Learning automata

Gabriele Puppis

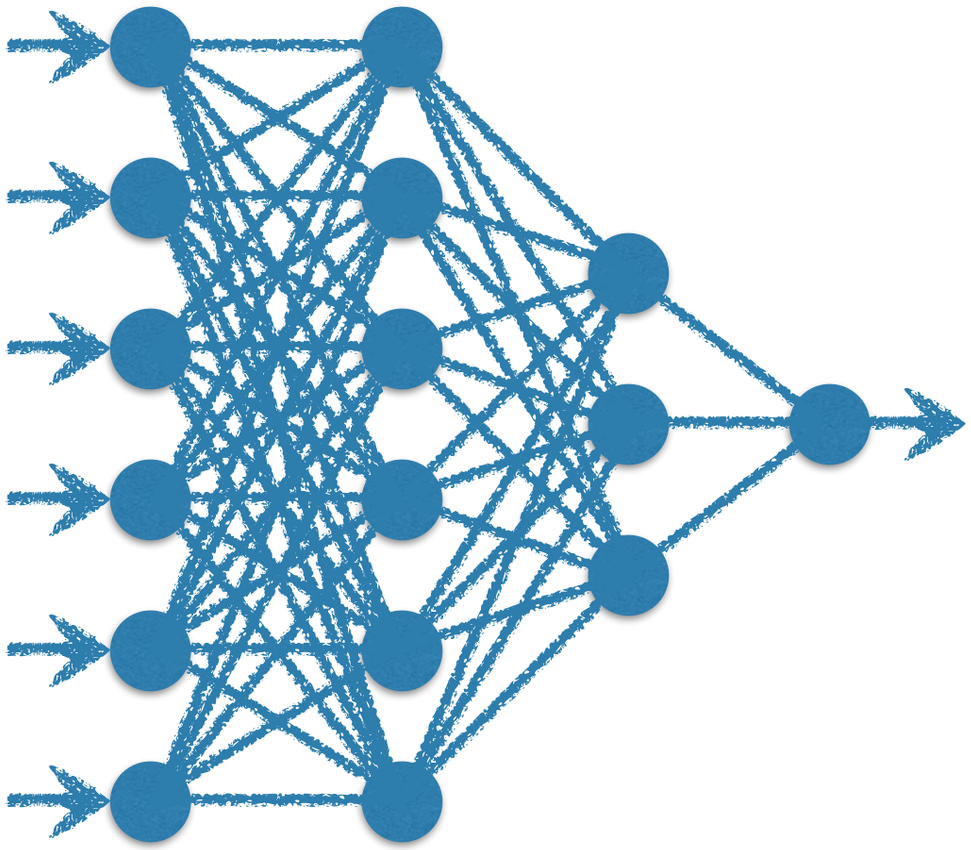
Learning automata …and generalisations!

Gabriele Puppis

Learning scenarios (supervised)

Learning scenarios (supervised)

NNs



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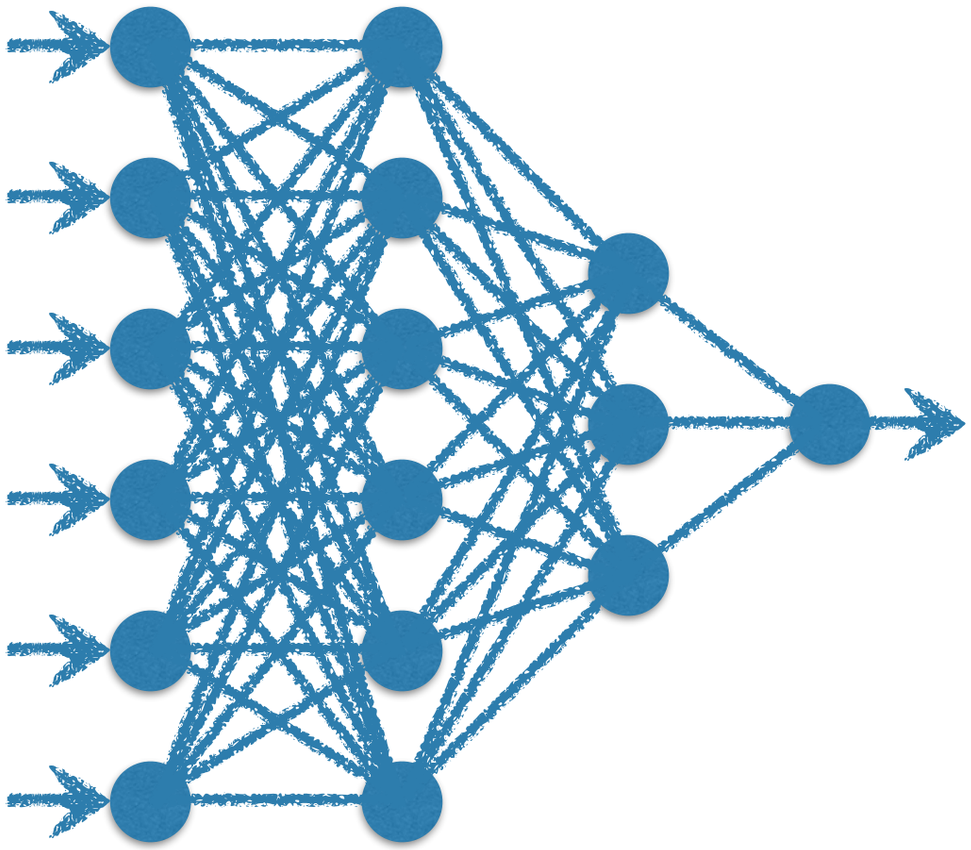
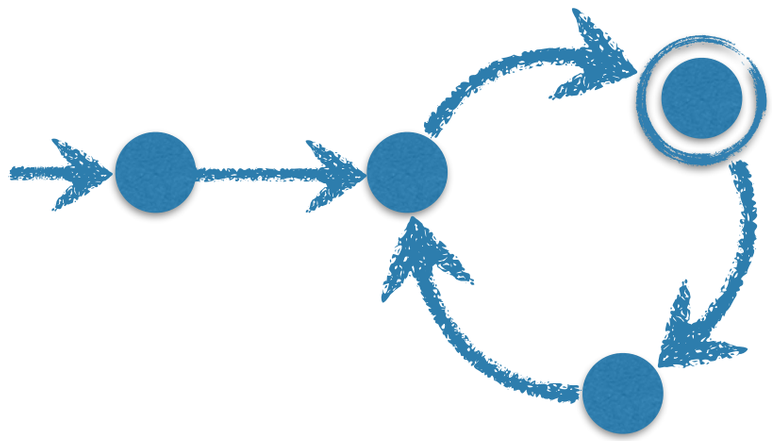


Automata



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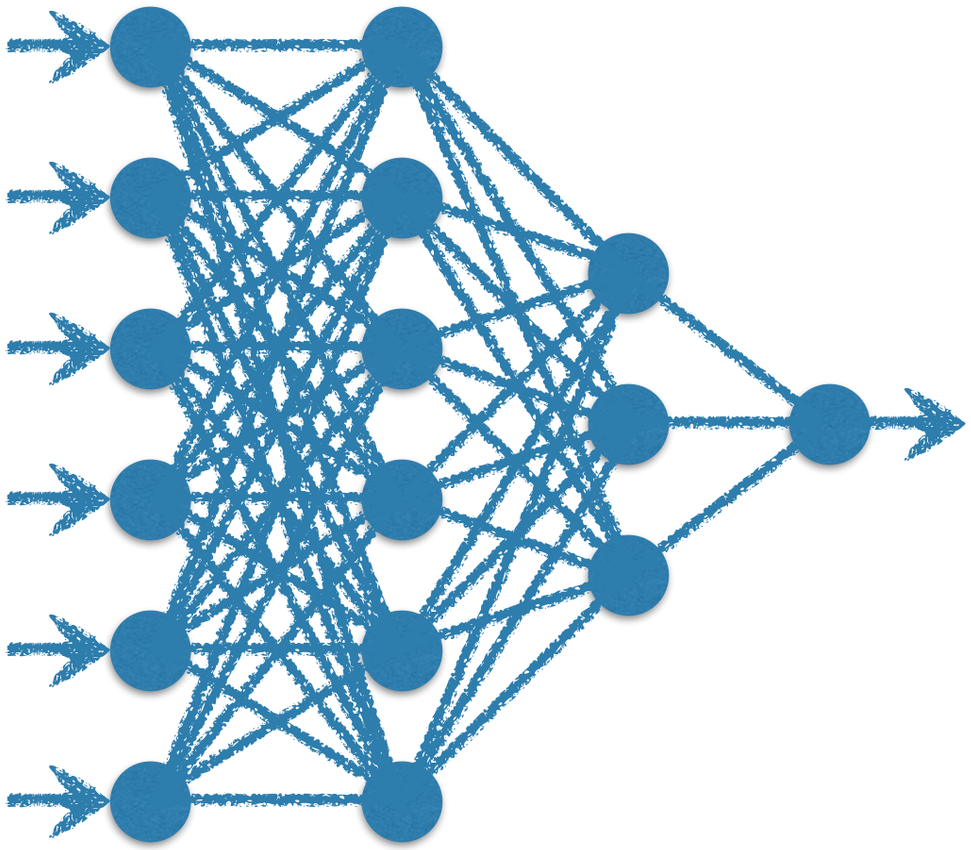
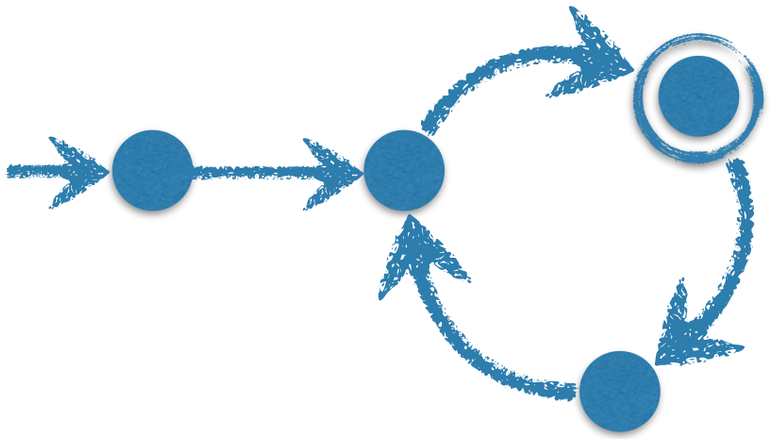


simple data, e.g. $\bar{x} \in \mathbb{R}^k$

Automata



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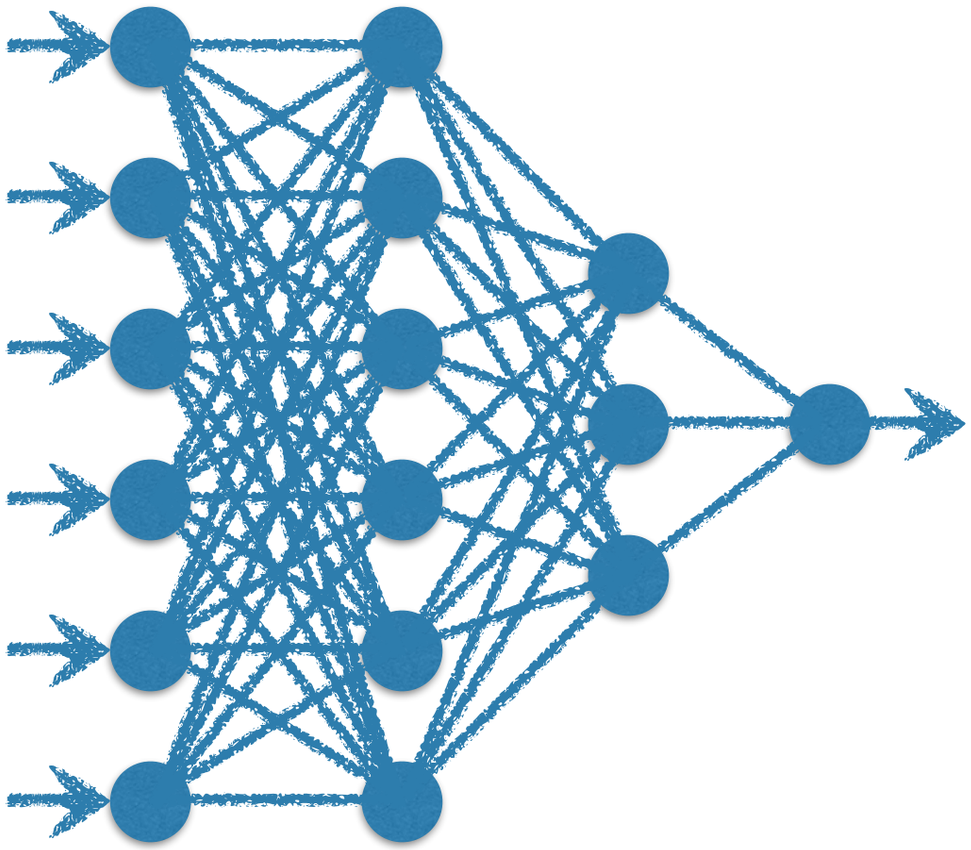

