For Whom the Booth Tolls

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Summary

We model traffic near a toll plaza with a combination of queueing theory and cellular automata in order to determine the optimum number of tollbooths. We assume that cars arrive at the toll plaza in a Poisson process, and that the probability of leaving the tollbooth is memoryless. This allows us to completely and analytically describe the accumulation of cars waiting for open tollbooths as an M|M|n queue. We then use a modified Nagel-Schreckenberg (NS) cellular automata scheme to model both the cars waiting for tollbooths and the cars merging onto the highway. The models offer results that are strikingly consistent, which serves to validate the conclusions drawn from the simulation.

We use our NS model to measure the average wait time at the toll plaza. From this we demonstrate a general method for choosing the number of tollbooths to minimize the wait time. For a 2-lane highway, the optimal number of booths is 4; for a 3-lane highway, it is 6. For larger numbers of lanes, the result depends on the arrival rate of the traffic.

The consistency of our model with a variety of theory and experiment suggests that it is accurate and robust. There is a high degree of agreement between the queueing theory results and the corresponding NS results. Special cases of our NS results are confirmed by empirical data from the literature. In addition, changing the distribution of the tollbooth wait time and changing the probability of random braking does not significantly alter the recommendations. This presents a compelling validation of our models and general approach.

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Introduction

A toll plaza creates slowdowns in two ways:

- If there are not enough tollbooths, queues form.
- If there are too many tollbooths, a traffic jam ensues when cars merge back onto the narrower highway.

We use queueing theory to predict how long vehicles will have to wait before they can be served by a tollbooth. Using cellular automata to model individual cars, we confirm this prediction of wait time. This vehicle-level model is used to predict how traffic merges after leaving the toll plaza.

Initial Assumptions

- **The optimal system minimizes average wait time.** We do not consider the cost of operating tollbooths.
- **Cars arrive at the toll plaza uniformly in time** (the interarrival distribution is exponential with rate λ). We can consider rush hour by varying the arrival rate λ.
- **Cars have a wait time at the tollbooth that is memoryless** (exponential distribution with rate μ). This assumption is confirmed by the study of tollbooths by Hare [1963].
- **Cars are indistinguishable**. All cars have the same length and the same maximum speed.
- **The toll plazas are not near on-ramps or exits.** We do not consider the possibility of additional cars merging, only those that were already on the main road.
- **Two-way highways are equivalent to two independent highways.** We consider only divided highways.

Delays Due to Too Few Tollbooths

Tollbooths As an M|M|n Queue

As a vehicle approaches the toll plaza, it has a choice of n tollbooths for service. Cars tend toward the shortest queue available. We simplify this behavior by supposing that all vehicles form a single queue, and that the next car in line enters a tollbooth as soon as one of the n booths becomes available. A real system would be less efficient, and therefore we expect longer times in a more detailed simulation.

We assume that vehicles arrive uniformly distributed in time. We additionally suppose that the length of service time is exponentially distributed as in Hare [1963]. This class of model is called a *memoryless arrivals, memoryless service times, n-server or "M|M|n" queue* (**Figure 1**).

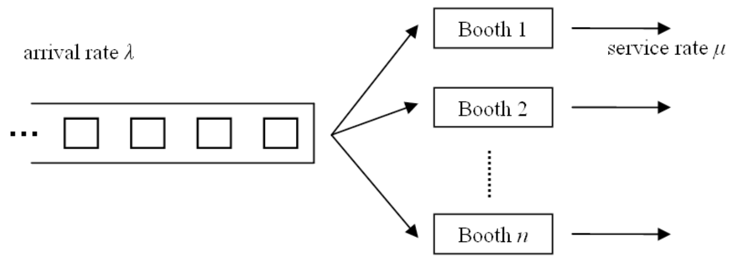


Figure 1. The M|M|n queue. Vehicles arrive at rate λ and are serviced at rate μ.

We define $X(t)$ as the number of vehicles either in the queue or at a tollbooth at time t. We also define the stationary probabilities p_k such that, in steady state, the probability that the queue has length k is p_k. From the input-output relationship of the M|M|n queue, the stationary probabilities must satisfy

$$\begin{aligned}
0 &= -\lambda p_0 + \mu p_1; \\
0 &= \lambda p_{k-1} - (\lambda + k\mu)p_k + (k+1)\mu p_{k+1}, && k = 1, \ldots, n; \\
0 &= \lambda p_{k-1} - (\lambda + n\mu)p_k + n\mu p_{k+1}, && k = n+1, n+2, \ldots .
\end{aligned}$$

The solution to this system is [Medhi 2003]:

$$p_0 = \left[\sum_{j=0}^{n-1} \frac{\rho^j}{j!} + \frac{\rho^n}{n!(1-\frac{\rho}{n})} \right]^{-1}, \qquad p_k = \begin{cases} \dfrac{\rho^k}{k!} p_0, & k = 0, \ldots, n; \\ \rho^{k-n} p_n, & k = n+1, n+2, \ldots, \end{cases}$$

where $\rho = \lambda/\mu$.

