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Safe Landings

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Abstract

We examine the physical principles of stunt crash pads made of corrugated cardboard boxes and build a mathematical model to describe them. The model leads to a computer simulation of a stunt person impacting a box catcher. Together, the simulation and model allow us to predict the behavior of box catchers from physical parameters and hence design them for maximum safety and minimum cost.

We present two case studies of box-catcher design, a motorcyclist landing after jumping over an elephant and David Blaine's televised Vertigo stunt. These demonstrate the ability of our model to handle both high-speed impacts and large weights. For each case, we calculate two possible box-catcher designs, showing the effects of varying design parameters. Air resistance is the dominant force with high impact speeds, while box buckling provides greater resistance at low speeds. We also discuss other box-catcher variations.

Basic Concept

Requirements

A falling stunt person has a large amount of kinetic energy,

$$U_{\mathrm{s}} = \tfrac{1}{2}\, m_{\mathrm{s}} u_{\mathrm{s}}^2 .$$

To land safely, most of it must be absorbed by a catcher before the performer hits the ground. Therefore, following Newton's Second Law, the catcher must exert a force to decelerate the performer,

$$F = m\,\frac{du_{\mathrm{s}}}{dt}.$$

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The total energy absorbed should equal the performer's initial energy (kinetic and potential),

$$\int F\,dz = U_{s_0} + V_{s_0}.$$

However, the catcher itself cannot exert too large a force or it would be no better than having the performer hit the ground in the first place. We therefore set a maximum force, F_{max}. The smaller we make F, the larger (and more expensive) the box catcher has to be. Therefore, to save both money and life, we would like to have

$$F \approx (1-\delta)F_{\text{max}},$$

where δ is a safety margin ($0 < \delta < 1$).

The Box Catcher

A box catcher consists of many corrugated cardboard boxes, stacked in layers, possibly with modifications such as ropes to keep the boxes together or inserted sheets of cardboard to add stability and distribute forces. When the stunt person falls into the box catcher, the impact crushes boxes beneath. As a box collapses, not only does the cardboard get torn and crumpled, but the air inside is forced out, providing a force resisting the fall that is significant but not too large. As the performer passes though the layers, each layer takes away some kinetic energy.

Modeling the Cardboard Box

We examine in detail the processes involved when a stunt person vertically impacts a single cardboard box. This analysis allows us to predict the effect of varying box parameters (shape, size, etc.) on the amount of energy absorbed by the box.

Assumptions: Sequence of Events

Although the impact involves many complex interactions—between the performer's posture, the structure of the box, the air inside the box, the support of the box, the angle and location of impact, and other details—modeling thin-shell buckling and turbulent compressible flow is neither cost-effective for a movie production nor practical for a paper of this nature. We therefore assume and describe separately the following sequence of events in the impact.

1. A force is applied to the top of the cardboard box.

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2. The force causes the sides of the box to buckle and lose structural integrity.
3. Air is pushed out of the box as the box is crushed.
4. The box is fully flattened.

We now consider the physical processes at play in each of these stages.

Table 1.
Nomenclature.

Property	Symbol	Units
Potential energy density	v	J/m^3
Young's modulus	Y	Pa
Strain in box walls	ϵ	none
Stress in box walls	S	Pa
Tensile strength	T_S	Pa
Total kinetic energy	U	J
Total potential energy	V	J
Volume of cardboard in box walls	$\mathcal{V}$	m^3
Distance scale over which buckling is significant	ΔH	m
Width of box	w	m
Thickness of box walls	τ	m
Height of box	ℓ	m
Surface area of the top face of the box	A	m^2
Proportion of top face through which air escapes	α	—
Velocity of stunt person	u_s	m/s
Mean velocity of expelled air	u_a	m/s
Mass of stunt person (and vehicle, if any)	m	kg
Density of air	ρ	kg/m^3
Acceleration due to gravity, $g = 9.8$ m/s^2	g	m/s^2

Stage 1: Force Applied

A force $F(t)$ is applied uniformly over the top surface. The walls of the box expand slightly, allowing the box to compress longitudinally (along the direction of the force). While this applied force is small enough (less than the force necessary to cause buckling), the box absorbs little of the force and transmits to the ground (or next layer of the box catcher) a large fraction of the applied force. The applied force increases until it is on the order of the buckling force $(F(t) \sim F_B)$. At this point, the box begins to buckle.

