Team Control Number

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2013 Mathematical Contest in Modeling (MCM) Summary Sheet

Summary

In order to find an ultimate shape of the pan to achieve the highest utility ratio of space without overcooking, two models are developed.

In the first part, the pan is preliminarily simplified into a 2-D figure. And to get a intermediate shape between rectangle and circle, curve chamfer is used, in which every corner of the initial rectangle is replaced by a quarter of a circle with a certain radius. As the radius increases, the shape of pan changes gradually from rectangle to circle. On condition that heat conduction is the primary factor, the heat distribution of pan with the change of time is illustrated based on *heat conduction equation*. To solve the equations, the *finite volume element method* (FVEM) is applied to get the numerical solution of the partial differential equation (PDE).

Furthermore, since a pan is three-dimensional, we develop our model into a 3-D one by the software—ANSYS. In both the cases, the same conclusion is drawn that the evenness reaches the maximum when circular and the minimum when square. And the heat distribution is quite similar in both the 2-D case and the 3-D case, which proves our simplification feasible and reasonable.

In the second part of optimization, the multi-objective programming is converted into a single-objective one with linear weighted method. Firstly, the utility ratio of space is defined to measure the number of pans in the oven and the variance of heat distribution to measure the evenness. Then, the two indexes are normalized to achieve the combination of objectives. In addition, *the dynamic programming* (DP) is applied to deal with the configuration of pans in the oven. Thus the optimal solution about the shape and the number of pans will be obtained once the weight p and the ratio W/L is given. In particular, when two objects are of the same importance, the optimal radius is always around the half of the maximum radius.

The Ultimate Brownie Pan

Team #17617

Contents

1 Introduction

1.1 Defining the Problem

If you are a chocolate lover there is nothing better than a delicious, home made brownie with a glass of ice cold milk when your sweet tooth is making you crave chocolate. However, the baking pans in our mom's kitchen are often rectangular, in which the brownies are often overcooked at the four corners. Meanwhile, if we choose a circular one to achieve even distribution of heat, the space in an oven will not be efficiently made use of.

In this case, it is necessary to find the best shape of pan to maximize both the utilization rate of the space in an oven and the even distribution of heat for the pan. Before this task, the distribution of heat across the outer edge of a pan should be illustrated. Thus, two tasks will be done in this paper:

- Develop a model to show the distribution of heat across the outer edge of a pan for pans of different shapes - rectangular to circular and other shapes in between.

- Develop a model that can be used to select the best type of pan (shape) under the following conditions:
  1. Maximize number of pans that can fit in the oven;
  2. Maximize even distribution of heat for the pan;
  3. Optimize a combination of conditions (1) and (2) where two weights are assigned to them.

1.2 Model Overview

At first,we use curve chamfer(**Figure1**) to describe the intermediate shapes between rectangular and circle.When the radius r changes from 0 to maximum,the shape changes from square(Here we simplify the rectangle into a square,and we will discuss this treatment in the part of assumptions) to circle.

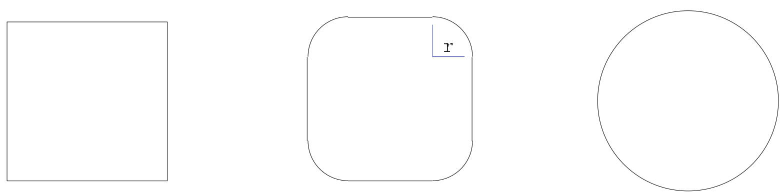

Figure 1: The sketch map of curve chamfer

Then, in order to solve the problems above,we build two models.The first model which is based on the *heat conduction equation* and solved by the means of *finite volume element method(FVEM)* can illustrate the distribution of heat across the pan;The second model which is based on *the linear weighted method* and *dynamic programming(DP)* that deals with the optimization of rectangular parts nesting.

In the first model,*the heat conduction equation* is used respectively in a 2-D case and a 3-D case,and in both the cases the distribution of heat across the pan can be illustrated.In the second model,the multi-objective programming is converted into a single-objective programming by using the *analysis of variance(ANOVA)*, data normalization and linear weighted method. Thus we can get the best shape of the pan to optimize the utilization rate of the space in an oven and the distribution of heat across the pan.

2 Assumptions

- **The pan is initially to a cuboid**
  In reality, the shape of the pan used in an oven is similar to a flat box without lid. Nevertheless, considering that the heat distribution of brownie baking in the pan is mainly influenced by the pan and the need of simplifying the model, we initially regard the pan as a homogeneous cuboid and determine it as our object of study.
- **The space in oven is constant temperature field**
  According to the technical parameters published by some household oven manufacturers, the temperature range in the oven is about 2 Degree Celsius, a relatively smaller value, so we decide to ignore the range and treat the inner space of an oven as a constant temperature field.
- **The process of the the thermal transmission is a transient process**
  Before the pan is put inside, the oven is often preheated[1]. Thus when the pan enters, there is a temperature difference between the pan and its surroundings. So it takes time for the pan to get heated gradually to a certain temp. And we believe that the heat distribution of cuboid in this process is most of the time non-uniform, which leads to the overcooking at the corners.
- **Only square pan is considered, or ignore the rectangle pan whose length is unequal to its width**
  In order to make calculation easier, we simplify the rectangle into square.
- **Using different chamfer to describe the different shape between the square and circle**
  As for the study of intermediate shape between rectangle and circle, basically two ways are available. One is to use regular polygons and the other is curve chamfer. However, the angle of polygons does no good to heat distribution because of tip thermal effect[2], we decide to abandon the disadvantageous regular polygon and to apply curve chamfer only.

