

MASTER THESIS IN  
ARTIFICIAL INTELLIGENCE AND CYBERSECURITY

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# *Cryptographic Primitives for Zero-Knowledge: Theory and Implementation*

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# Abstract

Zero Knowledge (ZK) proof systems have been a subject of study of increasing interest in the last 40 years. In the last decade, the efficiency of the proposed frameworks, along with the processing power of computing devices, has improved to the point of making ZK computation feasible in real-world scenarios. One of the primary applications lies in hash-tree commitment verification, and in this past five years there has been intense research in proposing ZK-friendly cryptographic primitives.

In this work, we begin by studying the history of ZK systems and reviewing the state of the art concerning ZK-friendly cryptographic permutations. We then apply a new generic algebraic framework, the Generalized Triangular Dynamical System, to design a cryptographic permutation called *Arion*. Finally, we implement our permutation together with the reviewed ones in the Groth16 ZK-SNARK framework and compare their efficiency for Merkle tree commitment verification.



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# 1

## Introduction

An important research branch of cryptography which emerged in the last forty years is the study of *Zero Knowledge Proof* (ZKP) systems [GMR89]. The main idea behind ZKP is to have two (or, in some cases, more) parties, where one is the *prover* and the other is the *verifier*: in a classical proof system, the prover must be able to convince the verifier that a certain statement is true, when this is indeed the case, but the verifier cannot be fooled if the statement is actually false. In a ZKP system we also require that the verifier does not get any useful additional information (i.e. knowledge) other than the truth, or lack thereof, of the statement. This additional requirement is particularly interesting when dealing with statements that are notoriously (believed to be) hard to prove, so that the verifier would not be realistically able to prove them in a reasonable amount of time. As a simple example, a prover would like to show that a propositional logic formula is satisfiable (an instance of the famous SAT problem) without revealing the satisfying assignment to the verifier.

Along the years, additional interesting and useful properties have been added to extend and improve the capabilities of ZKP systems. For example, we would like to have a *Non-interactive* protocol, to minimize the amount of required communication and have it happen only at the beginning and at the end of the protocol. We could also want to relax the soundness requirement so that it is guaranteed only against computationally bounded provers: in this case, instead of ‘proof’ we use the term *Argument of Knowledge*. More recently, there has been a research effort towards reducing the length of the ARK by ensuring that it is constant size or at most bounded by a logarithmic function in the length of the theorem statement: such systems are said to be *Succinct*. Implementations of ZK-SNARK system, like Pinocchio [PGHR13] or Groth16 [Gro16], represent the current state of the art of ZKP systems, and allow generating proofs to verify any computation representable by means of *bounded arithmetic circuits*. A major downside of ZK-SNARK protocols is their need of a trusted third party (TTP) to set up the system, hence current research is studying *Transparent* systems (ZK-STARK) to address this issue [BSBHR18].

An especially useful application of ZKP systems is proving knowledge of a preimage for a cryptographic hash function digest (a.k.a. commitment). Many data integrity systems, such as blockchains, rely on Merkle Trees [Mer79] to ensure efficient commitment validation, especially in dynamic environments. In Merkle Trees, a hash function is applied in a bottom-up fashion: the leaves will contain the data owned by some parties, while the root will contain the tree commitment. In a non zero-knowledge

setting, a prover would send the verifier his leaf together with the co-path, the verifier would then recompute the tree commitment and compare it with the public one and be convinced whether the prover does actually own the leaf. On the other hand, in a ZK-SNARK setting, we first have to represent the computation through a bounded arithmetic circuit, i.e. we are allowed to use exclusively a constant number of additions and multiplications over some suitable finite field. The circuit, together with a *proving key* provided by a TTP, and some private and public data, is then used by the prover to generate a proof which is sent to the verifier, who in turn uses a *verification key*, again provided by the same TTP, to assert whether the circuit computation was performed correctly.

While the various ZK-SNARK (or ZK-STARK) frameworks differ in the details, it is intuitive to see that the complexity of generating the proof (which dominates the cost of the protocol) must depend on the size of the circuit, which in turn depends on the amount of multiplications and additions performed in the computation: in the case of Merkle Tree commitment verification, most of the computation consists in iterating the underlying hash function. Since the finite field over which ZK-SNARK frameworks works is typically a huge prime field ( $\approx 2^{256}$  elements), traditional hash functions like MD5 [Riv90] or SHA [Dan15], which are designed to be extremely efficient on classical boolean circuits, become extremely inefficient in the ZK case.

It is no wonder then, that in the last years researchers began to study so-called ZK-friendly cryptographic permutation (ZKFCP) designs that exploit the features of large prime fields to be efficient when translated into arithmetic circuit, fundamentally resulting in a one-to-one mapping. Being a new research topic, these designs have seen a rapid series of improvements [AGR<sup>+</sup>16, GKR<sup>+</sup>21, GHR<sup>+</sup>22] in the last three years: in a two-part series of papers undergoing publication, we presented an algebraic framework, called *Generalized Triangular Dynamical System* (GTDS), which allows to express many of the existing cryptographic permutation designs and eases the construction of new ones, while at the same time giving strong security guarantees, and we then applied it to devise the Arion block cipher and the ArionHash hash function. Using the `libsark`<sup>1</sup> library (an implementation of the Groth16 framework), we implemented our hash function, along with other competitor hash functions and a hash-agnostic, variable-arity Merkle Tree circuit template, in a C++ project which we then used to compare their real-world performance for same-level security guarantees in various scenarios.<sup>2</sup>

## Structure of this thesis

This work is organized as follows: Chapter 2 presents the mathematical background for both current ZK-SNARK systems and our hash function. Chapter 3 presents the computational background of ZK-SNARK systems, and their origin. Chapter 4 presents the cryptographical background of ZK-SNARK systems, their latest evolutions and their most important real-world applications. In Chapter 5, we review the history and the state of the art concerning ZK-friendly hash functions; we present the GTDS framework and its instantiation in the form of the Arion block cipher and the ArionHash hash function; and we study the implementation and performance of the latter by comparing it to the state of the art. Finally, in Chapter 6, we draw our conclusion and explore future directions of the work.

<sup>1</sup><https://github.com/scipr-lab/libsark>

<sup>2</sup>This work was funded by the project "RampUp Cybersecurity" (KWF-3520 | 31870 | 45842) by the KWF.



# 2

## Mathematical Background

In this chapter we will introduce all the mathematical concepts behind ZKP and ZK-friendly functions. While we decided, for completeness, to include even some of the more fundamental notions, we still expect the reader to have at least a rough idea of these concepts. Section 2.1 will introduce prime fields, cyclic groups other related notions.

### 2.1 Finite algebra

In algebra, a *tuple* consisting of one or more *sets* together with one or more *operations* over the sets is called an *algebraic structure*. Such structures can be organized according to a quite wide taxonomy, depending on whether they satisfy certain properties or not. We will denote sets by capital letters (e.g.  $S, T, U, \dots$ ), a generic operation by a circled dot  $\odot$  and algebraic structures by blackboard bold letters (e.g.  $\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots$ ). We will also denote constants over a set by lowercase letters at the beginning of the alphabet (e.g.  $a, b, c, \dots$ ) and variables over a set by lowercase letters at the end of the alphabet (e.g.  $x, y, z, \dots$ ). Finally, we will often use the term algebra to mean algebraic structure, whenever we believe the meaning to be clear from the context.

*Remark 2.1.* Some symbols will be reserved to denote some common algebraic structures. In particular,  $\mathbb{B}$  will denote the boolean algebra, while  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  will denote, respectively, the natural, integer, rational, real and complex numbers.

We will denote the *cardinality* of set  $S$  by  $|S|$ , and use the same notation for the *order* of an algebraic structure and for the *arity* of an operation: for example, if  $\odot$  is a binary operation, like integer addition<sup>1</sup>, then  $|\odot| = 2$ . When an algebraic structure  $\mathbb{A}$  has exactly one *underlying set*  $A$ , we will identify the two, e.g. by writing  $x \in \mathbb{A}$  to mean  $x \in A$ .

**Definition 2.1** (Finite algebra). A *finite algebra* is an algebraic structure  $\mathbb{A}$  such that  $|\mathbb{A}| \in \mathbb{N}$ .

**Definition 2.2** (Subalgebra). An algebraic structure  $\mathbb{A} = (A, \odot_1, \dots, \odot_n)$  is a *subalgebra* of an algebraic structure  $\mathbb{A}' = (A', \odot'_1, \dots, \odot'_m)$ , for some  $n \leq m$ , if  $A \subseteq A'$  and  $\forall i \leq n: \odot_i \subseteq \odot'_i$ .

Elements of different algebraic structures can be associated through *morphisms*.

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<sup>1</sup>If considered as a relation, addition would be ternary.

**Definition 2.3** (Homomorphism). Given two algebras  $\mathbb{A} = (A, \odot_1, \dots, \odot_n)$ ,  $\mathbb{A}' = (A', \odot'_1, \dots, \odot'_n)$  such that  $\forall i \leq n: |\odot_i| = |\odot'_i| = a_i$ , a *homomorphism* is a map  $h: A \rightarrow A'$  such that:

$$\forall i \leq n, \forall x_1, \dots, x_{a_i} \in A: h(\odot_i(x_1, \dots, x_{a_i})) = \odot'_i(h(x_1), \dots, h(x_{a_i})) \quad (\text{linearity})$$

We say that  $\mathbb{A}$  is homomorphic to  $\mathbb{A}'$  through  $h$ .

**Definition 2.4** (Isomorphism). An *isomorphism* is a bijective homomorphism.

Given two algebras  $\mathbb{A}$  and  $\mathbb{A}'$ , if they are isomorphic through some map  $h$ , we write  $\mathbb{A} \cong_h \mathbb{A}'$ , or more succinctly  $\mathbb{A} \cong \mathbb{A}'$ .

**Definition 2.5** (Endomorphism, Automorphism). An *endomorphism* is a homomorphism from an algebraic structure  $\mathbb{A}$  to itself. An *automorphism* is an endomorphism which is also an isomorphism.

### 2.1.1 Groups

We will now introduce some important classes of algebraic structures equipped with one fundamental operation.

**Definition 2.6** (Monoid). A *monoid* is a pair  $\mathbb{M} = (M, \odot)$ , where  $M$  is the underlying set and  $\odot: M \times M \rightarrow M$  is the *composition* operation, such that the following properties are satisfied:

$$\begin{aligned} \forall x, y \in M: x \odot (y \odot z) &= (x \odot y) \odot z & (\text{associativity}) \\ \exists e \in M: \forall x \in M: x \odot e &= x & (\text{identity element}) \end{aligned}$$

$\mathbb{M}$  is a *commutative (or Abelian) monoid*, if it also holds that:

$$\forall x, y \in M: x \odot y = y \odot x \quad (\text{commutativity})$$

Finally, we can define *exponentiation* as follows:

$$\forall x \in \mathbb{M}, \forall k \in \mathbb{N}: x^k = \begin{cases} e & k = 0 \\ x^{k-1} \odot x & k > 0 \end{cases} \quad (2.1)$$

**Definition 2.7** (Cyclic Monoid). A *cyclic monoid* is a monoid  $\mathbb{M} = (M, \odot)$  which has a *generator element*  $g$  such that:

$$\mathbb{M} = \langle g \rangle = \left( \{g^k \mid k \in \mathbb{N}\}, \odot \right)$$

**Definition 2.8** (Group). A *group* is a monoid  $\mathbb{G} = (G, \odot)$ , such that:

$$\forall x \in G: \exists \bar{x} \in G: x \odot \bar{x} = e \quad (\text{inverse element})$$

With the notion of inverse element, we can extend exponentiation as follows:

$$\forall x \in \mathbb{G}, \forall k \in \mathbb{Z}: x^k = \begin{cases} x^{k-1} \odot x & k \geq 0 \\ x^{k+1} \odot x^{-1} & k < 0 \end{cases}$$

If  $\mathbb{G}$  is also a commutative (resp. cyclic) monoid, then it is a commutative (resp. cyclic) group.

We will sometimes use the notation  $e_{\mathbb{A}}$  to specify the algebra over which we intend to pick the identity element, dropping the subscript when  $\mathbb{A}$  is clear from the context.

*Remark 2.2.* Although we will strive, throughout this work, to be as unambiguous as possible, in some points we will likely be using a particular symbol to denote different operations. For example,  $+$  might denote addition between numbers, or polynomials, or vectors, and so on. This *overloading* will be done mostly in an effort to slim the notation and shift the attention from the operations themselves to the surrounding context. In any case, the semantics will always be clear from the operands and the context.

**Example 2.1.** The algebra  $(\mathbb{Z}, +)$  (i.e. integer numbers without multiplication) is an Abelian group: addition is associative and commutative, the identity element is  $e = 0$ , and every number  $x$  has an inverse  $\bar{x} = -x$  (e.g.  $\overline{42} = -42$ ).

**Example 2.2.** Given a commutative group  $\mathbb{G} = (G, \odot)$ , consider the algebra  $\text{End}(\mathbb{G})_+ = (H, +)$ , where  $H$  is the set of endomorphisms over  $\mathbb{G}$  and  $+: H \times H \rightarrow H$  is such that  $\forall h_1, h_2 \in H, \forall x \in G: (h_1 + h_2)(x) = h_1(x) \odot h_2(x)$ .

$\text{End}(\mathbb{G})_+$  is a commutative group:  $+$  is both associative and commutative, the identity element is  $e_{\text{End}(\mathbb{G})_+} = z$ , where  $z$  is the zero endomorphism (i.e.  $\forall x \in G: z(x) = e_{\mathbb{G}}$ ); finally, every homomorphism  $h \in H$  has an inverse  $\bar{h}$  such that  $\forall x \in G: \bar{h}(x) = \overline{h(x)}$ .

**Example 2.3.** Consider now the algebra  $\text{End}(\mathbb{G})_{\circ} = (H, \circ)$  where  $\mathbb{G}$  and  $H$  are defined as in Example 2.2, and  $\circ: H \times H \rightarrow H$  is defined as function composition:  $(h_1 \circ h_2)(x) = h_1(h_2(x))$ .

$\text{End}(\mathbb{G})_{\circ}$  is a monoid: function composition is associative, and the identity element is  $e_{\text{End}(\mathbb{G})_{\circ}} = \text{id}$ , where  $\text{id}$  is the identity endomorphism (i.e.  $\forall x \in G: \text{id}(x) = x$ ).

### 2.1.2 Fields

Many algebraic structures rely on two fundamental operations, called *addition* and *multiplication*: two important types of such structures are *rings* and *fields*.

**Definition 2.9 (Ring).** A *ring* is a triple  $\mathbb{O} = (O, \oplus, \otimes)$  where  $O$  is the underlying set,  $\oplus: O \times O \rightarrow O$  is the *addition* operation and  $\otimes: O \times O \rightarrow O$  is the *multiplication* operation, such that the following properties are satisfied:

$\mathbb{O}_{\oplus} = (O, \oplus)$  is an abelian group

$\mathbb{O}_{\otimes} = (O, \otimes)$  is a monoid

$\forall x, y, z \in O: x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$  (*left distributivity*)

$\forall x, y, z \in O: (y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$  (*right distributivity*)

If  $\mathbb{O}_{\otimes}$  is a commutative monoid, then  $\mathbb{O}$  is a *commutative (Abelian) ring*.

Given a ring  $\mathbb{O}$  and an element  $x \in \mathbb{O}$ , its additive (resp. multiplicative) inverse is denoted by  $\bar{x}_{\oplus}$  (resp.  $\bar{x}_{\otimes}$ ). Similarly, the additive (resp. multiplicative) identity element is denoted by  $e_{\oplus}$  (resp.  $e_{\otimes}$ ). In numeric algebras, the additive and multiplicative inverses are typically equivalent to the opposite  $-x$

and the reciprocal  $\frac{1}{x}$  (or  $x^{-1}$ ) respectively, while the additive and multiplicative inverses are typically equivalent to 0 and 1 respectively.

**Definition 2.10** (Field). A *field* is a ring  $\mathbb{F} = (F, \oplus, \otimes)$  such that  $e_{\oplus} \neq e_{\otimes}$  and  $\mathbb{F}_{\otimes} \setminus \{e_{\oplus}\}$  is a commutative group.

Fields are one of the most important and studied algebraic structures: the algebraic structure of the real numbers  $\mathbb{R}$  is a field, as are complex numbers  $\mathbb{C}$ . Given the set of integers  $Z_q = \{0, \dots, q-1\}$ , we denote by  $\oplus_q$  integer sum modulo  $q$ , and by  $\otimes_q$  integer multiplication modulo  $q$ . Furthermore, we will denote by  $\langle g \rangle_q$  the cyclic group generated by  $g$  under the operation  $\otimes_q$ . The algebra  $\mathbb{Z}_q = (Z_q, \oplus_q, \otimes_q)$  is a finite ring  $\forall q \in \mathbb{N}$ , and it is a finite field if and only if  $q$  is prime.

**Definition 2.11** (Discrete logarithm). The *discrete logarithm* over some cyclic group  $\langle g \rangle$  of order  $q$  is the function:

$$\log_g(g^x): \langle g \rangle \rightarrow \mathbb{Z}_q = x$$

When the group generator is clear from the context, we simply write  $\log$  instead of  $\log_g$ . Typically, cyclic groups are obtained as the subset of a larger finite field (see Example 2.6).

**Definition 2.12** (Euler totient). The *Euler totient* is the function

$$\phi(x): \mathbb{N} \rightarrow \mathbb{N} = |\{y \mid (y < x) \wedge (\gcd(x, y) = 1)\}|$$

For any prime number  $p$ , it is easy to see that  $\phi(p) = p-1$ , while for a composite number, its totient is the product of the totients of its factors.

**Theorem 2.1** (Euler's theorem).  $\forall x, y \in \mathbb{N}: \gcd(x, y) = 1 \implies x^{\phi(y)} \bmod y = 1$ .

It is important to remark that, when we need to invert the exponentiation  $x^y$  over a prime field  $\mathbb{Z}_p$ , we must consider the inverse in its underlying multiplicative group, which has order  $\phi(p)$ . The multiplicative inverse  $z$  of  $y$  modulo  $p$  does not invert the exponentiation since, by Euler's theorem,  $(x^y)^z = x^{yz} = x^{kp+1} = x^{k+1} \neq x^{y \otimes z} = x$ . Instead, if we use the inverse  $w$  of  $y$  modulo  $\phi(p)$ , we have that  $x^{yw} = x^{k\phi(p)+1} = x$ . Unfortunately, both  $z$  and  $w$  can be denoted by  $1/y$  in the standard literature, with the semantic depending on the context (when talking about exponentiations, we always intend the second meaning).

**Example 2.4.** Boolean circuits with XOR and AND gates behave like elements of the boolean field  $\mathbb{B} = (\{\perp, \top\}, \text{XOR}, \text{AND})$ . It is easy to show that  $\mathbb{B} \cong \mathbb{Z}_2$ .

**Example 2.5.** Given an Abelian group  $\mathbb{G}$ , the algebra  $\mathbb{H}_{\mathbb{G}} = \text{End}(\mathbb{G}) = \text{End}(\mathbb{G})_+ \cup \text{End}(\mathbb{G})_{\circ}$  is the *endomorphism ring* of  $\mathbb{G}$ :  $\text{End}(\mathbb{G})_+$  is an Abelian group,  $\text{End}(\mathbb{G})_{\circ}$  is a monoid (see Examples 2.2 and 2.3), and it is easy to show that  $\circ$  distributes over  $+$  both on the left and the right.

**Example 2.6.** Consider the cyclic group  $\mathbb{G} = \langle 2 \rangle_{23} = (\{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}, \otimes_{23})$  which has order  $|\mathbb{G}| = 11$ . Let's show that  $\log = \log_2$  is a homomorphism between  $\mathbb{G}$  and  $\mathbb{Z}_{11, \oplus}$ : for any two elements  $x, y \in \mathbb{G}$ , we have:

$$x \otimes_{23} y = 2^{\log(x)} \otimes_{23} 2^{\log(y)} = 2^{\log(x) \oplus_{11} \log(y)}$$

Since  $\log_2$  is a bijection, it is also an isomorphism. In fact, one can show that  $\forall q \in \mathbb{N}$  and  $\forall g < q$  such that  $\gcd(g, q) = 1$  (otherwise  $\langle g \rangle_q$  would not be a group), then  $\mathbb{G} = \langle g \rangle_q \cong_{\log_q} \mathbb{Z}_{|\mathbb{G}|, \oplus}$ .

### 2.1.3 Vector spaces

All the algebraic structures we have seen in the previous section operate on an underlying set whose elements we consider to be, in some sense, atomic. On the other hand, many objects interact with each other exhibiting a multidimensional behaviour (e.g. physical forces). The standard structure to deal with such objects are *vector spaces*.

**Definition 2.13** (Module). A *module* is a quadruple  $\mathbb{M} = (M, \mathbb{O}, +, \odot)$  where  $M$  is the underlying vector set,  $\mathbb{O} = (O, \oplus, \otimes)$  is the underlying scalar ring,  $+: M \times M \rightarrow M$  is the *module addition* operation and  $\odot: O \times M \rightarrow M$  is the *scalar multiplication* operation, such that  $\mathbb{M}_+ = (M, +)$  is a commutative group and  $\odot$  is a homomorphism between  $\mathbb{O}$  and  $\text{End}(\mathbb{M}_+)$ .

