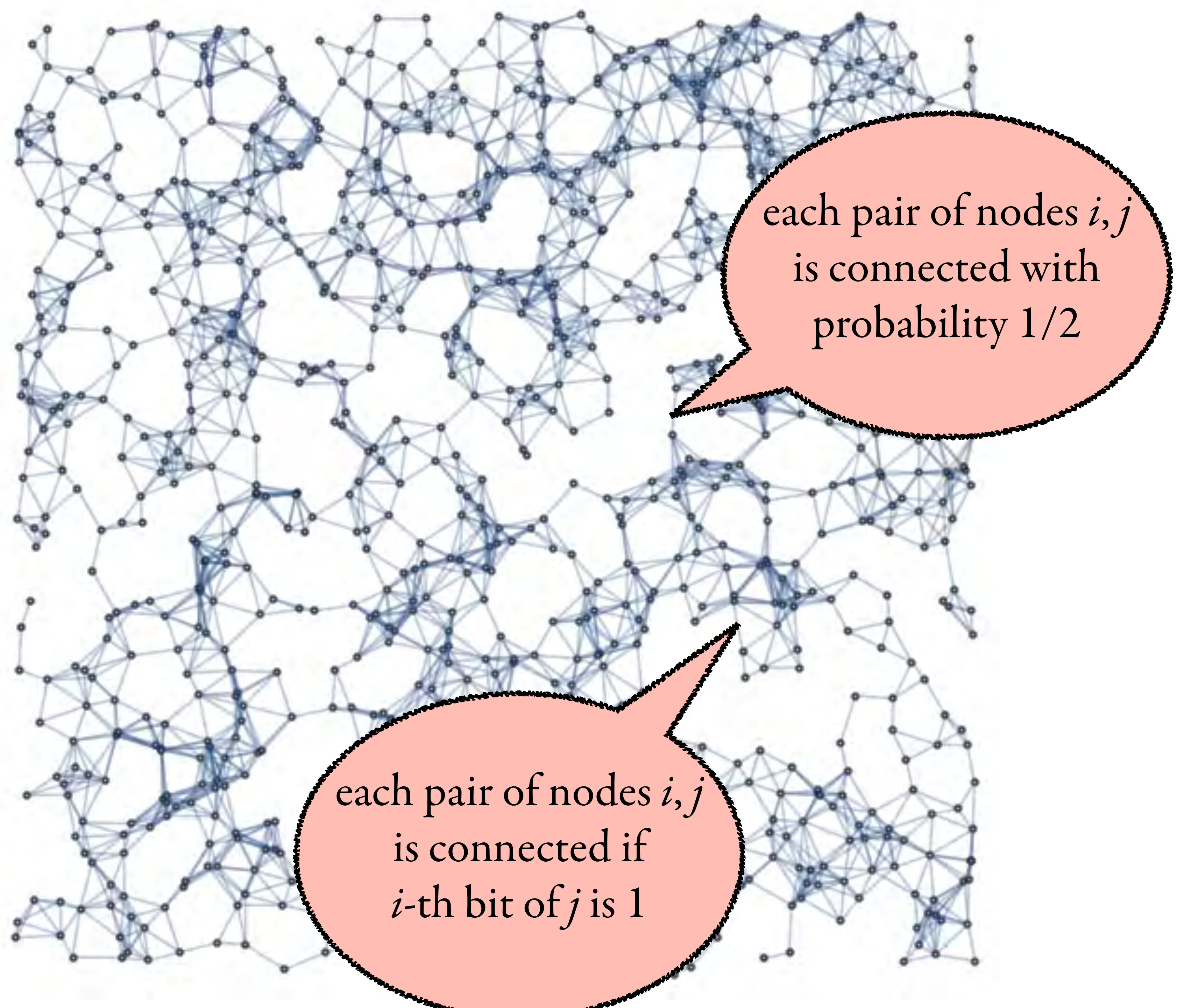

