

Summary

Aim to determine the location of “sweet spot” and the different “sweet spot” effects of the uncorked, corked and metal bats, we employ methods in dynamics to build models and generate data of batted ball speed by simulation, which exactly match the actual data obtained from experiments.

Based on classical mechanics, we firstly develop a model to describe the collision between ball and bat, from which we obtain the distribution of batted-ball speed (BBS) as the function of the impact location. Then we successfully deduce the location of “sweet spot”, where the BBS reaches the maximum. With all possible case studies, we conclude that “sweet spot” is about 140mm from the end.

Considering the more complex structure of the corked bat, we augment our basic model by building double-spring model and adopting three empirical formulas. Then we are able to adopt the foregoing analytical method to examine the trampoline effect. We carefully analyze the “sweet spot” effect corresponding to the geometrical parameters of “corked hole” and make simulations. The results are illustrated in mesh figures and demonstrate that “sweet spot” effect of the corked bat significantly depends on the density of stuffing materials-“corking” with rubber enhances the “sweet spot” effect while with cork is the opposite.

Based on our models, we design a special metal bat exhibiting faster BBS and similar controllability compared with ordinary wood bats. From the special case, we reach a general conclusion that metal bats tends to outperform wood bats through technical parameters optimization.

What’s more, on the basis of our models, we provide technique tips and formulas for the corked or metal bat design, which will significantly enhance the “sweet spot” effect or make it easier to control. In conclusion, our model successfully achieves our goals that building a useful model both for illumination and application.

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An Identification of “Sweet Spot”

1 Introduction

The baseball game not only gains great popularity among young people, but also exerts a certain fascination on physicists. Adair, Brody, Cross, Nathan and Russell all published a number of notable papers addressing both experimental and theoretical issues involved in the ball-bat collision. Their methods range from classical mechanics model to Finite Element simulation. The “sweet spot”, the corked bat, trampoline effect, vibration and metal bats all have been studied to different extent with different approaches. However, the researchers’ work focus on explaining the physical phenomena occurring before, during and after ball-bat collision. Therefore, their research results may not be appropriate to solve the proposed problems. We need to develop a model, neglecting the transient process of collision, possess concise and clear physical meaning. What’s more, the model should be problem-oriented.

The three proposed problems are:

- Problem I: Explain why the “sweet spot” is not at the end of the bat.
- Problem II: Confirm the fact that “corking” enhances the “sweet spot” effect and explain why MLB prohibits “corking”.
- Problem III: Predict different behavior for wood and metal bats and explain why MLB prohibits metal bats.

The bat-ball collision is complicated by the nonlinear compress behavior of the ball and the vibration behavior of the bat. With the help of the current literatures, we first expect to reasonably neglect the influences of the some factors for simplicity and build a classical mechanical model to find the location of “sweet spot” at which the maximum power is transferred to the ball in collision. We want to obtain the “sweet spot” location by solving the dynamic equation, while some more specific problems and case has to be examined. All parameters in the expression have clear physical meanings and could be easily measured. What’s more, we even expect that the expression only includes basic arithmetic operations in order to guide the exercise of baseball sport.

As for corked bats and metal bats, their differences in structure and material may result in better and worse performance compared with ordinary wood bats. So we plan to augment the basic model is to deeply examine the physical meanings of parameters and to modify these parameters in order to fully describe the corked bats and metal bats. On the basis of our model’s results, we provide the explanation for the MLB’s banning of corked bats and metal bats.

2 Assumption

1. No consideration for the failure of the bat. Our model is developed under the precondition that both the uncorked and corked bat are working well.
2. The loss of energy caused by bending vibration is neglected (except for *Bending Vibration Part*).
3. The uncorked bat is uniform in density and is symmetric to the parallel axis.
4. The stuffing object of the corked bat has the same perpendicular axis as the corked bat.

3 Symbols

COR	The coefficient of restitution
$BBCOR$	The coefficient of restitution of bat-ball
CM	The center of mass
MOI	The moment of inertia with respects to CM
BBS, v_f	The batted ball speed
J, J_{CM}	MOI
e^*	The equivalent $BBCOR$
$b_{sweet\ spot}$	The location of “sweet spot”
V_i, ω_i	The swinging speed and angular speed
z_{cm}, a	The location of CM from the knob

4 Details of the Model

4.1 Model Overview

To find out the “sweet spot”, we propose a classical mechanics model without consideration for bending vibration. We derive the expression of batted-ball speed (*BBS*) as the function of 1)impact location, 2)ball mass, 3)ball initial speed, 4) the moment of inertia of bat, 5)the swinging bat speed and 6) the coefficient of restitution(COR). We choose a standard wood bat and employ analytical method to obtain the “sweet spot” location, in which initial speed of ball and swinging bat speed vary over meaningful and practical ranges. The “sweet spot” lies at a point about 15cm away from the end of bat.

Then we build a double-spring (DS) model to modify *BBCOR* by elaborating the collision on corked bat. We combined the two models to calculate the *BBS* considering the effects of corking on three parameters. We prove that a typical cork “corking” will weaken the “sweet spot” effect, whereas a typical rubber “corking” will enhance the “sweet spot” effect.

To predict and compare the different behavior of wood and metal bats, we first design a special aluminum bat with the same mass, outline shape, mass center and moment of inertia as that of the standard wood bat; We demonstrate that the special metal bats will have faster *BBS* and same controllability. Based on the special case, we reach a general conclusion that metal bats tend to have faster *BBS* and same controllability by optimizing technical parameters. At last, we list some possible negative effects posed by metal bats and explain MLB’s prohibitions of metal bats.

4.2 Fundamental Model

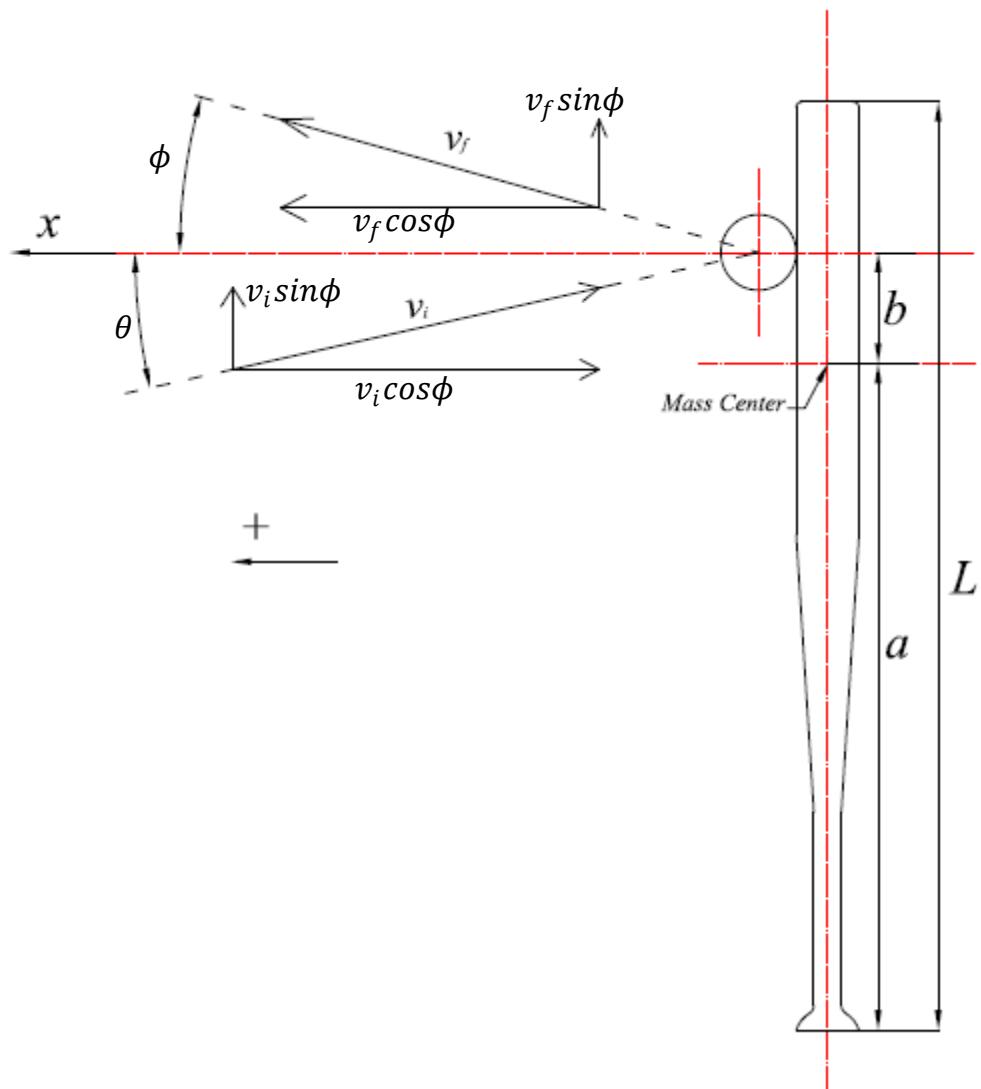


Figure 1. Illustration of theoretical derivation

Based on the Momentum Theorem, we obtain

$$I = M(V_f - V_i) + m(v_f \cos\phi - v_i \cos\theta) \quad (1)$$

In which

I is a linear impulse in x direction which the batter communicates to the ball-bat system during the period of contact, $kg \cdot m/s$;

M is the mass of the bat, kg ;

V_f is the velocity of the mass center of the bat at contact after collision (positive or negative), m/s ;

V_i is the velocity of the mass center of the bat at contact before collision (positive or negative), m/s ;

m is the mass of the ball, kg ;

v_f is the velocity of the mass center of the ball at contact after collision (positive or negative), m/s;

v_i is the velocity of the mass center of the ball at contact before collision (positive or negative), m/s;

ϕ, θ are the angles of the directions of the ball as illustrated in **Figure 1**.

Based on the Angular Momentum Theorem, we obtain

$$L = mb(v_f \cos\phi - v_i \cos\theta) + J(\omega_f - \omega_i) \quad (2)$$

In which

L is a angular impulse about the direction of ω which the batter communicates to the ball-bat system during the period of contact, m;

b is the distance from the hit spot to the center of mass of the bat, m;

J is the moment of inertia of the bat about the center of the mass, kg · m²;

ω_f is the angular velocity of the bat at contract after collision with respect to a vertical axis of rotation passing through its mass center, rad/s;

ω_i is the angular velocity of the bat at contract before collision with respect to a vertical axis of rotation passing through its mass center, rad/s.

By definition of the coefficient of restitution, we obtain

$$e[(V_i + \omega_i b) - v_i \cos\theta] = v_f \cos\phi - (V_f + \omega_f b) \quad (3)$$

In which e is the coefficient of restitution with respects to ball and bat (BBCOR, in this model only dependent on materials).

These three equations are sufficient for the determination of the three primed unknowns. In particular, we find the x component of the velocity for the departing ball

$$v_f \cos\phi = \frac{(1+e)(V_i + \omega_i b) + v_i \cos\theta \left(\frac{m}{M} + \frac{mb^2}{J} - e \right) + \frac{I}{M} + \frac{bL}{J}}{1 + \frac{m}{M} + \frac{mb^2}{J}} \quad (4)$$

4.2.1 Velocity for the Departing Ball

Case I: eliminating the impulse I, L

With a hard ball (high COR) the duration of contact is short and the impulse I and L is therefore small. When the collision has been initiated there is little more for the batter to do [P. Kirkpatrick 1963]. This is also supported by Howard Brody's experiment in which he observed the vibrations of a hand-held baseball bat [1989]. According to Howard Brody's experiment [1989], the hand-held bat

behaves as it were a free body, in other words, the impulse I and L are nearly 0, which is correct in both hardball and softball (more details in Appendix II).

Case II: eliminating the linear velocity $\omega_i b$

When the collision takes place near the mass center of the bat, as it very frequently does, all terms containing b may be deleted [P. Kirkpatrick 1963].

Therefore, in many cases, it is quite reasonable to regard the impulse I, L and $\omega_i b$ as zero. As simplified by these approximations, Eq. 4 can be reduced to

$$v_f = \frac{(1+e)(V_i + \omega_i b) + v_i \cos\theta \left(\frac{m}{M} + \frac{mb^2}{J} - e \right)}{\left(1 + \frac{m}{M} + \frac{mb^2}{J} \right) \cos\phi} \quad (5)$$

In order to make the expression simple and easy to understand, we import new symbols:

$$\text{recoil factor} \quad r = m/M + mb^2/J \quad (6)$$

$$\text{collision efficiency} \quad q = (e - r)/(1 + r) \quad (7)$$

$$\text{pitch speed (positive)} \quad v_{ball} = -v_i$$

$$\text{bat speed (positive)} \quad v_{bat} = V_i + \omega_i b$$

$$\text{batted ball speed} \quad BBS = v_f$$

the Eq.5 becomes the following two expressions, both of which will be in later analysis.

$$v_f = \frac{(1+e)(V_i + \omega_i b) + v_i \cos\theta(r - e)}{(1+r)\cos\phi} \quad (8)$$

$$BBS = \frac{qv_{ball} \cos\theta + (1+q)v_{bat}}{\cos\phi} \quad (9)$$

4.2.2 Find the Location of “Sweet Spot”

For the reason that the directions of velocity of ball merely affect the “sweet spot”, we only look for the expression for the location of “Sweet spot” in one simple situation in which $\theta = 0, \phi = 0$. If $\theta \neq 0$ or $\phi \neq 0$, the expression for “Sweet spot” location can be deduced in the same way and the result has no significant difference.

Substituting $m/M + mb^2/M$ for r in Eq. 8 gives

$$v_f = \frac{(1+e)(V_i + \omega_i b) + v_i \left(\frac{m}{M} + \frac{mb^2}{J} - e \right)}{1 + \frac{m}{M} + \frac{mb^2}{J}} \quad (10)$$

Note that when $\omega_i = 0$ (the bat has no initial rotational energy), this expression is reduced to

$$v_f = \frac{(1+e)V_i + v_i \left(\frac{m}{M} + \frac{mb^2}{J} - e \right)}{1 + \frac{m}{M} + \frac{mb^2}{J}} \quad (11)$$

And it is clear that in this case v_f will have a maximum at $b = 0$, which indicates that the “sweet spot” is at the center of mass (CM) when the bat is stationary before the collision.

To obtain the value of b at which v_f can reach the maximum in the general case (the bat is not initial stationary), we differentiated the expression of v_f with respect to b , set the result be zero and get

$$\omega_i b^2 - 2(v_i - V_i)b - \frac{\omega_i(M+m)J}{mM} = 0. \quad (12)$$

This equation can be solved for b :

$$b = \frac{v_i - V_i}{\omega_i} \pm \sqrt{\left(\frac{v_i - V_i}{\omega_i} \right)^2 + \frac{J(m+M)}{mM}}. \quad (13)$$

The “+” sign will be used here because the value of $\frac{v_i - V_i}{\omega_i}$ is negative, and thus we get

$$b = \frac{v_i - V_i}{\omega_i} + \sqrt{\left(\frac{v_i - V_i}{\omega_i} \right)^2 + \frac{J(m+M)}{mM}}. \quad (14)$$

It is clearly shown in this expression that this point is not the COP location, since the value of b is dependent upon the ball and bat velocity and the properties of the bat. Particularly, the values of m, M and J can be obtained experimentally and v_i as well as V_i, ω_i are determined respectively by the pitcher throwing the ball and the hitter swinging the bat.

If $\left[\frac{J(m+M)}{mM} \right] / \left(\frac{v_i - V_i}{\omega_i} \right)^2$ is less than one, then the square root can be expanded and the result is

$$b \cong \frac{(z+1)\omega_i k^2}{2(V_i - v_i)} \quad (15)$$

Where $z = M/m$, $k = \sqrt{J/M}$.

