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2013 Mathematical Contest in Modeling (MCM) Summary Sheet

The Best Rounded Rectangle for Ultimate Brownies

Team 22550

Abstract

We developed multiple models to create a principled approach to selecting the ultimate brownie pan shape that achieved an even heat distribution H and maximized that number of brownie pans N in an oven. For simplicity we restricted our search to the set of rounded rectangle shapes and divided the problem into two seperate parts, which we recombined at the end.

We first developed a naive method for finding the dimensions of rectangles that achieve the upperbound $N^* = \lfloor WL/A \rfloor$ and extend it to rounded rectangles. We also obtained a weak lower bound on N for any rounded rectangle.

We seperately developed a finite difference approximation to model the heat transfer baking process of the brownie. Our model was first compared to an analytic solution in a simple rectangular case, but we extended the model to all rounded rectangles and accounted for nonconstant diffusivity. We calculated the variance H via a cumulative varaince measure and a cumulative absolution deviation measure, which tended to agree. We noticed that the variance in the heat distribution decreased with respect to a shape's 'circleness', but was not greatly affected by a shape's 'stretchness,' so we come to the solution of a stretched rectangle with bounded corners. We also noticed a strong sensitivity to the brownie batter depth in our pans due to our fixed boundary conditons; however adjusting to convective boundary conditions did not appear to change this.

To combine our two seperate models via p, we considered two optimization approaches for $p < .5$ and $p \geq .5$. The first appraoch ($p < .5$) optimized the heat distribution of rounded rectangles restricted by a required number of pans. Similarly, the second approach ($p \geq .5$) optimized the number of pans restricted to a limited heat variance. The restricted sets were estimated via interpolation of data points from the weak lowerbound for bounded rectangles and our finite difference approximation. Both models agreed that stretched rounded rectangles are optimal. For example in a typical home oven, our method selects a pill-shaped 20cm by 40cm pan over a typical 23cm by 33cm brownie pan.

Our model selects stretched rectangles that are as round as possible, so our next step would be to consider elliptical shapes in order to explicitly account for the interdependence between N and H. We also noted that our model's boundary conditions need fine tuning to get a more realistic result.

Ultimately, our method selects the best rounded rectangle shape for a brownie pan based on a specified trade off between perfect edges (circles) and best packing (rectangles).

1 Introduction

We are interested in developing the pan shape to produce the ultimate brownies: evenly baked, using as much of the oven as possible. In order to accomplish this, the best brownie pan shape must maximize the space usage of the oven and minimize the variance in the heat distribution of the brownies.

Since we can hypothesize that the best shape for an evenly cooked brownie is a circle (no heat concentration at the corners) and the best shape for fitting brownies into a rectangular oven is a rectangle, we restrict our search domain of shapes to rounded rectangles. We break this problem into the three parts:

Identical shape packing problem We first consider the related problem of maximizing the number of pans N in an oven for a given shape. For simplicity we approximate rounded rectangles with rectangles and circles and model the packing problem in the two limiting cases. The two special cases of this problem that have been researched: the pallet loading problem (PLP) for identical rectangles and the circle packing problem (CPP) for identical circles. Both are non-trivial nonlinear optimization problems; therefore we developed a simple heuristic algorithm to bound N for any given shape.

Heat transfer problem Second, we modeling the heat transfer from the oven to the brownie as it cooks to measure the variance in the heat distribution. This heat distribution determine how overcooked the edges and other parts of the brownie may be. There has been previous work in baking cake that we used as a starting point. We first consider an analytic solution for fixed diffusivity and fixed boundary conditions, but later relax these assumptions in our finite difference approximation, which we also extend to all rounded rectangles.

Optimization Finally, we combining the first two components to select the ultimate brownie pan. We consider two approaches: maximizing the number of pans when constrained to a specific heat variance, and minimizing the heat variance when constrained to a minimum required number of pans.

After analyzing the problem, we have discovered that long thin rectangles with rounded ends are the best solution to the problem.

2 Definitions & Assumptions

2.1 Definitions

First some definitions.

- The width of the oven (depth from the oven door) is W
- The length of the oven is L
- The area of a pan is A
- The maximum number pans able to fit in the oven N
- The domain containing a pan of brownies and no air Ω
- The temperature of the brownies $T(x)$ for $x \in \Omega$

- The initial temperature of the brownies T_{init}
- The temperature of the oven T_{Oven}
- The baking time t_{Bake}.
- The measure of heat variance of the brownies H. We consider two possible measures for H which will be discussed below.
- The theoretic maximum number of brownie pans $N^* = \lfloor W \times L/A \rfloor$
- The thermal diffusivity α.
- The thermal conductivity of the pan k
- The heat transfer coefficient of air h
- The temperature T_b on the boundary of the domain $\partial\Omega$
- The spacings in the x, y, z, r, and θ directions Δx, Δy, Δz, Δr, $\Delta\theta$ for the finite difference method, respectively

2.2 Rounded Rectangle

The cross-section shape we consider is a rounded rectangle governed by parameters w, l, r. A rounded rectangle consists of a rectangle with quarter circle corners. An example is shown in Figure 1 , where r denotes the radius of the corners and w, l denote the inner width and length.

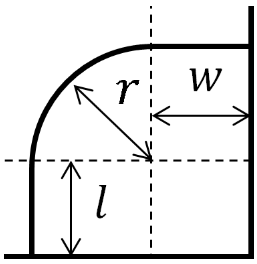


Figure 1: A quarter section of a rounded rectangle.

The area of a rounded rectangle is $A = \pi r^2 + 4wl + 4(w + l)r$.

The Bounding Rectangle: $A_{BR} = 4(w + r)(l + r)$. The extra space in this approximation is $W_{BR} = (4 - pi)r^2$. The error in this approximation is a function of r.

2.3 Assumptions for packing an oven rack (N)

Oven and Pan Dimensions

- For our examples we will use the following typical oven sizes:
  - small: 61cm by 47cm (24" by 18.5") (typical interior dimensions of Kenmore ® brand home ovens)
  - large: 61cm by 91.4cm (2' by 3') (very large commercial ovens)
- Typical pan sizes:

  - small: 23cm by 33cm (9" by 13")
  - large: 23cm by 15.3cm (9" by 6")

- We will ignore the second rack of the oven, as we will assume there is alway sufficient space between

2.4 Assumptions for Modeling Heat Distribution (H)

Oven Temperature

- Oven temperature is 177°C (350 °F), a common brownie baking temperature
- Assume a convection oven: temperature is constant and even, with perfect mixing in the air.
- We will ignore the interaction effects of multi pans in an oven at one time. If the uneven heat distribution is a concern, then creating a buffer of 2cm around the pan is a rule of thumb for dealing with this problem.

Pan

- We will assume an aluminum pan because aluminum has the highest diffusitivity, thus the greatest chance of brownie burning.
- Aluminum conducts heat much more readily than brownies (thermal conductivity, $k_{Al} \approx 200\frac{W}{m}K << k_{brownies} \approx 1\frac{W}{m}K$ and aluminum pans tend to be very thin, so temperature throughout the pan reaches T_{oven} much faster than the brownies. We can begin our model by assuming the brownies only interface with the air and we will ignore the pan. Later on, we will add the pan in to the model.

Brownie Size

- Brownies are about 3cm thick, and can range from 1.5cm to 4cm.
- We will assume that the depth of the pan is always greater than the the batter depth of the brownies.
- We will assume that the brownie batter does not decrease in size for simplicity. In real life some moisture will evaporate.

Brownie Batter

- Brownies behave similarly to cake, but are denser and have a higher fat content which implies a lower diffusivity.
- Thermodynamic properties of brownie batter, such as the diffusivity α do not vary appreciably with temperature.
- Our initial model uses a thermal diffusivity of dense cake from [7]. Later, we assume that the diffusivity of brownie batter can take on two different values, and change α during the cooking process accordingly: the value of α for water when batter is below 100°C, and the value of α for cake when it is above 100 °C.