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complex functions, e.g. $f: \mathbb{R}^k \to \{0,1\}$
representing pictures of dogs

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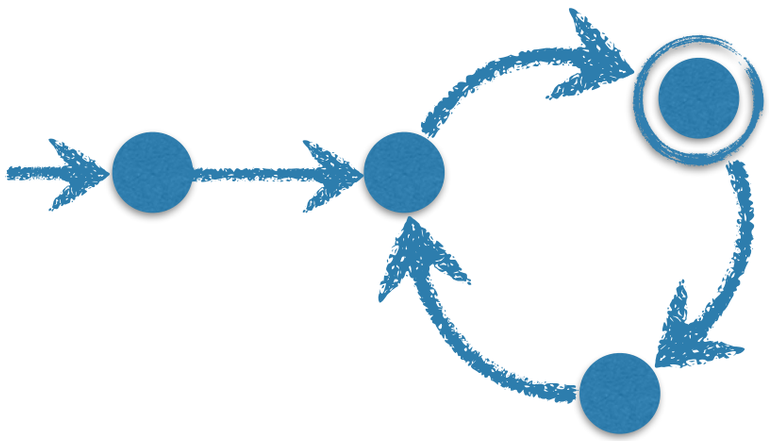

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representing regular $L \subseteq \Sigma^*$

Learning scenarios (supervised)

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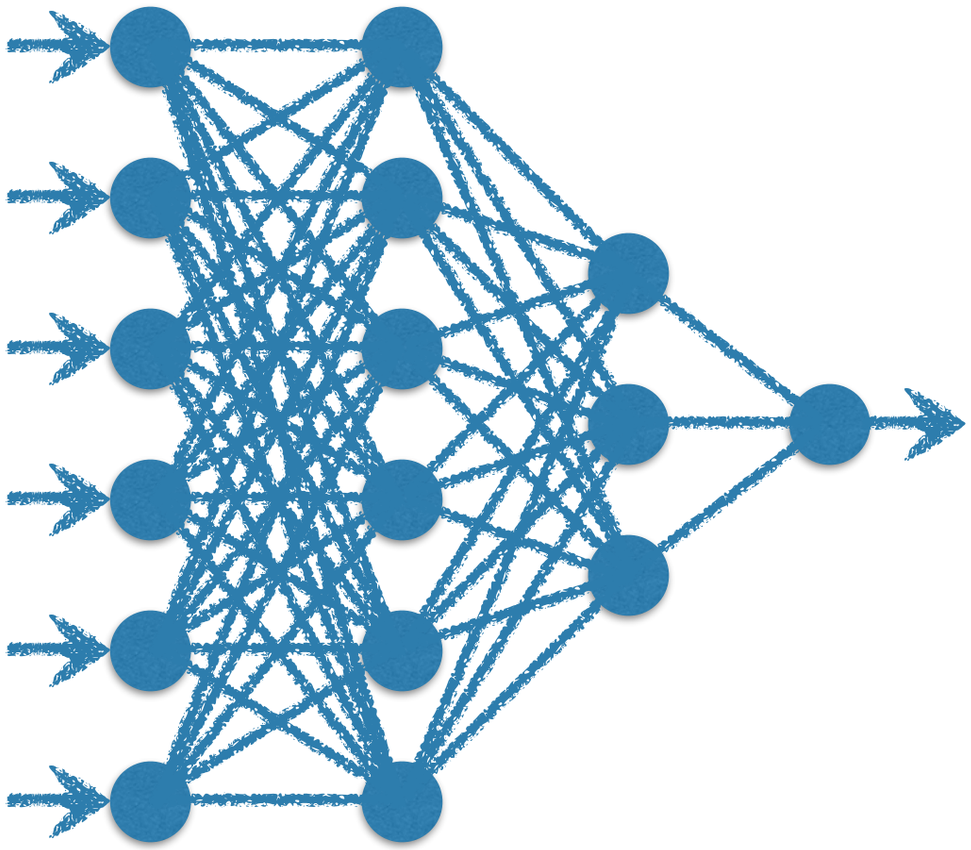

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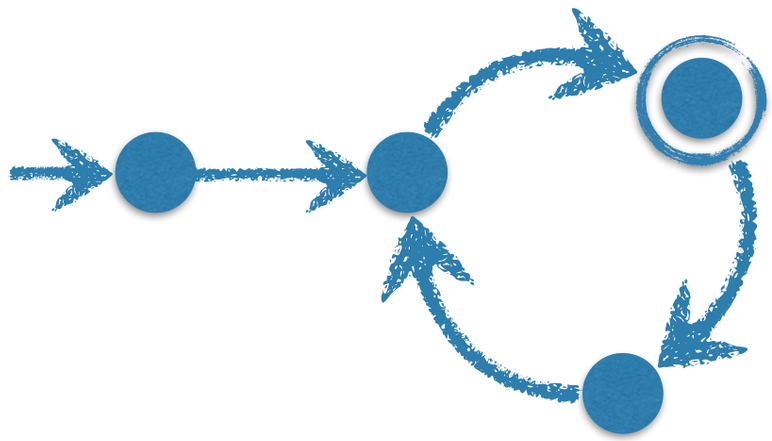
Automata



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structure (states+transitions) is learned

Learning scenarios (supervised)

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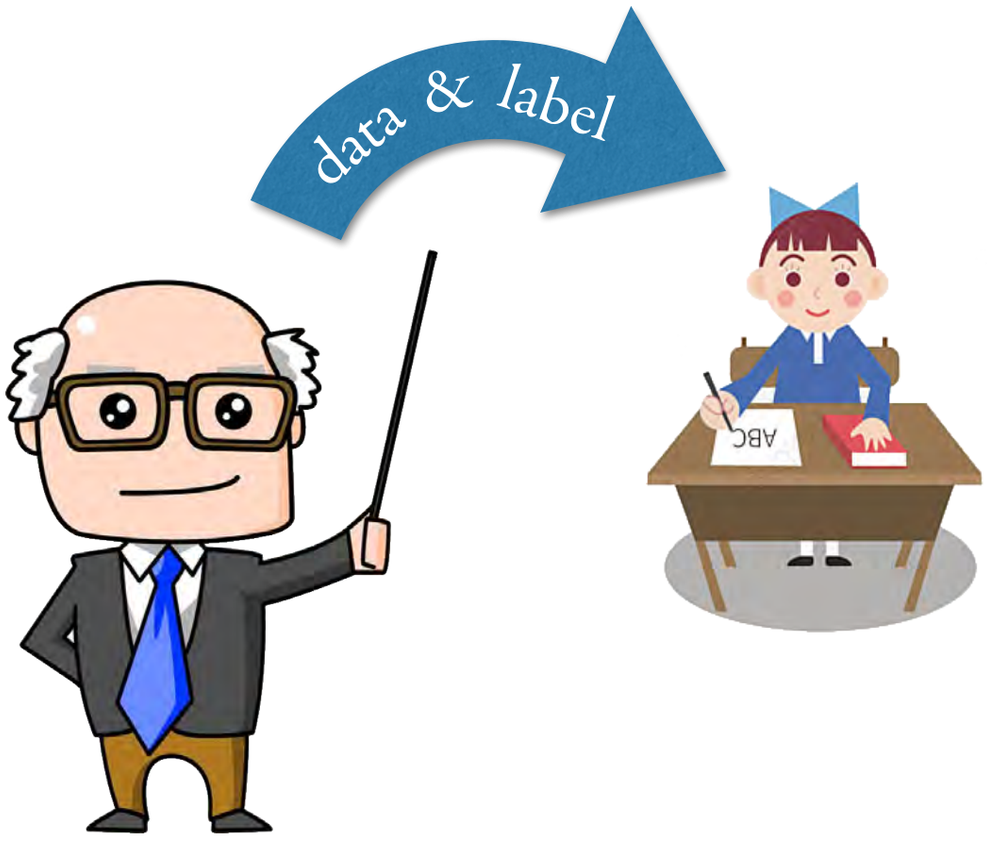
Passive



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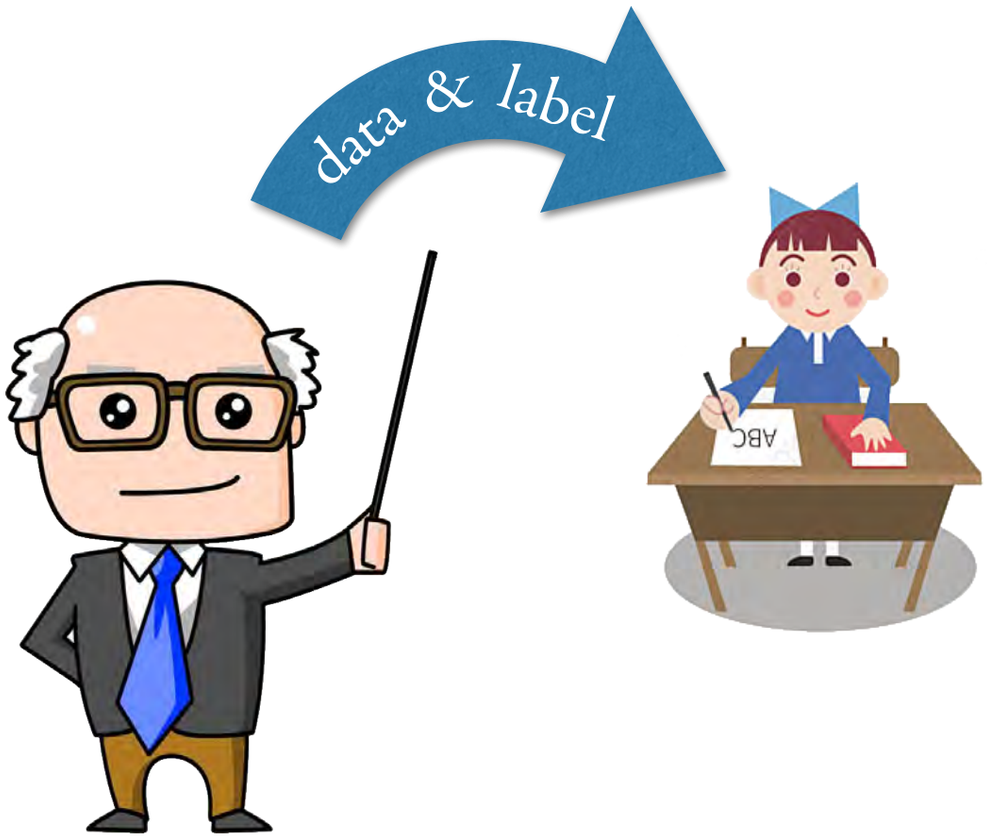
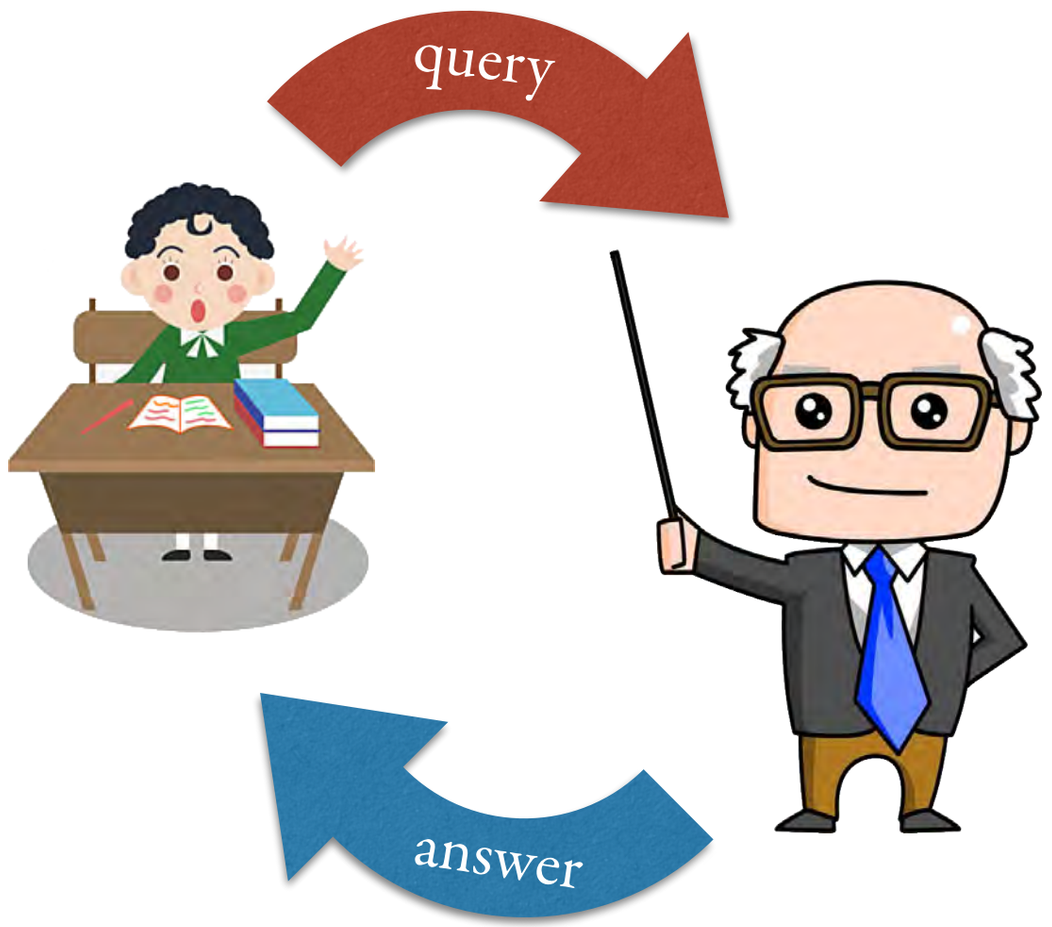