4.3 Advanced Model

4.3.1 Overview of Corking

A corked bat is one which has been hollowed a cylinder on the middle of its tip. The size of the drilling hole can be 1 inch in diameter, and 6 to 10 inch deep; the removed part can be replaced with cork, rubber or Styrofoam; a cap is lastly plugged [Nathan 2003]. Since only the “one piece of solid wood bat” is permitted in the MLB [official rules], ordinary wood corked bat is considered in the following parts.

With corked structure, not only is the bat lighter, but the center of mass, or balance point, of the bat moves closer to hands. In technical physics language, the moment of inertia (*MOI*) of the bat about the *CM* is reduced which leads to easy-control. What’s more, since the thickness of bat’s shell decreases after corking, the bat’s shell may be compressed during the collision with the ball and springs back, much like a trampoline, resulting in much less loss of energy than would be the case if the ball hit a completely rigid surface. In other words, the equivalent *BBCOR* (in model above) has decreased.

Summarily, the physics of corked bat and ball collision can be fully identified by the parameters of model above, however, in contrast with the original bat (uncorked), the corked bat exhibits different values of specific parameters, including *MOI*, M_{ball} , *COR*. By examining both the characteristics of the structure of corked bat and the new physical meanings of key parameters in the first model, we augment our first model on the basis of getting the equivalent *BBCOR* (relating to structure of corked bat) and varying values of parameters.

4.3.2 Double-Spring Model

During the collision between hollowed bat and ball, the “spring” on the bat is exited (the shell of bat is compressed) so that the trampoline effect could be observed in mini-scale. Because of the hoop structure, a radial standing wave and the mentioned bending wave add to a resultant vibration.

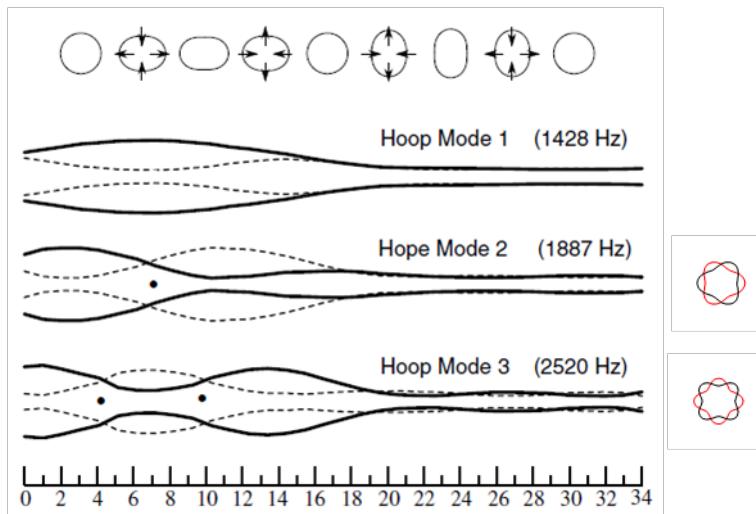


Figure 2. Longitudinal sections and cross sections for Hoop Modes of standard bat [Russell, 2004]

To both investigate the radial vibration and avoid developing a novel vibration model, we consider a most common elastic element in Physics, the springs, and use an uncorked bat with a spring attached to simulate the corked bat. What is worth mention is that the model incorporates the effects of elastic collision and energy loss completely in equivalent *BBCOR*; however, the advanced double spring model illuminates what new physical parameters make up equivalent *BBCOR*: they are k_{ball} , k_{bat} , e_{ball} , e_{bat} .

Figure 2 shows the scheme for model when colliding. Where

k_{ball} is the stiffness of the ball
 k_{bat} is the stiffness of the bat
 e_{ball} is the coefficient of restitution (COR) of the object with ball's material bouncing off a stationary completely elastic object (only dependent on material)

e_{bat} is COR of the object with bat's material bouncing off a stationary completely elastic object (only dependent on material)

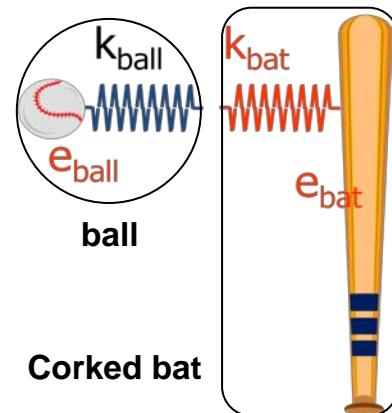


Figure 3. scheme for double spring model

4.3.3 Determining Equivalent *BBCOR* by DS Model

In a reference frame where center of mass of the system (CM_{BSB} , consisted of ball, spring and bat) remains at rest, the collision can be divided into the following four procedures:

- i) The ball and bat (with spring respectively) approach each other;
- ii) The two springs contact and compress until the velocity (in CM_{BSB}

- reference frame) of bat and ball turn to be zero;
- iii) The ball and bat are accelerated by the springs respectively. In procedure ii and iii, the loss of energy happens through the effect between bat (ball) and spring.
 - iv) The two springs is no longer at contact, and the ball and bat separate each other.

By definition of the coefficient of restitution (the ratio of the differences in velocities before and after the collision) and the physical meaning in energy form, we obtain the equivalent *BBCOR* (with trampoline effect and material effect inside) in a reference of home frame

$$e^* = \frac{v_f - V_f}{v_i - V_i} = \sqrt{\frac{E_f}{E_i}}. \quad (16)$$

Considering the conservation of momentum and the process of collision, the equivalent *BBCOR* in *Fundamental Model* is as following

$$e^{*2} = \frac{k_{bat}}{k_{ball} + k_{bat}} e_{ball}^2 + \frac{k_{ball}}{k_{ball} + k_{bat}} e_{bat}^2, \quad (17)$$

where

e^{*2} the fraction of energy restored in ball and bat (kinetic energy) after collision;

$\frac{k_{bat}}{k_{ball} + k_{bat}}$ the fraction of initial energy stored in ball;

e_{ball}^2 the fraction of stored energy returned to kinetic energy of ball;

$\frac{k_{ball}}{k_{ball} + k_{bat}}$ the fraction of initial energy stored in bat;

e_{bat}^2 the fraction of stored energy returned to kinetic energy of bat.

(Notes: Mathematical deduction see Appendix I)

4.3.4 Evaluating the “Sweet Spot” Effect

Parameters of bat analysis

■ Parameter affecting *BBS*

➤ Angular and linear velocity ω_i, V_i

According to Eq.4, when ω_i, V_i increase, *BBS* will increase. Considering the limit of athletes' biological energy, ω_i is affected by J and V_i by M .

➤ Recoil factor r

According to Eq.4, the increment of r leads to higher BBS.

➤ Moment of inertia and mass J, M

According to Eq. 4, when J, M increase, r will decrease which results in higher BBS. However, the increase of J, M will respectively leads to the decrease of ω_i and V_i , which will make BBS decrease.

➤ Location of “Sweet Spot” b

According to Eq. 4, we can obtain the location of “Sweet Spot” $b_{sweet\ spot}$, and

using $b_{sweet\ spot}$ to calculate the maximum of BBS.

➤ Equivalent BBCOR e^*

According to Eq. 4, the increment of e^* will leads to higher BBS. Since e_{bat} and e_{ball} are determined by materials, e^* will be only determined by k_{ball}/k_{bat} .

For clarity, we list the above analysis results in **Table 1**.

Table 1 The interrelationship between parameters and effects on BBS

EFFECTS	ω_i	V_i	r	e^*	BBS
$\omega_i \uparrow$					😊
$V_i \uparrow$					😊
$r \uparrow$					😢
$J \uparrow$	↓		↓		😢 + 😊 = ?
$M \uparrow$		↓	↓		😢 + 😊 = ?
$\frac{k_{ball}}{k_{bat}} \uparrow$				↑	😊

The independent parameters of corked bat only includes $M, J, e_{ball}, e_{bat}, k_{ball}, k_{bat}$, all of which are easily obtained:

M, J can be easily measured with basic experimental instruments;

e_{ball}, e_{bat} can be obtained in the tool book, like *Mechanical Design Handbook*.

k_{ball}, k_{bat} can be directly obtained through stiffness measure experiment, instead of other complex models.

■ Parameters affecting Easy Control

The lighter weight (smaller M) and smaller swing weight (small J) also lead to better bat control[Nathan, 2004], which has a beneficial effect for a contact-type hitter, who is just trying to meet the ball squarely rather than get the highest batted

ball speed. The batter can accelerate the bat to high speed more quickly with a corked bat, allowing the batter to react to the pitch more quickly, wait longer before committing on the swing, and more easily change in mid-swing.

■ Parameters Estimation

➤ Swinging Angular Speed Estimated by J

As showed above, the angular velocity ω_i and moment of inertia J have coupling effects on BBS . In order to determine the effect of J 's increment on BBS , we need an empirical equation illustrating the relationship between ω_i and J . We adopt the formula in Daniel A. Russell's paper [Russell, 2007]. Based on the analysis of the bat swing speed data from the Crisco-Greenwald field study and data fitting done by Alan Nathan, the empirical estimation is

$$\omega_i = 45.3 \left(\frac{J_{knob}}{16000} \right)^{-0.30769} \quad (18)$$

If we know the location of center of mass, we can use the parallel axis theorem to calculate I_{knob} and the conversion is roughly $I_{knob} = J + Ma^2$ with units of $oz \cdot in^2$.

➤ Swinging Speed Estimation by M

Like the interrelationship between ω_i and J , V_i and M have the same relationship and coupling effects on BBS . So an empirical equation of V_i and M is wanted. We adopt the model of A.Terry Bahill and Miguel Morna Freitas [1995]. Based on the data for Leah, a member of the University of Arimona, NCAA National Champion softball team, Bahill and Freitas use fitting method to examine the relationship between the mass(M with unit of oz) and swinging velocity of bat(V_i with unit of mph), getting the following formula

$$(M + 70.4)(V_i + 5.4) = 6032. \quad (19)$$

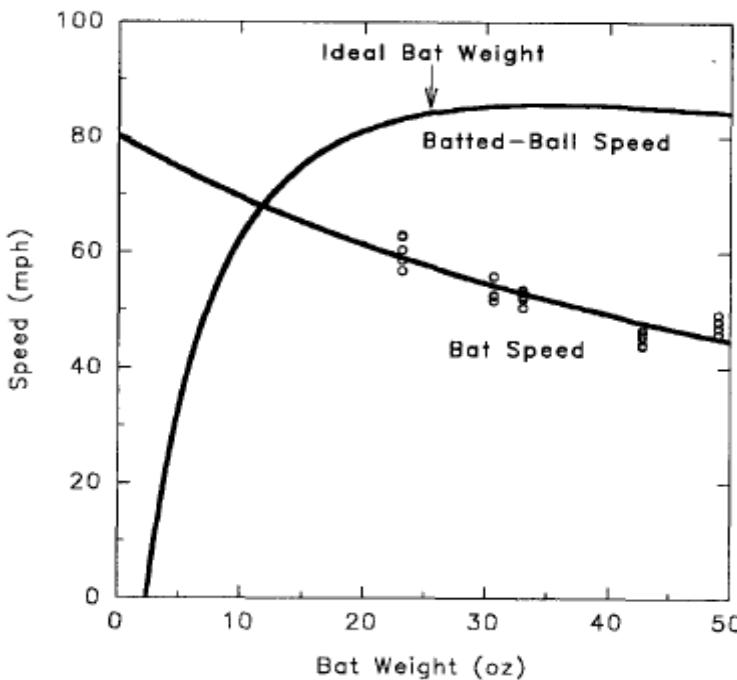


Figure 4 The Data used for Fitting Formula, cited from Bahill and Freitas's paper

➤ *Stiffness of Hoop Spring Estimation through thickness (t)*

The observed fundamental hoop vibration mode, which accounts for the majority of vibrant energy and is responsible for trampoline effect, has a frequency of about 1 kHz. It means that when the ball leaves the bat, taking place about 1ms after the ball touches the bat [Brody, 1985], the fundamental hoop vibration mode has not been set up. In other words, at the moment that ball exit, it has not “seen” either the knob or tip of the bat. This fact indicates us that we can view the bat as a hoop spring.

The hoop stress constant k_{bat} can be identified by an empirical relationship

$$k_{bat} \propto \left(\frac{t}{R}\right)^3, \quad (20)$$

Where

t is the thickness of the shell, m ;

R is the radius of the bat cross section, m .

Methods for Evaluating “Sweet Spot” Effects

■ Overview of corking effects

I) Effects on parameters by corking

➤ Since the hallowed wood bat was filled by cork/rubber which has a smaller/larger density, the bat has smaller/larger M and J (in comparison with the uncorked), which leads to smaller/larger r (recoil factor) and higher/lower V_i and ω_i .

- Since the shell has been thin, the trampoline effect of the bat will be more evident than the uncorked. Enlarging the trampoline effect will significantly increase *BBS*.

II) Evaluating “Sweet Spot” Effects

- Maximum of *BBS*

Since the *BBS* is the function of location of hitting point, which gets the maximum in “Sweet Spot” point, we calculate the maximum *BBS* to estimate the “Sweet Spot” Effect of corked bats. For the reason that there are both positive and negative effects on the maximum *BBS* after corking, quantitative method is needed.

- Easy Control

Since both M and J decrease, we make sure that the corked bat is better to control.

■ Methods for quantitative evaluation

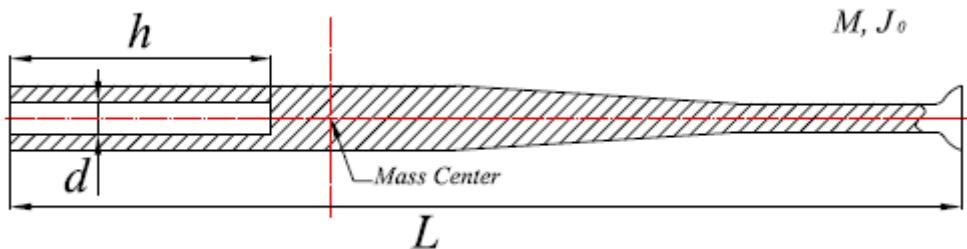


Figure 5. A typical example of cored bat

- i) Neglect the significance of trampoline effect

Through rough estimation, the effect of hoop mode of the cored wood bat could be neglected (See details of deduction in *Appendix II*). This estimation was also proved by the work of Alan M. Nathan [2003], who gives the quantitative results through experiments. He pointed out that there is nearly no trampoline effect from the hollowed-out wood bat or the cork filler, for the reason that “it requires much greater force to compress such a bat than it does to compress an aluminum bat”.

- ii) Calculating the *MOI* and mass of cored bat (J_{cored} , M_{cored})

Based on the definition of moment of inertia and *parallel axis theorem*, we obtain the *MOI* of the cored bat as following

$$J_{cored} = J^* + \left[M + \frac{\pi}{4}(\rho - \rho_0)d^2h \right] \left(x_{CM} - \frac{L}{2} \right)^2, \quad (21)$$

where

$$J^* = J_0 + \frac{\pi}{48}(\rho - \rho_0)d^2h\left(\frac{3d^2}{4} + h^2\right) + \frac{\pi}{4}(\rho - \rho_0)d^2h\left(L - a - \frac{h^2}{2}\right),$$

$$x_{CM} = \frac{M(L-a) + \frac{\pi}{8}(\rho - \rho_0)d^2h^2}{M + \frac{\pi}{4}(\rho - \rho_0)d^2h},$$

J_0 = MOI with respect to CM of the uncorked bat.

And the mass of corked bat is

$$M_{corked} = \frac{\pi}{4}\rho_0D^2L + \frac{\pi}{4}(\rho - \rho_0)d^2h. \quad (22)$$

iii) Measure the structure of corked bat and obtain the values of D, L, d, h .

Set ρ_0 = the density of wood,

ρ = the density of stuffed material (cork or rubber)

and calculate the $M_{uncorked}, L_{uncorked}, M_{corked}, L_{corked}$.