Stage 2: Box Buckles

The walls crumple, the box tears, and the top of the box is pressed down. Once the box has lost structural integrity, although the action of deforming the box may present some resistance, the force that the box itself can withstand is greatly diminished.

Consider the pristine box being deformed by a force applied to its upper face. To counteract this force, the walls of the box expand in the transverse

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directions (perpendicular to the force), creating a strain in the walls. For a given strain ϵ, the potential energy density v stored in the walls of the box is

$$v = \tfrac{1}{2} Y \epsilon^2 + \mathcal{O}(\epsilon^4),$$

where Y is the Young's modulus for boxboard. For this calculation, the effects of longitudinal (along the direction of the force) contractions in the box walls are negligible compared to the transverse expansion.

A stress S is created in the box walls as a result of the battle between the wall's pressure to expand and its resistance to expansion,

$$S = Y\epsilon + \mathcal{O}(\epsilon^2).$$

Only a small strain ($\epsilon \ll 1$) is necessary to cause the box to buckle, so we neglect higher-order terms. There is a point where the box stops expanding and begins to give way to the increasing stress placed on it. When the stress in the box walls reaches its tensile strength T_{S}, it loses its structural integrity and the box bursts as it continues to buckle. Thus, we can set a limit on the maximum strain that the box walls can endure,

$$\epsilon_{\mathrm{max}} = \frac{T_{\mathrm{S}}}{Y}.$$

The typical tensile strength of cardboard is on the order of a few MPa, while the Young's modulus is on the order of a few GPa. Thus, we may assume that $\epsilon \ll 1$. The maximum energy density allowed in the walls of the box is now

$$v_{\mathrm{max}} = \tfrac{1}{2} Y \epsilon_{\mathrm{max}}^2 = \frac{1}{2} \frac{T_{\mathrm{S}}^2}{Y}.$$

The total energy stored in the walls just before the box bursts is

$$V_{\mathrm{max}} = \frac{1}{2} \frac{T_{\mathrm{S}}^2}{Y} \mathcal{V},$$

where $\mathcal{V}$ = (lateral surface area) $\times$ (thickness) is the volume of cardboard in the box walls. If we assume that the force to deform the box is constant over the deformation distance, then by conservation of energy, we have

$$\frac{dU_{\mathrm{s}}}{dt} + \frac{dV_{\mathrm{s}}}{dt} = -\frac{dU_{\mathrm{box}}}{dt} = -\frac{dU_{\mathrm{box}}}{dt}\frac{dz}{dt} = -\frac{\Delta U_{\mathrm{box}}}{\Delta H}\frac{dz}{dt} = \frac{u_{\mathrm{s}}}{2\Delta H}\frac{T_{\mathrm{S}}^2}{Y}\mathcal{V},$$

where ΔH is the change in height of the box over the course of the buckling process. Since

$$m_{\mathrm{s}} u_{\mathrm{s}} \frac{du_{\mathrm{s}}}{dt} = \frac{dU_{\mathrm{s}}}{dt},$$

the change in velocity of the falling stunt person due to buckling the box is

$$\frac{du_s}{dt} = \frac{1}{2m_{\mathrm{s}}\Delta H}\frac{T_{\mathrm{S}}^2}{Y}\mathcal{V}. \tag{1}$$

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To determine ΔH, we assume an average force $F \approx F_B$, where F_B is the buckling force of the cardboard walls. The **Appendix** shows that the buckling force is related to the Young's modulus and other physical parameters by

$$F_B = Y \frac{\pi^2}{12} \frac{w\tau^3}{\ell^2}. \tag{2}$$

In this case, $\mathcal{V} = 4\ell w\tau$ for a square box. Then ΔH can be estimated by

$$\begin{aligned} F_B \Delta H &= \frac{1}{2} \frac{T_S^2}{Y} \mathcal{V}, \\ Y \frac{\pi^2}{12} \frac{w\tau^3}{\ell^2} \Delta H &= 2\ell w\tau \frac{T_S^2}{Y}, \\ \Delta H &= \frac{24}{\pi^2} \left(\frac{T_S}{Y}\right)^2 \left(\frac{\ell}{\tau}\right)^2 \ell. \end{aligned}$$

For a typical box, ΔH is on the order of a few centimeters.

