Sprinkle, Sprinkle, Little Yard

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Summary

We determine an optimal algorithm for irrigating an 80 m $\times$ 30 m field using a hand-move 20-m pipe set, using a combination of analytical arguments and simulated annealing. We minimize the number of times that the pipe is moved and maximize the Christiansen uniformity coefficient of the watering.

We model flow from a sprinkler as flow from a pipe combined with projectile motion with air resistance; doing so predicts a range and distribution consistent with data from the literature. We determine the position of sprinkler heads on a pipe to optimize uniformity of watering; our results are consistent with predictions from both simulated annealing and Nelder-Mead optimization.

Using an averaging technique inspired by radial basis functions, we prove that periodic spacing of pipe locations maximizes uniformity. Numerical simulation supports this result; we construct a sequence of irrigation steps and show that both the uniformity and number of steps required are locally optimal.

To prevent overwatering, we cannot leave the pipe in a single location until the minimum watering requirement for that region is met; to water sufficiently, we must water in several passes. The number of passes is minimized as uniformity is maximized.

We propose watering the field with four repetitions of five steps, each step lasting roughly 30 min. We place two sprinkler heads on the pipe, one at each end. The five steps are uniformly spaced along the long direction of the field, with the first step at the field boundary. The pipe locations are centered in the short direction. This strategy requires only 20 steps and has a Christiansen uniformity coefficient of 94, well above the commercial irrigation minimum of 80. Simulated annealing to maximize uniformity of watering re-creates our solution from a random initialization.

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The consistency between solutions from numerical optimization and from analytical techniques suggests that our result is at least a local optimum. Moreover, the solution remains optimal upon varying the sprinkler profile, indicating that the results are not overly sensitive to our initial assumptions.

Introduction

Maximizing the uniformity of irrigation reduces the amount of water needed [Ascough and Kiker 2002]. We use a 420-kPa, 150-L/min water source and a 20-m hand-move pipe set for a 80 m × 30 m field. We determine the number and placement of sprinkler heads on the pipe, together with a schedule of watering locations and times that maximizes uniformity and minimizes time required.

Initial Assumptions

- **We live in a boring place.** We have a flat, windless, weatherless field. Though wind is often an influential factor, uniformity can be corrected to compensate for wind [Tarjuelo Martín-Benito et al. 1992].
- **Time required = number of moves.** We attempt to minimize the number of moves; we do not consider any other kind of effort, such as minimizing the total distance that the pipe must be moved.
- **Our sprinkler heads are ideal.** The distribution of water from the sprinkler heads is radially symmetric and the same for every head.
- **Average, not instantaneous overwatering.** We can water an area for half an hour at a rate of 1.5 cm/h, then leave it for half an hour, and this would not constitute overwatering. Without this assumption, it is impossible to meet the constraints on watering.
- **The pipe must stay within the field.** Our pipe locations remain completely on the field, though we allow water to fall off the field.

Slow-Watering and Fast-Watering

We break watering techniques into two categories:

- **Slow-watering.** Keep the pipe in one section of the field until it has been watered sufficiently, that is, we water the field in one pass.
- **Fast-watering.** Make multiple short passes, waiting for the field to absorb the water between runs.

Slow-watering minimizes effort (the number of times we move the pipe) while fast-watering requires extra moves. Fast-watering also carries the physical risk of washing away the topsoil, but we ignore this.

With the given constraints, we cannot create a slow-watering solution. To irrigate the field in one pass, we must keep the pipe in one position until the minimum is met. We should water at a rate no greater than 0.75 cm/h. But the rate of water flow, $150\ \mathrm{L/min} = 9 \times 10^6\ \mathrm{cm}^3/\mathrm{h}$, amounts to 0.375 cm/h over the entire field, or half the field at the maximum rate. However, our sprinkler cannot cover so great an area, hence cannot help overwatering the area that it reaches if we water for an hour or more.

We are forced to choose a fast-watering technique, which involves several passes over the field.

Judging the Quality of Solutions for Fast Watering

What solution is best? We want to minimize the number of times that we move the pipe, which is number of passes required times the number of pipe locations in each pass.

How many passes? The number of passes is determined by the minimum watering criterion. If the minimum application rate is S_{min}, then to make sure that every location receives the minimum 2 cm of water, we need

$$S_{\text{min}} t \times (\text{number of passes}) = 2 \text{ cm},$$

where t is the watering time.

How long to water? We choose t so that we don't overwater. With, in one pass, a maximum application rate of S_{max} cm/h, we can water only long enough to apply 0.75 cm, the maximum possible in an hour:

$$S_{\text{max}} t = 0.75 \text{ cm}.$$

Combining the two equations, we find

$$\text{number of passes} = \left\lceil \frac{8}{3} \frac{S_{\text{max}}}{S_{\text{min}}} \right\rceil. \qquad \textbf{(1)}$$

The ratio $S_{\text{max}}/S_{\text{min}}$ decreases with increasing uniformity. In other words, *increase in uniformity decreases the number of moves required.*

Christiansen Coefficient of Uniformity

$S_{\text{min}}/S_{\text{max}}$ is not a typical measurement of uniformity. We also use the Christiansen coefficient of uniformity, the most broadly used and well-recognized criterion for uniformity of watering [Ascough and Kiker 2002; Tarjuelo Martín-Benito et al. 1992]:

$$CU = 100 \left(1 - \frac{\sigma_S}{\langle S \rangle} \right),$$

where σ_S is the standard deviation of the application rate during a pass and $\langle S \rangle$ is the mean.