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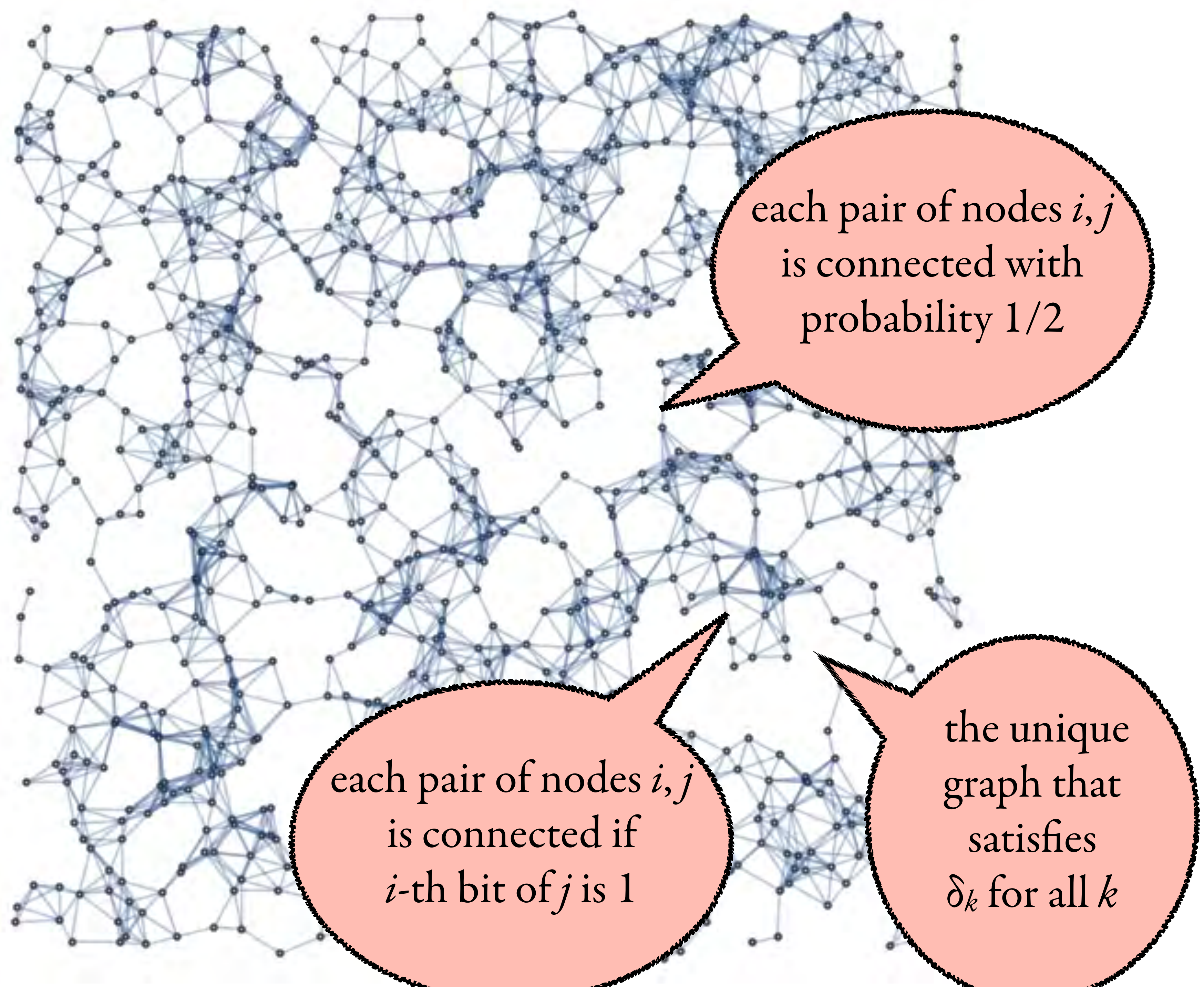

