Parameter Estimating of U.S. population

1. Malthus Model estimation

1.1 Malthus Model

$$\frac{dP(t)}{dt} = P_0P(t), P(0) = P_0$$

where r is the unknown parameter.

The solution of the Malthus model is

$$\mathrm{P(t)} = P_0e^{rt}$$

1.2 Use polyfit function

We use U.S. population data to estimate r and P_0 in the solution of Malthus model. We set 1790-2000 as 0-20. We change the exponential form of solution into a linear form by using log:

$$\log(\mathrm{P(t)}) = \log P_0 + rt = a + rt$$

We estimate the two parameter using both data from 1790-1900 and data from 1790-2000 with polyfit function in matlab. Then we plot the the fitting curve along with the data. The code is in appendix A. We also find that Malthus model is no longer suitable for 20th century America

The outcome is: r=0.2743, a=1.4323, P_0=4.1884. The curve is in Figure 1. We replace 0-11 with 1790-1900 when drawing the curve.

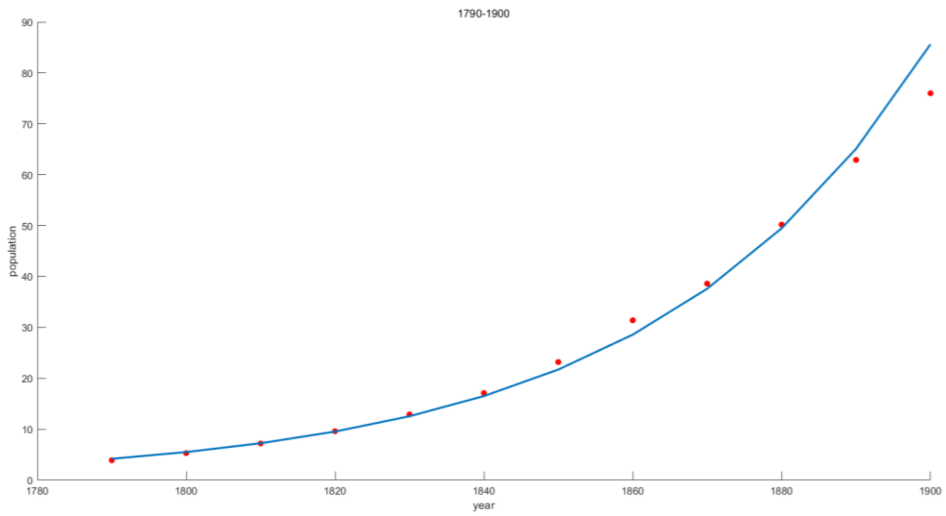

Figure 1. The estimated curve using polyfit and data from 1790-1900

When using data from 1790-2000, the outcome is: r=0.2022, a=1.7992, P_0=6.0450. The curve is in Figure 2. We replace 0-20 with 1790-2000 when drawing the curve. We can see that the curve cannot fit data well.