Summing over Sprinklers

We determine $S(\vec{x})$ by superimposing the water flows from the sprinkler heads, using the expression $\varphi(|\vec{x} - \vec{x_1}|)$ to denote application of 1 cm/h at position $\vec{x}$ from the sprinkler head at $\vec{x_1}$:

$$S(\vec{x}) = \sum_k t_k \varphi\left(|\vec{x} - \vec{x_k}|\right),$$

where t_k is the time spent at sprinkler head k.

Determining the Sprinkler Profile $\varphi(r)$

To optimize the layout, we must know how the sprinkler applies water as a function of distance, $\varphi(r)$. This is a complicated function that depends on the sprinkler type and pressure in the line; its form is not well-known and it is often is simulated numerically [Carrión et al. 2001].

The Linear Model

A first guess at the sprinkler function would be a simple decreasing linear function. In fact, this is reasonably consistent with measured data (**Figure 1**).

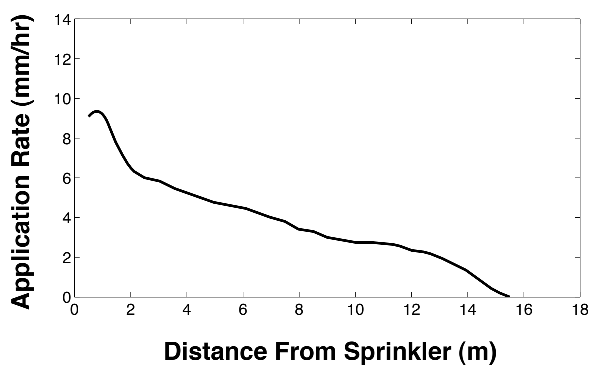


Figure 1. The sprinkler function is approximately linear (redrawn from Carrión et al. [2001]).

The linear approximation allows for simple solutions; in one dimension, it is possible to combine linear functions to lead to a uniform water distribution [Smajstrla et al. 1997]. Several other empirical models have been used for $\varphi(r)$; Mateos uses, among others, $\varphi(r) \sim \left(1 - r^2/R^2\right)$ [Mateos 1998].

Model of Water Distribution from a Sprinkler

Output Speed of Sprinklers

We model a sprinkler head, ignoring rotational effects and angle, as a hole in the pipe. Bernoulli's equation states that along a streamline,

$$P + \tfrac{1}{2}\rho v^2 + \rho g h = \text{constant},$$

where

P is the pressure,

ρ is the density of water,

v is the fluid velocity,

g is the acceleration of gravity at the Earth's surface, and

h is the height of the fluid.

We assume that the variation in height is negligible and consider a point within the pipe and a point at the hole. Then, with P_w the water pressure and P_a the atmospheric pressure, $P_w + \frac{1}{2}\rho v_w^2 = P_a + \frac{1}{2}\rho v_o^2$ implies that

$$v_o^2 = v_w^2 + \frac{2(P_w - P_a)}{\rho}$$

is the speed of outgoing water for one sprinkler. Typically, this result is stated as $v_o = \sqrt{2gH}$, where $H = 2(P_w - P_a)/\rho$ is the pressure head and the speed of the water is considered negligible. In our case, however, v_w is significant.

With n sprinklers, the continuity property requires that the flux in any one section of the tube is $J/A_i n$, where J is the total flux (150 L/min) and A_i is the cross-sectional pipe area. Therefore, the output speed is

$$v_n = \sqrt{\left(\frac{J}{A_i n}\right)^2 + \frac{2(P_w - P_a)}{\rho}},$$

which ranges from 25 to 40 m/s, depending on the number of sprinkler heads.

Sprinkler Range

The spray remains coherent for a while before breaking up into particles [Carrión et al. 2001; Kranz et al. 2005]. We treat the motion of the outgoing water drop as a projectile problem, first without air resistance, then with air resistance proportional to the square of the speed.

Without air resistance, direct integration of the equations of motion gives the range and flight time for initial speed v_o at angle θ to the horizontal:

$$\text{range} = \frac{v_o^2}{g}\sin 2\theta, \qquad \text{time of flight} = \frac{2v_o \sin\theta}{g}. \tag{2}$$

Air resistance is often represented using a damping force quadratic in speed; the resulting equations cannot be solved analytically in general [Marion and Thornton 1988; Tan and Wu 1981]. However, in our system, the droplets have very large horizontal speeds and only a small vertical distance to fall. In the limit, we can ignore the vertical drag force, writing

$$\frac{dv_x}{dt} = -kv_x^2, \qquad v_x(0) = v_o; \qquad \frac{dv_y}{dt} = -g, \qquad v_y(0) = 0.$$

The equation in x has solution $v_x(t) = 1/(kt + v_o^{-1})$, which yields

$$x(t) = \tfrac{1}{k} \ln\left(ktv_o + 1\right).$$

This gives us one solution, but we need to consider variations. Different drop sizes have different drag forces, according to the Prandtl expression [Carrión et al. 2001; Marion and Thornton 1988]:

$$k = \frac{C_d \rho_a A}{2m},$$

where

C_d is the dimensionless drag coefficient (on the order of 1),

ρ_a is the density of air,

A is the cross-sectional area of the drop, and

m is the mass of the drop.

