Quantified Boolean Formulas

QBF (Quantified Boolean Formulas)

Vocabulary	Propositional variables:	$\Sigma = \{p, q, r, \ldots\}$
	Boolean connectives:	$\vee, \wedge, \neg, \rightarrow, \leftrightarrow$
	Quantifiers:	$\exists, \forall$

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Syntax

$$\phi : \quad p \mid \ldots \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \phi \rightarrow \phi \mid \phi \leftrightarrow \phi$$

$$\exists p\, \phi \mid \forall p\, \phi \mid \ldots$$

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Semantics

Requires a model $M : \text{FreeVar}(\phi) \rightarrow \{\text{true}, \text{false}\}$

Describes when ϕ holds on $M \quad (M \vDash \phi)$

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Semantics

Requires a model $M : \mathit{FreeVar}(\phi) \rightarrow \{\text{true}, \text{false}\}$

Describes when ϕ holds on M $\;(M \models \phi)$

$M \models p$ iff $M(p) = \text{true}$

$M \models \phi_1 \vee \phi_2$ iff $M \models \phi_1$ or $M \models \phi_2$

…

$M \models \exists p\, \phi$ iff $M' \models \phi$ *for some* $M' \in \{M[p:=\text{true}], M[p:=\text{false}]\}$

$M \models \forall p\, \phi$ iff $M' \models \phi$ *for every* $M' \in \{M[p:=\text{true}], M[p:=\text{false}]\}$

Free variables and renaming

Syntax $\phi : \; p \mid \ldots \mid \phi \lor \phi \mid \phi \land \phi \mid \neg\phi \mid \phi \to \phi \mid \phi \leftrightarrow \phi$

$\exists p\, \phi \mid \forall p\, \phi \mid \ldots$

Free variables and renaming

Syntax $\phi :$ p | … | $\phi \vee \phi$ | $\phi \wedge \phi$ | $\neg\phi$ | $\phi \rightarrow \phi$ | $\phi \leftrightarrow \phi$

$\exists p\, \phi$ | $\forall p\, \phi$ | …

Occurrence p is <u>bound</u> in ϕ if it appears under $\exists p$ or $\forall p$

is <u>free</u> in ϕ if it is not bound

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$\phi(p_1,\ldots,p_k)$ means all free variables of ϕ are among $p_1,\ldots,p_k$

Free variables and renaming

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$\phi[p/\alpha]$	denotes		the formula obtained by replacing all *free occurrences* of p by α

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Notation abuse: given $\phi(p)$ $\phi[q]$ is shorthand for $\phi[p/q]$

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Lemma (compositionality)

Let ϕ, α be some QBF and p a variable

Suppose that $M \models p$ iff $M \models \alpha$
Then $M \models \phi$ iff $M \models \phi[p/\alpha]$
provided no free variable of α occurs in ϕ

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Lemma (compositionality)

Let ϕ, α be some QBF and p a variable

Suppose that $M \models p$ iff $M \models \alpha$
Then $M \models \phi$ iff $M \models \phi[p/\alpha]$
provided no free variable of α occurs in ϕ

Corollary (renaming)

$\exists p\, \phi$ is equivalent to $\exists q\, \phi[p/q]$
provided q does not occur in φ

Economy of syntax

Syntax $\phi :\ p \mid \ldots \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \phi \rightarrow \phi \mid \phi \leftrightarrow \phi$
$\quad \exists p \mid \forall p \mid \ldots$

Lemma $\forall$ can be defined using $\exists, \neg$

Economy of syntax

Syntax $\phi :\ p \mid \dots \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \phi \rightarrow \phi \mid \phi \leftrightarrow \phi$
$\exists p \mid \forall p \mid \dots$

Lemma $\forall$ can be defined using $\exists, \neg$

namely:

$\forall p\, \phi$ is equivalent to $\neg(\exists p\, \neg\phi)$

Algorithms

Model-check(φ, M)

```
if φ = p  then
    return M(p)
else if  φ = φ1 ∨ φ2  then
    return Model-check(φ1, M)  OR
           Model-check(φ2, M)
else if ...
...

else if  φ = ∃p φ'  then
    return Model-check(φ', M[p:=true])  OR
           Model-check(φ', M[p:=false])

else if  φ = ∀p φ' then
    return Model-check(φ', M[p:=true]) AND
           Model-check(φ', M[p:=false])
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```
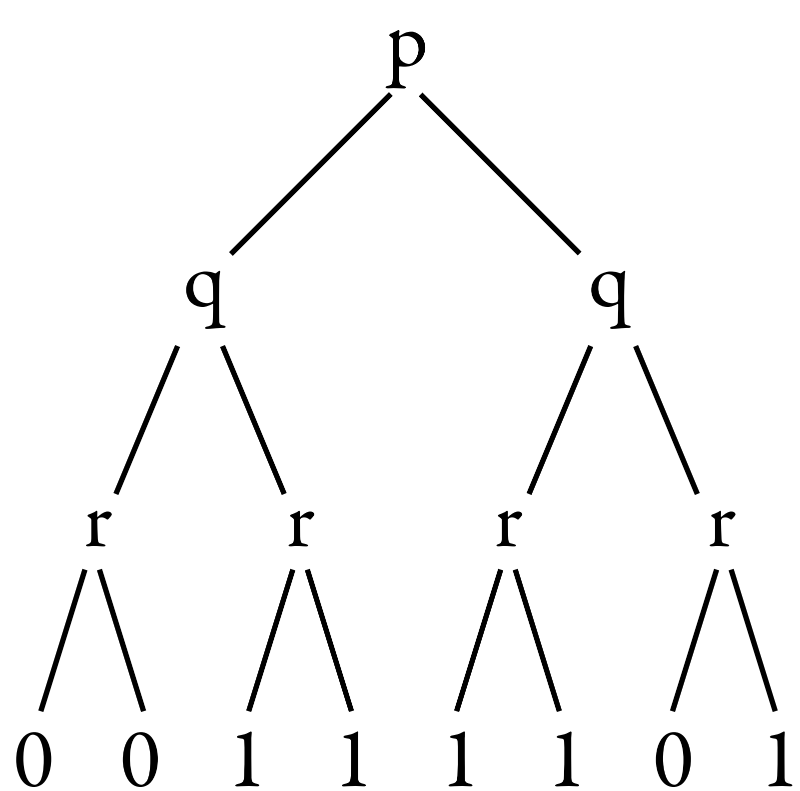

Complexity: **PSPACE**-complete

Propositional satisfiability vs QBF model-checking

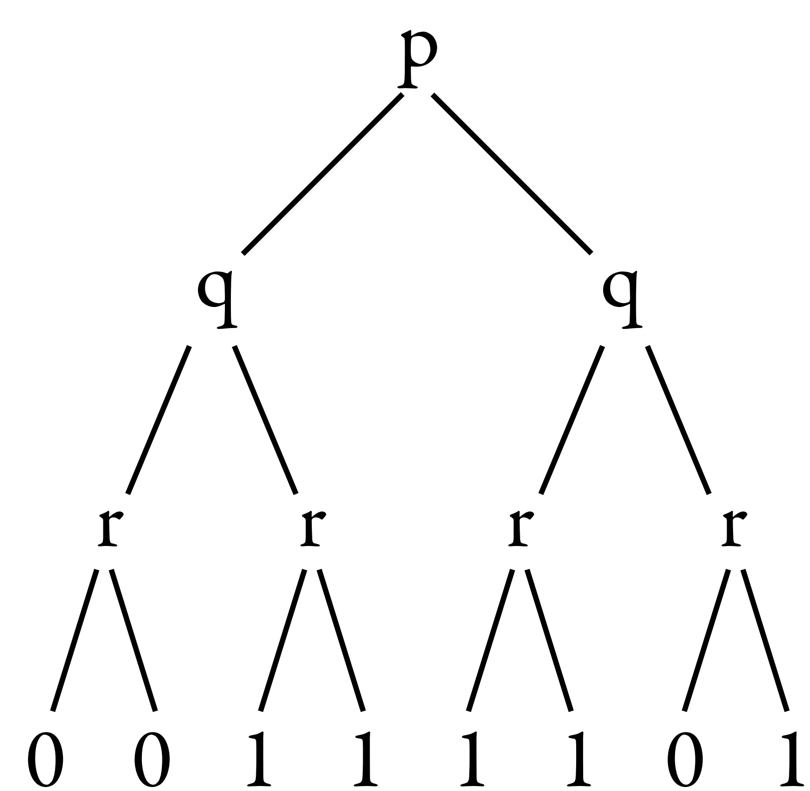
SAT

$[\exists p\ \exists q\ \exists r]\ \phi(p,q,r)$



QBF

$\forall p\ \exists q\ \forall r\ \phi(p,q,r)$



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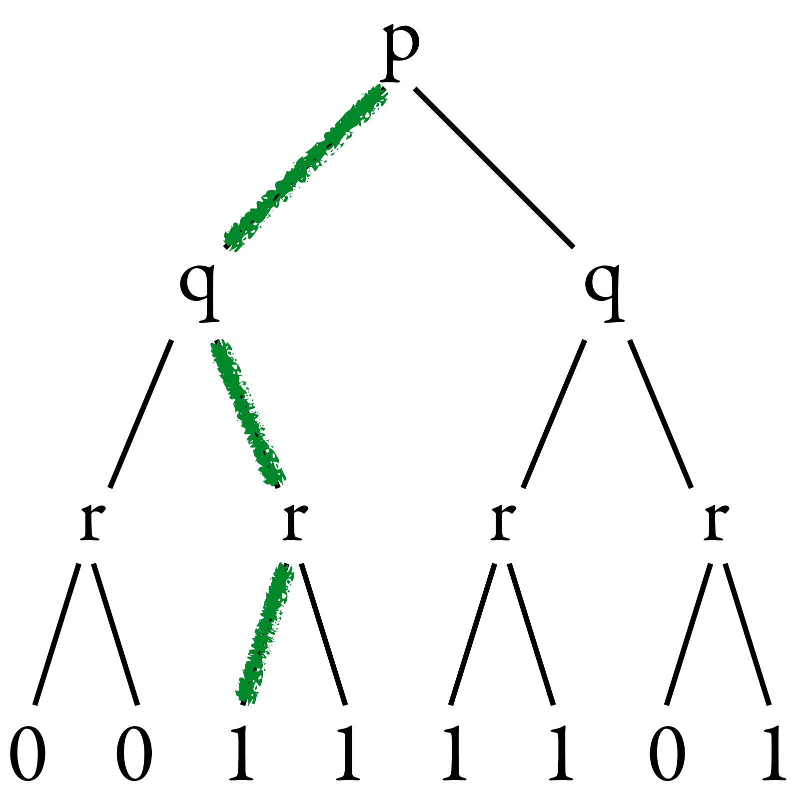

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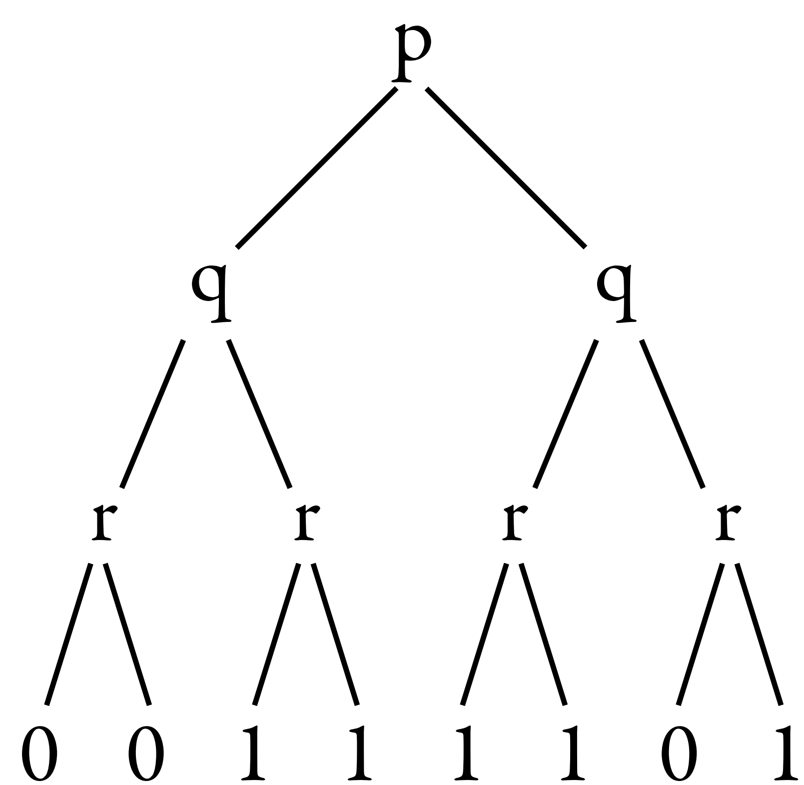

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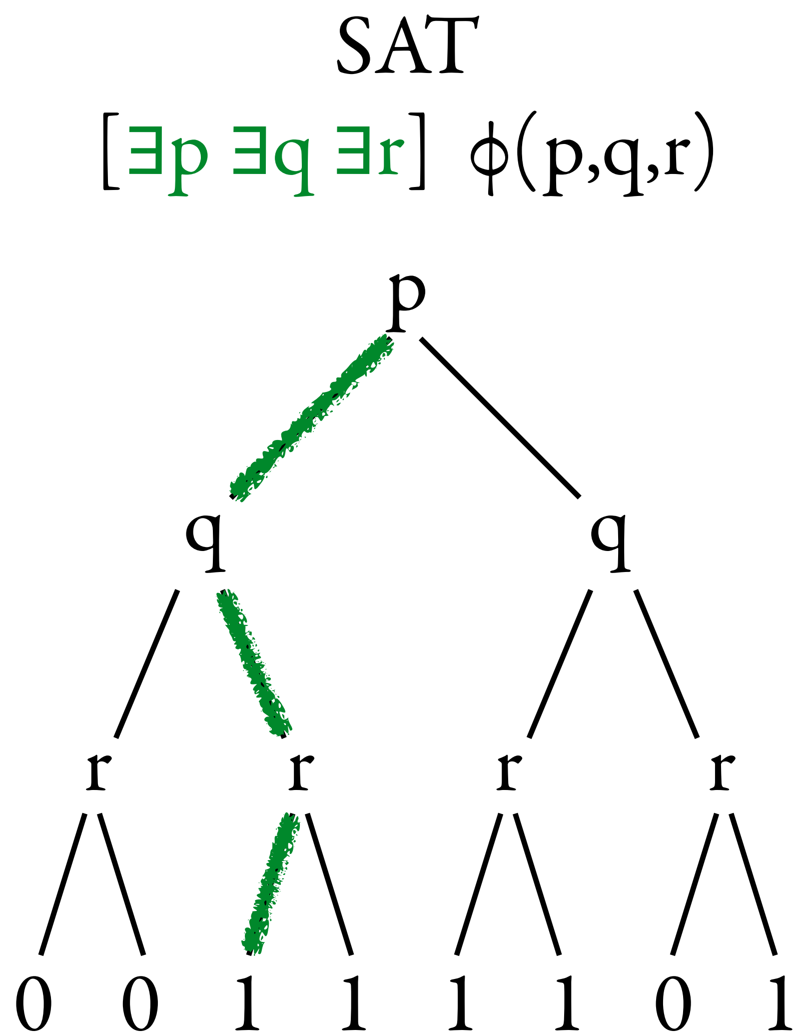