Stage 3: Box is Crushed

Without structural integrity, the box is crushed by the applied force. However, in crushing the box, the air inside must be pushed out.

Let A be the surface area of the top of the box. We make the ad hoc assumption that when the box buckled, it is torn such that air can escape through an area αA. Provided $\alpha \sim \mathcal{O}(1)$, we can assume incompressible flow, since the area of opening in the box is of the same order of magnitude as the area being pushed in. By conservation of mass, we obtain the the velocity of the air moving out of the box in terms of α and the velocity of the stunt person:

$$-u_{\mathrm{s}} A = u_{\mathrm{a}}(\alpha A) \qquad \Longrightarrow \qquad u_{\mathrm{a}} = -u_{\mathrm{s}}/\alpha.$$

Using conservation of energy, we equate the change in energy of the stunt person and the air leaving the box,

$$\frac{dU_{\mathrm{s}}}{dt} + \frac{dV_{\mathrm{s}}}{dt} = -\left(\frac{dU_{\mathrm{a}}}{dt} + \frac{dV_{\mathrm{a}}}{dt}\right).$$

The potential energy of the air does not change significantly as it is ejected, so the energy equation simplifies to

$$\frac{dU_{\mathrm{s}}}{dt} + \frac{dV_{\mathrm{s}}}{dt} = -\frac{dU_{\mathrm{a}}}{dt}. \tag{3}$$

The energy gain of air outside the box is due to air carrying kinetic energy out of the box:

$$\frac{dU_{\mathrm{a}}}{dt} = \frac{1}{2} \frac{dm_{\mathrm{a}}}{dt} u_{\mathrm{a}}^2 = \tfrac{1}{2} \left(\rho\alpha A u_{\mathrm{a}}\right) u_{\mathrm{a}}^2, \tag{4}$$

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while the energy loss of the stunt person is from deceleration and falling:

$$\frac{dU_{\mathrm{s}}}{dt} + \frac{dV_{\mathrm{s}}}{dt} = m_{\mathrm{s}} u_{\mathrm{s}} \frac{du_{\mathrm{s}}}{dt} + m_{\mathrm{s}} g u_{\mathrm{s}}. \tag{5}$$

Combining **(3–5)**, substituting for u_{a}, and rearranging, we obtain

$$\frac{\mathrm{d}u_{\mathrm{s}}}{\mathrm{dt}} = \frac{1}{2} \frac{\rho A}{m_{\mathrm{s}}} \frac{u_{\mathrm{s}}^2}{\alpha^2} - g. \tag{6}$$

Stage 4: Box Is Flattened

Once the cardboard box is fully compressed and all the air is pushed out, we assume that the box no longer has any effect on the stunt person.

Summary

In stages 1 and 4, the box is essentially inert—little energy goes into it. Therefore in our mathematical description, we ignore these and concentrate on stages 2 and 3. Stage 2 occurs in the top ΔH of box deformation, while stage 3 occurs in the remainder of the deformation. Combining **(1)** and **(6)**, we get

$$\frac{du_{\mathrm{s}}}{dt} = \begin{cases} \dfrac{1}{2m_{\mathrm{s}}\Delta H} \dfrac{T_{\mathrm{S}}^2}{Y} \mathcal{V}, & |z| < |\Delta H|; \\[2ex] \dfrac{1}{2} \dfrac{\rho A}{m_{\mathrm{s}}} \dfrac{u_{\mathrm{s}}^2}{\alpha^2} - g, & |z| \geq |\Delta H|, \end{cases} \tag{7}$$

where z is measured from the top of the box.

Modeling the Box Catcher

Cardboard Properties

We assume that each cardboard box is made of corrugated cardboard with uniform physical properties, using the data shown in **Table 2**.

Table 2.

Properties of corrugated cardboard [Bever 1986].

Property	Symbol	Value
Tensile strength	T_S	12.2 MPa
Thickness	τ	5 mm
Young's modulus	Y	1.3 GPa

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Assumptions

- Layers of boxes are comprised of identical boxes laid side-by-side over the entire area of the box catcher.
- The box catcher is large enough compared to the stunt person that edge effects are negligible.
- The boxes are held together so that there is no relative horizontal velocity between them.
- Loose cardboard is placed between layers of cardboard boxes, so that any force transmitted through the top layer of boxes is well distributed to lower boxes, ensuring that only one layer is crushed at a time. In other words, we treat each layer of boxes independently and the box catcher as a sequence of layers.