Let the random variable W be the time that a vehicle spends in the system (time in the queue + time in the tollbooth). From Medhi [2003], the distribution and expected value of W are

$$\begin{aligned}
P(W = w) &= \sum_{k=0}^{n-1} \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} p_0 \, \mu e^{-\mu w} + \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} p_0 \, \frac{n\mu^2}{(n-1)\mu - \lambda} \left(e^{-\mu w} - e^{-(1-\rho)n\mu w} \right), \\
E[W] &= \frac{1}{\mu} + \frac{p_n}{n\mu(1-\rho)^2}.
\end{aligned}$$

This result describes the first part of the general problem: how the cars line up depending on the number n of tollbooths.

To model the traffic merging after the tollbooths, it is important to describe how vehicles leave the M|M|n queue. For an M|M|n queue, the interdeparture times of the output of the queue are exponentially distributed with rate λ, and the output process has the same distribution as the input process. Because of the memoryless nature, interdeparture intervals are mutually independent (see Medhi [2003] or Bocharov et al. [2004] for proofs of these statements).

We define D as the number of cars departing the tollbooth during an interval Δt. Then the probability that d cars leave in that time is:

$$P(D = d) = \frac{e^{-\lambda \Delta t}(\lambda \Delta t)^d}{d!},$$

where λ is the mean number of cars that arrive at the toll plaza in a time step.

The M|M|n queue provides a simple and well-developed model of the tollbooth plaza. In particular, the average wait time and the output process are known, allowing us to verify simulation results.

Limitations of the M|M|n Queue

Though useful, the M|M|n queue is incomplete and oversimplifies the problem. Even though the M|M|n queue allows us to find the distribution of departures simply, its assumptions prevent it from being a complete solution. By using a single-queue theory, we assume that any car can go to any open server. This is overly optimistic, especially when the density is high. We would expect our predictions to be more valid for low density. Perhaps most importantly, the M|M|n queue only simulates half of the problem—the waiting times due to back-ups *in front of* the tollbooths.

Modeling Traffic with Cellular Automata

Overview

The complex system of traffic can be modeled by the simple rules of automata. We use cellular automata to model the traffic flow on a "microscopic" scale. In this scheme, we discretize space and time and introduce cars that each behave according to a small set of rules.

Cellular automata are well-suited for simulating our specific problem, since there are a large number of individual vehicles in the toll plaza, all of which are interacting. Continuous or macroscopic models could not capture this interaction and its role in causing jams that spontaneously form both before and after the toll plaza.

We first create a one-lane highway model and then add a delay for the time to pay the toll. As a one-lane simulation can allow no passing, cars accumulate behind the stopped car, creating a queue. We then extend this model into a multiple lane system, and then to a multiple lane system where the number of lanes is not constant, that is, where the road enters or leaves a toll plaza.

Single-lane Nagel-Schreckenberg Traffic

Most automata used to simulate traffic are generalizations of the Nagel-Schreckenberg cellular automata model (NS) [Chowdhury et al. 2000]. The NS model is a standard tool used to simulate traffic flow and has been shown to correspond to empirical results [Brilon et al. 1991; Chowdhury et al. 2000; Gray and Griffeath 2001; Knopse et al. 2004; Rickert et al. 1996; Schreckenberg et al. 1995].

We use this automaton to create a numerical model to confirm the queueing theory predictions.

In the NS model, a car is represented by an integer position x_n and an integer speed v_n. The vehicles are deterministically moved by their velocities, $x_n \rightarrow x_n + v_n$. The system evolves by applying the following procedure (**Figure 2**) *simultaneously* to all (x_n, v_n).

NS Algorithm

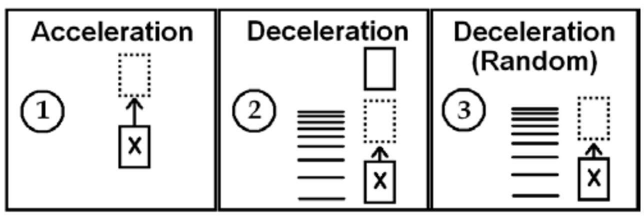


Figure 2. Rules of the NS algorithm.

1. **Acceleration**. If the vehicle can speed up without exceeding the speed limit $v_{\max}$, it adds one to its speed, $v_n \rightarrow v_n + 1$. Otherwise, the vehicle has constant speed, $v_n \rightarrow v_n$.

