A Physics Evoked Meshfree Method

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1 Introduction

The meshfree methods with kernel approximation have been widely used, such as:

SPH — Gingold & Monaghan, 1977,
RKPM — Liu, 1995,
CSPM — Chen, 1999,
MSPH — Zhang, 2004.

RKPM, CSPH, and MSPH proved the kernel approximation of SPH step by step, but they all recur to the correction for the kernel functions merely with mathematical ideas.

RKPM correct the kernel function as follows:

$$W^{(\mathrm{RKPM})}(\boldsymbol{r}-\boldsymbol{r}',\ h)=C^{(\mathrm{RKPM})}(\boldsymbol{r}-\boldsymbol{r}',\ h)W^{(\mathrm{SPH})}(\boldsymbol{r}-\boldsymbol{r}',\ h)$$

and $C^{(\mathrm{RKPM})}(\boldsymbol{r}-\boldsymbol{r}',\ h)$ is obtained from

$$\begin{cases}\int_{\Omega}W^{(\mathrm{RKPM})}(\boldsymbol{r}-\boldsymbol{r}',\ h)\mathrm{d}\boldsymbol{r}'=1\\ \int_{\Omega}W^{(\mathrm{RKPM})}(\boldsymbol{r}-\boldsymbol{r}',\ h)(\boldsymbol{r}-\boldsymbol{r}')^{m}\,\mathrm{d}\boldsymbol{r}'=0,\ m=1,2,\cdots \mathrm{n}_{\mathrm{c}}.\end{cases}$$

Although the accuracy of approximation is improved, it is found these methods do not overcome the deficiencies substantially as they are applied to dynamic problems.

For pure fluid dynamics

Governing equations based on the conservation law are:

Mass: $$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{v}$$

Momentum: $$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p$$

Energy: $$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{p}{\rho}\nabla\cdot\boldsymbol{v}$$

If use the following SPH discretizations:

Mass:

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \sum_{j=1}^{N} m_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \cdot \nabla_i W_{ij}^{(\mathrm{SPH})} \tag{1}$$

Momentum:

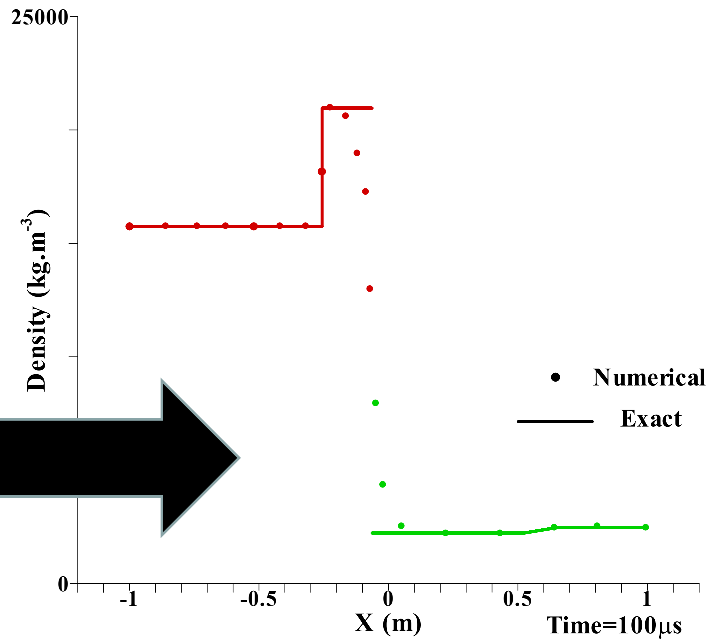
$$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_{j=1}^{N} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij}^{(\mathrm{SPH})} \right) \nabla_i W_{ij}^{(\mathrm{SPH})} \tag{2}$$

Energy:

$$\frac{\mathrm{d}e_i}{\mathrm{d}t} = \frac{1}{2} \sum_{j=1}^{N} m_j \left[\left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij}^{(\mathrm{SPH})} \right) \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) + H_{ij}^{(\mathrm{SPH})} \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \right] \cdot \nabla_i W_{ij}^{(\mathrm{SPH})} \tag{3}$$

We will have illusive results:

1) Interfaces separation;
2) Obvious wall heating;
3) Density/energy dissipation (or averaging phenomena).



One of the most serious problem is the non-physical interface separation when high-pressure gas pushes the low-pressure heavy metal.

