Two Tools for Tollbooth Optimization

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Summary

We determine the optimal number of lanes in a toll plaza to maximize the transit rate of vehicles through the system. We use two different approaches, one macroscopic and one discrete, to model traffic through the toll plaza.

In our first approach, we derive results about flows through a sequence of bottlenecks and demonstrate that maximum flow occurs when the flow rate through all bottlenecks is equal. We apply these results to the toll-plaza system to determine the optimal number of toll lanes. At high densities, the optimal number of tollbooths exhibits a linear relationship with the number of toll lanes.

We then construct a discrete traffic simulation based on stochastic cellular automata, a microscopic approach to traffic modeling, which we use to validate the optimality of our model. Furthermore, we demonstrate that the simulation generates flow rates very close to those of toll plazas on the Garden State Parkway in New Jersey, which further confirms the accuracy of our predictions.

Having the number of toll lanes equal the number of highway lanes is optimal only when a highway has consistently low density and is suboptimal otherwise. For medium- to high-density traffic, the optimal number of toll lanes is three to four times the number of highway lanes. Both models demonstrate that if a tollway has lanes in excess of the optimal, flow will not increase or abate.

Finally, we examine how well our models can be generalized and comment on their applicability to the real world.

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Statement of Problem

We are asked for a model that determines the optimal number of tollbooths in a toll plaza located on an n-lane tollway. The two criteria that we use for evaluating optimality are total throughput in number of cars and average transit time for individual cars to pass through the plaza.

Definitions

Number of highway lanes, n: The number of lanes on the highway entering and leaving the plaza.

Number of transit lanes, m: The number of tollbooths and lanes in the toll plaza.

Entry zone: The m-lane region of the toll plaza between the entry tollway and the tollbooths.

Merge zone: The m-lane region of the toll plaza between the tollbooths and the exit tollway.

Flow or throughput, q: Number of cars per second which pass through a given point x in our system.

Backlog B: Number of queued cars waiting to enter the tollbooths or exit the plaza.

Tollbooth processing time, τ_i: The number of seconds required, on average, for a car to pull into, pay, and exit a tollbooth i.

Density $\rho(x)$: number of vehicles per square meter in a given region.

m^*: the optimal number of tollbooth lanes.

Bottleneck capacity, q_b: the maximum number of cars per second that can pass through a given bottleneck b.

Assumptions

- A toll plaza consists of n highway lanes diverging into m toll lanes and converging back into n highhway lanes. The toll plaza is sufficiently long to permit cars to reach all of the m tollbooths.
- Each tollbooth controls one lane and can serve at most one car at a time.
- Exit from the tollbooths is not metered.

- Drivers seek to move through the toll plaza as quickly as possible while maintaining safety.
- Within the toll plaza, all vehicles move at the safest possible maximum speed for a given density, since drivers seek to avoid accidents.

Model Development

Motivations

There are two general approaches to modeling traffic motion:

Macroscopic approaches begin with some observations about aggregate traffic behavior and attempt to approximate traffic behavior as a continuous flow over some large region or large time period.

Microscopic approaches attempt to model driver and car behavior and use this information in aggregate as the basis for modeling the large-scale behavior of traffic.

The microscopic approach often hinges on a large number of parameters that may be difficult to model accurately. For example, driver decision-making strategies, driving styles, preferred following distances, and the physical parameters of individual vehicles are highly variable.

We first pursue the macroscopic approach. Such approaches are traditionally used for modeling traffic behavior over long stretches of highway, and to model traffic jams, so it may seem that such an approach is inapplicable to a setting where highway length is not large. However, there are two advantages:

- Properties that vary significantly between drivers and vehicles are averaged out if we let the system run for a sufficiently long time and it approaches a steady-state equilibrium.
- At equilibrium, we can use the total throughput of cars through the toll plaza over a given duration as a metric for the disruption it causes.

We construct a theoretical flow model of the tollbooth plaza to determine the effect of varying numbers of tollbooth lanes for given numbers of transit lanes, and use this to predict optimal conditions.

The downside of the macroscopic approach is that traffic flow is not necessarily continuous, so approximations made in the model may not reflect reality. The best way to check them is to contrast them with real traffic data. As a result, we eventually construct a full microscopic approach to generate realistic data to test our continuous model: We design a cellular-automata simulation, constructed with an independent set of driver behaviors, to verify our macroscopic model. To represent the effect of unknown variables, we introduce a small random component to the simulation.