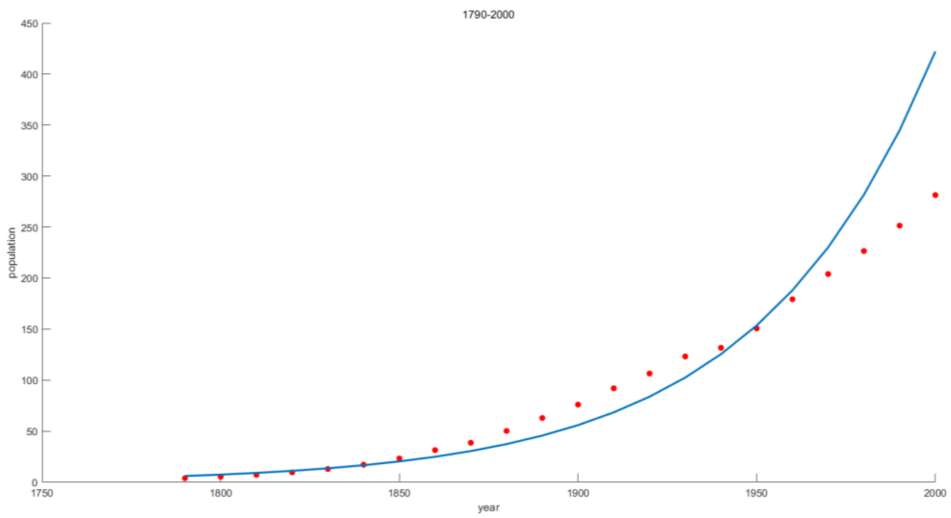

Figure 2. The estimated curve using polyfit and data from 1790-2000

1.3 Use gradient descent

We use least square method and gradient descent to estimate the two parameter using both data from 1790-1900 and data from 1790-2000.
The loss function is:

$$\mathrm{J} = \frac{1}{2m}\sum_{i=1}^{m}(\mathrm{a} + rt - \log P_i)^2$$

where m is the total number of data, P_i means the ith population data.
The gradient descent formula is:

$$\mathrm{r}' = \mathrm{r} - \alpha_1 \frac{\partial}{\partial r} J(r,a) = r - \frac{\alpha_1}{m}(\mathrm{a} + rt - \log P_i)t$$

$$\mathrm{a}' = \mathrm{a} - \alpha_2 \frac{\partial}{\partial a} J(r,a) = a - \frac{\alpha_2}{m}(\mathrm{a} + rt - \log P_i)$$

The program is in appendix B.

The outcome is: r=0.2579, a=1.5260, P_0=4.5997. We replace 0-11 with 1790-1900. The curve is in Figure 3.

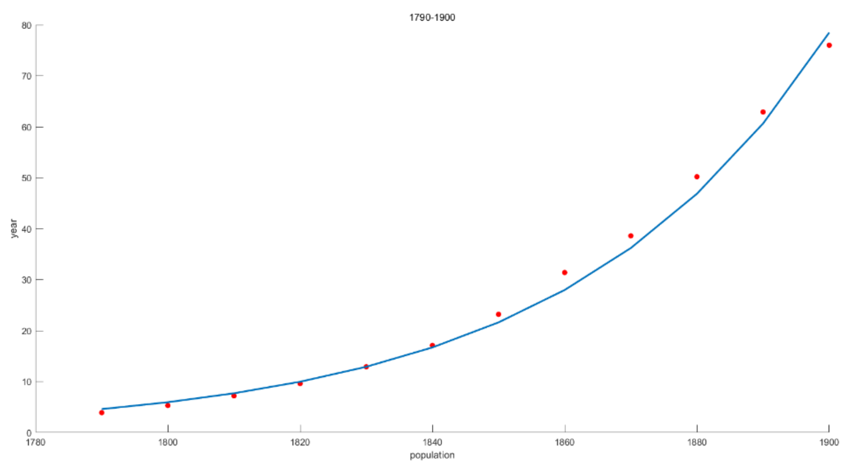

Figure 3. The estimated curve using gradient descent and data from 1790-1900

The outcome is: r=0.2022, a=1.7989, P_0=6.0428. We replace 0-20 with 1790-2000 when drawing the curve. The curve is in Figure 4. We can see that the curve cannot fit data well.