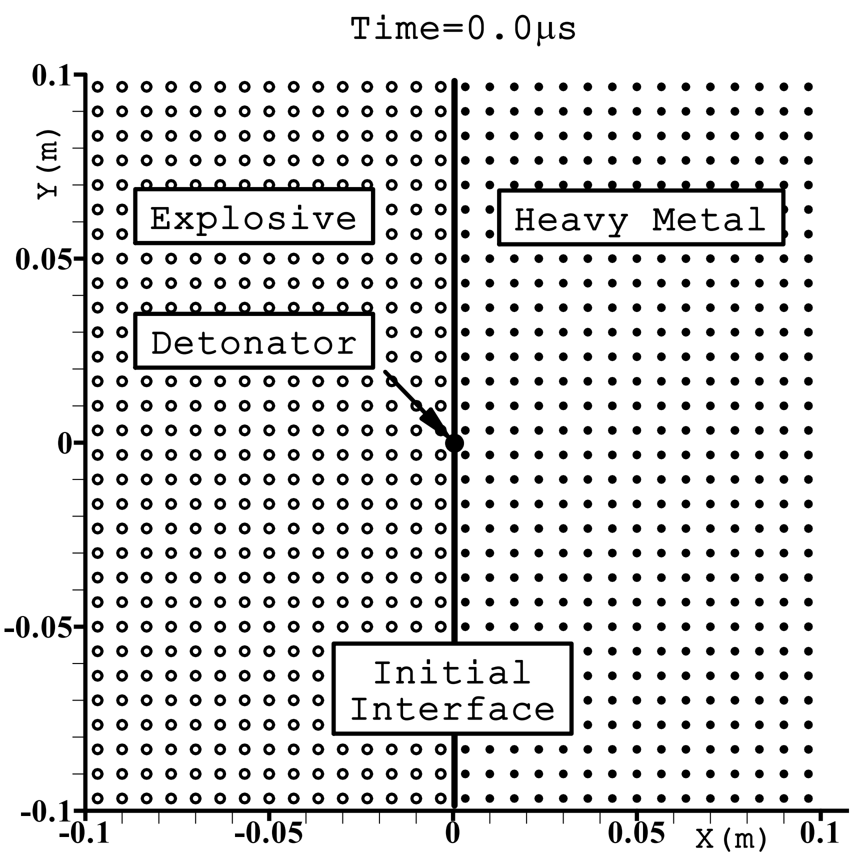


Fig. 1 Initial state of detonation model

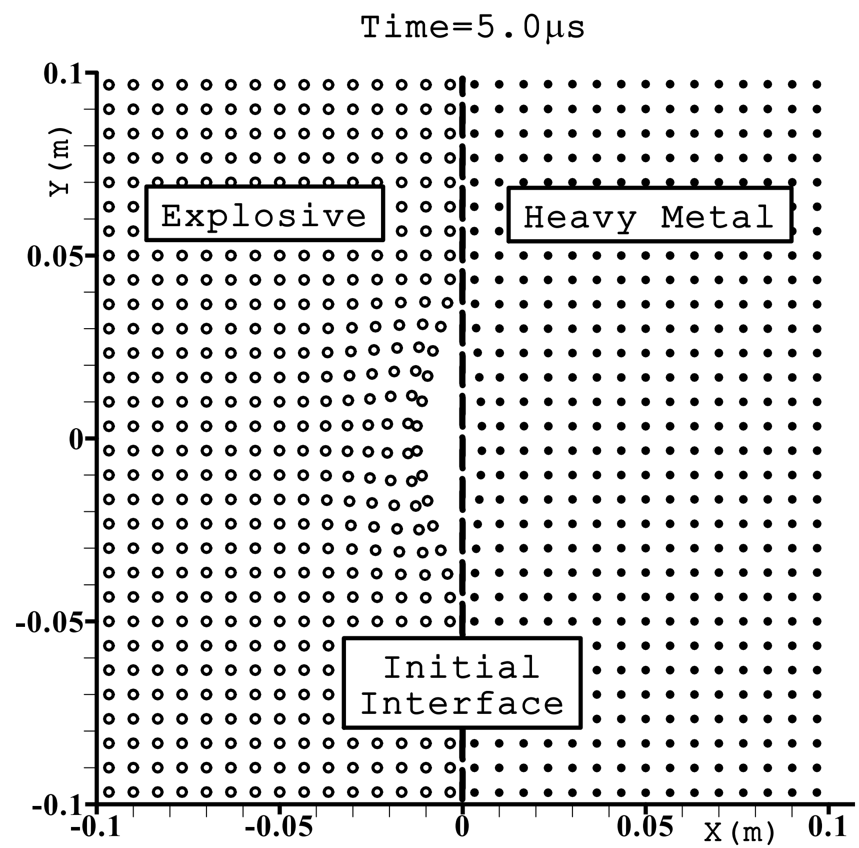


Fig. 2 Interface splitting after detonation

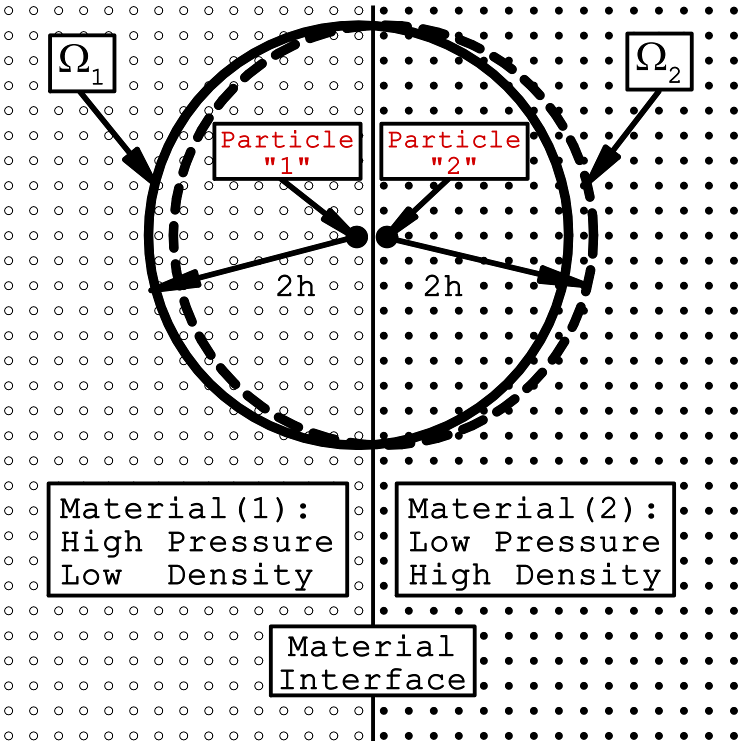


Fig. 3 Support domains of particles near interfaces

Momentum "1"

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\sum_j^N m_j \left(\frac{p_1}{\rho_1^2} + \boxed{\frac{p_j}{\rho_j^2} + \Pi_{1j}^{(\mathrm{SPH})}} \right) \nabla_1 W_{1j}^{(\mathrm{SPH})}$$

Momentum "2"

$$\frac{\mathrm{d}\boldsymbol{v}_2}{\mathrm{d}t} = -\sum_j^N m_j \left(\frac{p_2}{\rho_2^2} + \boxed{\frac{p_j}{\rho_j^2} + \Pi_{2j}^{(\mathrm{SPH})}} \right) \nabla_2 W_{2j}^{(\mathrm{SPH})}$$

Residual equations

Momentum "1"

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{p_1}{\rho_1^2} \sum_{j=1}^N m_j \nabla_1 W_{1j}^{(\mathrm{SPH})}$$

Momentum "2"

$$\frac{\mathrm{d}\boldsymbol{v}_2}{\mathrm{d}t} = -\frac{p_2}{\rho_2^2} \sum_{j=1}^N m_j \nabla_2 W_{2j}^{(\mathrm{SPH})}$$

$$\nabla_1 W_{1j}^{(\mathrm{SPH})} \approx \nabla_2 W_{2j}^{(\mathrm{SPH})}$$

$$p_1 / \rho_1^2 >> p_2 / \rho_2^2$$

"Interface Separation" can be easily revealed

To our opinion, this problem occurs in the kernel approximation for the momentum equation.