Flow in the Plaza

Initial Observations and Conservation of Flow

We begin by defining the flow q of traffic as the number of cars per second to pass through a perpendicular cross-section across all lanes of the highway, dx. By definition, flow follows the equation

$$q = \rho v,$$

where v is the average vehicle speed and ρ is the average vehicle density.

Two bottlenecks limit the flow in every toll plaza: the first is at the tollbooths, caused by the time required for cars to stop and pay the toll, and the second occurs when the lanes exiting the tollgates merge back into the highway.

All traffic that enters the plaza must eventually exit the plaza. Treating the motion of vehicles through the plaza as a continuous flow of traffic, we can represent this fact with the following lemma.

Lemma. *For any given cross-sectional slice dx of the system,*

$$\int_{all\ time} (q_{\text{out}} - q_{\text{in}})\, dx = 0.$$

Using the relation $q = \rho v$, we can arrive at the following relationship, following the method of Kuhne and Michalopoulos [2002, Ch. 5, 5–8]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{1}$$

This is the conservation of traffic flow equation given by Kuhne and Michalopoulos [2002, Ch. 5, 5–8], among others.

Bottlenecking Constraints on Flow

Bottlenecks along the highway restrict the maximum rate of traffic flow, as described in the following theorem:

Bottleneck Theorem. *The flow of vehicles through any system void of sources and sinks is bounded by the minimum of the bottleneck capacities along the path.*

Proof: Suppose that the flow at some point dx along the highway is in excess of the maximum of all bottleneck capacities ahead of it, i.e.

$$q(x) > \max\{q_{\text{bottleneck}_1}, \dots, q_{\text{bottleneck}_i}\},$$

where i is the number of bottlenecks ahead of dx. Since in the steady-state model all points flow at the same rate, this would mean that $\max\{q_{\text{bottleneck}}\} = q(x)$, which is a contradiction. $\square$

This result is used several times in the construction of our model.

A Queueing Model Based on Flow

Since the rate of flow is constrained by the bottleneck of minimum capacity, it follows that the throughput is determined by the relative rates of the bottleneck at the tollbooths and at the point where the highway retreats back to its original size (i.e., the "merge point").

This observation reduces the problem to examining throughput solely at the endpoints of the "problem zone," without need to consider the behavior of traffic flow between those points. Thus, we can proceed simply by modeling the behavior of traffic at these two points.

Calculating Backlogs

We find the number of cars at each of these bottlenecks at any given time, so as to determine when a backlog occurs. For an arbitrary segment of the m-lane section of the toll plaza, we integrate the conservation of flow equation **(1)** with respect to x over the length of the road segment (with m lanes). This gives the instantaneous number of vehicles within the segment, $B(t)$:

$$B(t) = \int_x m\rho(x,t)dx.$$

From this we obtain the rate at which the backlog or pileup in the merge zone is growing:

$$\frac{dB(t)}{dt} = \begin{cases} q_{\text{arrival}} - q_{\text{departure}}, & \text{if } q_{\text{arrival}} > q_{\text{departure}}; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where q_{arrival} is the rate (in cars/s) at which cars enter the segment and $q_{\text{departure}}$ is the rate at which they are exit. It then follows that:

Theorem (Flow Equilibrium). *To prevent congestion from building at a bottleneck (i.e., to keep $B(t) = 0$) while maintaining maximum system throughput, it must be that*

$$q_{\text{arrival}} \leq q_{\text{departure}_{\max}},$$

where $q_{\text{departure}_{\max}}$ is the bottleneck capacity and thus the maximum system throughput.

Proof : Consider the following three possible cases:

Case 1: Let $q_{\text{arrival}} > q_{departure_{\max}}$. Then by **(2)**, the number of cars building up in the system will increase at a rate of $q_{\text{arrival}} - q_{\text{departure}} > 0$.

Case 2: Let $q_{\text{arrival}} < q_{departure_{\max}}$. By **(2)**, the rate of increase in the number of cars in the system is 0, and system throughput is q_{arrival}.

Case 3: Let $q_{\text{arrival}} = q_{departure_{\max}}$. Similar to Case 2, a backlog does not grow, and the system throughput is q_{arrival}. Note that, however, q_{arrival}, and thus system throughput, is at a maximum while preventing congestion at the bottleneck; therefore, this is clearly the optimal case. □

Applications to Toll Plaza System

Adapting this general result to our model, we define

q_{in_i} to be the flow in cars/second entering the system in lane i,

q_{tolls_i} to be the flow or turnover rate of tollbooth i, and

q_{out_i} to be the flow leaving the system (at or after the merge point) in lane i.