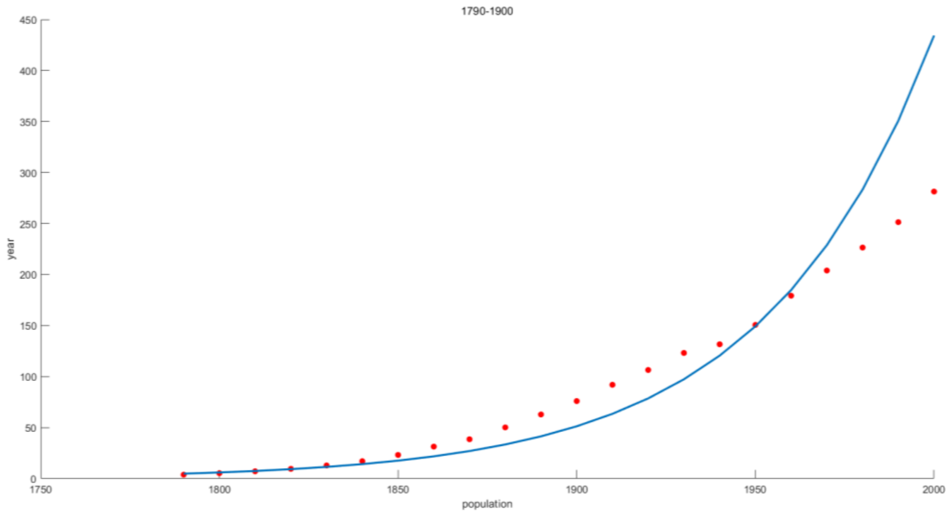

Figure 4. The estimated curve using gradient descent and data from 1790-2000

2. Logistic Model estimation

2.1 Logistic Model

$$\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{P_\infty}\right), P(0) = P_0$$

where r, pm are the unknown parameters and s=r/P_∞.

The solution of the logistic model is

$$\mathrm{P(t)} = \frac{P_\infty}{1 + (\frac{P_\infty}{P_0} - 1)e^{-rt}}$$

The solution is too complicated to be used to estimate unknown parameters. Therefore, we change the equation of logistic model into another form:

$$\frac{\frac{dP(t)}{dt}}{P(t)} = r - sP(t)$$

2.2 Forward diffenence

We use forward difference to compute dp/dt and at the last point, we use backward difference. We assume that t is 0-20, which means that dt=1. The formula of forward difference is

$$\mathrm{P'[i]} = \frac{P[i+1] - P[i]}{dt}$$

The formula of backward differnece is

$$\mathrm{P'[i]} = \frac{P[i] - P[i-1]}{dt}$$

Then we use polyfit to estimate r and s. The program is in appendix C and the curve is in Figure 5. We replace 0-20 with 1790-2000 when drawing the curve. The outcome is: r=0.3186, s=9.9524e-4, P_∞=r/s=320.0850. We assume that P_0 is 3.9. According to the estimation of parameters, we predict that the population of 2010 is 290.7706.

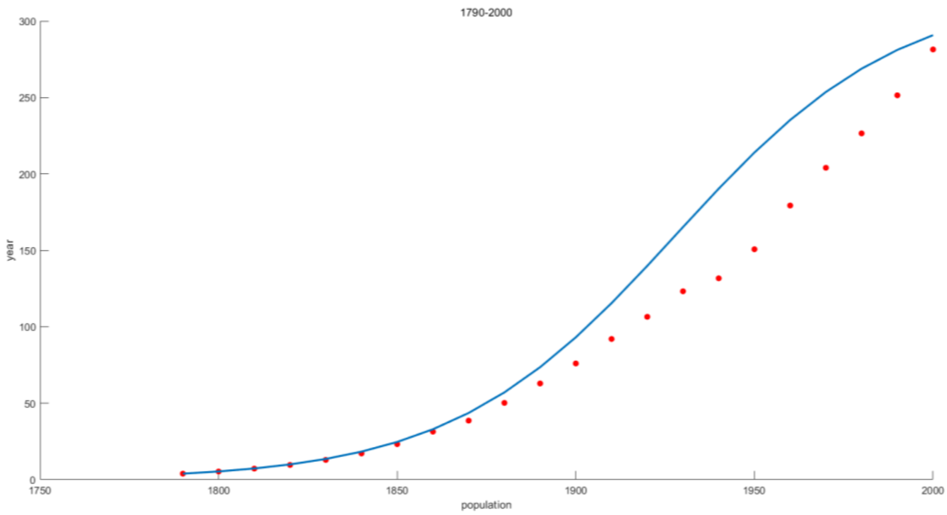

Figure 5. The estimated curve using forward difference and data from 1790-2000

2.3 Central difference

We use central difference to compute dp/dt. At the first point, we use forward difference and we use backward difference at the last pont. The formula of central differnece is

$$\mathrm{P}'[\mathrm{i}] = \frac{P[i+1] - P[i-1]}{2dt}$$

Then we use polyfit to estimate r and s and use the population of 1790 to compute P_0. The program is in appendix D and the curve is in Figure 6. The outcome is: r=0.2886, s=8.6108e-4, P_∞=r/s=335.2095. According to the estimation of parameters, we predict that the population of 2010 is 279.8024.