- **Each pan has the same area**
- **Initially two racks in the oven, evenly spaced and the distribution on each rack are the same**
- **The distribution of heat across the pans on the upper rack and on the under rack is the same. Therefore, we can only study the pans on one of the racks.**

3 Notations

Table 1: Notations and Descriptions

Notations	Descriptions
W/L	Width to length ratio of the oven
A	The area of pan
N	The number fit in the oven
H	The distribution of heat across the pan
r	The radius of the curve chamfer
p	Weight giving to the first optimization objective
T_t	The distribution of temperature on the pan at time t
T_{out}	The temperature out of the pan
f	Energy source inside the heat carrier
λ	Heat conductivity of the pan
h	coefficient of heat convection
ρ	The density of the pan
c	Specific heat capacity
$\sigma^2(H)$	The variance of the distribution of heat across the pan
ξ	Utilization rate of the area in the oven
μ	The standardized value of ξ
η	Evenness of the distribution of heat across the pan
OBJ	Objective function of the multiple objective programming

4 Part I

4.1 Model I:The Distribution of Heat

Newton articulated some principles of heat flow through solids, but it was Fourier who created the correct systematic theory. Inside a solid there is no convective transfer of heat energy and little radiative transfer, so temperature changes only by conduction, as the energy we now recognize as molecular kinetic energy flows from hotter regions to cooler regions. The basic principles of heat are:

- The heat energy contained in a material is proportional to the temperature, the density of the material, and a physical characteristic of the material called the specific heat capacity.

- The heat transfer through the boundary of a region is proportional to the heat conductivity, to the gradient of the temperature across the region, and to the area of contact.

Based on the two principles above, the heat conduction equation can be figured out

$$\frac{\partial T}{\partial t} - a^2\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial y^2}\right) = f$$

where $a^2 = \frac{\lambda}{\rho c}$ and λ, ρ, c, f respectively represent the heat conductivity, density , thermal capacity and energy source[3], and x, y, z are the coordinates.

- **Initial Condition**

$$T\,|_{t=0} = T(x, y, z)$$

- **Boundary Conditions**

  There are three kinds of boundary conditions of the heat conduction equation above:

  - **Dirichlet boundary condition**
    The temperature at the boundary is known.

$$T|_s = \alpha(x, y, z, t)$$

  - **The second boundary condition**
    Heat can flow into the body through the boundary.

$$\left.\frac{\partial T}{\partial n}\right|_s = \beta(x, y, z, t)$$

  - **The third boundary condition**
    There is heat exchange on the boundary.

$$\left.[\frac{\partial T}{\partial n} + hT]\right|_s = \gamma(x, y, z, t) = hT_{out}$$

    where s is the boundary, and T_{out} is the surrounding temperature.

Obviously, the heat conduction equation in 1-D and 2-D can also be figured out.

4.1.1 The evenness of the distribution of heat (H) across the pan

Based on the Fourier's theory of heat conduction, considering a certain volume element, the connection of $\mathrm{d}H$, $\mathrm{d}A$, $\mathrm{d}t$, $\frac{\partial T}{\partial n}$ satisfies the equation below:

$$\mathrm{d}H = \lambda \frac{\partial T}{\partial n} \mathrm{d}A\mathrm{d}t$$

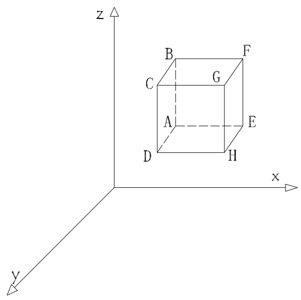

Figure 2: The sketch map of the volume element

For the sake of simplicity, we just study the heat inflow within small volume element. According to Fourier's law, the heat influx through the surface *ABCD* in a short time can be expressed as:

$$\mathrm{d}H|_x = -(\lambda \frac{\partial T}{\partial x})|_x \mathrm{d}t\mathrm{d}y\mathrm{d}z$$

The negative sign indicates the outer normal direction of the surface is opposite to the x-axis positive one, in other words,

$$(\frac{\partial T}{\partial n})|_x = -(\frac{\partial T}{\partial x})|_x$$

Similarly, the heat flowing into the volume is as follows:

$$\mathrm{d}H|_{x+\mathrm{d}x} = (\lambda \frac{\partial T}{\partial n})|_{x+\mathrm{d}x} \mathrm{d}t\mathrm{d}y\mathrm{d}z = (\lambda \frac{\partial T}{\partial x})|_{x+\mathrm{d}x} \mathrm{d}t\mathrm{d}y\mathrm{d}z$$

Accordingly, the inflow of the heat through these two surfaces, which are perpendicular to the x-axis, is

$$\mathrm{d}H|_x + \mathrm{d}H|_{x+\mathrm{d}x} = \left[(\lambda \frac{\partial T}{\partial x})|_{x+\mathrm{d}x} - (\lambda \frac{\partial T}{\partial x})|_x \right] \mathrm{d}t\mathrm{d}y\mathrm{d}z = \frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x}) \mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Similarly, heat fluxing into the tiny cube along y, z axis

$$\frac{\partial}{\partial y}(\lambda \frac{\partial T}{\partial y}) \mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z})\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

The increased heat make its temperature elevate. According to the law of conservation of energy, we can get the heat balance equation:

$$\left[\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+\frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z})\right]\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z=\rho c\frac{\partial T}{\partial t}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

or,

$$\left[\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+\frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z})\right]=\rho c\frac{\partial T}{\partial t}$$

Therefore, the heat accumulation of the volume element can be obtained by means of infinitesimal calculus.

$$H=\int_0^{t_0}\left[\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+\frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z})\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t=\int_0^{t_0}\rho c\frac{\partial T}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t=\Delta V\int_0^{t_0}\rho c\frac{\partial T}{\partial t}\mathrm{d}t$$

Thus, once variation of the temperature obtained, we can get the heat accumulation in the micro unit easily.

