# Linear and generalized linear models

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# Outline

- Simple linear regression.
- Fitting a model and extracting results.
- Predictions and diagnostics.
- Categorical factors and contrast matrices.
- Main effects and interactions.
- Generalized linear models.
- Modelling curved effects.

# Variables in generalized linear models

- The **outcome** or **response** variable must be numeric.
- Main types of response variables are
  - Metric or continuous (a measurement with units)
  - Binary (two values coded 0/1)
  - Failure (does the subject fail at end of follow-up)
  - Count (aggregated failure data, number of cases)
- **Explanatory** variables or **regressors** can be
  - Numeric or quantitative variables
  - Categorical factors, represented by class indicators or contrast matrices.

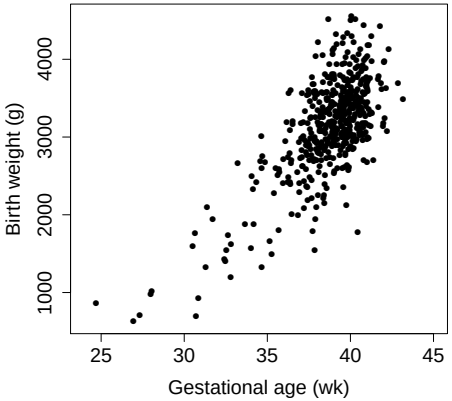
# The `births` data in `Epi`

| | |
|---|---|
| `id`: | Identity number for mother and baby. |
| `bweight`: | Birth weight of baby. |
| `lowbw`: | Indicator for birth weight less than 2500 g. |
| `gestwks`: | Gestation period in weeks. |
| `preterm`: | Indicator for gestation period less than 37 weeks. |
| `matage`: | Maternal age. |
| `hyp`: | Indicator for maternal hypertension (0 = no, 1 = yes). |
| `sex`: | Sex of baby (1 = male, 2 = female). |

Declaring and transforming some variables as factors:

```
> library(Epi) ; data(births)
> births <- transform(births,
+    hyp = factor(hyp, labels=c("N", "H")),
+    sex = factor(sex, labels=c("M", "F")),
+    gest4 = cut(gestwks,breaks=c(20, 35, 37, 39, 45), right=FALSE)  )
> births <- subset(births, !is.na(gestwks))
```

# Birth weight and gestational age




```
> with(births, plot(bweight ~ gestwks, xlim = c(24,45), pch = 16, cex.axis=1.5, cex.lab = 1.5,
+  xlab= "Gestational age (wk)", ylab= "Birth weight (g)" ) )
```

# Metric response, numeric explanatory variable

Roughly linear relationship btw `bweight` and `gestwks`

→ Simple **linear regression model** fitted.

```
> m <- lm(bweight ~ gestwks, data=births)
```

- `lm()` is the function that fits linear regression models, assuming **Gaussian** distribution for **error** terms.
- `bweight ~ gestwks` is the **model formula**
- `m` is a **model object** belonging to **class** "`lm`".

`> coef(m)` – Printing the estimated regression coefficients

```
(Intercept)     gestwks
    -4489.1       197.0
```

Interpretation of **intercept** and **slope**?

# Model object and extractor functions

Model object = **list** of different elements, each being separately accessible. – See `str(m)` for the full list.

Functions that extract results from the fitted model object

- `summary(m)` – lots of output
- `coef(m)` – beta-hats only (see above)
- `ci.lin(m)[,c(1,5,6)]` – $\widehat{\beta}_j$s plus confidence limits

```
             Estimate    2.5%   97.5%
(Intercept)  -4489.1 -5157.3 -3821.0
gestwks        197.0   179.7   214.2
```

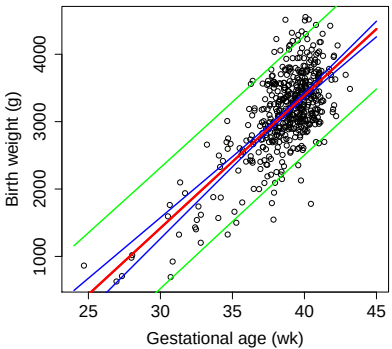
  This function is in `Epi` package
- `anova(m)` – Analysis of Variance Table