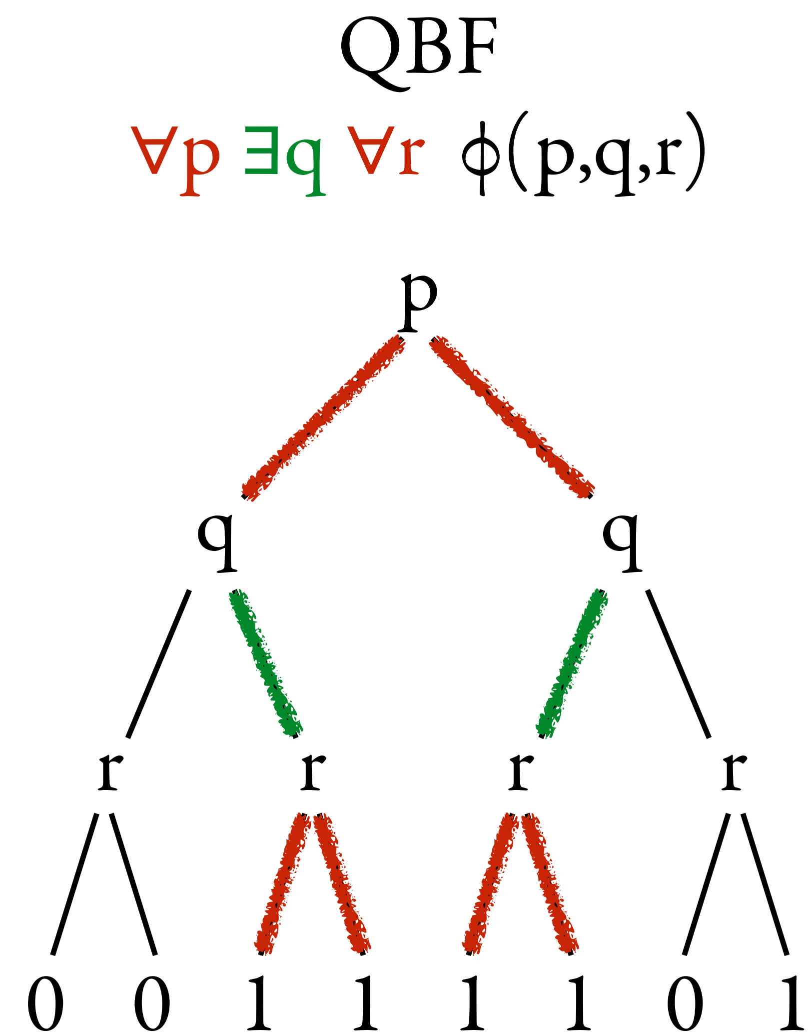

Normal forms

Prenex

$$\phi : \quad \exists p\, \phi \;\mid\; \forall p\, \phi \;\mid\; \alpha$$

$$\alpha : \quad p \;\mid\; \alpha \vee \alpha \;\mid\; \alpha \wedge \alpha \;\mid\; \neg\alpha \;\mid\; \ldots$$

Normal forms

<u>Prenex</u>

$$\begin{aligned} \phi &: \quad \exists p\, \phi \;\mid\; \forall p\, \phi \;\mid\; \alpha \\ \alpha &: \quad p \;\mid\; \alpha \vee \alpha \;\mid\; \alpha \wedge \alpha \;\mid\; \neg\alpha \;\mid\; \dots \end{aligned}$$

<u>Prenex-CNF/DNF</u>

when in addition α is in CNF/DNF

Normal forms

Prenex

$$\phi: \quad \exists p\, \phi \mid \forall p\, \phi \mid \alpha$$
$$\alpha: \quad p \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \ldots$$

Prenex-CNF/DNF

when in addition α is in CNF/DNF

Lemma 1 Prenex normal form can be computed in polynomial time

Normal forms

<u>Prenex</u>

$$\phi : \quad \exists p\, \phi \;\mid\; \forall p\, \phi \;\mid\; \alpha$$
$$\alpha : \quad p \;\mid\; \alpha \vee \alpha \;\mid\; \alpha \wedge \alpha \;\mid\; \neg\alpha \;\mid\; \ldots$$