Our model considers the interaction between the two bottlenecks. Upon investigation, two observations become apparent:

1. The maximum flow, $q_{\max}$ through a cross-sectional slice of the highway dx is independent of the road structure before that point. In other words, the maximum flow capacity is fixed solely by the number of lanes at that point and not the number of lanes merging or diverging into it.

2. The only cross-sectional slice dx at which the maximal flow can be varied (by the model) is at the tollbooths; this is done by changing the number of tollbooths, which directly results in a change in the number of cars that can be processed per unit time.

With this in mind we apply the Flow Equilibrium Theorem. We divide the system into two segments, the first from $(-\infty, x_{\text{tolls}})$ and the second from $(x_{\text{tolls}}, x_{\text{merge}})$, where x_{tolls} is the point x along the highway where the tollbooths are and x_{merge} is the point where the m lanes of the toll plaza merge into the n lanes of the highway.

The q_{arrival} of the first segment (into the tollbooths) is simply q_{in}, and $q_{\text{departure}} = q_{\text{tolls}}$. For the second segment, $q_{\text{arrival}} = q_{\text{tolls}}$ and $q_{\text{departure}} = q_{\text{out}}$.

Since $q_{\text{tolls}} = m/\tau$, only the number of tollbooths and their individual turnover rates τ determine the flow entering the merge zone. By observation (1), we note that the bottleneck capacity of the merge point, $q_{\text{out}_{\max}}$, is independent of m and q_{tolls}; it is merely a property of an n-lane highway.

We are therefore interested in how q_{tolls} affects the flow of cars through the merge zone. By observation (2), when $q_{\text{tolls}} > q_{\text{out}}$ the backlog increases at a rate of

$$\frac{dB(t)}{dt} = q_{\text{tolls}} - q_{\text{out}}.$$

The backlog continues to grow until the entire merge zone is filled, and then it spills out into the segment before the tolls. This buildup does not fully dissipate until q_{in} reduces to below $q_{out_{\max}}$, or in other words, until the incoming flow rate is below the bottleneck capacity of the tightest bottleneck (such as at the end of rush hour).

To prevent the occurrence of this effect, let $q_{\text{tolls}} \leq q_{\text{out}}$, which allows traffic to flow through the merge zone without causing backlog. However, when $q_{\text{tolls}} < q_{\text{out}}$, the merge point is not operating at maximum flow; therefore, letting $q_{\text{tolls}} = q_{\text{out}}$ is optimal. Surprisingly, however, we show later that this is actually a *lower bound* on q_{tolls}.

From this result, we get

$$q_{\text{out}} = nq_{out_{\max}} = q_{\text{tolls}} = \frac{m^*}{\tau},$$

where m^* is the optimal number of tollbooths for an n-lane highway. Solving for m^*, we get

$$\boxed{m^* = n\tau q_{\text{out}_{\max}}.}$$

Performance When the Number of Tollbooths Exceeds m^*

We now consider toll plaza performance when the number of tollbooths m exceeds the predicted optimum m^*. This investigation is necessary. For example, if our model were to predict m^* slightly above the actual value, a backlog would build within the merge zone, but it might build so slowly that by the time its size became critical, the rush-hour mass of vehicles would already have dissipated.

As previously shown, when $m > m^*$ and $q_{\text{in}} > q_{\text{tolls}_{\max}}$, a backlog builds in the merge zone at a rate of $(q_{\text{tolls}_{\max}} - q_{\text{out}})$. Until the merge zone fills completely with vehicles (when vehicle density is at a maximum), the tollbooths continue to process vehicles at their maximum rate, $q_{\text{tolls}_{\max}} = m/\tau$. In this case, the backlog at the tolls grows at a rate of $(q_{\text{in}} - q_{\text{tolls}_{\max}})$.

As a result, the effective backlog growth is the sum of the backlog growth rates at each bottleneck:

$$\begin{aligned}\frac{d}{dt}\,\text{backlog}_{\text{effective}} &= \frac{d}{dt}\,\text{backlog}_{\text{tolls}} + \frac{d}{dt}\,\text{backlog}_{\text{merge}} \\ &= (q_{\text{in}} - q_{\text{tolls}_{\max}}) + (q_{\text{tolls}_{\max}} - q_{\text{out}}) \\ &= (q_{\text{in}} - q_{\text{out}}).\end{aligned}$$

