

Bumping for Dollars: The Airline Overbooking Problem

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Introduction

We construct a model that expresses the expected revenue for a flight in terms of the number of reservations, the capacity of the plane, the price of a ticket, the value of a voucher, and the probability of a person showing up for the flight. When values are supplied for every variable but the first, the function can be maximized to yield an optimal booking that maximizes expected revenue.

We apply the model to three situations: a single flight, two flights in a chain of flights, and multiple flights in a chain of flights. We conclude that fewer flights will increase the value of the penalty or voucher and thus decrease the optimal number of reservations. Heightened security also lowers the optimal number of reservations. An increase in passengers' fear decreases the probability that a person will show up for a flight and thus increases the optimal number of reservations. Finally, the loss of billions of dollar in revenue has no effect on the optimal value of reservations.

We model the probability of a given number of people showing up as a binomial distribution. We express the average expected revenue of a flight in terms of the number of bookings made.

Starting with the Single-Flight case, we derive a model and revenue function for a flight unaffected by previous flights. From this situation, we expand the model to the Two-Flight case, in which the earlier flight affects the number of people who show up for the later flight. We generalize the model even further to the number of people showing up depending on many previous flights.

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The Model

In each of the three situations modeled, we derive two formulas. The first, $P(k)$, describes the probability that k people show up for a flight. The second, $\text{Revenue}(b, c, r, x, p)$, describes the expected revenue for a flight as a function of the number of reservations. We verified these theoretical equations by a Monte-Carlo simulation.

For the Single-Flight Model:

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

$$\text{Revenue}(b, c, r, x, p) = \sum_{k=0}^{c+(b-c)} P(k) [r \min(k, c) - x \max(k - c, 0)].$$

For the Two-Flight Model:

$$P_2(k) = P_1(k) \left[1 - \sum_{i=c+1}^b P_1(i) \right] + \sum_{j=1}^{b-c} P_1(k-j) P_1(c+j),$$

$$\text{Revenue}_2(b, c, r, x, p) = \sum_{k=0}^{c+2(b-c)} P(k) [r \min(k, c) - x \max(k - c, 0)].$$

For the n -Flight Model:

$$P_n(k) = P_1(k) \left[1 - \sum_{i=c+1}^{c+(n-1)(b-c)} P_{n-1}(i) \right]$$

$$+ \sum_{j=1}^{(n-1)(b-c)} P_1(k-j) P_{n-1}(c+j),$$

$$\text{Revenue}_n(b, c, r, x, p) = \sum_{k=0}^{c+n(b-c)} P_n(k) [r \min(k, c) - x \max(k - c, 0)].$$

The variables are:

b = number of reservations (or bookings) per flight

c = plane capacity

r = price of a ticket

x = value of a voucher

p = probability that a booked passenger shows up for a flight

Given p , c , r , and x , the method finds the b that maximizes revenue.

Derivation of the Single-Flight Model

The binomial distribution applies to calculating the probability that a number of passengers shows up for a flight:

- The probability involves repeated events (each trial calculates the probability of one person showing up) with only two possible outcomes (either the person is a show or no-show).
- We assume that people's actions do not influence one another; each person's chance of showing up is independent of another person's chance. This is not true in reality, as people often travel in groups; but this a necessary and appropriate simplification.
- We assume that the probability of a person arriving remains constant for each person.

We use the binomial distribution to calculate expected revenue. Airlines overbook their flights, knowing that some people will not take the flight. Given a certain overbooking strategy b (i.e., the maximum number of reservations taken for a particular flight, with $b > c$, the capacity of the plane), the expected revenue is

$$\text{Revenue}(b, r, p) = \sum_{k=0}^{c+(b-c)} P(k) r \min(k, c).$$

The function is incomplete, however, because it does not penalize the airline for the consequences of overbooking. The airline usually provides bumped passengers with either an airline ticket voucher or a cash reimbursement, valued at x per bumped person:

$$\text{Revenue}(b, r, p) = \sum_{k=0}^{c+(b-c)} P(k) [r \min(k, c) - x \max(k - c, 0)].$$

When $k \leq c$, the x term is zero; when $k > c$, the airlines is penalized for having to bump people.