Based on Eq. 4 and Eq. 14, we obtain the maximum BBS of the corked and uncorked bat, and therefore determine whether “corking” a bat enhances the “sweet spot” effect. (Calculation details in *Simulation and Results*)

5 Simulation and Analysis

5.1 Basic Data Used in Simulation

The uncorked wood bat is the one used by Cross in his extensive set of measurements, which is a 33 in/31 oz Louisville Slugger Model R161 [Cross 1998]. The detail data is shown in **Table 2**.

Table 2 Data used for simulation

Property	Description	Value
L	The length of the bat	0.84m
M	The mass of the bat	0.885kg
J_{cm}	The MOI of the bat	0.045kg · m ²
a	The distance of center of mass to from handle	0.564m
m	The mass of the ball	0.145kg
average of V_i	The average of V_i measured in a contest	24m/s
average of ω_i	The average of ω_i measured in a contest	51rad/s
average of v_i	The average of v_i measured in a contest	40m/s

5.2 Solution to Problem I

5.2.1 Simulation

As we have shown that the “sweet spot” location is determined by several parameters. The values of m, M and J are the constant properties of the baseball and the bat, while the value of v_i , V_i and ω are determined by the pitcher and the batter. In other words, the v_i , V_i and ω value will vary over a certain range, which will influence the “sweet spot” location. The properties of the bat we use to simulate are listed in Table 2. Herein, we do a series of calculation based on the variation of v_i , V_i and ω to investigate the value of b .

■ The effect of ball speed v_i

We analyze the effect of ball speed v_i on the “sweet spot” location $b_{sweet\ spot}$ in **Figure 6** by assigning v_i a series of values (31m/s ~ 50m/s) and calculating $b_{sweet\ spot}$. We vary v_i from 31m/s to 50m/s for practical concern. The calculation shows that the value of $b_{sweet\ spot}$ is lowered by increasing the ball speed v_i . However, the “sweet spot” location still has a distance of about 0.15m from the tip.

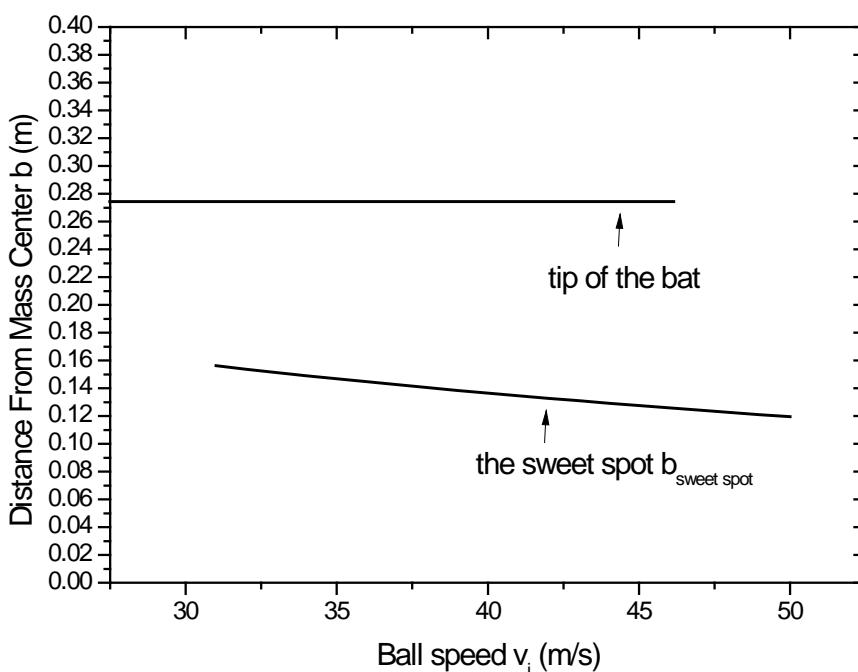


Figure 6. The plot of the “sweet spot” location $b_{sweet\ spot}$ as the ball speed v_i ranges from 31m/s to 50 m/s. The impact occurs on the standard wood bat, which has a CM speed of 54 mph and a rotational

speed about the CM of 51 s^{-1} .

■ The effect of swinging speed V_i

We analyze the effect of ball speed v_i on the “sweet spot” location $b_{\text{sweet spot}}$ in **Figure 7** by assigning V_i a series of values (19m/s ~29m/s) and calculating $b_{\text{sweet spot}}$. Considering the practical situation, we vary v_i from 19m/s to 29m/s. The calculation shows that the value of $b_{\text{sweet spot}}$ is lowered by increasing the swing speed V_i and the effect is smaller than that of the ball speed. In the same manner, the “sweet spot” location still has a distance of about 0.15m from the tip.

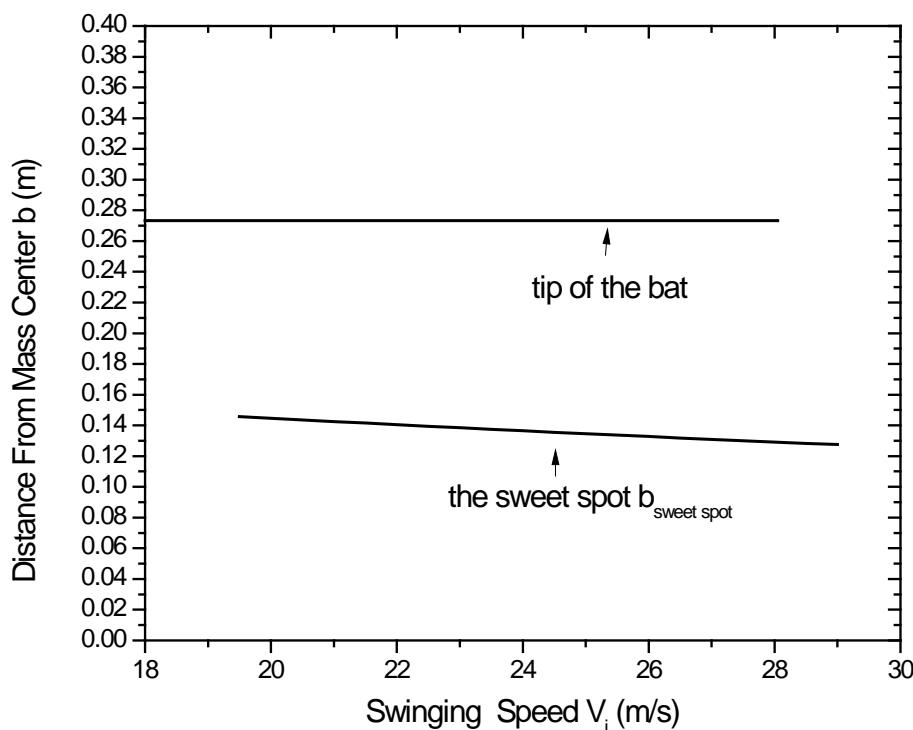


Figure 7. The plot of the “sweet spot” location $b_{\text{sweet spot}}$ as the swinging speed V_i ranges from 19m/s to 29 m/s. The impact occurs on the standard wood bat, which has a CM speed of 54 mph and a rotational speed about the CM of 51 s^{-1} .

■ The effect of bat rotational speed ω

We analyze the effect of ball speed ω on the “sweet spot” location $b_{\text{sweet spot}}$ in **Figure 8** by assigning ω a series of values ($43 \text{ s}^{-1} \sim 61 \text{ s}^{-1}$) and calculating $b_{\text{sweet spot}}$. Considering the practical situation, we vary

ω from 43 s^{-1} to 61 s^{-1} . The calculation shows that the value of $b_{\text{sweet spot}}$ is raised by increasing the swing speed ω . similarly, the “sweet spot” location still has a distance of about 0.15m from the tip.

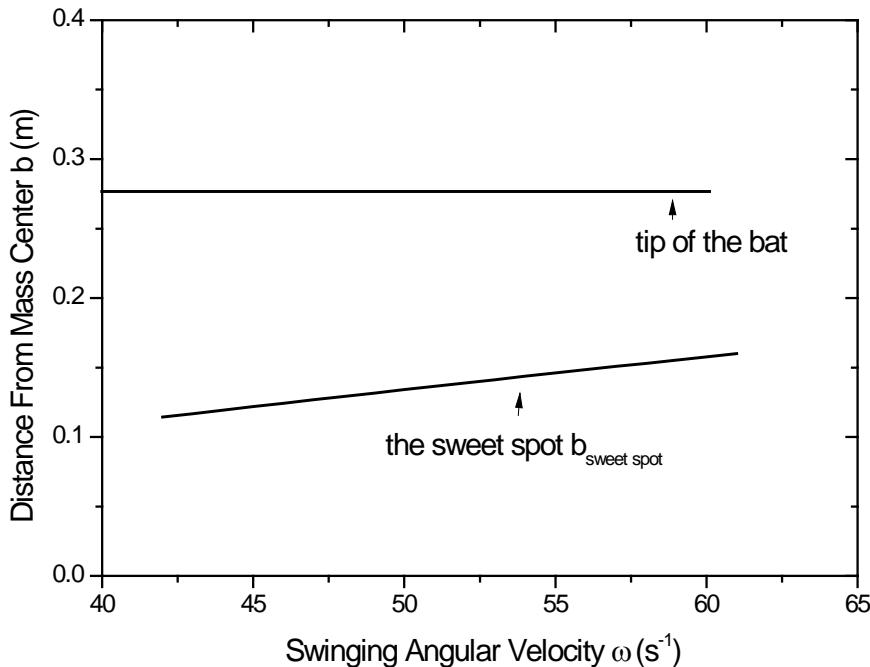


Figure 8. The plot of the “sweet spot” as swinging angular velocity ω ranges from 43 s^{-1} to 61 s^{-1} m/s. The impact occurs on the standard wood bat, which has a CM speed of 54 mph and a rotational speed about the CM of 51 s^{-1} .

■ the effect of impact location on the value v_f (BBS)

Figure 6, Figure 7 and **Figure 8** clearly illustrate that the optimum impact location is still far away from the tip of the bat, although the value of $b_{\text{sweet spot}}$ (the “sweet spot”) will vary with the value of v_i , V_i and ω . As the expression of v_f suggests, the impact location will influence the value of v_f . It is tempting to know how big the influence will be, for example, what is the exit velocity of the ball if the collision takes place at other locations other than the “sweet spot”. Herein, We analyze the effect of impact location on *ted ball speed* v_f , and put results in **Figure 9** by varying the impact location from 0.1m (from the knob) to 0.885m (the tip of the bat). The calculation shows that the value of v_f assumes a parabola-like curve and attains the maximum at

the “sweet spot” rather than the tip. This fact provides the direct evidence that the “sweet spot” is not at the tip of the bat.

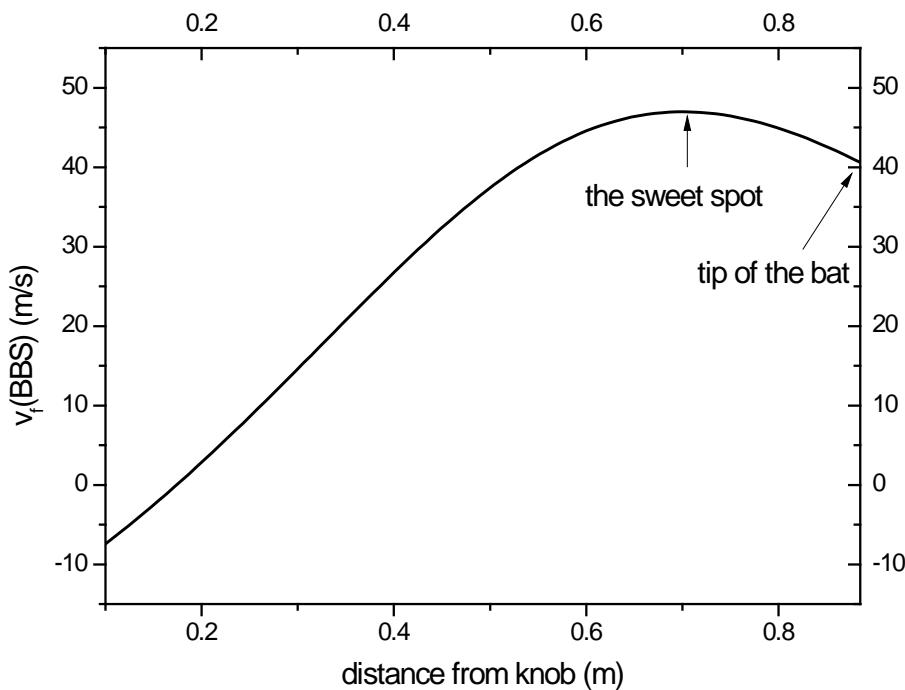


Figure 9. The plot of $v_f(\text{BBS})$ as a function of impact location for an impact of a 40m/s ball on the standard wood bat, which has a CM speed of 54 mph and a rotational speed about the CM of 51 s^{-1} .

5.2.2 Where is the “Sweet Spot”?

The simulation we perform demonstrates that the “sweet spot” is about 15 cm away from the end of the bat when parameters v_i , V_i and ω varies over the practical and meaningful ranges. So, we draw a conclusion from the above analysis that:

The “sweet spot” is not at the end of bat.

5.3 Solution to Problem II

5.3.1 BBS Formula Simulation

■ The Effects of Ratio of Stiffness ($k_{\text{ball}}/k_{\text{bat}}$) on Maximum BBS

Based on **Table 1**, we know that the increase of $k_{\text{ball}}/k_{\text{bat}}$ leads to bigger equivalent BBCOR and therefore leads to higher maximum BBS. In order to make quantitative analysis, at first, we determine the quantitative relationship between

k_{ball}/k_{bat} and equivalent BBCOR based on the Eq.17, as shown in **Figure 10**. Secondly, we deduced the relationship between k_{ball}/k_{bat} and maximum BBS using Eq.4, which is illustrated in **Figure 11**.

From semi-log figure of $\frac{k_{ball}}{k_{bat}} \sim v_{f,max}$ (**Figure 11**), we can clearly figure out that the increase of maximum BBS becomes quite slow, when k_{ball}/k_{bat} is bigger than 1. It illuminates that we should make the k_{ball}/k_{bat} smaller than one if we expect more significant trampoline effect which enhances the BBS.

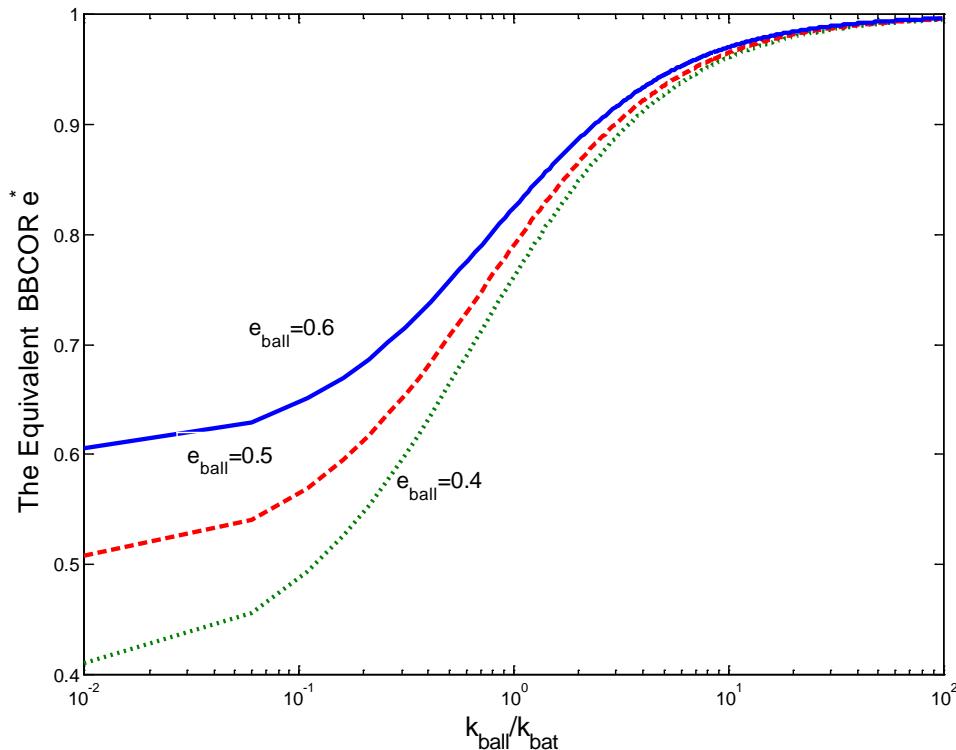


Figure 10 The equivalent BBCOR is a function of ration of stiffness with $e_{bat} = 1$.