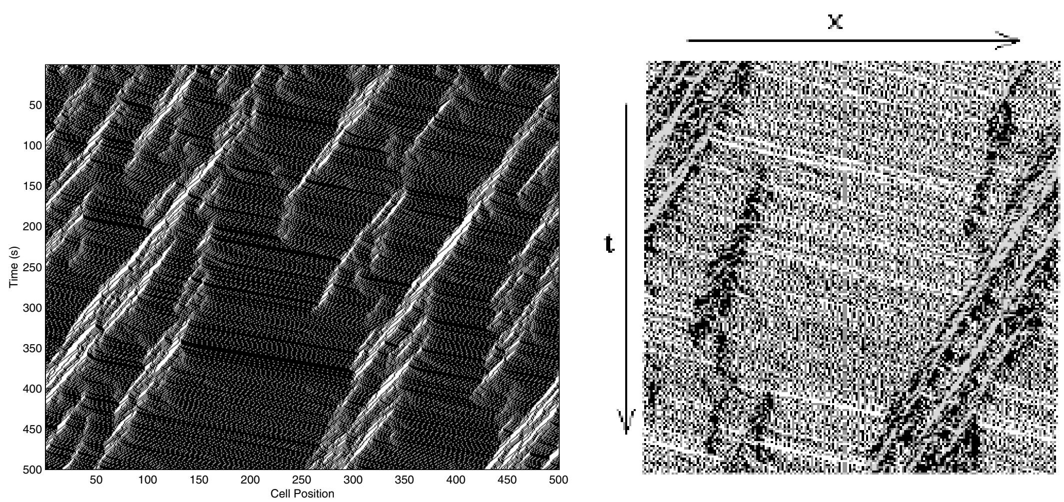
2. **Collision prevention**. If the distance between the vehicle and the car ahead of it, d_n, is less than or equal to v_n, that is, the nth vehicle will collide if it doesn't slow down, then $v_n \rightarrow d_n - 1$.

3. **Random slowing**. Vehicles often slow for nontraffic reasons (cell phones, coffee mugs, even laptops) and drivers occasionally make irrational choices. With some probability p_{brake}, we have $v_n \rightarrow v_n - 1$, presuming $v_n > 0$.

We choose the cell size to be 7.5 m to match Nagel and many others [Brilon et al. 1991; Chowdhury et al. 2000]. Since a typical maximum speed for cars is 30–35 m/s, choosing $v_{\max} = 5$ makes a single time step close to 1 s. We also use periodic boundary conditions for simplicity. (We later abandon these boundary conditions, since open boundary conditions—a Poisson generator and a sink—are consistent with the M|M|n model.) In addition, research results indicate that the periodic boundary may oversimplify the distribution of vehicles [Yang et al. 2004].

We apply the above algorithm to a random initial state with a given density. This system was created by assigning each cell a probability of occupation c, which is the vehicle density parameter. This matches Gray and Griffeath's approach [2001].

The NS model produces results similar to those cited as typical by Chowdhury [2000]. In **Figure 3**, the state of the system at time step i is drawn in the ith column, with a white pixel where there is a vehicle and a black pixel for open space. Both images show generally smooth flow interrupted by congestion.



a. Results from the NS model, $c = 0.2$, $p = 0.25$. **b.** Results from Chowdhury et al. [2000].

Figure 3. Typical results from two models.

Properties of and Support for the NS Model

The one-lane NS model is self-consistent, flexible, and matches known empirical data.

Some of the properties of the NS model can be predicted analytically [Nagel and Herrmann 1993]. We use this information as well as experimental results to test our model. In the limiting case where the random braking probability is zero, it is possible for vehicles to "cruise," moving at their maximum speed at all times, corresponding to a flux of $J = cv_{\max}$. This is possible only if there is sufficient space. Once the "hole density" or the remaining spaces, given by $(1 - c)$, is smaller than this flux, the lack of free space limits the speed of the vehicles. This relationship between flux and density is given by:

$$J(c) = \min\{cv_{\max}, 1 - c\}, \qquad \textbf{(1)}$$

where J is the flux of cars, the number of cars passing a cell in unit time, and c is the density of cars. We ran our NS automaton with $p_{\text{brake}} = 0$ for 20 trials with excellent agreement between our mean and the theory, as seen in **Figure 4**.