All of SPH, RKPM, CSPH, and MSPH allocate the weight by volume when exerting kernel approximation for momentum equation, but, according to the second law of Newton, the weight should be allocated by mass, i.e., density should be included into the kernel function.

System Θ as example, its theoretical value of acceleration should be

$$a_{\mathrm{x}}^{*} = \frac{(p_1 - p_3)}{s(\rho_1 + \rho_2)}$$

If allocate the weight by mass, we obtain the averaged acceleration as

$$a_{\mathrm{x}}^{mass} = \frac{s\rho_1}{s(\rho_1 + \rho_2)} \frac{(p_1 - p_2)}{s\rho_1} + \frac{s\rho_2}{s(\rho_1 + \rho_2)} \frac{(p_2 - p_3)}{s\rho_2} = a_{\mathrm{x}}^{*}$$

Otherwise, if allocate the weight by volume, the averaged acceleration is

$$a_{\mathrm{x}}^{volume} = \frac{s}{s+s} \frac{(p_1 - p_2)}{s\rho_1} + \frac{s}{s+s} \frac{(p_2 - p_3)}{s\rho_2} = \frac{(p_1 - p_2)}{2s\rho_1} + \frac{(p_2 - p_3)}{2s\rho_2} \neq a_{\mathrm{x}}^{*}$$

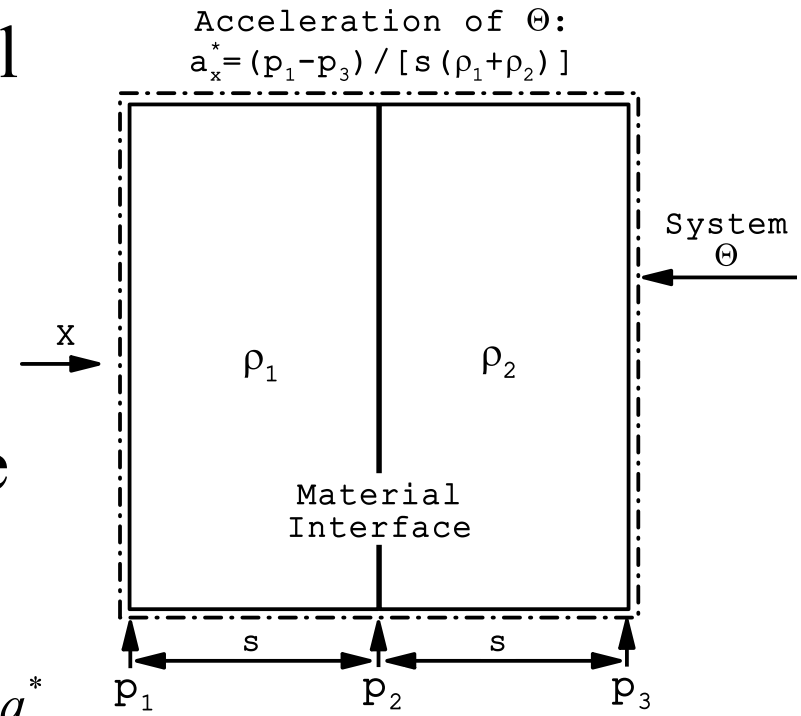


Fig. 4 Conjunct acceleration of adjacent particles

Instability maybe activated easily

Wall heating

Wall heating is caused by over-evaluating the negative divergence when computing the artificial viscosity. So, I think it can be expectantly restrained by adding a modularizing factor about velocity.

Density/Energy dissipation (or averaging phenomena)

In SPH schemes, variables that contain the information of density ρ_j exist inside the summation functor Σ, this may lead to density/energy dissipations.

Mass: $$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \sum_{j=1}^{N} \boxed{m_j} \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \cdot \nabla_i W_{ij}^{(\mathrm{SPH})}$$

Lead to density/energy dissipations

Energy: $$\frac{\mathrm{d}e_i}{\mathrm{d}t} = \frac{1}{2} \sum_{j=1}^{N} \boxed{m_j} \left[\left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\boxed{\rho_j^2}} + \Pi_{ij}^{(\mathrm{SPH})} \right) \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) + H_{ij}^{(\mathrm{SPH})} \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \right] \cdot \nabla_i W_{ij}^{(\mathrm{SPH})}$$

2 Main Ideas of PECM Method

M&S are confronted with various challenges that come from application:

1) Coupling of multiple matters;
2) Coupling of multiple physics;
3) Density from vacuum-very low-very high;
4) Large deformation or swirls;
5) Crack development with history experiences;
6) Mass transfer between discrete volumes.

Classify the matters as two types:

1) Substantial matters
   - High degree of assembling;
   - Evident macro-density;
   - Action consistency in discrete cloud;
   - Generally in thermal equilibrium;
   - Dominate virtual matters and physics procedures;
   - Actions influenced by previous experiences;
   - Exist as solid/liquid/gas/particles/drops, etc.;
2) Virtual matters
   - Low degree of assembling;
   - No evident macro-density;
   - Action inconsistency in discrete cloud;
   - Generally not in thermal equilibrium;
   - Dominated by substantial matters;
   - Actions independent with previous experiences;
   - Exist as magnetic field/rays of x,α,γ/neutron/electron, etc.;

Bring forward a mesh-free method, named PECM-Physics Evoked Cloud Method. It doesn’t have detailed descriptions in approximation, Modularizing Factor or discrete equation, but should observe six principles:

1) Unitary meshfree computing for all substantial matters;
2) Commonness and speciality of algorithms are respectively carried by inheriting and deriving of the matter-classes;
3) Exchange information without manufactured slip sides;
4) Trustily represent continuity or discontinuity in reality;
5) Can represent mass transfer between discrete clouds;
6) All the numerical schemes must observe physical laws.