<u>Prenex-CNF/DNF</u> when in addition α is in CNF/DNF

Lemma 1 Prenex normal form can be computed in polynomial time

Lemma 2 Model-checking prenex QBFs with n <u>quantifier alternations</u> is in

$$\mathbf{\Sigma_n = NP^{coNP^{\cdots}}} \quad \text{or} \quad \mathbf{\Pi_n = coNP^{NP^{\cdots}}}$$

Application example — compression

k bits

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0

$\phi_{init}(\bar{p})$ $\phi_{trans}(\bar{p},\bar{p}')$ $\phi_{goal}(\bar{p})$

Reachability in $n = 2^k$ steps:

$$\phi_{reach} = \exists \bar{p}_1,\ldots,\bar{p}_n \ \phi_{init}[\bar{p}_1] \wedge \phi_{goal}[\bar{p}_n] \wedge \bigwedge_{i=2,\ldots,n} \phi_{trans}[\bar{p}_{i-1},\bar{p}_i]$$

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0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
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Size: $|\phi_{reach}| \approx |\phi_{init}| + |\phi_{goal}| + k \cdot 2^k \cdot |\phi_{trans}|$ can we do better?

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0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0

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$$\forall \bar{s},\bar{s}' \left(\bigvee_{i=2,\ldots,n} \bar{s}\bar{s}' = \bar{p}_{i-1}\bar{p}_i \right) \rightarrow \phi_{trans}[\bar{s},\bar{s}']$$

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$\phi_{init}(\bar{p})$ $\phi_{trans}(\bar{p},\bar{p}')$ $\phi_{goal}(\bar{p})$

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$$k \cdot 2^k + |\phi_{trans}|$$

Size: $|\phi_{reach}| \approx |\phi_{init}| + |\phi_{goal}| + k \cdot 2^k \cdot |\phi_{trans}|$ can we do better?

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k bits

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$\phi_{\text{init}}(\bar{p})$ $\quad$ $\phi_{\text{trans}}(\bar{p},\bar{p}')$ $\quad$ $\phi_{\text{goal}}(\bar{p})$

Reachability in $n = 2^k$ steps:

$$\phi_{\text{reach}} = \exists \bar{p}_1,\ldots,\bar{p}_n\ \phi_{\text{init}}[\bar{p}_1] \wedge \phi_{\text{goal}}[\bar{p}_n] \wedge \bigwedge_{i=2,\ldots,n} \phi_{\text{trans}}[\bar{p}_{i-1},\bar{p}_i]$$

$$\forall \bar{s},\bar{s}'\ \Big(\bigvee_{i=2,\ldots,n} \bar{s}\bar{s}'=\bar{p}_{i-1}\bar{p}_i\Big) \rightarrow \phi_{\text{trans}}[\bar{s},\bar{s}']$$