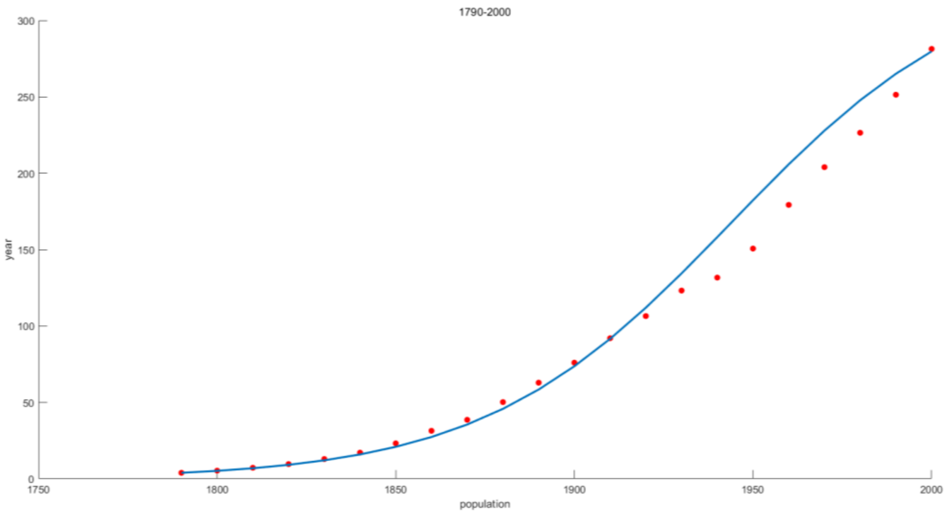

Figure 6. The estimated curve using central difference and data from 1790-2000

2.4 Use lsqcurvefit function

We use the results of central difference as the initial value of lsqcurvefit function to estimate the value of r and P_∞. P_0 is still 3.9. The code is in appendix E. The outcome is: r=0.2735, P_∞=342.4404. According to the estimation of parameters, we predict that the population of 2010 is 267.9673. The curve is in Figure 7.

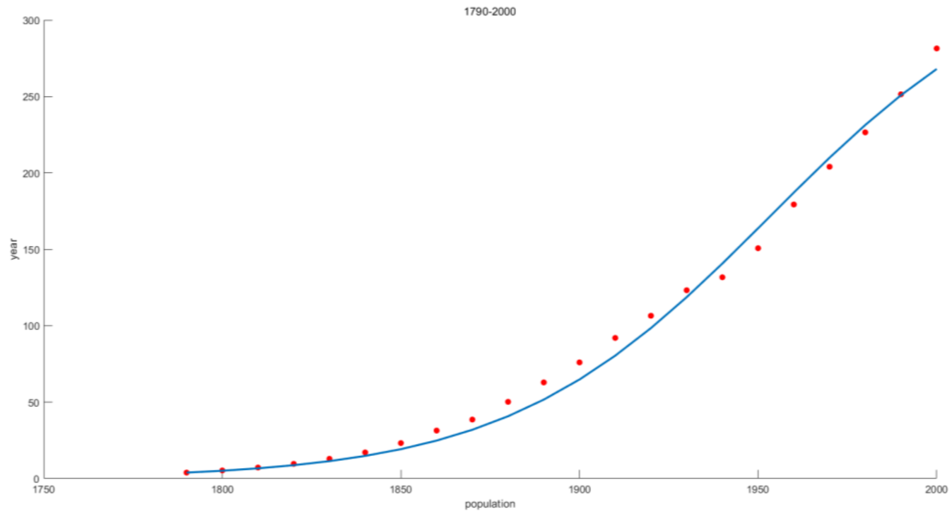

Figure 7. The estimated curve using lsqcurvefit and data from 1790-2000

We also try to take P_0 as a parameter. In other words, there are 3 parameters which needs to be estimated, which are r, P_0 and P_∞. The code is in appendix F. The outcome is: r=0.2155, P_0=7.6982, P_∞=446.5756. According to the estimation of parameters, we predict that the population of 2010 is 276.0479. The curve is in Figure 8.

