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TSINGHUA UNIVERSITY

A Two-Phase SPH Model for Sediment Transport in Free Surface Flows

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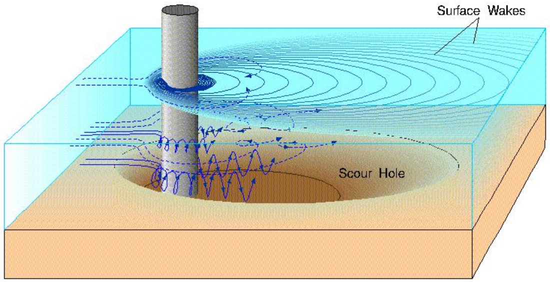
Sediment transport in violent free-surface flows

➢ Tsunami-induced beach evolution



➢ Scour around coastal structures

➢ Storm-induced beach evolution

Keys to modeling:

➢ Simulation of violent free surface flows

➢ Complex coastal topography and structures

➢ Two-phase modeling of sediment and water

SPH + Two-Phase Modeling

Contents

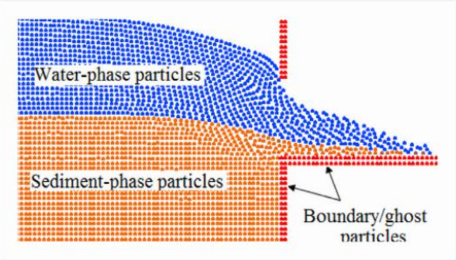
Model Development

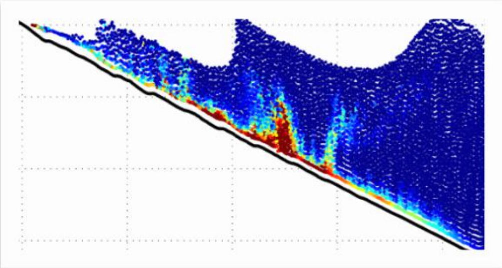
What kind of two-phase SPH model ?

SPH Two-Phase Models

Multi-density, Multi-viscosity Fluid Model

- SPH particles for both water and ‘solid’
- Solid particle is mixture
- Multi-immiscible-fluid



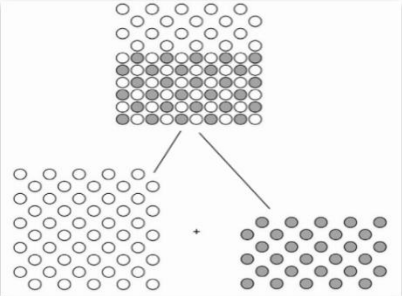


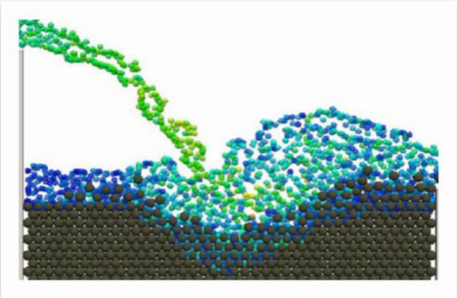
Mixture Model

- SPH particles for mixture
- phase difference

Interpenetrating-Fluid Model

- Two SPH-particle layers
- overlap
- variable smoothing length





SPH-DEM Model

- SPH particles for water DEM for solid
- solid-liquid interaction
- enormous computational cost

Needs

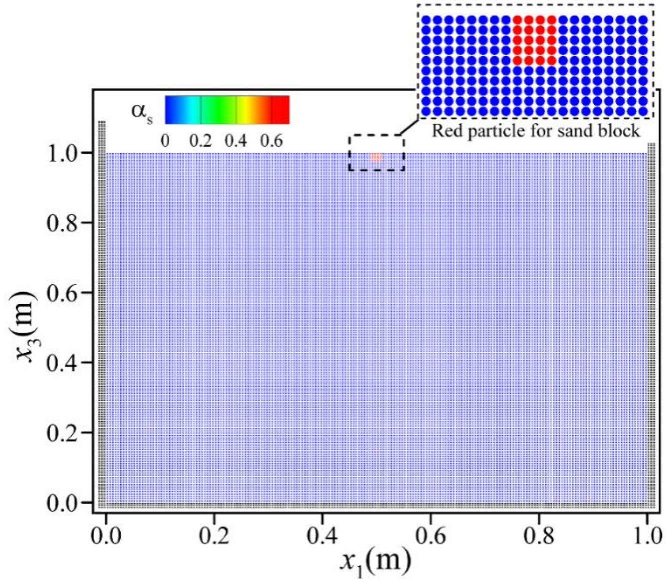
- Two-phase modeling;
- Model the water flow in granular bed materials;
- To model suspended load;
- Reduce the computational cost.

continuum two-phase formulation
+
single-SPH-particle-layer approach

What kind of two-phase SPH model ?

A continuum two-phase SPH model

- Continuum two-phase formulation of water-sediment flows
- A single set of SPH particles for the whole flow domain
- Each SPH particle moves with water velocity and carries properties of the two phases
- Use volumetric fraction for sand phase (consider suspended load)
- Fluid and solid phases are interpenetrating



Initial particle distribution for the problem of 2D sand dumping

Governing equations for two-phase flows

Continuum two-phase formulation

$$\frac{\partial \alpha_k \rho_k}{\partial t} + \frac{\partial\left(\alpha_k \rho_k u_{k,j}\right)}{\partial x_j} = 0$$

$$\frac{\partial\left(\alpha_k \rho_k u_{k,i}\right)}{\partial t} + \frac{\partial\left(\alpha_k \rho_k u_{k,i} u_{k,j}\right)}{\partial x_j} = -\alpha_k \frac{\partial p}{\partial x_i} + \frac{\partial\left(\alpha_k \tau_{k,ij}\right)}{\partial x_j} + \alpha_k \rho_k g_{k,i} + F_{k,i}$$