4.1.2 The Case of 2-D

•Extra Assumptions

1.In this model, we assume that the pan is a 2-D plane. In this case, the planar heat conduction equation can be used to calculate the distribution of heat across the idealized planar pan.

2.We assume that the heat diffusion of the pan is ignored, or the pan only absorbs heat from surroundings before its distribution of temperature reaches a stable status.

•The Heat Conduction Equation

In the 2-D case, the heat conduction equation can be described as

$$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+f \tag{1}$$

where x, y stands for the coordinates, $\rho, c, t, T, \lambda, f$ respectively represent areal density, specific heat capacity, time, temperature, the heat conductivity and energy source.

In the initial equation, f represents the energy source inside the conductive body. However, since there is no energy source inside the pan, i.e., it only absorbs heat from surroundings, the energy source $f = 0$.

Initial Condition

$$T\mid_{t=0}=T_0$$

Boundary Conditions

Under the assumptions, we regard the three-dimensional cuboid pan as two-dimensional rectangle pan. In this case, the heat out of the pan can only flow into it through the four sides of the rectangle(**Figure**3).The boundary conditions on the four sides of the pan can be described as

$$\frac{\partial T(x,y)}{\partial x} + hT(x,y)|_{x=\pm\frac{a}{2},-a\le y\le a} = hT_{out}$$

$$\frac{\partial T(x,y)}{\partial y} + hT(x,y)|_{y=\pm\frac{a}{2},-a\le x\le a} = hT_{out}$$

where λ is the heat conductivity.

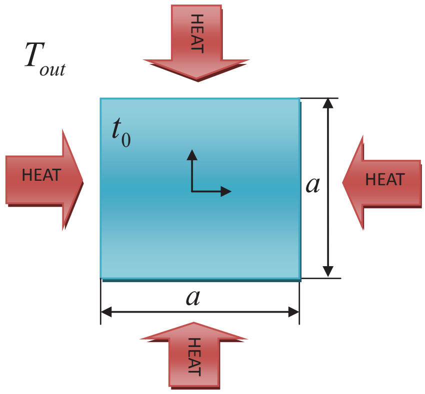

Figure 3: planar heat conduction in rectangle

When curve chamfer is considered(**Figure**4), the boundary conditions can be described as

$$\frac{\partial T(x,y)}{\partial x} + hT(x,y)|_{x=\pm\frac{b}{2},-\frac{b}{2}\le y\le \frac{b}{2}} = hT_{out}$$

$$\frac{\partial T(x,y)}{\partial y} + hT(x,y)|_{y=\pm\frac{b}{2},-\frac{b}{2}\le x\le \frac{b}{2}} = hT_{out}$$

$$\frac{\partial T(x,y)}{\partial x} + \frac{\partial T(x,y)}{\partial y} + hT(x,y)|_{\omega} = hT_{out}$$

where ω is the arc boundary.

4.1.3 The Case of 3-D

•Extra Assumptions

1.In this model, we consider the pan as a three-dimensional figure. So the heat conduction equation in this case is a three-dimensional one.

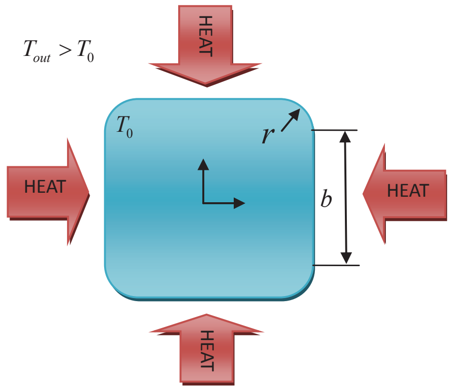

Figure 4: planar heat conduction in rounded rectangle

2.The heat diffusion of the pan is ignored.

•The Heat Conduction Equation

In this model, we consider the pan as a three-dimensional figure. So the heat conduction equation in this case is a three-dimensional one

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial z}) + \frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z}) + f \tag{2}$$

where x, y, z stands for the coordinates, $\rho, c, t, T, \lambda, S$ respectively stands for density, specific heat capacity, time, temperature, the heat conductivity and energy source. Because here is no energy source inside the pan, the energy source $f = 0$.

Initial Condition

$$T\,|_{t=0} = T_0$$

Boundary Conditions

In the 3-D case, except the boundary conditions on the four sides of the pan, the top surface and the bottom surface of the pan also have boundary conditions. And when curve chamfer is considered, the transfer of heat can be described in **Figure**5.

As illustrated in **Figure**5, the heat flow into the pan through the sides of the pan. As for the bottom surface of the pan, because it doesn't contact with either the energy heat source or the brownie to be cooked, we think of its boundary conditions the same as those on the four sides of the pan. And for the top surface of the pan, because it contacts with the brownie to be cooked, the heat flow into the bread from the top s surface,which makes its boundary conditions different from those with the four sides and the bottom. Thus, the boundary conditions of the three-dimensional heat conduction equation can be described as

$$\frac{\partial T(x,y,z)}{\partial x} + \frac{\partial T(x,y,z)}{\partial y} + \frac{\partial T(x,y,z)}{\partial z} + h_1 T(x,y,z)\Big|_{z=\frac{z_0}{2}} = h_1 T_{out}$$

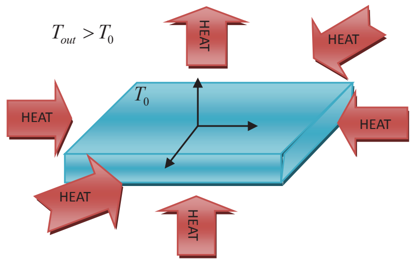

Figure 5: three-dimensional heat conduction in chamfer box

$$\frac{\partial T(x,y,z)}{\partial x} + \frac{\partial T(x,y,z)}{\partial y} + \frac{\partial T(x,y,z)}{\partial z} + h_2 T(x,y,z)\,|_{\Omega} = h_2 T_{out}$$

where Ω is the boundary except the top surface, z_0 is the thickness of the pan, h_1 is the the heat conductivity between pan and surrounding air and h_2 is the the heat conductivity between pan and brownie.