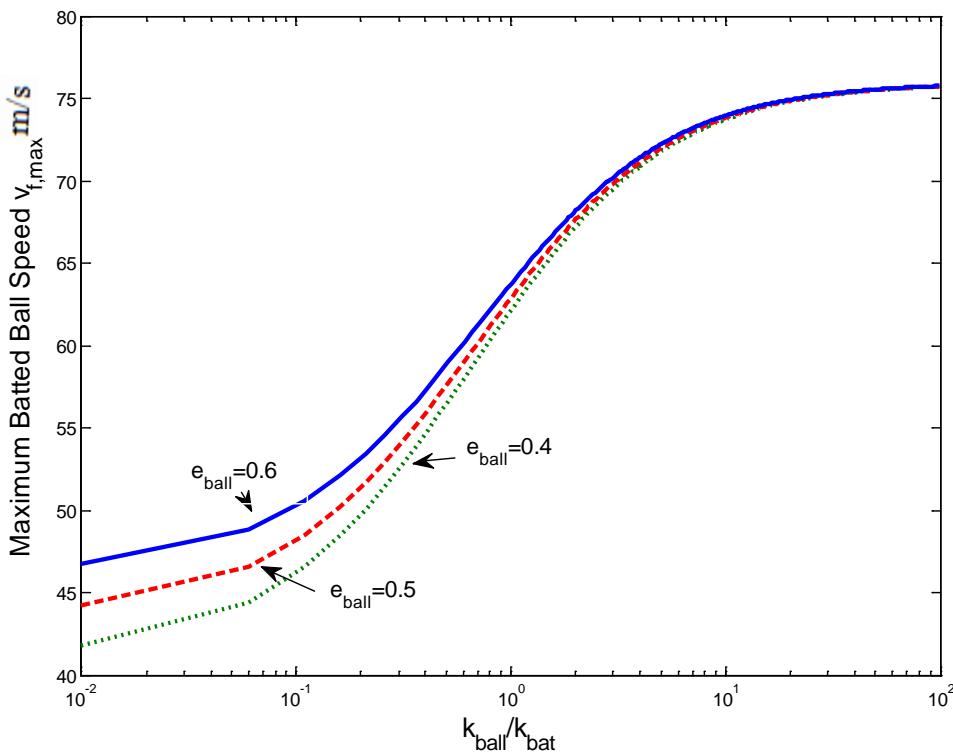


Figure 11 The maximum BBS ($v_{f,max}$) is a function of equivalent BBCOR which is defined by k_{ball}/k_{bat} . The bat is corked with $e_{bat} = 0.98$, whose CM speed is 24m/s and angular speed is 51rad/s.

■ The Effect of Moment of Inertia (J) on Maximum BBS

From the **Table 1**, we may be depressed by the fact that merely qualitative analysis based on the equation of BBS is unable to conclude whether the effect of MOI's increase on BBS is positive or negative. Here, we import the data to do the precise quantitative analysis, and then determine the specific relationship between them.

The procedure to simulate is shown as following

Figure 12

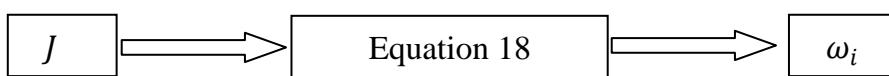


Figure 13

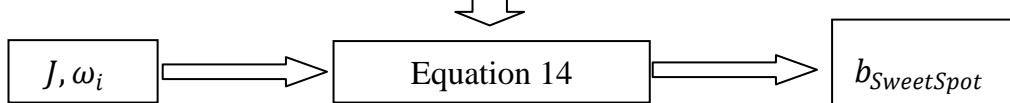
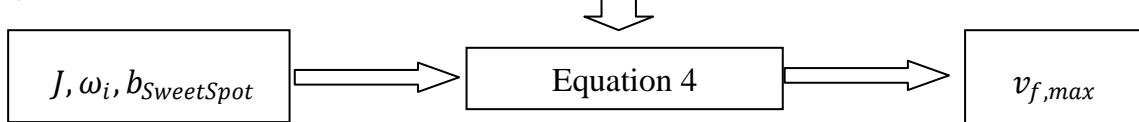


Figure 14



From **Figure 14**, we are able to easily make sure that the increase of J (limited in

possible field of variation) leads to higher maximum *BBS*, which is quite useful in comparing the “Sweet Spot” effect of the corked and uncorked bat and designing better corked bat.

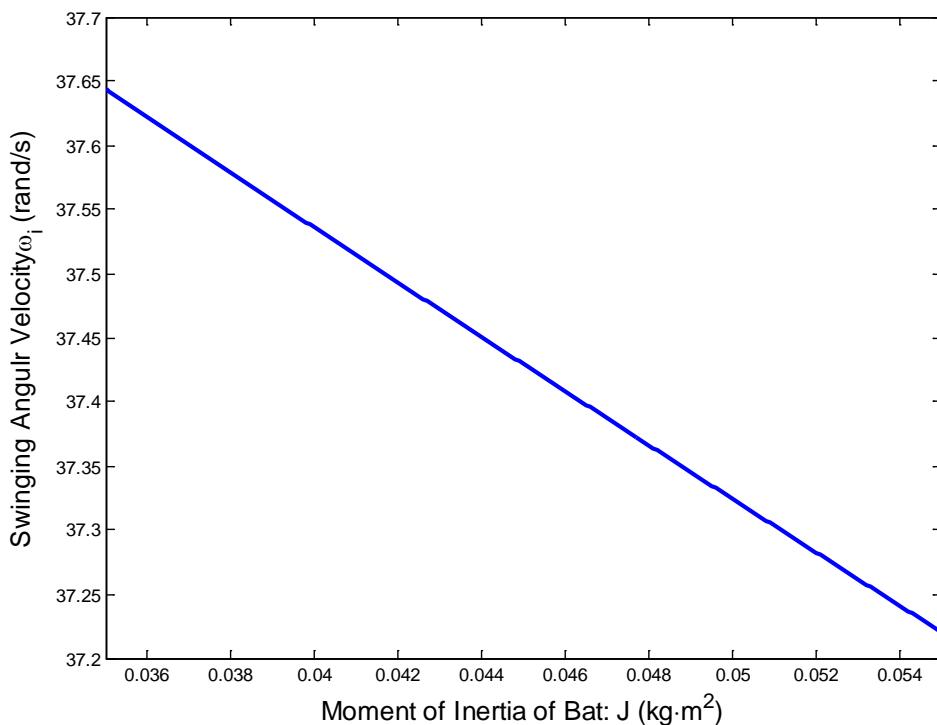


Figure 12 The relationship between ω_i and J is illuminated by an empirical equation.

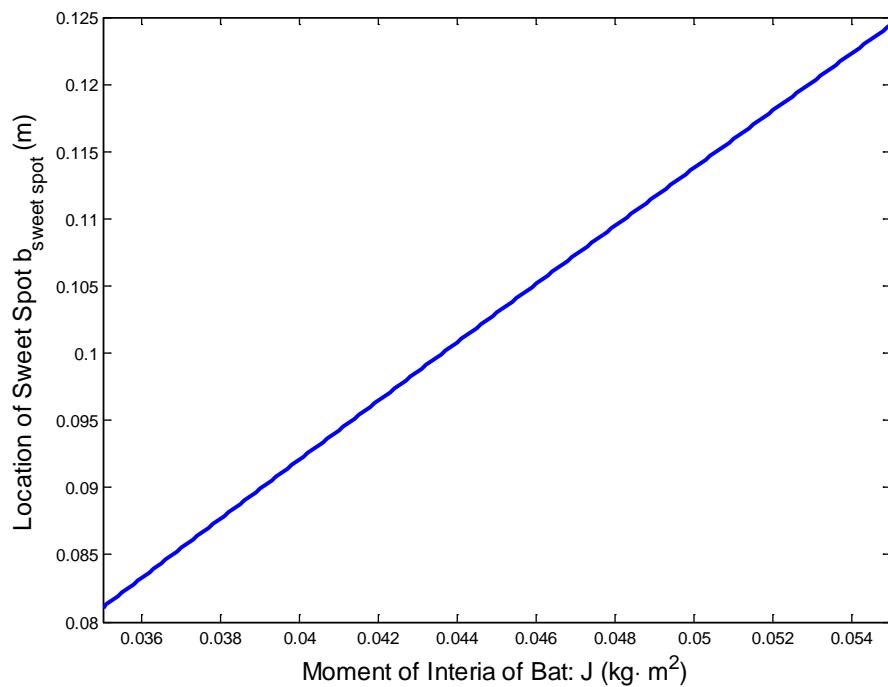


Figure 13 The Location of “sweet spot” is determined by moment of inertia, with $V_i = 24 \text{ m/s}$, $v_i = -40 \text{ m/s}$, $M = 0.885 \text{ kg}$.

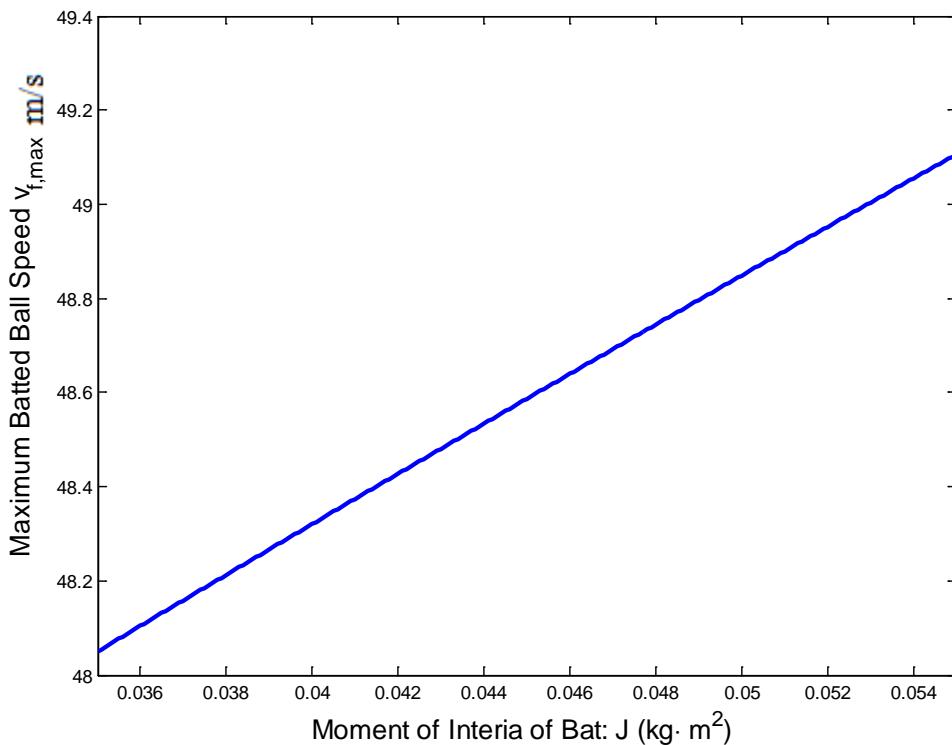


Figure 14. The figure i illustrating the relationship between MOI and Maximum BBS based on the data $\frac{k_{ball}}{k_{bat}} = 0.1$

■ The Effect of Mass (M) on Maximum BBS

Similar to the effect of MOI , the effect of Mass on BBS cannot be determined only by qualitative analysis. We make the quantitative analysis in the same procedure as MOI , only replacing ω_i with V_i , and J with M .

From **Figure 17**, we easily obtain the similar result that the increase of M (limited in possible field of variation) leads to higher maximum BBS . What's more, **Figure 17** also tells us that the increase speed of maximum BBS is getting slow with the increase of mass.

Especially, we draw the swinging bat speed V_i and maximum BBS v_i with respect to mass M in the same figure, as **Figure 18** shows. Comparing with **Figure 4** which is drawn with experimental data [Bahlil and Freitas, 1995], our data obtained from calculation of our model exactly match that experimental data, which proves the correctness of our model.

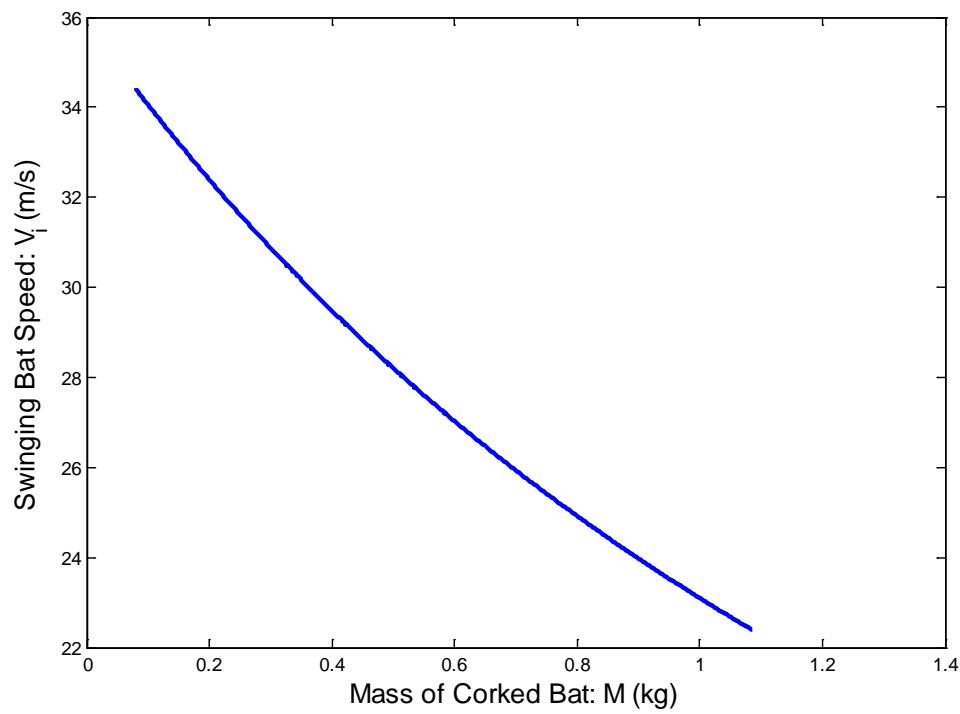


Figure 15. The plot of swinging bat speed V_i as the function of the corked bat mass M .