$$\phi_k[\bar{p}_1,\bar{p}_n]$$

$$\phi_1(\bar{p},\bar{q}) = \phi_{\text{trans}}[\bar{p},\bar{q}]$$

$$\phi_{i+1}(\bar{p},\bar{q}) = \exists \bar{r}\ \forall \bar{s},\bar{s}'$$

$$\bar{s}\bar{s}'=\bar{p}\bar{r} \vee \bar{s}\bar{s}'=\bar{r}\bar{q} \rightarrow \phi_i[\bar{s},\bar{s}']$$

$$k \cdot 2^k + |\phi_{\text{trans}}|$$

Size: $\quad |\phi_{\text{reach}}| \approx |\phi_{\text{init}}| + |\phi_{\text{goal}}| + k \cdot 2^k \cdot |\phi_{\text{trans}}|$ $\quad$ can we do better?

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$$k^2 + |\phi_{trans}|$$

$$k \cdot 2^k + |\phi_{trans}|$$

Size: $|\phi_{reach}| \approx |\phi_{init}| + |\phi_{goal}| + k \cdot 2^k \cdot |\phi_{trans}|$ can we do better?

Things to remember

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- QBF logic is still simple

- QBF model-checking *generalises satisfiability* & *validity* of prop. formulas

  (complexity is slightly higher and depends on quantifier alternation)

- There is a trick to *succinctly describe* a reachability property