4.1.4 Method to Solve the Heat Conduction Equation

Based on the principle of conservation of energy, finite volume element method is widely used to solve the complex differential equation in the field of thermology or physics[4]. Therefore, the *finite volume element method(FVEM)* is used to solve the heat conduction equation above.

According to **eq1**, the planar *FVEM* can be described[5] as

$$\int_{\tau}^{\tau+\Delta t} \int\int_{s} c\frac{\partial T}{\partial \tau} dxdyd\tau = \int_{\tau}^{\tau+\Delta\tau} \int\int_{s} (\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y}) + f)dxdyd\tau \qquad (3)$$

where s is the heat conduction region.

Thus, the discrete equation of the planar heat conduction equation can be derived[5] as

$$a_p T_p = a_E T_E + a_W T_W + a_N T_N + a_p^0 T_p^0$$

$$a_p = a_E + a_W + a_N + a_p^0$$

$$a_p^0 = \frac{\rho c \Delta V}{\Delta \tau}$$

$$a_E = \lambda_e \frac{\Delta y}{\delta x_e}, a_W = \lambda_w \frac{\Delta y}{\delta x_w}, a_N = \lambda_n \frac{\Delta y}{\delta x_n}, a_S = \lambda_s \frac{\Delta y}{\delta x_s}$$

$$\Delta V = \Delta x \Delta y$$

where $\lambda_e, \lambda_w, \lambda_n, \lambda_s$ are the heat conductivities on the four edges of the element(**Figure**6).

The flow chart of solving the heat conduction equation using *FVEM* is demonstrated in **Figure**7.

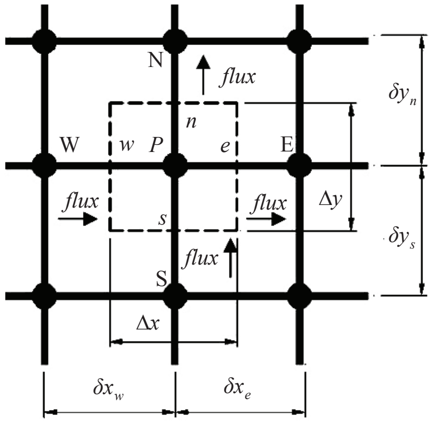

Figure 6: heat conduction in the finite element

4.2 Solutions to Model I

To solve the problem quantitatively, we searched for amounts of data about the size of an oven, the area of a pan, and material properties. Eventually, we consider the material of the pan is iron and the data to be used to solve the problem are listed in **Table2**

Table 2: Parameters and Values

Parameters	Values
the internal area of the oven	$310mm \times 252mm = 78120mm^2$
the area of the pan	$60mm \times 60mm = 3600mm^2$
thermal convection coefficient between iron and air	$45W/(m^2\cdot$°C)
thermal conductivity of iron	$63.5W/(m\cdot$ °C)
specific heat capacity of iron	$500J/(kg\cdot$°C)
density of iron	$7900kg/m^3$
initial temperature of the pan	20°C
temperature inside the oven	200°C
baking time	$1000s$

4.2.1 The Solution to the Case of 2-D

We use Matlab to simulate the heat distribution of the pan. In the simulation, pans of different shapes are put in the same temperature(200°C) and heated for 1000*sec*. From the output figures, we find that the temperatures at corners are much higher than those in the middle of the edge. And the temperature at the center of the pan is the lowest. By comparison, conclusion can be drawn that when the shape of the pan changes from square to circle, the sharper the corner is

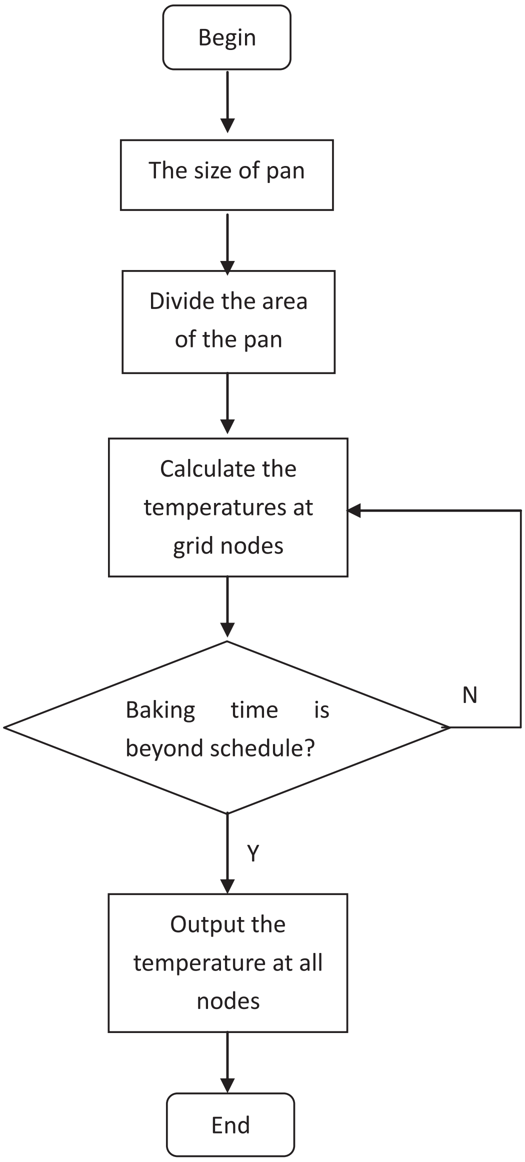

Figure 7: using the *FVEM* to solve heat conduction equation

the higher the difference of temperature is. Data in the table below can properly illustrate our conclusion.