$k = f, s$

α : volume fraction

ρ : density

u : velocity

p : pressure

τ : viscous stress

F : interphase force

Spatially filtering and Favre averaging

$$\frac{\partial\left(\overline{\alpha_k \rho_k}\right)}{\partial t} + \frac{\partial\left(\overline{\alpha_k \rho_k} \tilde{u}_{k,j}\right)}{\partial x_j} = 0$$

$$\tilde{f}_k = \frac{\overline{\alpha_k \rho_k f_k}}{\overline{\alpha_k \rho_k}}$$

$$\frac{\partial\left(\overline{\alpha_k \rho_k} \tilde{u}_{k,i}\right)}{\partial t} + \frac{\partial\left(\overline{\alpha_k \rho_k} \tilde{u}_{k,i} \tilde{u}_{k,j}\right)}{\partial x_j} = -\overline{\alpha}_k \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial\left(\tau^{0}_{k,ij} + \tau^{SPS}_{k,ij}\right)}{\partial x_j} + \overline{\alpha_k \rho_k} g_i + \overline{F}_{\mathrm{int},i}$$

Governing equations for two-phase flows

Substantial derivative

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u_{f,j}\frac{\partial}{\partial x_j}$$

☐ Lagrangian governing equations for SPH particles

- Water density $$\frac{d\left(\alpha_f\rho_f\right)}{dt}=-\alpha_f\rho_f\frac{\partial u_{f,j}}{\partial x_j}$$

- Water velocity $$\frac{du_{f,i}}{dt}=-\frac{1}{\rho_{f0}}\frac{\partial p_f}{\partial x_i}+\frac{1}{\alpha_f\rho_f}\frac{\partial\left[\alpha_f\rho_f\left(\tau^0_{f,ij}+\tau^{SPS}_{f,ij}\right)\right]}{\partial x_j}+g_i-\underbrace{\frac{\gamma\alpha_s}{\alpha_f\rho_f}\left(u_{f,i}-u_{s,i}\right)}_{\text{interphase force: drag}}$$
$$\underbrace{+\frac{\gamma\alpha_s}{\alpha_f\rho_f}\frac{\nu^t_f}{\alpha_f\mathrm{Sc}}\frac{\partial\ln\alpha_s}{\partial x_i}}_{\text{For numerical stability at interface}}$$

- Sand concentration $$\frac{d\alpha_s}{dt}=-\alpha_s\frac{\partial u_{f,j}}{\partial x_j}-\frac{\partial\left[\alpha_s\left(u_{s,j}-u_{f,j}\right)\right]}{\partial x_j}$$

- Sand velocity $$\frac{du_{s,i}}{dt}=-\frac{1}{\rho_s}\frac{\partial p_f}{\partial x_i}+\frac{1}{\alpha_s\rho_s}\frac{\partial\left[\alpha_s\rho_s\left(\tau^0_{s,ij}+\tau^{SPS}_{s,ij}\right)\right]}{\partial x_j}+g_i+\frac{\gamma}{\rho_s}\left(u_{f,i}-u_{s,i}\right)$$
$$-\frac{\gamma}{\rho_s}\frac{\nu^t_f}{\alpha_f\mathrm{Sc}}\frac{\partial\ln\alpha_s}{\partial x_i}-\underbrace{\left(u_{s,j}-u_{f,j}\right)\frac{\partial u_{s,i}}{\partial x_j}}_{\text{inter-particle sediment momentum flux}}$$

Governing equations for two-phase flows

- Turbulence formulation

$$\tau_{k,ij}^{SPS} = \nu_k^t \left(\frac{\partial u_{k,i}}{\partial x_j} + \frac{\partial u_{k,j}}{\partial x_i} \right)$$

ν_k^t : turbulent viscosity coefficient

Smagorinsky model considering effect of solid particles on turbulence

$$\nu_k^t = \left(C_k \Delta \right)^2 \left| \mathbf{S}_k \right| \left(1 - \frac{\alpha_s}{\alpha_{sm}} \right)^n \qquad \left| \mathbf{S}_k \right| = \sqrt{2 S_{k,ij} S_{k,ij}} \qquad S_{k,ij} = \frac{1}{2} \left(\frac{\partial u_{k,i}}{\partial x_j} + \frac{\partial u_{k,j}}{\partial x_i} \right)$$

- Constitutive relations

$$\tau_{f,ij}^0 = \nu_f^0 \left(\frac{\partial u_{f,i}}{\partial x_j} + \frac{\partial u_{f,j}}{\partial x_i} \right) \qquad \tau_{s,ij}^0 = \nu_s^0 \left(\frac{\partial u_{s,i}}{\partial x_j} + \frac{\partial u_{s,j}}{\partial x_i} \right) - \frac{p_s}{\rho_s} \delta_{ij}$$

ν_k^0 : kinetic viscosity

Pressure of solid phase in dense sediment-laden flows

$$p_s = \underbrace{Fr \frac{\left(\alpha_s - \alpha_* \right)^r}{\left(\alpha^* - \alpha_s \right)^s}}_{\text{enduring contact}} + \underbrace{\frac{b^2 \alpha_s^2}{\left(\alpha_{s0} - \alpha_s \right)^2} \left(\mu_f + a \rho_s d_s^{\ 2} \left| \mathbf{S}_s \right| \right) \left| \mathbf{S}_s \right|}_{\text{Rheology for collision/friction}}$$

$$\nu_s^0 = \frac{\eta p_s}{\rho_s \left| \mathbf{S}_s \right|}$$

η : friction coefficient

Novel EOS for water-sediment mixture

- Weakly Compressibility Assumption
  - water is weakly compressible
  - sediment is incompressible
- Novel Equation of State (EOS) for water pressure in the mixture

$$p = \frac{\rho_{f0} c_0^2}{\xi} \frac{\alpha_f \rho_f + \alpha_s \rho_{f0}}{\alpha_f \rho_f} \left[\left(\frac{\alpha_f \rho_f + \alpha_s \rho_{f0}}{\rho_{f0}} \right)^{\xi} - 1 \right] \qquad \xi = 7 \qquad \rho_{f0} = 1000\,\mathrm{kg/m^3}$$