**Definition 2.14** (Linear map). A *linear map* is a homomorphism between two modules.

**Definition 2.15** ( $k$ -linear map). Given  $k$  modules  $\mathbb{M}_1, \dots, \mathbb{M}_k, \mathbb{M}'$  over the same scalar ring  $\mathbb{O}$ , a  *$k$ -linear map* is a map  $f: \mathbb{M}_1 \times \dots \times \mathbb{M}_k \rightarrow \mathbb{M}'$  such that,  $\forall i \in \mathbb{N}$ , every map obtained by fixing all but the  $i$ th argument is a linear map.

Although  $k$ -linear maps are defined between modules, their definition can be extended to other algebraic structures.

**Definition 2.16** (Vector space). A *vector space* is a module such that the underlying scalar ring is a field.

An important vector space is the one of  $n$ -dimensional *column vectors* over a field  $\mathbb{F} = (F, \oplus, \otimes)$  such that  $\mathbb{F}^n = (F^n, \mathbb{F}, +, \odot)$ , where  $+$  is entry-wise field addition between column vectors and  $\odot$  is element-wise field multiplication of scalars with column vectors. We will denote elements of a column vector space  $\mathbb{V}$  by bold letters (e.g.  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \dots$ ), and elements of the dual *row vector* space  $\mathbb{V}^\top$  by superscripting a  $\top$  symbol (e.g.  $\mathbf{u}^\top, \mathbf{v}^\top, \mathbf{w}^\top, \dots$ ). Finally, we denote the  $i$ th element of a column vector  $\mathbf{v}$  by  $v_i$ .

**Definition 2.17** (Dot product). The *dot product* operation over an  $n$ -dimensional vector space  $\mathbb{V} = (V, \mathbb{F}, +, \odot)$ , with  $\mathbb{F} = (F, \oplus, \otimes)$ , is the map:

$$\mathbf{v} \cdot \mathbf{w}: \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F} = \bigoplus_{i=1}^n v_i \otimes w_i$$

**Definition 2.18** (Vector rotation). The *right rotation* operation over an  $n$ -dimensional vector space  $\mathbb{V}$  is the map:

$$\mathbf{v} \ggg k: \mathbb{V} \times \mathbb{N} \rightarrow \mathbb{V} = \left( v_{(0-k \bmod n)+1} \quad \cdots \quad v_{(n-1-k \bmod n)+1} \right)^\top$$

Another important vector space is the one of  $(n \times m)$ -dimensional *matrices* over some base field  $\mathbb{F}$ :  $\mathbb{F}^{n \times m} = ((F^n)^m, \mathbb{F}, +, \odot)$ , where  $+$  is element-wise field addition between matrices, and  $\odot$  is element-wise field multiplication of scalars with matrices. We will denote elements of a matrix space  $\mathbb{M}$  by bold capital letters (e.g.  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ ), we denote the  $i$ th row of a matrix  $\mathbf{M}$  by  $\mathbf{M}_i$ , and the  $j$ th element of the  $i$ th row with  $M_{i,j}$ .

In this work, we will only deal with column vector space extensions of the kind  $\mathbb{F}^n$  and row vector space extensions of the kind  $(\mathbb{F}^m)^\top$  for some base field  $\mathbb{F}$  and some  $n, m \in \mathbb{N}$ . Similarly, we will only deal with matrix space extensions of the kind  $\mathbb{F}^{n \times m}$ . Therefore, the  $i$ th column of a matrix will always be an element of  $\mathbb{F}^n \cong \mathbb{F}^{n \times 1}$ , and the  $j$ th row of a matrix will always be an element of  $(\mathbb{F}^m)^\top \cong \mathbb{F}^{1 \times m}$ .

**Definition 2.19** (Transpose matrix). The *transpose* of a matrix  $\mathbf{M} \in \mathbb{F}^{n \times m}$  is the matrix:

$$\mathbf{M}^\top \mid \forall i \leq n, \forall j \leq m: \mathbf{M}_{i,j}^\top = \mathbf{M}_{j,i}$$

**Definition 2.20** (Matrix concatenation). Given a matrix  $\mathbf{A} \in \mathbb{F}^{n_1 \times m_1}$  and a matrix  $\mathbf{B} \in \mathbb{F}^{n_2 \times m_2}$ , let  $n = \min(n_1, n_2)$ . The *row-wise concatenation* of  $\mathbf{A}$  and  $\mathbf{B}$  is the matrix  $\mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \in \mathbb{F}^{n \times (m_1+m_2)}$ , such that  $\forall i \leq n, \forall j \leq m_1: \mathbf{C}_{i,j} = \mathbf{A}_{i,j}$  and  $\forall j \leq m_2: \mathbf{C}_{i,j} = \mathbf{B}_{i,j}$ .

Their *column-wise concatenation* is the matrix  $\begin{pmatrix} \mathbf{A}; \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^\top & \mathbf{B}^\top \end{pmatrix}^\top$ .

Their *diagonal-wise concatenation* is the matrix  $\mathbf{D} = \text{diag}(\mathbf{A}) \in \mathbb{F}^{(n_1+n_2) \times (m_1+m_2)}$  such that:

$$\begin{aligned} \forall i \leq n_1, \forall j \leq m_1: \mathbf{D}_{i,j} &= \mathbf{A}_{i,j} & \forall i \leq n_1, \forall j \leq m_2: \mathbf{D}_{i,j} &= 0 \\ \forall i \leq n_2, \forall j \leq m_1: \mathbf{D}_{i,j} &= 0 & \forall i \leq n_2, \forall j \leq m_2: \mathbf{D}_{i,j} &= \mathbf{B}_{i,j} \end{aligned}$$

**Definition 2.21** (Diagonal vector). Given a matrix  $\mathbf{A} \in \mathbb{F}^{n \times m}$ , its *diagonal vector* is the vector  $\text{diag}(\mathbf{A}) \in \mathbb{F}^{\min(n,m)} \mid \forall i \leq \min(n,m): \text{diag}(\mathbf{A}) = \mathbf{A}_{i,i}$ .

**Definition 2.22** (Matrix multiplication). *Matrix multiplication* over a base field  $\mathbb{F}$  and some  $m, n_1, n_2 \in \mathbb{N}$ , is the map  $\mathbf{AB}: \mathbb{F}^{n_1 \times m} \times \mathbb{F}^{m \times n_2} \rightarrow \mathbb{F}^{n_1 \times n_2}$  such that  $\forall i \leq n_1, \forall j \leq n_2: (\mathbf{AB})_{i,j} = \mathbf{A}_i \cdot \mathbf{B}_j^\top$

**Definition 2.23** (Circulant matrix). A *circulant matrix* over a matrix space  $\mathbb{F}^{n \times m}$  is a matrix:

$$\mathbf{M} \in \mathbb{F}^{n \times m} \mid \forall i < n: \mathbf{M}_{i+1} = \mathbf{M}_i \ggg 1$$

We say that a circulant matrix is *induced* by the elements in its first row, and denote the circulant matrix induced by  $x_1, \dots, x_n$  by  $\text{circ}(x_1, \dots, x_n)$ . This definition can be extended to allow “circulant” matrices where instead of rotating single elements we rotate submatrices of size  $n \times m$ . We denote a “circulant” matrix induced in this way through the matrices  $\mathbf{M}_1, \dots, \mathbf{M}_k \in \mathbb{F}^{n \times m}$  by  $\text{circ}(\mathbf{M}_1, \dots, \mathbf{M}_k)$  (note that it is an element of  $\mathbb{F}^{nk \times mk}$ ).

## 2.1.4 Polynomials

The last fundamental object that we will need are polynomials and their relative algebras.

**Definition 2.24** (Monovariate polynomial ring). A *monovariate polynomial ring* over a field  $\mathbb{F}$  is the triple  $\mathbb{F}[x] = (F[x], +, \cdot)$  where  $F[x]$  is the set of monovariate polynomials with coefficients over  $F$  in the indeterminate  $x$ ,  $+: F[x] \times F[x] \rightarrow F[x]$  is the *polynomial addition* operation and  $\cdot: F[x] \times F[x] \rightarrow F[x]$  is the *polynomial multiplication* operation, such that all the properties of a ring are satisfied.

We will denote polynomials by lowercase letters (e.g.  $p, q, r, \dots$ ), and the degree of some polynomial  $p$  by  $\deg(p)$ . Given a field  $\mathbb{F}$  and its corresponding polynomial ring  $\mathbb{F}[x]$ , we will denote by  $\mathbb{F}[x]^n$  the  $n$ -

dimensional module<sup>2</sup> of column vectors of polynomials over  $\mathbb{F}$ , with addition and scalar product defined in the standard way.

**Definition 2.25** (Lagrange interpolation [War79]). *Lagrange interpolation* is the function:

$$\text{lag}(\mathbf{x}, \mathbf{y}): \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}[x] = \sum_i \mathbf{y}_i \prod_{j \neq i} \frac{x - \mathbf{x}_j}{\mathbf{x}_i - \mathbf{x}_j}$$

By using interpolation, we can build the unique polynomial of degree  $n - 1$  which,  $\forall i \leq n$  assumes value  $\mathbf{y}_i$  at point  $\mathbf{x}_i$ . We can extend Lagrange interpolation to a matrix space  $\mathbb{F}^{n \times m}$  as follows:

$$\text{lag}(\mathbf{X}, \mathbf{Y}): \mathbb{F}^{n \times m} \times \mathbb{F}^{n \times m} \rightarrow \mathbb{F}[x]^n = \left( \text{lag}(\mathbf{X}_1, \mathbf{Y}_1) \quad \cdots \quad \text{lag}(\mathbf{X}_n, \mathbf{Y}_n) \right)$$

## 2.2 Arithmetic Programs

Suppose we have some algebra  $\mathbb{A}$ : we can represent and deal with finite sequences of operations, called *expressions*, between elements of  $\mathbb{A}$  and/or variables over  $\mathbb{A}$ .

For example, given the expression  $x^2 + x + 1$  over  $\mathbb{A}[x]$ , we might be interested to know what is the *evaluation* of the expression given some value for  $x$ . We will limit our analysis to fields (and rings).

**Definition 2.26** (Arithmetic formula over a field). Given a field  $\mathbb{F}$ , an *explicit arithmetic formula* over  $\mathbb{F}$  is any expression  $\varphi$  of the kind:

$\varphi \equiv a$	with $a$ constant over $\mathbb{F}$
$\varphi \equiv x$	with $x$ variable over $\mathbb{F}$
$\varphi \equiv \varphi_1 \oplus \varphi_2$	with $\varphi_1, \varphi_2$ formulae over $\mathbb{F}$
$\varphi \equiv \varphi_1 \otimes \varphi_2$	with $\varphi_1, \varphi_2$ formulae over $\mathbb{F}$

Additionally, an *implicit arithmetic formula* also allows expressions involving exponentiations:

$$\varphi \equiv \varphi_1^k \quad \forall k \in \mathbb{N}, \text{ with } \varphi_1 \text{ formula over } \mathbb{F}$$

It is always possible to translate an implicit formula into an equivalent explicit one by unrolling exponentiations. We denote the explicit version of an implicit formula  $\varphi$  with  $\widehat{\varphi}$ . Note also that multiplication by constants can also be unrolled to a sequence of additions (in fact, they can be viewed as exponentiations w.r.t. addition). From now on, we will only deal with arithmetic formulae over some field (or ring)  $\mathbb{F}$ , in which case implicit arithmetic expressions are equivalent to multi-variate polynomials.

**Example 2.7.** Consider the finite field  $\mathbb{Z}_{13}$ . For ease of notation, we will use  $+$ , juxtaposition and superscripting to denote, respectively, field addition, field multiplication, and exponentiation w.r.t. field multiplication. A possible implicit arithmetic formula over  $\mathbb{Z}_{13}$  is the following expression:

$$\varphi = x_2 \left( x_1^3 + 4x_2 + 5 \right)$$

---

<sup>2</sup>Since  $\mathbb{F}[x]$  is a commutative ring,  $\mathbb{F}[x]^n$  is actually a module, not a vector space, but we will nevertheless call its elements ‘vectors’ for the sake of simplicity.

Since in a finite field multiplication by a constant is simply repeated addition, i.e.  $cx = +^c(x)$ , the explicit version of  $\varphi$  then is:

$$\hat{\varphi} = x_2(x_1x_1x_1 + x_2 + x_2 + x_2 + x_2 + 5)$$

### 2.2.1 Arithmetic circuits

It is possible to visually represent an arithmetic formula using a particular kind of labeled *directed acyclic graph* (DAG), called the *arithmetic circuit*.

**Definition 2.27** (Arithmetic circuit). An *arithmetic circuit* over a field  $\mathbb{F}$  and a set of variables  $X$  over  $\mathbb{F}$  is a triple  $\mathcal{G} = (V, E, L)$  where  $V$  is the set of *vertices*,  $E \subseteq V \times V$  is the set of *edges*, and  $L: V \rightarrow A \cup X \cup (\{\oplus, \otimes\} \times \mathbb{N})$  is the vertex *labeling map*, such that,  $\forall v \in V$ :

$$\begin{aligned} L(v) \in A &\implies \nexists w \in V: (w, v) \in E && \text{(no in-edges for constant nodes)} \\ L(v) \in X &\implies \nexists w \in V: (w, v) \in E && \text{(no in-edges for variable nodes)} \\ \forall \odot_i \in L(v) &\implies |\{(w, v)\}_{w \in V} \cap E| = |\odot| && \text{(exactly } |\odot| \text{ in-edges for } \odot_i \text{ nodes)} \end{aligned}$$

As an abuse of notation, we will sometimes identify a node  $v$  with its label  $L(v)$ . Given any explicit arithmetic formula  $\varphi$  over an algebra  $\mathbb{A}$  and a set of variables  $X$ , we can build the corresponding arithmetic circuit  $\mathcal{G} = (V, E, L)$  in the following way: for every distinct (i.e. ignoring repetitions) variable  $x$  appearing in  $\varphi$ , we add a vertex  $v$  with label  $L(v) = x$ ; for every distinct constant  $c$  appearing in  $\varphi$  we add a vertex  $v$  with label  $L(v) = c$ ; finally, for every occurrence  $i$  of some operation  $\odot$  in  $\varphi$ , we add a vertex  $v$  with label  $L(v) = \odot_i$ . We can partition  $V$  as follows:

- *Constant vertices*:  $\mathcal{G}_{const} = \{v \mid L(v) \in \mathbb{F}\}$ .
- *Variable vertices*:  $\mathcal{G}_{var} = \{v \mid L(v) \in X\}$ .
- *Addition vertices*:  $\mathcal{G}_{\oplus} = \{v \mid \oplus \in L(v)\}$ .
- *Multiplication vertices*:  $\mathcal{G}_{\otimes} = \{v \mid \otimes \in L(v)\}$ .
- *Operation vertices*:  $\mathcal{G}_{\odot} = \mathcal{G}_{\oplus} \cup \mathcal{G}_{\otimes}$ .
- *Input vertices*:  $\mathcal{G}_{in} = \mathcal{G}_{const} \cup \mathcal{G}_{var}$ .
- *Output vertices*:  $\mathcal{G}_{out} = \{v \mid \nexists w \in V: (v, w) \in E\}$ .
- *I/O vertices*:  $\mathcal{G}_{IO} = \mathcal{G}_{in} \cup \mathcal{G}_{out}$ .

To build the set of edges  $E$ , for every operation occurring in  $\varphi$ , we connect the vertices representing the operands to the vertex representing said operation, e.g. if we have the formula  $(x \odot y) \odot z$  we add the edges  $(x, \odot_1)$ ,  $(y, \odot_1)$  and  $(z, \odot_2)$ . We also consider operation nodes as holding the intermediate values of the computation: in the previous example, we will also have the edge  $(\odot_1, \odot_2)$ , where  $\odot_1$  represents



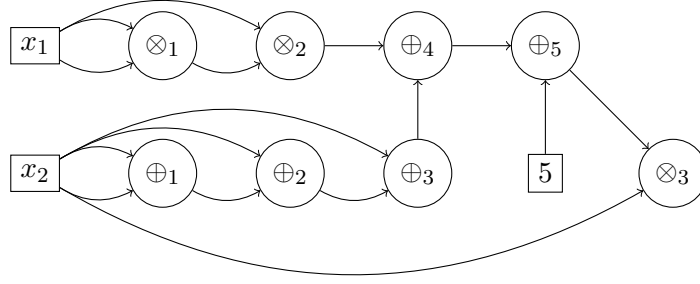


Figure 2.1: Circuit of the formula in Example 2.7. Rectangular nodes represent input vertices.

the intermediate value  $x \odot y$ . The fact that we store all the intermediate values of a computation can be greatly exploited when optimizing the design of a circuit for some formula.

Since arithmetic circuits contain no cycles, they can only be used to represent a fixed number of operations (aka *bounded computations*). In general though, this is not really a big issue, as oftentimes we can easily synthesize circuits *on-the-fly*.

**Example 2.8.** Figure 2.1 shows the arithmetic circuit derived from the formula shown in Example 2.7. We can see the two variable vertices  $x_1$  and  $x_2$  which are also input vertices, the constant vertex 5, which is an input vertex too, the addition vertices  $\oplus_1, \dots, \oplus_5$  and the multiplication vertices  $\otimes_1, \otimes_2, \otimes_3$ , of which the latter is also an output vertex.

**Definition 2.28** (Circuit input). A *circuit input* for an arithmetic circuit  $\mathcal{G} = (V, E, L)$  over a field  $\mathbb{F}$  is a triple  $\mathcal{I}_{\mathcal{G}} = (V, E, L')$  such that,  $\forall v \in \mathcal{G}_{in}: L'(v) \in \mathbb{F}$ .

In particular, since every vertex in  $\mathcal{G}_{in}$  has now a constant value,  $\mathcal{I}$  induces an *evaluation*  $\mathcal{E}$  over  $\mathcal{G}$  which associates to every operation vertex  $v \in \mathcal{G}_{\odot}$  the value  $\text{eval}(v)$ .

**Definition 2.29** (Circuit assignment). A *circuit assignment* for an arithmetic circuit  $\mathcal{G} = (V, E, L)$  over a field  $\mathbb{F}$  is a triple  $\mathcal{A}_{\mathcal{G}} = (V, E, L')$  such that,  $\forall v \in V: L'(v) \in \mathbb{F}$ .

An assignment  $\mathcal{A}$  for a circuit  $\mathcal{G}$  associates to every node of the circuit a value over the underlying field, but not necessarily a sensible one. However, as an assignment is also a circuit input, it induces an evaluation. Therefore, we say that  $\mathcal{A}$  is *valid* if and only if  $L'(v) = \text{eval}(v)$ .

**Example 2.9.** Figure 2.2 shows a valid assignment for the circuit  $\mathcal{G}$  shown in Example 2.8: we fix the inputs  $x_1 = 2$  and  $x_2 = 3$ , and label every operation vertex  $v$  with the correct value  $\text{eval}(v)$ .

### 2.2.2 Rank-1 Constraint Systems

An arithmetic circuit tells us two things: (i) how to compute the intermediate values, after fixing the inputs, and (ii) how the intermediate values are constrained, depending on the inputs.

**Definition 2.30** (Rank-1 Constraint System [CD98]). Given  $m, n \in \mathbb{N}$ , an  $m/n$  *Rank-1 Constraint System* (R1CS) over a field  $\mathbb{F}$  is a triple  $\mathcal{C} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$  such that  $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{F}^{m \times n}$ .

**Definition 2.31** (R1CS solution). A *solution* to an R1CS  $\mathcal{C}$  over a field  $\mathbb{F}$  is a vector  $\mathbf{s} \in \mathbb{F}^n$  such that  $(\mathbf{A}\mathbf{s})(\mathbf{B}\mathbf{s}) = \mathbf{C}\mathbf{s}$ .

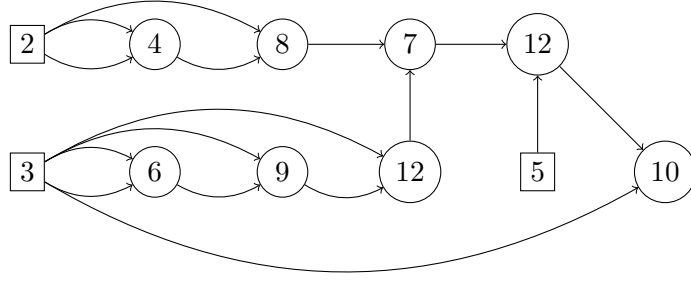


Figure 2.2: A valid assignment for the circuit in Figure 2.1. Remember that the underlying field is  $\mathbb{Z}_{13}$ .

Any explicit arithmetic circuit  $\mathcal{G}$  with  $m$  multiplicative gates (i.e.  $m = |\mathcal{G}_{\otimes}|$ ) and  $n$  input variables (i.e.  $n = |\mathcal{G}_{var}|$ ) can be associated with an  $m/(n + m + 1)$  R1CS  $\mathcal{C}$  representing the constraints in the circuit, as follows:

1. Add a new ‘constant variable’  $x_0$  which always assumes value 1.
2. For every multiplicative gate  $\otimes_i$  in the circuit, add a new *intermediate* variable  $t_i$ .
3. Define the column vector  $\mathbf{x} = (1 \ x_1 \ \cdots \ x_n \ t_1 \ \cdots \ t_m)^\top$ .
4. Express every multiplication gate  $\otimes_i$  as an equation in the canonical form  $(\mathbf{A}_i \mathbf{x})(\mathbf{B}_i \mathbf{x}) = \mathbf{C}_i \mathbf{x}$ .

The intermediate variable  $t_m$ , which corresponds to the ‘last’ multiplication gate, is often denoted  $y$  as it represents the output of the circuit.