As pretty as this graph is, it indicates only that the model is self-consistent and can be approximated; it does not show that it actually represents a real

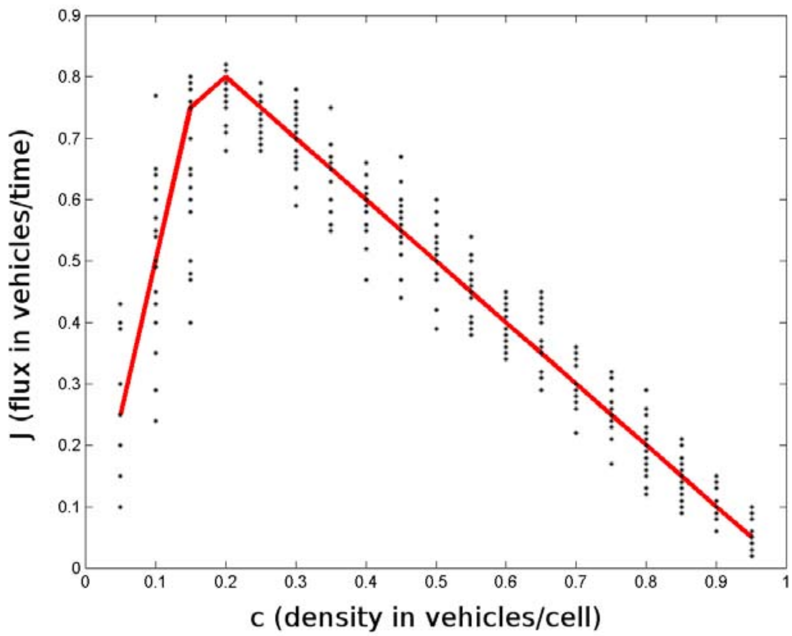


Figure 4. The flux equation **(1)** predicts the results of the NS model with very good accuracy.

system. We consult empirical data on vehicle flux (**Figure 5b**). Clearly, the NS model is an accurate approximation of the known data.

It is also possible to use mean field theory to describe the NS model. Even if $p_{\text{brake}} \neq 0$, the case of $v_{\max} = 1$ can be solved analytically with this technique [Schadschneider and Schreckenberg 1997]. For our system, with $v_{\max} = 5$, this becomes computationally difficult.

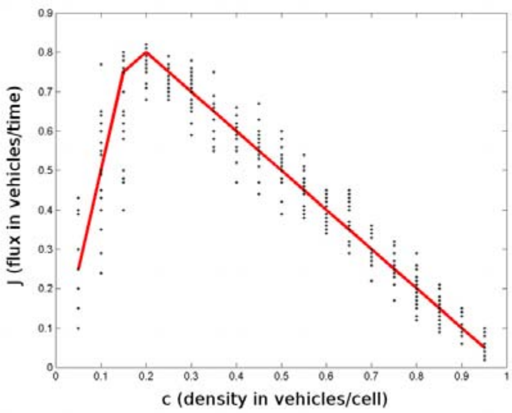
Adding Delays

Delays prevent the use of periodic boundary conditions.

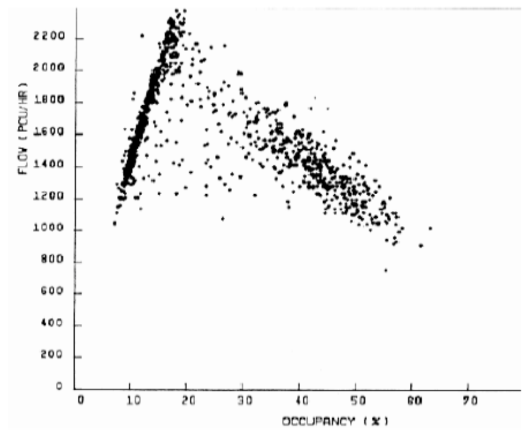
To simulate an encounter with a tollbooth, we must add a delay to the unobstructed system. Simon and Nagel model the NS automaton for a blockage but only with a fixed delay probability [1998]. We assume that the service time is exponentially distributed, with a probability of $1 - \exp(-\mu\Delta t)$ that any one tollbooth completes service in Δt, and so we use this assumption to describe the delay in our NS model as well.

Introducing this delay creates an asymmetry in the problem; particles to the right of the barrier have to loop around to reach the "tollbooth," whereas particles on the left will impact it immediately. Because of this, we measure the flux at both the one-quarter and three-quarter points of the lane (**Figure 6**).

The fundamental diagrams in **Figure 6** confirm our intuition regarding the interaction of flux and a bottleneck (the tollbooth). The flux at the quarter point experiences a decline due to congestion at a relatively low vehicle density compared with the three-quarter point, which is beyond the bottleneck. The



a. NS model with theory.

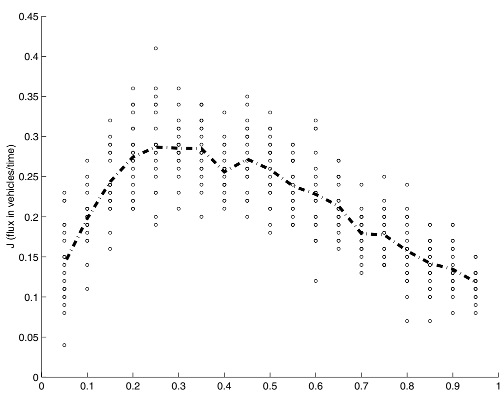


b. Empirical data from Canadian highway [Chowdhury et al. 2000].