Interestingly, this result implies that the total backlog of the system is entirely dependent on the rate at which vehicles enter and the maximum rate at which they can leave (i.e., the bottleneck capacity of the tightest bottleneck). Therefore, as long as the tollbooths do not limit the total flow capacity of the system, the exact rate at which the tollbooths process vehicles does not affect the system flow. This line of reasoning leads us to the following theorem:

Theorem (Lower bound on m^*). *The optimal number of tollbooths for an arbitrary n-lane highway is greater than or equal to $m^* = n\tau q_{out_{\max}}$.*

Maximum Flow on an n-Lane Highway

Garber and Hoel [1999] observe empirically an inverse relationship between ρ and v, and several models have been proposed to describe this behavior. It is generally accepted that any such model must exhibit the following properties:

- Flow is zero when density is zero.
- As density increases to some critical value, so does flow.
- Past this critical density the flow begins to decrease.
- Flow cannot decrease beyond some minimum value.

Let $v_{\max}$ be the maximum velocity of cars traveling freely on the highway (generally the speed limit), and let $\rho_{\max}$ be the maximum number of cars (i.e., jam-packed) per unit area of highway (a constant).

One of the more popular models, proposed by Greenshield [Garber and Hoel 1999], establishes a linear relationship between the two:

$$v = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) \qquad \Rightarrow \qquad q = \rho v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right),$$

where $v_{\max}$ is the maximum speed at which cars tend to travel when flowing undisrupted along the highway, generally taken as the speed limit.

A second popular model, introduced by Greenberg [Garber and Hoel 1999], proposes a logarithmic relationship:

$$v = v_{\max}\ln\left(\frac{\rho}{\rho_{\max}}\right) \qquad \Rightarrow \qquad q = \rho v_{\max}\ln\left(\frac{\rho}{\rho_{\max}}\right). \tag{3}$$

While accurate in certain cases, these models do not seem to represent effectively the motion of cars through toll plazas, because they both model the flow as zero when density has reached its maximum. Although density will tend to some maximum, flow will asymptotically approach but never reach zero, since some number of cars will still flow out of the system over a long enough time interval. This discrepancy is a direct result of the limitations inherent in treating traffic as a continuous flow. Determining the precise relationship between ρ and v is a relatively complex modeling task beyond the scope of this paper, so we accept Greenberg's model with the restriction that if $\rho = \rho_{\max}$, flow will approach some low constant value instead.

To determine a rough estimate of the maximum flow through the merge point (or any n-lane highway for that matter), we use Greenberg's model. To find the maximum flow $q_{\max}$, we differentiate **(3)**:

$$\frac{d}{d\rho}q(\rho) = v_{\max}\log\left(\frac{\rho_{\max}}{\rho}\right) - v_{\max} = 0$$

Solving for ρ, we get $\rho = \rho_{\max}/2$. Therefore, flow is maximized for density $\rho_{\max}/2$, which gives

$$q_{\max} = \frac{\rho_{\max} v_{\max}}{2}.$$

Streamlined Flow Model

We made the assumption in the previous section that flow is distributed uniformly among all lanes—that is, an equal number of cars pass through each lane per second. However, in real toll plaza systems there are conditions when this does not apply. For example, on New Jersey's Garden State Parkway users are restricted to movement between sets of individual highway lanes, before and after the tollbooths, by lane dividers. As a result, sets of lanes operate independently of each other. Moreover, one or more lanes may be reserved for low-speed vehicles (recreational vehicles or large trucks) or high-speed traffic (electronic toll collection, motorcycles, buses, or carpools).

To generalize our model, we relax this assumption to account for varying flows between lanes of traffic. We divide the total flow through the tollbooths, and the total outgoing flow through the exit, into individual lane flows, so that

$$q_{\text{tolls}} = \sum_{i=1}^{m} q_{\text{tolls}_i}, \qquad q_{\text{out}} = \sum_{j=1}^{n} q_{\text{exit}_j}.$$

As observed in the **Bottleneck Theorem**, no lane can flow faster than $q_{\max}$. However, by conservation of traffic flow **(1)**, total traffic flow through the system remains the same, even though streams of traffic may move at different rates.

We must also consider what happens when lanes merge towards the exit of the toll plaza. Whereas in the queueing theory model we were allowed to consider only the flow at two points—q_{exit} and q_{toll}—we must now also consider the flow rate at all points where two or more lanes merge into one.

The flow rate at a merge point can never exceed $q_{\max}$, the maximum flow for a single lane. According to the results derived earlier in the basic model, the rate of flow through each lane equals the rate of the slowest bottleneck ahead. However, it is the *combined* rate of two merging lanes that exceeds the lane bottleneck, not each individual lane, so naively setting $q_{\text{in}} = q_{\max}$ would incorrectly increase the rate of the premerged lane to match the bottleneck.