- Corresponding Shepard filtering to damp pressure oscillation

$$\left(\bar{\rho}_f\right)_a = \frac{\sum_b V_b \left(\rho_f\right)_b W_{ab}}{\sum_b V_b W_{ab}} = \frac{\sum_b \frac{\left(m_f\right)_b}{1 - \left(\alpha_s\right)_b} W_{ab}}{\sum_b \frac{\left(m_f\right)_b}{\left(\alpha_f \rho_f\right)_b} W_{ab}}$$

$$\left(\overline{\alpha_f \rho_f}\right)_a = \frac{\left(\alpha_f \rho_f\right)_a}{\left(\alpha_f \rho_f\right)_a + \left(\alpha_s\right)_a \left(\bar{\rho}_f\right)_a} \left(\bar{\rho}_f\right)_a$$

$$\left(\bar{\alpha}_s\right)_a = \frac{\left(\alpha_s\right)_a}{\left(\alpha_f \rho_f\right)_a + \left(\alpha_s\right)_a \left(\bar{\rho}_f\right)_a} \left(\bar{\rho}_f\right)_a$$

SPH formulation

- Position of particle a

$$\frac{d\left(X_i\right)_a}{dt}=\left(u_{f,i}\right)_a$$

$$\frac{\partial f}{\partial x}=\sum_b \frac{\phi_b}{\phi_a}\left(f_b-f_a\right)\nabla_a W_{ab}\frac{m_b}{\rho_b}$$

$$\frac{\partial f}{\partial x}=\sum_b\left(\frac{\phi_b}{\phi_a}f_a+\frac{\phi_a}{\phi_b}f_b\right)\nabla_a W_{ab}\frac{m_b}{\rho_b}$$

- Velocity of particle a

$$\frac{du_{f,i}}{dt}=-\frac{1}{\rho_{f0}}\frac{\partial p_f}{\partial x_i}+\frac{1}{\alpha_f\rho_f}\frac{\partial \tau_f}{\partial x_j}+g_i-\frac{\gamma\alpha_s}{\alpha_f\rho_f}\left(u_{f,i}-u_{s,i}\right)+\frac{\gamma\alpha_s}{\alpha_f\rho_f}\frac{\nu_f^t}{\alpha_f \mathrm{Sc}}\frac{\partial \ln\alpha_s}{\partial x_i}$$

$$\Downarrow$$

$$\frac{d\left(u_{f,i}\right)_a}{dt}=-\frac{1}{\rho_{f0}}\sum_b V_b\left[\left(p_f\right)_a+\left(p_f\right)_b\right]\left(\nabla_a W_{ab}\right)_i+\frac{1}{\left(\alpha_f\rho_f\right)_a}\sum_b V_b\left[\left(\tau_{f,ij}\right)_a+\left(\tau_{f,ij}\right)_b\right]\left(\nabla_a W_{ab}\right)_j+g_i$$

$$-\frac{\gamma_a\left(\alpha_s\right)_a}{\left(\alpha_f\rho_f\right)_a}\left(u_{f,i}-u_{s,i}\right)_a+\frac{\gamma_a\left(\alpha_s\right)_a}{\left(\alpha_f\rho_f\right)_a}\frac{\left(\nu_f^t\right)_a}{\left(\alpha_f\right)_a \mathrm{Sc}}\sum_b V_b\ln\frac{\left(\alpha_s\right)_b}{\left(\alpha_s\right)_a}\left(\nabla_a W_{ab}\right)_i$$

SPH formulation

$$\frac{\partial f}{\partial x} = \sum_b \frac{\phi_b}{\phi_a}\left(f_b - f_a\right)\nabla_a W_{ab}\frac{m_b}{\rho_b}$$

$$\frac{\partial f}{\partial x} = \sum_b \left(\frac{\phi_b}{\phi_a} f_a + \frac{\phi_a}{\phi_b} f_b\right)\nabla_a W_{ab}\frac{m_b}{\rho_b}$$

- Water density carried by particle a

$$\frac{d\left(\alpha_f \rho_f\right)}{dt} = -\alpha_f \rho_f \frac{\partial u_{f,j}}{\partial x_j}$$

$$\Rightarrow \quad \frac{d\left(\alpha_f \rho_f\right)_a}{dt} = -\left(\alpha_f \rho_f\right)_a \sum_b V_b \left[\left(u_{f,j}\right)_a - \left(u_{f,j}\right)_b\right]\left(\nabla_a W_{ab}\right)_j$$

- Sediment concentration carried by particle a

$$\frac{d\alpha_s}{dt} = -\alpha_s \frac{\partial u_{f,j}}{\partial x_j} - \frac{\partial\left[\alpha_s\left(u_{s,j} - u_{f,j}\right)\right]}{\partial x_j}$$

$$\Downarrow$$

$$\frac{d\left(\alpha_s\right)_a}{dt} = -\left(\alpha_s\right)_a \sum_b V_b \left[\left(u_{f,j}\right)_a - \left(u_{f,j}\right)_b\right]\left(\nabla_a W_{ab}\right)_j$$
$$-\sum_b V_b \left\{\left(\alpha_s\right)_a \max\left[\left(u_{s,j} - u_{f,j}\right)_a \left(\nabla_a W_{ab}\right)_j, 0\right] + \left(\alpha_s\right)_b \min\left[\left(u_{s,j} - u_{f,j}\right)_b \left(\nabla_a W_{ab}\right)_j, 0\right]\right\}$$