Active



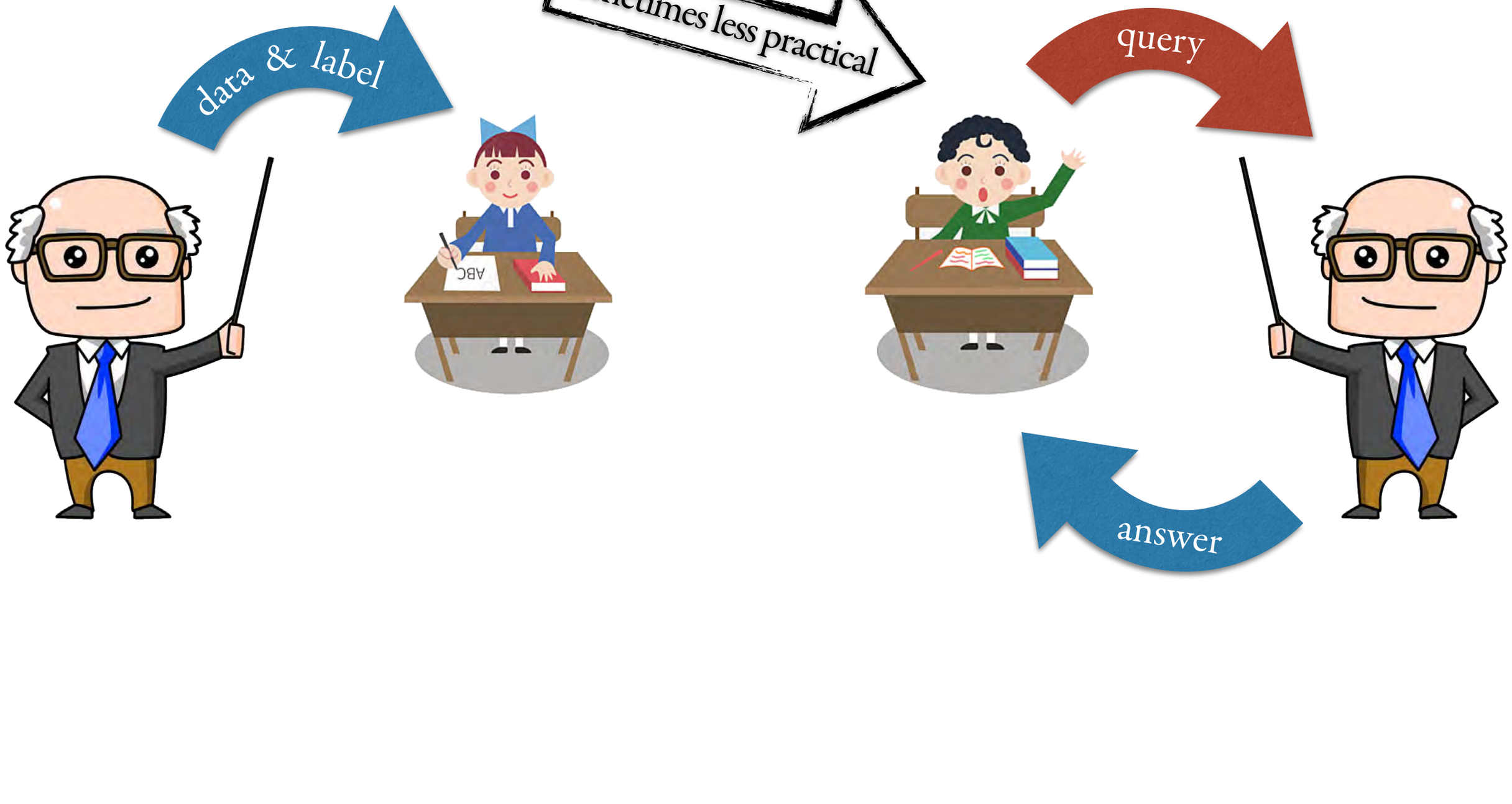
Learning scenarios (supervised)

Passive

more efficient

sometimes less practical

Active



A bit of history on automata learning

‘56	Moore	*Gedanken-experiments on sequential machines*
‘67	Gold	*passive learning in the limit*
‘87	Angluin	*active learning with queries*
’93...	Pitt et al.	*PAC-learning, cryptographic hardness*
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Edward F. Moore

INTRODUCTION

This paper is concerned with finite automata[1] from the experimental point of view. This does not mean that it reports the results of any experimentation on actual physical models, but rather it is concerned with what kinds of conclusions about the internal conditions of a finite machine it is possible to draw from external experiments. To emphasize the conceptual nature of these experiments, the word "gedanken-experiments" has been borrowed from the physicists for the title.

The sequential machines considered have a finite number of states, a finite number of possible input symbols, and a finite number of possible output symbols. The behavior of these machines is strictly deterministic (i.e., no random elements are permitted in the machines) in that the present state of a machine depends only on its previous input and previous state, and the present output depends only on the present state.

The point of view of this paper might also be extended to probabilistic machines (such as the noisy discrete channel of communication theory[2]), but this will not be attempted here.

EXPERIMENTS

There will be two kinds of experiments considered in this paper. The first of these, called a simple experiment, is depicted in Figure 1.

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Learning as a game

1) Teacher has a secret regular language L_0 (e.g. represented by DFA A_0)
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   a) yes if $w \in L_0$, no otherwise
   b) yes if $L_0 = L(A)$ then game ends and Learner wins,
      otherwise gives a (shortest) counter-example $w \in (L_0 - L(A)) \cup (L(A) - L_0)$
      then game continues from 2

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- membership queries alone are not sufficient for Learner to win

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Observations

- membership queries alone are not sufficient for Learner to win
- instead, equivalence queries alone are sufficient to win (why? how quick?)

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Theorem [Angluin '87] Learner has a strategy to win in a number of rounds that is *polynomial* in $|A_0|$

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in the form of an algorithm (L^* algorithm)

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<u>MAT (Minimally Adequate Teacher)</u>

Sometimes answering an equivalence query is not practical: if Teacher knew A_0, he could pass this information directly to Learner, so why bothering querying?

Equivalence queries can however be *approximated* by a series of membership queries: as long as membership in A matches membership in A_0, Learner assumes he made the correct guess.

This latter setting is often called
<u>Black-box learning</u>

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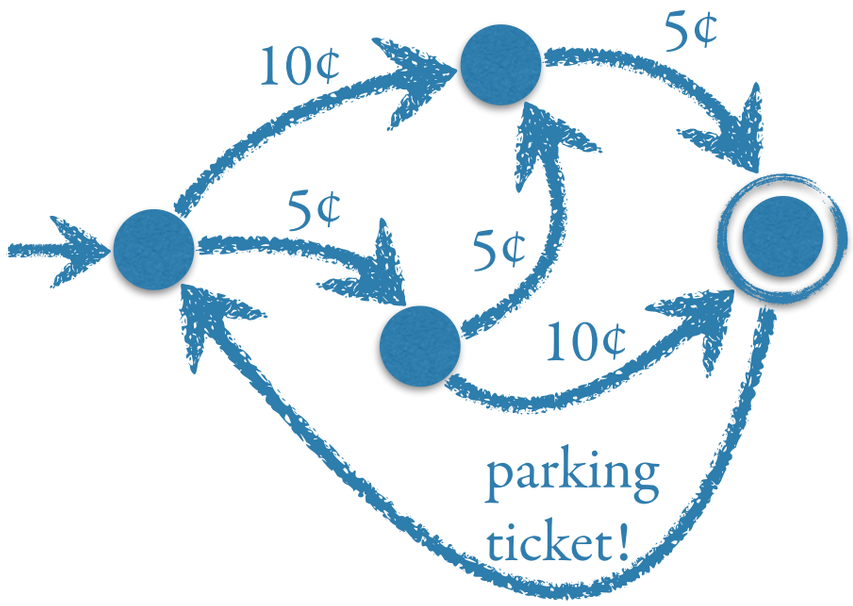

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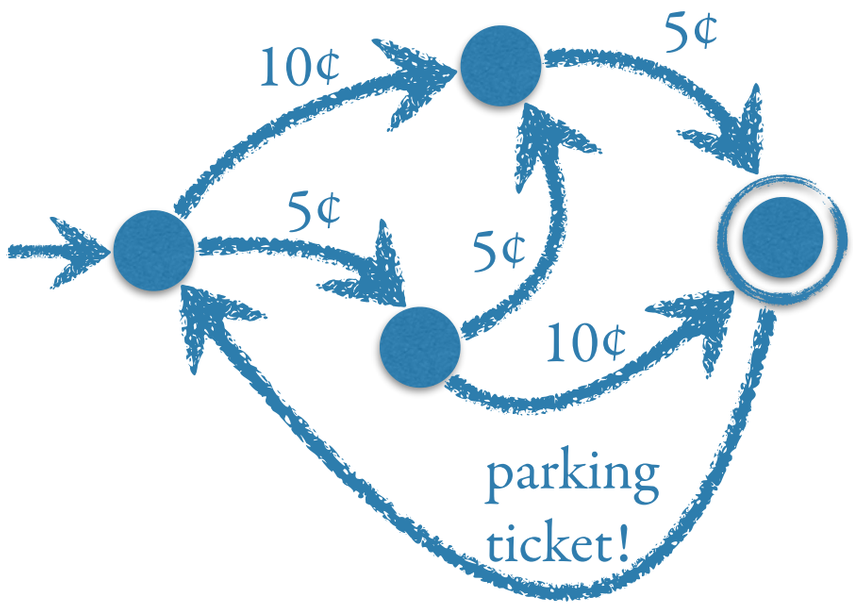

Verification: model-learning (in contrast to model-checking)

Control theory: system identification, diagram inference

Language theory: grammar inference, regular extrapolation

Security: protocol state fuzzing

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Learning as a killer app of Myhill-Nerode theorem

Myhill-Nerode equivalence: $u \sim_{L_0} v$ if ....?