**Example 2.10.** Consider the circuit of Example 2.8: there are three multiplications in total, and since we have two input variables, the associated R1CS  $\mathcal{C}$  will be a 3/6 R1CS. So, our variable vector will be:

$$\mathbf{x} = (1 \ x_1 \ x_2 \ t_1 \ t_2 \ y)^\top$$

Let us explicit all the intermediate variables, and transform the constraint equations in canonical form:

$$\begin{array}{lll} t_1 = x_1 x_1 & \iff & (x_1)(x_1) = t_1 \\ t_2 = t_1 x_1 + 4x_2 + 5 & \iff & (t_1)(x_1) = 8 + 9x_2 + t_2 \\ y = t_2 x_2 & \iff & (x_2)(t_2) = y \end{array}$$

We can now extract the  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 8 & 0 & 9 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let’s check the assignment that was given in Example 2.9, where  $x_1 = 2$  and  $x_2 = 3$ :

$$\begin{array}{ll} t_1 = x_1 x_1 = 2 \times 2 = 4 & \equiv 4 \pmod{13} \\ t_2 = t_1 x_1 + 4x_2 + 5 = 4 \times 2 + 4 \times 3 + 5 = 25 & \equiv 12 \pmod{13} \\ y = t_2 x_2 = 12 \times 3 = 36 & \equiv 10 \pmod{13} \end{array}$$

The solution vector  $\mathbf{s}$  will be:

$$\mathbf{s} = \begin{pmatrix} 1 & 2 & 3 & 4 & 12 & 10 \end{pmatrix}^\top$$

It is not hard to verify that indeed  $(\mathbf{A}\mathbf{s})(\mathbf{B}\mathbf{s}) = \mathbf{C}\mathbf{s}$ .

### 2.2.3 Quadratic Arithmetic Programs

The size of a solution for a R1CS grows linearly in the number of multiplication gates of the corresponding arithmetic circuit. This is an issue, as such solution is not *compact*: by compact, we mean that its size should be constant (i.e. independent of the circuit's size) or at most logarithmic (i.e. upper-bounded by a logarithmic function in the circuit's size).

**Definition 2.32** (Quadratic Arithmetic Program [GGPR12]). Given some  $m, n \in \mathbb{N}$ , an  $m/n$  *Quadratic Arithmetic Program (QAP)* over  $\mathbb{F}$  is a quadruple  $\mathcal{Q} = (t, \mathbf{v}, \mathbf{w}, \mathbf{y})$  where  $t \in \mathbb{F}[x]$  is the *target polynomial*, and  $\mathbf{v}, \mathbf{w}, \mathbf{y} \in \mathbb{F}[x]^m$  are, respectively, the *left input polynomial*, the *right input polynomial* and the *output polynomial*, such that:

$$\forall i \leq m: \deg(\mathbf{v}_i) = \deg(\mathbf{w}_i) = \deg(\mathbf{y}_i) = \deg(t) - 1 = n - 1$$

**Definition 2.33** (QAP solution). A *solution* to a QAP  $\mathcal{Q} = (t, \mathbf{v}, \mathbf{w}, \mathbf{y})$  over a field  $\mathbb{F}$  is a pair of polynomials  $h, p \in \mathbb{F}[x]$  such that  $p = ht$ .

It is possible to represent any  $m/n$  R1CS  $\mathcal{C}$  with an  $m/n$  QAP  $\mathcal{Q}$ . First, we choose any arbitrary row vector  $\mathbf{z} \in (\mathbb{F}^m)^\top$  such that  $\forall i, j: \mathbf{z}_i \neq \mathbf{z}_j$ ; typically  $\mathbf{z} = \begin{pmatrix} 1 & \dots & m \end{pmatrix}$ . Now, let  $\mathbf{Z} \in \mathbb{F}^{n \times m}$  be a matrix such that  $\forall i: \mathbf{Z}_i = \mathbf{z}$ , then the resulting QAP  $\mathcal{Q} = (t, \mathbf{v}, \mathbf{w}, \mathbf{y})$  will be:

$$t = \prod_i (x - \mathbf{z}_i) \quad \mathbf{v} = \text{lag}(\mathbf{Z}, \mathbf{A}^\top) \quad \mathbf{w} = \text{lag}(\mathbf{Z}, \mathbf{B}^\top) \quad \mathbf{y} = \text{lag}(\mathbf{Z}, \mathbf{C}^\top)$$

Then, given a solution  $\mathbf{s}$  for  $\mathcal{C}$ , we can build a solution for  $\mathcal{Q}$  by computing  $p = (\mathbf{v}\mathbf{s})(\mathbf{w}\mathbf{s}) - \mathbf{y}\mathbf{s}$ , which, by construction, is divisible by  $t$ , and  $h$  will be their common factor. One might see that  $p$  is not really more compact than  $\mathbf{s}$ : as a matter of fact, given an  $m/n$  arithmetic circuit,  $\mathbf{s}$  has size  $n$ , while  $p$  has degree (and therefore size) up to  $2n$ .

However, the fact that  $p = ht$  implies that, for every  $x \in \mathbb{F}$ ,  $p(x) = h(x)t(x)$ . If we are working over a big finite field  $\mathbb{F}$  (say,  $|\mathbb{F}| \approx 2^{256}$ ), it is very unlikely that, without knowing  $p$ , one is able to guess some  $y$  such that  $y = p(x)$  (the probability is  $1/|\mathbb{F}|$ ). This means that, if we wish to check that someone knows  $p$  with high confidence, we can just ask for a couple of values  $x, y$  and check whether  $y = h(x)t(x)$ . Note that we can exponentially increase our confidence by asking for more pairs.

**Example 2.11.** Consider the 3/6 R1CS  $\mathcal{C}$  of Example 2.10: we want to compute the corresponding 3/6 QAP  $\mathcal{Q} = (t, \mathbf{v}, \mathbf{w}, \mathbf{y})$ . First, we set:

$$\mathbf{z} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} \mathbf{z}; \mathbf{z}; \mathbf{z}; \mathbf{z}; \mathbf{z}; \mathbf{z} \end{pmatrix}$$

Then, we compute the target polynomial  $t$ :

$$t = (x - 1)(x - 2)(x - 3) = (x + 12)(x + 11)(x + 10) = x^3 + 7x^2 + 11x + 7$$

We can now compute the left and right input constraint polynomial vectors  $\mathbf{v}$  and  $\mathbf{w}$ , and the output constraint polynomial vector  $\mathbf{y}$ . Notice how the 2nd, 4th and 5th columns of  $\mathbf{A}$  form the canonical basis of  $\mathbb{Z}_{13}^3$ , and since  $\text{lag}$  is a linear function, we can express all other polynomials as linear combinations of  $\text{lag}(\mathbf{z}, \mathbf{A}_2^\top)$ ,  $\text{lag}(\mathbf{z}, \mathbf{A}_4^\top)$  and  $\text{lag}(\mathbf{z}, \mathbf{A}_5^\top)$ :

$$\mathbf{v} = \text{lag}(\mathbf{Z}, \mathbf{A}^\top) = \begin{pmatrix} \text{lag}(\mathbf{z}, \mathbf{A}_1^\top) \\ \text{lag}(\mathbf{z}, \mathbf{A}_2^\top) \\ \text{lag}(\mathbf{z}, \mathbf{A}_3^\top) \\ \text{lag}(\mathbf{z}, \mathbf{A}_4^\top) \\ \text{lag}(\mathbf{z}, \mathbf{A}_5^\top) \\ \text{lag}(\mathbf{z}, \mathbf{A}_6^\top) \end{pmatrix} = \begin{pmatrix} 0 \\ 7x^2 + 4x + 3 \\ 0 \\ 12x^2 + 4x + 10 \\ 7x^2 + 5x + 1 \\ 0 \end{pmatrix}$$

$$\mathbf{w} = \begin{pmatrix} 0 \\ \mathbf{v}_2 + \mathbf{v}_4 \\ \mathbf{v}_5 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6x^2 + 8x \\ 7x^2 + 5x + 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} 8\mathbf{v}_4 \\ 0 \\ 9\mathbf{v}_4 \\ \mathbf{v}_2 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} 5x^2 + 6x + 2 \\ 0 \\ 4x^2 + 10x + 12 \\ 7x^2 + 4x + 3 \\ 12x^2 + 4x + 10 \\ 7x^2 + 5x + 1 \end{pmatrix}$$

Recall that a possible solution to the R1CS was  $\mathbf{s} = (1 \ 2 \ 3 \ 4 \ 12 \ 10)^\top$ . Let's check if it is also a valid assignment for the QAP:

$$\begin{aligned} p &= (\mathbf{v}\mathbf{s})(\mathbf{w}\mathbf{s}) - \mathbf{y}\mathbf{s} = 8x^4 + 5x^3 + 4x^2 + 2x + 7 \\ h &= \frac{p}{t} = \frac{8x^4 + 5x^3 + 4x^2 + 2x + 7}{x^3 + 7x^2 + 11x + 7} = 8x - 51 + \frac{273x^2 + 507x + 364}{x^3 + 7x^2 + 11x + 7} = 8x + 1 \\ ht &= (8x + 1)(x^3 + 7x^2 + 11x + 7) = 8x^4 + 5x^3 + 4x^2 + 2x + 7 = p \end{aligned}$$

Since  $ht = p$ , clearly  $p$  is divisible by  $t$ , meaning that  $h$  and  $p$  are in fact a solution to the  $\mathcal{Q}$ .

# An Overview of Proof Systems

Proving theorems is an activity which has always been considered highly intellectual, and even today most theorems are proven “by hand”. But what does it really mean to prove a theorem? Is finding a proof for a theorem really harder than checking whether said proof is valid? And is it possible to prove to someone that a theorem is true, without revealing why that is so?

In this chapter, after introducing the required concepts *computation and complexity theory* in Section 3.1 and Section 3.2, we give an overview of *interactive proof systems* in Section 3.3, and the history of their *zero-knowledge* counterparts in Section 3.4.

## 3.1 Interactive Turing machines

A *computational model* (or model of computation) is any kind of system able to describe how to produce some *output* given some *input* [Sav97]. Different models do this in different ways, each one with its own strength and weaknesses in terms of *expressiveness*, *complexity* and *succinctness*. Two historically important models of computations are Alonzo Church’s  $\lambda$ -calculus [Chu41] and Alan Turing’s *Turing machine* (TM) [Tur50]. Among several equivalent models [Dav04], Turing machines became the standard model of computation.

**Definition 3.1** (Turing machine [Pap94]). A Turing machine is a quadruple  $\mathcal{M} = (\Sigma, Q, q_0, \delta)$ , where the *alphabet*  $\Sigma$  is a set of symbols such that  $\sqcup \in \Sigma$ , the *state set*  $Q$  is a set of symbols such that  $\{\perp, \top\} \subseteq Q$ ,  $q_0 \in Q$  is the *initial state*, and  $\delta: (Q \setminus \{\perp, \top\}) \times \Sigma \rightarrow Q \times \Sigma \times \{\leftarrow, \rightarrow\}$  is the *transition function*.

By only requiring  $\delta$  to be a relation instead of a function, we obtain the so-called *non-deterministic* Turing machine (NTM): given a state and an alphabet symbol, the machine can take different choices at every step. A TM  $\mathcal{M}$  manipulates a string  $\bar{\sigma}$  over the alphabet  $\Sigma \setminus \{\sqcup\}$  by placing it over an *infinite, discrete working tape*  $\mathfrak{W}$ , a total order isomorphic to  $\mathbb{Z}$ . The input string is positioned such that its first symbol is matched with the position 0 of the tape; all the positions before the first symbol and after the last symbol are filled with the *blank* symbol  $\sqcup$ . The computation  $\mathcal{M}(\bar{\sigma})$  starts in the initial state  $q_0$  with the *head* of the TM positioned over the position 0 of the tape, and proceeds according to the transition function: depending on the current state  $q$  and the symbol  $\sigma$  written in the current location of the head, it replaces  $\sigma$  with a new symbol  $\sigma'$ , it moves the head to the left ( $\leftarrow$ ) or to the right ( $\rightarrow$ )

and it transitions into a new state  $q'$ . The computation *terminates* whenever one of the two *halting* states is reached: if  $\mathcal{M}(\bar{\sigma}) = \perp$ , then the input string  $\bar{\sigma}$  is *rejected*, else if  $\mathcal{M}(\bar{\sigma}) = \top$ , then  $\bar{\sigma}$  is *accepted*. It can also happen that the computation does not terminate: in such cases, we write  $\mathcal{M}(\bar{\sigma}) = \uparrow$ , and we say that the computation *hangs*.

In many scenarios, it is useful to extend Turing machines to include additional features, for example to represent the ability to access some source of randomness, or to communicate with an external environment to read inputs and produce outputs in an interactive manner.

**Definition 3.2** (Input/Output Turing machine). An input/output Turing machine is a quadruple  $\mathcal{M} = (\Sigma, Q, q_0, \delta)$  where  $\Sigma$ ,  $Q$  and  $q_0$  are defined as in Definition 3.1, and:

$$\delta: (Q \setminus \{\perp, \top\}) \times \Sigma^2 \rightarrow Q \times \Sigma^2 \times \{\leftarrow, \rightarrow\}^2$$

The additional parameters in the transition function of an input/output Turing machine (I/O TM) account for two new tapes, namely the *read-only input tape*  $\mathfrak{I}$  and the *write-only output tape*  $\mathfrak{O}$ : now, depending on the state  $q$ , the input symbol  $\sigma_i$  and the working symbol  $\sigma_w$ , the machine overwrites  $\sigma_w$  with a new symbol  $\sigma'_w$  and moves left/right on  $\mathfrak{W}$ , it writes a new symbol  $\sigma_o$  on  $\mathfrak{O}$ , where it can move only to the right, and it moves to the left/right on  $\mathfrak{I}$ . Additionally, in an I/O TM, the input string is placed on  $\mathfrak{I}$  instead of  $\mathfrak{W}$ , which is instead blank at the beginning of the computation.

**Definition 3.3** (Probabilistic Turing machine). A probabilistic Turing machine is a quadruple  $\mathcal{M} = (\Sigma, Q, q_0, \delta)$  where  $\Sigma$ ,  $Q$  and  $q_0$  are defined as in Definition 3.1, and:

$$\delta: (Q \setminus \{\perp, \top\}) \times \Sigma \times \{0, 1\} \rightarrow Q \times \Sigma \times \{\leftarrow, \rightarrow\}$$

In a probabilistic Turing machine (PTM), we have an additional *read-only random tape*  $\mathfrak{P}$  which is populated with an infinite, uniformly random sequence of *coin tosses* (zeros and ones), that are used by the transition function to decide what to do. As for the writing tape of an I/O TM, the head on  $\mathfrak{P}$  can only move to the right.

**Definition 3.4** (Interactive Turing machine). An interactive Turing machine is a quadruple  $\mathcal{M} = (\Sigma, Q, q_0, \delta)$  where  $\Sigma$ ,  $Q$  and  $q_0$  are defined as in Definition 3.1, and:

$$\delta: (Q \setminus \{\perp, \top\}) \times \Sigma^2 \rightarrow Q \times \Sigma^2 \times \{\leftarrow, \rightarrow\}$$

An interactive Turing machine (ITM) is quite similar to an I/O TM, as it also has two additional tapes, the *send-tape*  $\mathfrak{S}$  and the *receive-tape*  $\mathfrak{R}$ . However, unlike for the input tape  $\mathfrak{I}$  of an I/O TM, the head on  $\mathfrak{R}$  cannot move backwards.

*Remark 3.1.* Our definition of ITM differs slightly from the standard one in the literature [GMW91, GMR89], but we find it to be more modular. In any case, from now on, we will say *interactive Turing machine* to actually mean an *interactive, probabilistic, input/output Turing machine*.

**Definition 3.5** (Interactive protocol). An interactive protocol is a pair  $\mathcal{I} = (\mathcal{M}, \mathcal{M}')$  where  $\mathcal{M}$  and  $\mathcal{M}'$  are interactive Turing machines such that  $\mathfrak{I} = \mathfrak{I}'$ ,  $\mathfrak{S} = \mathfrak{R}'$ ,  $\mathfrak{R} = \mathfrak{S}'$ , and their state sets  $Q, Q'$  contain the special *idle state*  $q_{idle} \in Q, Q'$ .

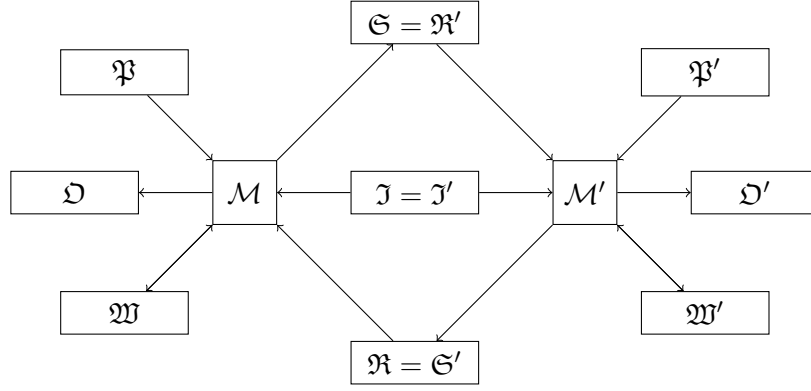


Figure 3.1: Visualization of an interactive protocol.

The computation of an interactive protocol (IP) over some string  $\bar{\sigma}$ ,  $\mathcal{I}(\bar{\sigma})$ , proceeds in the following manner: initially, the tapes  $\mathfrak{S}$ ,  $\mathfrak{R}$ ,  $\mathfrak{W}$ ,  $\mathfrak{W}'$ ,  $\mathfrak{D}$  and  $\mathfrak{D}'$  are all empty (i.e. filled with blank symbols), the tapes  $\mathfrak{P}$  and  $\mathfrak{P}'$  are filled with random bits, and the tape  $\mathfrak{I}$  contains  $\bar{\sigma}$ . The ITM  $\mathcal{M}$  is said to be *active* and works normally until it transitions in the special state  $q_{idle}$ , becoming *inactive*. When this happens, control passes to  $\mathcal{M}'$ , which becomes active and works normally until it reaches its own idle state; control goes back to  $\mathcal{M}$ , and the process repeats. When one of the two machines halts, control passes over the other one until it also halts. The protocol *succeeds* if both machines halt in the accepting state  $\top$ , and it *fails* if at least one of them halts in the rejecting state  $\perp$ . To denote the final states reached by one of the machines at the end of the computation, we write  $\mathcal{I}_{\mathcal{M}}(\bar{\sigma})$  and  $\mathcal{I}_{\mathcal{M}'}(\bar{\sigma})$  respectively. Figure 3.1 depicts the fundamental structure of an interactive protocol.

## 3.2 Problems and complexity

Historically, the most important class of problems that have been analyzed are so-called *decision problems*, i.e. problems whose solution is a binary *yes-or-no* answer [Sip13]. This perfectly suits Turing machines as we can interpret their acceptance or rejection of the input string respectively as a yes and a no answer.

**Definition 3.6** (Kleene's closure). The Kleene's closure of a set  $S$  is the set  $S^* = \bigcup_{n \in \mathbb{N}} S^n$ .

As Turing machines operate over strings in  $\Sigma^*$ , also called *words*, they partition  $\Sigma^*$  into three *languages* (a language is any set of strings): the language of accepted words, the language of rejected words and the language of hanging words.

**Definition 3.7** (Turing-recognizable language). A language  $L \subseteq \Sigma^*$  is recognized by some Turing machine  $\mathcal{M}$  if  $\forall w \in L: \mathcal{M}(w) = \top$ .

**Definition 3.8** (Turing-decidable language). A language  $L \subseteq \Sigma^*$  is decided by some Turing machine  $\mathcal{M}$  if it is recognized by  $\mathcal{M}$  and  $\forall w \notin L: \mathcal{M}(w) = \perp$ .

We denote the language recognized by a Turing machine  $\mathcal{M}$  with  $L(\mathcal{M})$ . To solve an *instance*  $\Pi$  of some decision problem **PROB**, we first encode the instance into a string  $\langle \Pi \rangle \in \Sigma^*$  such that  $\langle \Pi \rangle \in L(\mathcal{M})$  if and only if the answer to  $\Pi$  is ‘yes’.

The class of recognizable languages, called RE, strictly includes the class of decidable languages, called DEC [Tur37]. But even decidable languages are not all equal: their *computational complexity*, that is, the amount of some resource which is required by a Turing machine to decide membership words in function of their length, can vary wildly. In general, we are only interested in the *asymptotic behaviour* of the machine.

**Definition 3.9** (Big-O notation). Given two functions  $f, g: \mathbb{N} \rightarrow \mathbb{N}$ , then  $f = \mathcal{O}(g)$  if and only if  $\exists c, n$  such that  $\forall x \geq n \ f(x) \leq c \cdot g(x)$ .

We write  $f = \Omega(g)$  when  $g = \mathcal{O}(f)$ , and we write  $f = \Theta(g)$  when  $f = \mathcal{O}(g)$  and  $g = \mathcal{O}(f)$ . When there exists a TM  $\mathcal{M}$  for which some complexity metric  $C$  is upper-bounded at most by a polynomial function in the length  $n$  of the input word (i.e.  $C(\mathcal{M}) = \mathcal{O}(n^c)$  for some constant  $c \in \mathbb{N}$ ), we say that deciding the language is *feasible* w.r.t.  $C$ . On the other hand, if  $C$  is upper-bounded at least by an exponential function (i.e.  $C(\mathcal{M}) = \mathcal{O}(c^n)$  for some constant  $c > 1$ ), we say that the problem is *infeasible* w.r.t.  $C$ . The standard complexity metrics are *time* TIME, that is the amount of transition steps a TM performs before halting, and *space* SPACE, that is the amount of tape locations visited by a TM before halting<sup>1</sup>.