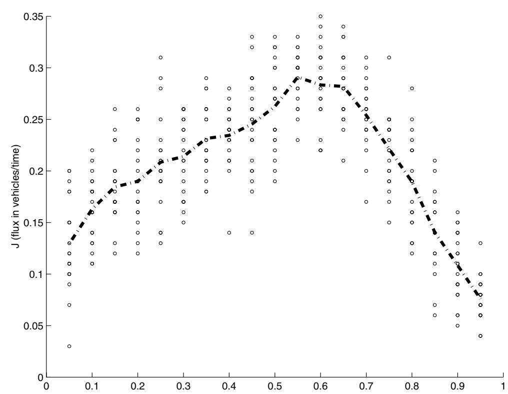
Figure 5. Comparison of model with data.

heavy incoming traffic therefore affects the accumulating queue faster than the vehicles past the tollbooth.

These fundamental diagrams show that the periodic boundary conditions are inappropriate for this calculation; with periodic boundaries, the input rate to the queue is limited by the (smaller) flux of vehicles wrapping around from the right. This is not representative of a true traffic jam; without periodic boundary conditions, jams cannot affect the flux upstream from them.



a. Fundamental diagram at one-quarter point, $p = 0$.



b. Fundamental diagram at three--quarters point, $p = 0$.

Figure 6. Fundamental diagrams.

Simulating the Complete System

Multiple Lanes

By adding a new rule to the one-lane automaton, we can model multilane highways. We use a single-lane model to ensure that our automaton is a proper representation of the real world, but the actual problem is a multiple-lane one. Two-lane system studies are less common than single-lane studies, and higher-lane models even rarer [Chowdhury et al. 2000; Nagel et al. 1998; Rickert et al. 1996]. We extend the rule set of the automaton to describe lane changes, using the one-lane NS rules with a single additional rule for lane changing (**Figure 7**).

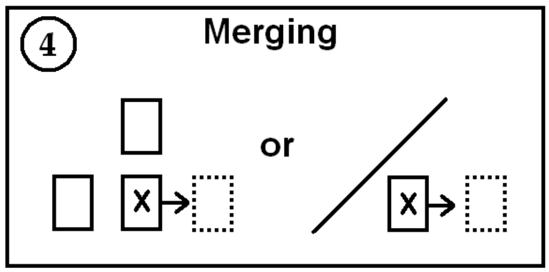


Figure 7. The multi-lane automaton rule.

4. **Merge to avoid obstacles**. The vehicle attempts to merge if its forward path is obstructed ($d_n = 0$). The vehicle randomly chooses an intended direction, right or left. If that intended direction is blocked, the car moves in the other direction unless both directions are blocked (the car is surrounded). This is consistent with the boundaries and tailgating rules proposed by Rickert et al. [1996].

Changing the Highway Shape

By using the multilane automaton, we can model a multilane highway that has realistic lane-changing behavior. This still does not, however, model a transition between highway and a number of tollbooths.

To create the toll plaza, we introduce borders that force the automata to change lanes to avoid hitting the boundary. The borders outline the edge of a ramp that moves from the highway onto a wider toll plaza and back again (**Figure 8**). This is the only aspect of this model that is not general; by imposing different boundaries, we could easily model a different problem. To simulate the wait at the tollbooth, we also add a delay at the center, as in the one-lane case.

The previous models [Gray and Griffeath 2001; Nagel et al. 1998; Rickert et al. 1996] assume a roadway of constant width—the number of lanes does not change. By restricting the geometry of our roadway to represent the toll plaza (note the “diamond” shape in **Figure 8** and introducing the behavior of

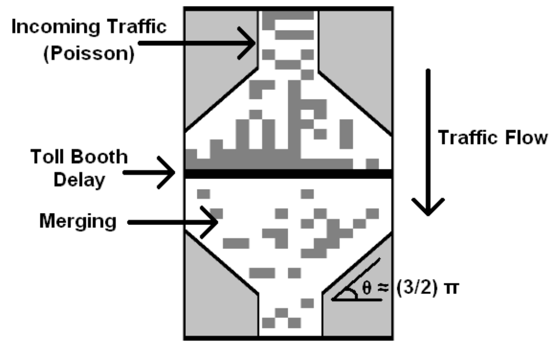


Figure 8. We introduce imaginary borders into the system to narrow and widen traffic.

merging away from obstacles, we have increased the flexibility of the model without making additional assumptions.