3 Discretized Equations of PECM

Kernel Approximation

For the right hand side of momentum equation, the kernel approximation of PECM is as:

$$\left\langle -\frac{1}{\rho}\nabla p \right\rangle_i = \int_{\Omega_i}\left(-\frac{1}{\rho}\nabla_{r'} p\right)\rho W^{(\mathrm{PECM})}(\boldsymbol{r}_i - \boldsymbol{r}',\ h_i)\mathrm{d}\boldsymbol{r}' = -\sum_{j=1}^{N} b_j p_j \nabla_i W_{ij}^{(\mathrm{PECM})}$$

Where $b_j = m_j / \rho_j$, $W^{(\mathrm{PECM})}(\boldsymbol{r} - \boldsymbol{r}',\ h) = C^{(\mathrm{PECM})}(\boldsymbol{r} - \boldsymbol{r}',\ h)\ W^{(\mathrm{SPH})}(\boldsymbol{r} - \boldsymbol{r}',\ h)$

$C^{(\mathrm{PECM})}(\boldsymbol{r} - \boldsymbol{r}',\ h)$ is obtained from

$$\int_{\Omega}\rho(\boldsymbol{r}')W^{(\mathrm{PECM})}(\boldsymbol{r} - \boldsymbol{r}',\ h)\mathrm{d}\boldsymbol{r}' = 1$$

$$\int_{\Omega}\rho(\boldsymbol{r}')W^{(\mathrm{PECM})}(\boldsymbol{r} - \boldsymbol{r}',\ h)(\boldsymbol{r} - \boldsymbol{r}')^m \mathrm{d}\boldsymbol{r}' = 0$$

$m = 1,\ \ 2, \cdots \mathrm{n_c}.$

$\Rightarrow$

$$\sum_{j=1}^{N} b_j \rho_j C_{ij}^{(\mathrm{PECM})} W_{ij}^{(\mathrm{SPH})} = 1$$

$$\sum_{j=1}^{N} b_j \rho_j C_{ij}^{(\mathrm{PECM})} W_{ij}^{(\mathrm{SPH})} \boldsymbol{r}_{ij}^m = 0$$

$m = 1,\ \ 2, \cdots \mathrm{n_c}.$

Modularizing Factors

(1) Viscous pressure

$$\Pi_{ij}^{(\mathrm{PECM})} = \begin{cases} \left(-\alpha c_{ij}\mu_{ij} + \beta\mu_{ij}^2\right)\rho_{ij} & \left(\mathbf{v}_i - \mathbf{v}_j\right)\cdot\left(\mathbf{r}_i - \mathbf{r}_j\right) < 0 \\ 0 & \left(\mathbf{v}_i - \mathbf{v}_j\right)\cdot\left(\mathbf{r}_i - \mathbf{r}_j\right) \geq 0 \end{cases}$$

(2) Heat flux

$$\boldsymbol{H}_{ij}^{(\mathrm{PECM})} = 2\zeta_{ij}\rho_{ij}\left(e_i - e_j\right)\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)/\left[\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)\cdot\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) + \eta h_{ij}^2\right]$$

(3) Velocity adjustment

$$\boldsymbol{\Phi}_{ij}^{(\mathrm{PECM})} = \delta h_{ij}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)\left(p_i - p_j\right)/\left[\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)\cdot\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) + \theta h_{ij}^2\right]/\,\overline{c}_{ij}\,/\,\overline{\rho}_{ij}$$

Discrete Equations of PECM

Mass:

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \rho_i \sum_{j=1}^{N} b_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j + \boldsymbol{\Phi}_{ij}^{(\mathrm{PECM})} \right) \cdot \nabla_i W_{ij}^{(\mathrm{RKPM})}$$

Momentum:

$$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_{j=1}^{N} b_j \left(p_j + \Pi_{ij}^{(\mathrm{PECM})} \right) \nabla_i W_{ij}^{(\mathrm{PECM})}$$

Energy:

$$\frac{\mathrm{d}e_i}{\mathrm{d}t} = \frac{1}{\rho_i} \sum_{j=1}^{N} b_j \left[\left(p_i + \frac{\Pi_{ij}^{(\mathrm{PECM})}}{2} \right) \left(\boldsymbol{v}_i - \boldsymbol{v}_j + \boldsymbol{\Phi}_{ij}^{(\mathrm{PECM})} \right) + \boldsymbol{H}_{ij}^{(\mathrm{PECM})} \right] \cdot \nabla_i W_{ij}^{(\mathrm{RKPM})}$$

4 Numerical Examples

Examples are five 1-dimensional models in which strong discontinuity exist!

Table 1 Entity models of examples

Zone	Examples	Coordinate-x (m)	Density-ρ_0 (kg.m^{-3})	Energy-e_0 (J.kg^{-1})	Pressure-p_0 (Pa)	Velocity-v_0 (m.s^{-1})
Left	1	[-1.0, 0.0]	2.400×10^3	6.000×10^6	2.88×10^{10}	-3.00×10^3
	2	[-1.0, 0.0]	2.400×10^3	6.000×10^6	2.88×10^{10}	-6.00×10^3
	3	[-1.0, 0.0]	2.400×10^3	6.000×10^6	2.88×10^{10}	-9.00×10^3
	4	[-1.0, 0.0]	1.575×10^4	15.87	1.000×10^6	0.000
	5	[-1.0, 0.0]	1.575	0.000	0.000	0.000
Right	1	[0.0, 1.0]	2.400×10^3	6.000×10^6	2.88×10^{10}	3.00×10^3
	2	[0.0, 1.0]	2.400×10^3	6.000×10^6	2.88×10^{10}	6.00×10^3
	3	[0.0, 1.0]	2.400×10^3	6.000×10^6	2.88×10^{10}	9.00×10^3
	4	[0.0, 1.0]	2.470×10^3	7.085×10^6	3.50×10^{10}	0.000
	5	[0.0, 1.0]	1.575×10^4	3.175×10^5	2.00×10^{10}	0.000