Doneness

- We will considers the Brownies as baked is when all Brownie batter reaches 100 °C. This is based on the rule-of-thumb of inserting a wooden skewer into the center of the brownie and remove it clean, with no wet batter clinging to it.
- Brownie batter "doneness" scales with temperature, and so does overcooked-ness. A good pan for heat distribution has a low variance in temperature across the pan, while a bad pan has a high variance–i.e., very hot edges and corners.
- Real world brownies will be thoroughly cooked after approximately 30 minutes

3 Maximizing Number of Pans N

We are interested in choosing a shape of brownie pan to use space in the oven most efficiently. Because the shape of pans that we are considering are rounded rectangles (ranging from rectangles to circles), and coming up with a good solution to the packing problem is a non-linear optimization problem that we choose not to approach, we develop heuristics and approximations for packing either rectangles or circles.

Note that an optimal for packing into one oven will not be optimal for any oven, and so in order to make a good general recommendation, we need an understanding of the general characteristics of a pan that packs well, which we can achieve without finding the optimal shape for a given oven.

3.1 Rectangles

3.1.1 The Pallet Loading Problem (Classic Rectangle Packing)

In the pallet loading problem, the task is to fit as many identical rectangles as possible into a set rectangular area, using only 90°rotations, a constraint which means that the number of useful packings shrinks as the rectangles approach squares. We first applied available algorithms for the PLP [5, 3] to our problem of packing rectangular pans (or rounded rectangular pans which we approximate as rectangles), but we noticed that for situations like the brownie pans, where the area of the oven is not particularly large compared to the area of the pan, PLP solutions do not achieve good packings.

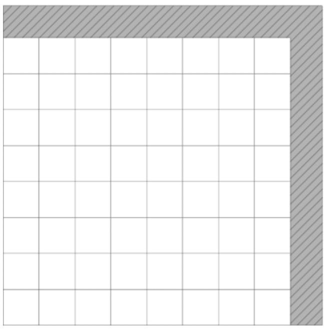


Figure 2: Packing of squares (white) with side length of 9 units into an oven (gray) with side length of 80 units. With this packing, more squares could fit if arbitrary rotation was allowed, making a PLP packing like this one (using only 90°rotation) both very naïve and suboptimal.

3.1.2 Packing vs Packaging

The solutions to the Pallet Loading Problem are far from optimal for our purposes, because of the lack of flexibility in placing relatively large boxes on a pallet. However, designing a pack-able pan is not exactly the same problem as packing any given pan. The biggest problem in packing pallets is that the shape of the pallet is given. If we can choose our pan shape to package our brownies in a pack-able way, we efficiently use space oven space.

Now we have a new problem. Instead of trying to find an algorithm that will produce the optimal packing for an arbitrary brownie pan we solve the simpler problem of finding a pan that results in an optimum packing. As a concrete example, we consider that the oven being used is approximately the size of a home oven with a maximum depth (W) of 47cm, and $W < L$. We will also start by assuming the brownie pan is reasonably large compared to the oven (suppose the required area for the brownie pan has an area of 760cm^2). Begin by noting that the theoretical maximum number of pans N^* of area A that can be put into the oven is

$$N^* = \left\lfloor \frac{WL}{A} \right\rfloor \tag{1}$$

This theoretical maximum is very easy to prove. Note that for N^*+1 pans the area of the pans in greater than the area of the oven. However, if each pan has width $w = W$ and length $l = A/w$ then N^* pans can trivially be put into the oven by lining the pans in one row so that the width w is parallel to the depth of the oven W. Now we are always guaranteed to be able to fit N^* pans into the oven. Note that with the example given of 760 cm^2 per pan and an oven depth of 47cm each pan would have dimension $l \approx$ 16cm by 47cm. Since many real brownie pans are already 15.2cm in wide, this example does not represent ludicrously thin pans, and we will address possible effect on heat distribution.

Now we show that the method of finding a long thin pan shape will generalize well. Suppose, for example, that the oven in question is a commercial oven, where the racks can reach sizes of $61cm \times 91.4cm$. If the desired brownie pan has an area of 760cm^2, this brownie pan would have $l \approx 12cm$ if it stretched to the back of the oven. A pan this long (more than half a meter long) and thin is impractical. To handle the situation where the area A for the pan is very small and/or the oven is extremely large, we extend the idea. Suppose instead of taking $w = W$ and $l = A/W$ we take $w = W/k$ and $l = A/W$ for $k \in \mathbb{N}$. Now keeping the length of pan $w = W/k$ parallel to the width W of the oven we are able to fit $n_w = k$ pans along the width of the oven. Along the length of the oven we can now only fit

$$n_l = \left\lfloor \frac{WL}{kA} \right\rfloor$$

Which means we can fit a total of

$$N_{wl} = k \left\lfloor \frac{WL}{kA} \right\rfloor$$

Using the identity $x mod(y) = x - y\lfloor x/y \rfloor$ we can rewrite this as

$$N_{wl} = \left\lfloor \frac{WL}{A} \right\rfloor - \left\lfloor \frac{WL}{A} \right\rfloor mod(k)$$

or

$$N_{wl} = N^* - N^* mod(k)$$

This implies that $N_{wl} = N^*$ for a natural number k if and only if k is a factor of N^*. Therefore, even for the case of a very large commercial oven that is 48″ deep we can set $k = 2$ and essentially cut our depth by 2, making the value l of the brownies twice as large and making the pan more practical for actual use while only losing at most 1 pan from the optimal packing, since $N^* mod(2)$ can only be equal to 0 or 1. Furthermore, k could even be 3 and then the largest number of pans that would be lost in the oven would be 2. Note also that if N^* has a factor of 2 or 3 in it then k can be set to this factor and no pans are lost.

An analytic lower bound can be demonstrated for fitting pans into the oven, with a lower bound given for each k with which to subdivide W:

Take w, l for the bounding rectangle with $l = \frac{A}{W}$ Then:

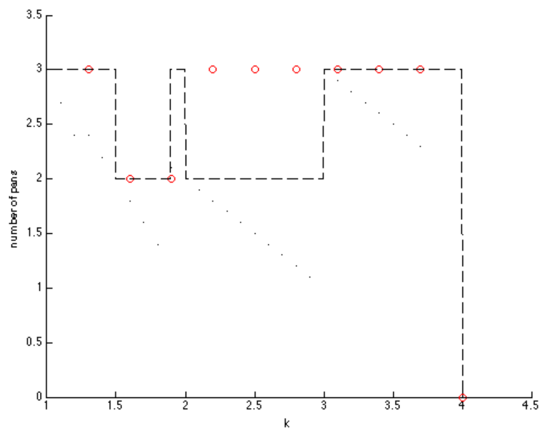
$$n_w = \left\lfloor \frac{W}{w} \right\rfloor = \lfloor k \rfloor = k - \{k\}$$

$$n_l = \left\lfloor \frac{L}{l} \right\rfloor = \left\lfloor \frac{W \times L}{kA} \right\rfloor$$

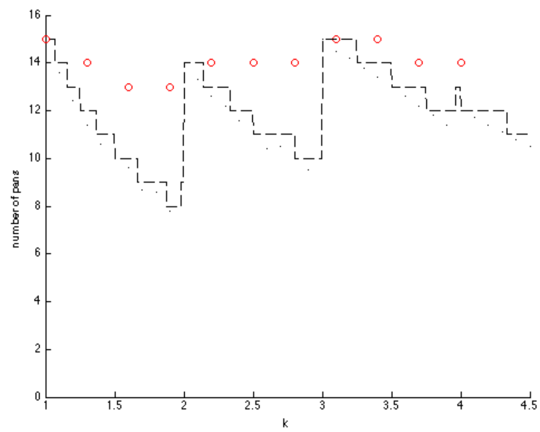
$$\text{so } n_w n_l = \lfloor k \rfloor \left\lfloor \frac{W \times L}{kA} \right\rfloor = \lfloor k \rfloor \left\lfloor \frac{W \times L}{(\lfloor k \rfloor + \{k\})A} \right\rfloor$$

$$n_w n_l = \lfloor k \rfloor = (k - \{k\}) \left\lfloor \frac{W \times L}{kA} \right\rfloor = N^* - N^* mod\, k - (\{k\} \left\lfloor \frac{W \times L}{kA} \right\rfloor)$$

$$\text{so } N_{w,l} \geq N^* - N^* mod\, k - \{k\} n_l$$



The large dimension pan ($23cm \times 33$) in the home oven ($61cm \times 47cm$).