The booking decision b and the capacity c of the plane are fixed before the model begins. This model considers just one flight in a complex network of flights; it does not allow for the possibility that passengers are bumped from a previous flight, since it assumes that the only passengers are those who made a reservation for this particular flight. The model also applies to just one flight: If the number of passengers who show up is greater than the capacity of the plane, those bumped passengers receive a voucher and—with a wave of the magic wand of assumption—disappear. Finally, regardless of the flight's destination (Hawaii or Death Valley), we assume that there is enough demand to fill the predetermined number of bookings.

Since p is constant throughout our model, the Revenue function is really dependent only on the number of bookings, the capacity of the plane, the cost of a ticket, and the cost of the penalty.

Application of the Single-Flight Model

We set $p = .9$. Since b must be an integer, the revenue function is not continuous. Thus, the analytic method of maximizing of the function (namely, differentiating and setting the derivative equal to zero) cannot be applied. Instead, we use Maple 6.

After setting values for the probability, plane capacity, and ticket and voucher values, we evaluate the function at $b = c$, then increment b until a maximum for Revenue is found.

We used three plane capacities: 10, 30, and 100. The values of the ticket price r , the voucher x , and the arrival probability p are held constant at \$300, \$300, and .9 for the examples in **Table 1**.

Table 1.
Results for the Single-Flight Model.

Capacity	Optimal overbooking	Revenue	Bump probability
10	11	\$2,782	31%
30	33	\$8,598	35%
100	111	\$29,250	44%

The probabilities of bumping are larger than the industry frequency of about 20%. Worse, the model ignores all the problems created by these bumped passengers. The model is further weakened in light of the post-September 11 issues proposed by the problem. Among the four issues—fewer flights, heightened security, passengers' fear, and losses—this model can account only for increased passenger fear (indicated by a change in probability that a passenger shows up). Clearly this Single-Flight Model is not a proper solution to the airline-overbooking problem.

Derivation of the Two-Flight Model

The Two-Flight Model begins with updating both the probability and revenue functions. Unlike the Single-Flight Model, the new functions reflect the possibility that passengers bumped from one flight fill seats on the next. By this assumption, the probability function for the second flight, $P_2(k)$, changes, because k may now also be expressed as a combination of people ticketed for the second flight and bumped passengers from the first flight. Since the revenue

function depends on the probability function, it too must change.

$$\begin{aligned}
 P_2(k) &= Pr(k \text{ people show up for flight 2}) \\
 &= Pr(k \text{ regular passengers arrive})Pr(\text{no one bumped from flight 1}) \\
 &\quad + Pr(k-1 \text{ passengers arrive})Pr(1 \text{ passenger bumped}) + \cdots \\
 &\quad + Pr(k-j \text{ arrive})Pr(j \text{ passengers bumped}) + \cdots \\
 &\quad + Pr(k-(b-c) \text{ arrive})Pr(b-c \text{ passengers bumped}) \\
 &= P_1(k) \left[1 - \sum_{i=c+1}^b P_1(i) \right] + \sum_{j=1}^{b-c} P_1(k-j)P_1(c+j).
 \end{aligned}$$

A maximum of $b-c$ people can be bumped from flight 1, since at most b people show up for it and we assume that no passengers are carried over from any previous flight. The probability that 1 passenger is bumped from flight 1 is exactly the probability that $c+1$ people are present for it. Thus we have $Pr(j \text{ passengers bumped}) = P_1(c+j)$. As long as b , p , and c remain the same, the probability that new (prebooked, non-bumped) passengers arrive never changes; it is independent of the number of bumped passengers from a previous flight. (We assume that there is no way for a passenger to know how many people have been bumped onto his or her flight from a previous one.) Thus, $Pr(k-j \text{ regular passengers arrive})$ will always be computed by $P_1(k-j)$, our original probability function for the Single-Flight Model.