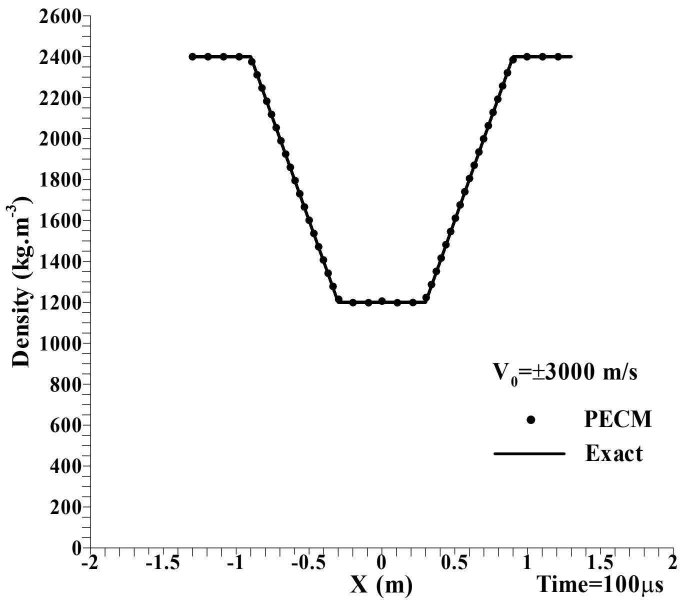
Table 2 Physics models of examples

Zone	Examples	Equation of state	Sound Speed equation	$\gamma(1)$	$c_0/(\mathrm{m.s^{-1}})$
Left	1	$p=(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6000.0
	2	$p=(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6000.0
	3	$p=(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6000.0
	4	$p=c_0^2(\rho-\rho_0)+(\gamma-1)\rho e$	$c=\sqrt{c_0^2+(\gamma-1)(e+p/\rho)}$	5.0	1290.0
	5	$p=c_0^2(\rho-\rho_0)+(\gamma-1)\rho e$	$c=\sqrt{c_0^2+(\gamma-1)(e+p/\rho)}$	1.4	760.0
Right	1	$p=(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6000.0
	2	$p=(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6000.0
	3	$p=(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6000.0
	4	$p=c_0^2(\rho-\rho_0)+(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	3.0	6520.0
	5	$p=c_0^2(\rho-\rho_0)+(\gamma-1)\rho e$	$c=\sqrt{\gamma(\gamma-1)e}$	5.0	2520.0

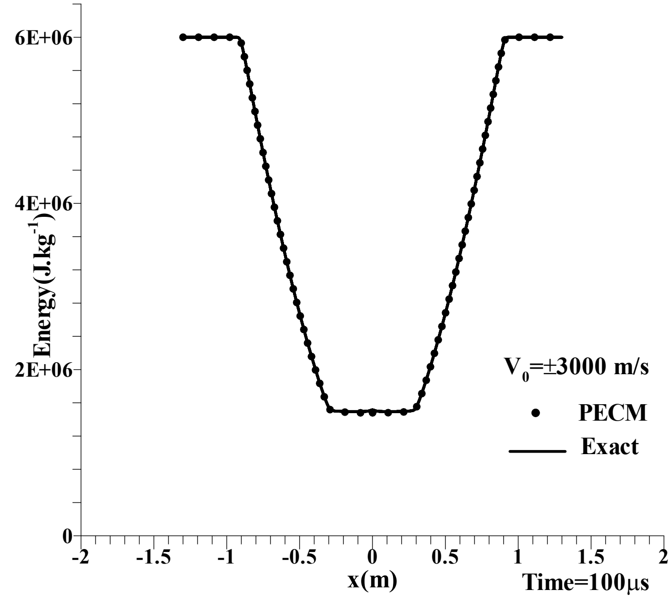
Table 3 Parameters of numerical computation

Δx/ mm	h_0/mm	τ	α	β	ε	g_1	g_2	η	δ	θ	n_c
0.5	0.75	0.5	1.0	2.0	0.01	0.1	0.2	0.01	0.5	0.01	2

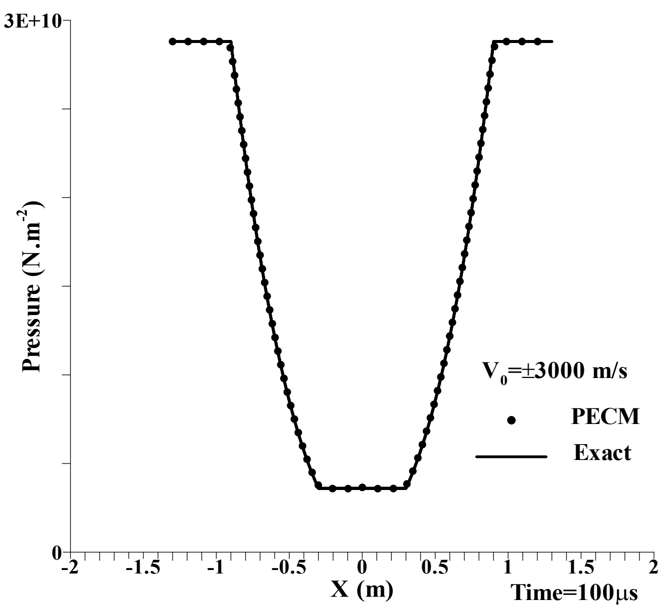
Example-1



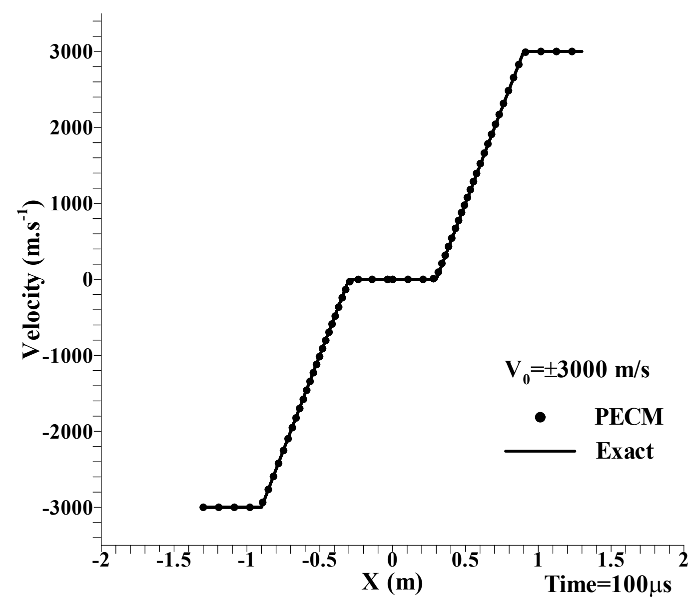
Density (kg.m-3)
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
0
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
X (m)
Time=100μs
V0=±3000 m/s
PECM
Exact