We model the drop as a sphere of water, so $m = \rho_w(4/3)\pi R^3$, where R is the radius of the drop. Using $A = \pi R^2$, we get

$$k = \tfrac{3}{8}\, C_d \,\frac{\rho_{\text{air}}}{\rho_{\text{water}}}\,\frac{1}{R}. \tag{3}$$

This means that the distribution of $1/k$ is *exactly* the distribution of R, the droplet size distribution. The distance $x(t)$ is, to first order, proportional to $1/k$, so *the size distribution of the drops directly controls the distribution of their distance!*

The probability that a drop flies a distance x, in terms of the radius, is approximately

$$P(X = x) \approx \frac{8}{3}\left(\frac{\rho_{\text{water}}}{\rho_{\text{air}}}\right)\frac{R}{C_d}\,\ln\left(ktV_o + 1\right) P(\mathcal{R} = R).$$

Unfortunately, the distribution $P(\mathcal{R} = R)$ is not known. Raindrops follow the empirical distribution $\lambda \exp\left(-\lambda R\right)$ [Marshall and Palmer 1948], but there is no a priori reason to assume that sprinkler droplets do. The droplets are roughly spherical because of their surface tension, so we could also assume a Maxwell-Boltzmann distribution based on surface tension energy, $P(\mathcal{R} = R) = 1/Z \exp\left(-J\pi r^2/kT\right)$, yielding a normal distribution. The drop-size distribution from fire sprinklers is described as log-normal [Sheppard 2002].

Since we are not certain about the exact distribution, we combine the physical intuition gained from this model with the simplicity of the linear model.

Making the Linear Model Physically Consistent

We use the simple linear distribution but choose its properties to conserve water volume. We choose a linear shape such that:

- the width of the shape depends on the speed, and
- the total leaving the sprinklers is the total water supplied.

These two conditions determine the slope and intercept of the linear approximation. To do this in a realistic way, we must estimate the drag force using **(3)**. We guess that C_d for a water drop in air is 0.2 and assume that the largest radius of a water droplet is 0.05 cm; these are reasonable values [Tan and Wu 1981] that lead to $k = 0.15\ \mathrm{m}^{-1}$.

We now develop the x-intercept and y-intercept of the linear system in terms of the physics. A water drop that falls for t seconds travels

$$x_n(t) = \tfrac{1}{k}\ln\left(ktv_n + 1\right)$$

meters horizontally. This fixes the width of our linear distribution. Now, we ask: What is t? Normally, we would just calculate the amount of time for a drop to fall. For the 10 cm from the top of the pipe to the bottom of the pipe, this would be 0.14 s; however, no sprinkler throws out drops horizontally. When we calculated the distance traveled, we assumed horizontal initial speed; we now correct for this by using the no-air-resistance theory. We choose t as the no-resistance flight time from **(2)** with the sprinkler at a 45° angle. This is physically reasonable because air resistance makes only a small correction to flight time [Marion and Thornton 1988].

Using this approximation, we find 18–21 m as typical values for x_n, the sprinkler "throw," depending on the number of heads on the pipe. These results are consistent with typical sprinklers at pressures around 400 kPa [Carrión et al. 2001; Tarjuelo Martín-Benito et al. 1992].

Conserving Volume

Now that we know x_n, we can determine the y-intercept for the linear function. We know that in any unit of time, the amount of water coming into the pipe (J) is the amount of water sprayed by the sprinklers. For a sprinkler profile $\varphi(r)$, we can write this assumption as:

$$n\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}\varphi(r)r\,dr\,d\theta = 2\pi n\int_{r=0}^{\infty}\varphi(r)r\,dr = J,$$

where J is the incoming flux of water, 150 L/min. For our linear $\varphi(r)$, with $\varphi(0) = h_n$ and $\varphi(x_n) = 0$, this is equivalent to

$$n\,(\text{volume of cone with height } h_n \text{ and radius } x_n) = n\left(\frac{\pi}{3}h_n x_n^2\right) = J.$$

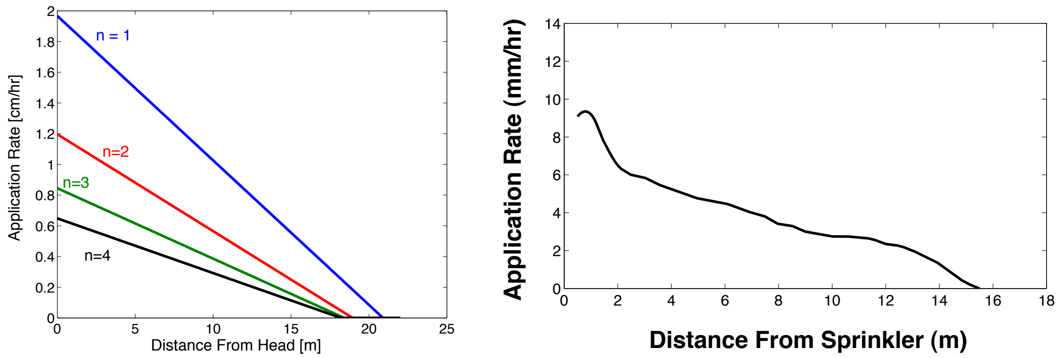


Figure 2. The modeled water distribution $\varphi(r)$ changes with the number of heads on the pipe (left) and is consistent with the experimental results of **Figure 1** (right).

This equation lets us fix the height, and completely determine the water distribution from a sprinkler,

$$h_n = \frac{3J}{\pi n x_n^2}.$$

The sprinkler profile $\varphi(r)$ is

$$\varphi(r) = \max\left(\frac{1}{k}\ln\left(ktv_n + 1\right) - r\left\{\frac{3Jk^2}{\pi n\left[\ln\left(ktv_n + 1\right)\right]^2}\right\}, 0\right). \tag{4}$$

We illustrate it for a few values of n in **Figure 2**.

Radial Approximation

For sprinkler heads along a pipe, we get a roughly elliptical distribution of water application rates. If we approximate this as a radial distribution, we get a one-dimensional profile function. We can then find a set of pipe locations by superimposing these functions and maximizing uniformity.

Determining the Radial Profile

Let L be the length of the pipe. We require $\varphi(r)$ to be monotonically decreasing, as is the case for our approximate linear sprinkler profile and for some other distributions, such as the exponential and the normal centered at zero (**Figure 3**).

