Thinking Outside the Box and Over the Elephant

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Abstract

We present a mathematical model of the collapsing of a box structure, which is to be used to protect a stunt motorcyclist who jumps over an elephant. Reasonable values of the model's parameters cause it to predict that we should construct the structure out of fifty $6\,\mathrm{in} \times 28\,\mathrm{in} \times 28\,\mathrm{in}$ boxes, stacked five high, two wide, and five long. In general, the model predicts that we should use boxes whose height is one-quarter of the harmonic mean of their length and width. We discuss the assumptions, derivation, and limitations of this model.

Introduction

A stunt motorcyclist jumps over an elephant; we use cardboard boxes to cushion the landing. Our goal is to determine how to arrange the boxes to protect the motorcyclist. We determine

- how many boxes to use,
- the size of the boxes.
- the arrangement of the boxes, and
- any modifications to the boxes.

In addition, our model must accommodate motorcyclists jumping from different heights and on motorcycles of different weights. Our goal is to reduce the impulse at landing, thus essentially simulating a much lower jump (of which we assume the rider is capable).

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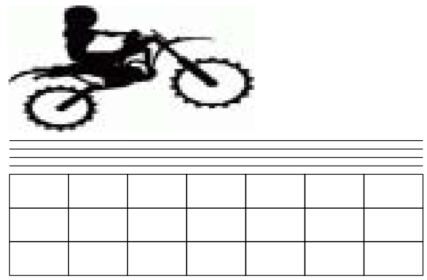

Figure 1. The landing platform (graphic from Just'In Designs [2001]).

As the rider breaks through the top layer of boxes, crashing through cardboard at a high horizontal speed, it will be difficult to maintain balance. It is too dangerous to rely on the cardboard to stop the horizontal motion of the rider, unless we use such a large pile that keeping it from being visible to the camera would be nearly impossible.

We are faced with how to cushion the rider's landing without creating merely a pit of boxes. Imagine jumping from a 10-ft roof. If the jumper lands on a large wooden platform resting on a deep foam pit, the risk for injury is much less; the foam spreads out the jumper's deceleration over a much longer time, simulating a much lower jumping height.

Our goal is to create a landing platform for the motorcyclist that behaves much like the wooden platform on the foam pit. We simulate the foam pit by stacks of boxes. Our "platform" is constructed from boxes unfolded into cardboard flats and placed in a layer on top of the "pit" (**Figure 1**). The idea is that the motorcyclist should never break through this layer of flats but should merely break the boxes underneath it.

Safety Considerations

- Once the motorcycle has landed on the stack of cardboard boxes, its deceleration should to be as uniform as possible as the structure collapses—the more uniform the deceleration, the easier to maintain balance.
- We want the platform to remain as level and rigid as possible. If it is not level, the rider may lose balance; it it is insufficiently rigid, it may bend and collapse into the pile of boxes.

Terminology

- The **flute type** of a gauge of cardboard refers to its corrugated core structure. Three sheets of linerboard compose one sheet of corrugated cardboard;the middle one is shaped into flutes, or waves, by a machine, and then the outer two sheets are glued on either side of it. For example, C-flute corrugated cardboard is the most common form [Mall City Containers n.d.].

- The **edge crush test** (ECT) value of a gauge of cardboard is the force per unit length that must be applied along the edge before the edge breaks. We make extensive use of the concept of such a value; however, the actual numbers given for gauges of cardboard apply to ideal situations that would not be replicated in the cases that we are considering [Boxland Online 1999].

- The **flatwise compression test** (FCT) value of a gauge of cardboard is the pressure that must be applied to collapse it. It does not directly correlate to the stress placed on boxes in practice and therefore is not used as an industry standard [Pflug et al. 2000].

- The **bursting strength** of a gauge of cardboard is the amount of air pressure required to break a sample. Because our model is concerned mainly with the strength of a box edge, but the bursting strength more accurately measures the strength of a face, we do not make use of it [Boxland Online 1999].

- The **stacking weight** of a box is the weight that can be applied uniformly to the top of a box without crushing it. In general, the stacking weight of a box is smaller than the ECT or bursting strength, because it takes into account the structural weaknesses of the particular box. We derive most of our numerical values for box strength from manufacturers' specified stacking weights [Bankers Box 2003].