Modified upwind scheme for inter-particle sediment mass flux

SPH formulation

- Sediment velocity carried by particle a

$$\frac{du_{s,i}}{dt}=-\frac{1}{\rho_s}\frac{\partial p_f}{\partial x_i}+\frac{1}{\alpha_s\rho_s}\frac{\partial\left[\alpha_s\rho_s\left(\tau^0_{s,ij}+\tau^{SPS}_{s,ij}\right)\right]}{\partial x_j}+g_i+\frac{\gamma}{\rho_s}\left(u_{f,i}-u_{s,i}\right)$$

$$-\frac{\gamma}{\rho_s}\frac{\nu^t_f}{\alpha_f\mathrm{Sc}}\frac{\partial\ln\alpha_s}{\partial x_i}-\left(u_{s,j}-u_{f,j}\right)\frac{\partial u_{s,i}}{\partial x_j}$$

$$\frac{\partial f}{\partial x}=\sum_b\frac{\phi_b}{\phi_a}\left(f_b-f_a\right)\nabla_a W_{ab}\frac{m_b}{\rho_b}$$

$$\frac{\partial f}{\partial x}=\sum_b\left(\frac{\phi_b}{\phi_a}f_a+\frac{\phi_a}{\phi_b}f_b\right)\nabla_a W_{ab}\frac{m_b}{\rho_b}$$

$$\Downarrow$$

$$\frac{d\left(u_{s,i}\right)_a}{dt}=-\frac{1}{\rho_s}\sum_b V_b\left[\left(p_f\right)_a+\left(p_f\right)_b\right]\left(\nabla_a W_{ab}\right)_i+\underline{\sum_b V_b\left[\left(\frac{\tau_{s,ij}}{\alpha_s\rho_s}\right)_a+\left(\frac{\tau_{s,ij}}{\alpha_s\rho_s}\right)_b\right]\left[1+\frac{1}{2}\ln\frac{\left(\alpha_s\right)_b}{\left(\alpha_s\right)_a}\right]\left(\nabla_a W_{ab}\right)_j}$$

For numerical stability at interface

$$+g_i+\frac{\gamma_a}{\rho_s}\left(u_{f,i}-u_{s,i}\right)_a-\frac{\gamma_a}{\rho_s}\frac{\left(\nu^t_f\right)_a}{\left(\alpha_f\right)_a\mathrm{Sc}}\sum_b V_b\ln\frac{\left(\alpha_s\right)_b}{\left(\alpha_s\right)_a}\left(\nabla_a W_{ab}\right)_i$$

$$\underline{+\sum_b V_b\min\left[\left(u_{s,j}-u_{f,j}\right)_b\left(\nabla_a W_{ab}\right)_j,0\right]\left[\left(u_{s,i}\right)_a-\left(u_{s,i}\right)_b\right]}$$

Upwind scheme for inter-particle sediment momentum flux

Numerical implementation

- Dynamic Boundary Condition for complex coastal structures

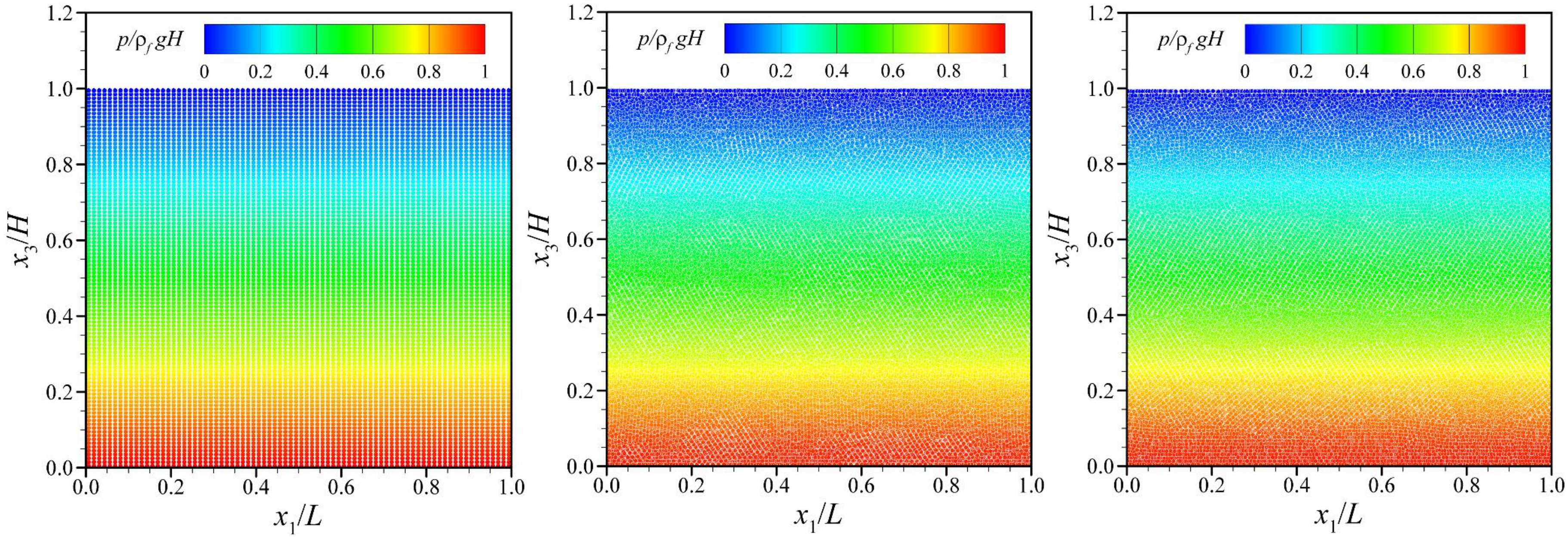
- Open-source package GPUSPH

  - Programmed with CUDA and C++
  - Parallel computation on Nvidia CUDA-enabled GPU
  - Nvidia Tesla K40c GPU, 2880 processor cores

Model Validation and Applications

Idealized cases

☐ Still water with neutrally buoyant sediment



(a) initial pressure and particle distribution

(b) uniform distribution of sediment

(c) linear distribution of sediment

- Still water: benchmark case for SPH models
- Pressure satisfies hydrostatic law and sediment concentration keeps unchanged
- Negligible unphysical rise of free surface

Idealized cases

☐ Settling of natural sand in still water

Control equations for idealized 1D problem of gravitational settling of natural sand in still water:

$$\frac{\partial \alpha_s}{\partial t} - \omega_s \left(1-\alpha_s\right)^{1.65} \frac{\partial \alpha_s}{\partial x_3} = 0$$