In the second summation of $P_2(k)$, the term $k-j$ could be negative for small k . If so, we define the probability of a negative number of people showing up from a previous flight to be 0 (empty seats on a flight cannot be filled by passengers from later flights!).

We now express the second revenue function in terms of the second probability function:

$$\text{Revenue}_2(b, c, r, x, p) = \sum_{k=0}^{c+2(b-c)} P(k) [r \min(k, c - x \max(k-c, 0))].$$

A passenger bumped from one flight is automatically booked on the next flight and seated before regular passengers, so as to have almost no chance of being bumped again. For the second flight, we assume that the number of people who show up is affected only by that flight and the previous flight, and that there is enough demand to fill the predetermined number of bookings.

The summation now has $c+2(b-c)$ as its maximum value. The second flight must not only account for b passengers but must also account for the number of people possibly bumped from the first flight.

Application of the Two-Flight Model

By introducing a second flight, we more accurately model the situation. The optimal overbooking strategy and maximum revenue should either remain the

same or slightly decrease.

Using the Revenue function for the Two-Flight Model, we now calculate maximum revenue and the associated overbooking strategy for the same plane capacities as for the Single-Flight Model. Again, the values of the ticket price r , the voucher x , and the arrival probability p are held constant at 300, 300, and .9. The results are in **Table 2**.

Table 2.
Results for the Two-Flight Model.

Capacity	Optimal overbooking	Revenue	Bump probability
10	11	\$2,745	34%
30	33	\$8,551	42%
100	111	\$29,107	57%

In each case, the optimal booking strategy is the same as the Single-Flight Model, but the maximum revenues is lower, and bump probability is higher. Since both flights are overbooked, the probability that someone is bumped should only increase.

The n -Flight Model

We generalize to n flights. We allow each flight to be influenced by the $(n - 1)$ flights before it. We still assume that a passenger bumped from one flight is given preferential seating on the next. However, giving seats to bumped passengers who are already at the airport decreases the number of seats for pre-booked passengers. The n -flight model allows this domino effect of bumping to ripple through $n - 1$ successive flights. As n gets large, our model becomes a better and better approximation of the real case, in which every flight is affected by many previous flights. Our probability function becomes recursive:

$$\begin{aligned}
 P_n(k) = P_1(k) & \left[1 - \sum_{i=c+1}^{c+(n-1)(b-c)} P_{n-1}(i) \right] \\
 & + \sum_{j=1}^{(n-1)(b-c)} P_1(k-j) P_{n-1}(c+j), \\
 \text{Revenue}_n(b, c, r, x, p) = & \sum_{k=0}^{c+n(b-c)} P_n(k) [r \min(k, c) - x \max(k - c, 0)].
 \end{aligned}$$

For the first summation, zero people show up from the previous flight, meaning that there are enough seats for everyone on that flight and anyone bumped from a previous flight. If the total possible number of people who can show up to the current flight is $b + (n - 1)(b - c)$ (as is explained in a moment),

then the total number of people who can show up for the previous flight must be $b + (n - 2)(b - c)$, which we use as the upper limit of the summation.

For the second summation, we use $(n - 1)(b - c)$ instead of $(b - c)$, since now there can be at most $(n - 1)(b - c)$ passengers bumped from flight $n - 1$. This upper bound for bumped passengers can be proved by mathematical induction. [EDITOR'S NOTE: We omit the authors' proof.]

The revenue function for the 2-flight model can also be extended to n flights in a straightforward way. Note that at most $n(b - c)$ people can be bumped from the n th flight. We have:

$$\text{Revenue}_n(b) = \sum_{k=0}^{c+n(b-c)} P_n(k) [r \min(k, c) - c \max(k - c, 0)].$$

We now consider booking strategies to optimize revenue.