The temperature is 200°C, the heating time is 1000*sec*.

In our model, we can find that because the elements at corners have three surfaces exposed to high temperature, through which the heat can easily enter the pan. But the other elements only have one (at the center) or two (at the edge) surfaces exposed, so the temperature increases much more slowly. As a result, the temperature at the corners is the higher than that at the edge, than that at the center of the pan.

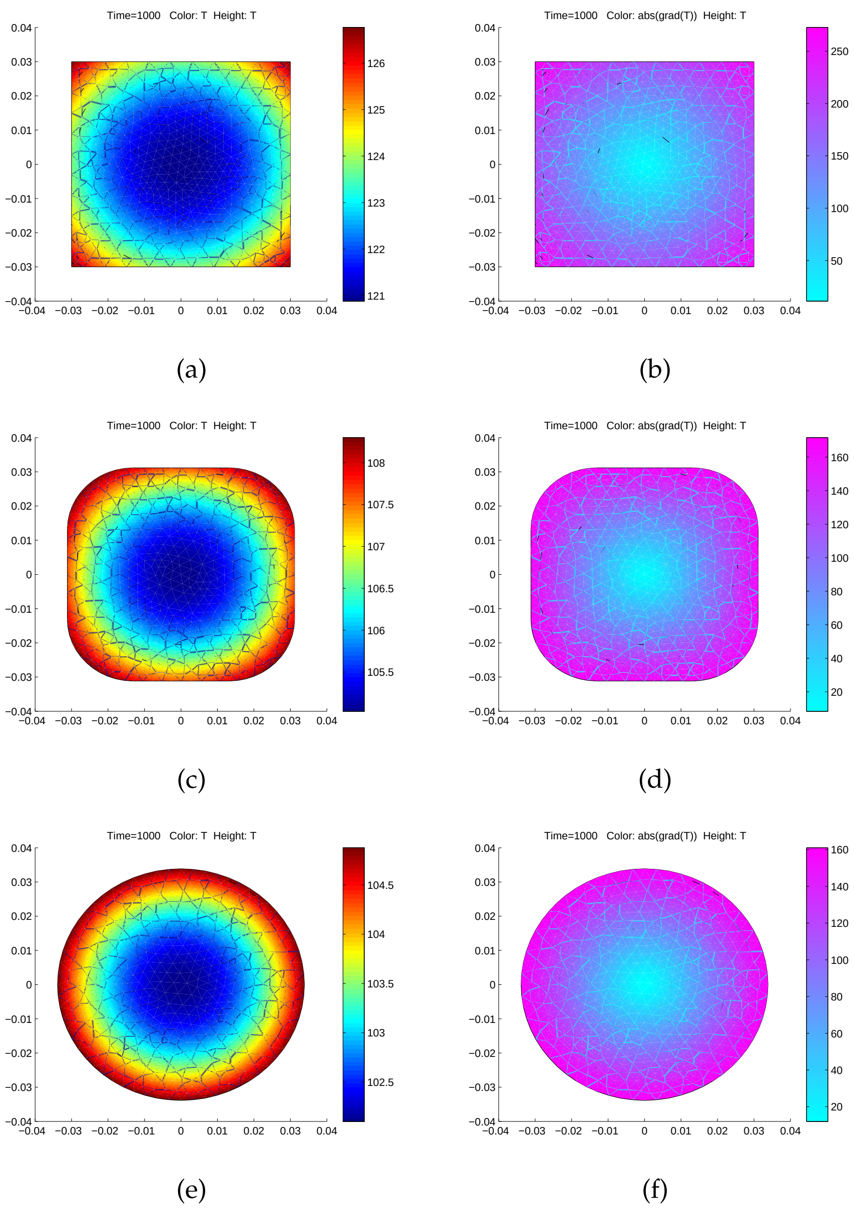

Figure 8: The distribution and gradient of heat of different shapes

As for the gradient of the temperature, we can find that the shaper the shape of the pan is, the higher is the gradient of the heat. In order to describe the evenness of the temperature distribution on the surface, we use variance to measure the evenness of the heat on the pan of different shapes . The results are in **Table3**.

4.2.2 The Solution to the Case of 3-D

It is difficult to figure out the analytical solution, even the numerical solution, to the three-dimensional heat conduction equation in this question. To simplify

Table 3: The data of r, $\sigma^2(H)$ and T

r	0	0.001	0.002	0.005	0.008	0.01	0.012
σ ²	868.0001	865.2482	808.3838	732.8263	685.739	698.1171	639.413
T	127	126	122	117	113	114	111
r	0.015	0.018	0.02	0.023	0.03	0.032	0.03385
σ ²	613.6184	585.228	576.5488	562.1477	539.387	545.9151	545.7515
T	108	106	105	104	102	102	103

this problem, we further make an assumption as follows: in our question about the pan, we use two-dimensional heat convention equation to take place of the 3-D one. As we know, when the pan is placed in a high temperature environment, it will quickly absorb heat through its top and bottom surface. However, since our goal is to maximize the evenness of temperature distribution on top surface, we can ignore this one-dimensional heat transfer for this can only affect the average temperature of the pan. It is obvious that when the temperature everywhere increases or decreases by the same value, the evenness of temperature distribution won't change when variance is used to describe the evenness of data.