ω_s : settling velocity of an individual particle

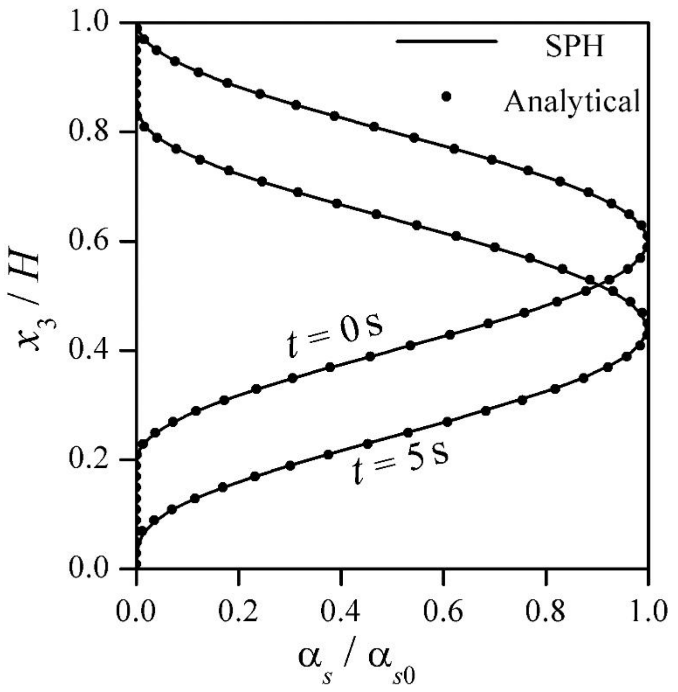
- Initial distribution of sediment concentration

$$\begin{cases} \alpha_s = \dfrac{\alpha_{s0}}{2}\left[1+\cos 2\pi\left(\dfrac{x_3-0.1}{0.4}-\dfrac{1}{2}\right)\right] & x_3 \geq 0.1 \\ 0 & x_3 < 0.1 \end{cases}$$

- Analytical solution by method of characteristics

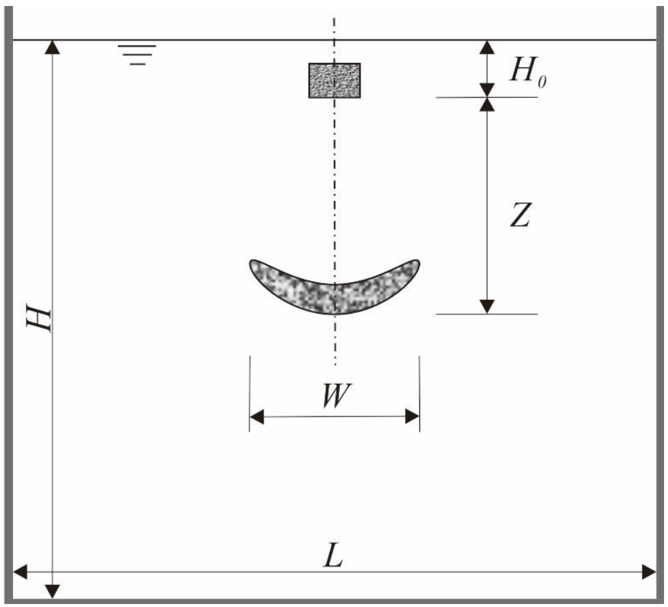
$$\begin{cases} \alpha_s\left(t, x_3\right) = f\left(\zeta\right) \\ \zeta = x_3 + \omega_s\left[1-f\left(\zeta\right)\right]^{1.65} t \end{cases}$$

- Good agreement between numerical results and analytical solutions.