Computation of the n -Flight Model

The Recursive Method

We can create documents in Maple to compute the probability and revenue functions. To compute $\text{Revenue}_n(b)$, we must evaluate $P_n(k)$ a total of $b + (n - 1)(b - c)$ times. In turn, $P_n(k)$ must evaluate $P_{n-1}(k)$ a total of $(2n - 1)(b - c)$ times, $P_{n-1}(k)$ must evaluate $P_{n-2}(k)$ a total of $[2(n - 1) - 1](b - c) = (2n - 3)(b - c)$ times, and so on. Thus, without even accounting for all the evaluations of $P_1(k)$ in each iteration, we make

$$\begin{aligned} & [b + (n - 1)(b - c)](2n - 1)(b - c)(2n - 3)(b - c) \cdots [2n - (2k + 1)](b - c) \cdots (1)(b - c) \\ & = [b + (n - 1)(b - c)] \frac{(2n - 1)!}{2(n - 1)!} (b - c)^{n-1} \end{aligned}$$

function calls. With $b = 105$ and $c = 100$, $\text{Revenue}_2(k)$ requires 1650 function calls, $\text{Revenue}_3(k)$ requires 86,250 calls, and $\text{Revenue}_4(k)$ requires more than 6.3 million function calls. The computation time is proportional to the number of function calls: $\text{Revenue}_2(105)$ takes less than 1 s, $\text{Revenue}_3(105)$ takes 13 s, and $\text{Revenue}_4(105)$ takes 483 s.

Of course, this is a very inefficient method. A more efficient method would be to store all probability values in an array, beginning with the values for $P_1(k)$ and working upwards to $P_n(k)$. However, Maple makes array manipulation difficult. Instead, we turn to another method.

[EDITOR'S NOTE: Mathematica (and perhaps Maple too) provides an easy-to-use capability for computation of such probabilities via dynamic programming. For an example of its use, see "Farmer Klaus and the Mouse," by Paul J. Campbell, *The UMAP Journal* 23 (2) (2002) 121–134.

Monte Carlo Simulation

We develop a Monte Carlo computer simulation coded in Pascal that runs the n -flight model numerous times and determines the average revenue for a large number of trials. Instead of obtaining precise probabilities using the functions developed above, we flip a (electronic) weighted coin to determine whether each individual passenger shows up for the flight. We tell the program how many trials to run, give it values for n , p , c , r , and x and tell it the largest value of b to check. The program begins with $b = c$. It flips numerous weighted coins to determine how many passengers show up for the first flight. It bumps any excess passengers to the second flight and flips coins again to see how many prebooked passengers arrive. The excess is bumped to the third flight and the process continues until the n th flight. Revenue is evaluated by adding an amount equal to the ticket price for each passenger who flies and deducting a penalty for each passenger who is bumped. The program iterates for successive values of b until it reaches the preassigned upper bound.

The output includes, for each b value, the mean revenue over all trials and the corresponding percentage standard error. Percentage standard error was usually less than 2% and often less than 1%.

Optimization Strategies for the n -Flight Model

We will never earn more than the ticket price (r) times the number of seats (c), so the gain from overbooking is limited—but the possible costs are not. At some point, the costs of overbooking outweigh the benefits; there should be a unique maximum for revenue.

To find the maximum revenue, we evaluate the revenue function at different booking values, beginning with $b = c$, until we find a b with $\text{Revenue}(b - 1) < \text{Revenue}(b)$ and $\text{Revenue}(b + 1) < \text{Revenue}(b)$. This will be our b_{opt} .

The obvious booking strategy is to book every flight with b_{opt} passengers. While this method maximizes flight revenue, it yields a high percentage of flights with bumped passengers. For a plane with 100 seats, the maximum revenue occurs at $b = 108$, with 34% of flights bumping passengers. Because our model does not account for changes in demand due to the airlines' behavior, this might not be the truly optimal value of b in the long run. Bumping large numbers of passengers will drive customers away; decreased demand will depress the price that we can charge and we reduce revenue in the long term. Similarly, an especially low percentage of bumped flights may increase demand, allow us to raise prices, and increase revenue. Thus, our model accounts only for short-term effects, not long-term ones.