Two of the most important *complexity classes*<sup>2</sup> are PTIME (P for short) and NPTIME (NP for short), which are the classes of languages decidable respectively by a deterministic TM and a nondeterministic TM using at most polynomial time. While we do not know if  $P \stackrel{?}{=} NP$ , it is widely believed that  $P \subset NP$ : for a deterministic Turing machine, deciding NP-COMPLETE problems (i.e. the hardest problems in NP) will generally take an exponential amount of time, and there is no known way in the physical world to build non-deterministic Turing machines. Although quantum computers have been shown to be able to crack problems which are believed to be infeasible for standard computers, like integer factorization [Sho94], NP-COMPLETE problems seem to be out of reach also for such powerful machines.

### 3.3 Interactive proof systems

Even though NP-COMPLETE problems are infeasible, they are *efficient to verify*: given an *instance*  $\Pi$  of some NP-COMPLETE problem, and an additional *witness* string, we can build a deterministic TM that checks whether the witness *proves* or not that the problem admits a positive answer.

**Example 3.1.** Consider the problem SAT of deciding whether a propositional logic formula  $\phi$  is satisfiable, which is the most famous NP-COMPLETE problem [Coo71]. If we had a TM  $\mathcal{M}$  with access to an *oracle* that, in  $\mathcal{O}(1)$  time, provides a valid assignment for the variables in  $\phi$ , it would be easy to verify that  $\phi$  is indeed satisfiable. However, if the provided assignment was not valid, while  $\mathcal{M}$  would reject it, there would be still no easy way to know whether  $\phi$  is actually satisfiable or not!

Now, let's say we want to prove some theorem  $\Pi$ : computationally, this is equivalent to deciding whether  $\Pi$  is word which belongs to the language of the valid propositions over some formal system (say, the ZFC set theory [FBHL73]). A *proof* of the theorem plays the same role of the *witness* we

<sup>1</sup>It is always the case that  $\text{SPACE} \leq \text{TIME}$ .

<sup>2</sup>[https://complexityzoo.net/Complexity\\_Zoo](https://complexityzoo.net/Complexity_Zoo)



discussed before: in general, verifying a proof for a theorem is (believed to be) much easier than finding the proof in the first place. Hence, we can extend the logical/mathematical concept of theorem to the more computational concept of language: for example, by NP theorem, we mean any language in NP.

**Definition 3.10** (NP proof system [GMR89]). An NP *proof system* for a language  $L$  is an interactive protocol  $\mathcal{I} = (\mathcal{P}, \mathcal{V})$ , where  $\mathcal{P}$  is the *prover* and  $\mathcal{V}$  is the *verifier*, such that:

$$\begin{aligned} \forall w \in L: \mathcal{I}_{\mathcal{V}}(w) &= \top && (\text{completeness}) \\ \forall w \notin L: \mathcal{I}_{\mathcal{V}}(w) &= \perp && (\text{soundness}) \\ \exists k \in \mathbb{N}: \text{TIME}(\mathcal{V}) &= \mathcal{O}(|x|^k) && (\text{boundness}) \end{aligned}$$

In an NP proof system (NPPS), the common input tape of  $\mathcal{P}$  and  $\mathcal{V}$  contains some word  $w$  representing some statement: in typical scenario, the statement is provided by the prover himself, who wants to convince the verifier of the truthness of such statement. During the protocol,  $\mathcal{P}$  and  $\mathcal{V}$  exchange messages through their communication tapes; at some point,  $\mathcal{P}$  sends to  $\mathcal{V}$  a candidate proof  $\pi$ : the verifier checks the proof and, if the proof is valid, it is always convinced of its validity, hence it will accept. On the other hand, if the proof happens to be wrong (e.g. if  $\mathcal{P}$  is trying to deceive  $\mathcal{V}$ ), then the verifier will never be convinced by such a proof, and it will reject. The polynomial bound on the execution time of  $\mathcal{V}$  is necessary to force cooperation, and avoid the case where  $\mathcal{V}$  simply ignores  $\mathcal{P}$  and computes the proof by itself.

**Definition 3.11** (Interactive proof system [GMR89]). An *interactive proof system* (IPS) for a language  $L$  is an interactive protocol  $\mathcal{I} = (\mathcal{P}, \mathcal{V})$  such that, for any arbitrarily big bound  $c \in \mathbb{R}_+$ :

$$\begin{aligned} \forall w \in L: \Pr(\mathcal{I}_{\mathcal{V}}(w) = \perp) &< |w|^{-c} && (\text{statistical completeness}) \\ \forall w \notin L: \Pr(\mathcal{I}_{\mathcal{V}}(w) = \top) &< |w|^{-c} && (\text{statistical soundness}) \\ \exists k \in \mathbb{N}: \text{TIME}(\mathcal{V}) &= \mathcal{O}(|x|^k) && (\text{verifier boundness}) \end{aligned}$$

The difference between a NP proof system and an IPS lies in the probabilistic nature of the latter: there is a (negligible) probability that a verifier will be convinced of the truthness of a false statement, or of the falseness of a true statement, although the latter is not relevant in most instances, as the prover *wants* the verifier to accept.

**Definition 3.12** (Interactive argument of knowledge system [BCC88]). An *interactive arguments of knowledge system* (IARK) is an interactive proof system  $\mathcal{I} = (\mathcal{P}, \mathcal{V})$  such that:

$$\exists k \in \mathbb{N}: \text{TIME}(\mathcal{P}) = \mathcal{O}(|x|^k) \quad (\text{prover boundness})$$

In the IARK setup, the two parties are modelled in a more ‘realistic’ way: a prover is not intrinsically more efficient than a verifier, what really gives it power is the knowledge of some *secret*, initially stored on its working tape, which allows the prover to quickly find a proof/witness (in fact, the secret is often the witness itself).

**Definition 3.13** (IP complexity class). The Interactive Polynomial class IP is the class of languages which are decided by some interactive proof system.

Of course  $\text{NP} \subseteq \text{IP}$ , as the verifier can check, with no error, a proof for any NP-COMPLETE problem statement in polynomial time. However, there are problems in IP which are not known to be in NP, such as the problems of deciding the order and the membership of a matrix group [BS84], and deciding whether two graphs are not isomorphic [GMW91].

### 3.3.1 Arthur-Merlin games

In an interactive proof system, the coin tosses of the two parties are *private*, and they can exchange any kind of message.

**Definition 3.14** (Arthur-Merlin games [BS84]). An Arthur-Merlin game is an interactive proof system  $\mathcal{I} = (M, A)$  such that Arthur can only send coin tosses over  $\mathfrak{S}$ .

In Arthur-Merlin games, the messages that the verifier (Arthur) can send to the prover (Merlin) are limited to the coin tosses from its random tape (in the sense that, whatever Arthur sends Merlin, he interprets it as a coin toss).

**Definition 3.15** (AM complexity class). The Arthur-Merlin complexity class AM is the class of languages which are decided by an Arthur-Merlin game.

While it is obvious that  $\text{AM} \subseteq \text{IP}$ , the converse was also shown to be true [BS84]: in particular, for any *fixed* number  $k$  of communications between Arthur and Merlin,  $\text{AM}[k] = \text{AM}[2]$ , meaning that it is possible to build an equivalent Arthur-Merlin game using only 2 communications. In fact, it was also shown that IP (and hence AM) is equal to PSPACE, i.e. the class of problems solvable by a Turing machine in polynomial space [Sha92].

## 3.4 Zero-Knowledge Protocols

Suppose that two parties are executing an IARK system for some hard problem: the instance is placed on the shared input tape, and also suppose that the secret in possession of the prover is simply a witness for the instance. All the prover has to do is send the witness to the verifier, which will in turn check it and decide whether to accept or not. In this process, the verifier gained more knowledge than just the solvability of the problem: it also learned a solution, and not just any solution, but exactly the one available to the prover, which should have been a secret. To address this issue, researchers started exploring the field of so-called zero-knowledge proofs [GMR89, GMW91].

Informally, two random variables  $U$  and  $V$  that map words of some language  $L \subseteq \{0, 1\}^*$  to words of  $\{0, 1\}^*$  are *perfectly indistinguishable* when no unbounded Turing machine is able to tell them apart, are *statistically indistinguishable* when no PSPACE Turing machine is able to tell them apart, and are *computationally indistinguishable* when no PTIME Turing machine  $\mathcal{M}$  is able to tell them apart. By ‘telling apart’, we mean that the distribution of the words that are accepted/rejected by Turing machines respecting the imposed bounds is independent of  $U$  and  $V$ : intuitively, this means that  $U$  and  $V$  are interchangeable with each other and using one over the other does not give an ‘edge’ to  $\mathcal{M}$  [GM84, GMR89, Yao82b].

**Example 3.2.** Consider the two random variables  $U, V : L \rightarrow \{0, 1\}^*$  for some  $L \subseteq \{0, 1\}^*$ , such that, for all words  $x \in L$  and all words  $w \in \{0, 1\}^{|x|}$ , it holds that:

$$\Pr(U(x) = w) = 2^{-|x|} \qquad \Pr(V(x) = w) = \begin{cases} 0 & x = 0 \dots 0 \\ 2^{-|x|+1} & x = 1 \dots 1 \\ 2^{-|x|} & \text{otherwise} \end{cases}$$

$U$  and  $V$  have *almost* the same distribution, with the  $1 \dots 1$  string happening twice as often in  $V$ . For increasingly longer strings, no Turing machine can tell the two distributions apart by collecting a polynomial amount of samples, since  $\sum_w |\Pr(U(x) = w) - \Pr(V(x) = w)| = 2^{-|x|+1}$ , hence  $U$  and  $V$  are statistically indistinguishable.

**Definition 3.16** (Tape view). A *tape view* is a random variable  $V_{\mathcal{M}}$  that models the concatenation of all the contents that are read/written by a halting Turing machine  $\mathcal{M}$  over its tapes.

For a deterministic, non-probabilistic Turing machine, the tape view variable is quite pointless, but it is a useful tool to model the behaviour of machines that exploit randomness, and especially for interactive protocols. For example, if we have a Turing machine with one tape  $\mathfrak{T}$ , then:

$$\Pr(V_{\mathcal{M}}(x) = w) = \Pr(V_{\mathcal{M}, \mathfrak{T}}(x) = w) = \Pr(\mathcal{M}(x) = w)$$

**Definition 3.17** (Approximability). A random variable  $U$  is (perfectly, statistically, computationally) *approximable* by a probabilistic Turing machine  $\mathcal{M}$  over some language  $L$  if  $U$  and  $V_{\mathcal{M}}$  are (perfectly, statistically, computationally) indistinguishable.

Note that for a random variable  $U$  and a halting PTM  $\mathcal{M}$  to be perfectly indistinguishable over some language  $L$ , it must be the case that  $\forall x \in L: \mathcal{M}(x) = V_{\mathcal{M}}(x) = \mathcal{U}(x)$ .

**Definition 3.18** (Zero-knowledge interactive protocol). A (perfectly, statistically, computationally) *Zero-knowledge interactive protocol* (ZKIP) over a language  $L \subseteq \{0, 1\}^*$  is an interactive protocol  $\mathcal{I} = (\mathcal{M}, \mathcal{M}')$  such that, for every  $\mathcal{M}'$ ,  $V_{\mathcal{M}'}$  is (perfectly, statistically, computationally) approximable by a Turing machine  $\mathcal{M}''$  over the language  $L' = \{(x, h) \mid x \in L \wedge h \in \{0, 1\}^*\}$ , where the string  $h$  represents the initial content of  $\mathfrak{W}'$ .

Naturally, a ZKIP which is also a proof system is a zero-knowledge proof system (ZKPS); similarly, if it is an interactive argument of knowledge system then it is a zero-knowledge interactive argument of knowledge system (ZK-IARK). From now on, by zero-knowledge we mean computational zero-knowledge, as assuming a polynomial-time bounded adversary is an acceptable restriction in the real world. The initial string  $h$  of a ZKIP can be interpreted as the *history* of previous interactions with the prover, or some eavesdropped information from the interactions that the prover had with other verifiers.

A proof system being zero-knowledge basically means that, even for curious or malicious verifiers, and even with additional knowledge on the behaviour of the prover, what can be computed is nothing more than what could have been computed in polynomial time, hence within the imposed computational

power limits, without communicating with the prover. While it is obvious that every problem solvable in probabilistic polynomial time (PP) has a zero-knowledge proof system (the prover does nothing and the verifier computes the solution by himself), it was proven that also all problems in NP have a ZKPS [GMW91]. By assuming the existence of secure probabilistic encryption, it was finally shown that also all Arthur-Merlin games, and hence all problems in IP, have a ZKPS [BOGG<sup>+</sup>90].

### 3.4.1 Non-interactive Zero-Knowledge

In many scenarios, especially ones involving multiple parties, interaction can be a problem as the communication cost of bidirectional  $n$ -to- $n$  grows quadratically. Such cases are in fact of great interest for zero-knowledge systems: multiple parties can be both provers and/or verifiers, and their number might be huge.

For this reason, researches explored the possibility of having zero knowledge *non-interactive* proof systems (ZK-NPS) or argument of knowledge systems (ZK-NARK). Unfortunately, only the languages in BPP, that is languages decidable in probabilistic polynomial time with a bounded error allow for zero-knowledge non-interactive proofs [Ore87, GK96]. Such languages are of course trivial, as the verifier has enough power to do all the computation by itself without the need of the prover.

However, by introducing an initial *preprocessing* phase, it is possible to regain the lost power [DSMP90], and the most prominent technique to achieve non-interaction is the *Common Reference String* (CRS) model [BFM88] (sometimes also called *common random string* model). The main idea of the CRS model is that, before engaging in the protocol, the prover and the verifier have both obtained access to a shared string of random bits. In the simplest case, the string is generated by a *trusted third party*, although in practice this is oftentimes not a viable solution as the whole point of zero-knowledge is having to deal with untrusted parties. To circumvent this problem, it is possible to generate the CRS by a *majority vote* between  $n$  authorities, which can be untrusted if picked singularly, but are assumed to be honest in their majority [GO06]. In fact, it was shown that it is possible, without losing zero-knowledge, to re-use multiple times a single CRS both by a single [BDSMP91] or multiple [FLS90] provers, although only for arguments of knowledge and not for proofs.

The first zero-knowledge systems, both interactive and non-interactive, were tailor-made for specific problems, such as the quadratic residuosity problem QR [GMR89], the Hamiltonian path problem HAM-PATH [LS91], or the 3-SAT problem [BDSMP91]. Although for any NP-COMPLETE problem PROB there is a polynomial-time algorithm [Kar72] that converts every instance of PROB to an instance of, say, 3-SAT, such reductions are often not trivial to devise and very expensive to apply. For this reason, researchers started devising constructions to prove arbitrary NP statements embedded in the form of boolean circuits [Dam93], which can neatly represent the computation of a Turing machine over any NP-COMPLETE problem [Coo71], and therefore remove the need to go through polynomial-time reductions.

The first of such systems [Dam93] required a CRS of size cubic in the length of the statement to be proven, although XOR and NOT gates didn't need to consume any bits from the CRS. In the following years, many improvements were proposed, reducing the complexity of the constructions from cubic to sub-quadratic [BBP95] and eventually linear [CD97].

# 4

## Hash Functions and Succinct Proofs

Cryptography, quite unsurprisingly, is the field in which zero-knowledge proof systems have got the most attention. The possibility of two or more parties to cooperate in performing joint computation without revealing any secret or compromising privacy is the fundamental idea behind many branches of cryptography such as *Multi Party Computation* (MPC) [Yao82a] and *Fully Homomorphic Encryption* (FHE) [ABC<sup>+</sup>15].

Indeed, an important application of zero-knowledge protocols lies in *verifiable computation*: a server (the prover) wants to convince some users (the verifiers) that some function has been executed properly, without revealing the inputs.

In Section 4.1, we review the fundamental notions concerning *cryptographic hash functions*, and some standard ways to construct them. Then, in Section 4.2, we go in more depth over *tree-like modes of hashing*, which constitute the basis of some of the most important use-cases of verifiable computation. In Section 4.3 we study the construction of zero-knowledge succinct NARK (ZK-SNARK) systems, with a focus on the *Pinocchio* protocol [PGHR13]. Finally, in Section 4.4 we describe `libsnark`, a C++ library which implements (a variant of) the Pinocchio protocol.

### 4.1 Secure Hash functions

Secure (cryptographic) hash functions are a fundamental tool of cryptography, as they can be used to produce short commitments for any message of length  $|n|$ .

**Definition 4.1** (Hash function). Given  $n \in \mathbb{N}$ , an  $n$ -bit hash function is a function  $H: \{0, 1\}^* \rightarrow \{0, 1\}^n$ .

The input of a hash function is called the *message*, while its output is called the *digest*. From the definition, it is immediate to see that there are an infinite number of messages which map to the same digest. Now, a very simple hash function might be truncation (i.e. take the first  $n$  bits and discard anything coming afterwards), but it is not of much interest for cryptography, as we require additional *security guarantees*.

*Remark 4.1.* When we say that it is *easy* (resp. *hard*) to compute a function, we mean that there is (resp. there is not) a probabilistic Turing machine which can compute such function in at most polynomial time. In particular, a *one-way function* is an easily computable function  $f$  whose inverse is hard to

compute, and a *trapdoor function* is a one-way function whose inverse becomes easy to compute given some additional knowledge, called the *key*.

**Remark 4.2.** The  $\oplus$  symbol used in this chapter can have different meanings: in the literature, it is mostly used to denote bitwise XOR. Since a message/string over  $\{0,1\}^*$  can be considered as a vector over  $\mathbb{Z}_2^n$ , bitwise XOR is the same as vector addition. However, there are instances, particularly in zero-knowledge systems, where messages are interpreted as elements of some prime field  $\mathbb{Z}_p$ : in such cases,  $\oplus$  will denote addition modulo  $p$ , as defined in Chapter 2. However, such cases will be explicitly specified, so we decided to keep the ambiguity to be coherent with the literature.

**Definition 4.2** (Cryptographic hash function [AKDB11]). Given  $n \in \mathbb{N}$ , an *n-bit cryptographic hash function* (CHF) is an  $n$ -bit hash function which satisfies the following properties:

- **Collision resistance:** It is hard to find two messages  $m_1, m_2$  such that  $H(m_1) = H(m_2)$ .
- **Preimage resistance:** Given some digest  $d$ , it is hard to find a message  $m$  such that  $H(m) = d$  ( $H$  is a one-way function).
- **Second preimage resistance:** Given some message  $m_1$ , it is hard to find a message  $m_2$  such that  $H(m_1) = H(m_2)$ .

Note that second preimage resistance implies first preimage resistance (if given any  $m_1$  we can find a colliding  $m_2$ , we can do it also without being given  $m_1$ ), and in turn preimage resistance implies second preimage resistance (if given any  $d$  we can find a colliding  $m$ , then given any  $m'$  we can compute  $d$  and find the collision).

A perfect  $n$ -bit CHF provides  $n/2$  bits of security for collision resistance (birthday paradox), assuming that the output is uniformly distributed, meaning that every possible adversary is expected to need at least  $\mathcal{O}(2^{n/2})$  time to find a collision, while for preimage resistance it provides  $n$  bits of security. A CHF can be built by applying some known secure construction to functions which are simpler to devise.

**Definition 4.3** (Padding function). An *n-bit padding function* is a function  $\text{Pad}: \{0,1\}^* \rightarrow (\{0,1\}^n)^*$ .

**Definition 4.4** (Pseudorandom permutation). Given  $l \in \mathbb{N}$ , an *l-bit pseudorandom permutation* is a permutation  $P: \{0,1\}^l \rightarrow \{0,1\}^l$  which is indistinguishable from a uniform random distribution.

**Definition 4.5** (Keyed permutation). Given  $l, n \in \mathbb{N}$ , an *l/n-bit keyed permutation* is a function  $F: \{0,1\}^l \times \{0,1\}^n \rightarrow \{0,1\}^l$  which is a permutation on its first parameter for every fixed second argument.

**Definition 4.6** (Block cipher). A *block cipher* is a trapdoor pseudorandom keyed permutation.

**Definition 4.7** (One-way compression function). Given  $l_1, n, l_2 \in \mathbb{N}$  such that  $l_1 + n > l_2$ , an *l<sub>1</sub>/n/l<sub>2</sub>-bit* (or *(l<sub>1</sub> + n)-to-l<sub>2</sub>-bit*) *one-way compression function* (OWCF) is a one-way function  $F: \{0,1\}^{l_1} \times \{0,1\}^n \rightarrow \{0,1\}^{l_2}$ .

A *pseudorandom keyed permutation* (PKP) is typically built by iterating a “somewhat pseudorandom” keyed permutation  $F$  for an adequate number of *rounds*, until inverting the function becomes hard.

A block cipher can be built from a PKP by following some secure construction scheme, like the Feistel-Luby-Rackoff construction [MvOV01]. A OWCF can also be derived from a PKP or a block cipher, by applying a secure construction scheme, like the Davies-Meyer construction [Pre05]. Finally, CHF can again be derived either by a OWCF, for example through the Merkle-Damgård construction [Mer79], or directly from a pseudorandom permutation, like in the sponge construction [BDPVA07, Tiw17].