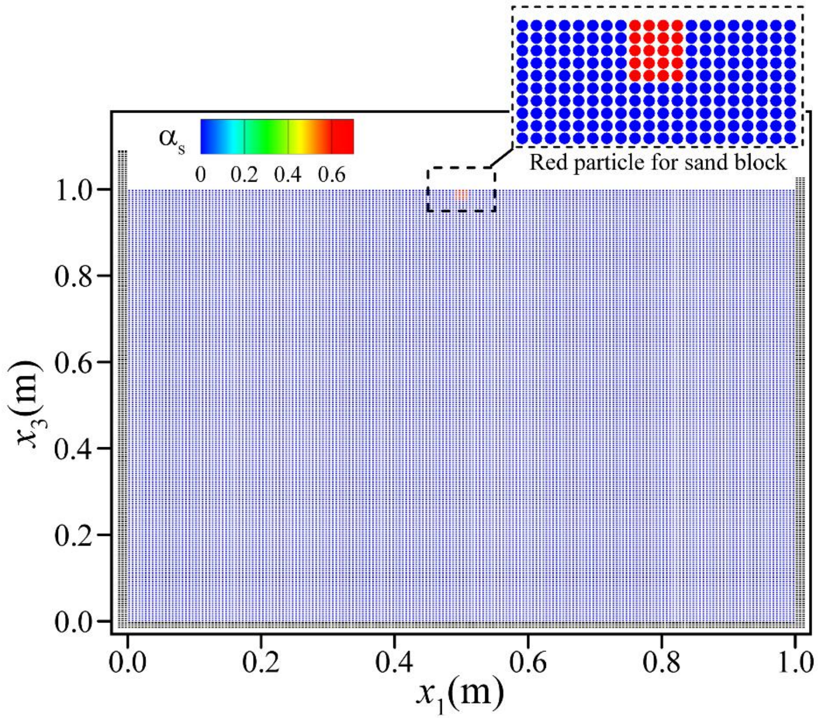


2D Sand dumping

- Sudden dumping of sediment from a line source into a water tank



(a) Sketch of the problem

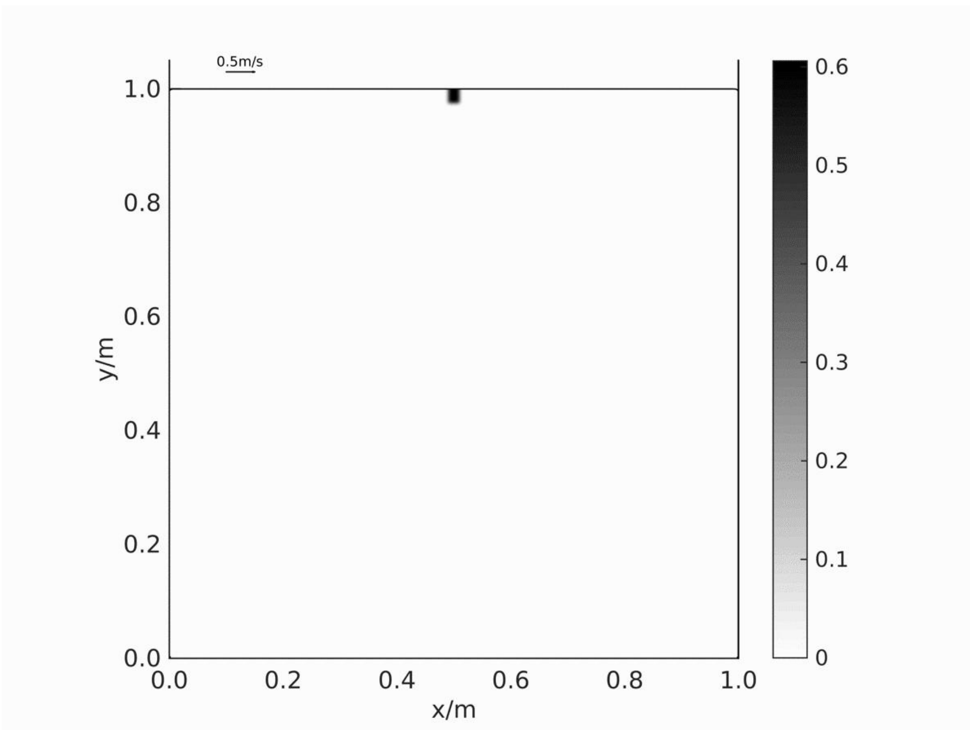


(b) Initial distribution of SPH particles

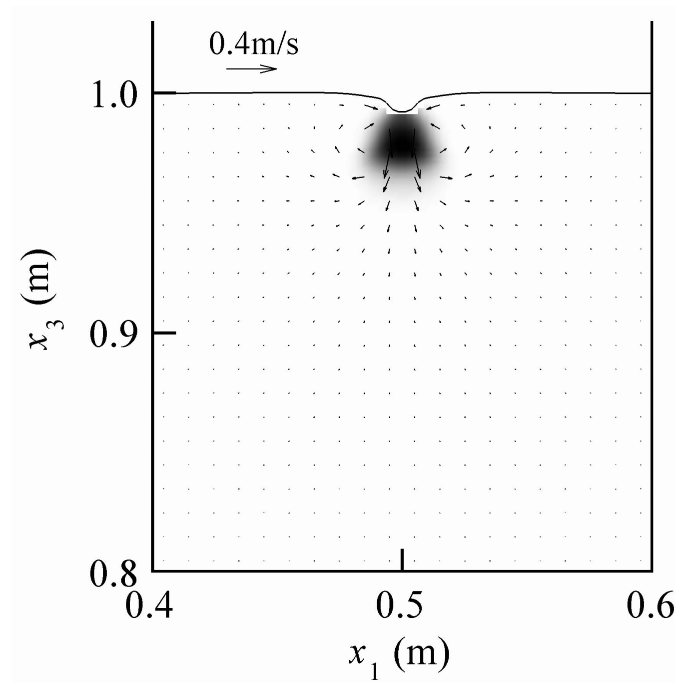
- Evolution of sand cloud: frontal velocity, width, sediment concentration
- Vibration of free surface
- Water vortex

2D Sand dumping

- Evolution of sand cloud and vibration of free surface



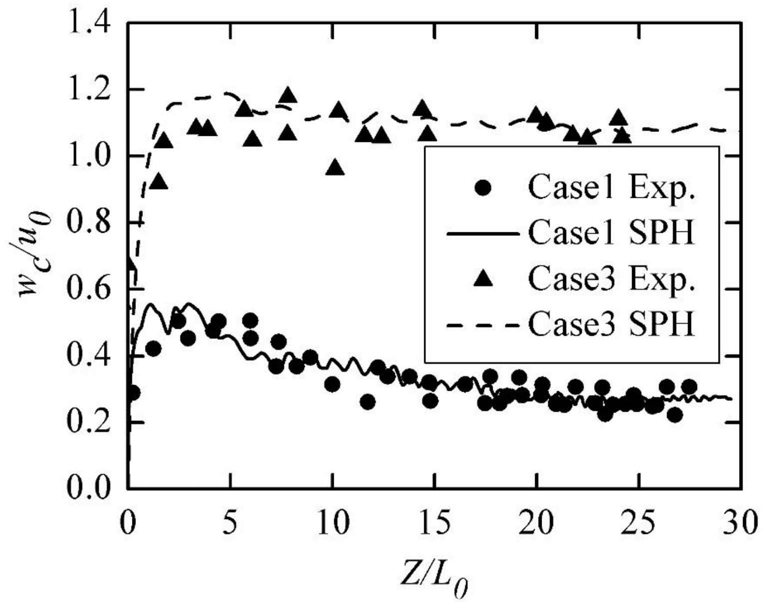
(a) Evolution of sand cloud in fine sediment case, t = 0 - 2.4s



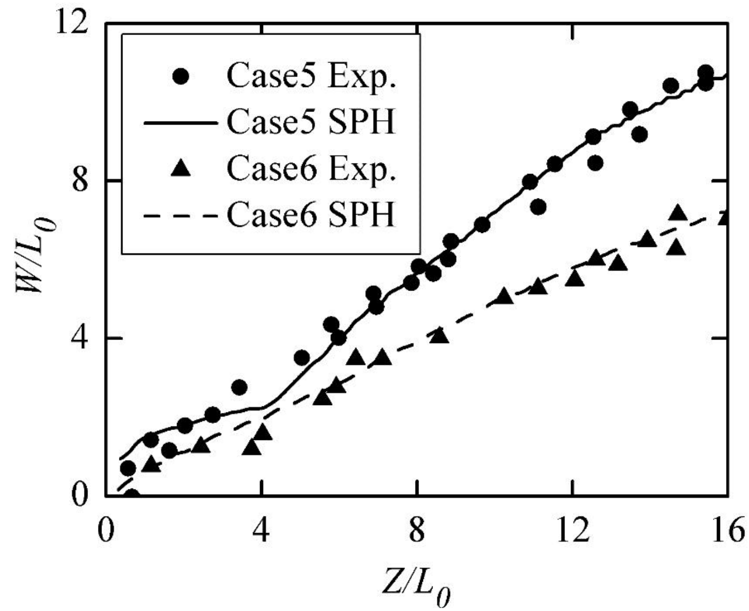
(b) Vibration of free surface at initial stage of settling in coarse Sediment case, t = 0 – 0.54s