Equations of Motion

Since each layer is independent, the equations of motion for the box catcher look similar to the equation of motion for a single box **(7)**. The equations depend on the dimensions of the boxes in each level, so we solve numerically for the motion of the stunt person. At each level of the box catcher, the performer impacts several boxes (approximately) at once. The number of boxes that act to decelerate the stunt person is the ratio A_s/A of the performer's cross-sectional area to the surface area of the top face of a cardboard box. Thus, we alter the equations of motion by this ratio, getting

$$\frac{du_{\mathrm{s}}}{dt} = \begin{cases} \dfrac{1}{2m_{\mathrm{s}}\Delta H}\dfrac{T_{\mathrm{S}}^2}{Y}\,\mathcal{V}\,\dfrac{A_{\mathrm{s}}}{A} & |z - z_{\text{top}}| < |\Delta H|; \\[2ex] \dfrac{1}{2}\dfrac{\rho A_{\mathrm{s}}}{m_{\mathrm{s}}}\dfrac{u_{\mathrm{s}}^2}{\alpha^2} - g, & |z - z_{\text{top}}| \geq |\Delta H|, \end{cases} \tag{8}$$

where

z is the vertical distance measured from the top of the stack,

z_{top} is the value of z at the top of the current box, and

A_{s} is the cross sectional area of the stunt person.

Now, given a suggested stack of boxes, we can integrate the equations of motion to see whether or not the stack successfully stops the falling stunt person, and if so, where in the stack.

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Analysis

Our model allows us to predict a stunt person's fall given parameters for the box catcher. Now we would like to find analytic results to guide box-catcher design.

Using the equations of motion **(8)**, we determine the force that the stunt person feels falling through the box catcher:

$$F = m_s \frac{du_\mathrm{s}}{dt} = \begin{cases} \dfrac{1}{2\Delta H} \dfrac{T_\mathrm{S}^2}{Y} \mathcal{V} \dfrac{A_\mathrm{s}}{A}, & |z - z_\mathrm{top}| < |\Delta H|; \\ \dfrac{1}{2} \rho A_\mathrm{s} \dfrac{u_\mathrm{s}^2}{\alpha^2} - m_\mathrm{s} g, & |z - z_\mathrm{top}| \geq |\Delta H|. \end{cases}$$

We want to make this force large enough to stop but not harm the performer. Therefore, we demand that

$$F \leq (1-\delta) F_\mathrm{max}. \qquad \textbf{(9)}$$

However, we wish to find solutions that both minimize cost (fewest boxes) and conform to spatial constraints (we don't want the box catcher to be taller than the obstacle that the stunt person is jumping over). Thus, it is in our best interest to maximize the force applied to the performer subject to **(9)**.

We are faced with two independent equations with three unknowns; we solve for two of them in terms of the third. With the simplifying assumption that the top face of each box is a square ($A = w^2$), we can solve for the optimal dimensions of the box given the stunt person's impact velocity:

$$\begin{aligned} \frac{\pi^2}{12} Y \tau^3 \frac{w}{\ell^2} \frac{A_\mathrm{s}}{A} &= F_\mathrm{max}(1-\delta) = F_\mathrm{thresh}, \\ \frac{1}{2} \frac{\rho}{\alpha^2} A_\mathrm{s} \tilde{u}^2 - m_s g &= F_\mathrm{thresh}, \end{aligned}$$

where $\tilde{u}^2$ is the stunt person's velocity after causing the box to buckle (but before expelling all the air). So we have

$$\tilde{u}^2 = u_o^2 - 4\tau \frac{T_\mathrm{S}^2}{m_s Y} w\ell \frac{A_\mathrm{s}}{A}.$$