**Proposition 4.1** (Feistel-Luby-Rackoff construction [MvOV01]). *Given an  $l/n$ -bit pseudorandom keyed permutation  $P$ , some message  $m = (m_1, m_2)$  such that  $m_1, m_2 \in \{0, 1\}^l$ , a number of rounds  $r > 3$ , and a set of keys  $k_1, \dots, k_r \in \{0, 1\}^n$ , then the function  $E_r$  is a  $2l/n$  block cipher, where:*

$$E_i(m, k_i) = (x_i, y_i) = \begin{cases} (m_1, m_2) & i = 0 \\ (y_{i-1}, x_{i-1} \oplus P(y_{i-1}, k_i)) & 1 \leq i \leq r \end{cases}$$

**Proposition 4.2** (Davies-Meyer construction [Pre05]). *Given an  $l/n$ -bit pseudorandom keyed permutation  $P$ , some number of blocks  $b \in \mathbb{N}$ , some initial value  $v \in \{0, 1\}^l$ , and some message  $m = (m_1, \dots, m_b)$  such that each  $m_i \in \{0, 1\}^n$ , then the function  $F_b$  is an  $l/bn/l$  one-way compression function, where:*

$$F_i(v, m) = \begin{cases} v & i = 0 \\ P(F_{i-1}(v, m), m_i) \oplus F_{i-1}(v, m) & 1 \leq i \leq b \end{cases}$$

**Proposition 4.3** (Merkle-Damgård construction [Mer79]). *Given an  $l/n/l$ -bit one way compression function  $F$ , some initial value  $v \in \{0, 1\}^l$ , some message  $m \in \{0, 1\}^*$ , a padding extension length  $k$  and an  $n$ -bit padding function  $\text{Pad}$  such that,  $\forall x, y \in \{0, 1\}^*$ :*

$$\begin{aligned} |\text{Pad}(x)| &= |x| + (-|x| \bmod n) + kn \\ |x| = |y| &\implies |\text{Pad}(x)| = |\text{Pad}(y)| \\ |x| \neq |y| &\implies \text{Pad}(x)_{|\text{Pad}(x)|} \neq \text{Pad}(y)_{|\text{Pad}(y)|} \end{aligned}$$

*then the function  $H_{|\text{Pad}(m)|}$  is an  $l$ -bit cryptographic hash function, where:*

$$H_i(m) = \begin{cases} v & i = 0 \\ F(H_{i-1}(m), \text{Pad}(m)_i) & 1 \leq i \leq |\text{Pad}(m)| \end{cases}$$

**Proposition 4.4** (Sponge construction [BDPVA07]). *Given an  $l$ -bit pseudorandom permutation  $P$ , a message  $m \in \{0, 1\}^*$ , a rate  $r$  and a capacity  $c$  such that  $r + c = l$ , an initial value  $v \in \{0, 1\}^b$  and an  $r$ -bit padding function  $\text{Pad}$ , the function  $H_{|\text{Pad}(m)|}$  is an  $r$ -bit cryptographic hash function, where:*

$$\begin{aligned} S_i(m) &= \begin{cases} v & i = 0 \\ P(S_{i-1}(m) \oplus \text{Pad}(m)_i) & 1 \leq i \leq |\text{Pad}(m)| \end{cases} \\ H_i(m) &= S_i(m)_{1, \dots, r} \end{aligned}$$

The sponge construction is particularly interesting as it is very flexible: given a pseudorandom



permutation, it can be used to build pseudorandom keyed permutations, cryptographic hash functions, random number generators and authenticated encryption [BDPVA12].

## 4.2 Tree-like modes of hashing

Consider an  $n$ -bit CHF  $H$ , and suppose that a prover claims to know some message  $m$ : the digest  $d = H(m)$  can be considered as a *short binding commitment* for  $m$ : By asking the prover to share the digest, whose size  $|d| = n$  is typically considered to be  $\mathcal{O}(1)$  (or  $\mathcal{O}(\log(|m|))$  in some cases), a verifier is convinced that the prover does know  $m$  with probability  $\approx 1 - 1/2^n$ . A modern standard CHF like SHA-256 [Dan15] produces digests of length at least 256 bits, making the  $1 - 1/2^n$  bound really hard to brute force through. Note that the verifier needs not to know  $m$  in advance: the commitment  $d$  is (temporarily) appended to a public *blockchain* and, at any point in the future, when the verifier becomes aware of some  $m'$  provided by the prover, if  $H(m') = d$ , the commitment can be approved or rejected.

Now, suppose that the prover wants to commit to a list of  $k$  messages: the simplest solution would be to publish the hash of every message, which would require appending  $\mathcal{O}(k)$  elements on the blockchain. Another way would be for the prover to share  $H((m_1, \dots, m_k))$ : the communication cost would only be  $\mathcal{O}(1)$  but, in general, not all the messages belong to the same prover, so this method would not work, and we need a better solution.

### 4.2.1 Merkle tree

**Definition 4.8** (Binary Merkle tree [Mer88]). A *binary Merkle tree* (MT) of height  $h \in \mathbb{N}$  over a  $2n$ -to- $n$  compression function  $C$ , is the complete binary tree of height  $h$  such that, given a sequence of input messages  $(m_1, \dots, m_{2^h-1})$  over  $\{0, 1\}^{2n}$ , produces an output digest  $d \in \{0, 1\}^n$  in the following way:

1. The leaf nodes  $\nu_1, \dots, \nu_{2^h-1}$  contain  $C(m_1), \dots, C(m_{2^h-1})$ .
2. Every other node  $\nu$  contains  $C(\nu_l, \nu_r)$ , where  $\nu_l$  is the left child of  $\nu$  and  $\nu_r$  is the right child of  $\nu$ .
3. The output digest  $d$  is the content of the root node.

The set of the sibling nodes visited in the path from a leaf of the tree to the root, including the leaf itself, is the *authentication path* of the leaf. By using Merkle trees, the prover only needs to send to the verifier, as a commitment for some message  $m_i$  among  $n = 2^h$  messages, the contents of the co-path from the leaf containing  $m_i$  to the root, in addition plus the hash of  $m_i$ : this requires just  $\mathcal{O}(\log(n))$  cost to validate the commitment. Merkle trees bottom-up construction is very easy to parallelize, and they can be used in the scenario with multiple provers: each prover only needs to commit to the path from its own leaf to the root of the tree. It is immediate to generalize the notion of binary Merkle tree to arbitrary arity.

**Proposition 4.5** (Security of Merkle tree mode of hash [Mer88]). *Given a one-way  $tn$ -to- $n$  compression function  $C$ , the  $t$ -ary Merkle tree over  $C$  is a cryptographic hash function.*

**Example 4.1.** Consider the sequence of pre-hashed messages  $S = (3, 4, 7, 7)$  and the compression function  $C(x, y) : (x, y) \mapsto (xy \bmod 13) + 1$  (for ease of exposition, we work over integers instead of bit



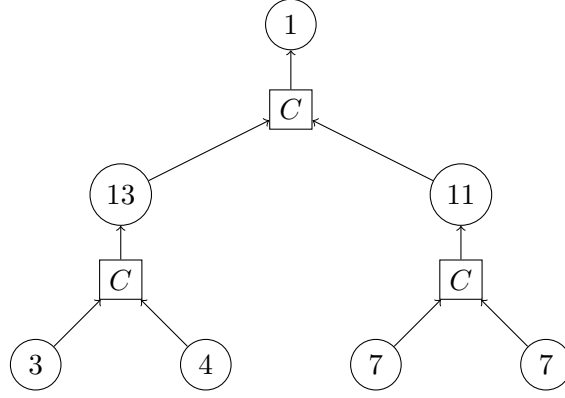


Figure 4.1: Merkle tree of Example 4.1.

strings, but the two can be readily converted into one another). Figure 4.1 shows the contents of the associated Merkle Tree. Note that the real message is not stored in the Merkle Tree, but only the ‘first level’ of hashes. The authentication path of the leaf labelled with 3 consists of the tuple  $(3, 4, 11)$ : by computing  $H(3, 4) = 13$  and then  $H(13, 11) = 1$  we can verify that the commitment is respected.

### 4.2.2 Augmented Binary Tree

The Merkle tree is the de-facto standard for blockchain applications, and basically for any scenario for which a ‘linear’ hash function cannot be used. In [Sta08], it was given a lower bound on the amount of queries necessary to obtain a collision for a  $(m + s)$ -to- $s$ -bit CHF  $H$  (the  $m$  is variable) built from a  $(n + c)$ -to- $n$ -bit OWCF  $C$ : if  $H$  makes  $r$  queries to  $C$ , it is possible to find a collision by making  $2^{\frac{nr+cr-m}{r+1}}$  queries to  $H$ . By combining this result with the  $2^{s/2}$  upper bound of the birthday paradox, one can immediately obtain a tight bound  $m = \frac{2nr+2cr-sr-s}{2}$  for the variable length  $m$  of the message.

**Definition 4.9** (Compactness [ABR21]). The *compactness* of an  $(m + s)$ -to- $s$ -bit hash function making  $r$  queries to an underlying  $(n + c)$ -to- $n$ -bit one-way compression function is the value  $\alpha = \frac{2m}{2nr+2cr-sr-s}$ .

**Example 4.2.** Consider a  $2n$ -to- $n$  bit OWCF and a Merkle Tree of height  $h$ : the computation of the tree is a  $(2^{h-1}n)$ -to- $n$ -bit hash function, and makes exactly  $r = 2^{h-1} - 1$  queries to  $C$ . We have  $s = c = n$  and  $m = 2^{h-1}n - n = nr$ , therefore the compactness of the Merkle Tree construction is:

$$\alpha = \frac{2m}{2nr + 2cr - sr - s} = \frac{2nr}{2nr + 2nr - nr - n} = \frac{2r}{3r - 1}$$

Which tends to  $2/3$  when  $r$  tends to infinity.

**Definition 4.10** (Augmented Binary tRee [ABR21]). An *Augmented Binary tRee* (ABR) of height  $h \in \mathbb{N}$  over a  $2n$ -to- $n$  compression function  $C$  is a complete binary tree of height  $h$  augmented with *middle* nodes such that, given a sequence of input messages  $S = (m_1, \dots, m_{2^{h-1}+2^{h-2}-1} \mid \forall i: m_i \in \{0, 1\}^*)$ , it produces an output digest  $d \in \{0, 1\}^n$  in the following way:

1. The leaf nodes  $\nu_1, \dots, \nu_{2^{h-1}}$  contain  $C(m_1), \dots, C(m_{2^{h-1}})$ .
2. There are no middle nodes in the leaf layer.

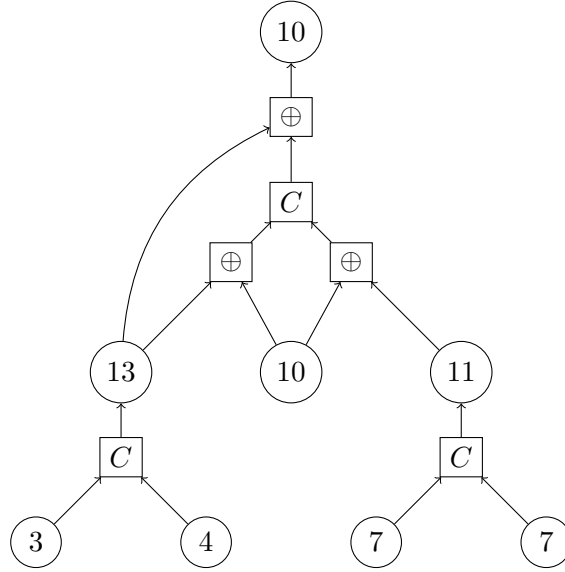


Figure 4.2: ABR of Example 4.4.

3. The middle nodes  $\nu_{2^{h-1}+1}, \dots, \nu_{|S|}$  contain  $C(m_{2^{h-1}+1}), \dots, C(m_{|S|})$ .
4. Every other node  $\nu$  contains  $C(\nu_l \oplus \nu_m, \nu_r \oplus \nu_m) \oplus \nu_r$ , where  $\nu_l$  is the left child of  $\nu$ ,  $\nu_r$  is the right child of  $\nu$ , and  $\nu_m$  is the middle child of  $\nu$ , or 0 if  $\nu$  doesn't have a middle child.

The authentication path of the ABR is similar to the one of the Merkle tree, but also includes the middle nodes encountered during the traversal.

**Proposition 4.6** (Security of ABR mode of hash [ABR21]). *Given a one-way  $2n$ -to- $n$  compression function  $C$ , the ABR over  $C$  is a cryptographic hash function.*

An ABR of height  $h$  can process 50% more messages than a Merkle Tree of the same height, while performing the same number of queries to the underlying compression function, with the additional cost introduced by the intermediate  $\oplus$  operations being negligible in most scenarios.

**Example 4.3.** Consider a  $2n$ -to- $n$  bit OWCF and an ABR of height  $h$ : the computation of the tree is a  $(2^{h-1} + 2^{h-2} - 1)n$ -to- $n$ -bit hash function, and makes exactly  $r = 2^{h-1} - 1$  queries to  $C$ . Like in Example 4.2, we have  $s = c = n$ , but this time  $m = (2^{h-1} + 2^{h-2} - 1)n - n = nr + nr/2 - n$ , so the compactness of the ABR construction is:

$$\alpha = \frac{2m}{2nr + 2cr - sr - s} = \frac{2nr + nr - 2n}{2nr + 2nr - nr - n} = \frac{3r - 2}{3r - 1}$$

Which approaches 1 as  $r$  approaches infinity, meaning that the ABR construction achieves optimal compactness.

It is worth of notice that, while the ABR hash mode achieves collision resistance, it does not achieve *indifferentiability* (a weaker notion of indistinguishability between Turing machines [MRH03]), hence a modified construction, called ABR+, was also proposed, although it does not achieve perfect compactness.

**Example 4.4.** Consider the same compression function  $C$  of Example 4.1, and consider the sequence of pre-hashed messages  $S' = (3, 4, 7, 7, 10)$ , in this case we interpret  $x \oplus y \equiv (x + y \bmod 13) + 1$ . Figure 4.2 shows the resulting ABR. The authentication path of the node labelled with 3 consists of the tuple  $(3, 4, 10, 11)$ : by computing  $H(3, 4) = 13$  and then  $H(13 \oplus 10, 11 \oplus 10) \oplus 13 = 10$  we can verify that the commitment is respected.

### 4.3 ZK-SNARK systems

As we saw in Section 3.4.1, researchers were able to construct ZK-NARK systems whose verification complexity was linear in the size of the problem instance, which is provided as a boolean circuit. Furthermore, in the CRS model, by using a block cipher, it is also possible to have *publicly verifiable* constructions [LS91], meaning that *any* verifier, not just the one that engages the protocol, is able to check the proof, which is encrypted with a *proving key*, by using a public *verification key*.

**Proposition 4.7** (Fiat-Shamir heuristic [FS87]). *Suppose a probabilistic I/O TM  $\mathcal{P}$  with access to a CHF  $H$  wants to prove its knowledge of the discrete logarithm  $x = \log(y)$  for some value  $y \in \mathbb{Z}_p$ , where  $p$  is a large prime number. Then  $\mathcal{P}$  can sample a random value  $v$  from  $\mathfrak{B}$ , compute the digest  $d = H(p, y, p^v)$ , the result  $r = v - dx \bmod (p - 1)$ , and finally output the quadruple  $(p, y, p^v, r)$ . Any PTIME TM  $\mathcal{V}$  with access to  $(p, y, p^v, r)$  and  $H$  can recompute  $d$  and check whether  $p^v = p^r y^d$  (If  $\mathcal{P}$  is not cheating, then  $p^r y^d = p^{v-dx} (p^x)^d = p^{v-dx} p^{dx} = p^{v-dx+dx} = p^v$ ). Assuming that the discrete logarithm is hard and that true CHF exist, if equality holds  $\mathcal{V}$  is convinced that  $\mathcal{P}$  knows  $x$  but is not able to retrieve it except with negligible probability.*

**Definition 4.11** (Succinct proof). A *succinct proof* for a statement  $\sigma$  over a language  $L \subseteq \{0, 1\}^*$  is a proof  $\pi$  such that  $|\pi| = \mathcal{O}(\log(|\sigma|))$ .

Similarly, one can define the notion of succinct argument of knowledge, and in particular, a succinct ZK-NARK system is called a ZK-SNARK system.

**Definition 4.12** (Probabilistically checkable proof system [BFLS91, FGL<sup>+</sup>91]). A *probabilistically checkable proof system* (PCP system) is an interactive proof system  $(\mathcal{P}, \mathcal{V})$  such that for any proof  $\pi$  provided by  $\mathcal{P}$ :  $\exists k: \text{TIME}(\mathcal{V}) = \mathcal{O}(\log^k(|\pi|))$ .

In a PCP system, the prover  $\mathcal{P}$  constructs a proof  $\pi$  of size polynomial in the length of the original statement  $\sigma$ ; since the verifier  $\mathcal{V}$  is polylogarithmically bound to the size of the proof, it can only query a small portion of it, however, it is enough to get statistical completeness and soundness.

In [Kil92], the author uses Merkle trees to have the prover commit to a proof  $\pi$ , (the bits of  $\pi$  are the leaves and the root, which has constant size, is sent to the verifier). The verifier then queries a certain number of authentication paths, which have length  $\mathcal{O}(\log(|\pi|))$ , and decides whether to accept or reject. In this sense, the protocol is therefore succinct. In [Mic00], the construction was extended and, by applying the Fiat-Shamir heuristic, it is possible to make the protocol non-interactive.

One of the first *succinct* ZK-NARK (ZK-SNARK) systems that didn't make explicit use of PCPs was devised in [Gro10], but had one important drawback: while the size of the proof is constant, the size of the CRS, and the computation that the prover has to perform is *quadratic* in the size of the input circuit (this bound was slightly improved in [Lip11]).

However, by first transforming the circuit into *quadratic span programs* (QSPs), the boolean equivalent of QAPs (Section 2.2.3), it was shown that it is possible to reduce both the size of the CRS and the prover’s computational complexity to linear, while still having succinct proofs [GGPR12]. Since all these constructions make use of encryption based on the hardness of finding the discrete logarithm of a number over a big finite field, dealing with boolean circuits and QSPs is not very efficient; although both polynomially sized boolean and arithmetic circuits are equivalent to PTIME Turing machines [PF79], working over arithmetic programs, and hence using QAPs over QSPs, can greatly reduce the constant factors involved in such constructions, although this depends on the kind of input problem (numeric problems can exploit arithmetic circuits much better than, say, propositional problems).

### 4.3.1 Pinocchio

An important application of ZK-SNARK systems is in *verifiable computation*. Consider a client (say, a mobile phone) that wants to delegate to a server (say, a cloud provider) some computation, for which several inputs are required: some are provided by the client, and some by the server:

- The client does not trust the server, so we would need a proof system, but since the server is not computationally unbounded, an *argument of knowledge* system will suffice.
- The server might have to provide the same computation to different clients, hence the system must be *non-interactive*.
- Verifying the computation must be cheaper than performing it, otherwise the client wouldn’t have to ask the server in the first place, the system must provide *succinct* proofs.
- The server is also interested in proving to the client that the computation was correct, for example to avoid legal liability, but it is not willing to share its own inputs, so our system must be *zero-knowledge*.

Clearly, among the various systems we saw up to now, ZK-SNARKs are the only ones that can reasonably fulfill all the requirements above. However, all the constructions we saw, due to the high overheads involved in generating the CRS, building the QSP/QAP, generating the proof, and even verifying it, were (much) slower than native execution by the client.

The first construction that was efficient enough to be usable in practice, and that in many cases broke the ‘native execution’ barrier for verification time, was *Pinocchio* [PGHR13].

**Definition 4.13** (Bilinear group [BF01]). A *bilinear group* is a group  $\mathbb{G}$  and  $\mathbb{G}'$  such that there is bilinear map (w.r.t. exponentiation)  $B: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}'$  such that  $\forall a, b \in \mathbb{Z}: B(x^a, y^b) = B(x, y)^{ab}$ .

The definition of maps linear w.r.t. exponentiation is a useful trick to avoid the need to introduce the notion of *elliptic curve* [Sil09], which is out of the scope of this thesis.

**Example 4.5.** Recall Example 2.6. Consider a group  $\mathbb{G} = \langle g \rangle$ , and define  $B: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{Z}_{|\mathbb{G}|}$  such that  $B(x, y) = g^{\log(x) \log(y)}$ . The map  $B$  is bilinear w.r.t. exponentiation, since:

$$\forall a, b \in \mathbb{Z}: B(x^a, y^b) = g^{\log(x^a) \log(y^b)} = g^{a \log(x) b \log(y)} = \left( g^{\log(x) \log(y)} \right)^{ab} = B(x, y)^{ab}$$

The two main ingredients of Pinocchio are QAPs and bilinear maps. Given in input an arithmetic formula  $\varphi$  over some field  $\mathbb{F}$  such that  $\mathbb{F}_\otimes \cong \langle g \rangle$ , equipped with a bilinear map  $B: \langle g \rangle \times \langle g \rangle \rightarrow \mathbb{F}$  defined as in Example 4.5, the Pinocchio protocol is organized in three phases: the *setup phase*, the *prover phase*, and the *verifier phase*.