Let P be the number of times that we move the pipe (which has n sprinkler heads on it). Let also

$\vec{p}_i$ and θ_i be the position and angle of the pipe,

t_i be the length of time that the pipe remains at the ith position, and

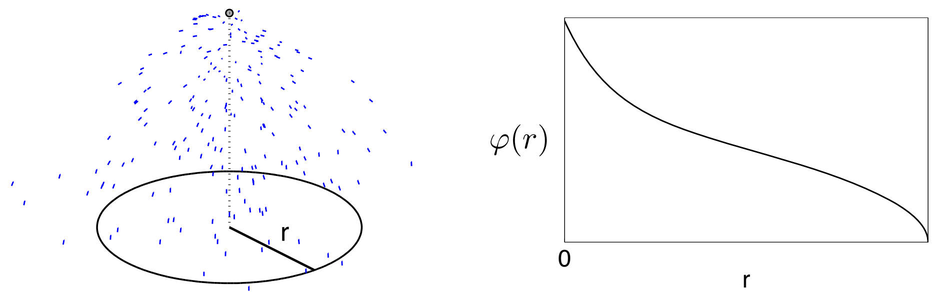


Figure 3. Sprinkler heads deliver water with a radially-symmetric distribution $\varphi(r)$.

s_j be the position of the jth head along the pipe, where $-\frac{L}{2} \leq s_j \leq \frac{L}{2}$.

We write a position on a pipe as the position vector of the pipe's center plus a part along the pipe's axis:

$$\vec{l}(\tau) = \vec{p} + \tau \hat{u}_\theta,$$

where $\vec{u}_\theta$ is an unit vector with angle θ from the positive x-axis,

$$\hat{u}_\theta = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta.$$

In this notation, the position of the jth sprinkler on the ith pipe location is $\vec{l}_i(s_j)$, and the sum of water over the field is

$$S(\vec{x}) = \sum_{i=1}^{P} t_i \sum_{j=1}^{n} \varphi\left(|\vec{x} - \vec{l}_i(s_j)|\right).$$

This $S(\vec{x})$ can be interpreted as a radial basis function interpolant [Powell 1987]. We write it as a sum over pipe locations rather than over sprinkler heads:

$$S(\vec{x}) = \sum_{i=1}^{P} t_i G_i(\vec{x}),$$

where $G_i(\vec{x})$ is the water distribution from the ith pipe position,

$$G_i(\vec{x}) = \sum_{j=1}^{n} \varphi\left(|\vec{x} - \vec{l}_i(s_j)|\right).$$

We define an approximation to $G_i(\vec{x})$ by breaking the sprinkler heads into infinitesimal pieces:

$$\begin{aligned} \widetilde{G}_i(\vec{x}) &= n \lim_{k\to\infty} \sum_{j=0}^{k-1} \tfrac{1}{k}\, \varphi\left(|\vec{x} - \vec{l}_i(s_j)|\right) \\ &= n \int_{-L/2}^{L/2} \varphi\left(|\vec{x} - \vec{l}_i(\tau)|\right) d\tau. \end{aligned} \qquad \textbf{(5)}$$

The second equality follows from assuming that the heads are sufficiently uniform to approximate an integral. The quantity $\widetilde{G}_i$ is approximately G_i; it is the limit of the water distribution as the number of heads becomes infinite while keeping constant the total volume of water that the pipe delivers.

If $\vec{x}$ is a point along the pipe, $\vec{x} = \vec{p} + t\vec{u}_\theta$, then **(5)** reduces to

$$\widetilde{G}_i(\vec{x}) = n \int_{-L/2}^{L/2} \varphi(|t-\tau|)\, \mathrm{d}\tau.$$

If φ is zero (or approximately zero) outside a radius r, and if $|t| < L - r$, then the integral is constant (or approximately constant) with respect to t. Hence the water distribution is dominantly characterized by the orthogonal distance from the pipe. Define $\mu(x)$, the pipe radial water distribution function, as

$$\mu(x) = \widetilde{G}(\hat{e}_x x) = n \int_{-L/2}^{L/2} \varphi(\sqrt{x^2 + \tau^2})\, d\tau.$$

The function $\mu(r)$ is the water distribution for the pipe, in analogy with the distribution $\varphi(r)$ for a sprinkler head. We use this function to approximate G_i, again using the analogy with radial basis functions:

$$G_i(\vec{x}) \approx \mu(|\vec{x} - \vec{z}|), \qquad \text{where } \vec{z} \text{ is the closest point to } \vec{x} \text{ on the pipe.}$$

We then use μ to approximate the total water sum,

$$\widetilde{S}(\vec{x}) = t \sum_{i=1}^{P} \mu(|\vec{x} - \vec{z}_i|), \qquad \text{where } \vec{z}_i \text{ is the closest point to } \vec{x} \text{ on the } i\text{th pipe.}$$

The function μ is proportional to a smoothed version of φ. With $\varphi(x) = \max(\lambda - |x|, 0)$, we obtain the profile

$$\begin{aligned}
\mu(x) &= n \int_{-L/2}^{L/2} (\lambda - \sqrt{x^2 + \tau^2})_+ \, \mathrm{d}\tau \\
&= \begin{cases} n \left[\lambda\sqrt{\lambda^2 - x^2} + \frac{x^2}{2} \ln \frac{\lambda - \sqrt{\lambda^2 - x^2}}{\lambda + \sqrt{\lambda^2 - x^2}} \right], & |x| \leq \lambda; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

The result is a unimodal function symmetric over the domain $[-\lambda, \lambda]$.