In order to demonstrate the reliability of our new assumption, we make two experiments to prove its feasibility. By using a powerful analysis software-Analysis, we get these results:

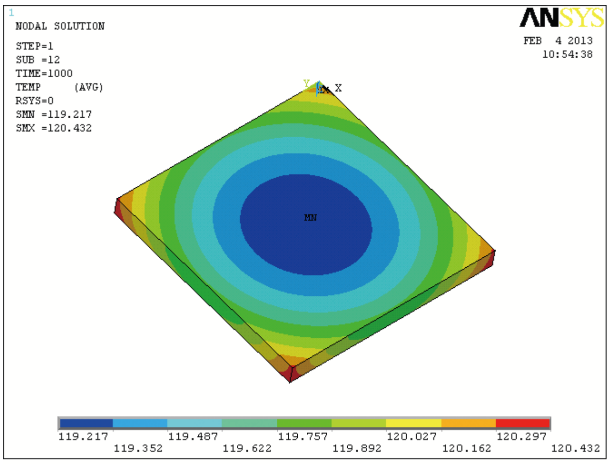

Figure 9: three-dimensional heat conduction in chamfer box

In our experiments, all the conditions are the same except that in **Figure9** the cuboid is totally exposed in the external high temperature(200°C) environment, while in **Figure10** the four sides are regarded as insulated. After the same period of time (800sec), the temperature distributions are showed in the above two figures. In **Figure10**, the heat transfer is one-dimensional, and the maximum temperature difference(the highest temperature minus the lowest temperature) is only 0.039°C, far less than the difference(0.757°C) in **Figure9**. We can regard it

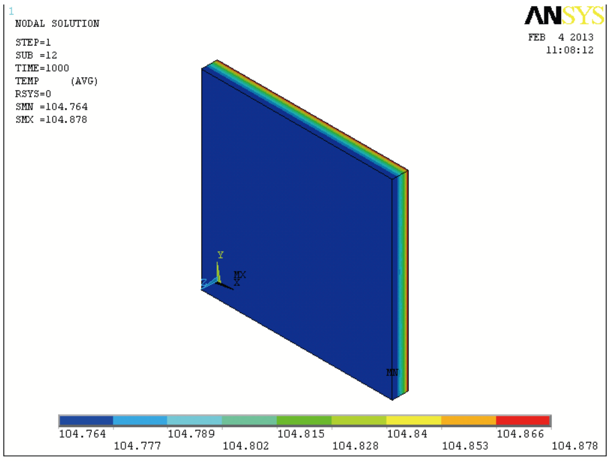

Figure 10: three-dimensional heat conduction in chamfer box

as a constant temperature field, which means it won't affect the evenness of the temperature distribution.

According to superposition principle, the heat in the pan consists of two parts, the one coming in through the four sides, and the other one coming in through the top and the bottom surface. Because the latter one's contribution to the unevenness can be ignored as well as the thickness of the pan, we simplify the three-dimensional heat conduction into a two-dimensional one, in which only the heat coming in through the four sides is considered.

5 Part II

5.1 Model II:Multi-objective Programming

•The number of pans that can fit in oven

When W/L and the area of each pan A are fixed(In fact, we can assign specific value to W and L on the basis of W/L. Thus the area of inner horizontal cross section of the oven $S = WL$ is also fixed), the maximum number of pans that can fit in the oven changes with the shape factor ω. In order to measure the utilization ratio of the space in the oven, we define a coefficient $\xi = \frac{NA}{S} = \frac{NA}{WL}$. Because A, $S = WL$ and W/L are all fixed, ξ is in direct proportion to N, thus we can use the maximum of ξ to describe the maximum of N.

In order to get the maximum number of the pan that can fit in the oven , we use the dynamic programming algorithm to fill the rack in the oven with the minimum bounding rectangle of the pan. We divide the inner area(S=WL) of the rack into same parts that are in the same size with the pan as many as possible. Each time we try to decide the direction in which the strip can be divided into the maximum number of required parts, until the remnant part of the oven can no more be divided. The flow chart is in **Figure11**.

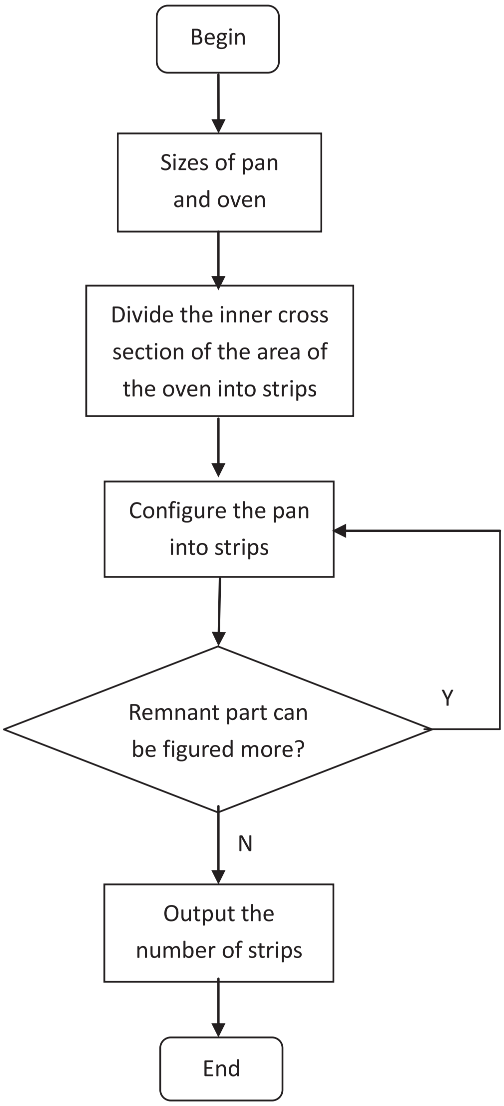

Figure 11: three-dimensional heat conduction in chamfer box

•The evenness of the distribution of heat across the pan

Through the analysis in model I, we can use the variance $\sigma^2(H)$ to measure the distribution of heat across the pan.