### Setup phase

In the setup phase, any of the parties derives the explicit formula  $\widehat{\varphi}$ , the circuit  $\mathcal{G}$  of  $\widehat{\varphi}$ , the R1CS  $\mathcal{C}$  of  $\mathcal{G}$  and finally the  $m/n$  QAP  $\mathcal{Q} = (t, \mathbf{v}, \mathbf{w}, \mathbf{y})$  of  $\mathcal{C}$ . Recall that  $m = |\mathbf{v}| = |\mathbf{w}| = |\mathbf{y}| = |\mathcal{G}_\otimes|$ <sup>1</sup>, that  $n = \deg(t) = m+1+|\mathcal{G}_{in}|$ , and that  $\forall i: \deg(\mathbf{v}_i) = \deg(\mathbf{w}_i) = \deg(\mathbf{y}_i) = n-1$ . The QAP (and eventually all the other components) are made public.

Either via a trusted third party or an ensemble of authorities [GO06], we generate a CRS  $\bar{\sigma}$ , which has length  $|\bar{\sigma}| = 8\lceil \log(|\mathbb{F}|) \rceil$  and interpret every  $\lceil \log(|\mathbb{F}|) \rceil$ -bit subsequence as elements of  $\mathbb{F}$ :

$$\bar{\sigma} = (r_v, r_w, s, \alpha_v, \alpha_w, \alpha_y, \beta, \gamma)$$

Finally, after fixing  $r_y = r_v r_w$ , we build the *proving key* and the *verification key*:

$$\begin{aligned} \mathbf{K}_\mathcal{P} &= \left( \begin{array}{ccc} \{g^{r_v \mathbf{v}_j(s)}\}, & \{g^{r_w \mathbf{w}_j(s)}\}, & \{g^{r_y \mathbf{y}_j(s)}\}, \\ \{g^{r_v \alpha_v \mathbf{v}_j(s)}\}, & \{g^{r_w \alpha_w \mathbf{w}_j(s)}\}, & \{g^{r_y \alpha_y \mathbf{y}_j(s)}\}, \\ \{g^{s^k}\}, & \{g^{r_v \beta \mathbf{v}_j(s)} g^{r_w \beta \mathbf{w}_j(s)} g^{r_y \beta \mathbf{y}_j(s)}\} & \end{array} \right) \\ \mathbf{K}_\mathcal{V} &= (g, g^{\alpha_v}, g^{\alpha_w}, g^{\alpha_y}, g^\gamma, g^{\beta \gamma}, g^{r_y t(s)}, \{g^{r_v \mathbf{v}_i(s)}\}, \{g^{r_w \mathbf{w}_i(s)}\}, \{g^{r_y \mathbf{y}_i(s)}\}) \end{aligned}$$

where  $i$  ranges over  $\llbracket 0, |\mathcal{G}_{in}| \rrbracket$ ,  $j$  ranges over  $\llbracket |\mathcal{G}_{in}|, n-1 \rrbracket$ , and  $k$  ranges over  $\llbracket 1, n \rrbracket$ .

It is important to remark that  $\bar{\sigma}$  must be deleted immediately after generating the two keys, as it can be exploited to tamper the protocol (in jargon, we say that it is *toxic waste*).

### Prover phase

After fixing the input vector  $\mathbf{x}$ , the prover creates a circuit input for  $\mathcal{G}$ , computes the induced evaluation and extracts the assignment  $\mathcal{A}$ . From  $\mathcal{A}$ , the prover derives the associated solution  $\mathbf{c}$  of  $\mathcal{C}$ , and computes the polynomials  $p = (\mathbf{v}\mathbf{c})(\mathbf{w}\mathbf{c}) - (\mathbf{y}\mathbf{c})$  and  $h = p/t$ . Now, to build a short proof  $\pi$ , the prover should sum the contributions of all the left input, right input, and output polynomials:

$$v_\otimes = \sum_{i=|\mathcal{G}_{in}|}^{n-1} \mathbf{c}_i \mathbf{v}_i(s) \quad w_\otimes = \sum_{i=|\mathcal{G}_{in}|}^{n-1} \mathbf{c}_i \mathbf{w}_i(s) \quad y_\otimes = \sum_{i=|\mathcal{G}_{in}|}^{n-1} \mathbf{c}_i \mathbf{y}_i(s)$$

and obtain the proof:

$$\pi = (g^{r_v v_\otimes}, g^{r_w w_\otimes}, g^{r_y y_\otimes}, g^{h(s)}, g^{r_v \alpha_v v_\otimes}, g^{r_w \alpha_w w_\otimes}, g^{r_y \alpha_y y_\otimes}, g^{r_v \beta v_\otimes} g^{r_w \beta w_\otimes} g^{r_y \beta y_\otimes})$$

<sup>1</sup>As we will see, it is possible to reduce this number in some special cases by exploiting the structure of R1CS constraints. Also note that the original protocol as in [PGHR13] does not make explicit use of R1CS, but we want to stress its importance as it is the standard way to represent arithmetic circuits in `libsnark` [BSCTV13].

Although  $r_v$ ,  $r_w$  or  $r_y$  have been deleted, and finding the discrete logarithm is hard, by using  $K_P$ , it is possible to compute:

$$g^{r_v v_\otimes} = \prod_{i=|\mathcal{G}_{in}|}^{n-1} \left(g^{r_v v_i(s)}\right)^{c_i} \quad g^{r_w w_\otimes} = \prod_{i=|\mathcal{G}_{in}|}^{n-1} \left(g^{r_w w_i(s)}\right)^{c_i} \quad g^{r_y y_\otimes} = \prod_{i=|\mathcal{G}_{in}|}^{n-1} \left(g^{r_y y_i(s)}\right)^{c_i} \quad (4.1)$$

and similarly for the other values (except for  $g^{h(s)}$  which can be computed directly). Finally, the prover publishes the pair  $(\phi(\mathbf{x}), \pi)$ .

### Verifier phase

At this point, any potential verifier with access to  $K_V$  and to the bilinear map  $B$  can check the alleged proof. First, the verifier should check whether  $p = ht$ , which, as we noted in Section 2.2.3, it is statistically equivalent to checking whether  $p(s) = h(s)t(s)$ . In the spirit of Equation (4.1), using  $K_V$ , the verifier can compute the input/output contributions  $g^{r_v v_{I/O}}$ ,  $g^{r_w w_{I/O}}$  and  $g^{r_y y_{I/O}}$ . Clearly, for any arbitrary  $k \in \mathbb{F} \setminus \{0\}$ , we have that  $p(s) = h(s)t(s) \iff k^{p(s)} = k^{h(s)t(s)}$  (we “work in the exponent”). If we fix  $k = B(g^{r_v}, g^{r_w})$ , with some algebraic effort the previous equation can be transformed into the equivalent one:

$$B(g^{r_v v_{I/O}} g^{r_v v_\otimes}, g^{r_w w_{I/O}} g^{r_w w_\otimes}) = B(g^{r_y t(s)}, g^{h(s)}) B(g^{r_y y_{I/O}} g^{r_y y_\otimes}, g)$$

Even if the divisibility check passes, it might still be the case that the alleged proof was not built using the polynomials in  $\mathcal{Q}$ . To address this eventuality, the verifier checks whether:

$$B(g^{r_v \alpha_v v_\otimes}, g) = B(g^{r_v v_\otimes}, g^{\alpha_v}) \quad B(g^{r_w \alpha_w w_\otimes}, g) = B(g^{r_w w_\otimes}, g^{\alpha_w}) \quad B(g^{r_y \alpha_y y_\otimes}, g) = B(g^{r_y y_\otimes}, g^{\alpha_y})$$

Finally, the prover might have used the correct polynomials, but didn’t use the same coefficients  $c_i$  when building  $v_\otimes$ ,  $w_\otimes$  and  $y_\otimes$ . This last concern is resolved by checking whether:

$$B(g^{r_v \beta v_\otimes} g^{r_w \beta w_\otimes} g^{r_y \beta y_\otimes}, g^\gamma) = B(g^{r_v v_\otimes} g^{r_w w_\otimes} g^{r_y y_\otimes}, g^{\beta \gamma})$$

The protocol as described is not zero-knowledge, but it is quite simple to make it so: the verification key is extended to include  $g^{r_v \alpha_v t(s)}$ ,  $g^{r_v \beta v t(s)}$ , and then the prover generates a random value  $\delta_v$  and replaces each polynomial  $v_i$  with  $v_i + \delta_v t$ . The same thing is done for  $w$  and  $y$ : the validity of the checks performed by the verifier is not affected by this change, but it can be shown that the scheme is now statistically zero-knowledge.

It is immediate to see that the proving key  $K_P$  contains  $8|\mathcal{G}_\otimes| + |\mathcal{G}_{in}| + 1$  field elements, the verification key  $K_V$  contains  $3|\mathcal{G}_{in}| + 7$  field elements (6 more in the zero-knowledge setting), and the proof  $\pi$  contains 8 field elements, meaning that its size is independent of the input circuit.

## 4.4 libsnark

In the last ten years, new improvements were put forward to reduce the size of the messages and the complexity of the computation, especially on the prover’s side [Lip13] (for example, in [Gro16] the

size of the proof was reduced to just 3 field elements). Furthermore, much effort has been put into making working implementations of ZK-SNARK systems, such as `libsnark`<sup>2</sup>, a C++ [Str13] library which implements and refines several ZK-SNARK protocols [DFGK14, GM17, BBFR14, BSCG<sup>+</sup>13], although the core component (pre-processing ZK-SNARK, or PPZK-SNARK) is based mostly on the Pinocchio protocol and on [BSCTV13, Gro16], which are all extensions and improvements of the QAP (and QSP) model of [GGPR12].

In order to implement some function  $f$  which can be expressed as an arithmetic expression  $\varphi$  (i.e. no variable-length loops or recursion) in `libsnark`, we must provide both the arithmetic circuit and the associated R1CS. In many cases, deriving the R1CS from the arithmetic circuit is quite trivial, but there are instances where having them separate allows for some quite nice optimizations. Although there has been work on *compilers* that translate high-level code to R1CS constraints [ET18, BMBDMT22], as it is often the case, this comes at a cost of flexibility. Intuitively, we can divide the usage of `libsnark` in same three phases of the Pinocchio system (or really any other SNARK).

### Preprocessing phase

The first thing to do in `libsnark` is choosing, at compile time (or *a priori*, in a theoretical interpretation), which bilinear group to use for the protocol. To obtain bilinear groups that are paired to an underlying prime field, the typical construction makes use of elliptic curves: for example, the curve/-group BN254 [BN05], which is the standard curve used `libsnark`, or the curve BLS12<sup>3</sup> [BLS01], which will also be of interest to us.

After choosing the group, we set up the *protoboard* which, as the name suggests, it is the object where one places the components of the circuit:

```
libsnark::protoboard<Field> board;
board.set_input_sizes(PUBLIC_N);
```

The template argument `Field` specifies the underlying field, and `PUBLIC_N` specifies the number of output (i.e. public) variables in the circuit. On the protoboard, we allocate *variables* that will carry the input/intermediate/output values of the circuit evaluation, together with an annotation (for debug). However, it is usually much more convenient to only declare the input and output variables, and delegate the wiring to *gadgets* which act as composable black-boxes:

```
libsnark::pb_variable<Field> output_var;
libsnark::pb_variable<Field> input_var;

output_var.allocate(board, "out");
input_var.allocate(board, "in");
FooGadget gadget{board, input_var, output_var};
```

Note that the first `PUBLIC_N` variables allocated will be considered public, while the remaining ones will be private. A typical gadget must provide two methods: one to generate the R1CS constraints, and one to compute the circuit evaluation given a circuit input, which will be used in the proving phase. First, we generate the constraints:

<sup>2</sup><https://github.com/scipr-lab/libsnark>, you can also find a nice empirical comparison of the various protocols.

<sup>3</sup><https://electriccoin.co/blog/new-snark-curve/>

```
gadget.generate_r1cs_constraints();
```

Internally, `gadget` allocates the required intermediate variables and constrains their linear combinations. A linear combination can be expressed quite naturally:

```
libsark::pb_linear_combination<Field> lc = a1*x1 + ... + an*xn;
```

An R1CS constraint of the type  $ab = c$  is expressed as:

```
libsark::r1cs_constraint<Field> constraint{lc_a, lc_b, lc_c};
board.add_r1cs_constraint(constraint);
```

Once all the constraints have been specified, the last thing to do is to convert the R1CS to a QAP and get the keys  $K_P$  and  $K_V$ . The whole process is done transparently by the library:

```
auto keypair = libsark::r1cs_ppzksnark_generator(board.get_constraint_system());
```

Now the constraint system and  $K_V$  can be made public, while  $K_P$  is only known by the prover.

## Proving phase

To generate a (valid) proof, the prover has to first provide a circuit input and compute the induced evaluation. At the highest level, this is done by setting the values of the input variables and letting the gadget generate the intermediate and the output values:

```
board.val(input_var) = x; // x is some field element
gadget.generate_r1cs_witness(circuit_inputs);
```

Now, he can generate the proof, which is done transparently by the library by using the proving key and the primary (i.e. public) and auxiliary (i.e. private) inputs:

```
auto proof = libsark::r1cs_ppzksnark_prover(keypair.pk, board.primary_input(),
board.auxiliary_input());
```

Finally, the prover can make public the primary inputs of the protoboard together with the proof.

## Verification phase

At this point, the verifier has to check the proof by using the public key, and there are two kinds of choices regarding how to perform such verification:

- *Weak* vs. *strong* verification: in the former case, it is ok for the prover to only provide some primary inputs, which will be zero-padded. In the latter case, this is not considered acceptable.
- *Offline* vs. *online* verification: in the former case, the verification key is used ‘as is’, in the latter, it is processed to get faster verification times when used multiple times.

In any case, the whole thing is managed transparently by the library. For example, if we choose strong, online verification:

```
auto proc_vk = libsark::r1cs_ppzksnark_verifier_process_vk(keypair.vk);
bool halt = libsark::r1cs_ppzksnark_online_verifier_strong_IC(proc_vk, board.
primary_input(), proof);
```



Listing 4.1: The libsnark gadget corresponding to the R1CS in Example 2.10.

```

template<typename FieldT>
class FooGadget : public gadget<FieldT>
{
    const pb_variable<FieldT> x1, x2, y;
    pb_variable<FieldT> t1, t2, t3;

    FooGadget(protoboard<FieldT> &board,
               const std::string &ann,
               const pb_variable<FieldT> &x1,
               const pb_variable<FieldT> &x2,
               const pb_variable<FieldT> &y) :
        gadget<FieldT>{board, ann}, x1{x1}, x2{x2}, y{y}
    {
        t1.allocate(board, "t1");
        t2.allocate(board, "t2");
        t3.allocate(board, "t3");
    }

    void generate_r1cs_constraints()
    {
        board.add_r1cs_constraint(r1cs_constraint<FieldT>{x1, x1, t1});
        board.add_r1cs_constraint(r1cs_constraint<FieldT>{t1, x1, 8 + 9*x2 + t2});
        board.add_r1cs_constraint(r1cs_constraint<FieldT>{x2, t2, y});
    }

    void generate_r1cs_witness()
    {
        board.val(t1) = board.val(x1) * board.val(x1);
        board.val(t2) = board.val(t1) * board.val(x1) + 4*board.val(x2) + 5;
        board.val(y)  = board.val(x2) * board.val(t2);
    }
};

```

The verifier will accept the proof if `halt` is `true`, while it will reject if it is `false`.

**Example 4.6.** Consider the R1CS in Example 2.10. Listing 4.1 shows the implementation of the corresponding gadget in libsnark.



# Cryptographic Primitives from Generalized Triangular Dynamical Systems

One of the most important applications of zero-knowledge verifiable computation lies in digital currency transactions over the blockchain infrastructure. An example of ZK-SNARK applied in the real world is the ZCash cryptocurrency [BSCG<sup>+</sup>14], which is inspired by the more famous Bitcoin [NBF<sup>+</sup>16], and was devised by the authors of `libsnark` (which frames the zero-knowledge backend of the currency).

As we discussed in Section 4.2, the fundamental component of a blockchain is the Merkle tree, which uses one-way compression functions in order to produce the binding commitment. In a digital currency scenario, the leaves of the Merkle tree consist of the details of some transaction, typical information include the ID of the sender, the ID of the recipient, and the amount of currency to be transferred. Without a zero-knowledge framework in place, when one wants to verify whether a user did abide to their commitment, the only possible solution is to ask the user to disclose his transaction, together with the authentication path, and check that the tree commitment is respected. When using currencies like Bitcoin or Ethereum<sup>1</sup>, anyone can see the details of every single transaction being performed on the relative blockchain, meaning that there is no privacy whatsoever<sup>2</sup>. However, if we translate the Merkle tree computation in an equivalent circuit, it is possible to apply a zero-knowledge scheme that allows a verifier to be sure (with overwhelming probability) of the validity of a transaction without actually having to see it! Since a Merkle tree applies over and over the underlying compression function, the problem of creating a circuit for the former immediately reduces to the problem of creating a circuit for the latter.

In Section 5.1 we will review the evolution of the state of the art concerning zero-knowledge friendly compression functions. Then, in Section 5.2, we present a new algebraic framework to represent cryptographic primitives, the *Generalized Triangular Dynamic System*, and apply it to construct the Arion block cipher and the ArionHash hash function. Finally, in Section 5.3, we compare our new construction to the state of the art using the `libsnark` library, showing extremely competitive results.

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<sup>1</sup><https://ethereum.org/>

<sup>2</sup>For example, on <https://etherscan.io/> you can see the transactions on the Ethereum blockchain.

## 5.1 State of the art

The standard compression function used in Merkle trees is usually one of the SHA-2 or SHA-3 functions [Dan15]: this is certainly the most sensible choice in a *native* environment (i.e. no zero-knowledge), as SHA is specifically designed to be fast in both software and hardware [DMO04, MAK<sup>+</sup>12] implementations, and is the most studied hash function from a security standpoint (e.g. for SHA-2 see [KRS12, GLRW10, DEM16]).

However, when working with arithmetic circuits over a prime field  $\mathbb{F}_p$ , using SHA poses a fundamental problem: the underlying operations are bit-based: e.g. bitwise XOR, bitwise AND, bit shifts/rotations, additions modulo  $2^{32}$ . . . . While shifts and rotations come at no cost, as they basically consist of renaming the circuit wires/variables, bitwise operations like XOR and modular addition have to be simulated bit-by-bit, and the overhead introduced in such a translation is huge. For example, for SHA-256, over a bilinear group like BN254 for which  $|\mathbb{F}_p| \approx 2^{256}$ , we would need 256 input variables each holding a 256-bit integer to simulate the behaviour of every single bit during the SHA computation; clearly, this is decidedly suboptimal.

**Example 5.1.** Suppose we are given two strings  $a, b \in \{0, 1\}^n$ , and we want to compute  $a \text{ XOR } b$ . By interpreting them as vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{F}_p^n$ , (we interpret a string  $\sigma$  as a vector  $\mathbf{x}$  by setting  $\mathbf{x}_i = 0$  when  $\sigma(i) = 0$  and  $\mathbf{x}_i = 1$  when  $\sigma(i) = 1$ ), we can simulate bitwise XOR by computing,  $\forall i \leq n$ :

$$\mathbf{v}_i \text{ XOR } \mathbf{w}_i = \mathbf{v}_i + \mathbf{w}_i - 2\mathbf{v}_i\mathbf{w}_i$$

that is, every XOR operation requires one multiplication gate. Similarly, bitwise AND and non-native addition also require multiplications to be simulated. Furthermore, we must guarantee that the values  $\mathbf{v}_i$  and  $\mathbf{w}_i$  are boolean, as in principle they could assume any value in  $\mathbb{F}_p$ , so we must also add constraints of the kind  $\mathbf{v}_i(\mathbf{v}_i - 1) = 0$ .

### 5.1.1 MiMC

In an effort to find secure cryptographic designs that could be efficient in zero-knowledge settings, called *zk-friendly* designs, researchers began to study the properties of permutations that make use of a low number of multiplications (*multiplicative complexity*) [ARS<sup>+</sup>16].

One of the first constructions over prime fields was the *Minimal Multiplicative Complexity* (MiMC) family of cryptographic permutations [AGR<sup>+</sup>16]. The idea of MiMC, reprising an older proposal [NK95], is to use a very simple polynomial permutation as its core component, and by repeating it for an adequate number of rounds, obtain a secure construction.

**Definition 5.1** (MiMC keyed permutation [AGR<sup>+</sup>16]). Given a finite field  $\mathbb{F}_p$ , a number of rounds  $r \geq \left\lceil \frac{\log(p)}{\log(3)} \right\rceil$ , some constants  $c_1, \dots, c_r \in \mathbb{F}_p$  and a set of functions  $f_1, \dots, f_r: \mathbb{F}_p \times \mathbb{F}_p \rightarrow \mathbb{F}_p$  such that  $\forall i \leq r: f_i(x, k) = x^3 + k + c_i$ , the MiMC *keyed permutation* is defined as:

$$E_{\text{MiMC}}(x, k): \mathbb{F}_p \times \mathbb{F}_p \rightarrow \mathbb{F}_p = (f_r \circ \dots \circ f_1)(x, k) + k$$

The MiMC keyed permutation is also called MiMC- $n/n$ . By applying the Feistel construction on

the MiMC permutation, one obtains Feistel MiMC, or MiMC- $2n/n$ . Finally, by applying the sponge construction, one can obtain the MiMCHash hash function. In alternative, it is also possible to build a hash function using first the Davies-Meyer construction to obtain a one-way compression function, and then the Merkle-Damgård construction to obtain a hash function.