Probability of a formula - application

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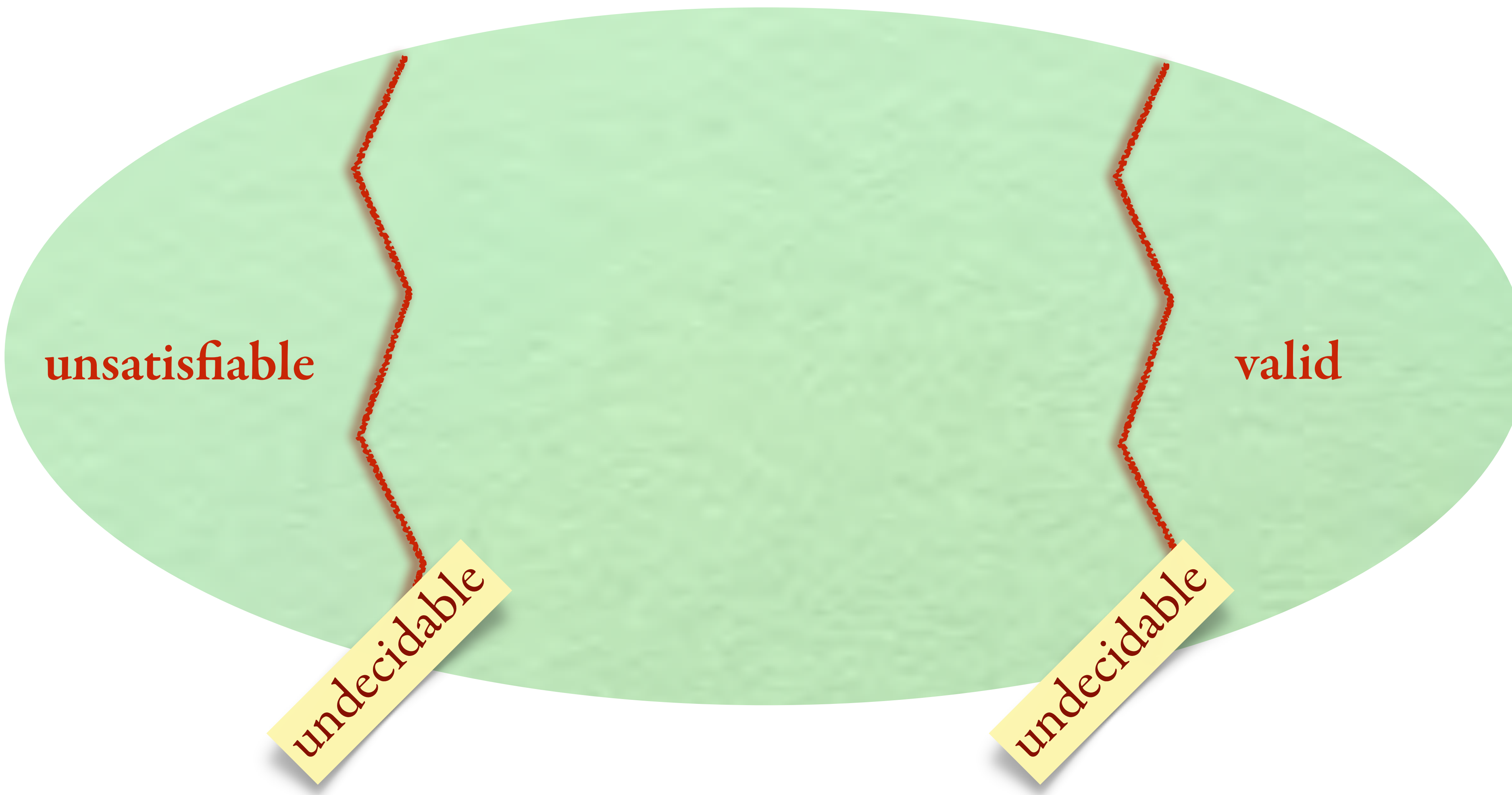