Moving away from maximum revenue lowers expected revenue by a small amount but decreases the bump probability by a large amount. For convenience, we set both the price and penalty to \$1, to avoid large numbers. While \$1 is unrealistic, the value does not change the optimal booking strategy from the case where both price and penalty are both \$300, because it is the ratio of

price to penalty—and not their actual values—that changes the optimal booking. Our example considers a 50-flight sequence of planes with capacity 100 each; if everyone showed up and there was no overbooking, the revenue would be \$5,000. At the optimal $b = 108$ for $p = .9$, the expected revenue is \$4,806 with bump probability of 33%. If we move down just 1 to $b = 107$, the revenue is \$4,791 and the bump probability drops to 21%.

Table 3.

Results of simulation: for each number for bookings, 100 trials with 50 flights per trial.

Bookings	Revenue	% Bump	Delta(%Bump)	Delta(%Rev)	D(Bmp/D(Rev)
100	\$4,501	0.00%	0.00%	0.02%	0.00
101	\$4,539	0.02%	0.02%	0.84%	0.02
102	\$4,589	0.04%	0.02%	1.09%	0.02
103	\$4,631	0.04%	0.58%	0.92%	0.63
104	\$4,677	0.62%	2.24%	0.97%	2.30
105	\$4,722	2.86%	5.66%	0.97%	5.82
106	\$4,758	8.52%	12.50%	0.76%	16.45
107	\$4,791	21.02%	12.64%	0.68%	18.64
108	\$4,807	33.66%	22.10%	0.34%	65.67
109	\$4,800	55.76%	17.04%	−0.15%	−115.28

We adopt as a criterion to compare two values of b the ratio of the relative change in the bump probability and the relative change in revenue:

$$\frac{\frac{\Delta P_{\text{bump}}}{P_{\text{bump}}}}{\frac{\Delta \text{Revenue}}{\text{Revenue}}} = \frac{\Delta(\% \text{Bump})}{\Delta(\% \text{Revenue})}.$$

The process goes: A maximum revenue is found, along with its high bump probability. The optimizer now considers a lower value of b and looks at the ratio of the change in the bump probability to the change in revenue. If this ratio is above a certain value k , the optimizer accepts the lower b . The optimizer continues to do this until the ratio is no longer greater than the constant. In **Table 3**, with $k = 20$, the new optimum b would be 107, because the ratio 18.64 is not greater than $k = 20$.

Table 4 shows three different optimization values for different plane capacities.

Application of the n -Flight Model

The problem specifically mentions four issues to be addressed by our model: fewer flights, heightened security, passengers' fear, and revenue losses.

Why are airlines offering fewer flights? If the airlines had kept offering the same number of flights, the question of an optimal overbooking strategy would be moot, because the planes would not fill. The huge drop in demand

Table 4.

Optimal bookings using different criteria ($p = .9$, $r = 1$, $x = 1$). For each number for bookings, 100 trials with 50 flights per trial.

c	Maximum revenue			$k = 20$			$k = 1$		
	b	Rev	$P_{\text{bump}}(\%)$	b	Rev	$P_{\text{bump}}(\%)$	b	Rev	$P_{\text{bump}}(\%)$
10	10	450	0	10	450	0	10	450	0
30	32	1390	45	31	1385	12	30	1350	0
50	54	2360	50	53	2360	25	51	2290	0.8
100	109	4810	50	107	4790	19	104	4670	0.7
150	163	7270	37	161	7220	14	157	7070	0.3
200	219	9740	47	215	9660	9	211	9490	0.4
280	307	13690	46	303	13610	13	299	13450	1.3

has reduced supply but could also result in slashed prices. Since the value of the compensation involuntarily bumped ticket-holders is tied to the ticket price (though with a ceiling), changes in ticket prices should affect the optimum booking level little if at all.