To indicate the flexibility of the method, we note that if φ follows a normal distribution, so does μ; or if φ is monotonically decreasing, then so is μ.

Optimality of Periodic Solutions

Consider an irrigation sequence where the pipe is oriented at the same angle θ and for the same duration t at each step for P steps. At each step, the pipe is

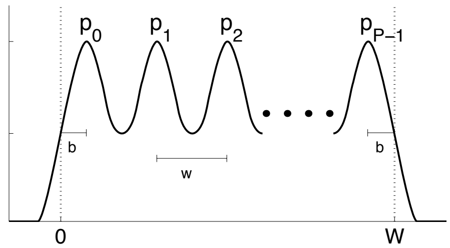


Figure 4. $S(\hat{e}_x x)$.

moved laterally w meters, so $\vec{p}_i = \vec{p}_0 + iw\vec{u}_{\theta+\pi/2}$. If, in the direction parallel to the pipe, $\vec{x}$ is not beyond the endpoints, then

$$S(\vec{x}) = t\sum_{i=1}^{P} \mu(|\vec{u}_{\theta+\pi/2} \cdot (\vec{p}_0 - \vec{x}) + iw|). \qquad \textbf{(6)}$$

Consider watering a $W \times L$ rectangular area. Define the boundary margin b as the distance between the boundary and the first pipe and the step widths $w = (W - 2b)/(P - 1)$ (**Figure 4**). Let $\vec{p}_0 = b\hat{e}_x$, $\theta = \frac{\pi}{2}$, and $\vec{p}_1 = p_0 + iwu_0$, such that the sequence irrigates the region $R = [0, W] \times [-\frac{L}{2}, \frac{L}{2}]$ (**Figure 5**).

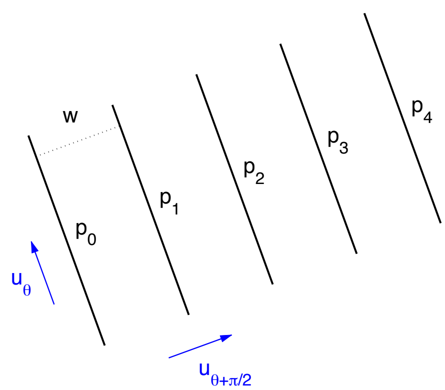


Figure 5. A five-step irrigation sequence with parallel pipe orientations.

We apply the "number of passes" uniformity criterion to $S(\vec{x})$. We maximize this criterion by maximizing its approximation in terms of $\mu(r)$:

$$\mathcal{C}_u = \frac{\min_{\vec{x}\in R} S(\vec{x})}{\max_{\vec{x}\in R} S(\vec{x})}.$$

A higher criterion value indicates more uniform irrigation of R.

Since $S(\vec{x})$ is nearly constant in the direction along the pipe, $\mathcal{C}$ is closely approximated by

$$\widetilde{\mathcal{C}}_u = \frac{\min_x S(\hat{e}_x x)}{\max_x S(\hat{e}_x x)}, \qquad \textbf{(7)}$$

that is, $\mathcal{C}_u$ restricted to the east-west line through the middle of R. Suppose that w is wide enough relative to the decay of μ such that the overlap between non-adjacent pipe terms in **(6)** is negligible. Then for $\lfloor \frac{y-b}{w} \rfloor = k$, $k = 1, 2, \ldots, (P-1)$, we have

$$\widetilde{S}(\hat{e}_x x) = \mu\big(x - b - (k-1)w\big) + \mu(b + kw - x). \qquad \textbf{(8)}$$

Since

$$\frac{\partial}{\partial x}\widetilde{S}\left(\hat{e}_x\left[b + \left(k + \tfrac{1}{2}\right)w\right]\right) = \mu'\left(\tfrac{w}{2}\right) - \mu'\left(\tfrac{w}{2}\right) = 0,$$

the midpoints $x = b + \left(k + \frac{1}{2}\right)w$ between pipe positions are local extrema of the water sum $\widetilde{S}$. Furthermore, by **(8)**, each extremum attains the same value

$$\widetilde{S}\left(\hat{e}_x\left[b + \left(k + \tfrac{1}{2}\right)w\right]\right) = 2\mu\left(\tfrac{w}{2}\right).$$

Since μ is monotonically decreasing, another set of extrema are the pipe positions $x = b + kw$, each attaining the value $\mu(0)$. At the ends of the field,

$$\widetilde{S}(\hat{e}_x x) = \begin{cases} \mu(b - x), & 0 \le x \le b; \\ \mu\big(x - b - (P-1)w\big), & b + (P-1)w \le x \le W. \end{cases}$$

For the physically-derived sprinkler distribution **(4)**, μ is simple enough that overlaps do not produce any other extrema. Therefore, $\widetilde{\mathcal{C}}_u$ is maximized by the choice of b and w such that all minima are equal and all maxima are equal.

For wider w, the pipe positions are maxima, the midpoints are minima, and the boundary margin b is selected such that $\widetilde{S}(0\hat{e}_x) = \widetilde{S}(W\hat{e}_x) = 2\mu(\frac{w}{2})$ (as in **Figure 4**). For narrower w, there is more overlap and the midpoints become maxima and b is set to zero.

Thus, the periodic solution maximizes the approximate criterion $\widetilde{C}_u$: *the periodic watering is locally optimal in uniformity of water delivery.*

Complete Solution

We restrict ourselves to periodic solutions; by symmetry, we place the pipe locations in the center of the field. We must now optimize over the number of pipe locations in one sweep of the field, the number of sprinkler heads, and the distribution of sprinkler heads along the pipe.