Assumptions

- *The force exerted on every layer beneath the top platform is horizontally uniform*. The force that the motorcycle exerts on the top platform is concentrated where the wheels touch. Ideally, however, this top platform is perfectly flat and rigid, so it distributes the force evenly to all lower layers. We approach this ideal by adding additional flats to the top platform.

- *The stacking weight of a box is proportional to its perimeter and inversely proportional to its height*. This assumption is physically reasonable, because weight on a box is supported by the edges and because the material in a taller box on average is farther from the box's points of stability than in a shorter box. This claim can be verified from data [Clean Sweep Supply 2002].

- *Nearly all of the work done to crush a box is used to initially damage its structural integrity*. After the structure of a box is damaged, the remaining compression follows much more easily; indeed, we suppose it to be negligible. We denote by d the distance through which this initial work is done and assume for simplicity that the work is done uniformly throughout d. Through rough experiments performed in our workroom, we find that $d \approx 0.03$ m. We assume that this value is constant but also discuss the effect of making it a function of the size of box and of the speed of the crushing object.

Figure 2. Tire before and after landing, acting as a shock absorber.

- *When the motorcycle lands, we ignore the effects of any shock absorbers and assume that the motorcyclist does not shift position to cushion the fall.* This is a worst-case scenario. To calculate how much force the tires experience per unit area, we consider a standard 19-inch tire of height 90 mm and width 120 mm [Kawasaki 2002]. It compresses no less than 50 mm (**Figure 2**). A simple geometry calculation then tells us that the surface area of the tire touching the platform is approximately 3000 mm^2. We assume that the force exerted on the motorcycle on landing is uniformly distributed over this area.
- *The pressure required to compress a stack of cardboard flats completely is the sum of the pressures required to compress each individual flat.*
- *In a uniformly layered stack of boxes, each layer collapses completely before the layer beneath it begins to collapse.* This is probably an oversimplification; however, it is reasonable to suppose that the motorcycle is falling nearly as fast as the force that it is transmitting.
- *The motorcyclist can easily land a jump 0.25 m high.*

The Model

Crushing an Individual Box

For a cardboard box of height h, width w, and length l, by the assumptions made above, the stacking weight S is

$$S(h, l, w) = \frac{k(l + w)}{h},$$

where k is a constant (with units of force).

Once a box is compressed by a small amount, its spine breaks and very little additional force is required to flatten it. Thus, most of the force that the box exerts on the bike is done over the distance $d \ll h$, and we assume that the work is done uniformly over this distance; this work is

$$W = \text{(force)(distance)} = \frac{k(l + w)d}{h}.$$

Crushing a Layer of Boxes

To ensure that a layer of boxes collapses uniformly, we build it out of n identical boxes. The total amount of work required to crush such a layer is

$$W_T = n\,\frac{k(l+w)d}{h}.$$

Once the structure starts to collapse, we want the rider to maintain a roughly constant average deceleration g' over each layer. It follows that the layer should do total work $m(g+g')h$, so

$$W_T = \frac{nkd(l+w)}{h} = m(g+g')h\,. \qquad \textbf{(1)}$$

Define $A = nlw$ to be the cross-sectional area of a layer of boxes; rearranging **(1)** produces

$$B \equiv \frac{Adk}{m(g+g')} = \frac{h^2lw}{l+w}. \qquad \textbf{(2)}$$

The constant B gives a necessary relationship among the dimensions of the box if we wish to maintain constant deceleration throughout the collision.

Finally, we would like to minimize the total amount of material, subject to the above constraint. To do so, consider the *efficiency* of a layer with a given composition of boxes to be the ratio of amount of work done to amount of material used. If the motorcyclist peaks at a height h_0, we must do work mgh_0 to stop the motorcycle. We minimize the total material needed by maximizing the efficiency of each layer.

The amount of material in a box is roughly proportional to its surface area, $S = 2(hl + lw + wh)$. Thus the amount of material used by the layer is proportional to $nS = 2n(hl + lw + wh)$. It follows that the efficiency E of a layer composed of boxes of dimensions $h \times l \times w$ is

$$E \propto \frac{W_T}{nS} = \frac{nkd(l+w)}{2nh(hl+lw+wh)} = \frac{kd(l+w)}{2h(hl+lw+hw)}.$$