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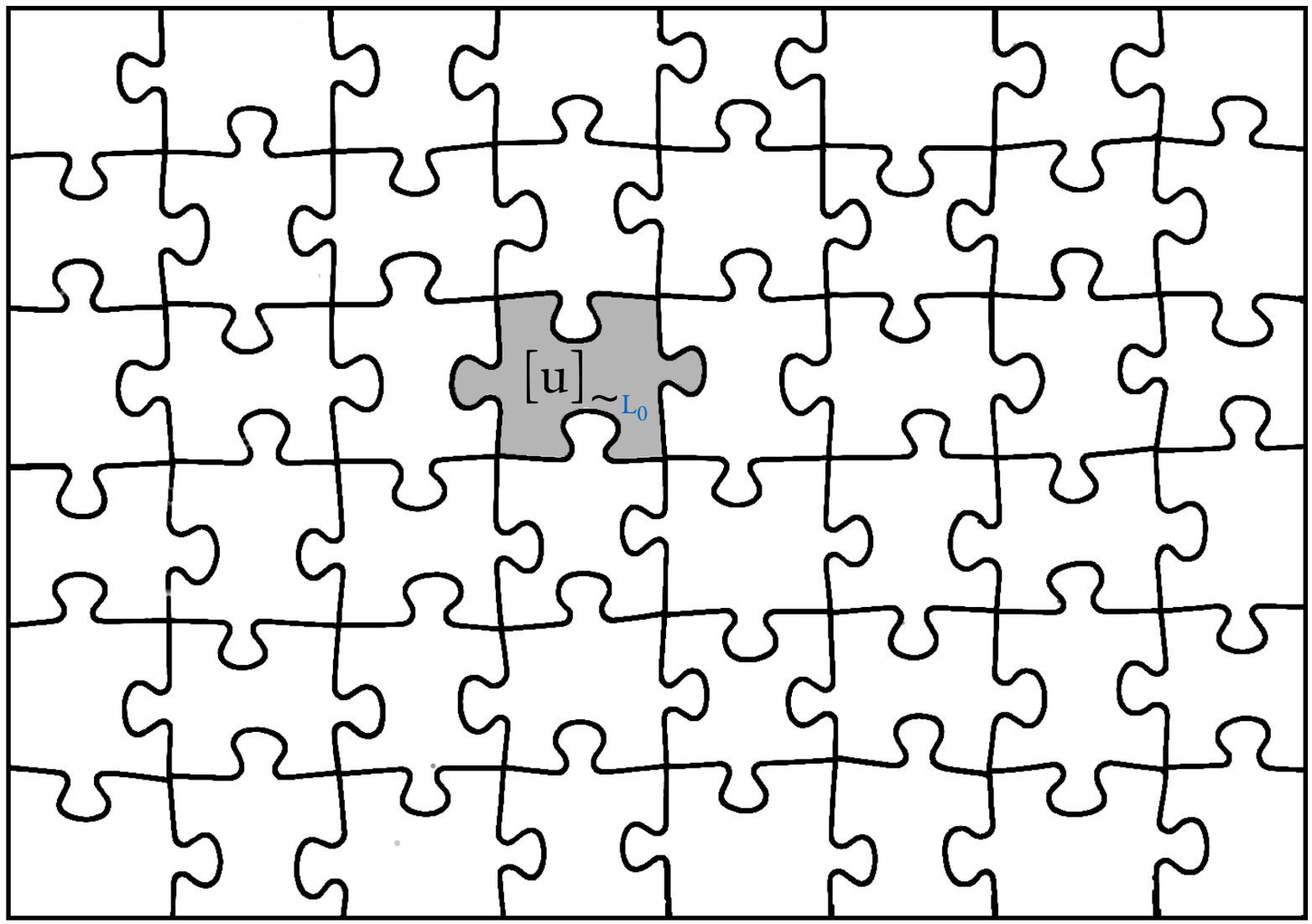

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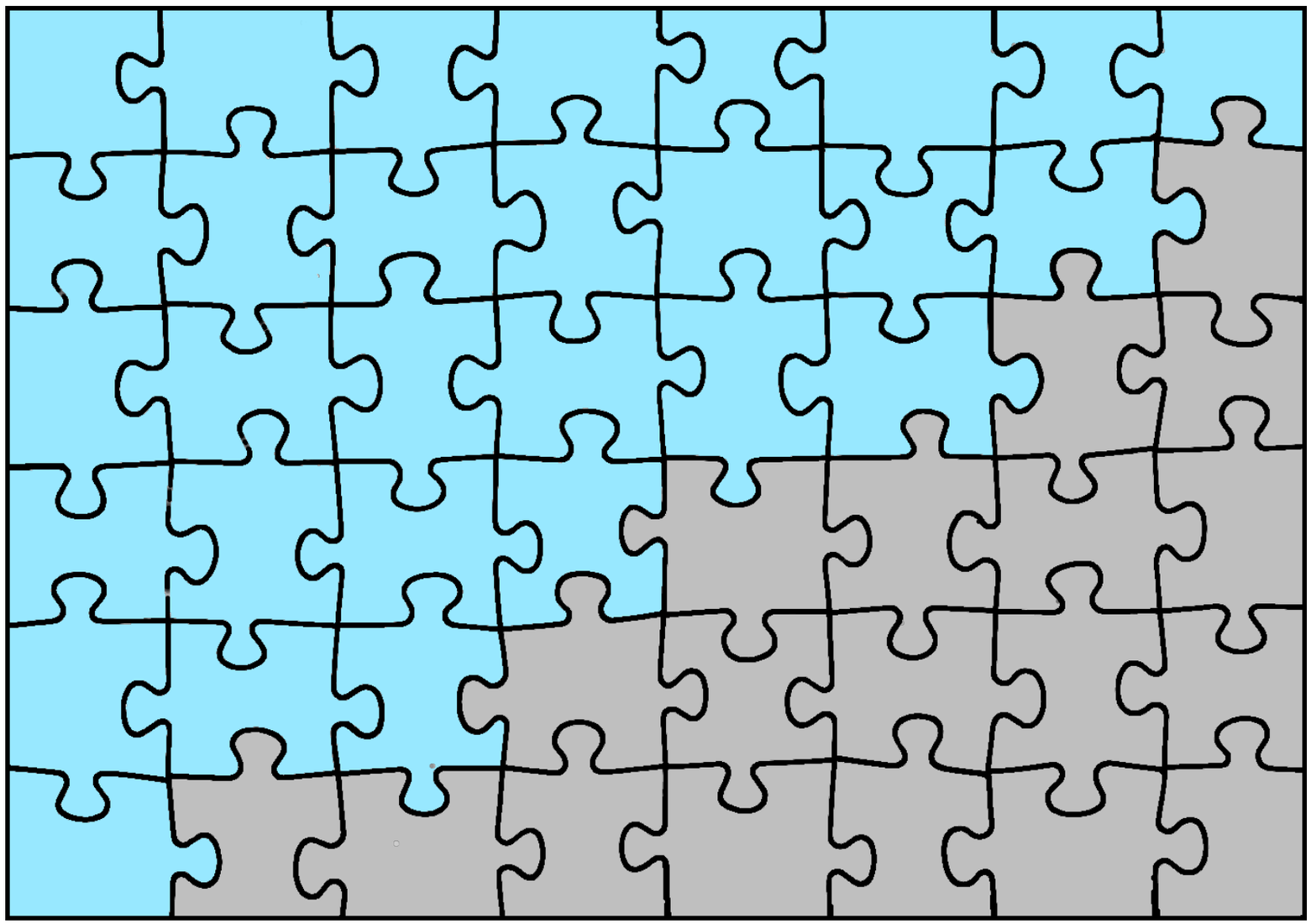
L_0



Σ^* - L_0

Properties:

- $\sim_{L_0}$ refines $=_{L_0}$

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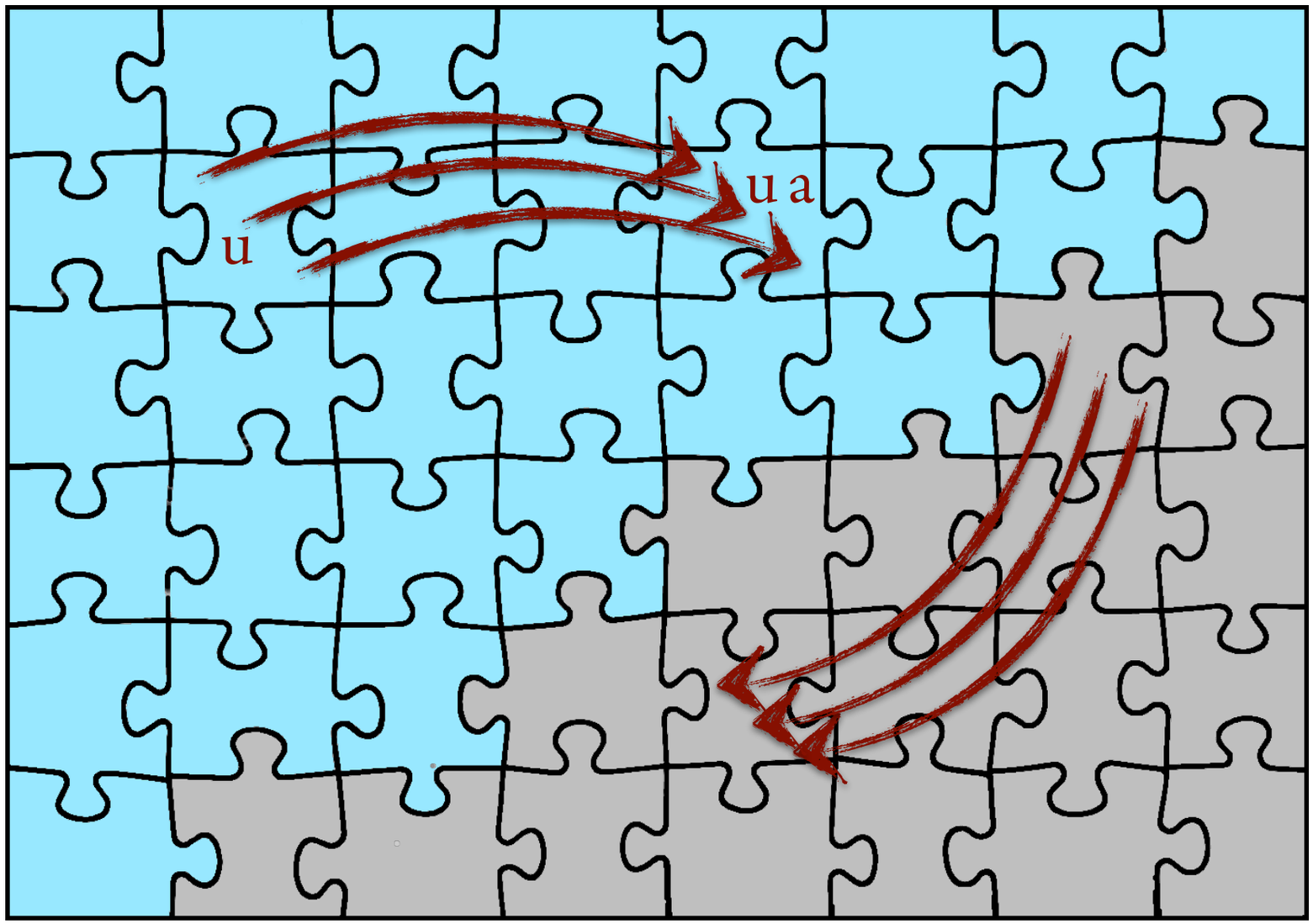

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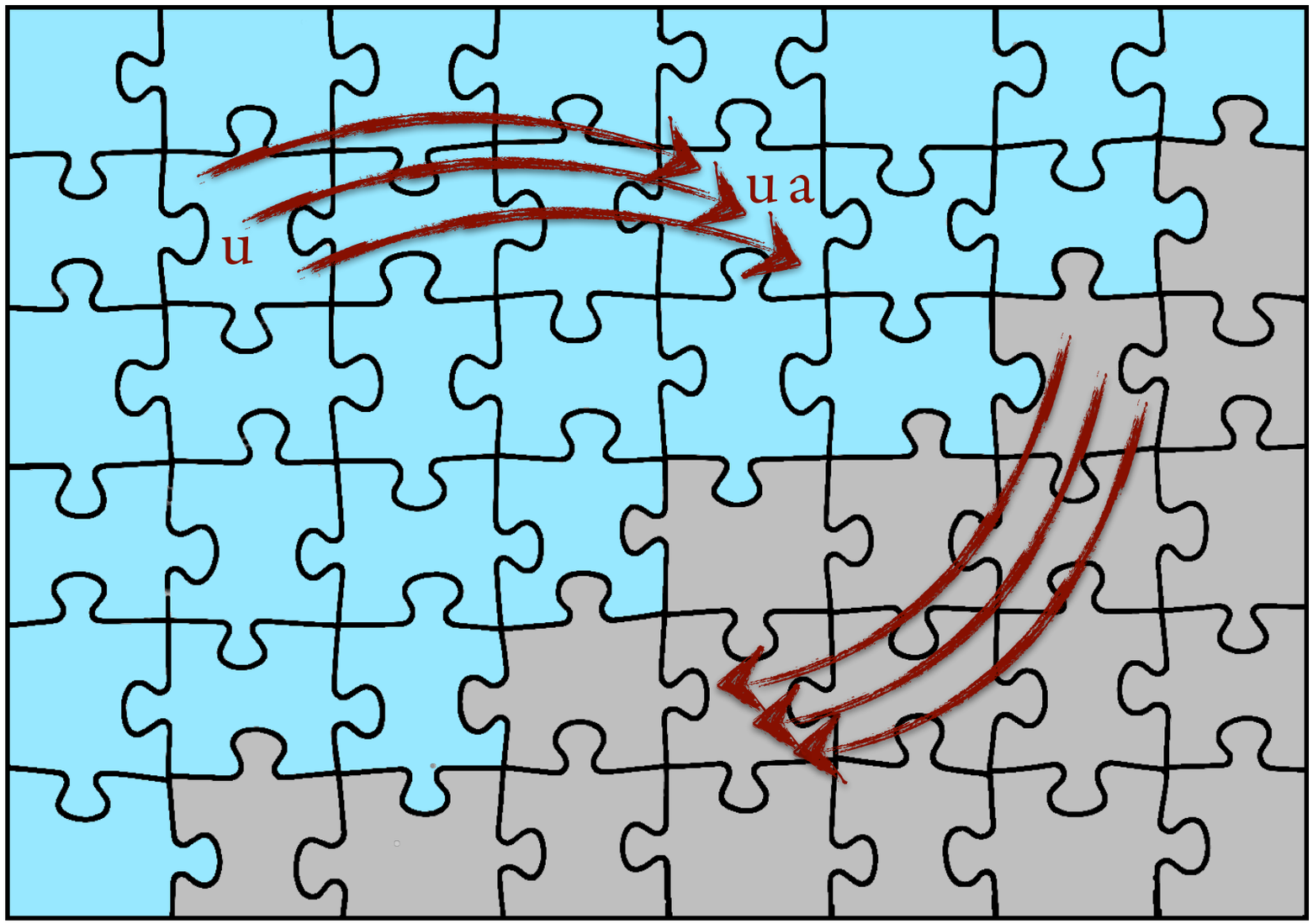