Consistency of M|M|n Queue and NS Model

The M|M|n queue is an idealized system. It predicts a shorter wait time than a real toll plaza, because it fails to account for inefficiencies in the queue (**Figure 9**). The M|M|n queue does, however, predict the correct distribution. In addition, the stability of the queue is very different from the stability of the NS model. An M|M|n queue achieves a steady state if $\lambda/\mu < n$, with λ the arrival rate, μ the service rate, and n the number of servers [Gross and Harris 1974]. We observe in the NS simulation that traffic in front of the tollbooths could create a growing backlog even when the corresponding M|M|n queue would be stable.

Despite these apparent inconsistencies, there is a very strong agreement between the queueing theory predictions and the observed results. From queueing theory analysis, we know the probability distribution of the number of cars leaving the tollbooths:

$$P(D = d) = \frac{\mathrm{e}^{-\lambda \Delta t}(\lambda \Delta t)^d}{d!}.$$

This equation provides a good deal of information about the queue, and this probability can easily be measured in the cellular automata model. We compare the simulated and theoretical probability distribution in **Figure 10**, and the two distributions are very similar: The difference in their means is decreasingly small and is less than 2% after 10^4 iterations of the NS model.

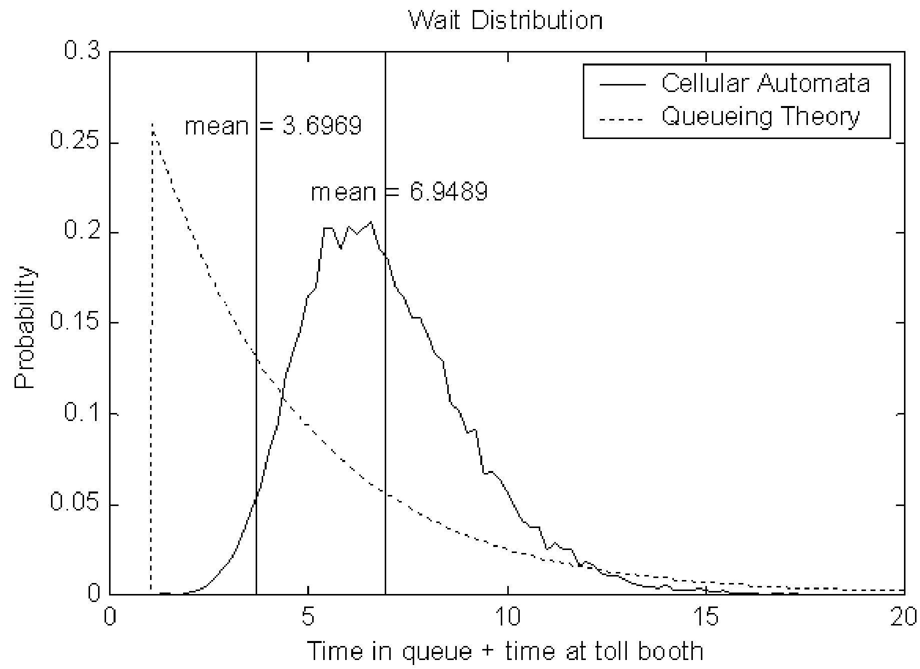


Figure 9. The M|M|n distribution and NS distribution are similar, but the M|M|n has a smaller mean wait time due to its optimistic assumptions.