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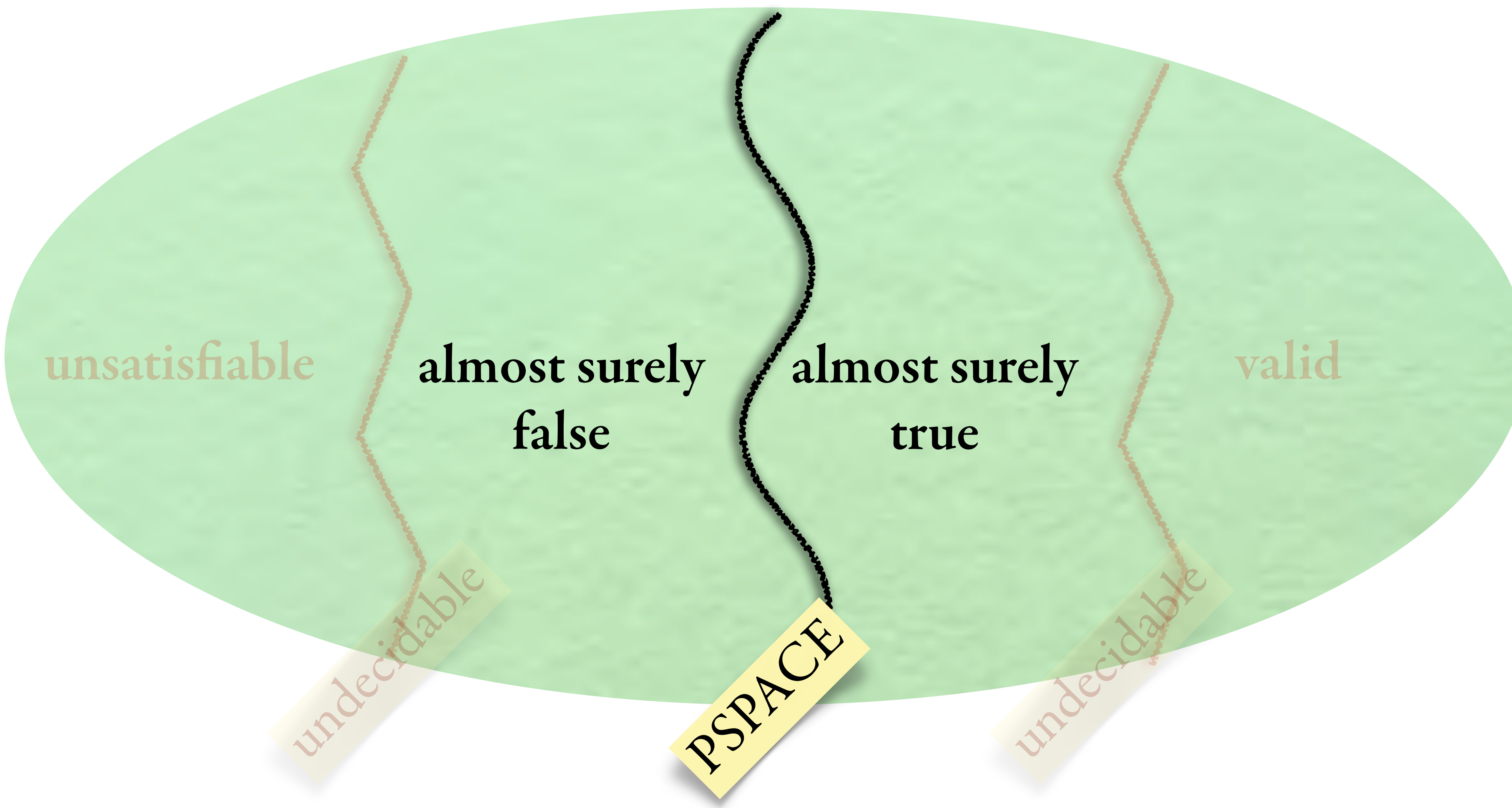