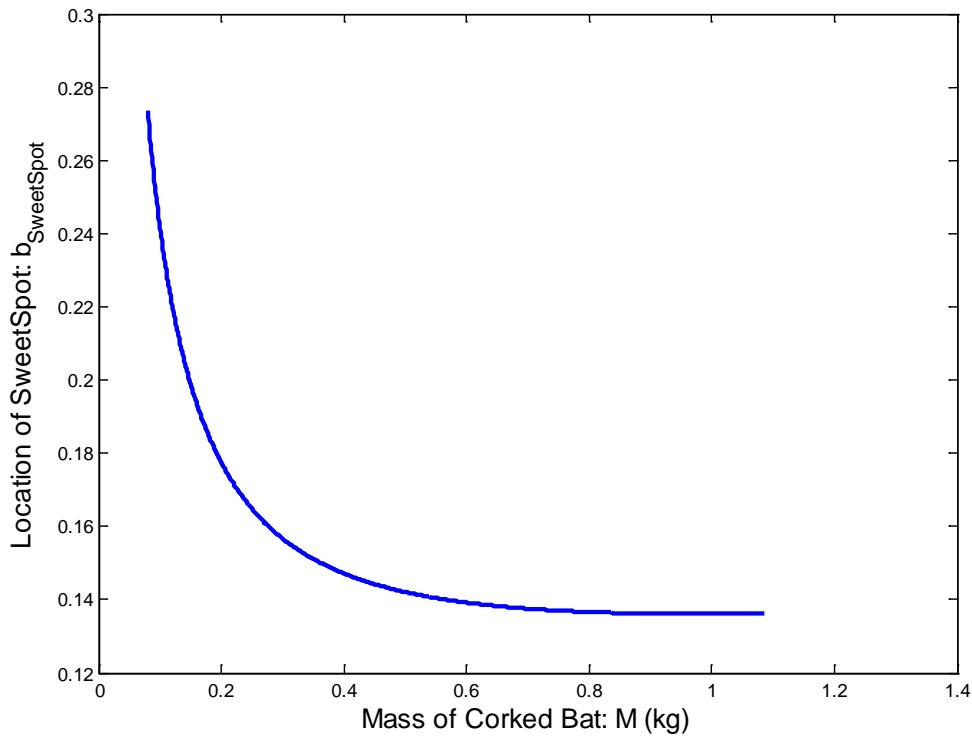


Figure 16. The plot of “sweet spot” location $b_{sweet\ spot}$ as the function of the corked bat mass M .

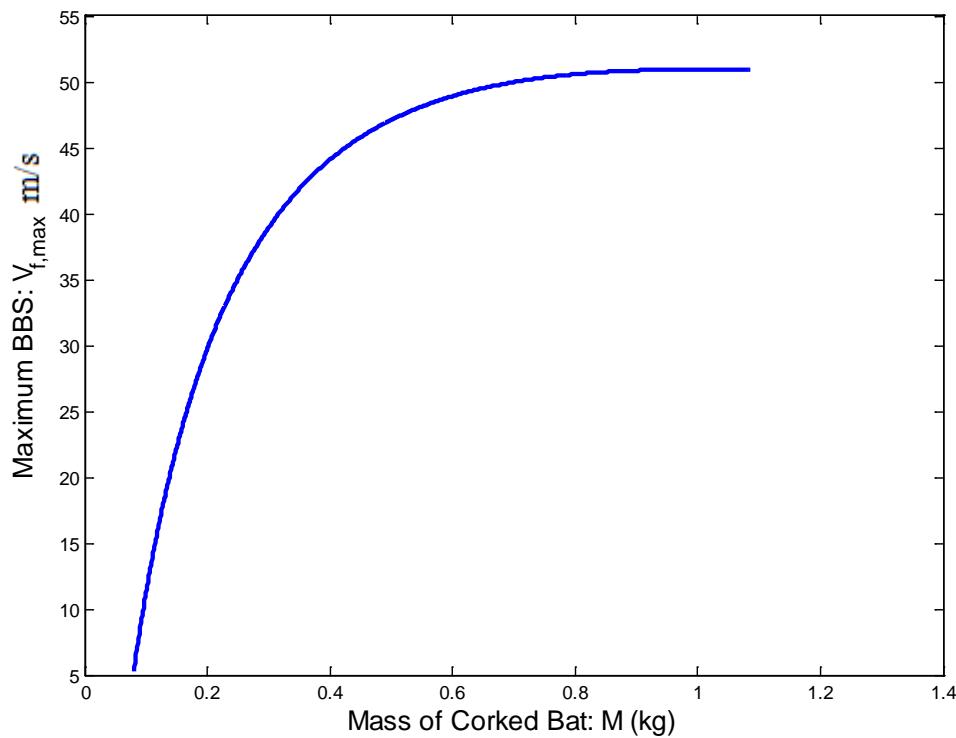


Figure 17. The maximum BBS $v_{f,\max}$ as the function of the corked bat mass M .

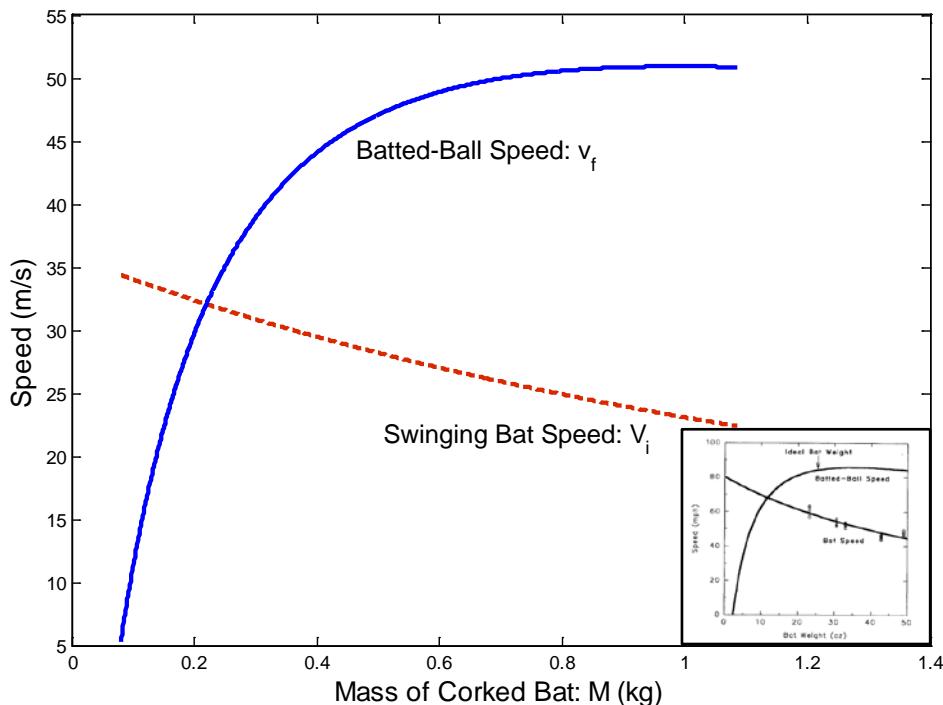


Figure 18 The plot of batted-ball speed v_f and swinging bat speed V_i as the function of the corked bat mass M . This figure derived from the calculation of our model exactly matches the actual data obtained by experiments.

5.3.2 “Sweet Spot” Effects of Corking

■ Simulating details

The corked bat can be fully described with the shape of a hallowed cylinder (“Corked Hole”, mathematical described with d, h) and the density of the stuffed materials (cork or rubber in this problem). Based on Eq.19 and Eq.18, we are able to calculate the mass and moment of inertia of a specific corked bat, and then obtain the values of maximum BBS which is the key indicator to evaluate “Sweet Spot” effect.

In **Figure 19**, we set the density of stuffed material $\rho_{cork} = 450\text{kg/m}^3$ to evaluation the “Sweet Spot” effect of different corked bats with the different “Corked Holes”, whose depth and diameter is in the actually possible field of variation ($h \in [0, 0.2]\text{m}$, $d \in [0, 0.05]\text{m}$) [Russel, 2004].

In **Figure 21**, the density of stuffed material $\rho_{rubber} = 1100\text{kg/m}^3$ to evaluation the “Sweet Spot” effect of different corked bats with the same variation of d and h .

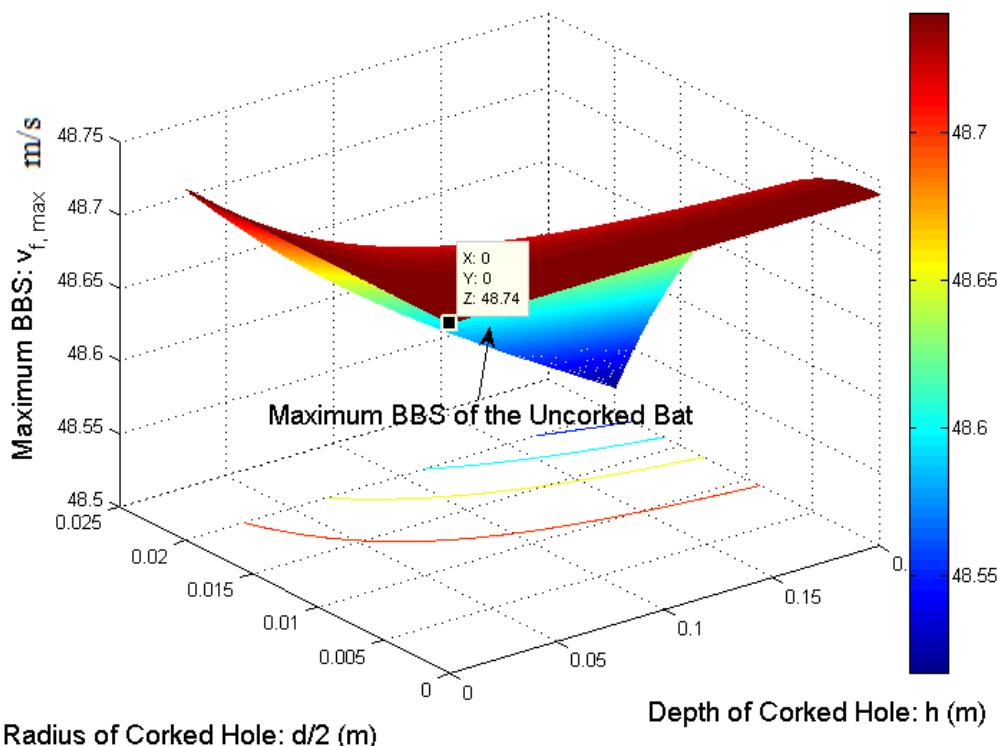


Figure 19 Figure in a wireframe mesh style. Based on the structure of the corked bat: depth and radius of corked hole, the density of stuffing cork $\rho_{cork} = 450\text{kg/m}^3$, we determined the J, M , and then obtain maximum BBS . Notes: $v_i = -40\text{m/s}$, $L = 840\text{mm}$, $a = 564\text{mm}$, $m = 0.145\text{kg}$.

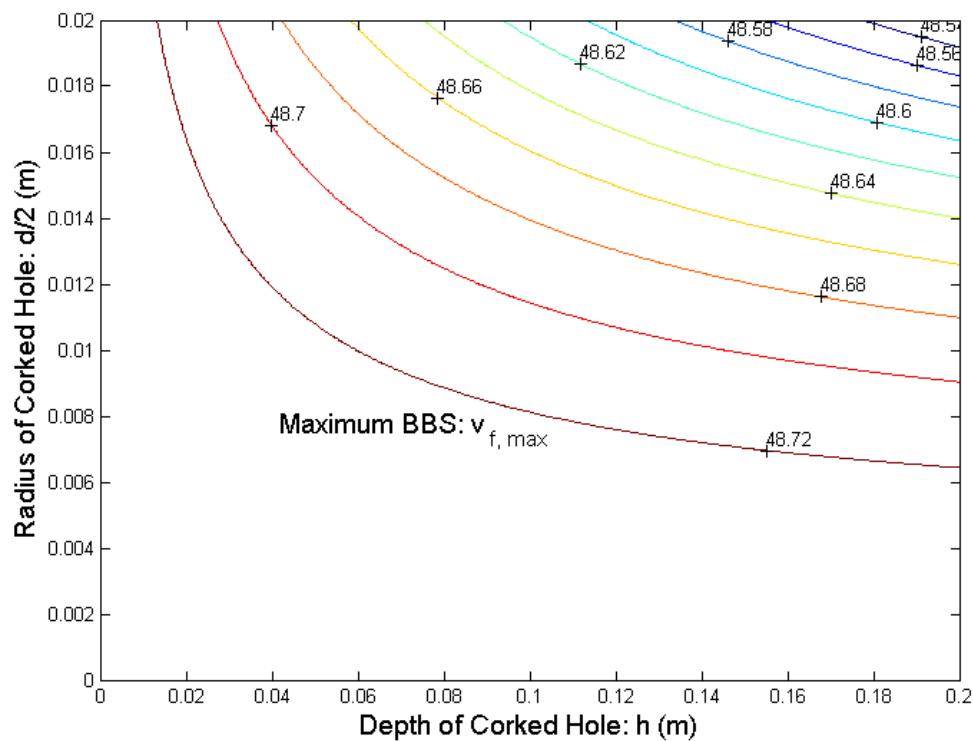


Figure 20 Same Figure in contour style.

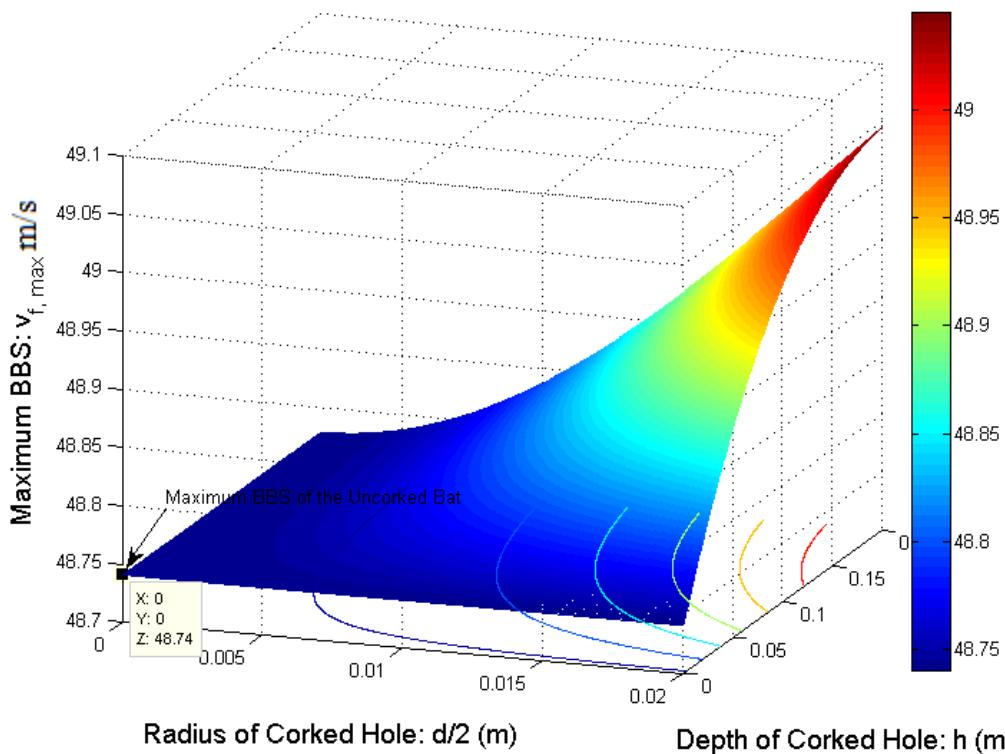


Figure 21 Based on the structure of the corked bat: depth and radius of corked hole, the density of stuffing rubber $\rho_{rubber} = 1100 \text{ kg/m}^3$, we determined the J, M , and then obtain maximum BBS. Notes: $v_i = -40 \text{ m/s}$, $L = 840 \text{ mm}$, $a = 564 \text{ mm}$, $m = 0.145 \text{ kg}$.