Energy(J.kg-1)
6E+06
4E+06
2E+06
0
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
x(m)
Time=100μs
V0=±3000 m/s
PECM
Exact

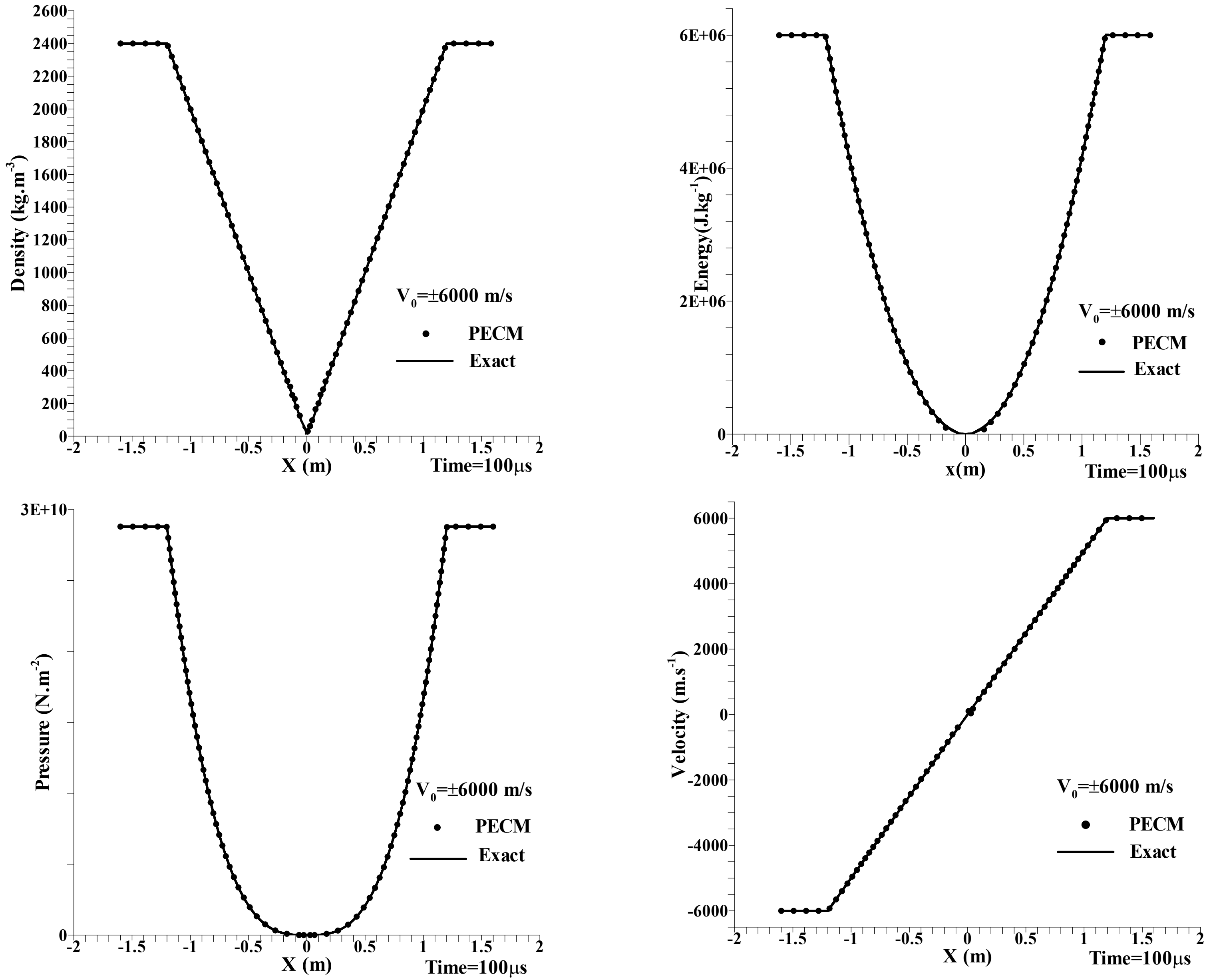


Pressure (N.m-2)
3E+10
0
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
X (m)
Time=100μs
V0=±3000 m/s
PECM
Exact