# Other extractor functions, for example

- `fitted(m)`, `resid(m)`, `vcov(m)`, ...
- `predict(m, newdata = ..., interval=...)`
  - Predicted responses for desired combinations of new values of the regressors – `newdata`
  - Argument `interval` specifies whether **confidence** intervals for the *mean* response or **prediction** intervals for *individual* responses are returned.
- `plot(m)` – produces various diagnostic plots based on residuals (raw or standardized)

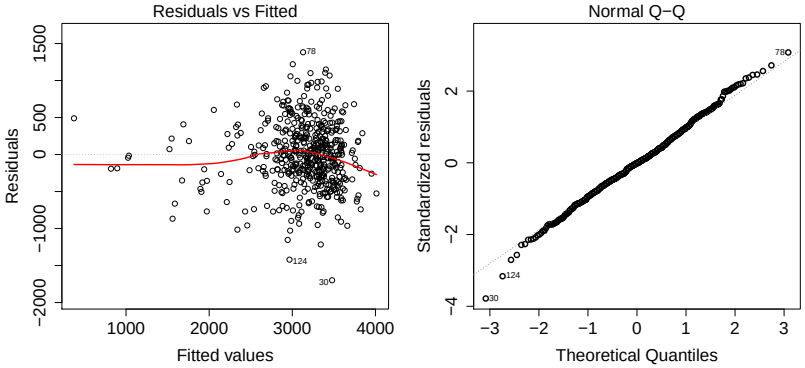
Many of these are special **methods** for certain **generic functions**, aimed at acting on objects of class "`lm`".

# Fitted values, confidence & prediction intervals




```
> nd <- data.frame( gestwks = seq(24, 45, by = 0.25 ) )
> pr.c1 <- predict( m, newdata=nd, interval="conf" )
> pr.p1 <- predict( m, newdata=nd, interval="pred" )
> with(births, plot(bweight ~ gestwks, xlim = c(24,45), cex.axis=1.5, cex.lab = 1.5, xlab = 'Gestation
> matlines( nd$gestwks, pr.c1, lty=1, lwd=c(3,2,2), col=c('red','blue','blue'))
> matlines( nd$gestwks, pr.p1, lty=1, lwd=c(3,2,2), col=c('red','green','green'))
```

# A couple of diagnostic plots




```
> par(mfrow=c(1,2))
> plot(m, 1:2, cex.lab = 1.5, cex.axis=1.5, cex.caption=1.5, lwd=2)
```

- Some deviation from linearity?
- Reasonable agreement with Gaussian error assumption?

# Factor as an explanatory variable

- How `bweight` depends on maternal hypertension?

```
> mh <- lm( bweight ~ hyp, data=births)

             Estimate   2.5%  97.5%
(Intercept)   3198.9 3140.2 3257.6
hypH          -430.7 -585.4 -275.9
```

- Removal of intercept $\rightarrow$ mean `bweights` by `hyp`:

```
> mh2 <- lm( bweight ~ -1 + hyp, data = births)
> coef(mh2)
  hypN   hypH
3198.9 2768.2
```

- Interpretation: `-430.7 = 2768.2 - 3198.9` = difference between level 2 *vs.* reference level 1 of `hyp`

# Additive model with both gestwks and hyp

- Joint effect of `hyp` and `gestwks` under additivity is modelled e.g. by updating a simpler model:

```
> mhg <- update(mh, . ~ . + gestwks)
             Estimate    2.5%   97.5%
(Intercept)  -4285.0 -4969.7 -3600.3
hypH          -143.7  -259.0   -28.4
gestwks        192.2   174.7   209.8
```

- The effect of `hyp`: `H` *vs.* `N` is attenuated (from $-430.7$ to $-143.7$).
- This suggests that much of the effect of hypertension on birth weight is mediated through a shorter gestation period among hypertensive mothers.

# Model with interaction of hyp and gestwks

- `mhgi <- lm(bweight ~ hyp + gestwks + hyp:gestwks, data = births)`
- Or with shorter formula: `bweight ~ hyp * gestwks`

| | Estimate | 2.5% | 97.5% |
|---|---|---|---|
| (Intercept) | -3960.8 | -4758.0 | -3163.6 |
| hypH | -1332.7 | -2841.0 | 175.7 |
| gestwks | 183.9 | 163.5 | 204.4 |
| hypH:gestwks | 31.4 | -8.3 | 71.1 |

- Estimated slope: 183.9 g/wk in reference group N and $183.9 + 31.4 =$ 215.3 g/wk in hypertensive mothers.