Analytical Prediction of Sprinkler Head Distribution

We analytically determine the location of sprinklers on a pipe to maximize uniformity. Let $\varphi(r)$ be the radial water distribution

$$\varphi(r) = h_n\left(1 - \frac{r}{x_n}\right)_+ = \max\big(h_n(1 - \tfrac{r}{x_n}), 0\big).$$

The sprinkler delivers a maximum of h_n cm/h at its center and delivery decays linearly to zero at radius x_n m. For $n = 2$, we have $h_n = 1.20$ cm/h and $x_n = 19.0$ m. The variables h_n and x_n decrease as the number of sprinkler heads n increases. Asymptotically, $x_n \to 17.9$ m (we need this lower bound later). The water distribution of the pipe is, as before,

$$G_i(\vec{x}) = \sum_{j=1}^{n} \varphi\left(|\vec{x} - \vec{l}_i(s_j)|\right).$$

We select the sprinkler head locations s_j that minimize our uniformity criterion $S_{\text{max}}/S_{\text{min}}$ of **(1)**:

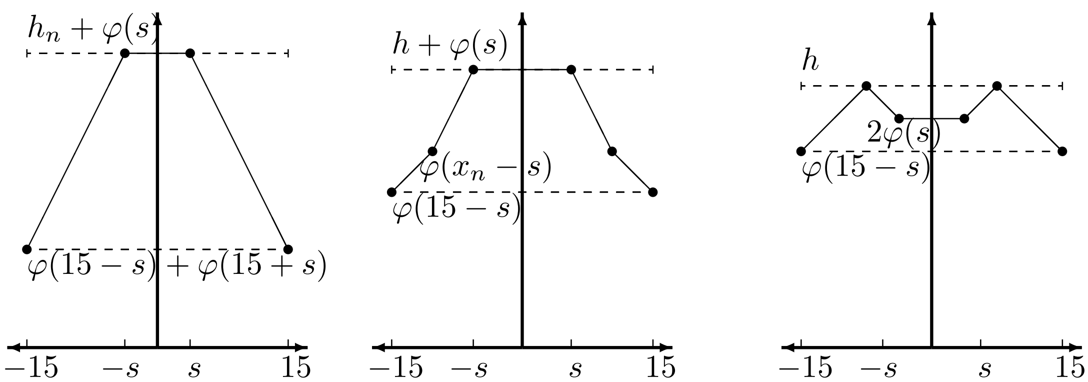
$$\mathcal{C}_u = \frac{\min_{|x|\le 15} G(x\hat{e}_x)}{\max_{|x|\le 15} G(x\hat{e}_x)}.$$

For $n = 2$, the pipe water distribution is

$$G(x\hat{e}_x) = \varphi(|x - s_1|) + \varphi(|x - s_2|).$$

The symmetry of the optimization problem implies that the heads are best placed symmetrically on the pipe. Let $s = s_1 = -s_2$, $s \ge 0$. Since the heads must be on the pipe, $s \le 10\,\text{m} < x_n$. Evaluating $G(x\hat{e}_x)$ reduces to three cases:

Case 1: $s \le x_n - 15$ Case 2: $x_n - 15 < s \le \frac{x_n}{2}$ Case 3: $\frac{x_n}{2} \le s \le 10$



In the first two cases, $\mathcal{C}_u$ improves as s increases. In the third case, G increases at the endpoints and the value in the middle decreases as s increases. Therefore, the uniformity criterion is optimized when

$$\varphi(15 - s) = 2\varphi(s),$$
$$h_n\left(1 - \tfrac{1}{x_n}(15 - s)\right) = 2h\left(1 - \tfrac{1}{x_n}s\right),$$
$$s = 5 + \tfrac{1}{3}x_n;$$

but this s places the heads beyond the endpoints of the pipe. Since $x_n > 17.9$ m, s is greater than 10.95 m.

The optimal choice is $s = 10$ m, placing the heads at the ends of the pipe.

For n even, $n > 2$, the solution is the same. Since $17.9\,\text{m} < x_n < 21.0\,\text{m}$ for all n, the same restrictions apply and the optimal choice is $s_j = (-1)^j 10$ m. For odd n, the symmetry requirement places the last sprinkler head in the center.

Optimization of Sprinkler Head Distribution

Sprinkler heads should be positioned as close as possible to the end of the pipe. For a fixed number of heads, we determine the distribution of the sprinkler heads that minimizes the number of passes required, that is, minimizes $S_{\max}/S_{\min}$ and thus maximizes uniformity. Simulated annealing and the downhill simplex (Nelder-Mead) method [Hiller and Lieberman 2005; Press et al. 1992] determine essentially the same results as we predicted above from analytical considerations!

Pipe Number, Initial Position, Number of Heads

We vary three parameters:

- **Pipe number** P controls the density of pipe positions in the field;
- **Initial position** b is the offset, how the solution interacts with the boundaries; and
- **Number of sprinkler heads** n.

The allowable ranges for these parameters are narrowly restricted. For instance, with fewer than three pipe locations per pass, we always have a region that never gets watered. Also, solutions requiring more than ten pipe locations per pass are suboptimal by the limitations on fast-watering.

The small range of parameters allows us to brute-force the optimization, calculating all possible cases (quantizing the variable b, which controls the distance of the first pipe from the boundary).

Results of Brute-Force Variation

We propose using five steps in a pass, with two sprinkler heads per pipe and periodic spacing of pipe locations, with the first pipe location on the boundary ($b = 0$). This requires four passes, or 20 moves, and has a uniformity coefficient of 94 (**Figures 6–8**); this solution is at least locally optimal.

We determine the watering time from the constraints. We make four passes, at each pass staying 32 min at each of the locations in **Figure 7**; the total amount of water applied in 96 h is given in **Figure 8**. The total watering time required is around 11 h, though the farmer does not need to be present for all of this time. The steps could also easily be split up over a four-day period.