There are some important observations to be made on the MiMC construction. First, the round permutation uses a low degree polynomial, but it is repeated for a high number of rounds: for example, if the size of the underlying field is  $\approx 2^{256}$ , the number of rounds will be  $r = 162$ . Note that  $x^3$  might not actually induce a permutation over  $\mathbb{F}_p$ , as in general 3 is not coprime with  $p - 1$  (in fact, in the underlying fields of both BN254 and BLS12, 3 is a factor of  $p - 1$ ). In such cases, one should modify the definition to consider the smallest number  $d$  such that  $\gcd(d, p - 1) = 1$ , and change the number of rounds to  $r \geq \left\lceil \frac{\log(p)}{\log(d)} \right\rceil$ .

A second observation is that  $r$  must be chosen to thwart many types of cryptanalysis techniques: since the MiMC permutation corresponds to the polynomial  $P = (\dots (x^3 + k + c_1)^3 + \dots)$  (which has degree  $\deg(P) = 3^r$ ), in addition to the traditional *brute-force*, *meet-in-the-middle* [DH77], *differential* [BS91] and *linear* [Mat94] attacks, one must also consider *algebraic attacks*, which exploit the inherent nature of this type of constructions.

In fact, traditional attacks don't tend to pose a major threat to these kinds of constructions: brute force is clearly too expensive and meet-in-the-middle is also infeasible both due to the high number of rounds and to the huge degree of the inverse permutation (usually  $1/3 \gg 3$ ). The permutation  $x^3$  is not approximable by a linear function [AABL12], hence linear attacks are not a threat, and since it can be easily shown that any arbitrary input difference  $\delta_{in}$  propagates to any arbitrary output difference  $\delta_{out}$  with a probability of at most  $2/2^n$ , differential attacks are also ineffective [Nyb94].

On the side of algebraic cryptanalysis, one might attempt an *interpolation attack*, which uses Lagrange interpolation to find a polynomial  $\tilde{P}$  which behaves like a keyless version of  $P$  [JK97]. This attack's complexity depends solely on  $\deg(P)$  (in fact, an interpolation can be computed in  $\mathcal{O}(n \log(n))$ , where  $n = \deg(P)$  [Sto85]), hence we must be sure that the degree of  $P$  also grows exponentially round by round (as it is the case). Another kind of algebraic attack is the *GCD attack*: by using two plaintext/ciphertext pairs, one can compute their greatest common divisor which will allow to easily retrieve the secret key. Again, computing the GCD depends almost linearly on the degree of the polynomial, hence one must again be sure that the degree grows exponentially.

## MiMC vs. SHA

Performing computations over large prime fields is extremely expensive, as addition and multiplication cannot be performed by a single CPU instruction, but must be emulated (multiplication in particular is extremely costly, even with clever implementations like the Montgomery form [Mon85]): for example, on an x86 architecture, a standard software implementation of SHA-256 is more than 100 times faster than MiMC over the BN254 curve (implemented using `libff`<sup>3</sup>, the underlying arithmetic library of `libsbnark`) to compress a 512 bit input to a 256 bit output. However, when translated into an arithmetic circuit, SHA-256 requires about 25000 multiplications, while MiMC requires only 640, making it about 40 times (furthermore, designing and optimizing the SHA-256 circuit is much harder than for MiMC).

<sup>3</sup><https://github.com/scipr-lab/libff>

### 5.1.2 Poseidon

After the design of MiMC, a natural question that followed was whether it was possible to increase the complexity of the round function and reduce the number of rounds without compromising on security. In particular, researchers started exploring the design of *algebraic frameworks* that could abstract from the details involved in the construction of a specific primitive, by still providing security guarantees for their instantiations.

In [GLR<sup>+</sup>19], the authors present the HADES design, a generalization of the *substitution-permutation network* (SPN) approach used in many famous cryptographic primitives such as AES [DR99]. The HADES design has the same three main steps of the classic SPN, where the message and the key are interpreted as vectors in  $\mathbb{F}_p^t$  for some prime field  $\mathbb{F}_p$  and some arbitrary  $t \in \mathbb{N}$ :

1. *AddKey*: add the key to the message.
2. *SubWords*: apply a substitution function  $S: \mathbb{F}_p \rightarrow \mathbb{F}_p$  to the elements of the message (*non-linear layer*).
3. *MixLayer*: apply a permutation function  $M: \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t$  to the message (*linear layer*).

The function  $S$  is defined as  $S(x) = x^d$ , where  $d$  is the smallest integer such that  $\gcd(d, p-1) = 1$ , while the permutation function  $M$  is defined as  $M(\mathbf{x}) = \mathbf{M}\mathbf{x}$ , where  $\mathbf{M}$  is a *maximum distance separable* (MDS) matrix (i.e. the determinant of all its submatrices are non-zero, this for example minimizes the effectiveness of differential attacks [MS77, RDP<sup>+</sup>96]).

The peculiarity of HADES is that only the first and last few rounds apply the substitution function to all elements: the middle rounds are *partial*, meaning that  $S$  is only applied to some elements of the message. This allows to use the first and last part of the construction as an argument for dealing with classical attacks, while also considering the middle rounds when dealing with algebraic attacks. Additionally, in the design of HADES, new kind of algebraic attacks were considered [BCD<sup>+</sup>20], the most important of which is certainly the *Gröbner basis attack* (a generalization of the concept of Gaussian elimination [CLO15, Laz83]). In a Gröbner attack, we view the function under attack as a system of polynomial equations for which we want to find a solution. This kind of attack, and in particular its probabilistic version [FGHR14], is (at the moment of this writing) the most efficient known attack against algebraic constructions (note however that the security of HADES was only proven against the deterministic version).

A family of hash functions that builds on the HADES design is POSEIDON [GKR<sup>+</sup>21].

**Definition 5.2** (POSEIDON permutation). Given a finite field  $\mathbb{F}_p$ , a number of branches  $t \in \mathbb{N}$ , a number of full rounds  $r_f \in \mathbb{N}$ , a number of partial rounds  $r_P \in \mathbb{N}$ , an MDS matrix  $\mathbf{M} \in \mathbb{F}_p^{t \times t}$ , and some constants  $\mathbf{c}_1, \dots, \mathbf{c}_r \in \mathbb{F}_p^t$ , where  $r = 2r_f + r_P$ , let the *full SBOX function* be:

$$S_f(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \begin{pmatrix} x_1^d & \dots & x_t^d \end{pmatrix}^\top$$

and the *partial SBOX function* be:

$$S_P(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \begin{pmatrix} x_1^d & x_2 & \dots & x_t \end{pmatrix}^\top$$

where  $d$  is the smallest integer such that  $\gcd(x, p-1) = 1$ . Then, let the full  $i$ th round function be  $f_i(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \mathbf{MS}_f(\mathbf{x} + \mathbf{c}_i)$  and the partial  $i$ th round function be  $P_i(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \mathbf{MS}_P(\mathbf{x} + \mathbf{c}_i)$ . The POSEIDON *permutation* is the function:

$$P_{\text{POSEIDON}} = f_r \circ \cdots \circ f_{r_f+r_P+1} \circ P_{r_f+r_P} \circ \cdots \circ P_{r_f+1} \circ f_{r_f} \circ \cdots \circ f_1$$

The POSEIDON permutation can be easily used in a sponge construction to obtain a one-way compression function. For example, by setting  $t = 3$ , we immediately obtain a 2/1 compression function, or by setting  $t = 5$  we get a 4/1 compression function. To get the same level of security  $s$ , the POSEIDON- $s$  arithmetic circuit is much cheaper than MiMC in terms of multiplications. For reference, the POSEIDON-128 2/1 compression function requires 276 multiplications, compared to the 640 required by MiMC.

Furthermore, POSEIDON was designed with other systems in mind other than the traditional ZK-SNARK constructions, such as  $\mathcal{P}\text{lonK}$  [GWC19], Bulletproofs [BBB<sup>+</sup>17] and ZK-STARKs [BSBHR18] (*transparent* ZK-SNARKs, i.e. they do not require a trusted setup phase). For example, in the  $\mathcal{P}\text{lonK}$  system the cost of the ZK-SNARK construction depends also on the number of additions, while in ZK-STARKs it depends on the *depth* of the circuit. These systems, which are much more recent, do not have consolidated libraries which support them, and their analysis is out of the scope of this work.

### 5.1.3 Griffin

Many new designs were proposed in the last few years, such as CIMINION [DGGK21], which exploits *Toffoli gates* [Tof80] to improve the construction, and *Rescue* [AABS<sup>+</sup>19], which uses inverse exponentiations (i.e.  $x^{1/d}$ ) in the substitution layer.

An extremely recent design is GRIFFIN [GHR<sup>+</sup>22], which deviates from the established SPN approach by instead using an extension of the Feistel construction, called the *Horst scheme*: instead of the mapping  $(x, y) \mapsto (x, y \oplus F(x))$  for some function  $F$  over the underlying field  $\mathbb{F}_p$ , it uses the mapping  $(x, y) \mapsto (x, y \otimes G(x) \oplus F(x))$ , where  $G$  is an additional function such that  $\forall x \in \mathbb{F}_p: G(x) \neq 0$ , in order to maintain invertibility (note that if  $\forall x: G(x) = 1$ , the *Horst* scheme degenerates into the Feistel construction).

A quirk of GRIFFIN is the matrix used in the linear layer (which is basically the same of POSEIDON): it is a  $t \times t$  MDS circulant matrix, where  $t$  is the number of branches, but is well-defined only when  $t = 3$  or  $t \equiv 0 \pmod{4}$ .

**Definition 5.3** (Griffin permutation). Given a finite field  $\mathbb{F}_p$ , a number of branches  $t \in \mathbb{N}$  such that  $t = 3$  or  $t \equiv 0 \pmod{4}$ , a number of rounds  $r \in \mathbb{N}$ , some constants  $\mathbf{c}_1, \dots, \mathbf{c}_r \in \mathbb{F}_p^t$ , some pairwise-distinct constants  $\gamma_3, \dots, \gamma_t \in \mathbb{F}_p$ , and some constants  $\alpha_3, \beta_3, \dots, \alpha_t, \beta_t \in \mathbb{F}_p$  such that  $\forall i \in [3, t]: \alpha_i \neq \beta_i$  and  $\alpha_i^2 - 4\beta_i$  is a quadratic non-residue modulo  $p$ , let the linear functions  $L_3, \dots, L_t$  be such that:

$$L_i(x, y, z): \mathbb{F}_p^3 \rightarrow \mathbb{F}_p = \gamma_i x + y + z$$

Furthermore, let the *non-linear layer* be the function:

$$S(\mathbf{x})_i: \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \mathbf{y}_i = \begin{cases} x_1^{1/d} & i = 1 \\ x_2^d & i = 2 \\ x_i \left( L_i(\mathbf{y}_0, \mathbf{y}_1, 0)^2 + \alpha_i L_i(\mathbf{y}_0, \mathbf{y}_1, 0) + \beta_i \right) & i = 3 \\ x_i \left( L_i(\mathbf{y}_0, \mathbf{y}_1, \mathbf{x}_{i-1})^2 + \alpha_i L_i(\mathbf{y}_0, \mathbf{y}_1, \mathbf{x}_{i-1}) + \beta_i \right) & 4 \leq i \leq t \end{cases}$$

where  $d$  is the smallest prime number such that  $\gcd(x, p-1) = 1$ .

Finally, let the matrix  $\mathbf{M}_t \in \mathbb{F}_p^{t \times t}$  be:

$$\mathbf{M}_t = \begin{cases} \text{circ}(2, 1, 1) & t = 3 \\ \text{circ}(3, 2, 1, 1) & t = 4 \\ \text{diag} \left( \underbrace{2, \dots, 2}_t \right) \text{diag} \left( \underbrace{\mathbf{M}_4, \dots, \mathbf{M}_4}_s \right) \text{circ} \left( \underbrace{\mathbf{I}_4, \dots, \mathbf{I}_4}_s \right) & t = 4s \end{cases}$$

and the  $i$ th round function be  $f_i(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \mathbf{M}_t S(\mathbf{x}) + \mathbf{c}_i$ . The **GRIFFIN permutation** is the function:

$$P_{\text{GRIFFIN}} = f_r \circ \dots \circ f_1 \circ \mathbf{M}_t$$

The quadratic non-residuosity requirement for the pairs  $\alpha_i, \beta_i$  is a condition imposed in order to make sure that the non-linear layer  $S$  does not output zero from the third branch onwards, unless the input is also zero.

Another important observation about the non-linear layer is the usage of the inverse exponentiation. As we already observed when discussing MiMC, usually  $1/d \gg d$ , which means that the degree of the permutation grows much faster. While this also entails that computing  $x^{\frac{1}{d}}$  will be slower, from the constraint system point of view there is absolutely no difference: since  $y = x^{\frac{1}{d}} \iff x = y^d$ , one can simply write the constraints in reverse, introducing no extra cost in the R1CS and, in turn, in the QAP!

Similarly to POSEIDON, the GRIFFIN permutation can be very easily extended to a compression function due to the embedded sponge mechanism. The fact that  $t$  must be either 3 or a multiple of 4 is a bit unfortunate though, as usually one would like to use powers of 2 as input size of a compression function: except for 2-to-1, GRIFFIN does not allow any such instantiation, (when  $t = 4s$ , we have a  $(4s - 1)$ -to-1 compression function).

From a performance point of view, GRIFFIN is extremely competitive: for reference, the 2-to-1 instantiation only needs 96 R1CS constraints, compared to the 276 required by POSEIDON. In fact, although it is a very recent design, i.e. it is subject to changes, GRIFFIN will be our target competitor in terms of pure performance in ZK-SNARK settings.

## 5.2 Arion: A new ZK-friendly permutation

The constructions that we saw in Section 5.1 are prominent examples of different *generations* of *Arithmetization Oriented* (AO) designs. For example, MiMC is an example of a Gen-I design: its main purpose



was mostly to demonstrate that it was indeed possible to construct secure and efficient cryptographic primitives by stacking simple, low-degree round functions. On the other hand, the HADES framework, its derivative POSEIDON and other similar constructions like *Rescue* are examples of Gen-II designs: by tweaking the SPN and Feistel constructions, their purpose was to massively improve the efficiency over Gen-I designs. Finally, GRIFFIN is an example of a Gen-III design: the underlying *Horst* scheme is neither a “pure” Feistel nor SPN design and, by deviating from such standard constructions, its authors were able to improve the efficiency even further. Furthermore, in third generation designs it was shown that one does not necessarily need to use round functions with a low degree, as long as the resulting constraint system is not affected negatively<sup>4</sup>.

Other from GRIFFIN, there is another very recent Gen-III construction, called *Anemoui* [BBC<sup>+</sup>22] and based on the *Flystel* design, which has some common points with the *Horst* construction. A very interesting fact used in *Flystel* was the notion of CCZ-equivalence [CCZ98], a concept which generalizes the intuition behind the idea that there is “no difference” between, say, using  $x^d$  and  $x^{\frac{1}{d}}$ .

**Definition 5.4** (Affine function). An *affine function* over an  $n$ -dimensional vector space  $\mathbb{F}^n$  is a function  $f(x): \mathbb{F}^n \rightarrow \mathbb{F}^n = \mathbf{M}x + q$ , where  $\mathbf{M} \in \mathbb{F}^{n \times n}$  and  $q \in \mathbb{F}^n$ .

**Definition 5.5** (Function graph). Given a set  $S$ , the *function graph* of a function  $f: S \rightarrow S$  is the pair  $\Gamma = (S, E)$  where  $E = \{(x, f(x)) \mid x \in S\}$ .

**Definition 5.6** (Induced permutation). Given a *function graph*  $\Gamma = (S, E)$  and a permutation  $P: S^2 \rightarrow S^2$ , the *induced permutation* of  $\Gamma$  by  $P$  is the function graph  $P(\Gamma) = (S, E')$  where  $E' = \{P(e) \mid e \in E\}$ .

**Definition 5.7** (CCZ equivalence [BBC<sup>+</sup>22, CCZ98]). Given a vector space  $\mathbb{V}$  and two functions  $f, g: \mathbb{V} \rightarrow \mathbb{V}$ ,  $f$  and  $g$  are *CCZ-equivalent* if there is an affine permutation  $L: \mathbb{V}^2 \rightarrow \mathbb{V}^2$  such that  $\Gamma_f = L(\Gamma_g)$ .

Clearly, CCZ-equivalence is an equivalence relation over a vector space  $\mathbb{V}$ , hence it induces a partitioning of  $\mathbb{V}^2$  into equivalence classes. An interesting fact is that all CCZ-equivalent functions share the same linear and differential properties. Even more importantly for our purposes is that CCZ-equivalent functions are indistinguishable under constraint verification: checking the constraint system of any member of the class verifies the validity of the computation of any other member. However, one thing that CCZ-equivalent functions do not share in general is their degree, hence we can use the one with the highest degree in the actual computation to provide the most strict security guarantees against algebraic attacks, while using the one with the lowest degree when building the constraint system for the verification in the SNARK framework.

In a high-level, intuitive way we can say that CCZ-equivalence allows us to “ignore the order” in which the witnesses of a certain computation are obtained: reprising our usual example, the verifier does not care that the prover first must know  $x$  in order to obtain  $y = x^{1/d}$ , all it matters is that the prover knows both of them and that their relationship is correct. In fact, even more generally, there is no way to know in which order someone actually got hold of the intermediate values of a computation, hence we might as well exploit this to our advantage in order to reduce the complexity of the protocol.

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<sup>4</sup>As it is often the case in research, the separation line is a bit blurry, as *Rescue*, which we said to be a Gen-II design, already used inverse exponentiations

### 5.2.1 The Generalized Triangular Dynamical System

As we said, the design of third generation AO cryptographic primitives diverts from the plain SPN or Feistel constructions. For this reason, we introduce the *Generalized Triangular Dynamical System* [RS22], GTDS for short, an algebraic framework which generalizes many previous designs (such as Feistel, SPN, Horst, ...) and their instantiations (MiMC, POSEIDON, GRIFFIN, ...), and enables us to provide a systematic security analysis of the constructions derived from it. In particular:

- The input is split in branches like in earlier designs.
- The round function offers the strength of all the incorporated designs.
- It is secure against classical attacks like differential cryptanalysis.
- It is secure against interpolation attacks already at the first round.
- Its linear layer mixes all the branches through a circulant matrix with no zero entries.
- It uses inverse exponents, but decouples them from the direct exponent.

**Definition 5.8** (GTDS of Arion [RS22]). Given a prime field  $\mathbb{F}_p$ , a number of branches  $t \in \mathbb{N}$ , the smallest integer  $d_1$  such that  $\gcd(d_1, p-1) = 1$ , an arbitrary integer  $d_2$  such that  $\gcd(d_2, p-1) = 1$ , some constants  $\alpha_1, \beta_1, \gamma_1, \dots, \alpha_{t-1}, \beta_{t-1}, \gamma_{t-1} \in \mathbb{F}_p$  such that  $\forall i < t: \alpha_i^2 - 4\beta_i$  is a quadratic non-residue modulo  $p$ , let  $e = 1/d_2$  and, for all  $i < t$  let:

$$\begin{aligned} g_i(x) &: \mathbb{F}_p \rightarrow \mathbb{F}_p = x^2 + \alpha_i x + \beta_i \\ h_i(x) &: \mathbb{F}_p \rightarrow \mathbb{F}_p = x^2 + \gamma_i x \end{aligned}$$

Then, the GTDS of Arion is the function  $F_{GTDS}(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t$  such that:

$$F_{GTDS}(\mathbf{x})_i = \mathbf{y}_i = \begin{cases} \mathbf{x}_i^{d_1} g_i(\sigma_{i+1,t}) + h_i(\sigma_{i+1,t}) & 1 \leq i < t \\ \mathbf{x}_i^e & i = t \end{cases}$$

where  $\sigma_{i,k} = \sum_{j=i+1}^k \mathbf{x}_j + \mathbf{y}_j$ .

It can be shown [RS22] that the GTDS is function is invertible. An interesting detail which came up only in the later phases of the GTDS design is the decoupling of the inverse exponentiation from the direct exponentiation, in the sense that, instead of using  $x^d$  and  $x^{1/d}$ , we use  $x^{d_1}$  and  $x^{1/d_2}$ . The rationale of this choice is that some security considerations about cryptographic constructions over the GTDS depend on the size of  $d_2$ : if it is too small, we would need more rounds to achieve the desired level of security, hence increasing the circuit complexity. If  $d_2$  were to be equal to  $d_1$ , the reduction in terms of number of rounds would be overcompensated by the increase of the complexity of a single round, but since  $d_2$  is only used in the last branch, the trade-off becomes more convenient, especially for bigger branch sizes.

Of course,  $d_2$  should require an optimal number of constraints. Note that, since we have at our disposal intermediate results, the optimal number of operations required to exponentiate a number

is not determined by the classical *binary exponentiation* algorithm [Gue11], but rather by *addition chains* [BC90].

### 5.2.2 Arion and ArionHash

Depending on our security and efficiency needs, we can instantiate the GTDS in many ways. We design Arion and ArionHash [RST23] to work over fields of size  $\approx 2^{256}$ . In order to achieve a *degree overflow* in the first round, it can be shown that  $4e$ , where  $e = 1/d_2$ , should be greater than  $p$ . For BN254 and BLS12, this can be achieved by  $d_2 \in \{121, 123, 125, 161, 257\}$ . For example, the optimal way to compute  $x^{121}$  is:

$$y = (x^2)^2 \quad z = (y^2y)^2 \quad x^{121} = (z^2)^2 zx$$

It is not hard to see that numbers of the type  $2^k + 1$ , for  $k \in \mathbb{N}$ , are the most efficient to compute, and 257 is a particularly attractive candidate as it is also a prime number, and requires the same number of multiplications to be computed as the other candidates, (unfortunately,  $129 = 43 \cdot 3$  is not invertible neither in BN254 nor in BLS12).

