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NOTE

THE COMPLEXITY OF EVALUATING INTERPOLATION POLYNOMIALS

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Abstract. It is known that computing all coefficients of the Lagrangian interpolation polynomial, given n nodes and n values, has nonscalar complexity of order $n \log n$. We show that the evaluation of a single coefficient or of the value of the interpolation polynomial at some new point already has complexity of order $n \log n$. These results are consquences of a more general theorem.

In this note we study certain computational tasks related to Lagrangian interpolation such as computing a single coefficient of the interpolation polynomial or its value at some further point.

To fix the model of computation, let F be a field of characteristic 0 and $x_0, \ldots, x_n, y_0, \ldots, y_n$ indeterminates over F. We consider the Ostrowski complexity $L(f_1, \ldots, f_m)$ of rational functions $f_1, \ldots, f_m \in F(x, y)$, i.e., the minimum number of nonscalar multiplications or divisions sufficient to compute $f_1, \ldots, f_m$ in $F(x, y)$ from $x_0, \ldots, x_n, y_0, \ldots, y_n$ by a straight line program. (In this model, F-linear operations are thought to be free of charge; for details, see, e.g., [2].)

Let us denote by

$$\sum_{j=0}^{n} a_j(x, y)t^j \in F(x, y)[t] \tag{1}$$

the Lagrangian interpolation polynomial defined by the equations

$$\sum_{j=0}^{n} a_j(x, y)x_i^j = y_i \quad (i = 0, 1, \ldots, n). \tag{2}$$

It is well known [3, 5] that the complexity of computing all coefficients $a_0, \ldots, a_n$ is of order $n \log n$; more precisely,

$$(n+1) \log n \leq L(a_0, \ldots, a_n) \leq 13(n+1) \log(n+1). \tag{3}$$

The complexity bounds of the tasks of

- computing one single coefficient a_j,
- evaluating the interpolation polynomial at an arbitrary point $\omega \in F$, i.e., computing $\sum_{j=0}^{n} a_j(x, y)\omega^j$,

and likewise corresponding tasks for the derivatives with respect to t are all covered by the following general result.

Theorem. *Let $\lambda_0, \dots, \lambda_n \in F$, not all λ_i equal to 0, and let $\sum_{j=0}^{n} a_j(x, y)t^j \in F(x, y)[t]$ be the Lagrangian interpolation polynomial defined by $\sum_{j=0}^{n} a_j(x, y)x_i^j = y_i$ $(i = 0, 1, \dots, n)$. Then*

$$\tfrac{1}{3}(n+1)\log\left(\frac{n+1}{e}\right) \leq L\left(\sum_{j=0}^{n} \lambda_j a_j(x, y)\right) \leq 13(n+1)\log(n+1).$$

As pointed out by Strassen [6], this theorem has (by a transitivity argument) the following consequence: Computing a representation of the interpolation polynomial (or its derivative, etc.) which allows an evaluation in linear time is itself a process with complexity of order at least $n \log n$. In this sense, the representation by coefficients and the barycentric representation [4] are both optimal.

Proof of the Theorem. The *upper bound* is an immediate consequence of the upper bound in (3) since we get $\sum_{j=0}^{n} \lambda_j a_j$ from the $a_0, \dots, a_n$ by F-linear operations only.

For the proof of the *lower bound* we assume w.l.o.g. that F is algebraically closed. For the sake of clearness let us write

$$\sum_{j=0}^{n} \lambda_j a_j(x, y) = l^{\mathrm{T}} V(x)^{-1} y$$

where $l = (\lambda_0, \dots, \lambda_n)^{\mathrm{T}}$, $y = (y_0, \dots, y_n)^{\mathrm{T}}$, and $V(x) = (x_i^j)_{0 \leq i,j \leq n}$ is the Vandermonde matrix. Differentiating with respect to $y_0, \dots, y_n$ we get rational functions $b_0, \dots, b_n \in F(x)$ defined by

$$\sum_{j=0}^{n} b_j(x)x_j^i = \lambda_i \quad (i = 0, \dots, n), \tag{4}$$

and by the Baur–Strassen derivation theorem [1] it follows that

$$L\left(\sum_{j=0}^{n} \lambda_j a_j\right) \geq \tfrac{1}{3}L(b_0, \dots, b_n). \tag{5}$$

As the subsequent lemma implies, the degree of the graph of the rational mapping

$$F^{n+1} \to F^{n+1} : (\xi_0, \dots, \xi_n) \mapsto (b_0(\xi), \dots, b_n(\xi))$$

defined for all ξ with pairwise different components (that is $\det V(\xi) \neq 0$), is at least $(n+1)!$. Thus, by (5) and the Strassen degree bound [5],

$$L\left(\sum_{j=0}^{n} \lambda_j a_j\right) \geq \tfrac{1}{3}\log((n+1)!) \geq \tfrac{1}{3}(n+1)\log\left(\frac{n+1}{e}\right)$$

proving the theorem. □

Lemma. *There exists some bivariate polynomial $h \in F[v, t]$ such that*

(i) *for any $\delta \in F$ a point $\xi \in F^{n+1}$ with pairwise different components is a solution of*

$$b_j(x) = x_j - \delta \quad (j = 0, \dots, n) \tag{6}$$

iff $h(\delta, \xi_0 - \delta) = \cdots = h(\delta, \xi_n - \delta) = 0$,

(ii) *for almost all $\delta \in F$ the polynomial $h(\delta, t)$ is separable.*

Proof. (i) Let $\delta \in F$, $\xi = (\xi_0, \dots, \xi_n) \in F^{n+1}$ with pairwise different components and $\eta_j := \xi_j - \delta$ $(j = 0, \dots, n)$. ξ is a solution of (6) iff $\sum_{j=0}^{n} (\xi_j - \delta)\xi_j^i = \lambda_i$ $(i = 0, \dots, n)$ or, equivalently,

$$\sum_{j=0}^{n} \eta_j \left(\sum_{k=0}^{i} \binom{i}{k} \delta^{i-k} \eta_j^k \right) = \lambda_i \quad (i = 0, \dots, n).$$