Simulated annealing methods reproduce these values quickly, indicating that the solution space is reasonable.

We also calculated the Christiansen uniformity coefficient for these states (**Figure 9**), which shows that our best solution maximizes uniformity as well as minimizing the number of moves.

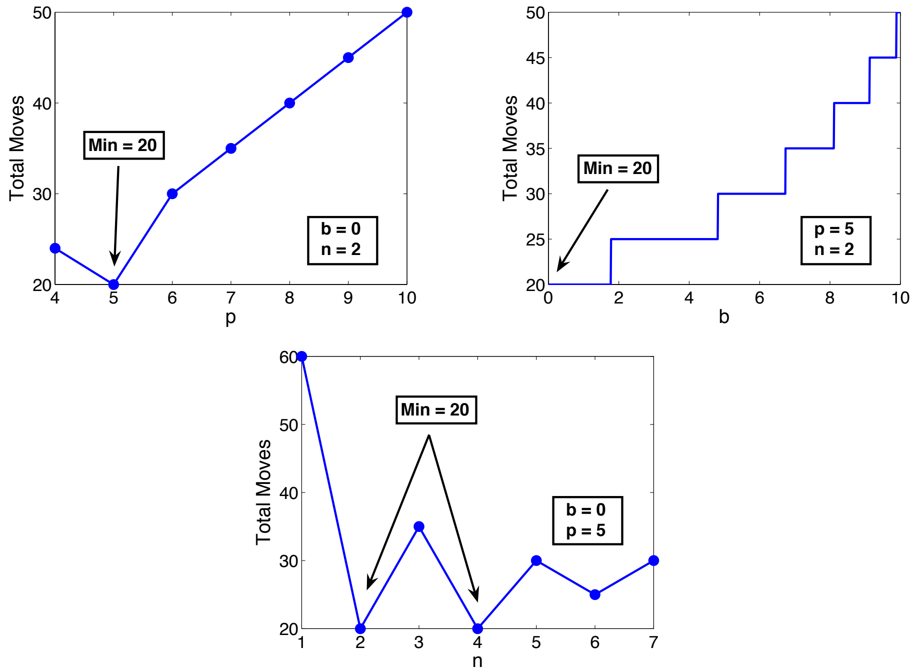


Figure 6. The number of moves required is minimal for $P = 5$, $b = 0$, and $n = 2$.

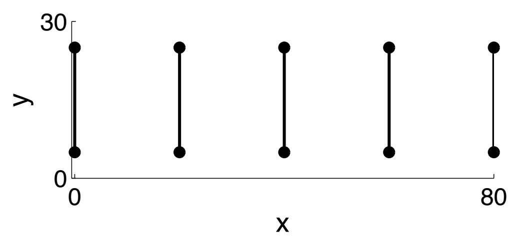


Figure 7. The layout of moves for our solution $P = 5$, $b = 0$, and $n = 2$.

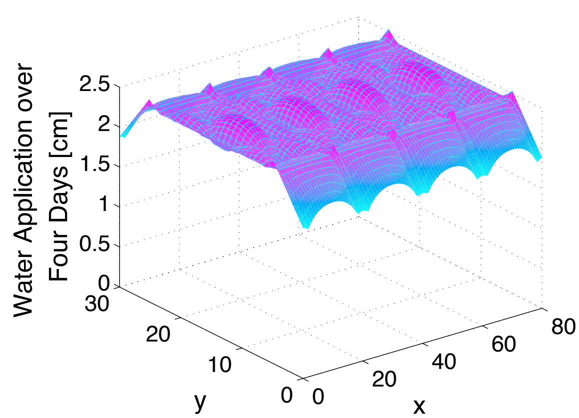


Figure 8. Our best solution has high uniformity (CU = 94), and meets the minimum watering criterion.

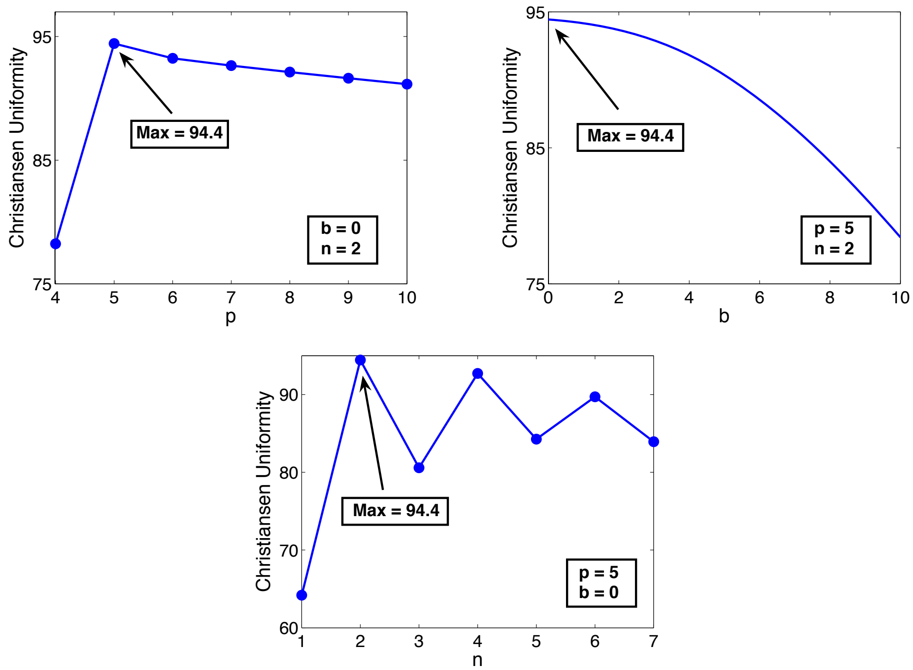


Figure 9. The uniformity of watering is at a maximum when the number of steps is at a minimum.