To introduce mixing between the various branches, we use an *affine layer* which employs a circulant matrix that has no zero entries and that is efficiently computable.

**Definition 5.9** (Affine layer of Arion). The *affine layer* of Arion over a vector space  $\mathbb{F}_p^t$  is the function:

$$L(\mathbf{x}, \mathbf{c}): (\mathbb{F}_p^n)^2 \rightarrow \mathbb{F}_p^n = \text{circ}(1, \dots, t)\mathbf{x} + \mathbf{c}$$

It is worth noting that the very simple matrix  $\text{circ}(1, \dots, t)$  is an MDS matrix for any prime field  $\mathbb{F}_p$  such that  $p \geq 2^{39}$  and for values of  $t \in \llbracket 2, 12 \rrbracket$ . Furthermore, computing the matrix-vector product using  $\text{circ}(1, \dots, t)$  can be done in  $\mathcal{O}(t)$  time instead of the typical  $\mathcal{O}(t^2)$  required by a standard matrix-vector multiplication algorithm, by using Algorithm 1.

---

**Algorithm 1** Efficient evaluation of the matrix-vector product with  $\text{circ}(1, \dots, t)$

---

```

function CIRC_MUL( $\mathbf{v} \in \mathbb{F}_p^t$ )
   $\mathbf{w} \leftarrow \mathbf{0} \in \mathbb{F}_p^t$ 
   $\sigma \leftarrow \sum_{i=1}^t \mathbf{v}_i$ 
   $\mathbf{w}_1 \leftarrow \sigma + \sum_{i=1}^t (i-1)\mathbf{v}_i$ 
  for  $i \in \llbracket 2, t \rrbracket$  do
     $\mathbf{w}_i \leftarrow \mathbf{w}_{i-1} - \sigma + n\mathbf{v}_{i-1}$ 
  return  $\mathbf{w}$ 

```

---

**Definition 5.10** (Arion keyed permutation [RST23]). Given a prime field  $\mathbb{F}_p$ , a number of branches  $t \in \mathbb{N}$ , a number of rounds  $r \in \mathbb{N}$ , some constants  $\mathbf{c}_1, \dots, \mathbf{c}_r \in \mathbb{F}_p^t$ , the *Arion keyed permutation* is the function  $\text{Arion} = \text{Arion}_r$ , where:

$$\text{Arion}_i(\mathbf{x}, \mathbf{k}_0, \dots, \mathbf{k}_r): (\mathbb{F}_p^t)^{r+2} \rightarrow \mathbb{F}_p^t = \mathbf{y}_i = \begin{cases} L(\mathbf{x}, \mathbf{0}) + \mathbf{k}_i & i = 0 \\ L(F_{\text{GTDS}}(\mathbf{x}), \mathbf{c}_i) + \mathbf{k}_i & 1 \leq i \leq r \end{cases}$$

Table 5.1: Instantiation parameters of Arion and  $\alpha$ -Arion for 128 bits of security and primes  $p \geq 2^{60}$ .

$d_1$	$t$	Arion	$\alpha$ -Arion
Rounds			
3	3	6	5
5	3	6	4
3	4	6	4
5	4	5	4
3	5	5	4
5	5	5	4
3	6	5	4
5	6	5	4
3	8	4	4
5	8	4	4

**Definition 5.11** (Arion- $\pi$  unkeyed permutation). The Arion *unkeyed permutation* is the function:

$$\text{Arion-}\pi(\mathbf{x}): \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t = \text{Arion}(\mathbf{x}, \mathbf{0}, \dots, \mathbf{0})$$

We can instantiate the hash function **ArionHash** by applying the sponge construction to the **Arion- $\pi$**  permutation. Since it has been shown that, for a pseudorandom permutation  $P$  over a vector space  $\mathbb{F}_p^t$ , a rate  $r$  and a capacity  $c$  such that  $t = r + c$ , the sponge construction is indifferentiable from a random distribution up to  $\min(p^r, p^{c/2})$  queries [BDPVA08], to provide  $\kappa$  bits of security, we must require that  $r \geq \kappa/\log(p)$  and  $c \geq 2\kappa/\log(p)$ .

As a padding scheme for a message  $m \in \{0, 1\}^*$  whose length is not a multiple of the rate  $r$ , we use  $\text{Pad}(m) = m \parallel 0^{(-|m| \bmod r)}$ , and we replace the initial value  $v = 0^{(t|\langle p \rangle|)}$  (where  $\langle x \rangle$  denotes the binary encoding of an object  $x$ ) with  $v' = \langle |m| \rangle \parallel 0^{(t-1)|\langle p \rangle|}$ . Note that we assume, as basically any constructions does, that  $|m| < p$ , which should not be a problem as typically  $p \approx 2^{256}$ , and we don't expect to hash messages of length  $|m| > 2^{256}$ .

In Table 5.1 you can find the parameters for Arion and its aggressive variant  $\alpha$ -Arion, the parameters for the respective hash functions are respectively the same. The aggressive variants have been parametrized in a way to provide the desired security level against all attacks but probabilistic Gröbner basis attacks. The choice to provide them anyway was dictated by the fact that, to the best of our knowledge, none of the competitor designs has been proved secure against such attacks, hence the aggressive versions provide the same guarantees in terms of security as the current state of the art. For a detailed security analysis of Arion, ArionHash and their aggressive variants, refer to [RST23].

### 5.3 Performance evaluation of Arion

As we saw in Section 5.2, the concept of CCZ-equivalence introduced in the **Anemoi** proposal [BBC<sup>+</sup>22] plays an important role to build high degree permutations verifiable with low degree constraint systems. In fact, it is possible to extend the concept of CCZ-equivalence by allowing any permutation.

**Definition 5.12** ( $\pi$ -equivalence). Given a vector space  $\mathbb{V}$  and two functions  $f, g: \mathbb{V} \rightarrow \mathbb{V}$ ,  $f$  and  $g$  are  $\pi$ -equivalent if there is a permutation  $\pi: \mathbb{V}^2 \rightarrow \mathbb{V}^2$  such that  $\Gamma_f = \pi(\Gamma_g)$ .

Any function  $\pi: \mathbb{V}^2 \rightarrow \mathbb{V}^2$  can be decomposed into its *projections*  $\pi_{\hat{x}}, \pi_{\hat{y}}: \mathbb{V}^2 \rightarrow \mathbb{V}$  such that:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{V}: \pi(\mathbf{x}, \mathbf{y}) = (\pi_{\hat{x}}(\mathbf{x}, \mathbf{y}), \pi_{\hat{y}}(\mathbf{x}, \mathbf{y}))$$

Thus, for any function  $f: \mathbb{V} \rightarrow \mathbb{V}$ , we have that  $\pi(\mathbf{x}, f(\mathbf{x})) = (\pi_{\hat{x}}(\mathbf{x}, f(\mathbf{x})), \pi_{\hat{y}}(\mathbf{x}, f(\mathbf{y})))$ . Now, let  $f_{\hat{x}}(\mathbf{x}) = \pi_{\hat{x}}(\mathbf{x}, f(\mathbf{x}))$  and  $f_{\hat{y}}(\mathbf{x}) = \pi_{\hat{y}}(\mathbf{x}, f(\mathbf{x}))$ , then  $\pi(\Gamma_f) = \Gamma_{f'}$  if and only if  $\pi_{\hat{x}}$  is a permutation, and in particular  $f' = f_{\hat{y}} \circ f_{\hat{x}}^{-1}$  is  $\pi$ -equivalent to  $f$ .

Just like CCZ-equivalence,  $\pi$ -equivalence is also an equivalence relation, denoted  $\simeq_{\pi}$ , which allows us to identify equivalence classes of functions over  $\mathbb{V}$ .

**Lemma 5.1** ( $\pi$ -equivalence of permutations). *All permutations over a vector space  $\mathbb{V}$  are  $\pi$ -equivalent.*

*Proof.* We just have to prove that, for every permutation  $f: \mathbb{V} \rightarrow \mathbb{V}$ , it is the case that  $f \simeq_{\pi} \text{id}$ , and the result will follow. Clearly, the function  $\pi(\mathbf{x}, \mathbf{y}): \mathbb{V}^2 \rightarrow \mathbb{V}^2 = (\mathbf{x}, f^{-1}(\mathbf{y}))$  is a permutation, and  $\pi(\Gamma_f) = \Gamma_{\text{id}}$ .  $\square$

**Definition 5.13** (Alternative GTDS of Arion). Given the same parameters as in Definition 5.8, the *alternative GTDS of Arion* is the function  $\tilde{F}_{GTDS}: \mathbb{F}_p^t \rightarrow \mathbb{F}_p^t$  such that:

$$\tilde{F}_{GTDS}(\mathbf{x})_i = \mathbf{y}_i = \begin{cases} \mathbf{x}_i^{d_1} g_i(\tilde{\sigma}_{i+1,t}) + h_i(\tilde{\sigma}_{i+1,t}) & 1 \leq i < t \\ \mathbf{x}_i^{d_2} & i = t \end{cases}$$

where  $\tilde{\tau}_{i,k} = \mathbf{x}_k + \mathbf{x}_k^e + \sum_{j=i}^{k-1} \mathbf{x}_j + \mathbf{y}_j$ .

**Proposition 5.1** ( $\pi$ -equivalence of GTDS).  $F_{GTDS} \simeq_{\pi} \tilde{F}_{GTDS}$ .

*Proof.*  $F_{GTDS}$  and  $\tilde{F}_{GTDS}$  are both permutations over the vector space  $\mathbb{F}_p^t$ , therefore by Lemma 5.1 the claim follows.  $\square$

As a corollary, we have that verifying the constraint system of  $F_{GTDS}$  is equivalent to verifying the constraint system of  $\tilde{F}_{GTDS}$ . Computing the number of multiplicative constraints for ArionHash is quite straightforward:

**Lemma 5.2** (R1CS constraints for ArionHash). *Given the ArionHash function over a prime field  $\mathbb{F}_p$  with branch size  $t$  and rate  $r$ , let  $\text{mul}(x): \mathbb{F}_p \rightarrow \mathbb{N}$  be the minimum number of field multiplications required to compute  $y^x$  for any  $y \in \mathbb{F}$ . The number of R1CS constraints required by ArionHash is:*

$$N_{\text{ArionHash}} = r((n-1)(\text{mul}(d_1) + 2) + \text{mul}(d_2))$$

*Proof.* Consider the alternative GTDS  $\tilde{F}_{GTDS}$ : we need  $\text{mul}(d_2)$  multiplicative constraints in the last branch. In the remaining  $n-1$  branches, we need  $\text{mul}(d_1)$  multiplications to compute  $\mathbf{x}_i^{d_1}$ , one multiplication for computing  $g_i$ , one for computing  $h_i$ , and one to multiply  $g_i$  with  $\mathbf{x}_i^{d_1}$ .  $\square$

For reference, the number of R1CS constraints required by POSEIDON and GRIFFIN over their re-

Table 5.2: R1CS constraint comparison over 256-bit prime fields and 128 bits of security with  $d_2 \in \{121, 123, 125, 161, 257\}$ .

$d_1$	$t$	ArionHash	$\alpha$ -ArionHash	GRIFFIN	Anemoi	POSEIDON
Rounds						
3	3	6	5	12		$r_f = 4, r_P = 84$
5	3	6	4	12		$r_f = 4, r_P = 56$
3	4	6	4	11	12	$r_f = 4, r_P = 84$
5	4	5	4	11	12	$r_f = 4, r_P = 56$
3	5	5	4			$r_f = 4, r_P = 84$
5	5	5	4			$r_f = 4, r_P = 56$
3	6	5	4		10	$r_f = 4, r_P = 84$
5	6	5	4		10	$r_f = 4, r_P = 84$
3	8	4	4	9	10	$r_f = 4, r_P = 84$
5	8	4	4	9	10	$r_f = 4, r_P = 56$
R1CS Constraints						
3	3	102	85	72		216
5	3	114	76	96		240
3	4	126	84	88	96	232
5	4	120	96	110	120	264
3	5	120	100			248
5	5	125	116			288
3	6	145	116		120	264
5	6	170	136		150	312
3	8	148	148	144	160	296
5	8	176	176	162	200	360

spective parameters (see Definition 5.2 and Definition 5.3) are given by:

$$N_{\text{POSEIDON}} = \text{mul}(d)(2tr_f + r_P)$$

$$N_{\text{GRIFFIN}} = 2r(\text{mul}(d) + t - 2)$$

Table 5.2 shows the number of constraints required for specific instantiations of Arion,  $\alpha$ -Arion, Anemoi, POSEIDON and GRIFFIN over  $\approx 256$ -bit prime fields for a target 128 bits of security.

### 5.3.1 Implementations and Experiments

In order to experiment with the cryptographic primitives and the constructions discussed throughout this thesis, we wrote the C++ `zkp-hash`<sup>5</sup> library, by building on `libsnark`. In particular, our library is meant to improve and extend `libsnark`, by providing:

- A native and ZK-SNARK implementation of SHA-512.
- Native and ZK-SNARK implementation of MiMCHash-256 and MiMCHash-512.
- Templates for native and ZK-SNARK implementations of POSEIDON, GRIFFIN and ArionHash.

<sup>5</sup>[https://github.com/SPAstef/zkp\\_hash](https://github.com/SPAstef/zkp_hash)

- A template for native and ZK-SNARK fixed-size Merkle tree authentication paths with arbitrary arity and node size, for arbitrary hash functions.
- A template for native and ZK-SNARK fixed-size ABR authentication paths with arbitrary node size over arbitrary  $2n/n$  hash functions.
- Additional utilities to simplify working with `libsnark`.

We would like to discuss a bit more in detail our Merkle tree implementation, both to highlight some important security considerations that might not be obvious at first, and as an optimization for its construction, which we are not aware of being applied in any other implementation.

Let's consider, for the sake of simplicity, a binary Merkle tree over a field  $\mathbb{F}_p$  where each node contains only one field element. The circuit of the tree takes in input the value  $x$  contained in the node that the prover claims to hold, together with the values  $y_1, \dots, y_{h-1}$  along the alleged authentication path, and by using the sub-circuit (gadget) of the underlying compression function  $C$ , it constrains the middle values  $m_1, \dots, m_h$  to be:

$$(m_1 = x) \wedge \forall i < h: m_{i+1} = \begin{cases} C(m_i, y_i) & m_i \text{ is a left leaf} \\ C(y_i, m_i) & m_i \text{ is a right leaf} \end{cases} \quad (5.1)$$

where  $m_h$  will be the root value. However, there is a big issue in this naive construction: the left/right choice is made transparently and the verifier, by simply inspecting the circuit layout, would be able to know the location of the prover's node and therefore its value! So we must *embed* the choice in the circuit (in the `libsnark` implementation this is left to the user, which we believe to be unnecessarily complicated and error-prone).

In our construction, each node is labeled with a left-to-right bottom-to-top 0-based index  $n$ : the leftmost leaf will have index  $n = 0$  and the root will have index  $n = 2^h - 2$ , where  $h$  is the height of the tree. The index will be an additional input variable to the circuit. Note that the base-2 representation of the index, from the LSB to the MSB, shows the path from the leaf to the root: if the  $i$ th bit of  $n$  is 0, it means that we are on a left node, if it is 1 it means that we are on a right node. By introducing a set of *selection variables*  $s_1, \dots, s_{h-1}$ , a set of *left variables*  $l_1, \dots, l_{h-1}$  and a set of *right variables*  $r_0, \dots, r_{h-1}$ , we can then replace Equation (5.1) with the following:

$$\begin{aligned} n &= \sum_{i=1}^h s_i 2^{i-1} \\ m_1 &= x \\ \forall i < h: &\begin{cases} l_i = (1 - s_i)m_i + s_i y_i \\ r_i = s_i m_i + (1 - s_i)y_i \\ m_{i+1} = C(l_i, m_i) \end{cases} \end{aligned}$$

Where the  $i$ th selection variable will have to be set to the value of the  $i$ th bit of the index. If any of the selection variables is set to an incorrect value, the verification will fail, hence we have been able to hide the path from the verifier without compromising the correctness of the construction.

Table 5.3: Performance of various hash functions for generating a proof of membership in a Merkle tree accumulator over BN254. Proving times are in milliseconds.

		Time (ms)										
Height	ArionHash	$\alpha$ -ArionHash			GRIFFIN			POSEIDON				
$d = 5$	$n = 3$	$n = 4$	$n = 8$	$n = 3$	$n = 4$	$n = 8$	$n = 3$	$n = 4$	$n = 8$	$n = 3$	$n = 4$	$n = 8$
4	101	103	142	73	87	143	88	99	133	186	212	274
8	211	216	294	145	177	294	181	209	270	386	417	566
16	392	401	554	278	334	553	338	387	505	745	805	1095
32	730	751	1046	509	646	1047	622	727	980	1422	1550	2111

### From binary to $t$ -ary

To work with  $t$ -ary trees of height  $h$  using the *minimal amount of constraints*, we need to make some changes to the previous reasoning. The authentication path will now consist of the node  $x$  and a set of  $t - 1$ -dimensional vectors  $\mathbf{y}_1, \dots, \mathbf{y}_{h-1}$ , where  $\mathbf{y}_i$  will contain the sibling nodes at the  $i$ th level of the tree, and the compression function will be  $t$ -to-1. The base- $t$  representation of the index, from LSD to MSD, will again show the path from the leaf to the root, where the  $i$ th digit will tell us which of the  $t$  children we are on. We introduce a set of  $t$ -dimensional selection variables  $\mathbf{s}_1, \dots, \mathbf{s}_{h-1}$ , a set of  $t$ -dimensional children variables  $\mathbf{c}_1, \dots, \mathbf{c}_{h-1}$ , and introduce the following constraints:

$$\begin{aligned}
 n &= \sum_{i=1}^h t^{i-1} \sum_{j=1}^t j s_{i,j} 2^{t-1} \\
 \mathbf{c}_{1,1} &= x \wedge \forall i \in \llbracket 2, t \rrbracket: \mathbf{c}_{1,i} = \mathbf{y}_{i-1} \\
 \forall i < h, \forall j \leq t: &\begin{cases} \mathbf{s}_{i,j}(m_{i+1} - \mathbf{y}_{i,j}) = \mathbf{c}_{i,j} - \mathbf{y}_{i,j} \\ m_{i+1} = C(\mathbf{c}_i) \end{cases}
 \end{aligned}$$

To better understand the constraints, let's focus on a single level  $i$ : the vector  $\mathbf{s}_i$  will contain all zeros except in the position dictated by the  $i$ th digit of  $n$  when represented in base  $t$ . In an electronic jargon, we can say that the selector blocks all the signals coming from the previous layer but in one position, and the dangling wires will be powered by the sibling signals provided by the prover. To make sure that the prover cannot cheat, the  $i$ th digit of  $n$  will have to match the value of the  $i$ th bit of  $\mathbf{s}_i$  weighted by its position  $j$ . Further extending the scheme to allow for single nodes to contain multiple field elements is quite straightforward.

### Arion vs. GRIFFIN vs. POSEIDON

Using our library, we compared Arion with GRIFFIN and POSEIDON. The comparative result of our experiments are given in Table 5.3. All experiments were run on a system with an Intel Core i7-11800H CPU and 32 GB of RAM on a Clear Linux instance, using the g++-12 compiler with `-O3 -march=native` flags. Our result shows that ArionHash significantly outperforms POSEIDON showing 2x efficiency improvement, and  $\alpha$ -ArionHash is considerably faster than GRIFFIN for the most common real-world parameter choices (i.e.  $n = 3$  in Merkle tree hashing mode).



# 6

## Conclusions and Future Work

In this work we have studied the history of zero-knowledge, succinct and non-interactive argument of knowledge protocols, a highly interdisciplinary field of study which involves advanced concepts of mathematic, computer science and cryptography. We have seen how such systems have seen an incredible improvement over the last ten years, to the point of finally becoming useful in many real-world scenarios, from private verifiable cloud computing to anonymous blockchain commitments.

We have seen how the best ZK-SNARK systems known today work over arithmetic models of computation, much different from the boolean ones algorithms are usually designed for. Since the most prominent application of such systems is in cryptography, and particularly in Merkle tree commitment verification, we revised the very recent history and the state of the art of arithmetization-oriented cryptographic primitives.

By distilling the core ideas in this field and introducing new ones, we then formalized them in the so-called Generalized Triangular Dynamical System framework, which we used to extract a new family of cryptographic functions: **Arion**. We finally compared **Arion** with the competitor designs, showing that we beat the state of the art when providing the same level of security guarantees, and by matching it when providing additional guarantees.

This work has been a very exciting journey, which is however just a glimpse of what could lie ahead. New proof systems like the circuit-depth oriented transparent ZK-STARKs, or the Lagrange-bases oriented **PlonK** offer new challenges, as they divert from the standard metrics used to measure the complexity of a good ZK-friendly function.

Another interesting direction of research is the application of ZK-SNARK systems and the related cryptographic functions in a wide range of hardware devices: from high performance computing (HPC) machines (e.g. verifiable delegated computation), to personal computers (e.g. anonymous transactions) to embedded internet-of-things (IoT) chips (e.g. private household monitoring).



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