The small dimension pan ($23cm \times 15.3cm$) in the commercial oven ($47cm \times 91.4cm$).

Figure 3: The black circles show points $n = N^* - N^* mod\, k - \{k\} n_l$, and the dashed line is the ceiling of n. Red data points show how many pans of area A and dimension $W/k \times (Ak)/W$ fit in the oven, with 90°rotation considered.

Overall, the idea of simply fitting the pans into the oven so that they stretch to depth of the oven (or such that an integral multiple of pans stretches the depth of the oven) is very practical. It makes loading very easy since a cook does not have to reach all the way to the back of the oven to get to the brownies in the back and there are no brownie pans that need to be placed at an obscure angle so that all N^* brownie pans can be placed into the oven. Finally, we know this size of brownie pan will always achieve N^*, and while another brownie pan shape may allow for N^*

brownie pans to be placed into the oven, we would need to consider arbitrary rotation for packing problems to find whether N^* brownie pans can indeed be placed into the oven for another value of N^*.

3.1.3 Considering Rotations

While we were considering rectangle packing, we used available recursive partitioning algorithms to find solutions to the pallet packing problem, which, as shown above, did not find optimal packings. In order to find truly optimal packings for pans, we would have needed to consider rectangle packing with arbitrary rotations. When considering rotation in rectangles, the packing problem becomes a non-linear optimization problem that we could not solve.

For practical purposes, our solution to fitting rectangles into an oven is reasonable, because we want pans that slide in easily, without a great deal of maneuvering, we feel that we do not need to consider packing with rotations.

3.2 Circles

The other shapes that we wish to consider for our ultimate brownie pan are circles and rounded rectangles with very large radius r compared to w and l, so they approach circles.

A true dense packing solution for the circle packing problem is also a non-linear optimization. While algorithms do exist, they are very complicated and difficult to develop, so we chose to use two simple heuristics to estimate circle packing. We will estimate circle packing by choosing the best of two tilings in the space, hexagonal or square. Since most dense packings of circles using either tiling for some of the space (hexagonal packing density is the best possible density of circles), this gives a reasonable estimate and a good lower bound. [6]

We note while running test on circular pans packed in this way, naÏ've circle packing is less dense than naÏ've square packing, and far worse than a rectangular packing where the pan area differs from the oven area by only one or two pans.

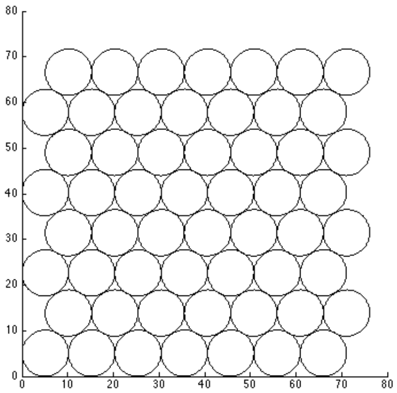


Figure 4: Packing of 56 circles (vs 64 naively packed squares of the same area) with an area of 81cm^2 into an oven with side length of 80 units. In this case, out of the two possible grids, the hexagonal grid is best (which is always true when pans are appropriately small compared to the oven).

3.3 Results and Analysis

From the intuition developed while working with the pallet loading problem (allowing 90°rotation) that a more "stretched" rectangle allows for more usefully different packings than squares of the same area, and the analysis offered in 1 demonstrating that elongated rectangles offer provably optimal packing in many cases, we conclude that the optimal shape for a pan is a rectangle elongated as much as is practical–all the way to the depth of a home oven.

When we apply this analysis to the larger problem of packing rounded rectangles, we will can expect a rounded pan with small r to pack in a similar way as its bounding rectangle, and a relatively square pan with very large r to pack with density similar to a bounding circle.

3.3.1 Extension to Rounded Rectangles

Our general method of finding a best rectangle to fit in a given oven can be extended to find how rounded that rectangle can be.

As is clear in Figure **??**, the problem is necessarily discrete–we can vary the ratio of w and l within some range without changing the number of pans we can pack. Also because there is almost always some area left in the oven $\frac{WL}{A} \neq N^*$, we can also increase the area of the rectangle to be packed (i.e., the bounding rectangle) within some range with changing the maximum number.

We exploit this fact to find the maximum radius of curvature of the corner of the pan (as we increase r the area of the bounded rectangle increases) that achieves the same packing as the best packing for that pan with $r = 0$.

We can round the corners of the stretched rectangle while keeping the total width $(2w+2r) = W$. This will let the rounded rectangles still stretch the entire width. Note however that as the corners become rounded, the total length $(2l + 2r)$ of the brownie pan will increase. So we are interested in finding the maximum value for r while still allowing for N^* pans to be placed in the oven. This value can be found by the following steps. Recall that N^* is given by

$$N^* = \left\lfloor \frac{WL}{A} \right\rfloor \tag{2}$$

Also recall that if we inscribe a rounded rectangle inside another rectangle the extra area is

$$W_{BR} = (4 - \pi)r^2 \tag{3}$$

Now note that if N^* pans of area A are placed in the oven then the total unused area in the oven will be

$$A_{extra} = WL - AN^* \tag{4}$$

Now we wish to make it so that the extra area in the bounding rectangles W_{BR} across all pans N^* is equal to the total area. Therefore, we also want

$$A_{extra} = N^*(4 - \pi)r^2 \tag{5}$$

or, solving for our desired value of r we have

$$r = \left(\frac{A_{extra}}{N^*(4 - \pi)} \right)^{.5} \tag{6}$$

Finally, note that it is possible that this method gives a radius so large that the N^* pans placed next to each other are longer than the length of the oven. This is because once the radius becomes large enough that $l = 0$ the width must decrease to compensate for the increase in radius. This means that the assumption that all the unused area will be in the corners of the bounding rectangle is no longer valid since the total width $(2 * w + 2 * r)$ will actually decrease. Therefore, we must propose an alternative solution in this case. If the value of r found satisfies the inequality

$$r \leq \frac{L}{2N^*} \tag{7}$$

then we are fine and continue to calculate the length and width as follows.

$$w = \frac{W - 2r}{2} \tag{8}$$

$$l = \frac{A - \pi r^2 - 4wr}{4w + 4r} \tag{9}$$

Otherwise, that is if the calculated value of r satisfies

$$r > \frac{l}{2N^*} \tag{10}$$

then this means that the radius has actually become so large that our assumptions are no longer valid. Therefore, the new values become

$$r = \frac{l}{2N^*} \tag{11}$$

$$l = 0 \tag{12}$$

$$w = \frac{A - \pi r^2}{4r} \tag{13}$$

The ideas shown here allow us to maximize the roundedness of our rectangles while still allowing for the very simple packing algorithm where the brownie pans go along the width of the oven, each one being placed next to the previous one.

When we begin to weigh optimal packings against roundedness (to improve heat distribution) will be able to use the fact that the number n of pans in the oven will be discrete. We can keep the same n while maximizing r, or chose to loose some number of pans p $(n = N^* - p)$ and maximize r.

3.4 Results and Analysis

From the intuition developed while working with the pallet loading problem (allowing 90°rotation) that a less square (more "stretched") rectangle allows for more usefully different packings than squares of the same area, and the analysis offered in 1 demonstrating that elongated rectangles offer optimal packing in many cases, we conclude that the optimal shape for a pan is a rectangle elongated as much as is practical–all the way to the depth of a home oven. When we set $w = W$ or $N^* mod\, k = 0$, we have a provably optimal shape to pack into the oven.

When we apply this analysis to the larger problem of packing rounded rectangles, we will can expect a rounded pan with small r to pack in a similar way as its bounding rectangle, and a relatively square pan with very large r to pack with density similar to a bounding circle.

Furthermore, we are able to use given dimensions and a given maximum packing to maximize r, and prove a lower bound for any given pan, which we can use to find an estimate of how many pans of an arbitrary (heat-distribution optimizing) shape fit into the oven.

3.4.1 Strength + Weaknesses

This model of packing does not consider rotations of the pans, nor does it take into account the fact that for any given rounded shape there may be other methods to arranging them, which would be useful if our problem was to pack any given rounded rectangle in an oven. That problem, however, is a level of complexity that we choose not to approach.