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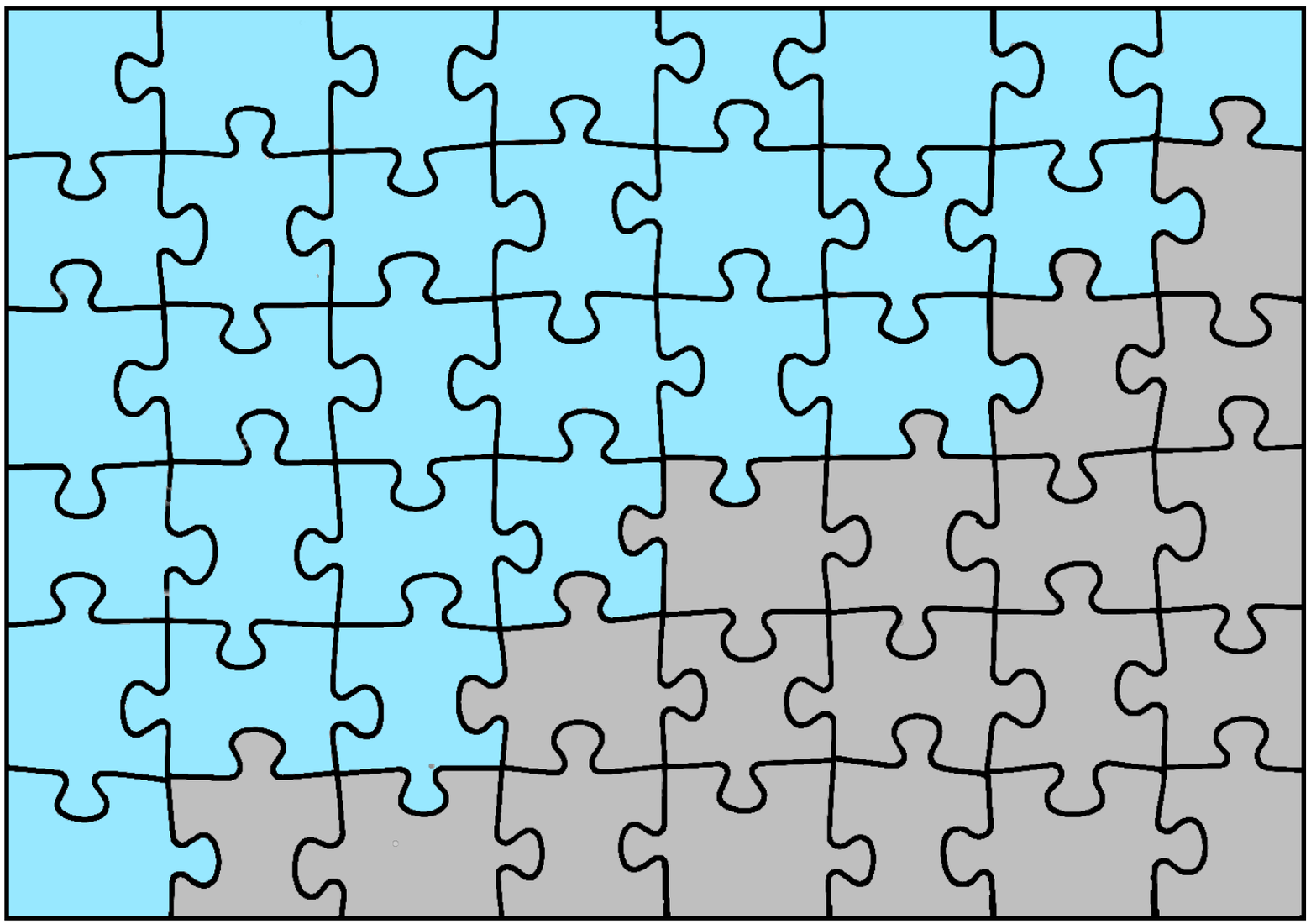

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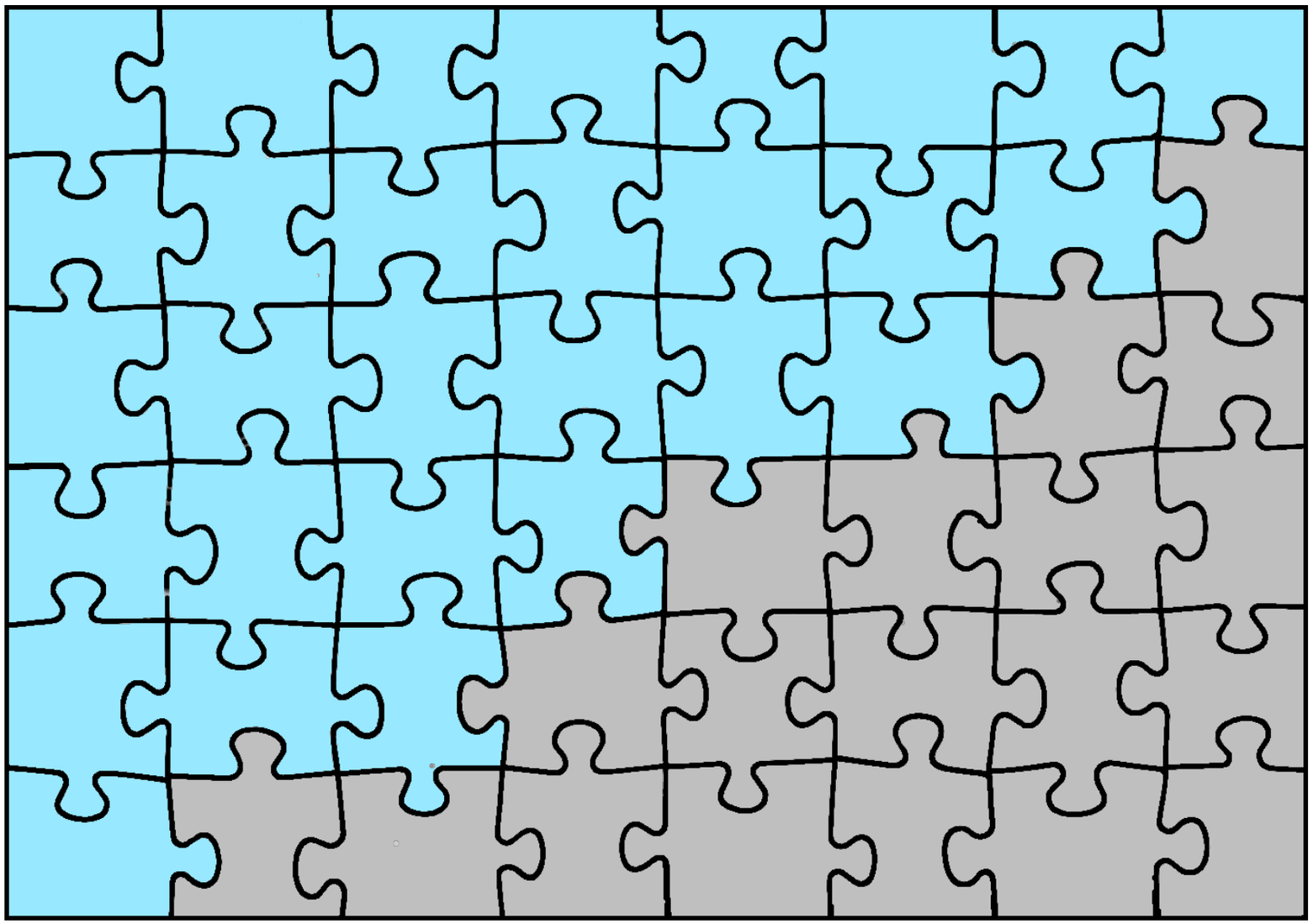
Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T$ $u\,t \in L_0 \leftrightarrow v\,t \in L_0$



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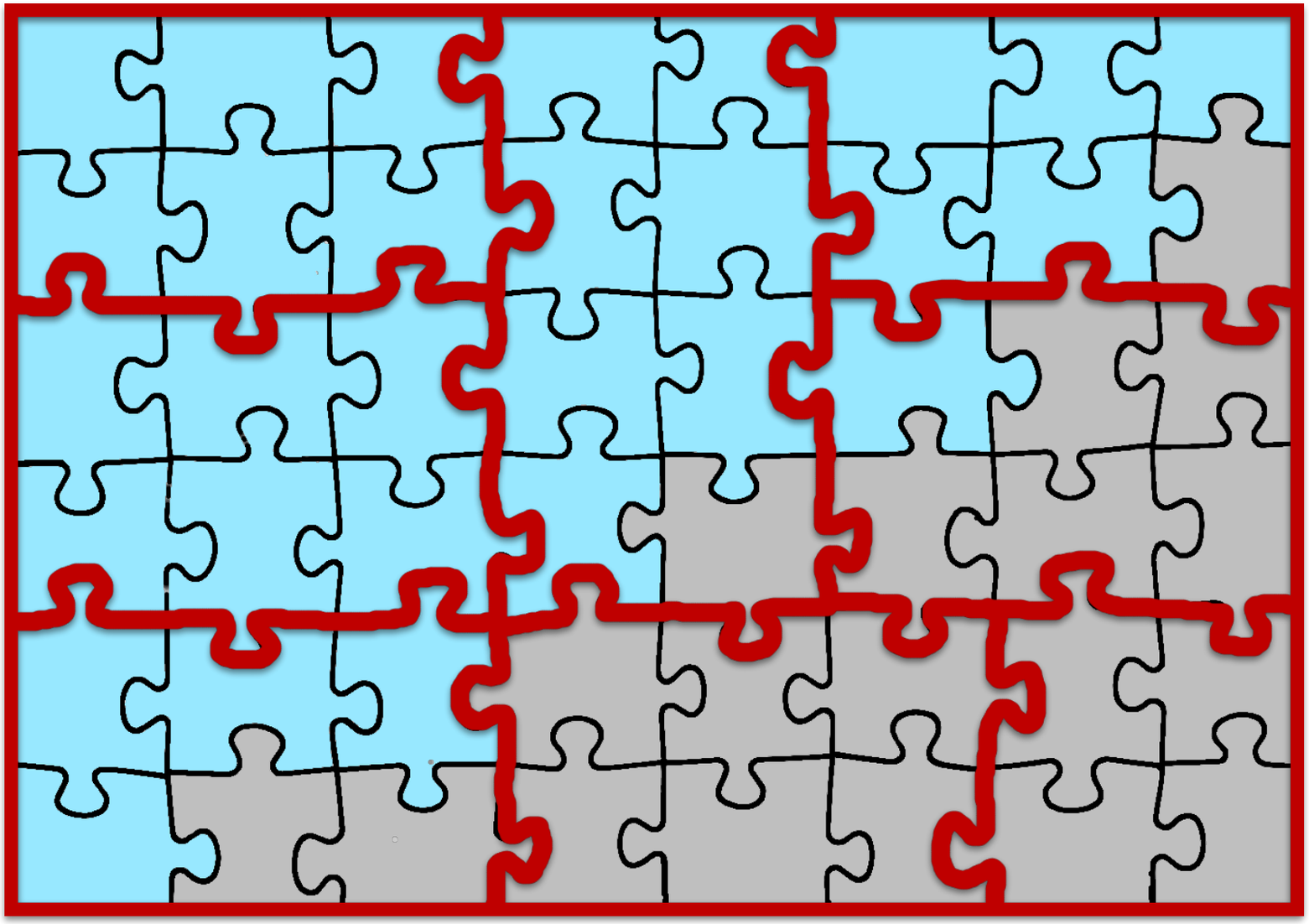

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- $\approx_{L_0,T}$ is coarser than $\sim_{L_0}$