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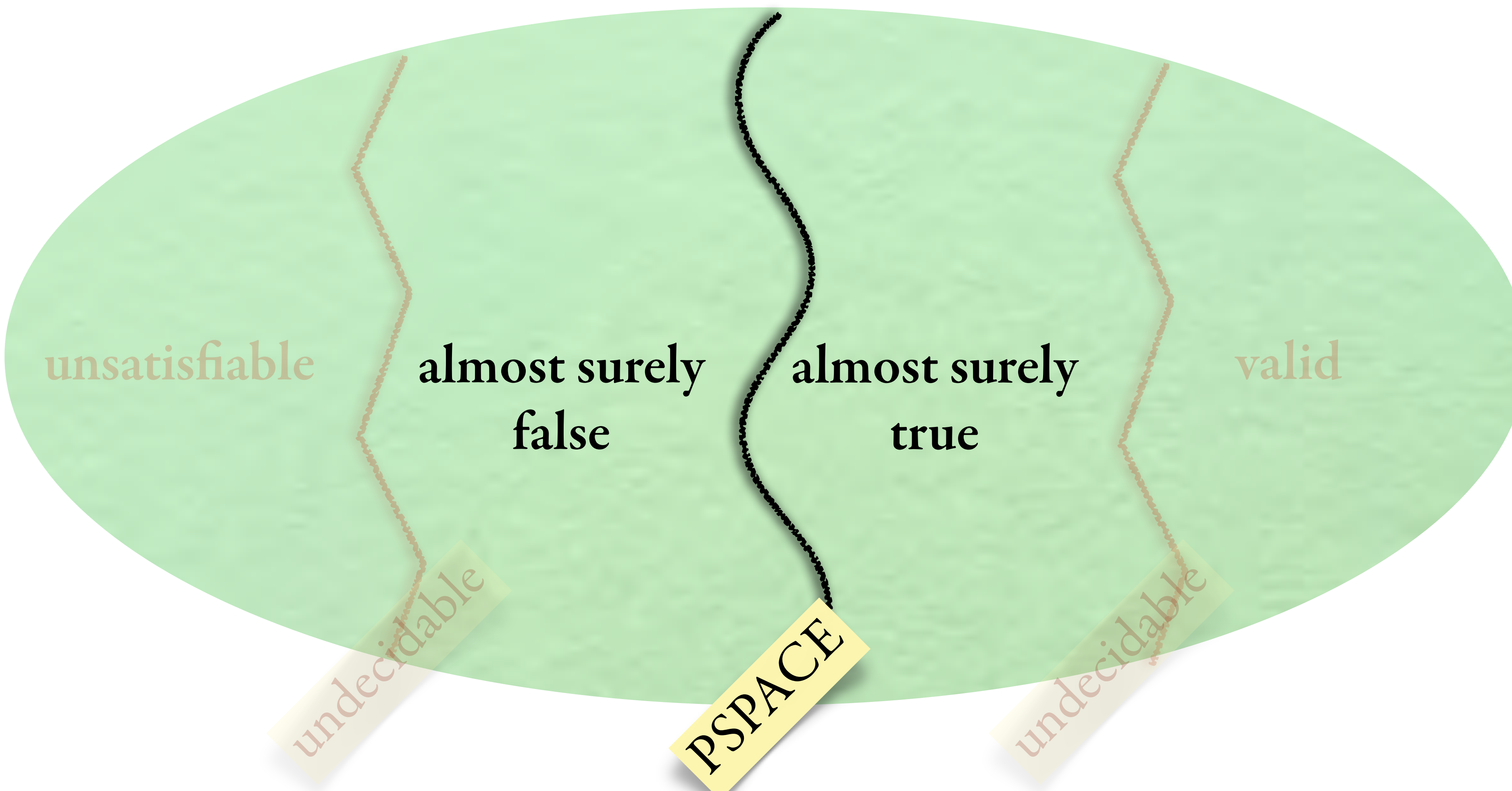