This relation between the power sums $s_i(\eta) = \sum_{j=0}^{n} \eta_j^i$ may be rewritten as

$$s_{i+1}(\eta) = \lambda_i - \sum_{k=0}^{i-1} \binom{i}{k} \delta^{i-k} s_{k+1}(\eta) \quad (i = 0, \dots, n),$$

and, using the identity

$$\sum_{k=j}^{i-1} (-1)^k \binom{i}{k}\binom{k}{j} = (-1)^{i-1}\binom{i}{j},$$

this becomes equivalent to

$$s_i(\eta) = \sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} \delta^{i-1-j} \lambda_j \quad (i = 1, \dots, n+1). \tag{7}$$

The power sums are connected with the elementary symmetric functions $\sigma_k(\eta_0, \dots, \eta_n)$ $(k = 0, \dots, n+1)$ by the Newton relations (see, e.g., [7, p. 102])

$$\sigma_0 = 1,$$

$$\sigma_i(\eta) = \frac{1}{i} \sum_{k=0}^{i-1} (-1)^{i-1-k} s_{i-k}(\eta) \sigma_k(\eta) \quad (i = 1, \dots, n+1).$$

If we replace here the $s_{i-k}(\eta)$'s by the right-hand side of (7), we get

$$\begin{aligned} &\sigma_0 = 1, \\ &\sigma_i(\eta) = \sum_{k=0}^{i-1} \left(\sum_{j=0}^{i-k-1} (-1)^j \frac{1}{i} \binom{i-k-1}{j} \delta^{i-1-k-j} \lambda_j \right) \sigma_k(\eta) \quad (i = 1, \dots, n+1), \end{aligned} \tag{8}$$

equivalent to (7).

From this recursion we get the desired polynomial h of the Lemma:

We define polynomials $c_i \in F[v]$ by

$$\begin{aligned} &c_0(v) := 1, \\ &c_i(v) := \sum_{k=0}^{i-1} \left(\sum_{j=0}^{i-k-1} (-1)^j \frac{1}{i} \binom{i-k-1}{j} v^{i-1-k-j} \lambda_j \right) c_k(v) \quad (i = 1, \dots, n+1) \end{aligned} \tag{9}$$

(in (8) we have replaced $\delta \in F$ by an indeterminate v and σ_i by c_i) and set

$$h(v, t) := \sum_{i=0}^{n+1} (-1)^i c_i(v) t^{n+1-i}. \tag{10}$$

By construction we have: ξ is a solution of (6) iff $\sigma_i(\eta) = c_i(\delta)$, and this is equivalent to

$$h(\delta, t) = \sum_{i=0}^{n+1} (-1)^i \sigma_i(\eta) t^{n+1-i} = \prod_{j=0}^{n} (t - \eta_j) = \prod_{j=0}^{n} (t - (\xi_i - \delta))$$

proving the first part of the Lemma.

(ii) To prove the second part, we look at the asymptotic behaviour for $v \to \infty$ of the polynomials $c_i(v)$ defined by (9).

Let r $(0 \leqslant r \leqslant n)$ be the smallest index such that $\lambda_r \neq 0$. Then, in (9), the second sum actually ranges over $j = r, \ldots, i-k-1$ only. This immediately yields

$$c_0(v) = 1, \qquad c_i(v) = 0 \quad (i = 1, \ldots, r). \tag{11}$$

and an easy inductive argument shows that

$$c_i(v) = (-1)^r \frac{1}{i} \binom{i-1}{r} \lambda_r v^{i-1-r} + \mathrm{O}(v^{i-2-r}) \quad (i = r+1, \ldots, n+1), \tag{12}$$

where $\mathrm{O}(v^k)$ collects all terms of degree $\leqslant k$.

Equations (10), (11), and (12) yield

$$h(v, t) = t^{n+1} + \sum_{i=r+1}^{n+1} \left((-1)^{r+i} \frac{1}{i} \binom{i-1}{i} \lambda_r v^{i-1-r} + \mathrm{O}(v^{i-2-r}) \right) t^{n+1-i}.$$

This representation of h implies that the discriminant with respect to t of h has the form

$$\mathrm{Dis}_t(h(v, t)) = (n+1)^{n+1} \left((-1)^{n+1+r} \frac{1}{n+1} \binom{n}{r} \right)^n v^{n(n-r)} + \mathrm{O}(v^{n(n-r)+1}).$$

Since $0 \leqslant r \leqslant n$, the discriminant is a nontrivial polynomial in v and hence $h(\delta, t)$ is separable for almost all $\delta \in F$. $\square$

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