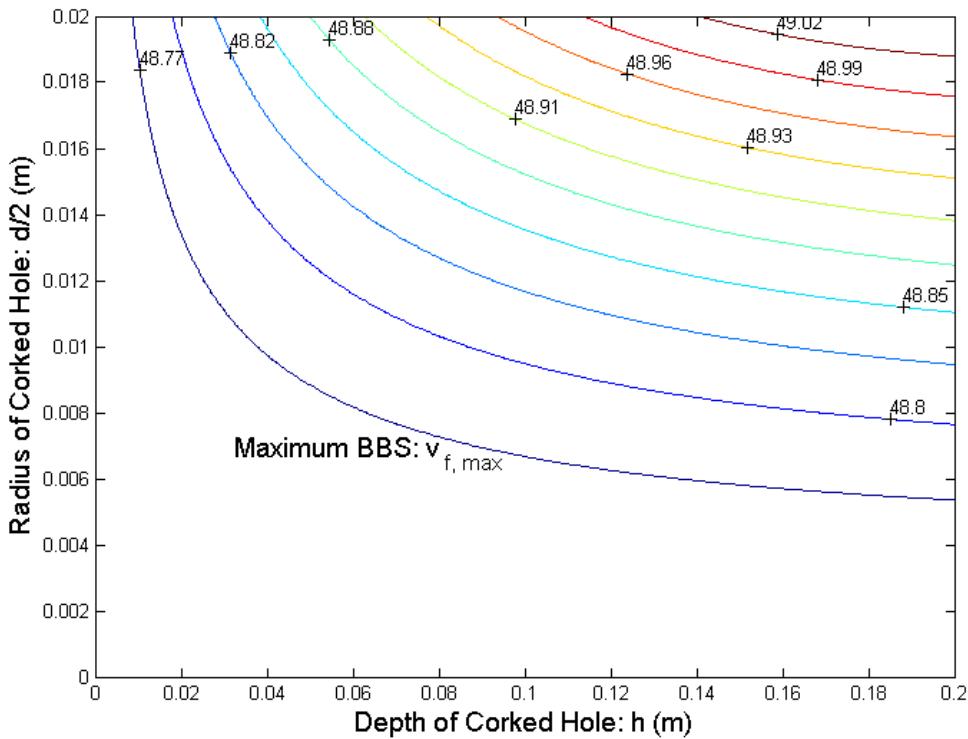


Figure 22 Figure in contour style

■ Results

I. Whether corking enhances “Sweet Spot” effect?

- The corked bat with cork stuffed ($\rho_{cork} < \rho_{ash}$)

In **Figure 19** or **Figure 20**, considering all possible values of d and h , the maximum BBS ($v_{f,max}$) of the corked bat with cork stuffed reaches the maximum at the point of $d = 0, h = 0$. In other words, the maximum BBS of all possible corked bat with cork stuffed is lower than that of the uncorked. However, since both M and J decrease, the corked bat is better to control than before.

Therefore, if “Sweet Spot” effect only concerns the maximum BBS (i.e. maximum power transferred to the ball), we can evidently conclude that “corking” a bat with cork stuffed cuts down the “Sweet Spot” effect, no matter what shape of “Corked Hole” is adopted.

- The corked bat with rubber stuffed ($\rho_{rubber} > \rho_{ash}$)

In **Figure 21** or **Figure 22**, considering all possible values of d and h , the maximum BBS ($v_{f,max}$) of the corked bat with rubber stuffed reaches the minimum at the point of $d = 0, h = 0$. In other words, the maximum BBS of all possible corked bat with rubber stuffed is higher than that of the uncorked. However, since both M and J increase, the corked bat becomes more difficult to

control than before.

Therefore, if “Sweet Spot” effect only concerns the maximum *BBS* (i.e. maximum power transferred to the ball), we can evidently conclude that “corking” a bat with rubber stuffed enhances the “Sweet Spot” effect, no matter what shape of “Corked Hole” is adopted.

II. Why MLB prohibits “corking”

As discussed above, we know that corking the bat with high density material stuffed will significantly enhance the “Sweet Spot” effect which will lead to higher batted ball speed. However, corking the bat with low density material stuffed will lead to better control which may satisfy some part of athletes, like contact-type hitters.

If the Major League Baseball (MLB) does not prohibit “corking”, the athletes can always improve their scores by using higher quality “corking” materials and more sophisticated design of coked bat. If so, the contest is not only the contest on athletes’ baseball skills, but also on the quality of bat, in other words, the technology and money, which will definitely leads to inequality. Aimed to protect the principle of sport-equality, the MLB has to list “corking” technology into taboo.

5.3 Solution to Problem III

Metal bat (usually aluminum bat) has received great popularity for its wider range of “sweet spot”, more power, better feel, and higher performance than the wood one (usually ash bat). What factors contribute to metal bat’s better performance? Why MLB prohibits the metal (aluminum) bat? Herein, we will answer those questions based on conclusions drawn from our previous models.

5.3.1 An Illustration of a Typical Metal Bat

Figure 23 is an illustration of a typical metal bat. Generally speaking, the outer shape of a metal bat and a wood bat is much alike. However, a large amount of metal will be hollowed out to ensure it has an appropriate mass. The shell of bat is about 0.24cm in thickness, about 1/7 of that of a coked wood bat [Nathan, 2004]. It must be mentioned that the shell structure leads to trampoline effect which enhance the performance of metal bats.

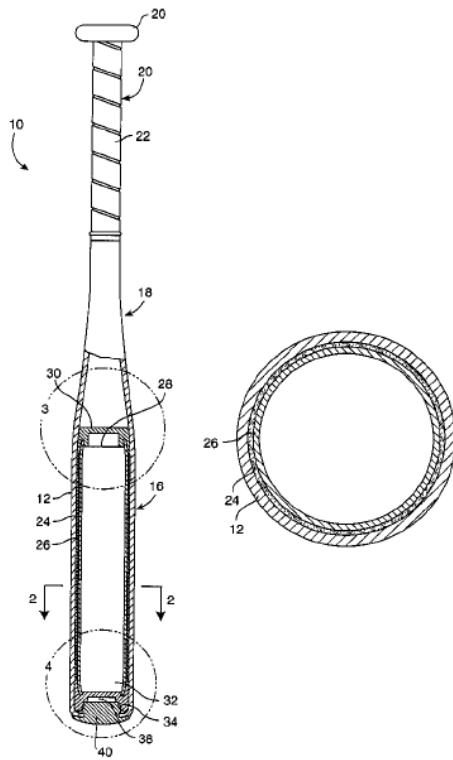


Figure 23. Illustration of a typical metal bat; metal bats are often designed to have special cavity to enhance its performance.[Nguyen, 2004]

5.3.2 Predicting Behavior of Metal and Wood Bats

■ Review of established models

We have developed two models to investigate how structural properties (J_{cm} , M , z_{cm} , coarking) influence the performance of a bat. We picture a logic relationship corresponding to our foregoing models in **Figure 24** to facilitate the prediction. In this figure, controllability and BBS are introduced to describe the behavior of a bat. **Figure 24.** clearly shows that both metal and wood bats can be characterized by parameters J_{cm} , M , z_{cm} and outer shape, which affects the bats' controllability and BBS . Apart from that, metal bats generally have trampoline effect which increases BBS as proved in our DS model. To predict and compare the behavior the two kinds of bats, we first start with a special case.

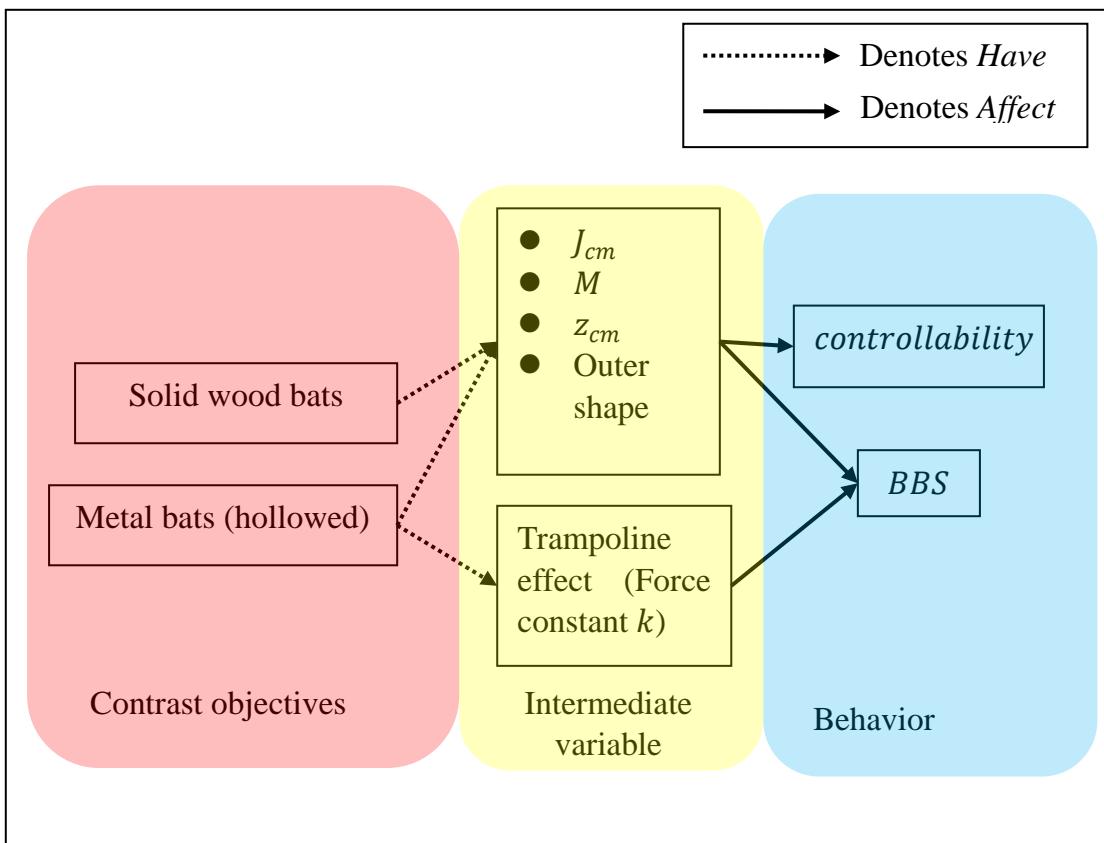


Figure 24. Illustration of useful conclusions drawn from foregoing models. Both metal and wood bats can be characterized by parameters J_{cm} , M , z_{cm} and outer shape, which affects the bats' controllability and BBS ; Metal bats have trampoline effect which increases BBS .

■ A special case: the same J_{cm} , M , z_{cm} , and outer shape

● Can such a special metal bat exist?

Before the comparison and prediction, one may cast doubt on the existence of such a special metal bat meeting satisfying these conditions. We do some calculation and obtain such a special metal design parameters and graph as listed below.

Table 3. The design parameters of the special aluminum bat. The solid wood bat is the one used by Cross in his measurements. It is a 33 in/31 oz Louisville Slugger Model R161 [Cross, 1998]; and the relevant properties are also listed

	Solid Wood Bat	Aluminum Bat	same or not
M	0.885kg	0.885 kg	yes
z_{CM}	0.564m	0.564m	yes
L	0.840m	0.840m	yes
J_{CM}	$0.045kg \cdot m^2$	$0.045kg \cdot m^2$	yes
density	$670kg/m^3$	$2700kg/m^3$	no
Outline Shape	Totally same		yes
Trampoline Effect	No	Yes	yes

The technique to construct a required bat is stated in the part of *technical tips* in the paper.

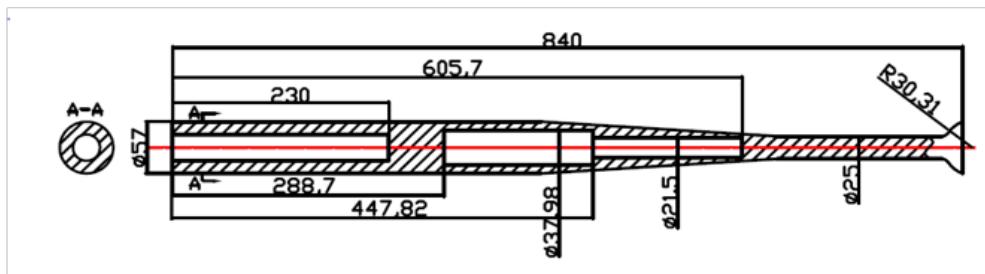


Figure 25. Illustration of the special metal bat design.

● Prediction and Comparison of behavior for wood and metal bats

As Figure 24 demonstrates, such a special metal bat will have the same controllability as the wood bat concerned. However, this special metal bat will have a faster *BBS* due to the contribution of trampoline effect. In a word, our model could predict that a metal bat will outperform the wood bat with the same J_{cm} , M , z_{cm} , COR and outer shape.

A quantitative analysis on the trampoline effect can be got on *Appendix II* of this paper. We can see that *BBS* could experience a gap from 45m/s to about 75m/s theoretically.

■ General cases

From the special case, we can conclude that in common situation, metal bats will possess both better *BBS* and controllability. The reason is that a better metal bat can be manufactured through the following steps on the basis of the special bat design:

- 1) Reduce J_{cm} and M by remove a certain amount of metal, thus increase the controllability.
- 2) Streamline the inner cavity to enhance the strength of the bat.

After the two steps, the newly-made bat still has strong trampoline effect while its controllability is raised.

An experimental example

We present another experimental example, in **Table 4**, which verifies our argument that the aluminum bat exhibit both faster *BBS* and better controllability.

Table 4. Contrast of wood and metal bats including Crisco-Greenwald's Cage Study[Greenwald, 2001; Crisco,2002] experiment result and model calculation result.

Bat	Length (in)	Weight (oz)	z_{cm} (in)	J_{cm} (oz-in ²)	Swing Speed (mph)	<i>BBS</i> (mph)
Wood	34	30.9	23	11516	67.9	98.6

bat						
Metal bat	33	29.2	20.7	9282	70.9	106.5

5.3.3 Reason for MLB's Prohibition of Metal Bats

Our above analysis points out metal bats generally have faster *BBS* and better controllability than wood bats do. This is also supported by Crisco and Greenwald Batting Cage Study[Greenwald, 2001; Crisco,2002], which shows that an average batted ball speed for wood bats is around 98.6-mph, while the average batted balls speed for metal bats lies between 100.3 mph and 106.5-mph. The faster travelling ball poses two negative effects on the athletes.

- The faster the ball travels, the less time athletes, such as the pitcher, have to react, thus resulting in injuries more easily. It is generally believed that the pitcher's reactionary time drops from 0.4s to 0.3s when the ball is hit by a metal bat [Russell, 2008].
- The faster the ball travels, the more severe injuries it tends to cause.

Apart from safety concern, we derive from our model that

- the advent of the metal bat will weaken the fairness of the game;
- powerful bats make athletes dependent upon the tool, not the game itself.

From the above analysis, we conclude the reason that MLB should prohibit metal bats is for **concerns of safety and the nature of sport**.

6 Technique Tips for Bat Design

One of the important implementation of our model is to instruct the design of a perfect bat.

6.1 Optimum Mass for Better “Sweet Spot” Effect

In order to simplify the process of estimation, we firstly simplify the Eq.4. When the collision takes place near to the mass center of the bat, as it very frequently does, all terms containing b may be deleted [P. Kirkpatrick 1963]. Therefore, $r = m/M$ and Eq.4 becomes

$$v_f = \frac{(1+e)(V_i + \omega_i b) + |v_i| \cos\theta(r - e)}{(1+r)\cos\phi}. \quad (23)$$

It shows that the velocity of the batted ball speed (*BBS*) v_f is dependent upon the ratio of m/M or r . According to the equation, the value of v_f will be raised if we merely lower the value of M and keep all the other parameters unchanged. However, as is often the case, the swing speed V_i will be decreased when the mass of the bat M becomes larger, which in turn leads to the decrease of v_f . One can anticipate that there exists an optimum bat mass on the assumption that the best bat requires that least energy input to impart a given velocity to the ball. We estimate the best mass of bat on the assumption that “that bat is best which requires the least energy input to impart a given velocity to the ball [P. Kirkpatrick 1963]”. Rearrangement of Eq.4 gives

$$V_f = \frac{(1+r)v_f \cos\phi + (e-r)v \cos\theta}{1+e}. \quad (24)$$

The kinetic energy of the bat (neglecting ω_i) is

$$W = \frac{1}{2}MV_f^2 = \frac{M}{2} \left[\frac{(1+r)v_f \cos\phi + (e-r)v \cos\theta}{1+e} \right]^2 \quad (25)$$

By differentiating the expression of W and setting it to zero, we obtain the minimum of W when

$$r = \frac{v_f \cos\phi + ev \cos\theta}{v_f \cos\phi - v \cos\theta} \cong \frac{v_f + ev}{v_f - v} \quad (26)$$

If the $v_f = -v_i$, then

$$r = \frac{1-e}{2}. \quad (27)$$

For example, when $e^* = 0.5$, then $r = 0.25$, $M = 4m$.