However, the fewer flights, the longer people who are denied boarding must wait for the next flight; being denied boarding is less convenient. Since compensation is usually offered in a kind of auction to induce voluntary relinquishing of seats, the airline will have to offer more. Therefore, longer delays between flights will increase the ratio of compensation amount to ticket price, tending to decreasing the optimal booking level.

How do heightened security measures affect our model? They mean more security checks, longer lines, longer waits, and an increased chance of missing a flight, particularly a connecting flight. Unfortunately, people who miss their connecting flight and thus are guaranteed a spot on the next flight are not included in our model explicitly; but they do have an implicit effect. If more people miss connecting flights, they put additional stress on the system: They increase the chance that the next and subsequent flights will have too many people. Therefore, in our booking strategy, we want a low bump probability. To attain it, we should decrease the ratio k , which decreases optimal booking level b .

Passenger fear leads not only to decreased demand (which we have already considered above) but also to a decreased probability p of a passenger showing up, which in turn increases the optimal booking level b .

However, the hardest to deal with is the huge revenue loss. Less profitable airlines may fold; but presumably if there is excess demand, other airlines will either add flights or raise the price. Hence, though the huge financial loss may change the industry as a whole, it doesn't affect the optimal booking strategy. It merely leads to fewer flights (already addressed) and may change prices (which we argued would have no effect).

We summarize these effects in **Table 5**.

Table 5.
Effects of post-September 11 factors.

Factor	Direct effect	Effect on optimal booking level
Fewer flights	$g \uparrow$	$b_{\text{opt}} \downarrow$
Heightened security measures	$\Delta P_{\text{bump}} / \Delta \text{Revenue} \downarrow$	$b_{\text{opt}} \downarrow$
Passenger fear	$p \downarrow$	$b_{\text{opt}} \uparrow$
Financial losses	—	—

Verification and Sensitivity of the Model

Since at least 100 trials were used per calculation, the Central Limit Theorem assures us that the distribution of the sample mean approximates well a normal curve and we can be 95% confident that the true value we are approximating is within two standard errors of the sample mean. Often this means we cannot be completely sure of the optimal b , because the maximum revenue is within two standard errors of the revenues of the values for b immediately above and below.

Convincing for us is that for small n and small c , the simulation provides values very close to those from the exact solutions processed in Maple. Because of the agreement, we are confident that our simulation is coded correctly and that the simulations are accurate, even for higher n and c .

That the simulation may be off by 1 for the optimal value of b is not much of a problem. For large c , though Maple may be too slow to calculate over a large range of values for b , Maple can be used to spot-check the value of b from the simulation, along with the ones immediately above and below.

In fact, we need not be much concerned about $n \geq 5$. A bumped person affects a second flight and may also affect a third and possibly a fourth flight. But the effect diminishes, so while the effect on flights close by cannot be discounted, ignoring her effect on a tenth flight does no great damage.

One might expect that changes in both booking level b and capacity c would significantly change the behavior of the model. But around b_{opt} , the revenue curve is fairly flat. For example, for $n = 50$, $c = 100$, and $r = x$, using $b_{\text{opt}} + 1$ instead of b_{opt} decreases revenue by only 0.12%, whereas adopting $b_{\text{opt}} - 1$ instead decreases revenue by only 0.21%. This insensitivity is important because one of our more limiting assumptions is constant p . Since slightly changing b only slightly changes revenue, the effect of varying p should not be too detrimental.

What is sensitive to changes in b is the bump probability. Using the same example as before, moving to $b_{\text{opt}} + 1$ increases the bump probability by 15 percentage points, while moving to $b_{\text{opt}} - 1$ decreases it 11 percentage points. While the smallest percentage changes in revenue are grouped around b_{opt} , the largest percentage changes in bump probability are grouped there.

Strengths, Weaknesses, and Extensions

Strengths

- The strong correspondence between the Maple calculations and the data from the simulation is quite heartening.
- Around b_{opt} , the revenue is insensitive compared to the bump probability. Variations on the n -flight mode provide a small range of near-optimal bs with similar results for revenue and a fairly wide range bump probability. The range allows an airline a choice.