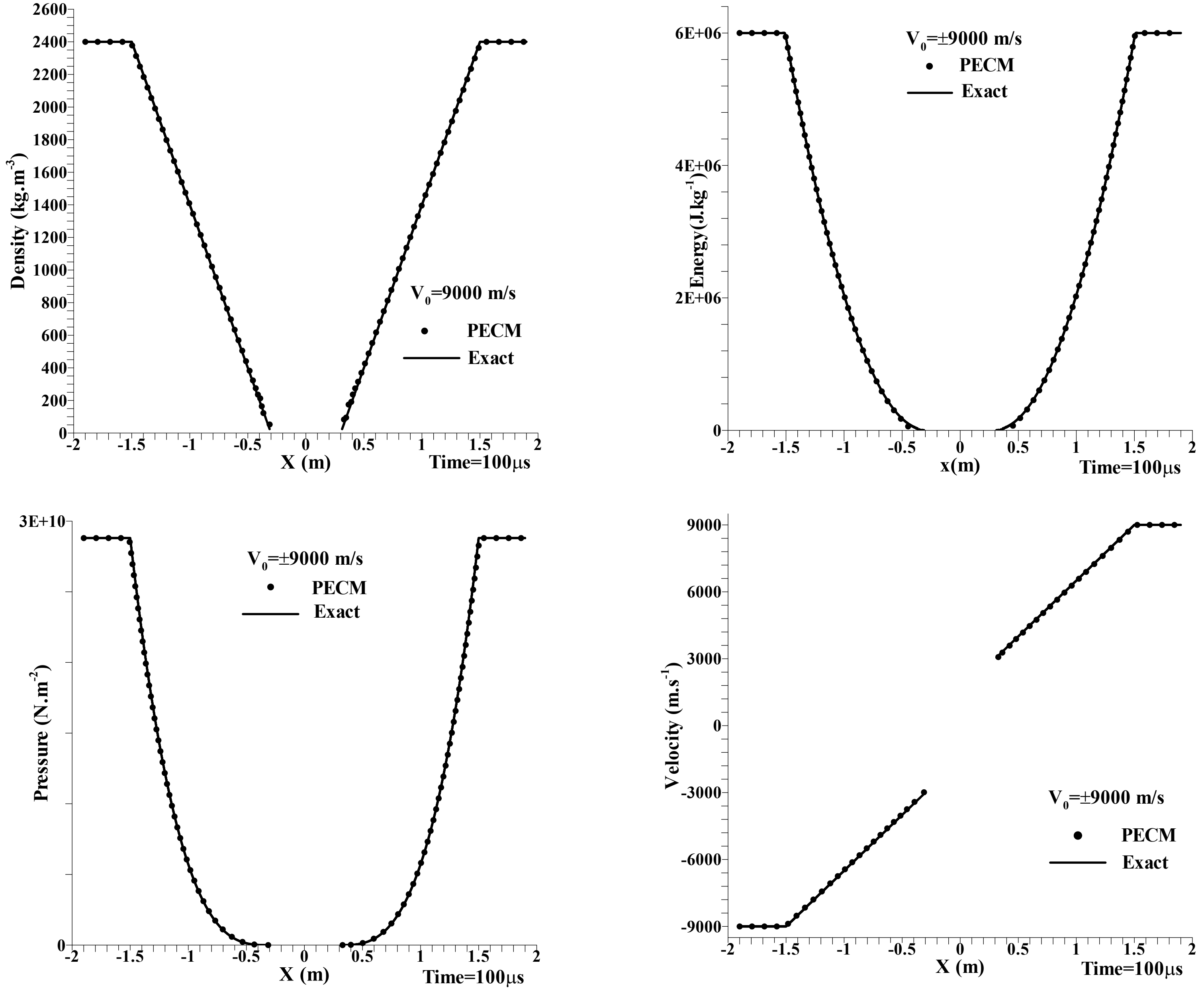


Velocity (m.s-1)
3000
2000
1000
0
-1000
-2000
-3000
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
X (m)
Time=100μs
V0=±3000 m/s
PECM
Exact

Example-2

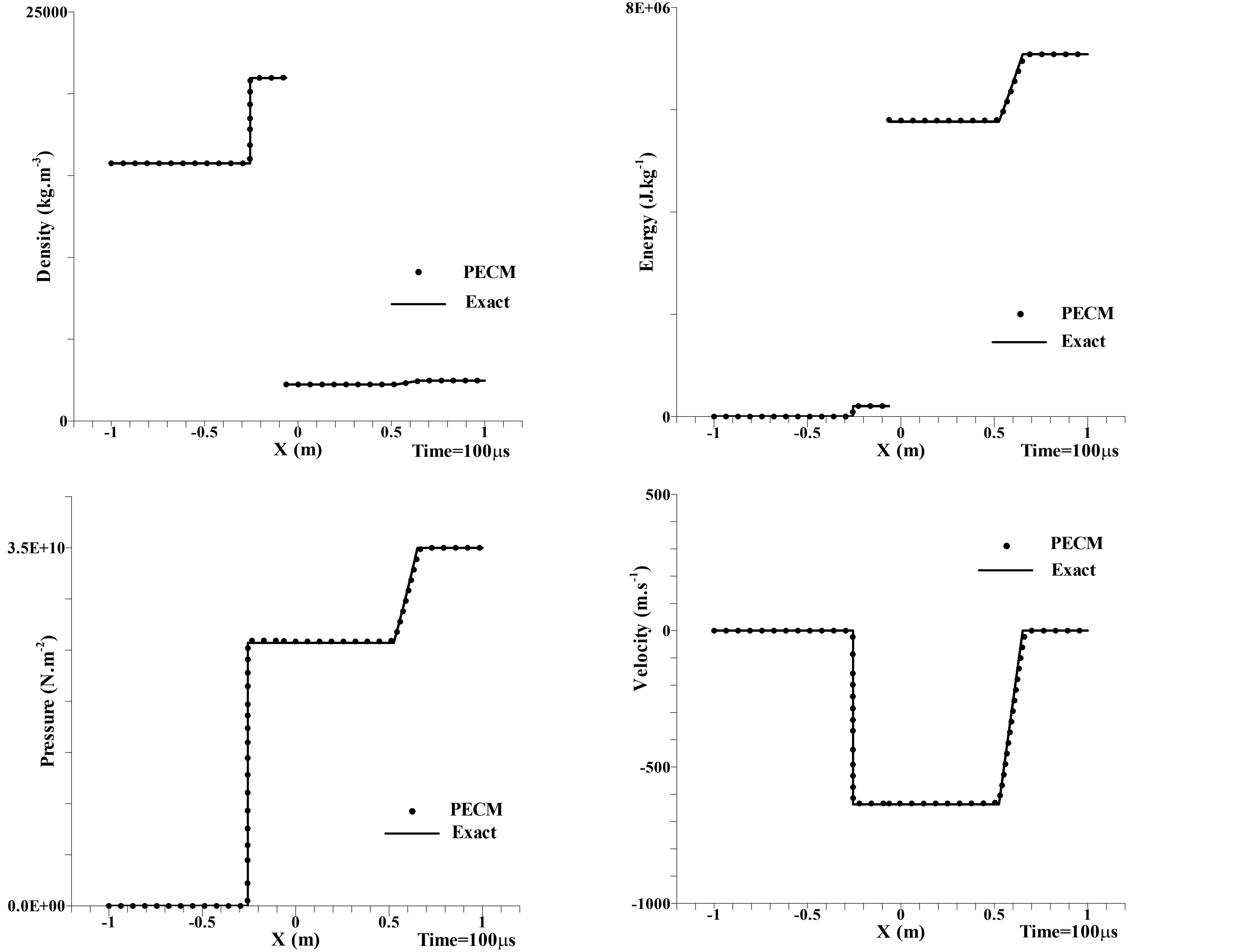

Density (kg.m^{-3})
X (m)
Time=100μs
V_0=±6000 m/s
PECM
Exact
Energy(J.kg^{-1})
x(m)
Time=100μs
V_0=±6000 m/s
PECM
Exact
Pressure (N.m^{-2})
X (m)
Time=100μs
V_0=±6000 m/s
PECM
Exact
Velocity (m.s^{-1})
X (m)
Time=100μs
V_0=±6000 m/s
PECM
Exact