•Optimization Model

It is a Multiple Objective Programming problem aiming to maximize both the number of pans that can fit in the oven and the even distribution of heat for the pan. Linear weighting method[6] can be used to solve this problem.

Combine the two conditions with weights p and $(1 - p)$, and the Multiple Objective Programming can be converted to a Single Objective Programming which can be solved with a conventional method.

•Data Normalization

However, the two objectives can not be added simply, because they have different dimensions. Data normalization is necessary before adding.

The utility ratio of space is defined as ξ

$$\xi = \frac{NA}{S} = \frac{NA}{WL}$$

μ is the standardization of ξ, which varies from 0 to 1.

$$\mu = \begin{cases} \frac{\xi - \xi_{\min}}{\xi_{\max} - \xi_{\min}} & N_{\max} \neq N_{\min} \\ 1 & N_{\max} = N_{\min} \end{cases} \tag{4}$$

The variance $\sigma^2(H)$, measuring the evenness of heat distribution, should be normalized to $(0, 1)$. Under the assumption that the evenness of heat distribution reaches a maximum in the shape of a circle is maximum while minimum in the shape of square, we think that the variance $\sigma^2(T)$ reaches maximum when the pan is square and minimum when circle. Thus, the standardization formula can be described as

$$\eta = \frac{\sigma^2_{\max}(H) - \sigma^2_i(H)}{\sigma^2_{\max}(H) - \sigma^2_{\min}(H)}$$

where $\sigma^2_{\max}(H)$ is the variance in the case of square, $\sigma^2_{\min}(H)$ when the pan is circle and $\sigma^2_i(H)$ is the variance when the shapes are between square and circle.

In this way, the $\sigma^2(H)$ can be normalized into $[0, 1]$, and we use this η to measure the evenness of the heat distribution.

•Objective Function Derivation

Combine the two index μ, η with weights p and $(1 - p)$, and the Multiple Objective Programming can be converted to a Single Objective Programming which can be solved with a conventional method.

$$OBJ = p \cdot \mu + (1 - p) \cdot \eta \tag{5}$$

Eq5 is the objective function to the Multiple Objective Programming.

5.2 Solutions to Model II

To get the optimal solution, we fix one of the three unknown parameters (W/L, p, and r) and assign the other two with various values in order to obtain the relationship between the objectives and to find the optimal point. To integrate theory with practice, the probable range of these parameters (mainly A and W/L) can be found in market. By analyzing the data of ovens, we finally decide that the range of W/L is [0.5,1], the pan's area $A = 0.0036m^2(60mm \times 60mm)$, the size of oven is $310mm \times 252mm$, the inner temperature of the oven is 200°C, and the initial temperature of the pan is 20°C.

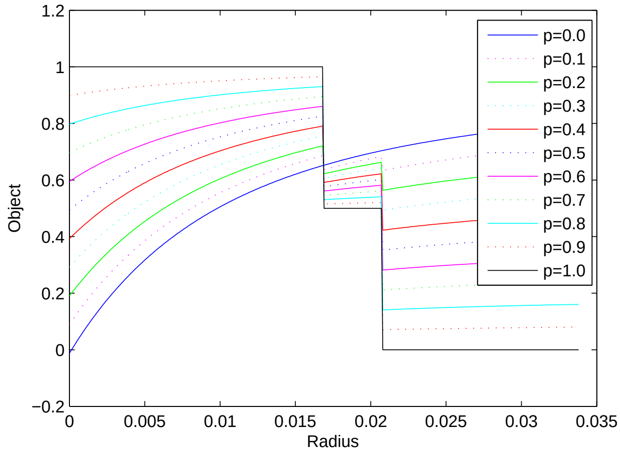

Figure 12: object-radius relationship

When the weight p is around 0.5 (the curve in red, pink in **Figure1**), the objective value reaches the maximum on condition that the radius is about half of the maximum or a quarter of the length of square side.

In **Figure12**, we set W/L as a constant and get the relationship between r and p. When $p = 0$, we only take the evenness of temperature into consideration, ignoring the number of pans. Thus the object-radius function is an increasing function. With the radius increasing, the shape of the pan is more and more close to a circle and the objective value also increases but the growth rate is decreasing. Represented in the figure, the slope of the curve is decreasing, which fits practical experience well. When p=1.0, we only take the number of pans into consideration, using the utilization ratio of area to describe it. In the model, when the square is chamfered, to simplify the configuration of pan, we ignore the chamfer part, i.e., the chamfered square is regarded as its circumscribed square. Since the area of the pan A is a constant, when the chamfer radius increases, the side length of the pan's externally circumscribed square's area increases. When the side length increases, the number of pans may suddenly plummets, for the side length of the horizontal cross section of oven is no longer enough to hold the previous number. So the curve shows stair-step shape. As the parameter p varies from 0 to 1, the shapes of the curves changes between the two. We can find that all the curves have its maximum point, but the optimal radius varies. When p is small, the evenness of pan is in the dominant role. So the optimal object is gained when the radius reaches its maximum value. But when p takes the other value (for example $p = 0.7$), the curve is serrate. All the curves have the same potential points at which the object may reach its maximum value. Here when $W/L = 0.8129, A = 0.0036mm^2$ and p is also set a certain value, the optimal potential radius may be 0, 0.0168, 0.0215, 0.0339.