4 Minimizing Temperature Variance H

4.1 Heat Transfer

Heat transfer, or diffusion, in a medium is governed by the heat equation:

$$\frac{\partial T}{\partial t} = \nabla \cdot (\alpha \nabla T) \tag{14}$$

Where diffusivity α is a physical parameter which controls the rate of heat transfer through the medium.

Our goal is to use the heat equation to model the diffusion of heat in a pan of brownie batter so that we can analyze the heat distribution in a cooked pan of brownies. We consider heat diffusing through homogeneous medium (brownie), calling α a fixed constant, and also consider α as a function of time and space heat diffusing through batter which changes (looses water) as it cooks.

We used data tables to estimate constant parameters for our model.

Material	conductivity (k) $\frac{W}{m}K$	diffusivity (α) $\frac{m^2}{s}$
Water_{lq}	0.65	1.5×10^{-7}
Brownies	.1	1×10^{-7}
Pan (Al alloys)	200	8×10^{-5}

Table 1: Model parameters. Note that thermodynamic constants are estimated over a wider temperature range (20°C to 177°C) and variation of composition. [9, 7]

And we estimate the heat transfer coefficient of Air as 10 $\frac{W}{m}K$ (for free convection)

4.1.1 Convective Boundary Condition

When brownies are placed in an oven, energy, in the form of heat, begins to transfer from the air in the oven to the brownies. In the case of brownies cooking in a convection oven, the dominant mode of heat transfer is convection from the air to the brownies. This is termed the *convective boundary condition.* In convective models of heat transfer, we try to estimate the effects of energy exchange between the air and the brownies/pan. Convective heat transfer between a fluid (air) and a plate (in this case, the sides of the brownies) can be expressed as:

$$\frac{\partial T}{\partial x} = -\frac{h_{air}}{k_{plate}}(T_{Oven} - T)$$

Where the constant $\frac{h_{Oven}}{k_{plate}}$ scales how quickly heat is transferred. If we think of h_{Oven} as a constant, then the conductivity of the plate (k_{plate}) scales how quickly energy is transferred. Here, thermal diffusivity corresponds to how sharply the temperature inside the plate decreases compared to the temperature of the air.

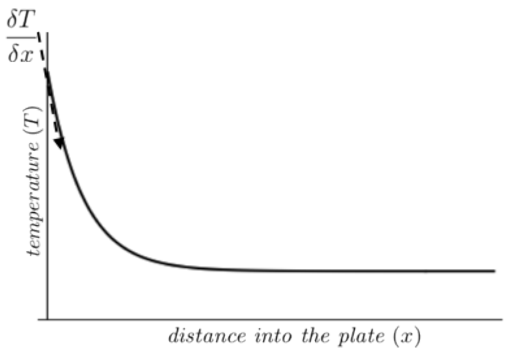


Figure 5: Temperature profile inside the plate when the thermal conductivity of the plate is very large (as for metal) i.e. $\frac{h_{fluid}}{k_{plate}} >> 1$.

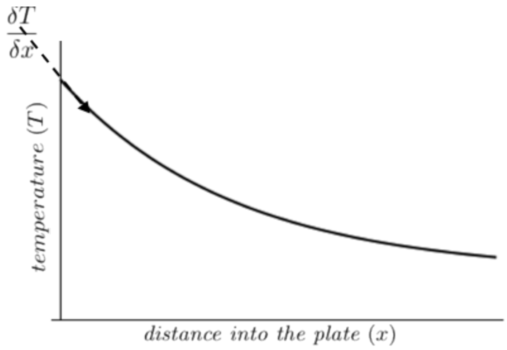


Figure 6: Temperature profile inside the plate when the thermal conductivity of the plate is very small, i.e. $\frac{h_{Oven}}{k_{plate}} << 1$.

4.1.2 Physical Equations

4.2 Analytic Solution

We can use Fourier series to solve Eq (14) with fixed boundary conditions by separation of variables. While separation of variables can not be used for all rounded rectangular shapes of brownie pans, it can be used to verify the numerical solution for the case of a rectangular and cylindrical pan. We investigate the rectangular case where all boundaries are fixed at a temperature of $T_b = 177°C$ and an initial uniform temperature across the boundaries of $T_i = 21°C$.

$$\alpha \nabla^2 T(\mathbf{x}, t) = \frac{\partial T(\mathbf{x}, t)}{\partial t} \qquad \text{for} \qquad \mathbf{x} \in \Omega$$
$$T(\mathbf{x}, t) = T_b \qquad \text{for} \qquad \mathbf{x} \in \partial\Omega$$
$$T(\mathbf{x}, 0) = T_i \qquad \text{for} \qquad \mathbf{x} \in \Omega$$

with Ω given by the region

$$0 \le x \le W = .2$$
$$0 \le y \le L = .3$$
$$0 \le z \le H = .03$$

To obtain homogenous boundary conditions we use a combination of solutions, $T(\mathbf{x}, t) = u(\mathbf{x}, t) + T_b$ so that the problem for u is

$$\alpha \nabla^2 u(\mathbf{x}, t) = \frac{\partial u(\mathbf{x}, t)}{\partial t} \qquad \text{for} \qquad \mathbf{x} \in \Omega$$
$$u(\mathbf{x}, t) = 0 \qquad \text{for} \qquad \mathbf{x} \in \partial\Omega$$
$$T(\mathbf{x}, 0) = T_i - T_b \qquad \text{for} \qquad \mathbf{x} \in \Omega$$

Now the heat equation can be solved easily for u using separation of variables. Recall that the heat equation with Dirichlet boundary conditions equal to zero gives a solution of the form

$$u(\mathbf{x}, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} A_{m,n,l} \sin\left(\frac{m\pi x}{W}\right) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{l\pi z}{H}\right) e^{-\lambda^2_{m,n,l}\alpha t}$$

where

$$\lambda^2_{m,n,l} = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{l\pi}{H}\right)^2$$

Note that there are an infinite number of unknown constants $A_{m,n,l}$ in this equation. Recall that setting $t = 0$ and using the orthogonality of the sine functions allows us to solve for our constants. Using

$$u(\mathbf{x}, t=0) = T_i - T_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{m,n,l} \sin\left(\frac{m\pi x}{W}\right)\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{l\pi z}{H}\right)$$

and then utilizing the orthogonality of the sine functions we have

$$\int_0^W (T_i - T_b)\sin\left(\frac{m_2\pi x}{W}\right)dx = \int_0^W \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{m,n,l}\sin\left(\frac{m\pi x}{W}\right)\sin\left(\frac{m_2\pi x}{W}\right)\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{l\pi z}{H}\right)dx$$

$$\int_0^W (T_i - T_b)\sin\left(\frac{m_2\pi x}{W}\right)dx = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{m,n,l}\frac{W\delta(m,m_2)}{2}\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{l\pi z}{H}\right)$$

And continuing in this fashion with the other two variables we have

$$\int_0^H\int_0^L\int_0^W (T_i - T_b)\sin\left(\frac{m_2\pi x}{W}\right)\sin\left(\frac{n_2\pi y}{L}\right)\sin\left(\frac{l_2\pi z}{H}\right)dxdydz = A_{m,n,l}\frac{W\delta(m,m_2)}{2}\frac{L\delta(n,n_2)}{2}\frac{H\delta(l,l_2)}{2}$$

Noting that the integrals on the left hand side are very easy to calculate and taking $m_2 = m$, $n_2 = n$, and $l_2 = l$, we have

$$(T_i - T_b)\left(\frac{W(\cos(m\pi)-1)}{m\pi}\right)\left(\frac{L(\cos(n\pi)-1)}{n\pi}\right)\left(\frac{H(\cos(l\pi)-1)}{l\pi}\right) = A_{m,n,l}\frac{WLH}{8}$$

And noting that having any of the terms m, n, or l even will result in the respective cosine term $\cos(m\pi)-1$, $\cos(n\pi)-1$, or $\cos(l\pi)-1$ be zero, we are only interested in the times when all three terms m, n, and l are all odd. This gives

$$A_{m,n,l} = (T_i - T_b)\frac{-64}{mnl\pi^3} \qquad \text{for m, n, l all odd}$$

4.3 Numerical Solution

To deal with more complicated boundary conditions, shapes, and nonconstant diffusivity, we model the heat equation by finite differences

4.3.1 Finite Difference Approximation

To apply the finite difference algorithm described as Algorithm 1 to rounded rectangles, we first discretize the state space. For simplicity, we split the rounded rectangle into a rectangular region parameterized by $x, y, z \in \Omega_R$ and a cylindrical region parameterized by $r, \theta, z \in \Omega_C$. Where Ω_C are the corner circles of the rounded rectangle and $\Omega_R = \Omega / \Omega_C$.