Define γ, the maximum number of gees of acceleration felt by the stunt person, by $F_\mathrm{thresh} = (1-\delta)F_\mathrm{max} = \gamma m_\mathrm{s} g$. We get

$$\ell^3 = \frac{\pi^2}{48\gamma} \left(\frac{Y}{T_\mathrm{S}}\right)^2 \frac{\tau^2}{g} \left(u_0^2 - 2(\gamma+1)\alpha^2 \frac{m_\mathrm{s} g}{\rho A_\mathrm{s}}\right) \qquad \textbf{(10)}$$

and

$$\frac{A(w)}{w} = w = \frac{\pi^2}{12\gamma} \frac{A_\mathrm{s} Y}{m_\mathrm{s} g} \frac{\tau^3}{\ell^2}. \qquad \textbf{(11)}$$

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Thus, we could write a routine that would integrate the equations of motion of the stunt person falling through each level of boxes and for each layer use the incoming velocity to calculate the optimal dimensions for the next layer of boxes. This would yield a box-catcher structure that would safely stop the stunt person in the fewest levels of boxes, minimizing the cost of the box catcher.

General Solutions

The most difficult aspect of finding a general solution is the need to know the speed of the stunt person at impact, which depends on the height of the box catcher. Given sufficient computing time, we could solve the equations with condition that the height above the ground at impact is the height of the box catcher. However, this is unnecessary for a paper of this scope.

Instead, we use the model to shed light on the qualitative aspects of building a box catcher. Equation **(10)** tells us that for higher speeds of the stunt person, we need taller boxes (larger ℓ) to keep the force on the stunt person at its maximum allowable level. This means that it is necessary to place the tallest boxes at the top of the stack, followed by shorter ones, for both cost effectiveness and safety. Inspecting **(11)** shows that the optimal width of the box is inversely proportional to the height ($w \propto \ell^{-2}$). Thus, the box catcher should have the thinnest boxes on top and the widest ones on the bottom.

It follows that

> *the optimal box catcher has short, wide boxes at the bottom and tall, narrow boxes at the top.*

Furthermore, the equations of motion **(8)** contain u^2 in the air expulsion stage but no velocity-dependent term in the buckling stage. Hence, for high impact speeds, air expulsion provides the dominant deceleration, while for smaller impacts, box buckling is the more important stage.

Results

Motorcycle Stunt

A stunt person on a motorcycle is preparing to jump over an elephant for the filming of a blockbuster movie. What should be going through a stunt coordinator's mind?

The average height of an African elephant is 4 m; the mass of a top-of-the-line motorcycle is 230 kg [Encyclopedia Britannica Online 2003].

We assume that the stunt person clears the elephant by about 2 m, so the impact velocity is 8 m/s. Choosing the parameters $\ell = \frac{1}{3}$ m and $w = \frac{1}{2}$ m, a 3 m-tall homogeneous box catcher stops the stunt person and motorcycle after they fall about 2.33 m (**Figure 1**).

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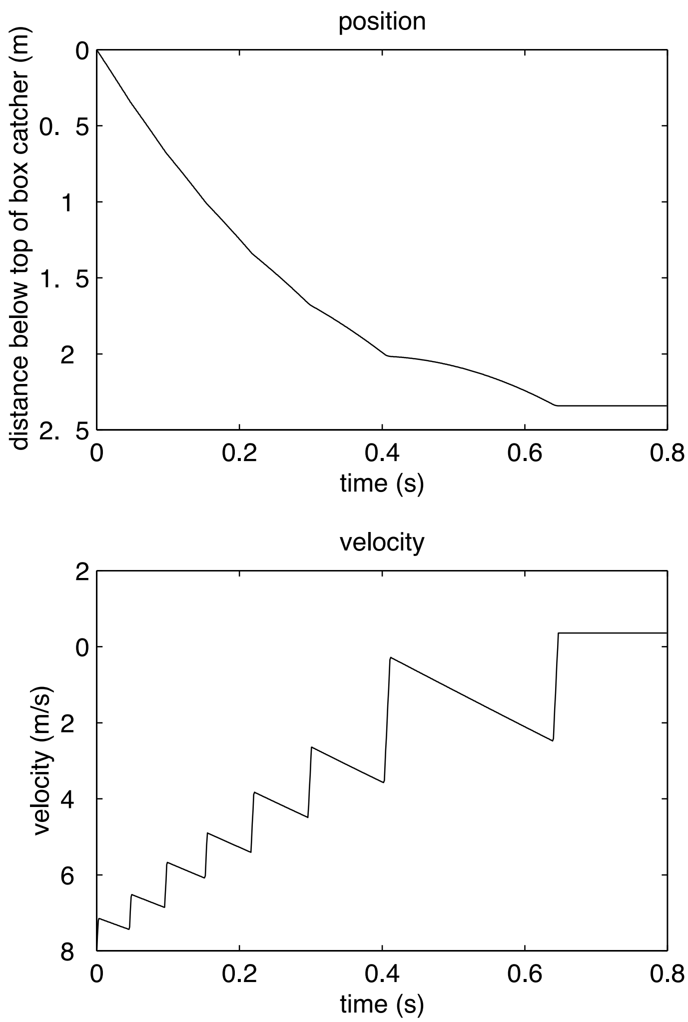