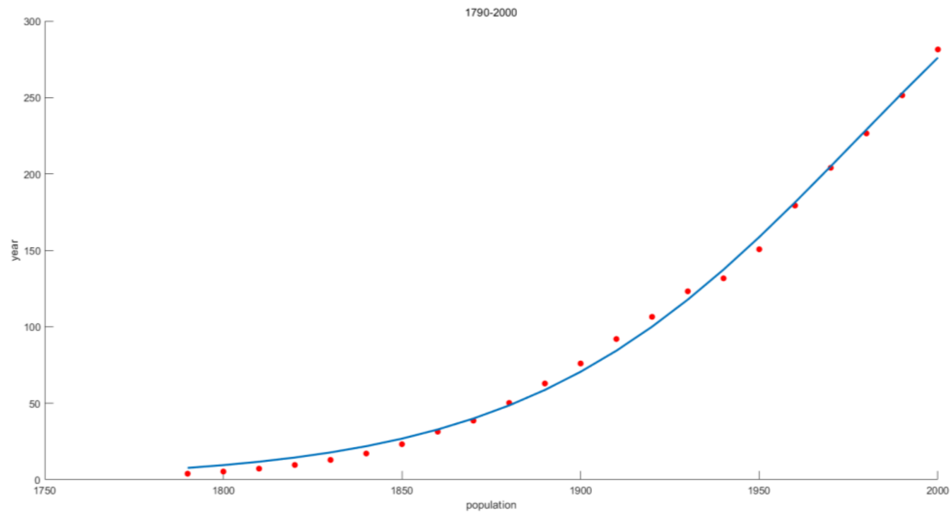

Figure 8. The estimated curve using lsqcurvefit with 3 parameters

Appendix

A. Code of polyfit estimation

```
clear;clc;
pop=[3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, ...
    92.0, 106.5, 123.2, 131.7, 150.7, 179.3, 204.0, 226.5, 251.4, 281.4];%population
pop=pop(1:12);%when using 1790-2000, omit this line
year=0:length(pop)-1;
%P(t)=3.9exp(rT) was changed to linear form: log(P(t))=log3.9+rT
pop1=log(pop);
[a,b]=polyfit(year,pop1,1)
scatter(year*10+1790,pop,'red','filled');
hold on;
p0=exp(a(2));
pred=p0*exp(a(1)*year);
plot(year*10+1790,pred,'LineWidth',2);
xlabel('year');
ylabel('population');
title('1790-1900');
```

B. Code of gradient descent

```
clear;clc;
pop=[3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, ...
    92.0, 106.5, 123.2, 131.7, 150.7, 179.3, 204.0, 226.5, 251.4, 281.4];%population
%pop=pop(1:12);%when using 1790-2000, omit this line
year=0:length(pop)-1;
%P(t)=3.9exp(rT) was changed to linear form: log(P(t))=log3.9+rT
pop1=log(pop);
inum=50; %iteration_number
%learning rate
lr=0.008;%when 1790-1900, lr=0.001; 1790-2000, lr=0.008
lr1=0.0008; %when 1790-1900, lr1=0.3; 1790-2000, lr1=0.0008
r=0.25; %initialize
lp0=1.7992;
m=length(pop);
loss1=double(zeros(inum,1));
scatter(year*10+1790,pop,'red','filled');
hold on;
for i=1:inum
     sum=double(0); %sum of square error
     sum1=double(0); %sum of gradient
```

```
        sum2=double(0);
        for j=1:m
                sum=sum+(lp0+r*year(j)-pop1(j))^2;
                sum1=sum1+year(j)*(lp0+r*year(j)-pop1(j));
                sum2=sum2+lp0+r*year(j)-pop1(j);
        end
        sum1;
        loss1(i)=sum/(2*m);
        r=r-lr/m*sum1;
        lp0=lp0-lr1/m*sum2;
end
p0=exp(lp0);
pred=exp(lp0)*exp(r*year);
plot(year*10+1790,pred,'LineWidth',2);
xlabel('population');
ylabel('year');
title('1790-1900');
```