We are at a minimum in all three orthogonal directions in parameter space, since $b > 0$ by our constraint that all pipe locations must be within the field. This is the halting condition for Powell's optimization method, so we have, once again, a local optimal point [Press et al. 1992].

General Simulated Annealing Results

We start with a completely random distribution of pipe locations and use simulated annealing to optimize for maximum uniformity. This, in many cases, reproduces our geometrically determined distribution (**Figure 10**).

Stability of Model under Alterations

What if we change $\varphi(r)$? For $\varphi(r) = A_n \exp(r/x_n)$, where x_n is the "throw" of the sprinkler and A_n is determined so as to conserve volume, simulated annealing to maximize uniformity gives results very similar to that of our linear φ (**Figure 11**). The same occurs with φ a normal distribution.

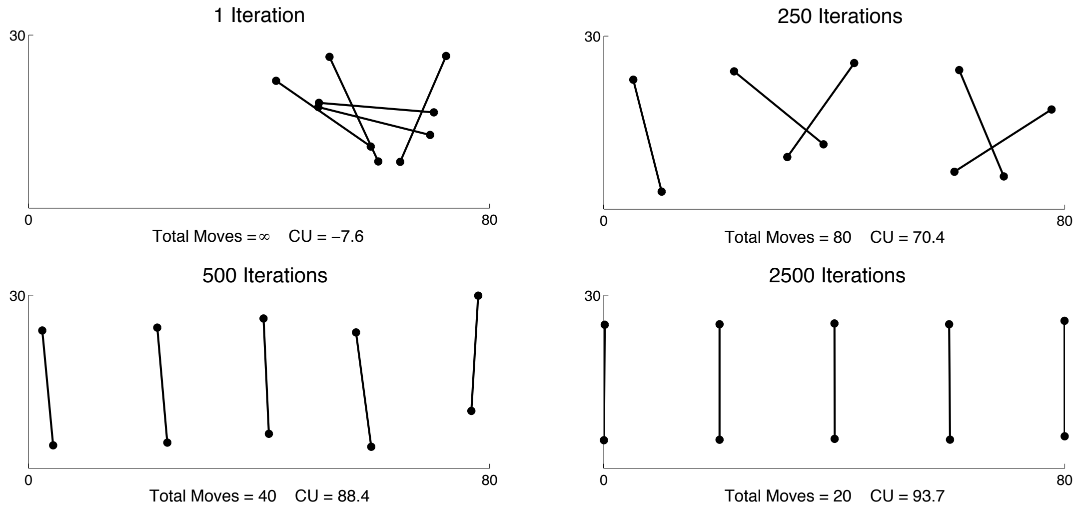


Figure 10. Simulated annealing converging to our geometrically-proposed solution.

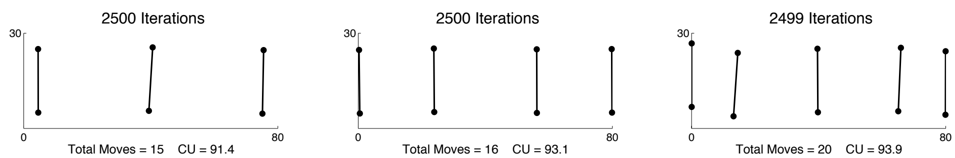


Figure 11. $\varphi(r) = A_n \exp\left(-r/x_n\right)$, for $P = 3, 4, 5$. Even when $\varphi(r)$ is changed, the general form of the optimum solution remains.

A Slow-Watering Solution

If we relax the restrictions, we can create a slow-watering solution. We take our optimum solution and treat it in the case where the flow rate is cut in half, $J = 75$ L/min. This leads to a watering solution with CU of 95 that requires moving the pipe only five times, with watering shifts of around 2 h instead of four shifts of 30 min each.

Conclusion

- We develop a physical model of sprinklers on a "hand move" pipe.
- Our model of the sprinklers predicts values for the range and distribution that are consistent with experiment.
- Using an averaging argument reduces the problem to one dimension and predicts a periodic solution.

- Through a combination of optimization and geometric arguments, we develop a fast-watering solution that requires only 20 moves with uniformity coefficient of 94, far better than the "market-worthy threshold" of 80.
- We have shown that a periodic solution is a local maximum in uniformity of watering.
- The optimum distribution of sprinkler heads has the heads near the end of the pipe.
- Simulated annealing recreates our best solution.
- Reducing the water flow would allow a slow-watering solution that required only five total moves, with CU = 95.

Strengths and Weaknesses

Strengths

- **Solution quality.** We have created a solution that works, with a relatively small number of moves and with a high uniformity.
- **Consistency.** Our analytical predictions are consistent with results from numerical optimization. Both approaches show that our solution is at least locally optimal.
- **Stability.** Searching the parameter space with simulated annealing reproduces our solution.
- **Feasibility and simplicity.** Our solution can be easily implemented by a rancher.
- **Physical consistency.** Our physical arguments produce sprinkler profiles very close to those measured experimentally.
- **Flexibility.** Our optimization techniques do not depend strongly on the sprinkler profile $\varphi(r)$.

Weaknesses

- **Constant attention.** Our system cannot be left for long periods of time—the water must be shut off, or the pipe moved, every 30 min.
- **Lack of geometric flexibility.** Though our general simulated annealing approach can adapt to different boundary conditions, our best solution depends strongly on the symmetry of the problem.
- **No global optimum.** Despite our extensive simulations, we cannot guarantee a global optimum.

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