As a result, each lane can contribute at most a decreased quantity such that the sum of the two lanes equals the bottleneck capacity. A simple way to represent this behavior is to allow each to contribute a proportion of flow relative to their combined size:

$$q1_{\text{reduced}} = q_{\max}\frac{q_1}{q_1 + q_2}, \qquad q2_{\text{reduced}} = q_{\max}\frac{q_2}{q_1 + q_2}.$$

Thus, when both flows would normally overfill the lane into which they merge, each lane's contribution to that lane's flow, q_{out}, will be proportional to its percentage of the total amount of flow present, without ever exceeding $q_{\max}$.

This observation lends itself to a simple recursive function in modeling toll plaza traffic as an aggregate of independent flows. Therefore, predictions of system behavior with introduction of electronic toll collection lanes and other flow-monitored lanes are significantly simplified, by applying this model at all steps of the merging process (i.e., by first merging every set of two lanes, and then merging the following two, etc.).

Simulation

Motivations for a Discrete Simulation

To validate the continuous model, we create a discrete simulation using cellular automata to generate traffic behavior. Whereas our earlier models approximate car flux as continuous, the cellular automata simulation treats individual vehicles as distinct entities that behave according to well-defined simple rules. Since the discrete model is based on an independent set of intuitions about the system and how it behaves, any agreement between the two models will suggest a high degree of accuracy in our modeling efforts.

Overview

The simulation runs on a two-dimensional grid of points, each of which corresponds to a width and length slightly greater than the average car size. The simulation takes in parameters that determine the geometry of the toll plaza as well as a probability, p, that a car enters the toll plaza in a given lane. When populating the grid, each cell can be one of four types and behaves according to its corresponding set of rules:

Free: a transient place holder when block is unoccupied.

Barrier: a boundary point of the toll plaza; this cell never changes.

Toll block: a tollbooth.

Car: a cell occupied by a vehicle.

Rules of State Evolution

The rules that create the next generation of cells (next state of the grid) from time t to $t + 1$ are:

- All cars travel at a constant speed of 1 forward cell per time step.
- A car can change lanes and move forward on the next step if there is an open adjacent cell to its side and an open cell along the appropriate diagonal.
- A car can stay in the same lane if the cell ahead is free, if the car in front of it moves forward on that step, or if it is in front of the toll and its toll delay has expired.
- As the cells in the entry of the toll plaza are freed by the evolution of the grid, at each time step new cars arrive in these with probability p, the density of incoming cars.

Stochastic State Evolution

We use random variables to make the system nondeterministic. This more closely represents real-world simulations where exact car path is unpredictable and also attempts to account for the wide range of parameters relating to driver psychology, variations in vehicles, and variations in service time in paying the toll. To generate this effect, we implement the following rules:

1. Each toll processes a car at a random rate each time, with distribution centered at $\tau = 3$:

$$p(x) = \begin{cases} 0.25, & x \in \{2, 4\}; \\ 0.5, & x \in \{3\}. \end{cases}$$

2. Cars switch lanes at random with some assigned probability, but their decision is influenced by the desirability of the target lane.

3. The arrival of cars into a lane is a Poisson process with rate λ, a parameter to the simulation.

Reporting

We run the simulation for 1,000 time steps. It returns total car throughput, total waiting time, average waiting time, total transit time, and the density of cars in each section of the system (i.e., before, within, and after the toll plaza).