We maximize E for each layer, subject to the constraint **(2)**. The calculations are easier if we minimize $1/E$. Neglecting constant factors, we minimize

$$f(h,l,w) = \frac{h}{l+w}(hl+hw+lw)$$

subject to the constraint

$$\frac{h^2lw}{l+w} = B,$$

where B is the constant defined in **(2)**. However, as long as we are obeying this constraint (that each layer does the same total work), we can write

$$f(h,l,w) = h^2 + \frac{hlw}{l+w} = h^2 + \frac{B}{h},$$

and thus f depends only on h. The function f is minimized at

$$h = \sqrt[3]{\frac{B}{2}} = \sqrt[3]{\frac{Adk}{2m(g+g')}}. \qquad \textbf{(3)}$$

At this value of h, the constraint reduces to

$$\frac{lw}{l+w} = \frac{B}{h^2} = \sqrt[3]{4B}.$$

This implies that the harmonic mean of l and w should be

$$H \equiv \frac{2lw}{l+w} = 2\sqrt[3]{4B} = 4h.$$

So, in the optimal situation, the box should be roughly four times as long and wide as it is tall. However, there are other considerations.

- For commercially available boxes, we must choose among some discrete set of realizable box dimensions.
- That the number n of boxes in a layer must be an integer affects the possible values of many of the parameters on which h is based (most notably A, the cross-sectional area of the layer). We select the box among the potential candidates which most nearly compensates for this change in parameters.

The Entire Structure

We need to determine the gross parameters of the entire structure. We determine the cross-sectional area of the structure by considering how much space the motorcyclist needs to land safely. The motorcycle has length about 1 m; we should leave at least this much space perpendicular to the direction of travel. We need substantially more in the direction of travel, to ensure that the motorcyclist can land safely with a reasonable margin of error. So we let the structure be 1 m wide and 3 m long, for a cross-sectional area of 3 m^2.

How many corrugated cardboard flats should we use? We do not want them all to crease under the weight of the motorcycle and rider, for then the motorcyclist could be thrown off balance. So we must determine how much we can expect the flats to bend as a result of the force exerted by the motorcycle.

To calculate the number of flats required, we use the flatwise compression test (FCT) data in Pflug et al. [2000] for C-flute cardboard. Though our goal is to prevent substantial creasing of the entire layer of flats, we note that if we have a reasonable number of flats, creasing the bottom flat requires completely crushing a substantial area along most of the remaining layers. Less than 20% of this pressure is required to dimple a sheet of cardboard to the point where it may be creased. Since we assume that the pressure required to crush the flats scales linearly with the number of flats, we find the maximum pressure that the motorcycle will ever exert on the flats and divide it by the FCT value for a single C-flute cardboard flat order to obtain the total number of sheets needed.

A brief examination of a piece of cardboard demonstrates that bending it in the direction parallel to the flutes is much easier than bending it in the direction perpendicular to the flutes. Hence, it would be risky to orient all flats in the same direction; if force is applied along a strip of the surface, all of the flats may easily give way and bend in succession. So, it would be wise to alternate the orientation of the flats in the stack. To make sure that we have the full strength in any direction, we use *twice* the number of flats calculated, alternating the direction of the flutes of each flat as we build the stack. Then, no matter how the motorcycle is oriented when it lands on the stack, at least the required strength exists in every direction.

Finally, we determine the overall height h_1 of the structure as follows. The motorcycle accelerates downward at a rate of g from the peak of its flight to the top of the structure; this distance is $h_0 - h_1$. It then decelerates constantly at a rate of g' until it reaches the ground, over a distance h_1. Since the motorcycle is at rest both at the apex of its flight and when it reaches the ground, the relation

$$(h_0 - h_1)g = h_1 g'$$

must hold. It follows, then, that

$$h_1 = \frac{h_0 g}{g + g'}. \qquad \textbf{(4)}$$

Finding the Desired Deceleration

To determine g', we need the height H of the jump. We discuss how to build the platform so that the rider does not experience a deceleration greater than for a 0.25-m jump.

We assume that most of the cushion of the landing is in the compression of the tires, which compress $\Delta x = 5$ cm. The vertical velocity of the rider on impact is $\sqrt{2gH}$. We also make the approximation that the motorcycle experiences constant deceleration after its tires hit the ground. So the rider travels at an initial speed of $\sqrt{2gH}$ and stops after 0.05 m. We determine the acceleration:

$$v_0^2 + v_f^2 = 2ax,$$
$$\left(\sqrt{2gH}\right)^2 + 0 = 2a(0.05).$$

Solving yields $a = 20gH$. That is, if the motorcyclist jumps to a height of 4 m, on landing the ground exerts a force on the motorcycle that feels like 80 times the force of gravity; if the jump were from 0.25 m, this force would be only $5mg$. To simulate a 0.25-m fall, we should have the motorcyclist decelerate at a rate of $5g$.