Figure 1. Simulation of box catcher stopping stunt person and motorcycle.

However, we can do better. We take our previous solution, remove one layer of boxes, and change the top layer to boxes with height $\frac{1}{6}$ m and width $\frac{2}{3}$ m (for a $\gamma \sim 1.3$). We've removed a row without compromising the safety of the stunt person. In this way, we can make changes until we arrive at an optimal arrangement, one with the fewest boxes necessary.

David Blaine's Vertigo

On May 22, 2002, David Blaine leaped from a ten-story pole onto a cardboard box catcher below. We consider his fall of more than 25 m onto approximately 4 m of boxes. His impact velocity was approximately 36.5 m/s (using the approximation $v_f = \sqrt{2gh}$, which is valid since the terminal velocity for a person falling in air is about 60 m/s [Resnick et al. 1992]).

Viewing his fall in the TV special [David Blaine's Vertigo 2002], we estimate the boxes to have height $\ell = \frac{1}{3}$ m and width $w = \frac{2}{3}$ m, with 12 layers of boxes in

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the catcher. According to our simulation, Blaine's momentum is finally stopped by the last box. A larger margin of safety could be accomplished by decreasing the value of either ℓ or w. By changing the width of the boxes to $w = \frac{1}{2}$ m, our simulation shows that Blaine would be stopped 1 m from the ground.

Other Factors

Landing Zone Considerations

The stunt person must land on the box catcher for it to be of any use. We suggest a probabilistic calculation of the landing zone; but since the principles involved are so dependent on the fall setup and conditions, we do not attempt this.

Box and Box Catcher Construction

We made a few key assumptions about design:

- *The boxes in the box catcher are held together so that there is no relative horizontal velocity between the boxes. This assumption is essential.* If the boxes could shift horizontally, they would likely shift out of the way of the falling stunt person. The catcher must also be large enough so that the structure that holds the boxes together doesn't interfere with energy dissipation.
- *Loose cardboard forms intervening layers between the different levels of cardboard boxes.* This assumption is necessary so that any force retransmitted by the top-level box does not cause another box to buckle; requiring one layer of boxes to buckle at a time allows us to optimize box parameters at each layer independently. A layer of cardboard on top of the catcher is essential to guarantee that the impact force of the falling stunt person is well-distributed.

Following these principles, and using a variety of sizes of boxes (as in equations **(10)** and **(11)**), we can tailor the box catcher to each stunt situation.

Conclusion

We present a model of energy dissipated by the collapse of a single box, noting two key stages:

- The walls of the box buckle and give way to the overwhelming force applied to it. Shearing forces cause tears in the walls, eventually leading to collapse.
- The air within the box is expelled, absorbing kinetic energy of the falling stunt person in the process.

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We consider each mechanism acting independently of the other. By solving the equations of motion in each stage, we predict the trajectory of a stunt person moving through a single box, and by extension, through the box catcher as a whole.

Whether or not the performer is stopped before reaching the bottom (and being injured) depends on the structure of the box catcher and the size of the boxes. Taller is safer but shorter is cheaper (and may be necessary to remain off-camera); we must balance safety with cost and size.

We present a solution where 9 layers of boxes are used to break the fall of the elephant-leaping motorcyclist, each with a height of $\frac{1}{3}$ m and a width of $\frac{1}{2}$ m. This is a poor solution, for the force on the falling stunt person is not optimized. We also present a slightly better solution with only 8 layers of boxes and outline the principles for continued optimization.

Finally, we observe that air resistance is the dominant force with high speeds, while box deformation provides greater resistance at low speeds.