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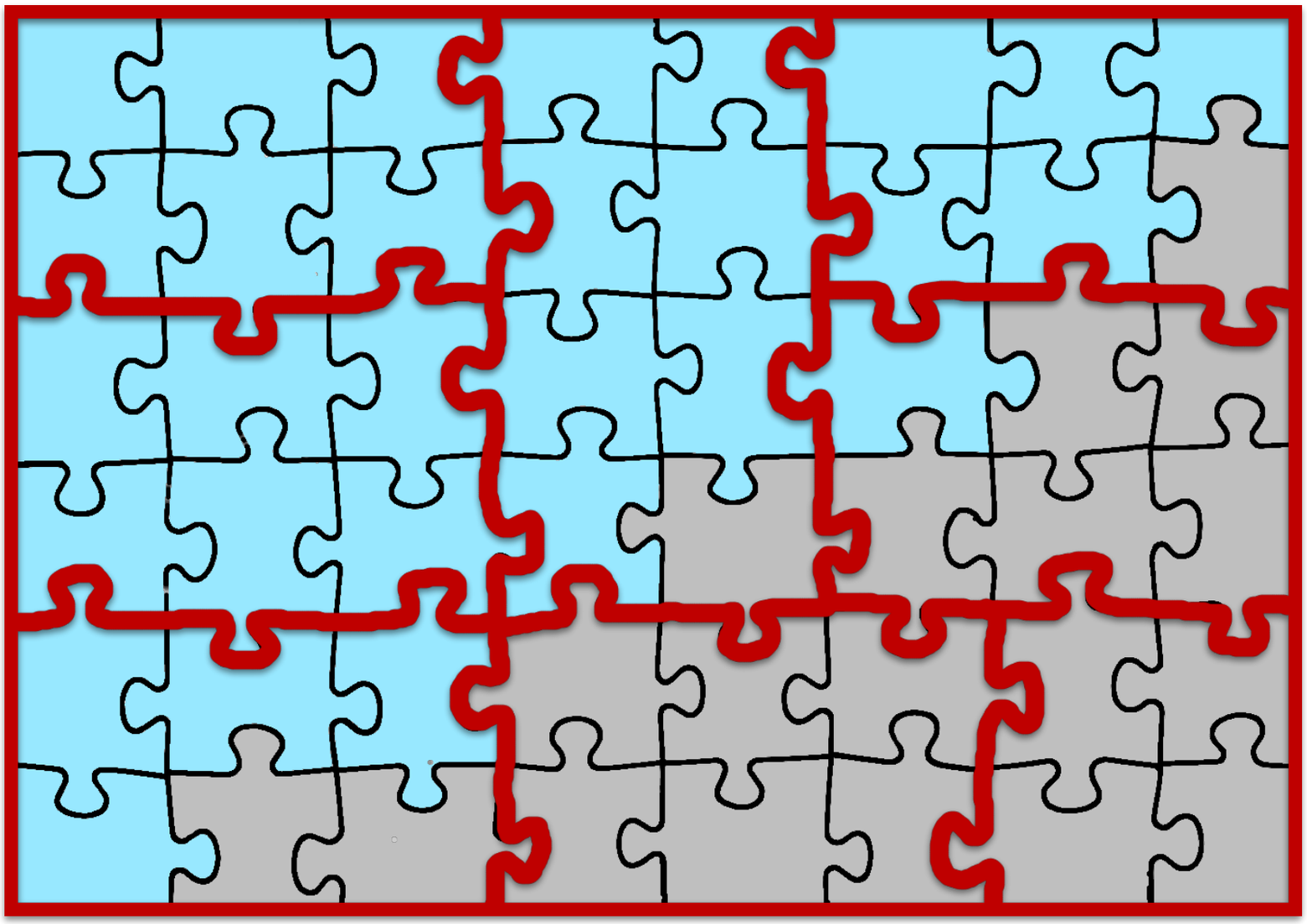

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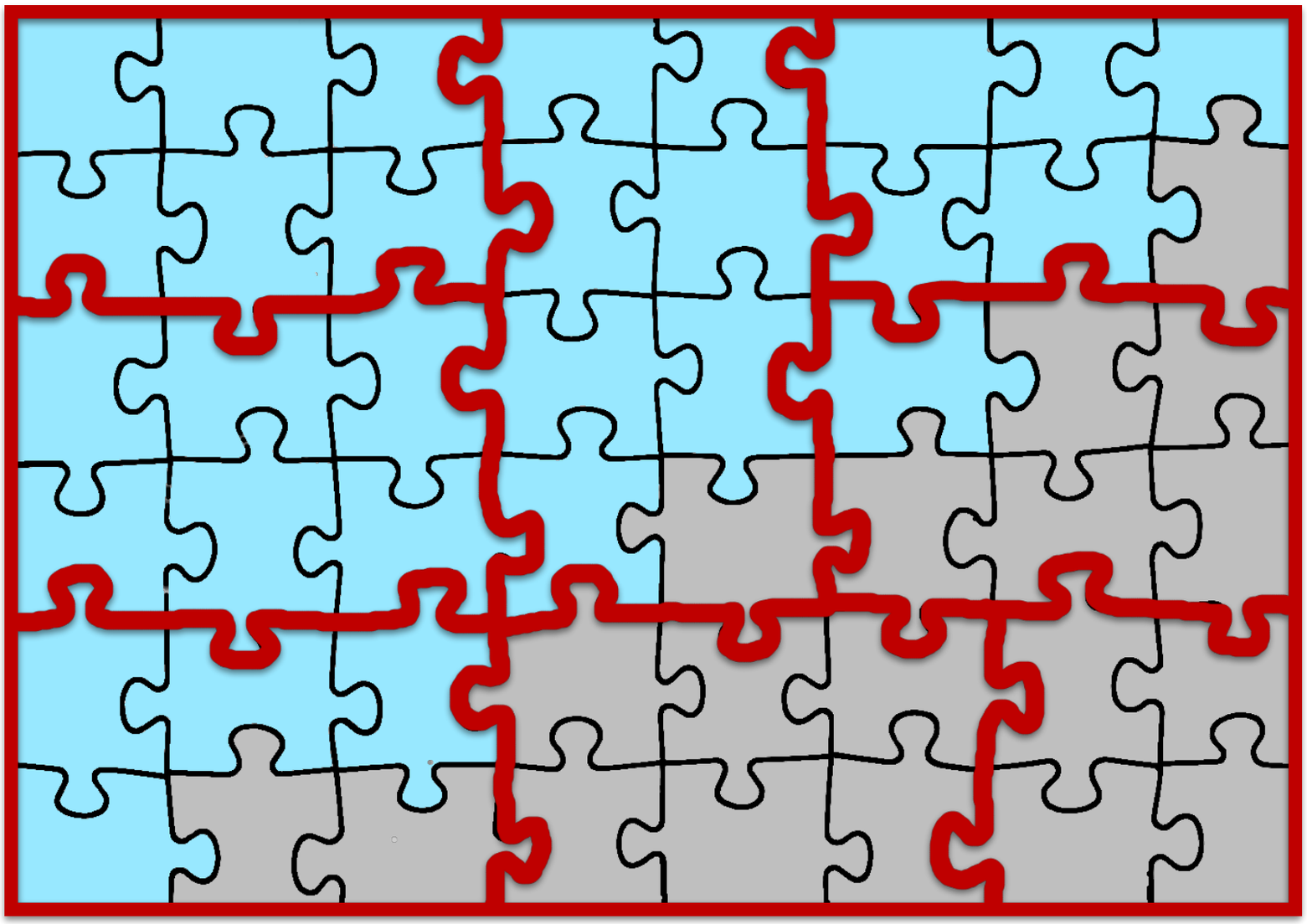

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

Learning as a killer app of Myhill-Nerode theorem

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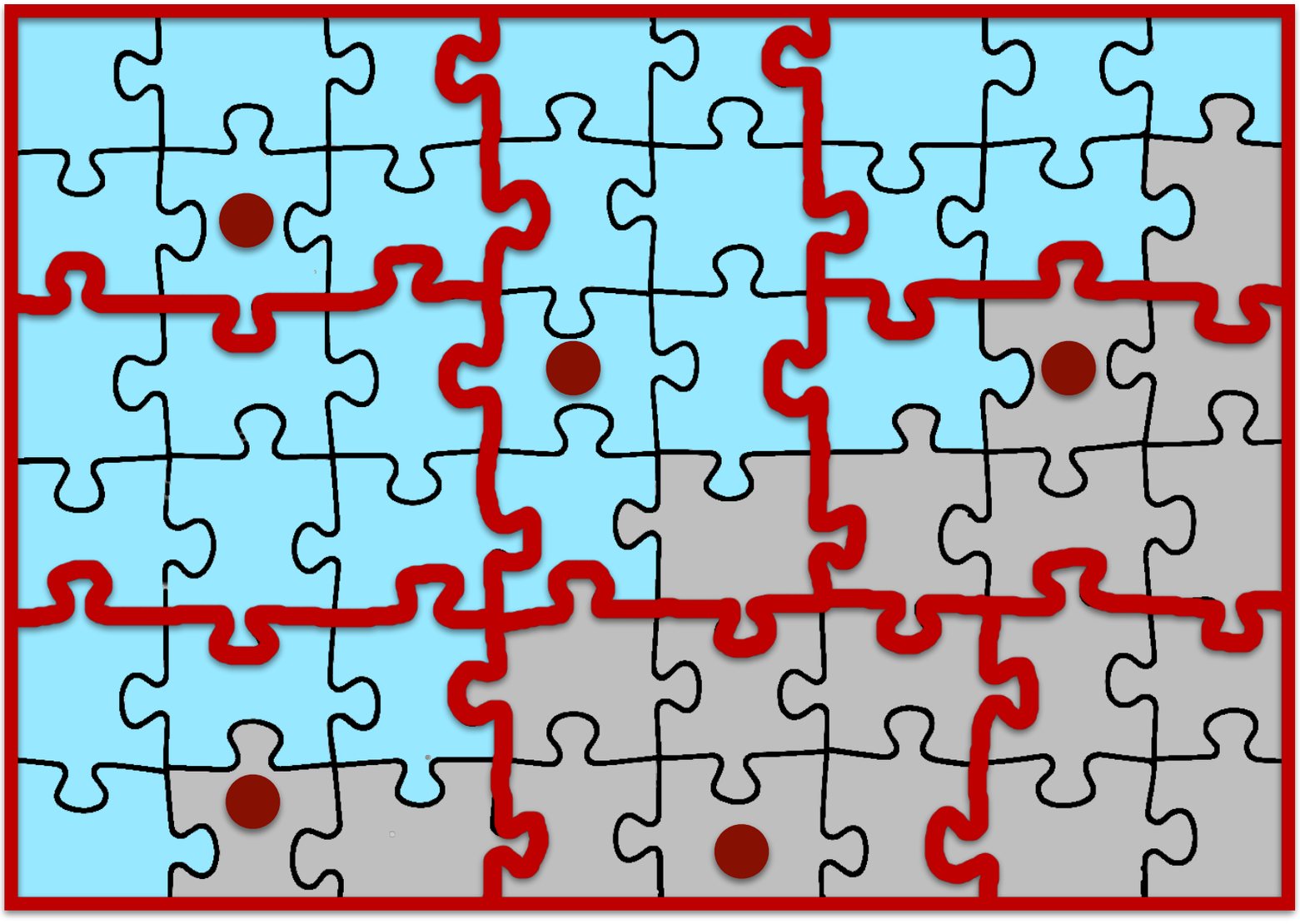