In **figure13**, we set p=0.5 as a constant to find the relationship between r and W/L. The result is showed above. When the parameter W/L varies, the

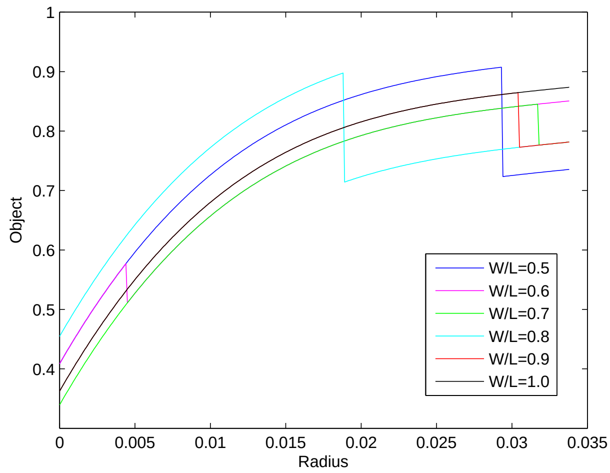

Figure 13: object-radius relationship

optimal radius also varies. From the above figure, we can find that the larger the W/L, the larger the objective value we can get. In our example, the maximum objective value is reached when $W/L = 0.5, r = 0.0289m, p = 0.5$ and $A = 0.0036mm^2$.

To optimize our model solution, we further analyze the objective value by illustrating the relationship in a figure where only one parameter is fixed. The results are as follows:

1.The radius r is given

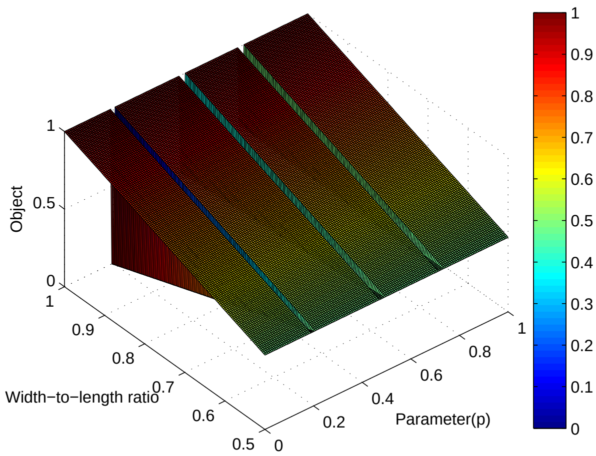

Figure 14: object-W/L-p relationship

In **figure14**($r = 0.01m$), we can draw the conclusion that as a whole, the objective value increases when the width-to-length ratio increases. When the value of width-to-length is 1.0, thus the oven inner shape is a square, the object can reach the maximum value 1.0. So the square-shape of oven is more efficient and practical. We get the object-radius curve which is like stair-step shape when the other parameters are given before, so as this plane.

2.The parameter p is given

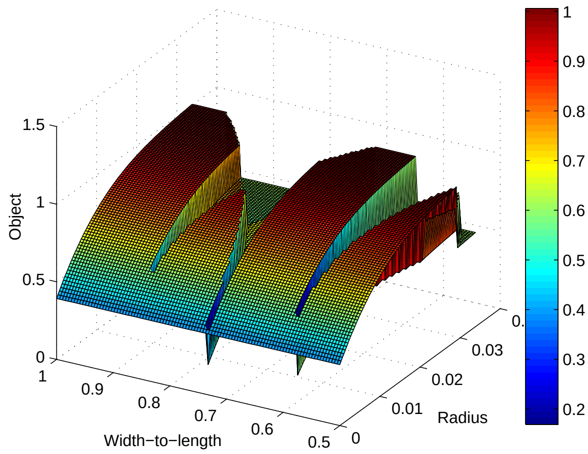

Figure 15: object-w/L-p relationship

We set the parameter p as a constant and its value is 0.7. In this situation, the arguments are only width-to-length ratio and radius of the chamfered square. It is more complicated because they are the most fundamental and essential parameters. The whole trend is that the objective value and the W/L are positively correlated, but the optimal solution is no longer gained at the maximum value of W/L. In our example, the optimal solution is 0.9213 when $(W/L, r) = (0.5543, 0.012m)$.

6 Strengths and Weaknesses

6.1 Strengths

- Begin with a simplified 2-D case, and then deal with a more complex 3-D case, thus having a more realistic model to solve the problems.
- Because of the difficulty to get the analytical solution to the heat conduction equation, we use *the finite volume element method(FVEM)* to get the numerical solution, and then get the distribution of heat across the pan.
- The Dynamic Programming algorithm is used to facilitate the optimization of configuration of pans.

6.2 Weaknesses

- Only heat conduction way is considered to solve the problem of the distribution of heat across the pan.
- Only square with rounded corners is considered as the intermediate shape between rectangle and circle.

7 Future Work

- In the model above, we use curve chamfer to get the intermediate shape between rectangle and circle, and only square, a special case of rectangle, is taken into consideration. However, other transitional shapes such as "square $\rightarrow$ regular polygon $\rightarrow$ square" or "rectangle $\rightarrow$ arbitrary polygon $\rightarrow$ ellipse" can also be tried to discover the best solution to the problems.
- Only heat conduction is considered when calculating the distribution of heat across the pan; Heat radiation and convection are ignored. In fact, the three thermal transmission ways are simultaneous, so the other two thermal transmission ways should be considered when dealing with the problems, even if they may have a smaller effect
- Maybe other better method can be used to measure the evenness of the distribution of the heat across the pan.
- When the area of the pan A changes, the results may also change.

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