$\Leftrightarrow$ For each additional week the difference in mean bweight between H and N group increases by 31.4 g.

- *Interpretation of* `Intercept` *and "main effect"* `hypH`*?*

# Model with interaction (cont'd)

More interpretable parametrization obtained if `gestwks` is **centered** at some reference value, using e.g. the **insulate** operator `I()` for explicit transformation of an original term.

- `mi2 <- lm(bweight ~ hyp*I(gestwks-40), ...)`

| | Estimate | 2.5% | 97.5% |
|---|---|---|---|
| (Intercept) | 3395.6 | 3347.5 | 3443.7 |
| hypH | -77.3 | -219.8 | 65.3 |
| I(gestwks - 40) | 183.9 | 163.5 | 204.4 |
| hypH:I(gestwks - 40) | 31.4 | -8.3 | 71.1 |

- Main effect of `hyp` $= -77.3$ is the difference between `H` and `N` at `gestwks` $= 40$.
- `Intercept` $= 3395.6$ is the estimated mean `bweight` at the reference value 40 of `gestwks` in group `N`.

# Factors and contrasts in R

- A categorical explanatory variable or **factor** with $L$ **levels** will be represented by $L-1$ linearly independent columns in the **model matrix** of a linear model.
- These columns can be defined in various ways implying alternative **parametrizations** for the effect of the factor.
- Parametrization is defined by given type of **contrasts**.
- Default: **treatment** contrasts, in which 1st class is the **reference**, and regression coefficient $\beta_k$ for class $k$ is interpreted as $\beta_k = \mu_k - \mu_1$
- Own parametrization may be tailored by function `C()`, with the pertinent **contrast matrix** as argument.
- Or, use `ci.lin(mod, ctr.mat = CM)` after fitting.

# Two factors: additive effects

- Factor $X$ has 3 levels, $Z$ has 2 levels – Model:

$$\mu = \alpha + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \gamma_1 Z_1 + \gamma_2 Z_2$$

- $X_1$ (reference), $X_2, X_3$ are the indicators for $X$,
- $Z_1$ (reference), $Z_2$ are the indicators for $Z$.
- Omitting $X_1$ and $Z_1$ the model for mean is:

$$\mu = \alpha + \beta_2 X_2 + \beta_3 X_3 + \gamma_2 Z_2$$

with predicted means $\mu_{jk}$ $(j = 1, 2, 3; k = 1, 2)$:

| | | $Z = 1$ | $Z = 2$ |
|---|---|---|---|
| $X$ | 1 | $\mu_{11} = \alpha$ | $\mu_{11} = \alpha + \gamma_2$ |
| | 2 | $\mu_{21} = \alpha + \beta_2$ | $\mu_{22} = \alpha + \beta_2 + \gamma_2$ |
| | 3 | $\mu_{31} = \alpha + \beta_3$ | $\mu_{32} = \alpha + \beta_3 + \gamma_2$ |

# Two factors with interaction

- Effect of $Z$ differs at different levels of $X$:

| | | $Z=1$ | $Z=2$ |
|---|---|---|---|
| $X$ | 1 | $\mu_{11}=\alpha$ | $\mu_{12}=\alpha+\gamma_2$ |
| | 2 | $\mu_{21}=\alpha+\beta_2$ | $\mu_{22}=\alpha+\beta_2+\gamma_2+\delta_{22}$ |
| | 3 | $\mu_{31}=\alpha+\beta_3$ | $\mu_{32}=\alpha+\beta_3+\gamma_2+\delta_{32}$ |

- How much the effect of $Z$ (level 2 *vs.* 1)
  changes when the level of $X$ is changed from 1 to 3:

$$\begin{aligned}\delta_{32} &= (\mu_{32}-\mu_{31})-(\mu_{12}-\mu_{11}) \\ &= (\mu_{32}-\mu_{12})-(\mu_{31}-\mu_{11}),\end{aligned}$$

  $=$ how much the effect of $X$ (level 3 *vs.* 1)
  changes when the level of $Z$ is changed from 1 to 2.