Numerical Results

We now return to the question of determining the number of flats needed for the top layer. By Pflug et al. [2000], the flatwise compression test (FCT) result for C-flute board is 1.5×10^5 Pa. We expect the motorcyclist to experience an acceleration of approximately $5g$ upon landing, distributed over a surface area of 3000 mm^2 = 0.003 m^2. We assume that the motorcycle has mass 100 kg [Kawasaki 2002] and the rider has mass 60 kg. The pressure exerted on the cardboard is

$$P = \frac{(160\,\text{kg})(5)(9.8\,\text{m/s}^2)}{0.003\,\text{m}^2} = 2.61 \times 10^6\ \text{Pa}.$$

For the cardboard at the bottom of the stack of flats to be bent significantly, enough pressure must be applied to crush most of the cardboard above it. Thus a lower bound on the number of flats is $\lceil (2.61 \times 10^6\,\text{Pa})/(1.50 \times 10^5\,\text{Pa}) \rceil = \lceil 17.4 \rceil = 18$ flats. To be perfectly safe, we double this figure and cross-hatch the flats; that is, we want 36 flats in the top platform, the flutes of which alternate in direction.

Next, we calculate the total mass of cardboard for these flats. We assume that the flats are $1\,\text{m} \times 3\,\text{m}$. From Gilchrist et al. [1999], we know that the density of C-flute corrugated cardboard is 537 g/m^2; we obtain a mass of 1.611 kg per flat, or about 60 kg for 36 flats, which is comparable to the weight of a second person. The thickness of a C-flute flat is 4.4 mm [Mall City Containers n.d.]; with 36 flats, the height of the stack is 158.4 mm.

We now plug some reasonable values into **(3)** and get a good approximation of the desired height of the boxes. Let the stacking weight constant $k = 800$ N; this is roughly the mean value found in Clean Sweep Supply [2002]. These values, along with $g' = 5g$, give an optimal h of roughly

$$h = \sqrt[3]{\frac{(3\ \text{m}^2)(0.05\ \text{m})(800\ \text{N})}{2(220\ \text{kg})\left[9.8 + 5(9.8)\ \text{m/s}^2\right]}} \approx 0.17\ \text{m}.$$

So the harmonic mean of l and w must be on the order of $4h = 0.67\,\text{m}$.

Converting these values into inches gives $h = 6.5\,\text{in}$ and a value of roughly $26.5\,\text{in}$ for the harmonic mean of l and w. The two commercially available box sizes that most closely approximate these values are $6\,\text{in} \times 26\,\text{in} \times 26\,\text{in}$ and $6\,\text{in} \times 28\,\text{in} \times 28\,\text{in}$ [Uline Shipping Supplies 2002]. Note that we must increase the cross-sectional area beyond what was calculated in order to keep the number of boxes per layer an integer. Doing so increases the total value of $B = Akd/m(g + g')$; thus, we ideally want a somewhat larger box than calculated. Since we cannot increase h (any commercially available box of this rough shape and size has a height of $6\,\text{in}$), we increase l and w. This choice increases the amount by which the cross-sectional area is larger than previously calculated. Since optimum values of l and w only change as $A^{1/3}$, however, the larger box is the closer to the optimum.

To have a landing pad of size at least $1\,\mathrm{m} \times 3\,\mathrm{m}$, we need two of these boxes lengthwise and five widthwise, for a total of 10 boxes per layer. We determine the total height h_1 of the pile as follows The average height of an adult male African elephant is $3.2\,\mathrm{m}$ [Theroux 2002]; the motorcyclist could easily clear such an elephant with a jump of $h_0 \approx 4\,\mathrm{m}$. From this value, and **(4)**, we obtain

$$h_1 = \frac{h_0 g}{g + g'} = \frac{h_0}{6}\,;$$

thus, we want $h_1 \approx 0.67\,\mathrm{m} = 26.2\,\mathrm{in}$. To exceed this number with 6-inch layers, we need 5 layers.