2D Sand dumping

- Comparison between computed and measured results



(a) Frontal velocity of sand cloud

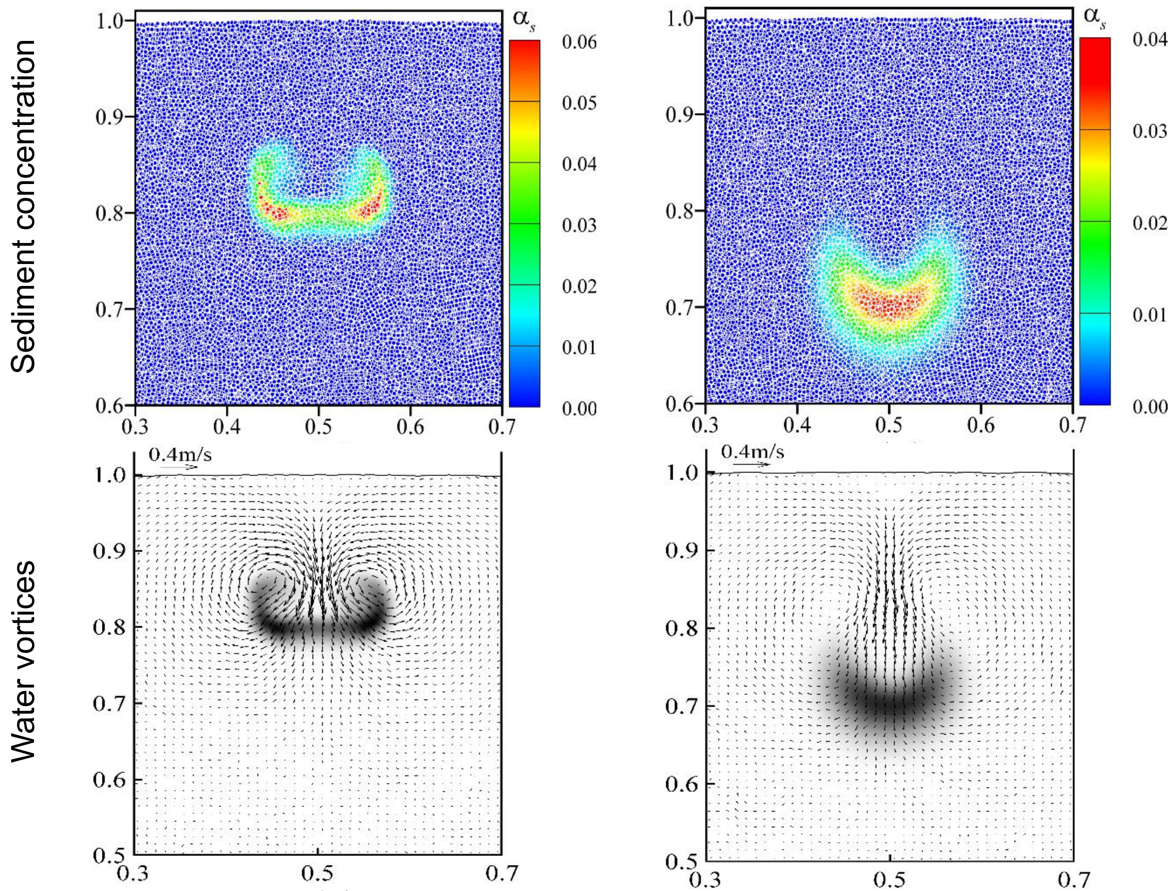


(b) Cloud width

- Good agreement between computational and experimental characteristics of sand cloud

2D Sand dumping

➢ Sediment concentration of sand cloud and water vortices



(a) Fine sediment case

(b) Coarse sediment case

◆ Fine sand cases

double peaks in distribution of sediment concentration

fully developed symmetric water vortices

◆ Coarse sand cases

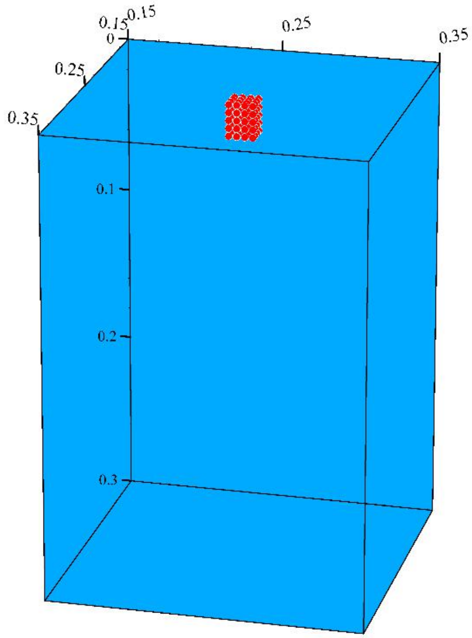
one peak of concentration at the center of sand cloud