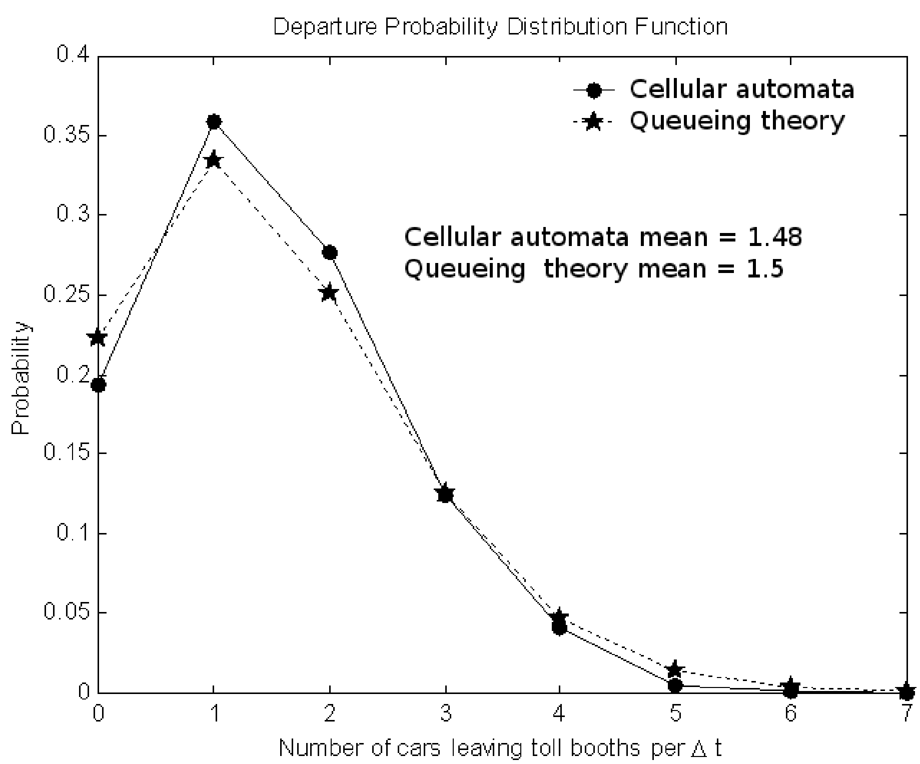


Figure 10. The simulated distribution of leaving vehicles is very close to that predicted by queueing theory.

Time Predictions of the Automata Model

The Optimal Number of Tollbooths

The optimal configuration minimizes the wait time; so to determine the correct number of tollbooths, we need to measure the average time for a vehicle to enter the toll plaza, wait in line, and then exit out onto the main road again. We do this by tracking automata and averaging the time that passes between entering and leaving the system.

Calculating Average Times

The average time required to pass through our system depends on the arrival rate (which controls congestion), the number of lanes, and the number of tollbooths. We consider the mean service rate to be fixed, at 5 s; Hare uses 9 s [1963]. However, though changing the service rate does change the average time, this change does not affect which value of n is optimal.

We fix the number l of incoming lanes and search over the number n of tollbooths and the arrival rate λ. We calculate the average wait time for a range of n and λ by using our cellular automata model and averaging over a long period of time to eliminate transient effects.

What, though, should these ranges be? We presume that n is not larger than three or four times the number of lanes. We placed this restriction after noticing that the wait time increases sharply when n is much larger than l. The range of λ is determined by commonsense restrictions. If λ is the mean number of cars arriving in a time step, then λ should be no more than the number of lanes of incoming traffic as this is the physical capacity of the road.

Optimal Results for 2 Lanes

We allow n to range from 2 to 8 and λ from 0 to 2. We plot the average time against n and λ in **Figure 11**.

The clear minimum in this graph lies along the line $n = 4$, even for different values of λ. This indicates that even for different arrival rates, the optimal number of tollbooths is 4. This is a very stable solution that does not require changes with traffic rates, at least for typical values.

Though 4 is the optimal number of booths, 2 is very near optimal and 3 is very bad. For $n = 3$, the additional lane adds more traffic jam than throughput. For small n, when there is one more booth than lanes of traffic, the wait time is a local maximum.

Optimal Results for 3 Lanes

By varying λ from 0.6 to 2.7 and n from 3 to 9, we find that the minimum occurs, once again independent of λ, at $n = 6$ (**Figure 12**).

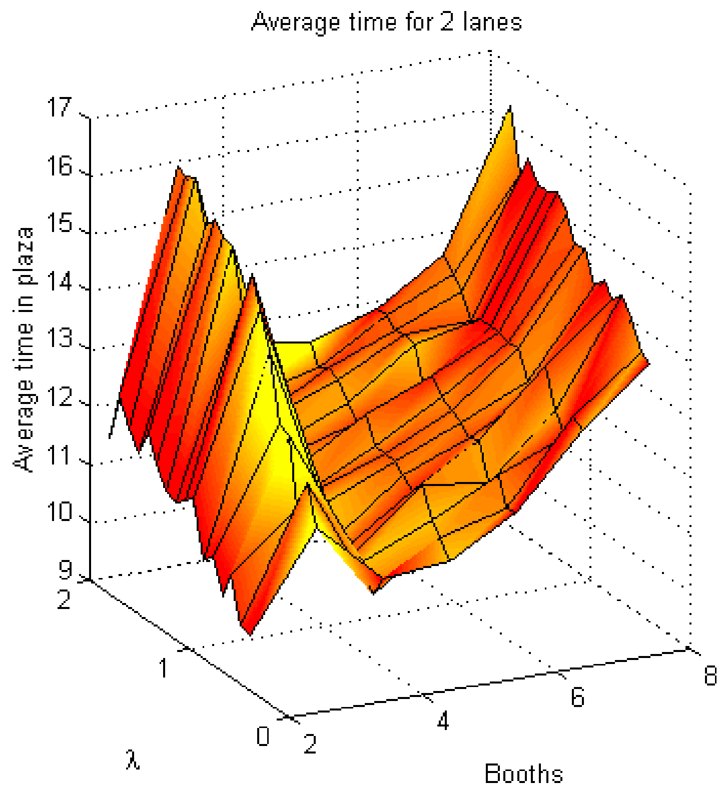


Figure 11. For a 2-lane system, the minimum time occurs for 4 tollbooths.