To summarize: We use $6\,\mathrm{in} \times 28\,\mathrm{in} \times 28\,\mathrm{in}$ boxes. There are 10 boxes in each layer of the stack, in a 2×5 grid, and the stack consists of 5 layers of boxes with 36 layers of cross-hatched cardboard flats piled on top.

Changing the Parameters

Because the dimensions of the boxes used vary as the cube root of B, which is either linear or inversely linear in most of our parameters, our results are fairly resistant to change in any parameter. For example, changing one of them by a factor of 2 changes the optimal box dimensions by only 25%; even increasing a parameter by an order of magnitude only doubles the dimensions.

The one exception is the jumping height. Since B is independent of that, so are the optimum box dimensions, except insofar as the height affects the amount of area. However, the jumping height does affect the height of the box pile; we want that to increase linearly with the jumping height, in a ratio given by the desired deceleration g'. Additionally, a very high or very low jumping height will cause our model to break down completely.

For C-flute cardboard in the flats, the total amount of cardboard needed essentially depends only on the weight of the motorcycle and rider and on the net deceleration. This may at first glance appear counterintuitive: Should we not expect the jumping height to affect the stress put on the flats? In fact, the height is irrelevant as long as the assumptions of the model are justified. Since the boxes below the flats are calculated to break upon experiencing a force mg', the force transmitted through the flats by the motorcyclist is never larger than this. The only exception is the initial force that the flats experience on first being hit by the motorcycle. If the jumping height is sufficiently large, the assumption—that this initial force is dominated by the normal force exerted by the boxes underneath—will break down, and we may need to increase the thickness beyond the value calculated. However, for jumping heights this large, it is probable that other parts of our model will break down.

Certain predictions of our model are independent of its parameters, so long as our assumptions are justified. Most notable is the observation that to best conserve material for a given result, the height of a box should be one-quarter the harmonic mean of its other two dimensions, the ratio of which can be specified arbitrarily.

Finally, for very large jumps, we may need to revise our calculation of g'. While a rider may be comfortable with a deceleration of $5g$ for a small fraction of a second, it is less reasonable to assume the same comfort level if the deceleration is to last several seconds. However, we shouldn't be too concerned about this, since for jumps that are high enough for it to be an issue, our model is likely to break down in other ways.

Weaknesses of the Model

The primary weakness of our model is its dependence on the adjustable parameter d, the distance through which the work of crushing the box is done. We assume that over the box sizes that we are concerned with, d is roughly independent of the dimensions of the box; verifying the truth of this assertion would require experimentation. However, so long as d varies at most linearly with h (which is reasonable, since for any box we must have $d < h$), our method still works; we can still find a uniform optimal box size for the pile. If d depends substantially on the amount of force applied (or the velocity of the motorcycle), this will no longer be the case: the optimal box size will vary with position in the stack of boxes. However, we think this hypothesis is unlikely.

Our assumption that each layer collapses in a reasonably uniform manner is also a weakness in the model, at least for some parameter values. If the motorcycle hits the structure with too much velocity, or the desired cross-sectional area of the structure is too large, it may not be possible to layer enough flats on top of the structure to ensure uniform collapse, especially if restricted to commercially available cardboard sizes.

Finally, it is unlikely that we could find easily a supply of boxes whose faces are as large as $1\,\mathrm{m} \times 3\,\mathrm{m}$, to create the flats we want. While we could custom-order such flats, this would likely drive the price of construction up substantially. The other alternatives are to use several smaller flats in the place of each large one, or to unfold large cardboard boxes to make the flats. Doing this could weaken the structure, but this problem could likely be circumvented by varying the positions of the weak spots in each layer of flats (and possibly by slightly increasing the safety factor in the number of flats used).

Conclusion

We have designed a landing platform out of cardboard boxes for a stunt motorcyclist who will jump over an elephant. Our model of this platform predicts that we can minimize the material used by using boxes with dimensions $6\,\mathrm{in} \times 28\,\mathrm{in} \times 28\,\mathrm{in}$.

The size of the boxes used depends neither on the mass of the motorcyclist and jumper nor the height of the jump. To accommodate a higher jump, just use more layers of boxes. The calculations are based on simulating the landing of a 0.25-m jump. To simulate a lower jump, we would use slightly taller boxes.

An advantage to our model is the small size of the cardboard stack—less than 3 m^3—and specifically its short height. It should be easy to hide such a structure from the camera.

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