Strengths of the Model

- All parameters in the program are flexible. Whether humid weather tends to weaken the cardboard (and lower both the tensile strength and Young's modulus), or the height of the jump changes, the stunt coordinator can test the safety of the box catcher with our program.
- The algorithm is robust, in that small changes in initial conditions do not cause drastic changes in the end result. We have seen results for extreme cases, such as extreme impact velocities (falling from great heights) or extreme weight considerations (a person on a motorcycle).
- Our algorithm takes just a few seconds to compute the trajectory of the stunt person through the box catcher, so it is practical for a stunt coordinator to test a variety of box-catcher configurations.

Weaknesses of the Model

- We were unable to derive an optimal box configuration.
- We could not get a good estimate of the magnitude of the force experienced by the stunt person. We present a plausible mechanism for dissipation of energy by a box catcher, but the force function that we use is discontinuous between box layers, though the actual physics of the situation has a continuous force.

Future Research

Future research on this project should develop an algorithm that not only tests the safety of a configuration of boxes but suggests an optimal configuration

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of boxes. Another challenge is that cardboard boxes probably have a cost proportional to the surface area of cardboard used, not to the number of boxes.

Appendix: On Buckling

We derive **(2)**,

$$F_B = Y\,\frac{\pi^2}{12}\,\frac{w\tau^3}{\ell^2}. \tag{2}$$

Consider one wall of a cardboard box with external forces acting on both ends. Considering the effects of shearing and inertial dynamical acceleration, the displacement of the box wall from its equilibrium (unstressed) position obeys the equation of motion

$$-D\,\frac{\partial^4\eta}{\partial z^4} - F\,\frac{\partial^2\eta}{\partial z^2} = \Lambda\,\frac{\partial^2\eta}{\partial t^2},$$

where $\Lambda \equiv W/g_e$ is the mass per unit length of the box wall [Thorne and Blandford 2002].

Table A1.

Nomenclature.

Property	Symbol	Units
Horiz. displacement from equilibrium (unstressed) location	η (z, t)	m
Flexural rigidity of the box wall	D	J · m
Young's modulus of the box wall	Y	Pa
Length of the box wall in z-direction	ℓ	m
Width of the box wall	w	m
Thickness of the box wall	τ	m
Force applied to each end of the box wall	F	N
Weight per unit length of the box wall	W	N/m
Acceleration due to gravity	g_e	m/s^2
Buckling force	F_B	N

We seek solutions for which the ends of the box wall remain fixed. This is a good assumption for our model, since we require the box catcher to be held together so that the boxes remain horizontally stationary with respect to one other. Thus, we set the boundary conditions to be $\eta(0,t) = \eta(\ell,t) = 0$. Using the separation ansatz $\eta(z,t) = \zeta(z)T(t)$, we get the linear ordinary differential equations

$$-D\frac{d^4\zeta}{dz^4} - F\frac{d^2\zeta}{dz^2} = \kappa_n\zeta \qquad \text{and} \qquad \Lambda\frac{d^2T}{dt^2} = \kappa_n T,$$

where κ_n is the separation constant. The normal-mode solutions are thus

$$\eta_n(z,t) = A\sin\left(\frac{n\pi}{\ell}\,z\right)e^{-i\omega_n t},$$

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where $\omega_n \in \mathbb{C}$ satisfies the dispersion relation

$$\omega_n^2 = \frac{1}{\Lambda}\left(\frac{n\pi}{\ell}\right)^2 \left[\left(\frac{n\pi}{\ell}\right)^2 D - F\right].$$

Consider the lowest-order normal mode ($n = 1$). For $F < F_{\text{crit}} \equiv \pi^2 D/\ell^2$, we have $\omega_1^2 > 0$, so $\omega_1 \in \mathbb{R}$, and the normal mode just oscillates in time, giving stable solutions. However, if $F > F_{\text{crit}}$, then $\omega_1^2 < 0$ and $\omega = \pm i\varpi$, $\varpi \in \mathbb{R}$, i.e. there are exponentially growing solutions ($\propto e^{\varpi t}$), and the normal-mode solution becomes unstable. Thus, the box wall buckles for applied forces $F > \pi^2 D/\ell^2$. The buckling force is

$$F_B = \frac{\pi^2 D}{\ell^2} = Y\,\frac{\pi^2}{12}\,\frac{wt^3}{\ell^2},$$

which is **(2)**.

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