We discretize Ω_R by using a grid spaced $(\Delta x, \Delta y, \Delta z)$ and discretize Ω_C using a grid spacing of $(\Delta r, \Delta\theta, \Delta z)$. This discretization around a point $T(x, y, z)$ is shown in Figure 7.

To relate our spatial information to the changing dynamics, we approximate the heat equation (14).

The time derivative is approximated in time steps of Δt by the forward difference

$$\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^n}{\Delta t}$$

Because we have two different parameterizations and are now considering varying diffusivity, the heat equation becomes

$$\frac{\partial T}{\partial t} = (\alpha_x T_x + \alpha T_{xx}) + (\alpha_y T_y + \alpha T_{yy}) + (\alpha_z T_z + \alpha T_{zz})$$

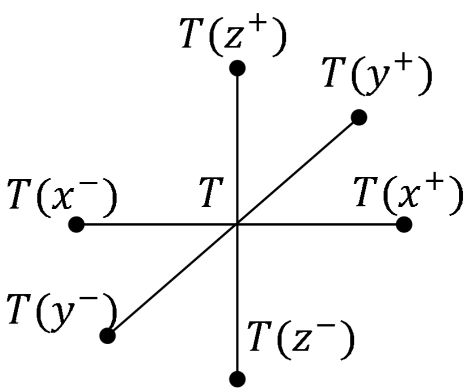


Figure 7: Zoomed-in look of the finite difference approximation grid. Points are spaced by $\Delta x, \Delta y, \Delta z$.

$$\frac{\partial T}{\partial t} = \left(\frac{1}{r}\alpha T_r + \alpha_r T_r + \alpha T_{rr}\right) + \frac{1}{r^2}\left(\alpha_\theta T_\theta + \alpha T_{\theta\theta}\right) + + \left(\alpha_z T_z + \alpha T_{zz}\right)$$

For rectangular and cylindrical coordinates respectively.

Using three-point finite difference approximations to the spatial derivatives, we obtain the following update equations

4.3.2 Rectangular Update Equations

$$\begin{aligned}
T(t^+) = T&\left(1 - 2\Delta t\alpha \cdot \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right)\right) \\
&+ T(x^+)\frac{\Delta t}{\Delta x^2}\left(\alpha + \frac{\alpha(x^+) - \alpha(x^-)}{4}\right) + T(x^-)\frac{\Delta t}{\Delta x^2}\left(\alpha - \frac{\alpha(x^+) - \alpha(x^-)}{4}\right) \\
&+ T(y^+)\frac{\Delta t}{\Delta y^2}\left(\alpha + \frac{\alpha(y^+) - \alpha(y^-)}{4}\right) - T(y^-)\frac{\Delta t}{\Delta y^2}\left(\alpha - \frac{\alpha(y^+) - \alpha(y^-)}{4}\right) \\
&+ T(z^+)\frac{\Delta t}{\Delta z^2}\left(\alpha + \frac{\alpha(z^+) - \alpha(z^-)}{4}\right) + T(z^-)\frac{\Delta t}{\Delta z^2}\left(\alpha - \frac{\alpha(z^+) - (z^-)}{4}\right)
\end{aligned} \tag{15}$$

4.3.3 Cylindrical Update Equations

$$\begin{aligned}
T(t^+) = T&\left(1 - 2\Delta t\alpha \cdot \left(\frac{1}{\Delta r^2} + \frac{1}{\Delta \theta^2} + \frac{1}{\Delta z^2}\right)\right) \\
&+ T(r^+)\frac{\Delta t}{\Delta r^2}\left(\alpha \cdot \left(\frac{1 + \Delta r}{2r}\right) + \frac{\alpha(r^+) - \alpha(r^-)}{4}\right) \\
&+ T(r^-)\frac{\Delta t}{\Delta r^2}\left(\alpha \cdot \left(\frac{1 - \Delta r}{2r}\right) - \frac{\alpha(r^+) - \alpha(r^-)}{4}\right) \\
&+ T(\theta^+)\frac{\Delta t}{\Delta \theta^2}\left(\alpha + \frac{\alpha(\theta^+) - \alpha(\theta^-)}{4}\right) + T(\theta^-)\frac{\Delta t}{\Delta \theta^2}\left(\alpha - \frac{\alpha(\theta^+) - \alpha(\theta^-)}{4}\right) \\
&+ T(z^+)\frac{\Delta t}{\Delta z^2}\left(\alpha + \frac{\alpha(z^+) - \alpha(z^-)}{4}\right) + T(z^-)\frac{\Delta t}{\Delta z^2}\left(\alpha - \frac{\alpha(z^+) - (z^-)}{4}\right)
\end{aligned} \tag{16}$$

4.3.4 Rectangle-Cylinder Boundary Equations

Points along the boundary $T(x, y, z, t) = T(r, \theta, z, t)$ between the quarter circle and the rectangular region must be treated seperately. Assuming that $\Delta\theta$ is small allows us to replace $T(x^-)$ with $T(\theta^-)$ since $\cos(\theta) \approx 1$, therefore we update the boundaries using

$$\begin{aligned}
T(t^+) = T&\left(1 - 2\Delta t\alpha \cdot \left(\frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right)\right) \\
&+ \Delta t\left(\frac{\alpha(x^+) - \alpha(\theta^-)}{r\Delta\theta + \Delta x}\frac{T(x^+) - T(\theta^-)}{r\Delta\theta + \Delta x}\right) + 2\alpha \cdot \left(\frac{T(x^+)}{\Delta x(r\Delta\theta + \Delta x)} + \frac{T}{r\Delta\theta \cdot \Delta x} + \frac{T(\theta^-)}{r\Delta\theta(r\Delta\theta + \Delta x)}\right) \\
&+ T(y^+)\frac{\Delta t}{\Delta y^2}\left(\alpha + \frac{\alpha(y^+) - \alpha(y^-)}{4}\right) + T(y^+)\frac{\Delta t}{\Delta y^2}\left(\alpha + \frac{\alpha(y^+) - \alpha(y^-)}{4}\right) \\
&+ T(z^+)\frac{\Delta t}{\Delta z^2}\left(\alpha + \frac{\alpha(z^+) - \alpha(z^-)}{4}\right) + T(z^-)\frac{\Delta t}{\Delta z^2}\left(\alpha - \frac{\alpha(z^+) - \alpha(z^-)}{4}\right)
\end{aligned} \tag{17}$$

4.3.5 Outer Boundary Conditions

There are two boundary conditions we consider:

Algorithm 1 Finite Difference

Input: Rounded Rect w, l, r, Δ Parameters, BC's, α Conditions
Create grids T, D for the Rounded Rect + BC's
Set $t = 0$.
repeat
  Update T in the rectangular region according to (15).
  Update T in the cylindrical region according to (16).
  Update T along the inner boundary according to (17).
  Update T along the outer boundaries according to the BC's.
  Update α according to (20).
  Update $t = t + \Delta t$.
until $T < T_{Cook}$
return T Distribution and baking time t

The first boundary condition is the Fixed Boundary Conditions. This case is very simple; we set $T(x) = T_{Oven} \forall x \in \partial\Omega$.

The second boundary condition is Convective Boundary Condition. We implement $\frac{\partial T}{\partial x} = -\frac{h}{k}(T_{Oven} - T)$ by two different approximations, because the derivative $\frac{\partial T}{\partial x}$ can be approximated by either

$$\frac{\partial T}{\partial x} \approx \frac{T_x - T_{x^-}}{\Delta x}$$

Giving,

$$T \approx \frac{T_{Oven}(h/k)\Delta x + T_{x^-}}{(h/k)\Delta x + 1} \tag{18}$$

Or,

$$\frac{\partial T}{\partial x} \approx \frac{T_{Oven} - T_{x^-}}{\Delta x}$$

Giving,

$$T \approx T_{Oven}(1 - (h/k)\Delta x) + T_{x^-}(h/k)\Delta x \tag{19}$$

Equation (18) corresponds to calculating $T(x)$ via the tangent line though $T(x^-)$, where as equation (19) corresponds to calculating $T(x)$ via the tangent line through T_{Oven}. However, theoretically both methods ended up being poor approximations. Equation (18) is not consistent as $\Delta x \to 0$: fixing $T(x)$ to $T(x^-)$; whereas equation (19) becomes the fixed boundary condition in the limit. We see this in practice as well.