cloud descends faster and moves Away from vortex cores

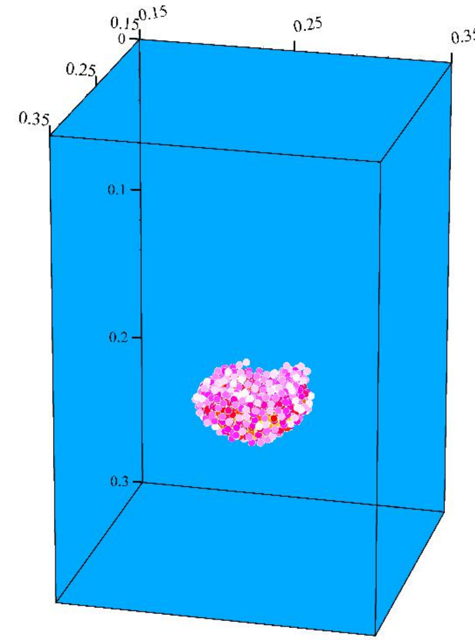
less affected by water vortices

3D Sand dumping

- 3D Sudden dumping of sand block into a water tank



(a) Initial distribution of SPH particles for the sand block

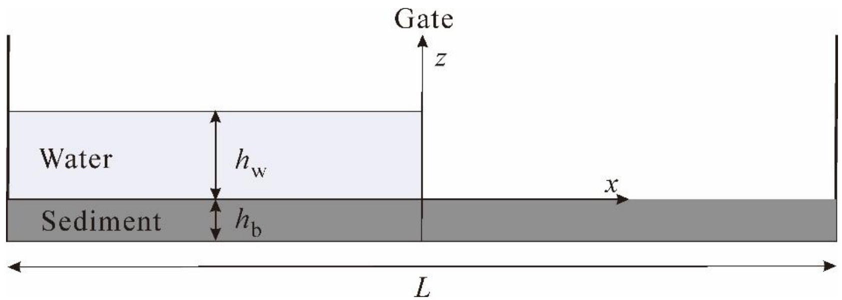


(b) Bowl-like configuration of SPH particles for the sand cloud

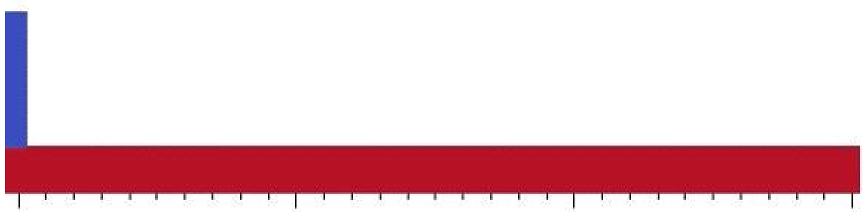
- The sand cloud has a shape of bowl with the maximum concentration located at the center of the cloud, consistent with the experimental results.

Bed erosion by dam-break flows

☐ Erodible sand bed

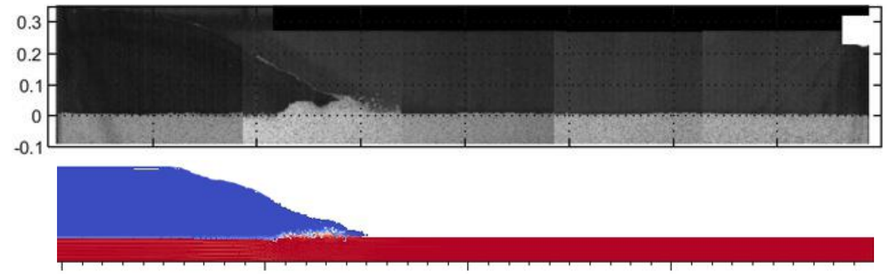


(a) Sketch of the problem

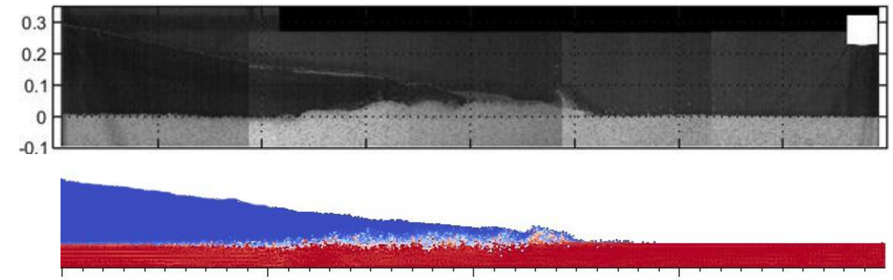


(b) Bed erosion

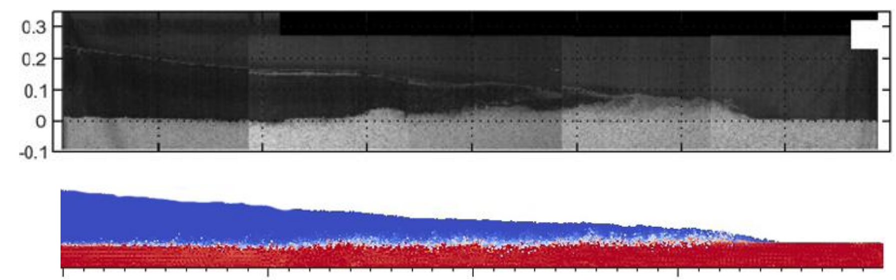
- Complex motion of free surface
- Appropriate constitutive laws for sediment stress



(c) t = 0.25 s



(d) t = 0.75 s



(e) t = 1.25 s

Conclusions

Conclusions

- A continuum two-phase SPH model for sediment transport in free surface flows
  - Continuum two-phase formulations
  - Single set SPH particles for sediment-water mixture
  - Novel EOS for solid-fluid mixture
  - Constitutive laws for sediment stress

- Validation of the model in idealized two-phase problems, 2D and 3D sand dumping, and bed erosion by dam-break flows

- **Further application of the model to storm-wave and tsunami induced sediment transport in coastal areas**

Thanks for your kind attention

- Related papers:

1. Huabin Shi, Xiping Yu, Robert A. Dalrymple, 2017. Development of a two-phase SPH model for sediment laden flows. *Computer Physics Communications*, 221, 259-272.
2. Huabin Shi, Xiping Yu, 2017. A two-phase SPH model for sediment transport in free surface flows. Proceeding of SPHERIC Beijing Workshop.
3. Huabin Shi, Pengfei Si, Xiping Yu, Robert Dalrymple. Simulation of sediment transport by dam break flows using a continuum two-phase SPH model. (to be submitted)

- Looking for a postdoc position on sediment transport, fluid dynamics, coastal morphology, or application of SPH method.