Example-3



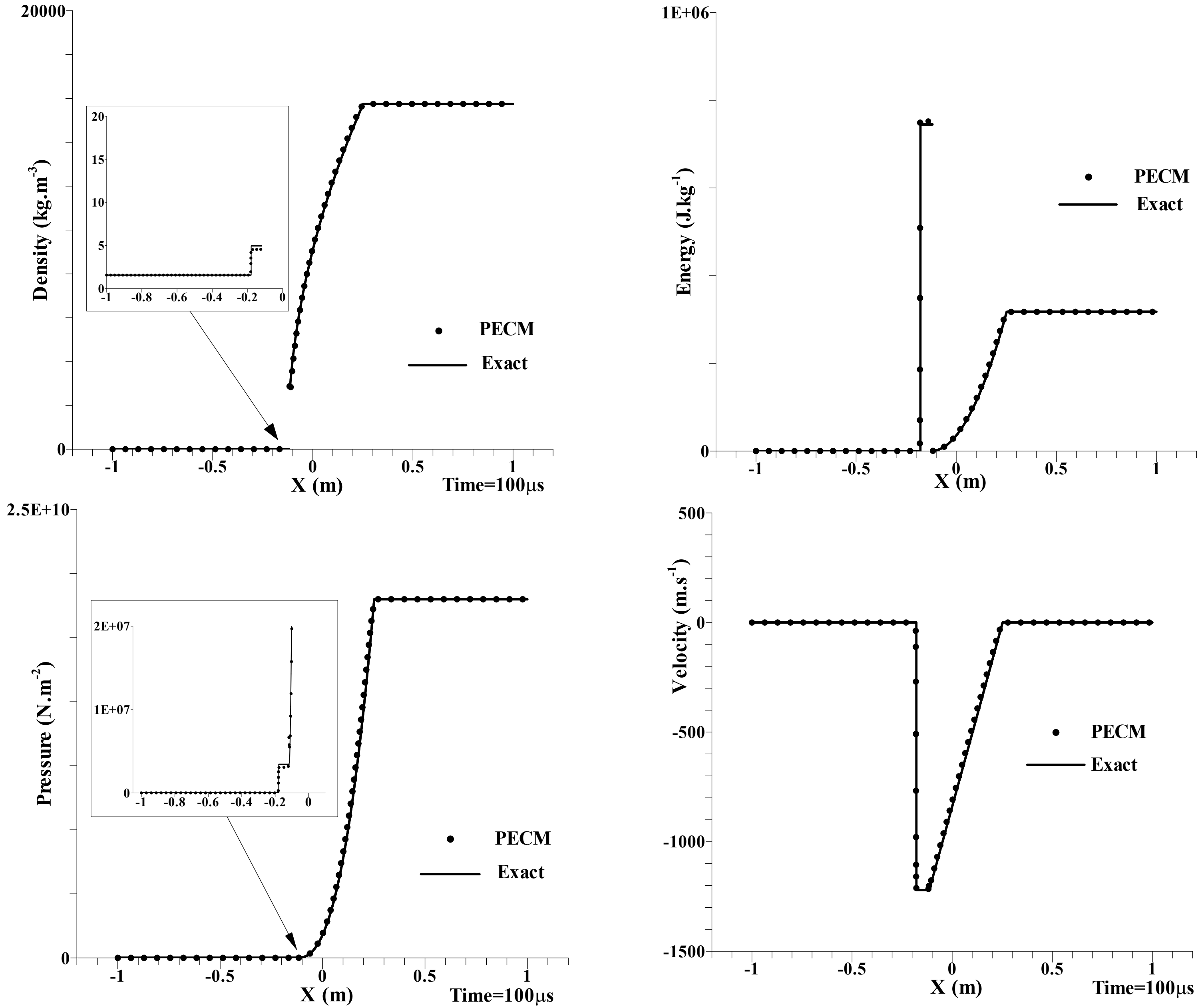
Density (kg.m^{-3})
X (m)
Time=100μs
V$_0$=9000 m/s
PECM
Exact
Energy(J.kg^{-1})
x(m)
Time=100μs
V$_0$=±9000 m/s
PECM
Exact
Pressure (N.m^{-2})
X (m)
Time=100μs
V$_0$=±9000 m/s
PECM
Exact
Velocity (m.s^{-1})
X (m)
Time=100μs
V$_0$=±9000 m/s
PECM
Exact

Example-4



25000
0
Density (kg.m-3)
PECM
Exact
-1
-0.5
0
0.5
1
X (m)
Time=100μs
8E+06
0
Energy (J.kg-1)
PECM
Exact
-1
-0.5
0
0.5
1
X (m)
Time=100μs
3.5E+10
0.0E+00
Pressure (N.m-2)
PECM
Exact
-1
-0.5
0
0.5
1
X (m)
Time=100μs
500
0
-500
-1000
Velocity (m.s-1)
PECM
Exact
-1
-0.5
0
0.5
1
X (m)
Time=100μs

Example-5



Density (kg.m^-3)
20000
0
-1
-0.5
0
0.5
1
X (m)
Time=100µs
PECM
Exact
20
15
10
5
0
-0.8
-0.6
-0.4
-0.2
Energy (J.kg^-1)
1E+06
Pressure (N.m^-2)
2.5E+10
2E+07
1E+07
Velocity (m.s^-1)
500
-500
-1000
-1500

5 Conclusions

- **In order to deal with the complicated problems including multi-matters and multi-physics, we have brought forward 6 principles to specify the new method PECM;**

- **According to the principles, we have improved the SPH and obtained a series of excellent results;**

- **It is important to design numerical schemes enlightened by physical laws.**


Thank you !