4.3.6 Non-constant α

While initially we held α fixed, to more accurately model the baking process, we considered varying α as a function of temperature: For simplicity in modeling the diffusivity of brownies we chose to use the diffusivity as water when the temperature was below boiling and the diffusivity of cake above boiling.

$$\alpha(T) = \begin{cases} \alpha_{\text{water}} & \text{if } T < 100 \\ \alpha_{\text{cake}} & \text{if } T \geq 100 \end{cases} \tag{20}$$

This approximation is better than assuming α is constant, throughout; however it is not as accurate as capturing the true chemical dynamics that would require modeling moisture transfer across the brownie.

4.3.7 Stability Analysis for Finite Difference Approximation

Recall from basic numerical analysis that finite difference schemes for solving partial differential equations have associated stability regions. The stability region will guarantee that the solution is bounded as time increases. This stability property is very important for the heat equation because we know from basic thermodynamics that energy will naturally flow from high temperatures to low temperatures and the temperature of our brownies should be bounded. One of the easiest ways to derive the stability condition is to use Von Neumann stability analysis. For simplicity in this section, we treat α as a constant, which is reasonable since we have only used two values of α for our model depending on the temperature of the brownies at the given point. We also show the Von Neumann stability analysis for one dimension in the spatial direction and note that the method easily generalizes to the three spatial dimensions used in this model. For the Von Neumann stability analysis the idea is to take a general Fourier mode and plug this into the finite difference scheme and show that this general node will be bounded. So taking the Fourier mode given as

$$T(x_j, t_n) = \xi^n e^{-ilj\Delta x}$$

and substituting into the finite difference scheme gives

$$\frac{\xi^{n+1}e^{-ilj\Delta x} - \xi^n e^{-ilj\Delta x}}{\Delta t} = \alpha \frac{\xi^n e^{-il(j+1)\Delta x} - 2\xi^n e^{-ilj\Delta x} + \xi^n e^{-il(j-1)\Delta x}}{\Delta x^2}$$

Dividing by the term $\xi^n e^{-ilj\Delta x}$ gives

$$\frac{\xi - 1}{\Delta t} = \alpha \frac{e^{-il\Delta x} - 2 + e^{il\Delta x}}{\Delta x^2}$$

Using a trigonometric representation for the right hand side we have

$$\frac{\xi - 1}{\Delta t} = \frac{\alpha}{\Delta x^2}(2\cos(l\Delta x) - 2)$$

Finally, recall that we want to make sure the Fourier mode is bounded as time increases. This means we need ξ^n to be bounded, and therefore we must have $|\xi| \leq 1$. So finally, we solve for ξ

$$\xi = 1 + \frac{2\Delta t\alpha}{\Delta x^2}(\cos(l\Delta x) - 1)$$

Note that Δt, α, and Δx^2 are all positive. Also note that $-2 \leq \cos(l\Delta x) - 1 \leq 0$. So since this equation takes the form $\xi = 1 + x$ where $x \leq 0$, we know $|\xi| \leq 1$ if $x > -2$. Furthermore, since the goal of this stability analysis is to show that $|\xi| \leq 1$ for all Fourier modes, we must consider the extreme case where $\cos(l\Delta x) = -1$. Therefore, we must have

$$1 - \frac{4\Delta t\alpha}{\Delta x^2} \geq -1$$

or

$$\frac{2\Delta t\alpha}{\Delta x^2} \leq 1$$

This final equation gives a condition on our step sizes Δx and Δt in terms of α. Generalizing the Von Neumann stability analysis to 3 dimensions is very easy and can be seen almost directly from the steps shown above. With a step size of Δx in the x and y direction and a step size of Δz in the z direction we simply obtain the $cos(l_1\Delta x) - 1$, $cos(l_2\Delta x) - 1$, and $cos(l_3\Delta z) - 1$ term for each variable and obtain

$$1 - 4\Delta t\alpha\Big(\frac{1}{\Delta x^2} + \frac{1}{\Delta x^2} + \frac{1}{\Delta z^2}\Big) \geq -1$$

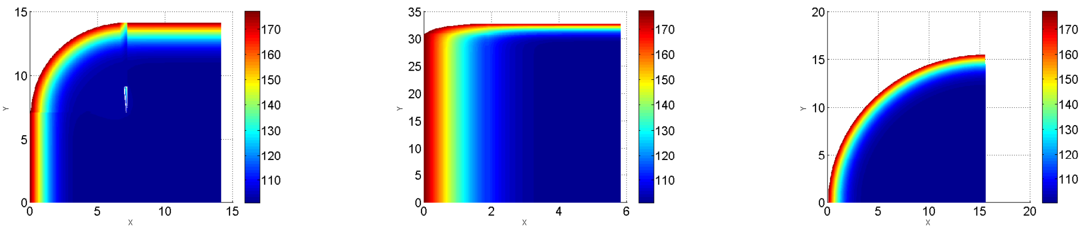
or

$$2\Delta t\alpha\Big(\frac{2}{\Delta x^2} + \frac{1}{\Delta z^2}\Big) \leq 1$$

4.4 Measures of H

Our ultimate goal in this section is to calculate the uneveness in the heat distribution for a given brownie pan shape.

Measures we considered are:



(a) Rounded Rectangle (b) Stretched (c) Circle

Figure 8: Example heat maps for various brownie pan shapes. There is a small interpolation glitch due to the step size of $\Delta\theta$ in meshing together the cylindrical and rectangular regions.

- Cumlative Variance:

$$H_V = \sum_{x\in\Omega}\left(\sum_{t<t_{bake}} T(x,t) - \bar{T}\right)^2$$

- Cumlative Absolute Deviation:

$$H_A = \sum_{x\in\Omega}\left|\sum_{t<t_{bake}} T(x,t) - \bar{T}\right|$$

Where $\bar{T}$ is the mean of $\sum_t T(t)$ for all x in Ω. The Cumlative Variance and Cumaltive Absolute Deviation are both measures of how much the cumulative temperature over time varies across the pan. The measure of deviation from the mean could have used alternative L^p norms, but for simplicity we only considered $p = 1$ or 2.

4.5 Results and Analysis

Given a rounded rectangle our algorithm tells us that the rectangles that behave like circle have the best heat distributions. However streching a rectangle out does not appear to have negative effects.

4.5.1 Strechness and Circleness

We are interested in two properties of rounded rectangles: strechness and circleness. We define strechness γ_1 to be the ratio of the shape's bounding rectangles length to width $\gamma_1 = \frac{l+r}{w+r}$. A rounded rectangle with a high strechness is very elongated and has higher packing potential.

We define circleness γ_2 to represent how close a shape resembles a circle $\gamma_2 = \frac{r}{l+h}$.

From the previous data table we can analyze the trade offs between strechness and circleness. We interpolated the data to obtain Figure 9. Note that the data there typically is a tradeoff between circleness and strechness therefore the interpolation doesn't carry much weight away from the data points. However it does help visualize how circleness has diminishing returns and how strecheness does not have a major impact on the heat distribution.