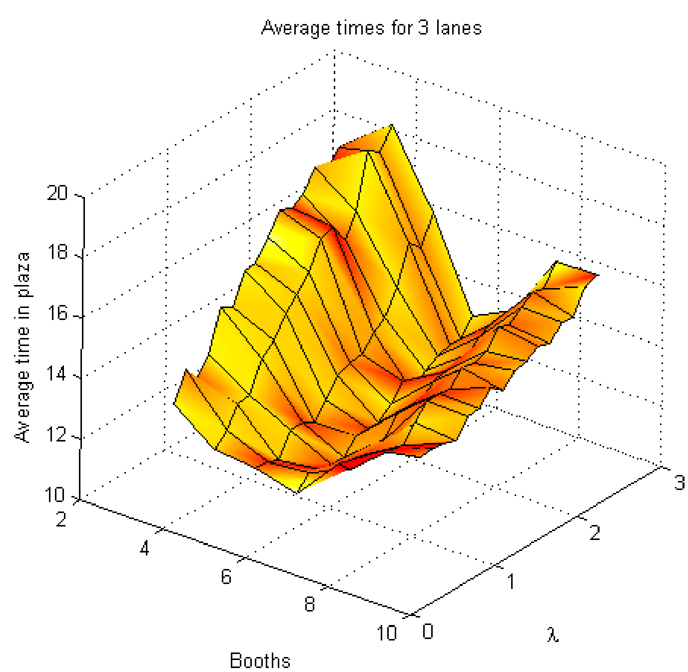


Figure 12. For a three-lane system, the minimum time occurs when there are six tollbooths.

Higher Numbers of Lanes

The results so far would suggest the naive solution of always having twice as many tollbooths as incoming lanes. Unfortunately, the 4-lane case disproves this guess. For different values of the arrival rate λ, the optimal number of tollbooths changes, from a low of 6 for small λ. The tollbooth owners could measure traffic flow, estimate λ, and open or close tollbooths as needed.

Generalizing These Results

To determine the stability of these results, we made calculations with the probability of random braking changed to 0.1 rather than 0. For each λ, even though the wait times change, the optimal numbers of tollbooths do not.

This process could be repeated for any required setup; we have illustrated a general technique for determining the optimal number of tollbooths.

One Tollbooth per Lane

If cars were not allowed to change lanes, the case with one tollbooth per lane of traffic would just reduce to l independent one-lane models, and this would be equivalent to our single-lane highway. We know that this is not the case. Cars move into the lane with the shortest queue.

In our results, the $n = l$ case is typically nonoptimal. The one exception to is the two-lane highway; here, the time for $n = 2$ is only barely longer than for $n = 4$. However, this is not because $n = l$ is always "bad," but because there is usually a better case. As the number of lanes increases, the number of tollbooths increase significantly. If we consider cost, the $n = l$ case could be very important, since for low l ($l < 5$), the $n = l + 1$ case is a local maximum and $n = l$ is a local minimum.

Re-examining Assumptions

Though our calculations assume an exponential service probability, using a Gaussian service distribution does not change the model's recommendations significantly.

Conclusions

We build a model of traffic flow near a toll plaza by using cellular automata modified from the one-lane Nagel-Schreckenberg automaton.

- Our model's predictions match empirical data on vehicle distribution and are confirmed by our queueing theory analysis of the problem.

- Changing the service rate and service distribution of the tollbooths does not significantly alter the recommendations for the optimal number of booths
- We establish a general technique for determining the optimal number of tollbooths to put on a given highway.
- Though in general the optimal tollbooth results are complex and depend on the arrival rate, there are simple cases: a 2-lane highway should have 4 tollbooths and a 3-lane highway should have 6, independent of the amount of traffic.
- In general, the case of as many tollbooths as lanes is suboptimal.

Strengths and Weaknesses

Strengths

- **Consistency.** Our queueing theory model and the cellular automata model agree on the distribution of cars leaving the booths. Both models match theoretical results and past empirical results. In addition, under small changes, like adjusting the probability of braking, the recommendations of the model do not change significantly.
- **Minimal assumptions required**. By using the automata, we reduce the number of parameters and assumptions. For our queueing theory, we assume that the probability of leaving the tollbooth is exponential, but altering this distribution does not affect the recommendations.
- **Flexibility.** Our model easily adapts to problems with different geometries, such as different numbers of lanes or even different boundaries.
- **Ease of implementation.** A complex problem is simulated using very simple rules.

Weaknesses

- **No closed-form solution.** For the complete model, we must actually calculate the simulation.
- **Calculation time.** To get an accurate average time for vehicles, we need to average over a number of time steps on the order of 10,000. As the number of lanes increases, computation slows.

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