Sample simulation run

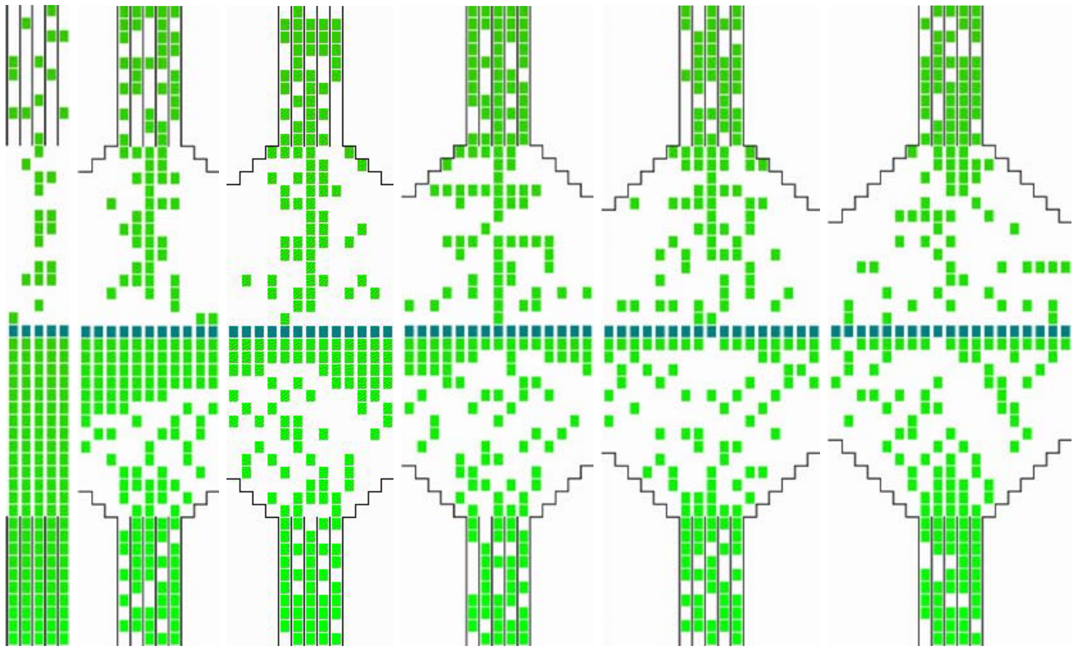
Cars are released at $t = 0$ and proceed toward the tollbooths. Upon entering the diverge zone, cars change lanes and spread out to minimize their total wait time at the tolls.

Depending on the number n of highway lanes and the number m of tollbooths, traffic eventually reaches an equilibrium flow, or else flow is reduced by the bottlenecks and backlog begins to grow. In our figures, the color of the vehicle designates the amount of time spent waiting in the system, with bright green the least, dark green moderate time, and bright red the most.

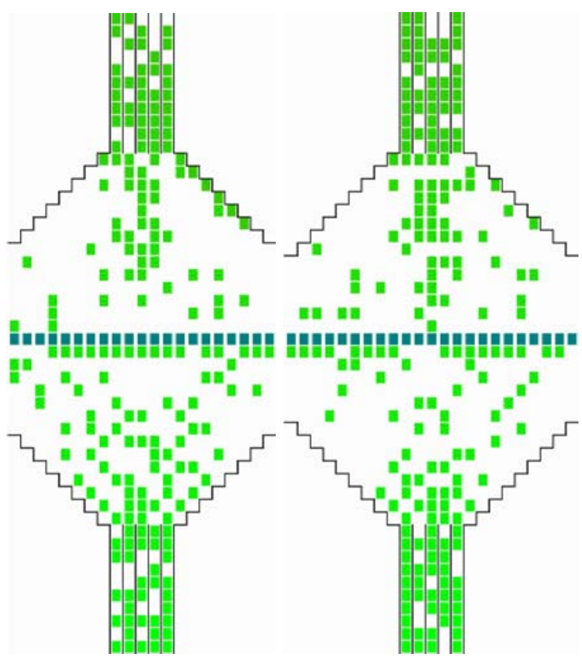
The model supports varying traffic density over time; we ran our simulations with arrival densities of 15%, 50%, and 85% percent.

Simulation Sample Images

Simulation sample activity is illustrated in **Figure 1**.



(a) m=5 (b) m=11 (c) m=13 (d) m=15 (e) m=17 (f) m=19



(g) m=21 (h) m=23

Figure 1. Simulation at step 1000 for n=5.

Running the Simulation

We ran the simulation for $t = 1000$ timesteps, with $\tau = 3$ steps, for all possible tollway sizes from $n = 1$ to $n = 8$, and for $m = n$ to $4n$ (for small n) or to $3n$ (for large n). For each highway size, we ran the simulation at 85%, 50%, and 15% density. We repeated this process 5 times for the sake of statistical significance. For a timestep of 1 s, a single run of the simulation corresponds to about 17 min of traffic.

Over each set of conditions, we track the total throughput of cars. We then compare the throughput achieved using m tollbooth lanes on a given n-lane tollway over the entire 1000-s period.

Results and Analysis

Model Predictions vs. Simulation Results

To compare further the accuracy of our theoretical model and our discrete simulation, we compare experimental data collected from the simulation with our model's predictions for the corresponding number of highway lanes n. To do this, we must make the following parameter assumptions.