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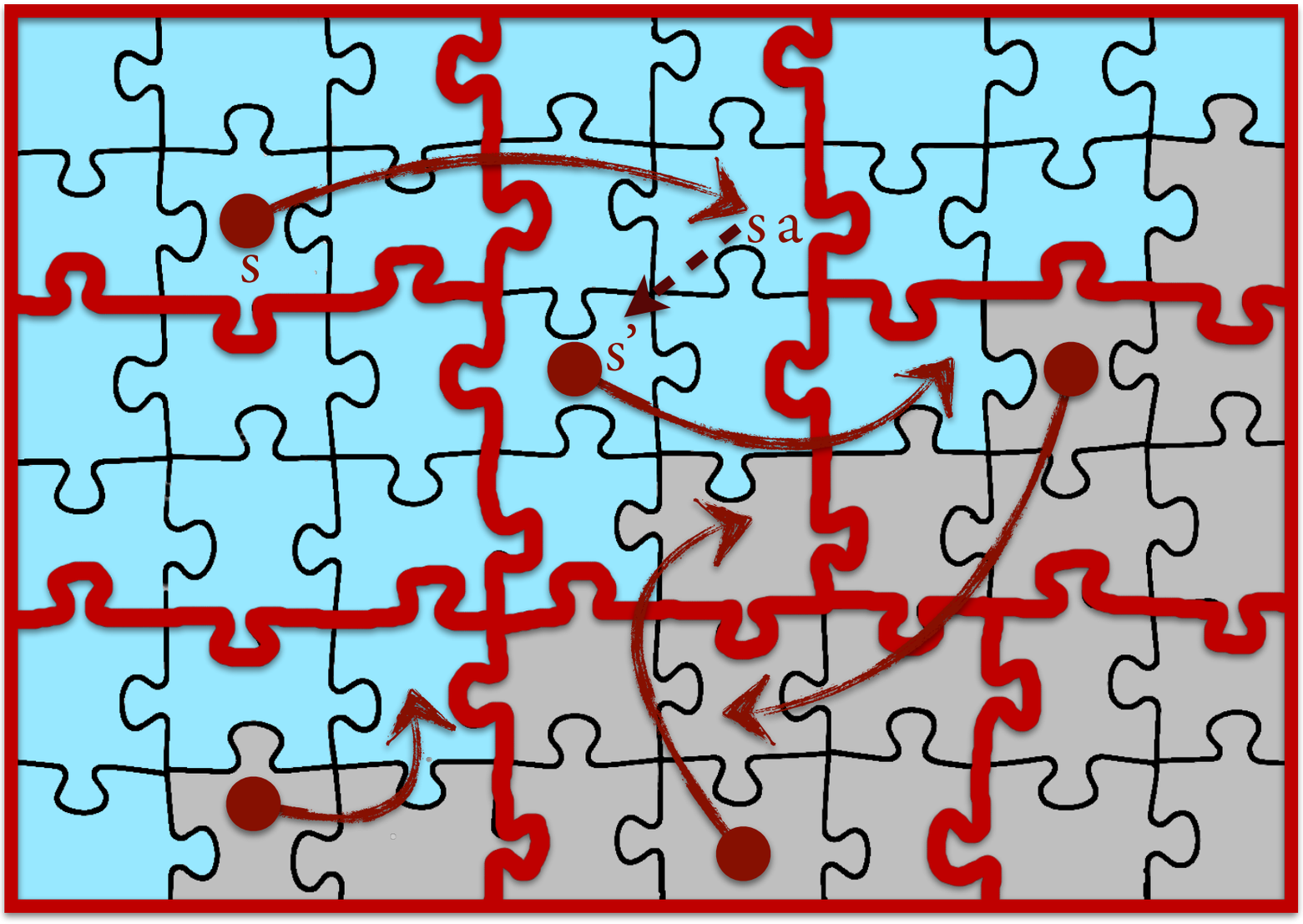

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is <u>T-minimal</u>
if $\forall s \neq s' \in S$ $s \not\approx_{L_0,T} s'$

$S \subseteq \Sigma^*$ is <u>T-complete</u>
if $\forall s \in S$ $\forall a \in \Sigma$
$\exists s' \in S$ $s' \approx_{L_0,T} sa$

Learning as a killer app of Myhill-Nerode theorem

Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall\, t \in T \quad u\,t \in L_0 \leftrightarrow v\,t \in L_0$

Learner-strategy

$S = T = \{\varepsilon\}$

loop // S is T-minimal here

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

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Learner-strategy

$S = T = \{\varepsilon\}$

loop // S is T-minimal here

while S NOT T-complete

choose $s \in S$ and $a \in \Sigma$ such that $\nexists s' \in S \;\; s' \approx_{L_0,T} s\,a$

$S = S \cup \{s\,a\}$

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is T-minimal if $\forall\, s \neq s' \in S \;\; s \not\approx_{L_0,T} s'$

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$S = S \cup \{s\,a\}$

$\forall t \in T$

Membership$(s'\,t)$=Membership$(s\,a\,t)$

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Learning as a killer app of Myhill-Nerode theorem

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while S NOT T-complete

choose $s \in S$ and $a \in \Sigma$ such that $\nexists s' \in S \;\; s' \approx_{L_0,T} s\,a$

$S = S \cup \{s\,a\}$

// S is now T-minimal and T-complete

A = DFA with state set S, initial state ε,
final states s'' such that Membership(s''),
transitions $\delta(s,a) = s' \approx_{L_0,T} s\,a$

$\forall t \in T$

Membership$(s'\,t)$=Membership$(s\,a\,t)$

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

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$\forall t \in T$

Membership(s' t)=Membership(s a t)

S T-complete → s' exists
S T-minimal → s' unique

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is T-minimal
if $\forall\, s \neq s' \in S \;\; s \not\approx_{L_0,T} s'$

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall\, t \in T \quad u\,t \in L_0 \leftrightarrow v\,t \in L_0$

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  while S NOT T-complete
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  return A
else
  let w be the counter-example of equivalence
  T = T ∪ {suffixes of w}
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else
  let w be the counter-example of equivalence
  T = T ∪ {suffixes of w}  // Claim: S becomes T-incomplete
                           // and will grow at next iteration...
```

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      transitions  δ(s,a) = s' ≈L0,T s a

if Equivalence(A)     // this surely happens
  return A            // when |S| = index(~L0)
else
  let w be the counter-example of equivalence
  T = T ∪ {suffixes of w}  // Claim: S becomes T-incomplete
                           // and will grow at next iteration...
```