Weaknesses

- The most obvious defect of our model is that many overbooking strategies are in use—and none of the them is ours! Our model is very restrictive because it assumes a constant booking strategy, as well as constant levels of p and c . In reality, most airlines use a dynamic system in which the overbooking level is not constant but instead is varied based on conditions that change from day to day and flight to flight.
- We replace the nation's vastly complicated network of intermeshing flights with a single flight path.
- We simplify the oligopoly of airlines to a single airline.
- We do not account for no-shows such as missed connections that are the fault of the airline or due to circumstances beyond its control (e.g., weather). In such circumstances, a flight's chance of being full is influenced by previous flights even if there is no overbooking.
- In assuming a binomial distribution, we assume people do not travel in groups, and thus their showing up are independent events.

Potential Extensions

- The bump probability could affect revenue in a way that we have not allowed for, namely, in terms of price. An airline that consistently offers better service should be able to charge a higher price. A way to incorporate this effect is to make price a function of bump probability, perhaps inversely proportional to it.
- It might be desirable to make the compensation x a function of the percentage of people that must be excluded from the plane. If 50% of the ticket-holders had to be excluded, then the incentives would have to be greater than if only 5% had to be excluded. At some point the airline would stop raising the

incentive and resort to involuntary denied boarding, but these would also have costs resulting from customer satisfaction. One could experiment with setting x equal to some constant times the ratio of those to be bumped, $m - c$, to the total number of people m .

- The probability function could easily be generalized to variable p ; in that case, $P(k)$ would become $P(k, p_n)$. The equation could be generalized to the planes having different values of c and b by changing the upper limit of the summations from $(n - 1)(b - c)$ to $\sum_{i=1}^{n-1} (b_i - c_i)$.

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Memo

To: CEO, TopFlight Airways
 From: Models R Us
 Re: Optimal Overbooking Strategy

Dear Sir/Madam:

We have heard of your company's financial hardships in the wake of September 11. We offer you our assistance. We are a team of students who have dedicated four intense days to understand the problem of airline overbooking. While many have been working on this problem for years, we feel our approach will give your company the extra edge you are seeking.

Because only 90% of passengers arrive for their scheduled flights, an overbooking strategy is necessary to maximize revenue. However, there is a penalty for overbooking. As you know, airlines offer vouchers and other incentives to

passengers to entice them to give up their seats. The airline is also responsible for finding bumped passengers a later flight.

Our model incorporates these features. We consider the effect on a given flight of any number of preceding flights. If too many passengers arrive from a previous flight, they can set off a domino effect; when these passengers are rescheduled on a later flight, they increase the chance that this flight, too, will be overbooked.

Our model allows you to combat this effect by finding the optimal booking level for a plane of a given capacity. We did computations for the model in two different ways: once using the mathematical software package Maple and again using a Monte Carlo simulation developed in Pascal. We found the values for these two computational approaches to be in very close agreement.

We also allow for the fact that maximizing revenue is not enough. If you maximize your revenue now but bump too many passengers, you could find demand for your services decreasing. You could be forced to charge a lower price, and your revenue might decrease in the long run. We offer you a way to establish a trade-off between revenue and percentage of flights with bumped passengers. You tell us how important it is to you to have few bumped flights, and we can tell you how many passengers to book.

Even using three different optimization strategies to account for the effects of fluctuating demand, we find that optimal values fall in a very narrow range. For a 100-seat plane, this range is 104 to 108.

We also evaluated the effect of the September 11th crisis on the airline industry. Our model predicts that, with a decreased number of flights, you should decrease the level of overbooking. If security delays many passengers from reaching their flights on time, you should also decrease the number of bookings. Increased passenger fear will decrease the probability that passengers show up for their flights, so in this case you should increase your booking number.

We have given you only a taste of what our model can do. We hope you will agree that contracting for our services will be of the highest benefit to your esteemed company.

Sincerely,

Models R Us