- The average vehicle length plus its separation distance from the vehicle ahead of it is approximately 15 ft.
- At high density, vehicles travel in the merge zone on average at 15 mph.
- In correspondence with the simulation, the average processing time at the tolls is $\tau = 3$ s/car.

Computing the model's predictions based on these parameters and running the simulation for from one to seven highway lanes yields the results in **Table 1**. The values for m^* are the minimum values for which adding additional lanes does not alter performance significantly. [EDITOR'S NOTE: The experimental graphs used to derive these values are omitted here.]

Figure 3 shows that our flow model is validated as an accurate long-term predictor of traffic behavior for high density scenarios ($p = .85$). We believe that the difference between the continuous model prediction and the observed value in the simulation stems from uncertainty in the value of the parameter $q_{\max}$. Our prediction of $q_{\max} = 4$ was accurate only for the high-density case. For the low-density case, we almost never experience the conditions caused by $q_{\max}$—clogging at the merge points—and so our model does not apply. The continuous flow model does not apply when traffic flow has significant variations in speed, when density cannot be considered a regular flow.

Our flow model accurately predicts the observed optimum to within 3 lanes for high-density traffic. From this, we see a high level of agreement between

Table 1.
Experimentally observed optimal number of lanes for various traffic densities vs. predicted optimal number $\lfloor m^* \rfloor$ of lanes. There is a rough one-to-four relationship between the number of highway lanes and the optimal number of tollbooths.

Highway lanes	Simulation density ρ			$\lfloor m^* \rfloor$
	.15	.50	.85	
1	1	4	5	4
2	2	6	10	8
3	3	9	13	12
4	4	10	16	16
5	5	13	18	20
6	6	14	21	24
7	7	16	25	28

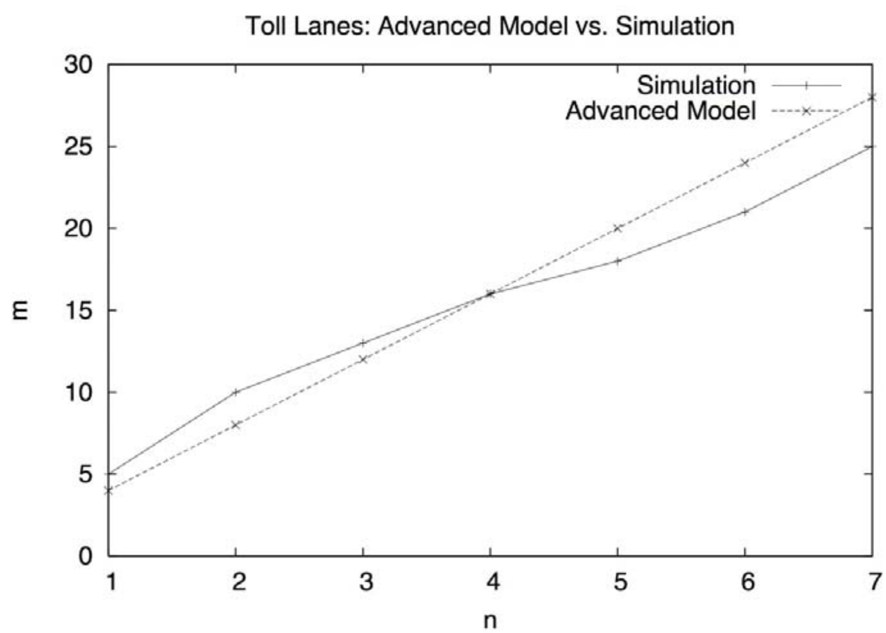


Figure 3. Observed optimal number m^* of toll lanes for given number n of highway lanes.

the two models, even though they operated on completely independent assumptions. These results suggest that our model predicts the optimal number of tollbooths for an arbitrary n-lane tollway fairly accurately.

Total Throughput vs. Average Wait Time

For every simulation that we ran, whenever throughput is higher or lower for a given pair of (m, n, ρ), average transit time is correspondingly higher or lower. We conclude that total throughput and average wait time are highly correlated. As a result, optimizing either of them results in good performance relative to the other criterion.

Accuracy of Simulation

Sensitivity of Parameters

We examine the effect of changing parameters. Altering the processing time τ, length of the tollway, length of the merging area, and the probability distribution for random behavior all affect the absolute throughputs achieved for different numbers of tollbooths—but do not affect the optimal number of lanes.

The only parameter with a significant effect on the optimal number of lanes is length of the merging area when set to an extremely low value, so that cars couldn't switch lanes in time to utilize all of the lanes. However, this condition contradicts an initial assumptions of our model and condition is unlikely to occur in the real world.