C. Forward difference

```
clear;clc;
pop=[3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, ...
    92.0, 106.5, 123.2, 131.7, 150.7, 179.3, 204.0, 226.5, 251.4, 281.4];%population
year=0:length(pop)-1;
m=length(year);
fordpop=zeros(m,1);
for i=1:m-1
        fordpop(i)=(pop(i+1)-pop(i))/1;
end
fordpop(m)=(pop(m)-pop(m-1))/1; %backward difference
[a,b]=polyfit(pop,fordpop'./pop,1)
p8=-a(2)/a(1);
r=a(2);
p0=3.9;
scatter(year*10+1790,pop,'red','filled');
hold on;
pred=p8./(1+(p8/p0-1)*exp(-r*year));
plot(year*10+1790,pred,'LineWidth',2);
xlabel('population');
ylabel('year');
title('1790-2000');
pred1=p8./(1+(p8/p0-1)*exp(-r*21))
```

D. Central difference

```
clear;clc;
pop=[3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, ...
    92.0, 106.5, 123.2, 131.7, 150.7, 179.3, 204.0, 226.5, 251.4, 281.4];%population
year=0:length(pop)-1;
m=length(year);
cendpop=zeros(m,1);
cendpop(1)=(pop(2)-pop(1))/1;
for i=2:m-1
     cendpop(i)=(pop(i+1)-pop(i-1))/2;
end
cendpop(m)=(pop(m)-pop(m-1))/1;
[a,b]=polyfit(pop,cendpop'./pop,1)
p8=-a(2)/a(1);
r=a(2);
p0=3.9;
scatter(year*10+1790,pop,'red','filled');
hold on;
pred=p8./(1+(p8/p0-1)*exp(-r*year));
plot(year*10+1790,pred,'LineWidth',2);
xlabel('population');
ylabel('year');
title('1790-2000');
pred1=p8./(1+(p8/p0-1)*exp(-r*21))
```

E. lsqcurvefit

```
clear;clc;
pop=[3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, ...
    92.0, 106.5, 123.2, 131.7, 150.7, 179.3, 204.0, 226.5, 251.4, 281.4];%population
year=0:length(pop)-1;
x0=[335.2095,3.9,0.2886];%initial
F=@(x,xdata)x(1)./(1+(x(1)/3.9-1)*exp(-x(2)*xdata)); %x(1)=pm; x(2)=p0; x(2)=r;
[x,resnorm] = lsqcurvefit(F,x0,year',pop')
scatter(year*10+1790,pop,'red','filled');
hold on;
pred=x(1)./(1+(x(1)/3.9-1)*exp(-x(2)*year));
plot(year*10+1790,pred,'LineWidth',2);
xlabel('population');
ylabel('year');
title('1790-2000');
pred1=x(1)/(1+(x(1)/3.9-1)*exp(-x(2)*21))
```

F. lsqcurvefit (3 parameters)

```
clear;clc;
pop=[3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, ...
    92.0, 106.5, 123.2, 131.7, 150.7, 179.3, 204.0, 226.5, 251.4, 281.4];%population
year=0:length(pop)-1;
x0=[335.2095,3.9,0.2886];%initial
F=@(x,xdata)x(1)./(1+(x(1)/x(2)-1)*exp(-x(3)*xdata)); %x(1)=pm; x(2)=p0; x(2)=r;
[x,resnorm] = lsqcurvefit(F,x0,year',pop')
scatter(year*10+1790,pop,'red','filled');
hold on;
pred=x(1)./(1+(x(1)/x(2)-1)*exp(-x(3)*year));
plot(year*10+1790,pred,'LineWidth',2);
xlabel('population');
ylabel('year');
title('1790-2000');
pred1=x(1)/(1+(x(1)/x(2)-1)*exp(-x(3)*21))
```