6.2 Designing a Special Aluminum Bat

This part primarily serves as the guidance for the case study in Problem III.

The solid wood bat we choose is the one used by Cross in his extensive set of measurements. It is a 33 in/31 oz Louisville Slugger Model R161 [Cross, 1998]; and the relevant properties are listed in **Table 5**. The parameters of special aluminum bat we design are also listed in **Table 5**.

Table 5. A Comparison between Solid Wood Bat & Aluminum Bat (customized)

	Solid Wood Bat	Aluminum Bat	same or not
M	0.885kg	0.885 kg	yes
z_{CM}	0.564m	0.564m	yes
L	0.840m	0.840m	yes
J_{CM}	$0.045\text{kg} \cdot \text{m}^2$	$0.045\text{kg} \cdot \text{m}^2$	yes
density	670kg/m^3	2700kg/m^3	no
Outline Shape	Totally same		yes
Trampoline Effect	No	Yes	yes

To customize such a bat, we need to reconfigure the mass distribution transversely. One of the possibilities is shown in Figure 26.

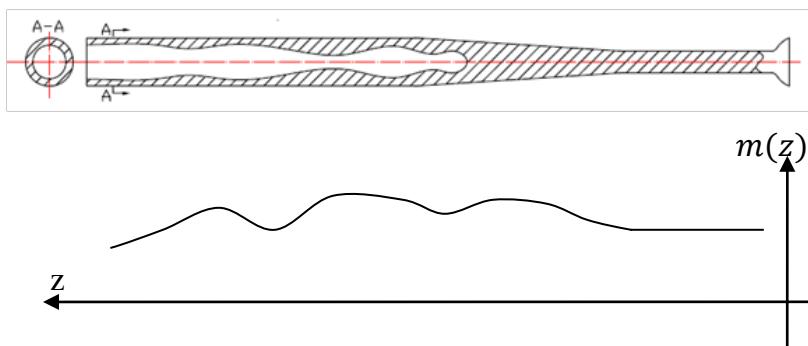


Figure 26. Cross section of typical inner cavity style of aluminum bat and its estimated according mass distribution function.

According to **Table 5** and Figure 26, we derive three constrain equations and assume that the bat is a spinning body.

- The mass of the wood bat and the customized bat are equal gives

$$M^w = \int_0^L dm(z), \quad (28)$$

where M^w is the mass of the wood bat, kg;

$m(z)$ is the mass distribution function of the customized bat;

L is the length of the wood bat, L.

- The mass center of the wood bat and the customized bat are the same gives

$$z_{cm}^w = \frac{\int_0^L z \cdot dm(z)}{M^w}, \quad (29)$$

where z_{cm}^w is the mass center of the wood bat,,kg.

- The moment of inertia of the wood bat and the customized bat with respect to

the mass center are the same gives

$$J_{cm}^w = \int_0^L (z - z_{cm})^2 \cdot dm(z), \quad (30)$$

where J_{cm}^w is the moment of inertia of the wood with respect to the mass center, $kg \cdot m^2$.

Any mass distribution function $m(z)$ satisfying the above three equations can be used to customize an aluminum bat.

In fact, there exists an infinite number of $m(z)$ functions, yet many of them are complicated or technical impractical. To facilitate the control test and analysis, we adopt a simple and special $m(z)$ as demonstrated by **Figure 27**.

- Derivation of special case

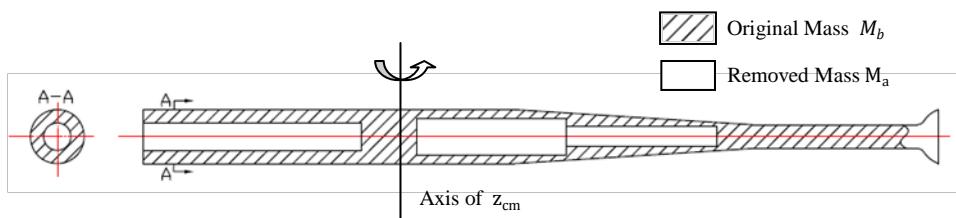


Figure 27. Cross section of this special and simple case of aluminum bat (geometrical parameters unknown); Shadowed part and white part are distinguished to help the according theoretical derivation of mass distribution $m(z)$ and the inner shape of bat.

We let M_a be the mass of the removed part, and M_b be the mass of the rest part. Because the outline shape of two bats are same, which means the two bats have the same volume if stuff the removed part M_a back, we have

$$\frac{M_a + M_b}{M^w} = \frac{\rho_{Al}}{\rho_{wood}}, \quad (31)$$

where

M^w is the mass of the wood bat, kg ;

ρ_{Al} is the density of aluminum, kg/m^3 ;

ρ_{wood} is the density of wood, kg/m^3 .

For the same reason, the mass center equation and the moment of inertia equation¹, we also have

$$M_a \cdot z_{cm,a} + M_b \cdot z_{cm,b} = (M_a + M_b)z_{cm}^w \quad (32)$$

¹ The moment of inertia and mass center are both linear quantities in terms of density.

$$\frac{J_{cm}^a + J_{cm}^b}{J_{cm}^w} = \frac{\rho_{Al}}{\rho_{wood}} \quad (33)$$

Using the three constrain equations, we can get the mechanics parameters of the removed part.

$$M_b = M^w \text{ & Eq. (31)} \Rightarrow M_b = \left(\frac{\rho_{Al}}{\rho_{wood}} - 1 \right) M^w \quad (34)$$

$$z_{cm,a} = z_{cm}^w \text{ & Eq. (32)} \Rightarrow z_{cm,b} = z_{cm}^w \quad (35)$$

$$J_{cm}^b = J_{cm}^w \text{ & Eq. (33)} \Rightarrow J_{cm}^b = \left(\frac{\rho_{Al}}{\rho_{wood}} - 1 \right) J_{cm}^w \quad (36)$$

Through calculation we obtain a diagram of the demo bat in **Figure 25**.

7 Discussion and Conclusion

We develop two models to elaborate the ball-bat collision. It has been shown that the models' simulation well match the empirical data and well explain some phenomena. It seems that some physical issues including the vibration of the bat and the nonlinear compress behavior of ball have been neglected in our paper. However it's not that case. We will validate our concise model in the following part.

7.1 Model Validation

Literature review indicates that some researchers have developed more complicated models (see **Figure 28**) to show the dynamic features and vibrant features. With advanced models, such phenomena as bending vibration, hoop vibration, and so on. As listed four models in **Figure 28**, the latter two, #3 and #4 do no good to answer the proposed problems. The energy stored in the wave on the bat after the collision is a part of consuming energy considered in modification of parameter, *BBCOR*, in the first two models.

Such a technique is quite useful in practice, because we can do experiment to determine *BBCOR*.

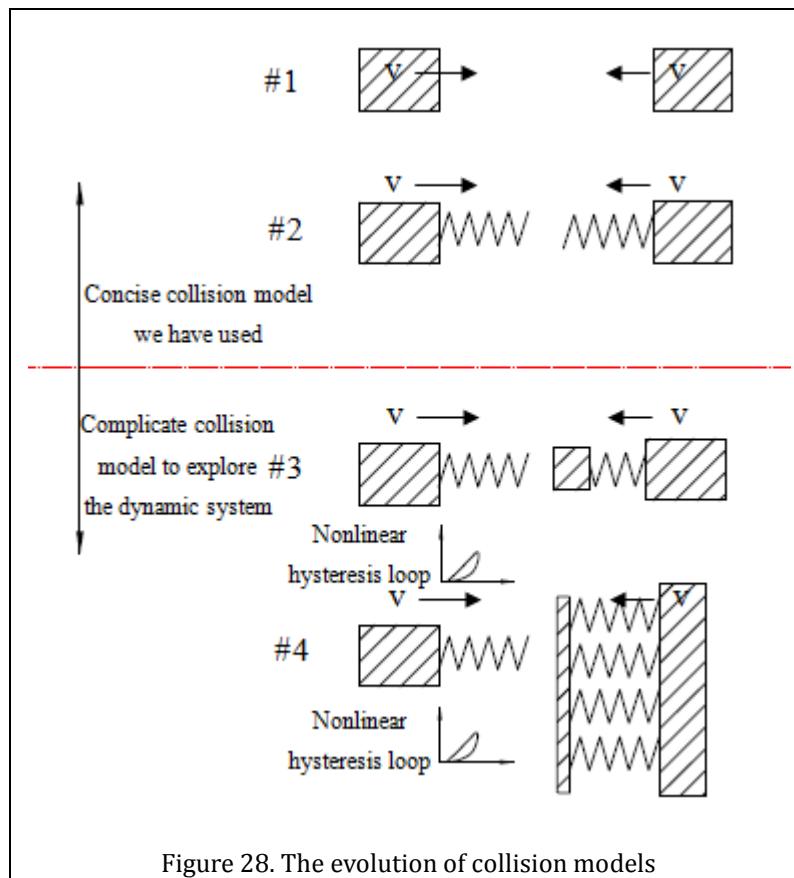


Figure 28. The evolution of collision models

7.2 Bending Vibration

In the fundamental model, we treat the bat as a rigid body and neglect the potential vibration excited by the collision. Actually, during and after collision, the baseball bat might exhibit several flexural bending modes of vibration, and the energy stored in the vibration movement is dependent on impact location. It is possible that at the “sweet spot” obtained in the fundamental modal will shift if intensive vibration consuming a large amount of energy is excited. Therefore it is worthwhile to enhance the fundamental model by taking into account the influence of the potential vibration on the performance of the bat.

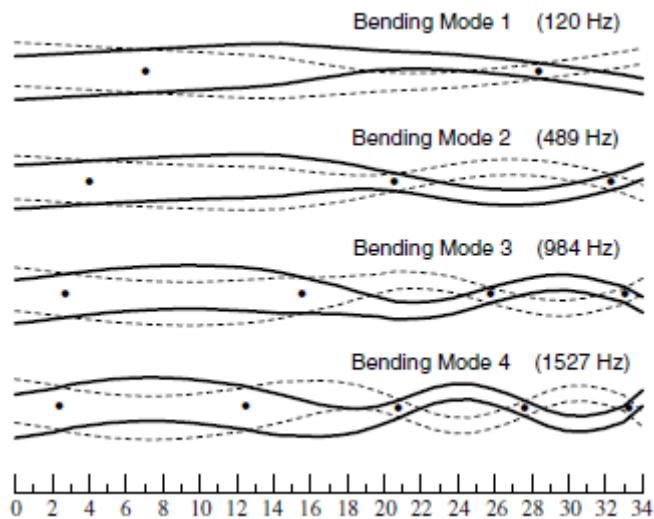


Figure 29. Illustration of bending vibration of the bat in different modes or frequencies [Russell,2000]

It is natural that the existence of the bending vibration will give rise to the change of v_f , which is determined by the properties of the bat and the baseball and the characteristics of the collision. According to the research work of Alan M. Nathan [2000], the expression of v_f can be modified by replacing e with e_{eff} , as stated below:

$$v_f = \left[\frac{e_{eff} - r}{1 + r} \right] v_{ball} + \left[\frac{e_{eff} + 1}{1 + r} \right] v_{bat}. \quad (37)$$

In this equation, e_{eff} is an effective coefficient of restitution for the collision of the ball with a flexible bat, contains all the dynamical information about the collision and has the desired properties that it reduces to e in the limit that vibration are neglected[Alan M. Nathan 2000]. It has been shown that e_{eff} depends strongly on the impact location yet weakly on the impact speed in Nathan's [2000] work.

Nathan investigated the baseball-bat collision from the perspective of vibration, two important results in his work are cited below:

- the vibration energy fraction as a function of impact location

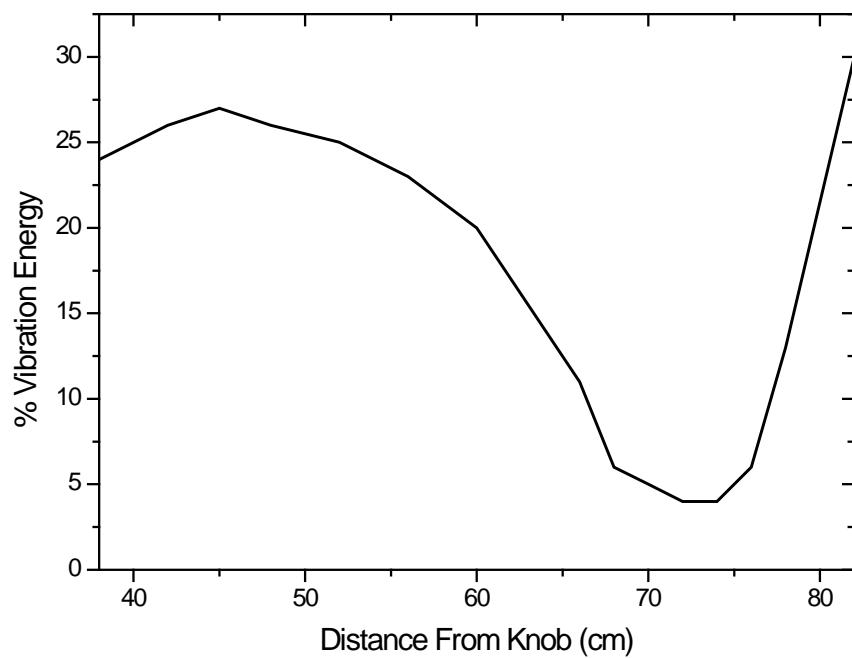


Figure 30. The distribution of vibration energy for an impact of a 90-mph ball on the standard wood bat, which has a CM speed of 54 mph and a rotational speed about the CM of 51 s^{-1} .

- the value of e_{eff} as a function of impact location

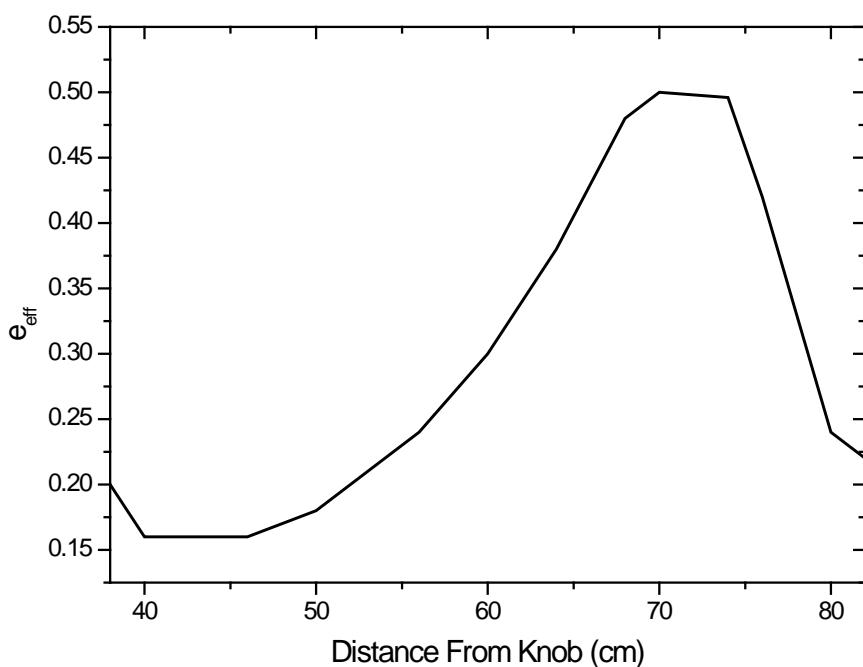


Figure 31. The plot of e_{eff} for an impact of a 90-mph ball on the standard wood bat, which has a CM

speed of 54 mph and a rotational speed about the CM of 51 s^{-1} .