Our simulation is therefore very robust with respect to parameter variation.

Faithfulness to Real-World Behavior

We use the cellular automata simulation to validate the effectiveness of the flow model; however, this verification is only accurate to the extent that the cellular automata is a realistic description of real-world traffic flow through tollbooths.

To validate our simulation, we examine real-world flow rates of the Union toll plaza of the Garden State Parkway at several peak flow times, where $n = 5$ and $m = 13$ [New Jersey Institute of Technology 2001, 11]. Examining the seven hours of data, we arrive at a throughput of 2393 cars/hr; our model predicts 2530 cars/hr. Our simulation matches the empirical results surprisingly well for peak density.

According to our simulation, the Garden State Parkway's performance is fairly suboptimal. The best results, according to the simulation, are obtained for $m = 20$ tollbooth lanes, enabling an increase in average throughput by almost a factor of two.

Extensions of Base Model

Electronic Toll Collection

Under electronic fare payment systems such as Fastrack and EZ-Pass, drivers attach an electronic device to their vehicles, which is scanned automatically as they pass through a special tollbooth lane with little or no reduction in speed.

Both our model and simulation can analyze inclusion of special "fastlanes." In the streamlined-flow model, fastlanes are simply lanes with a much higher rate of flow q_{toll_i} through the tollbooth. Since congestion still occurs later as a result of the narrow bottleneck caused by merges and $q_{\max}$, fast progress may still be impeded by slow merging. This possibility explains the common practice of having separate fastlane toll lanes running alongside the outside of the toll plaza, so that merging happens far enough down the road.

Because merge rates are proportional to ratio of the rates of the lanes merging and the maximum possible rate, we have that

$$q_{\text{fastlane-at-mergepoint}} = q_{\max} \left(\frac{q_{\text{fastlane}}}{q_{\text{fastlane}} + q_{\text{other}}} \right).$$

Since q_{fastlane} is potentially much greater than q_{other}, cars in the fastlane flow at a rate close to the maximum. As a result, users who choose a fastlane still move through the toll plaza faster than other cars, even when forced to merge with slower traffic. The more use of fastlanes, the higher overall average throughput, and the recommended number of toll lanes for regular use can drop.

Final Recommendations

- Our model predicts that the findings in **Table 1** provide the best results for high-density situations. For traffic density at or above 85% of the maximum bumper-to-bumper density, our model should be used. Lanes can be closed when density is lower.
- Our model provides a lower bound on the recommended number of tollbooth lanes. Running more tollbooth lanes than the optimal predicted value does not hinder throughput.
- The case $m = n$ suffices exactly when a road has consistently low-density traffic. For medium- and high-density traffic, this case causes suboptimal performance.

Model Assessment

Model Strengths

- The discrete and continuous models agree well at peak densities.
- We generate plausible traffic behavior through partially random behavior in our models. In particular, our models match well effects we observe in the Union toll plaza of the Garden State Parkway.
- Our model can scale successfully to represent the impact of electronic toll-taking and variable tollbooth speeds.

Model Weaknesses

- The primary shortcoming of the theoretical model is the assumption that car flux is continuous. Fractional values of car flux do not realistically represent low-density traffic.
- Our model is too sensitive to variations in the average amount of time to process a car at the tolls.
- The simulation accounts for many unknown factors with random choices. Validation of our model is accurate only insofar as this randomness accurate reflects driver behavior.
- We don't consider cost as a component of our solution.

References

Garber, Nicholas J., and Lester A. Hoel. 1999. *Traffic and Highway Engineering*. Pacific Grove, CA: Brady/Cole Publishing Company.

Jiang, Rui, and Qing-Song Nu. 2003. Cellular automata for synchronized traffic flow. *Journal of Physics A: Mathematical and General*.

Kuhne, R.D., and Panos Michalopoulos. 2002. *Revised Monograph on Traffic Flow Theory: Flow Models*. Turner-Fairbank Highway Research Center.

Mihaylova, Lyudmila, and Rene Boel. 2003. Hybrid stochastic framework for freeway traffic flow modeling. *ACM Proceedings of the 1st International Symposium on Information and Communication Technologies*.

New Jersey Institute of Technology. 2001. *Ten Year Plan to Remove Toll Barriers on the Garden State Parkway*.

Rastorfer, Robert L., Jr. 2004. Toll plaza concepts. ASCE Fall Conference, Houston, TX.

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