4.5.2 Comparing Analytic with Finite Difference

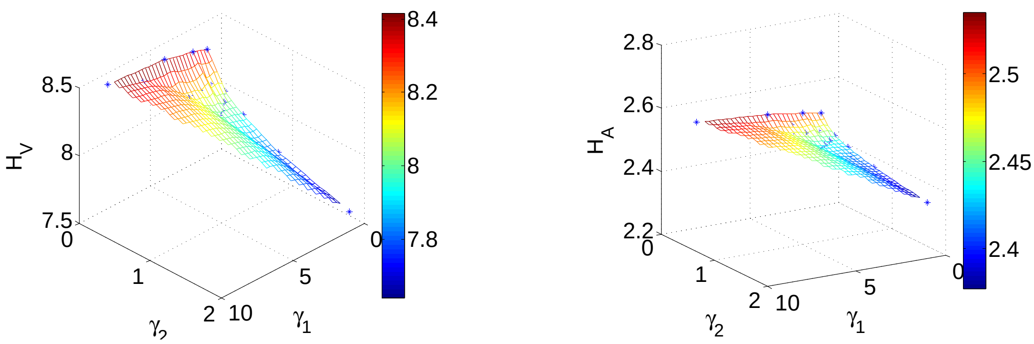
We compared the analytic solution with our finite difference algorithm on a rectangle with width 20, length 30, and depth 3 in Figure 10.

In both cases, we obtained a baking time within a stepsize of 863 seconds.

In addition, both algorithms are producing the heat maps with similar qualities; however due to our discretization, our finite difference algorithm overestimates the temperature at the edges and therefore underestimates the temperature at the center.

$\tilde{w}$	$\tilde{l}$	$\tilde{r}$	t_{bake}	$H_V(10^8)$	$H_A(10^4)$
0.00	0.00	1.00	751.30	5.09	1.91
1.00	1.00	4.00	830.30	7.64	2.38
1.00	2.00	4.00	830.35	7.74	2.39
1.00	4.00	4.00	830.35	7.87	2.42
1.00	8.00	4.00	830.35	8.00	2.44
1.00	1.00	2.00	830.35	7.80	2.41
1.00	2.00	2.00	830.35	7.89	2.42
1.00	4.00	2.00	830.35	8.00	2.44
1.00	8.00	2.00	830.35	8.14	2.47
1.00	1.00	1.00	830.35	7.94	2.43
1.00	2.00	1.00	830.35	8.02	2.44
1.00	4.00	1.00	830.35	8.12	2.46
1.00	8.00	1.00	830.30	8.25	2.50
1.00	1.00	0.50	830.35	8.05	2.44
1.00	2.00	0.50	830.35	8.12	2.46
1.00	4.00	0.50	830.35	8.22	2.49
1.00	8.00	0.50	830.30	8.34	2.52
1.00	1.00	0.00	830.35	8.29	2.50
1.00	2.00	0.00	830.35	8.33	2.50
1.00	4.00	0.00	830.35	8.38	2.52
1.00	8.00	0.00	830.05	8.42	2.53

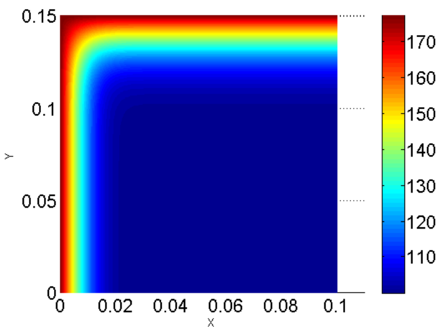
Table 2: Data table of trials for various ratios of w, l, r. All shapes are normalized to the same area. The measures of uneveness are normalized by 10^8 and 10^4 to make the table readable.



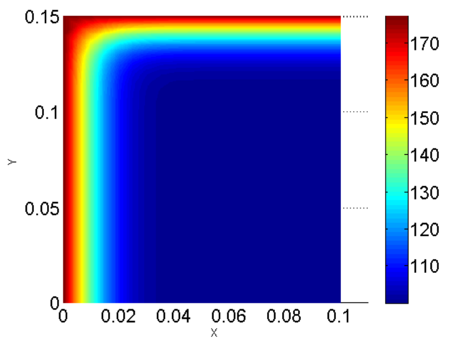
(a) $H_V(10^8)$ verses γ_1, γ_2

(b) $H_A(10^4)$ verses γ_1, γ_2

Figure 9: Plot of interpolations of H_V and H_A in the γ_1,γ_2 space. The mesh grid is the interpolation and the blue stars are the original data points from Table 2



(a) Analytic



(b) Finite Difference

Figure 10: Comparison of the middle cross-sections. The finite difference data is interpolated to create the heat map.

4.5.3 Sensitivity

We tested the finite difference algorithm's sensitivity to adjustments in various paramters. We do not include the stepsize parameters Δ that are discussed in the stability section of the algorithm above. In what follows we restricted the sensitivity analysis to the two limiting cases of a 9" by 11" rectangle and the cylinder of equivalent area. This should allow us to get a feel for the behavior of all rounded rectangles.

- Varying the Depth of the Brownie:z We noticed that our model is very sensitive to brownie depth. Increasing Z leads to a substantial increase in t_{bake}. The following short table outlines this flaw in our algorithm,

  Baking Time for Varying T

Shape	$Z = 1.5$	$Z = 3$	$Z = 4.5$
Rectangle	114	863	$2,485$
Circle	95	860	$1,932$

  We had believed that this corresponded to our fixed boundary condtions more rapidly warming up the center; however we will explain below that due to the poor derivative information at the boundary our Convection Boundary conditions are not much better than fixed boundary conditions.

- Varying Temperatures: T_{init} and T_{oven}
  Changing the oven temperature varied the results slightly (161 °C to 191 °C: a hotter oven baked the brownies more quickly with a more uneven heat distribution and a cooler oven baked the brownies slower, but more evenly. Increasing the initial temperature was equivalent to increasing the temperature of the oven. Both effects were less significant than varying the brownie depth Z or the shape.

- Varying Diffusivity: α For fixed diffusivity, there was no change in the heat distribution and only the time taken for the brownies to bake changed. The baking time for brownies with water's diffusivity constant was 863 seconds This is what we expected, because for constant diffusivity, α behaves as a scaling factor.

  When we adjusted α to be a piecewise function of temperature we obtained an intermediate result for the baking time, but more importantly, we also noticed an increase in the variance of the heat distribution. This make sense since flipping from lower α to higher α increases the rate of heat absorbtion, thereby making hot sections hotter. In our data presented outside of the sensitivity analysis we used this improved non-homogenous α.

It should be noted that the stability guarentees we showed no longer necessarily hold; however it behaves well in practice. It would be interesting to look into other possible functional relationships between $\alpha(x)$ and $T(x)$ or do something with moisture transfer.

- Fixed vs Convection Boundary Conditions (This should be moved to Strength vs Weakness) As stated above, fixed boundary conditions cause high sensitivity to Z and it is possible that this is drowning out the importance of shape, by causing a faster baking time.

  However our Convection Boundary Conditions did not noticiably improve the model. Equation (18) increases the time required to bake across all z due to diminishing the effect of T_{oven}; however it has bad limiting properties. On the other hand, equation (19) is basically equivalent to the fixed boundary conditions.

4.5.4 Strength + Weaknesses

The cumlative measure captures how much certain sections of the brownie are burnt. This measure places emphasis on having an even temperature distribution over time, which is our goal of even baking. In real life, brownie batter bubbles and is able to transfer heat to help facilitate even baking.

Uniform Heat distribution is not realistic. We can think of the uniform heat distribution as a best average case. If the distribution is uneven, then it could be possible that a particular shape is more robust to non-uniform heat distribution. Due to time constraints we did not investigate these properties and instead assumed that non-uniform heat would hurt all shapes similarly.

Perhaps we should have done something with modeling the crust to fix the boundrary condition issues we had

5 Combining N and H

The final step in our algorithm is to balance the trade off between maximizing the oven area used and minimizing the burnt brownie edges. Because optimizing both N and H simultaneously is difficult, we present two options for determining the ultimate brownie pan for a given tradeoff input $p \in [0, 1]$. The first emphasizes maximizing the number of pans N over minimizing H while the alternative method minimizes H and then maximizes N.

5.1 Max N Then Min H ($p < 0.5$)

Given an oven and pan area, we create some relatation function $f : p \mapsto \{1, \dots, N^*\}$ that is monotonically decreasing. The relaxation functions relaxes the constraint of selecting w, l, r to maximize N to selecting w, l, r that achieves the relaxed threshold $f(p)$. This creates a set $\mathcal{S}_1(f(p))$ of possible w, l, r combinations that can fit $f(p)$ pans into the oven. This is calculated by finding a lower bound on the number of bounding rectangles or circles that can fit into the oven. From among this set $\mathcal{S}_1$ we select the shape which minimizes H_A or H_V. Ideally a shape will minimize both measures. This algorithm is clearly described in Algorithm 2.