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                           // and will grow at next iteration...
```

Proof by contradiction:

Assume:

- $w = a_1 \dots a_n$ counter-example
- $t_i = a_{i+1} \dots a_n$ suffixes of w
- s_i state reached by A after reading prefix $a_1 \dots a_i$ of w
- S is $(T \cup \{t_0,\dots,t_n\})$-complete

Verify by induction on i that

$$w \in L_0 \text{ iff } s_i\, t_i \in L_0$$

Conclude $w \in L_0$ iff $w \in L(A)$

An alternative view - Hankel matrices

Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall\, t \in T \quad u\,t \in L_0 \leftrightarrow v\,t \in L_0$

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

$H(s,t) = \text{Membership}(st)$

(infinite, highly redundant, but nicely structured)

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	**0**	**1**	**0**	**0**	**0**	**...**	**0**	**...**
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

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	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	**1**	0	...	0	...
a	0	1	0	**0**	0	...	0	...
b	0	1	0	**0**	0	...	0	...
aa	1	0	0	**1**	0	...	0	...
ab	0	0	0	**0**	0	...	0	...
...	...	...	...	**...**	...	...	...	...
aba	0	0	0	**0**	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

$H(s,t) = \text{Membership}(s\,t)$

(infinite, highly redundant, but nicely structured)

a row = a $\sim_{L_0}$-class

a column = a test word t

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall\, t \in T \quad u\,t \in L_0 \leftrightarrow v\,t \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	**0**	**1**	0	**0**	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

$H(s,t) = \text{Membership}(s\,t)$

(infinite, highly redundant, but nicely structured)

a row = a $\sim_{L_0}$-class

a column = a test word t

a subrow = a $\approx_{L_0,T}$-class

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

An alternative view - Hankel matrices

Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

$S = \{\varepsilon\}$ $T = \{\varepsilon\}$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

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b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

An alternative view - Hankel matrices

Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	**1**	0	0	1	0	...	0	...
a	**0**	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

$S = \{\varepsilon\}$ $T = \{\varepsilon\}$

expand S to make it T-complete...

$S = \{\varepsilon, a\}$ $T = \{\varepsilon\}$

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

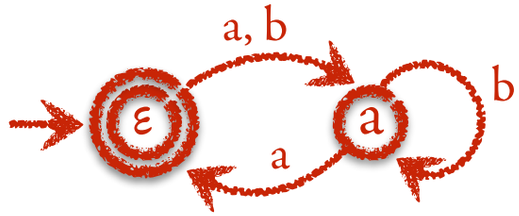
	ε	a	b	aa	ab	...	aba	...
ε	**1**	0	0	1	0	...	0	...
a	**0**	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

build candidate automaton...



An alternative view - Hankel matrices

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S = {ε} T = {ε}

expand S to make it T-complete...

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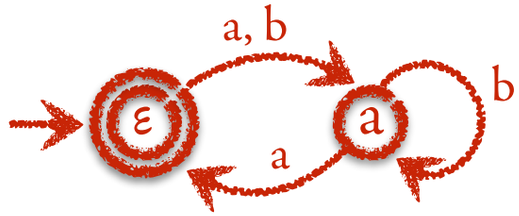
build candidate automaton...



expand T by counter-example w = aba

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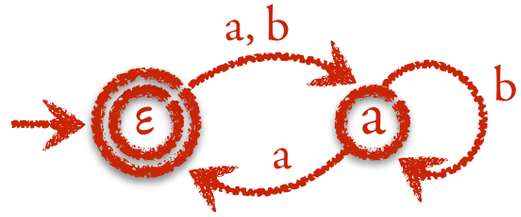
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

build candidate automaton...



expand T by counter-example w = aba

S = {ε, a} T = {ε, a, ba, aba}

An alternative view - Hankel matrices

Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

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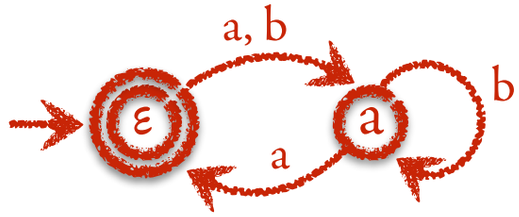
build candidate automaton...



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ε	**1**	**0**	0	1	0	...	**0**	...
a	**0**	**1**	0	0	0	...	**0**	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	**0**	**0**	0	0	0	...	**0**	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

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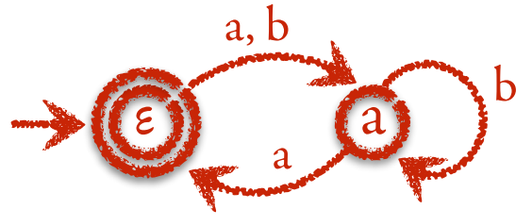

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expand S to make it T-complete...

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...	...	...	...	...	...	...	...	...
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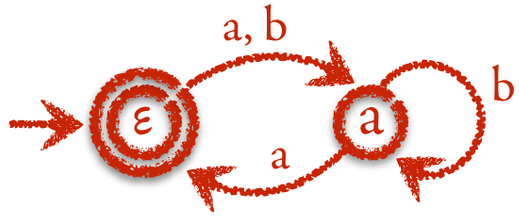

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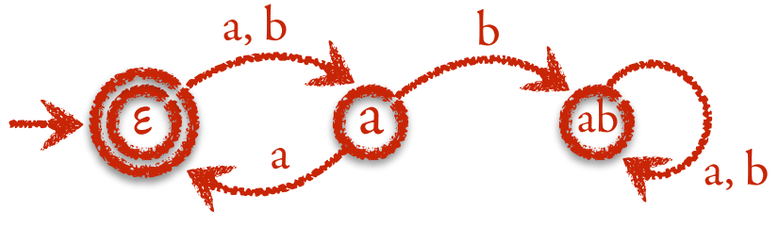

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...	...	...	...	...	...	...	...	...
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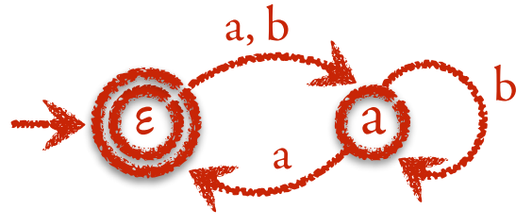

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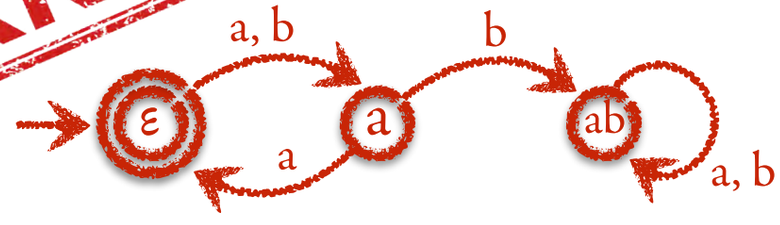
CORRECT



From word languages to word functions

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Automata can be used to represent not only languages, but also *functions on words*

$$f: \Sigma^* \to \Gamma^*$$

e.g. $f(abaab) = abcaacb$

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Many variants of automata with outputs. Simplest one is <u>sequential transducer</u>

i.e. $A = (\Sigma, \Gamma, Q, q_0, \delta)$ with $\delta: \ Q \times \Sigma \to Q \times \Gamma^*$ (e.g. $\delta(q, a)=(q', ac)$)

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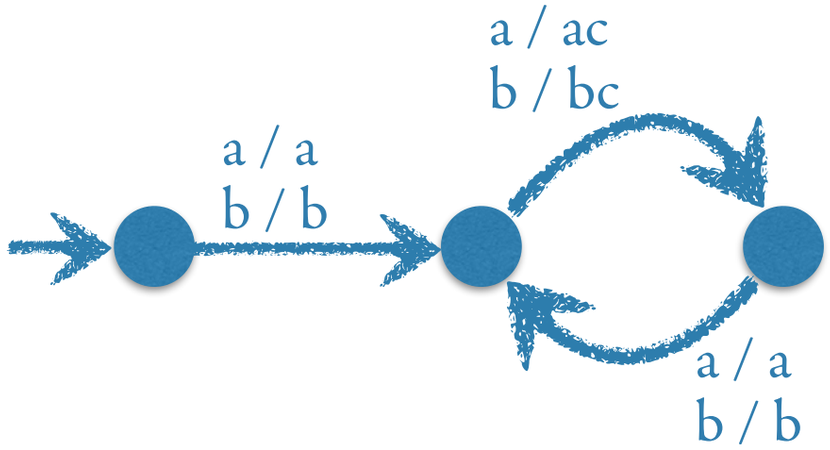

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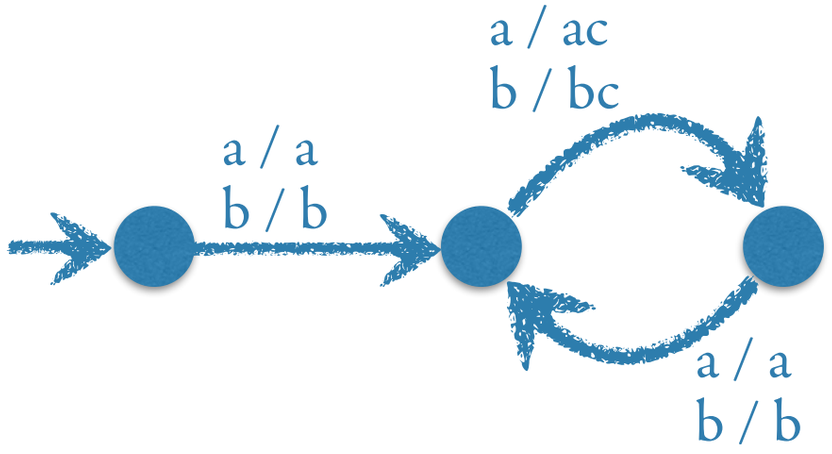

Note: sequential transducers can only compute *total, monotone* functions
(f is monotone if whenever w is prefix of w' then f(w) is prefix of f(w'))

Learning monotone word functions

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1) Teacher has a secret function f_0: $\Sigma^* \to \Gamma^*$, computed by a seq. transducer A_0
   Learner initially only knows the input alphabet Σ

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   b) or an <u>equivalence query</u> "Is f_0 computed by seq. transducer A ?"

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3) Teacher answers accordingly:
   a) gives value of $f_0(w)$
   b) yes if f_0 is computed by A (requires an algorithm for testing equivalence),
      otherwise gives a shortest <u>counter-example</u> w such that $f_0(w) \neq A(w)$

Learning monotone word functions

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Myhill-Nerode <u>equivalence</u>: $u \sim_{f_0} v$ if $\forall\, t \in \Sigma^*$ ...

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Example

f_0 inserts c between every two positions

$f_0(\varepsilon) = \varepsilon$, $f_0(a) = a$,
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$\sim_{f_0}$ has only two equivalence classes:
$[\varepsilon]_{\sim_{f_0}}$ $[a]_{\sim_{f_0}}$

E.g. $a \sim_{f_0} bba$ because $f(a???...) - f(a) = a?c??c... - a$
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- $\sim_{f_0}$ is right-invariant
  (i.e. $u \sim_{f_0} v \rightarrow ua \sim_{f_0} va$)

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  iff f is computed by
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<u>Hankel matrix</u> of f_0:

$H \in (\Gamma^*)^{\Sigma^* \times \Sigma^*}$

$H(s,t) = f_0(s\,t) - f_0(s)$

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	ε	a	b	aa	ab	...	aaa	...
ε	ε	a	b	aac	abc	...	aaca	...
a	ε	ac	bc	aca	acb	...	acaac	...
b	ε	ac	bc	aca	acb	...	acaac	...
aa	ε	a	b	aac	abc	...	aaca	...
ab	ε	a	b	aac	abc	...	aaca	...
...	...	...	...	...	...	...	...	...
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aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
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ab	ε	a	b	aac	abc	...	aaca	...
...	...	...	...	...	...	...	...	...
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Learner maintains (S,T)

S T-minimal if $\forall s \neq s' \in S$ $s \not\approx_{f_0,T} s'$

S T-complete if $\forall s \in S\ \forall a \in \Sigma$ $\exists s_a \in S$ $s_a \approx_{f_0,T} sa$

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start with

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ab	ε	a	b	aac	abc	...	aaca	...
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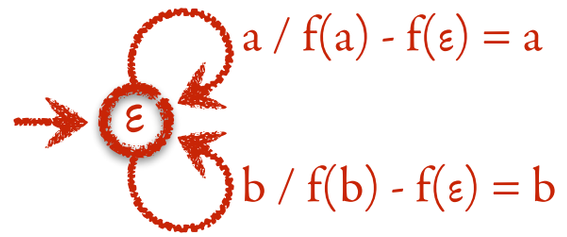
build candidate transducer...



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aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
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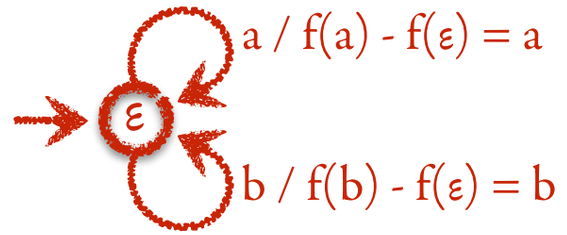

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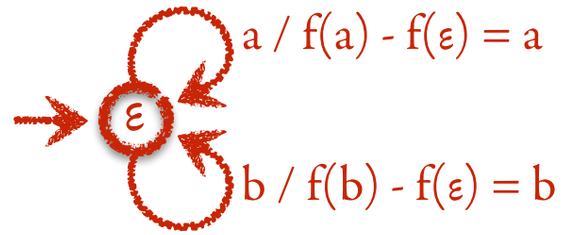
b	ε	ac	bc	aca	acb	...	acaac	...
aa	ε	a	b	aac	abc	...	aaca	...
ab	ε	a	b	aac	abc	...	aaca	...
...	...	...	...	...	...	...	...	...
aaa	ε	ac	bc	aca	acb	...	acaac	...
...	...	...	...	...	...	...	...	...

start with

S = {ε} T = {ε}

build candidate transducer...



expand T by counter-example w = ab

Learning monotone word functions

Myhill-Nerode equivalence: $u \sim_{f_0} v$ if $\forall t \in \Sigma^*$ $f_0(ut) - f_0(u) = f_0(vt) - f_0(v)$

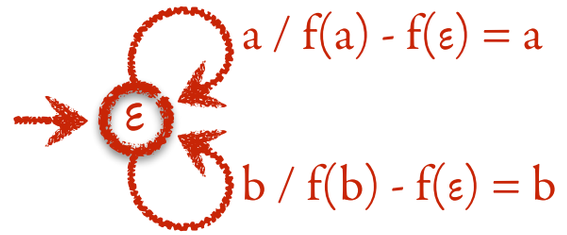
Relativization to test set T: $u \approx_{f_0,T} v$ if $\forall t \in T$ $f_0(ut) - f_0(u) = f_0(vt) - f_0(v)$

	ε	a	b	aa	ab	…	aaa	…
ε	**ε**	a	**b**	aac	**abc**	…	aaca	…
a	ε	ac	bc	aca	acb	…	acaac	…
b	ε	ac	bc	aca	acb	…	acaac	…
aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
aaa	ε	ac	bc	aca	acb	…	acaac	…
…	…	…	…	…	…	…	…	…

start with

S = {ε} T = {ε}

build candidate transducer…



expand T by counter-example w = ab

S = {ε} T = {ε, b, ab}

Learning monotone word functions

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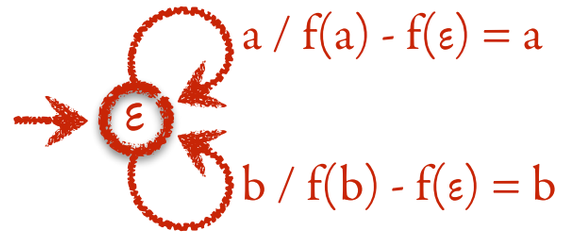
	ε	a	b	aa	ab	…	aaa	…
ε	**ε**	a	**b**	aac	**abc**	…	aaca	…
a	ε	ac	bc	aca	acb	…	acaac	…
b	ε	ac	bc	aca	acb	…	acaac	…
aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
aaa	ε	ac	bc	aca	acb	…	acaac	…
…	…	…	…	…	…	…	…	…

start with

S = {ε} T = {ε}

build candidate transducer…



expand T by counter-example w = ab

S = {ε} T = {ε, b, ab}

expand S to make it T-complete…

Learning monotone word functions

Myhill-Nerode equivalence: $u \sim_{f_0} v$ if $\forall t \in \Sigma^*$ $f_0(ut) - f_0(u) = f_0(vt) - f_0(v)$

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	ε	a	b	aa	ab	...	aaa	...
ε	**ε**	a	**b**	aac	**abc**	...	aaca	...
a	**ε**	ac	**bc**	aca	**acb**	...	acaac	...
b	ε	ac	bc	aca	acb	...	acaac	...
aa	ε	a	b	aac	abc	...	aaca	...
ab	ε	a	b	aac	abc	...	aaca	...
...	...	...	...	...	...	...	...	...
aaa	ε	ac	bc	aca	acb	...	acaac	...
...	...	...	...	...	...	...	...	...

start with

S = {ε} T = {ε}

build candidate transducer...

expand T by counter-example w = ab

S = {ε} T = {ε, b, ab}

expand S to make it T-complete...

S = {ε, a} T = {ε, b, ab}

Learning monotone word functions

Myhill-Nerode equivalence: $u \sim_{f_0} v$ if $\forall t \in \Sigma^*$ $f_0(ut) - f_0(u) = f_0(vt) - f_0(v)$

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	ε	a	b	aa	ab	...	aaa	...
ε	**ε**	a	**b**	aac	**abc**	...	aaca	...
a	**ε**	ac	**bc**	aca	**acb**	...	acaac	...
b	ε	ac	bc	aca	acb	...	acaac	...
aa	ε	a	b	aac	abc	...	aaca	...
ab	ε	a	b	aac	abc	...	aaca	...
...	...	...	...	...	...	...	...	...
aaa	ε	ac	bc	aca	acb	...	acaac	...
...	...	...	...	...	...	...	...	...

start with

S = {ε} T = {ε}

build candidate transducer...

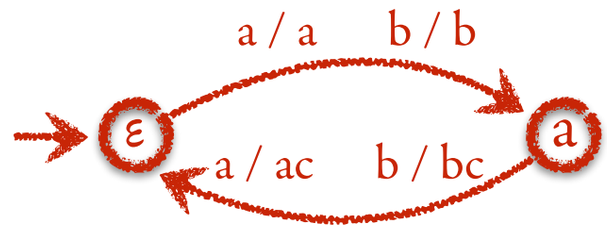
expand T by counter-example w = ab

S = {ε} T = {ε, b, ab}

expand S to make it T-complete...

S = {ε, a} T = {ε, b, ab}

build candidate transducer...



Learning monotone word functions

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	ε	a	b	aa	ab	…	aaa	…
ε	**ε**	a	**b**	aac	**abc**	…	aaca	…
a	**ε**	ac	**bc**	aca	**acb**	…	acaac	…
b	ε	ac	bc	aca	acb	…	acaac	…
aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
aaa	ε	ac	bc	aca	acb	…	acaac	…
…	…	…	…	…	…	…	…	…

start with

S = {ε} T = {ε}

build candidate transducer…

expand T by counter-example w = ab

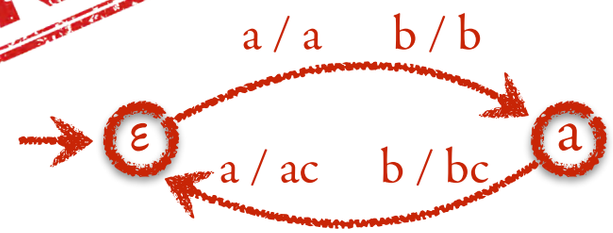
S = {ε} T = {ε, b, ab}

expand S to make it T-complete…

S = {ε, a} T = {ε, b, ab}

CORRECT

build candidate transducer…



From word languages to word functions… and beyond

This learning technique with Hankel matrices has been successfully applied to:

- Weighted automata (i.e. outputs given by products of weights along transitions)

  and, partially, to probabilistic automata

- Büchi automata

- Tree transducers (e.g. for learning XSLT transformations of XML documents)

- Timed & register automata (e.g. for processing strings with timestamps/data)

A simple yet useful real application (surprisingly, not yet done :/)

Implementing a learning algorithm for automata/transducers

would allow to automatically derive *RegEx* expressions like

```
/^(0?[1-9]|[12][0-9]|3[01])([ \/\-])(0?[1-9]|1[012])
\2([0-9][0-9][0-9][0-9])(([ -])([0-1]?[0-9]|2[0-3]):
[0-5]?[0-9]:[0-5]?[0-9])?$/
```

from positive and negative examples examples like

```
01/01/2000
18/10/1985
...
```

```
ab/01/2000
1/01/2000
18/10/200x
...
```