From the two figures, we can draw a conclusion that in the region 68-74 cm the vibration energy is small and e_{eff} is relatively large. This conclusion also enhances the reasonability of the fundamental model that does not consider the possible bending vibration effect, since the vibration energy is about 5% of the total energy when collision occur at the “sweet spot”. It is quite clear that the “sweet spot” is not on the bat end.

7.3 Problems Review

The three proposed problems have been well answered in our model *simulation and results* part.

- Where is the “sweet spot”?

We find out that the “sweet spot” is located at a point about 15 cm from the end of the tip, which supports the empirical finding.

- How about the “corking” effects?

We find out that the “corking” effect mainly depends on the shape hollowed cylinder and the filing material density. Corking rubber (density larger than ash) enhances the “sweet spot” effect, while corking cork (density smaller than ash) weakens the “sweet spot” effect. For the same corking material, the enhancing/weakening effect is dependent upon the shape of the hollowed cylinder. This model explains the penalty MLB gave to some players using specially-corked bat.

- How about metal bats?

We find out that a well-designed metal bat will possesses good qualities of faster *BBS* and better controllability than that of the wood one (usually ash bat). Our analysis exhibits the reason that MLB prohibits metal bats.

7.4 Strengths

- Our concise model is in good agreement with the experiment data. That's to say our model is practical to some extent, especially the model possess clear mode.
- We have well analyzed the performance of a bat, and controllability along with “sweet spot” effect is separately analyzed. Thus the analysis of the calculation results will be closer to the actuality.
- Our model has been developed to show the interrelationship of geometrical attributes, mechanical attributes and the performance. This can help the design of bat, and simultaneously, we have a in-depth understanding on the ball-bat interaction.

7.5 Weaknesses

- We haven't elaborately modeled the transverse wave and hoop vibration; the error introduced is about 5% (see details in the foregoing bending mode part).
- We have used three empirical formulas and each has its own limitation and error.

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Appendix I

Mathematical Process for deducing equivalent BBCOR

In a reference frame where center of mass of the system (CM_{BSB} , consisted of ball, spring and bat) remains at rest, we define new symbols

$$v_i^c = v_i - v_c \quad v_f^c = v_f - v_c$$

$$V_i^c = V_i - v_c \quad V_f^c = V_f - v_c$$

in which the speed of the center of mass of the system

$$v_c = \frac{mv_i + MV_i}{m + M}.$$

The collision in CM_{BSB} reference frame can be divided into the following four procedures:

- v) The ball and bat (with spring respectively) approach each other;
- vi) The two springs contact and compress until the velocity (in CM_{BSB} reference frame) of bat and ball turn to be zero;
- vii) The ball and bat are accelerated by the springs respectively. In procedure ii and iii, the loss of energy happens through the effect between bat (ball) and spring.
- viii) The two springs is no longer at contact, and the ball and bat separate each other.

By definition of the coefficient of restitution (the ratio of the differences in velocities before and after the collision), we obtain the equivalent BBCOR (with trampoline effect and material effect inside) in a reference of home frame

$$e^* = \frac{v_f - V_f}{v_i - V_i}$$

Considering the conservation of momentum, we obtain

$$mv_i^c + MV_i^c = mv_f^c + MV_f^c = 0 \Rightarrow V_f^c = -\frac{m}{M}v_f^c, V_i^c = -\frac{m}{M}v_i^c.$$

Then the equivalent BBCOR becomes

$$e^* = \frac{v_f - V_f}{v_i - V_i} = \frac{v_f^c - V_f^c}{v_i^c - V_i^c} = \frac{\frac{m}{M}v_f^c + \frac{m}{M}v_f^c}{\frac{m}{M}v_i^c + \frac{m}{M}v_i^c} = \frac{v_f^c}{v_i^c}.$$

We may also understand the physical meaning of BBCOR in the way of energy

as following:

$$\therefore E_i = \frac{1}{2}m(v_i^c)^2 + \frac{1}{2}M(V_i^c)^2 = \frac{1}{2}m(v_i^c)^2(1 + \frac{m}{M})$$

$$E_f = \frac{1}{2}m(v_f^c)^2 + \frac{1}{2}M(V_f^c)^2 = \frac{1}{2}m(v_f^c)^2(1 + \frac{m}{M})$$

$\therefore e^{*2} = \frac{E_f}{E_i}$ means that the BBCOR² is the fraction of energy restored in system after collision.

The total energy in the BSB system at contact before collision is E_i . If no loss of energy happens when the kinetic energy transfers to the spring, the total energy fully converts to the potential energy in spring at the end of procedure ii.

Since the total momentum remains zero all times, the point where two springs contact remains at rest in BSB reference frame.

Therefore, if no loss of energy happens,

$$\begin{cases} k_{ball} \Delta x_{ball} = k_{bat} \Delta x_{bat} \\ \Delta x_{ball} + \Delta x_{bat} = \Delta x \end{cases} \Rightarrow \begin{cases} \Delta x_{ball} = \frac{k_{bat}}{k_{ball} + k_{bat}} \Delta x \\ \Delta x_{bat} = \frac{k_{ball}}{k_{ball} + k_{bat}} \Delta x \end{cases}$$

the energy stored in the spring attached on ball is

$$E_i^{ball} = \frac{1}{2}k_{ball} \Delta x_{ball}^2 = \frac{1}{2}k_{bat} \Delta x_{ball}^2 \left(\frac{k_{ball} k_{bat}}{(k_{ball} + k_{bat})^2} \right)$$

$$E_i^{bat} = \frac{1}{2}k_{bat} \Delta x_{bat}^2 = \frac{1}{2}k_{ball} \Delta x_{bat}^2 \left(\frac{k_{ball} k_{bat}}{(k_{ball} + k_{bat})^2} \right)$$

Considering conservation of total energy $E_{ball} + E_{bat} = E$ and the physical meaning of BBCOR in energy perspective $e^2 = E_f/E$, we obtain

$$E_f^{ball} = E_i^{ball} e_{ball}^2 = \frac{k_{bat}}{k_{ball} + k_{bat}} e_{ball}^2 E$$

$$E_f^{bat} = E_i^{bat} e_{bat}^2 = \frac{k_{ball}}{k_{ball} + k_{bat}} e_{bat}^2 E$$

Therefore, in the collision of ball and corked bat, the different part of Eq. 16 is as following

$$e^{*2} = \frac{E_f}{E_i} = \frac{E_f^{ball} + E_f^{bat}}{E_i} = \frac{k_{bat}}{k_{ball} + k_{bat}} e_{ball}^2 + \frac{k_{ball}}{k_{ball} + k_{bat}} e_{bat}^2.$$

where

e^{*2} the fraction of energy restored in ball and bat (kinetic energy) after collision;

$\frac{k_{bat}}{k_{ball} + k_{bat}}$ the fraction of initial energy stored in ball;

e_{ball}^2 the fraction of stored energy returned to kinetic energy of ball;

$\frac{k_{ball}}{k_{ball} + k_{bat}}$ the fraction of initial energy stored in bat;

e_{bat}^2 the fraction of stored energy returned to kinetic energy of bat.

Appendix II

The Stiffness of the Corked Bat's Shell

Trampoline effect can be observed in the hollowed bat [Russell, 2004]. We have augmented our model from fundamental model to DS model, by which we know how the relative stiffness between ball and bat " k_{ball}/k_{bat} " influences the values of BBS and v_f . However, going a further step, we will discuss, to what extent does hollowing affects " k_{ball}/k_{bat} ", and then *the equivalent BBCOR e**, as well as the "sweet spot" effect.

- Hoop Spring

The observed fundamental hoop vibration mode, which accounts for the majority of vibrant energy and is responsible for trampoline effect, has a frequency of about 1 kHz. It means that when the ball leaves the bat, taking place about 1 ms after the ball touches the bat [H. Brody, 1985], the fundamental hoop vibration mode has not been set up. In other words, at the moment that ball exit, it has not "seen" either the knob or tip of the bat. This fact indicates us that we can view the bat as a hoop spring.

The hoop stress constant k_{bat} can be identified by an empirical relationship

$$k_{bat} \propto \left(\frac{t}{R}\right)^3,$$

Where

t is the thickness of the shell;

R is the radius of the bat cross section;

- Calculation of an Instance

A typical wood bat has a cross section diameter of 2.3 in, and the removed cylinder has a diameter of 1 in; accordingly, the maximum and minimum k_{bat} proportion can be estimated by:

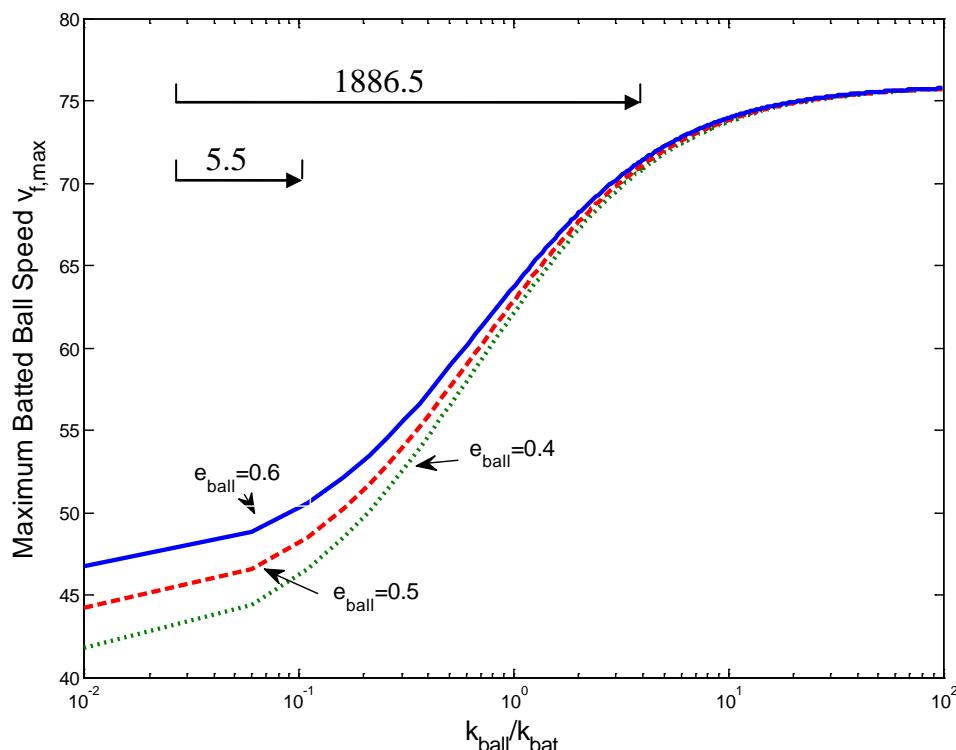
$$\frac{k^{max}}{k^{min}} = \left(\frac{t_{max}}{t_{min}}\right)^3 = \left(\frac{2.3}{1.3}\right)^3 \approx 5.5$$

$$\frac{k_{origin}^W}{k_{coarked}^W} = \left(\frac{t_{origin}^W}{t_{coarked}^W}\right)^3 = \left(\frac{2.3}{1.3}\right)^3 \approx 5.5$$

A typical aluminum bat can be extremely hollowed; practically, the rest hoop thickness of aluminum bat can be 1/7 of that of the wood one [Nathan, 2004]; accordingly, the maximum and minimum k_{hoop} proportion can be estimated by:

$$\frac{k^{max}}{k^{min}} = \left(\frac{t_{max}}{t_{min}} \right)^3 = \left(\frac{2.3}{1.3} \times 7 \right)^3 \approx 1886.5$$

$$\frac{k_{origin}^M}{k_{hollowed}^M} = \left(\frac{t_{origin}^M}{t_{hollowed}^M} \right)^3 = \left(\frac{2.3}{1.3} \times 7 \right)^3 \approx 1886.5$$



Based on the figure above, we can figure out that the trampoline effect of thin metal shell bat is quite significant while wood corked bat not.

Appendix III

Codes for MATLAB

```
% ****
% codes for drawing mesh and contour figures
% ****
[h,R]=meshgrid(0:0.0005:0.2);
R = R/10;
% ****
d=R*2; L=.84; a=.564; M=.885; rho1=450;rho2=1100;
% ****
J1=0.045+pi/48.* (rho1-670).* (d.^2.*h).* (.75*d.^2+h.^2)+pi/4.* (rho1-670).* d.^2.*h.* (L-a-h/2).^2+(M+pi/4.* (rho1-670).* d.^2.*h).* ((M.* (L-a)+pi/8.* (rho1-670).* d.^2.*h.^2)./(M+pi/4.* (rho1-670).* d.^2.*h)-L+a).^2;
J2=0.045+pi/48.* (rho2-670).* (d.^2.*h).* (.75*d.^2+h.^2)+pi/4.* (rho2-670).* d.^2.*h.* (L-a-h/2).^2+(M+pi/4.* (rho2-670).* d.^2.*h).* ((M.* (L-a)+pi/8.* (rho2-670).* d.^2.*h.^2)./(M+pi/4.* (rho2-670).* d.^2.*h)-L+a).^2;
% ****
%J5=-1/12*(3*670^2*h.^2*3.14^2.*R.^6+10*h.^4*670^2*3.14^2.*R.^4-12*h.^3*0.274*670^2*3.14^2.*R.^4-3*R.^4*0.885*670.*h.*3.14-13*h.^3*0.885*670*3.14.*R.^2+24*0.274*0.885*h.^2*670*3.14.*R.^2-12*R.^2*0.045*670.*h.*3.14-12*0.274^2*0.885*670*h.*3.14.*R.^2+12*0.045*0.885)./(-0.885+670*h.*3.14.*R.^2);
M1 = 0.885+(rho1 - 670)* 3.14* h.* R.^2;
M2 = 0.885+(rho2 - 670)* 3.14* h.* R.^2;
vil=(6032./(M1./0.02835+70.4)-5.4).*0.447;
vi2=(6032./(M2./0.02835+70.4)-5.4).*0.447;
JJ1=(J1+0.885*0.564).*54674.7;
JJ2=(J1+0.885*0.564).*54674.7;
w1=45.3.* (JJ1/16000).^ -0.30769;
w2=45.3.* (JJ2/16000).^ -0.30769;
b1=-(vi1+40)./w1+((vi1+40)./w1).^2+J1.* (0.145+M1)./(0.145.*M1)).^0.5 ;
b2=-(vi2+40)./w2+((vi2+40)./w2).^2+J2.* (0.145+M2)./(0.145.*M2)).^0.5 ;
vmax1=((1+0.564).* (vil+w1.*b1)-40.* (0.145./M1+0.145.*b1.^2./J1-0.564))./(1+0.145./M1+0.145.*b1.^2./J1);
vmax2=((1+0.564).* (vi2+w2.*b2)-40.* (0.145./M2+0.145.*b2.^2./J2-0.564))./(1+0.145./M2+0.145.*b2.^2./J2);
% drawing 1
figure(1);
meshc(h,R,vmax1);
```

```
figure(2);
%ma=max(max(vmax1));
%mi=min(min(vmax1));
C1=contour(h,R,vmax1,10);
clabel(C1);
% drawing 2
figure(3);
meshc(h,R,vmax2);
figure(4);
%ma=max(max(vmax2));
%mi=min(min(vmax2));
C2=contour(h,R,vmax2,10);
clabel(C2);
```