- See the exercise: interaction of `hyp` and `gest4`.

# Contrasts in R

- All contrasts can be implemented by supplying a suitable **contrast function** giving the **contrast matrix** e.g:

```
> contr.cum(3)          > contr.sum(3)
1 0 0                    1    1   0
2 1 0                    2    0   1
3 1 1                    3   -1  -1
```

- In model formula factor name `faktori` can be replaced by expression like `C(faktori, contr.cum)`.
- Function `ci.lin()` has an option for calculating CI's for linear functions of the parameters of a fitted model `mall` when supplied by a relevant contrast matrix

```
> ci.lin(mall, ctr.mat = CM)[ , c(1,5,6)]
```

→ No need to specify contrasts in model formula!

# From linear to generalized linear models

- An alternative way of fitting our 1st Gaussian model:

```
> m <- glm(bweight ~ gestwks, family=gaussian, data=births)
```

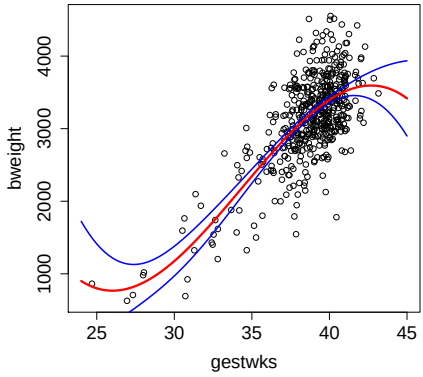
- Function `glm()` fits **generalized linear models** (GLM).
- Requires specification of the
  - **family** – i.e. the assumed "error" distribution for $Y_i$s,
  - **link** function – a transformation of the expected $Y_i$.
- Covers common models for other types of response variables and distributions, too, e.g. **logistic** regression for binary responses and **Poisson** regression for counts.
- Fitting: method of **maximum likelihood**.
- Many extractor functions for a `glm` object similar to those for an `lm` object.

# More about numeric regressors

What if dependence of $Y$ on $X$ is non-linear?

- **Categorize** the values of $X$ into a factor.
  - Continuous effects violently discretized by often arbitrary cutpoints. – Inefficient.
- Fit a low-degree (e.g. 2 to 4) **polynomial** of $X$.
  - Tail behaviour may be problematic.
- Use **fractional polynomials**.
  - Invariance problems. Only useful if $X = 0$ is well-defined.
- Use a **spline** model: smooth function $s(X; \beta)$. – See Martyn's lecture
  - More flexible models that act locally.
  - Effect of $X$ reported by graphing $\widehat{s}(X; \beta)$ & its CI

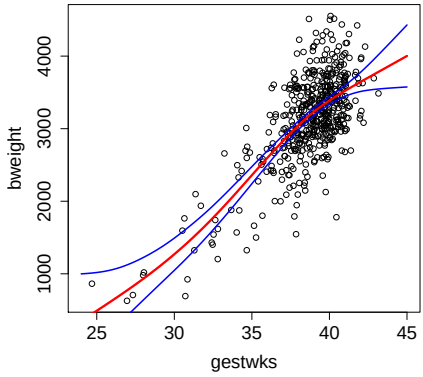
# Mean bweigth as 3rd order polynomial of gestwks




```
> mp3 <- update( m, . ~ . - gestwks + poly(gestwks, 3) )
```

- The model is linear in parameters with 4 terms & 4 df.
- Otherwise good, but the tails do not behave well.

# Penalized spline model with cross-validation




```
> library(mgcv)
> mpen <- gam( bweight ~ s(gestwks), data = births)
```

- Looks quite nice.
- Model df $\approx 4.2$; close to $4$, as in the 3rd degree polynomial model.

# What was covered

- A wide range of models from simple linear regression to splines.
- R functions fitting linear and generalized models:
  `lm()` and `glm()`.
- Parametrization of categorical explanatory factors; contrast matrices.
- Extracting results and predictions:
  `ci.lin()`, `fitted()`, `predict()`, `...` .
- Model diagnostics:
  `resid()`, `plot.lm()`, `...` .