Model-checking on large graphs/databases

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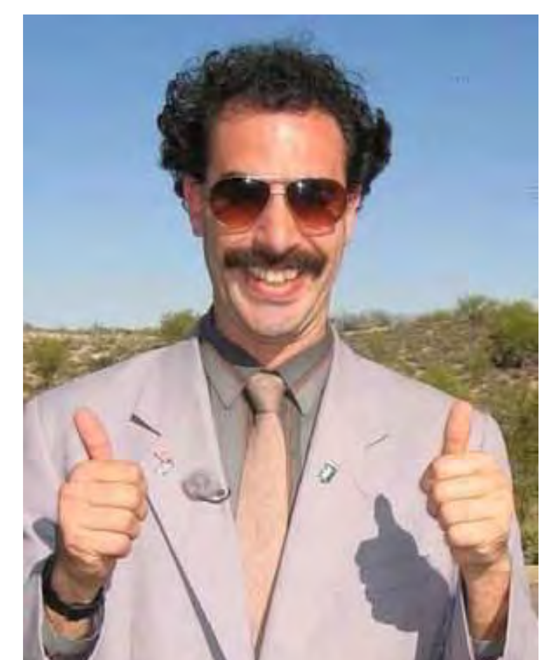

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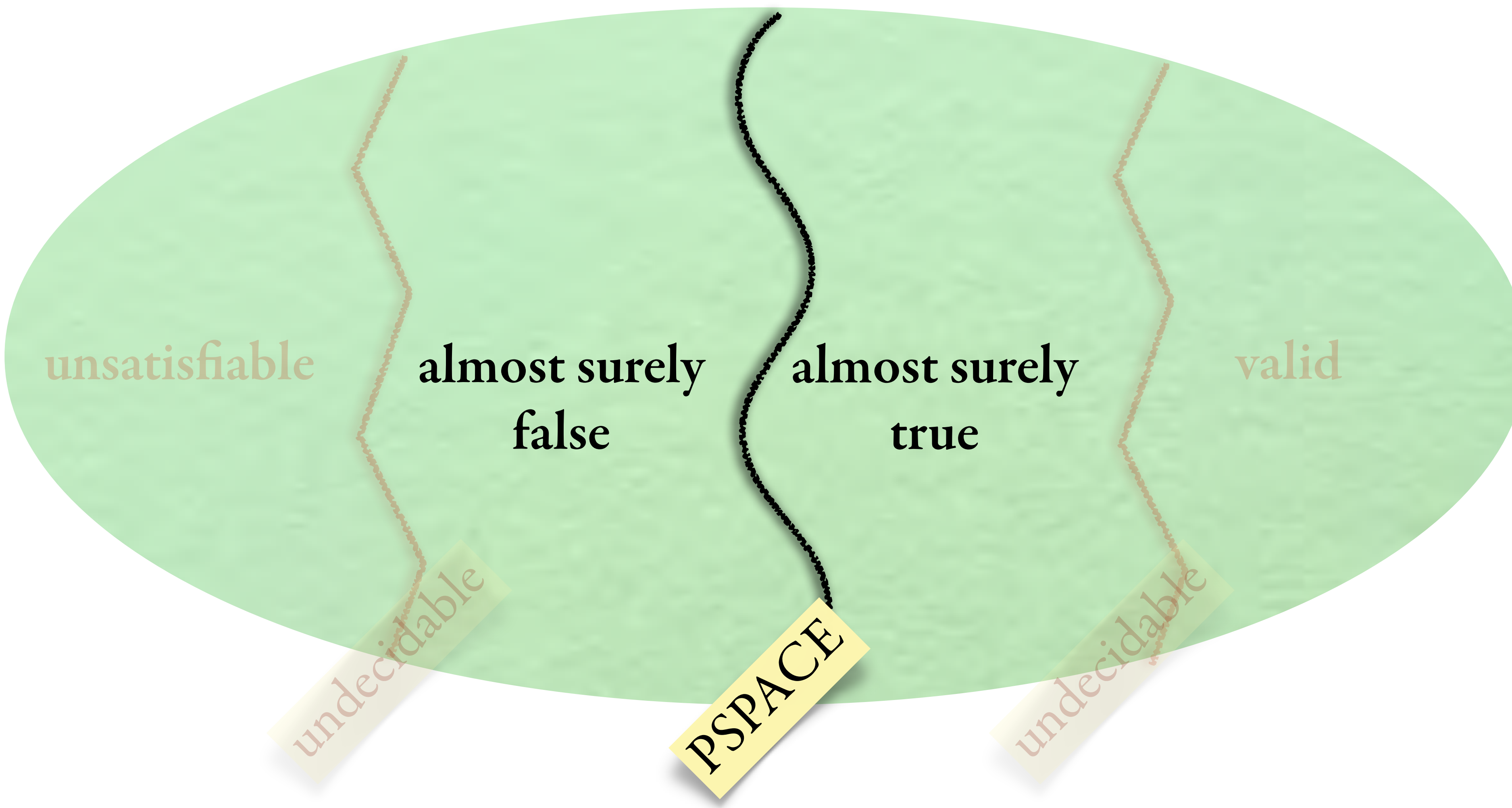

Disclaimer:

0/1 Law only applies applies to unconstrained graphs

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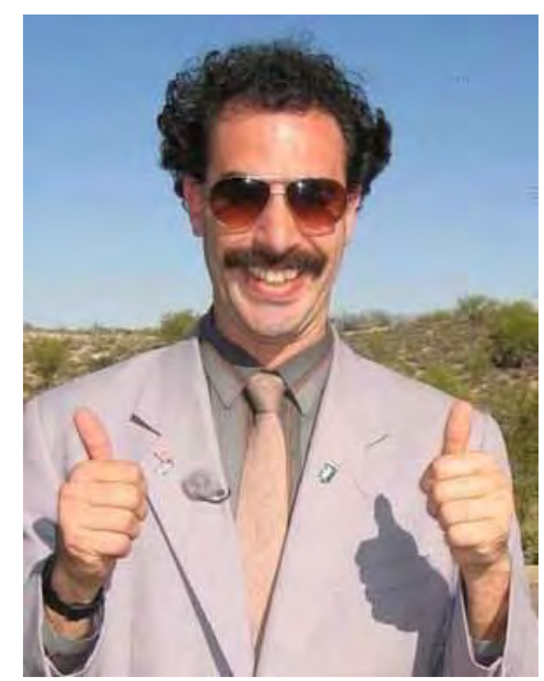

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Things to remember

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- FO is cool and quite expressive
- Model-checking is decidable (in **PSPACE**) when the universe is finite
  Satisfiability, validity, equivalence are all undecidable (reduction from Domino)
- For infinite universes, one can fix a model and study its FO theory
  Some FO theories are decidable, some are not
- Some FO theories can be reduced to others via FO interpretations