5.2 Min H Then Max N ($p \geq 0.5$)

The alternative method is to find the class of shapes that minimize H and then maximize N. This is formalized by defining another relaxation function $g : p \mapsto \mathbb{R}$ that defines an upperbound on the variance of the heat distribution. We can calculate over a range of data points and interpolate over the strechness-circleness space to obtain a subset $\mathcal{S}_2(g(p))$ of w, l, r values that will satisfy our constraint in H. From this subset $\mathcal{S}_2$ we select the brownie pan that maximizes the number of pans that can be fitted in N. This algorithm is described in Algorithm 3. To determine N we use the conservative lower bound estimate.

Algorithm 2 N then H

Input: Oven W, L, Pan Area A, tradeoff parameter p.
Calculate $f(p)$ the number of required pans
Define

$$\mathcal{S}_1 = \{\Omega : N_{LB}(\Omega) \geq f(p)\}$$

Where N_{LB} is a lower bound function for Ω.
Select

$$\Omega^* = \arg\min_{\Omega} \{H(\Omega) : \Omega \in \mathcal{S}_1\}$$

return The Ultimate Brownie Pan Ω^*.

Algorithm 3 H then N

Input: Oven W, L, Pan Area A, tradeoff parameter p.
Calculate $g(p)$ the maximum heat variance
Define

$$\mathcal{S}_2 = \{\Omega : H(\Omega) \leq g(p)\}$$

Select

$$\Omega^* = \arg\max_{\Omega} \{N_{LB}(\Omega) : \Omega \in \mathcal{S}_2\}$$

return The Ultimate Brownie Pan Ω^*.

5.3 Results and Analysis

From our examples in the previous sections we have shown that near specific integer ratios of W/w, $N(\Omega)$ reaches the optimal bound. Of this set, the rounded rectangles with higher circleness have a more even heat distribution. Therefore using Algorithm 2, the best pan shapes are strechted with rounded corners.

On the other hand, the best shapes are circles and we also noted that stretchness did not vary the heat variance H. Maximizing for N over this space gives us the stretched pan shapes. Therefore the stretch rounded rectangles are the best pan shapes according to Algorithm 3 as well.

5.3.1 Strengths

This method does create a principled way of balancing the trade off between N and H. The algorithms are almost duals (in a linear programming sense) of each other. If this were not a nonlinear optimization problem, then if both algorithms agreed, we would have found the optimal brownie pan. However, although we cannot make such guarantees, it is conforting that both approaches point towards stretched rounded rectangles.

5.4 Weaknesses

- The definitions of the relaxation function f, g are arbitrary; therefore the results we obtain by varying p may not be very smooth/continuous.
- Calculation of $\mathcal{S}$ in both examples is very difficult. The models involve exponential search spaces. Both algorithms describe what we would like to do holistically. Our approach is to sample multiple points and interpolate between them to estimate N and H for various w, l, r combinations.
- Both algorithms use $N_{LB}(\Omega)$ as a conservative estimate of the number of pans able to fit. This combined method is dependent on the correctness of our previous two sections; our lowerbound on N is not tight in most cases and our calculation of H suffers from poor boundary conditions.

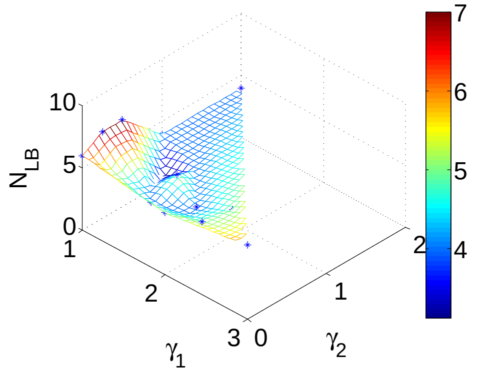


Figure 11: Example of the lower bound N_{LB} for a Large Oven and Fixed A.
Where $N_{LB} = N^*_{BR} - N^*_{BR} mod k_{BR} - \{k_{BR}\} \lfloor (WL)/(k_{BR}A_{BR}) \rfloor$

5.5 Future Work

Our analysis of brownie pans that are rounded rectangles and made of aluminum has shown that stretched rounded rectangles are the best type of brownie pan. Reviewing our analysis, we can see patterns that suggest useful future analysis for an even more ultimate pan:

Heat Transfer & Oven Dynamics

- Our boundary conditions are the areas that could be most improved. Further research and fine tuning of convective and radiative boundary condition implementation would improve our model for heat transfer by more accurately modeling.
- We could further improve our heat transfer model by allowing diffusivity α to change over space in addition to time. This could more accurately model the moisture transfer in the brownie. We could possibly run another simultaneous partial differential equation for α.
- Extending our analysis to traditional electric ovens (no convection) where the placement of the pan in the oven plays a larger role in the uniformity of the heat distribution across the final product.
- We modeled our pan in a convection oven, and ignored the interaction effects of multiple pans near each other. Although we explained that our model is robust to this changes, it would be worth double checking if this may adversely affect cooking times as well as temperature uniformity.

Pan

Our models have given us insight into more pan shapes and qualities that could help achieve brownie perfection.

- We concluded that long, relatively thin rounded rectangles are the best. It may be that another shape that combines both the qualities of circles and of stretched rectangles is an ellipse shaped pan.
- Heat transfers through an aluminum pan fairly quickly. Perhaps a thicker pan, or a less diffusive material (such as glass) would significantly improve evenness. This hypothesis is supported in literature that examines the effects of glass versus metal pans [4].
- Heat still concentrates at corners, but perhaps rather than rounding corners, making the pan very thick at corners would solve the overcooked corners problem.

6 Conclusions

In this problem, we needed to design a brownie pan that minimized the variance in the heat distribution H and maximized the number of pans N fitting in the oven. We concluded that the optimal pan shape was a stretched rounded rectangle. For example, in the case of a home oven (61cm by 47 cm) we recommend a 40cm by 20cm stadium shaped pan instead of a typical 23cm by 33cm pan.

This was a challenging problem for many reasons; optimal packing is a challenging non-linear optimization problem and modeling the heat distribution involved partial differential equations. For tractability we restricted our search to rounded rectangles and approached the two problems separately and then combined our results together at the end.

We first developed a method to find a set of rounded rectangles (ignoring) that achieved the packing upper bound $N^* = \lfloor W \cdot L/A \rfloor$. This partly solved our pan packing problem for maximizing the area used; we found optimal shapes, but we didn't necessarily find all optimal shapes. (Something about circles). In addition, we obtained a weak lower bound for N for any rounded rectangle w, l, r that is used in deciding the trade off between N and H.

Second, we developed a finite difference approximation to the heat equation in rounded rectangles for non-constant diffusivity. We noted that our approximation was extremely sensitive to the brownie batter depth in the pan due to our boundary conditions; however our approach for accounting for the pan and the convective boundary did not change our results by much. To capture the moisture transfer within the brownies as it was baking, we allowed the diffusivity of the brownies to change as a function of temperature; however we did not model moisture transfer and crust formation in our model. Under these conditions we determined that the more 'circle like' γ_2 a rounded rectangle was, the better the heat distribution obtained. We also noted that there was less of a tradeoff in stretching γ_1 the rounded rectangle, which implied that trading γ_2 for γ_1 is useful in our comparison between N and H.

In our final combined model, we developed two dual approaches to determine the ultimate brownie pan shape. The first optimized the heat distribution for a required number of pans in the oven and the second optimized the number of pans in the oven for a limited heat variance. Both approaches indicated that stretched rounded rectangles are the best possible pan shape within the search space of rounded rectangles. We assumed that increasing the number of pans in the oven did not affect the heat distribution among the pans. This is partly justified because the pans we selected, the more circle like stretched rectangle are penalized less due to air though their rounded corners. This means that accounting for varying nonuniform heat should help prove our results.

The two next major steps we would take would be extending the search space of rounded rectangles to included elliptical corners and to better model the boundary conditions of the pan.

Brownie Gourmet Magazine Ad

Brownie Gourmet Magazine Presents:
The ULTIMATE* Brownie Pan
(*Statement not evaluated by the FDA)

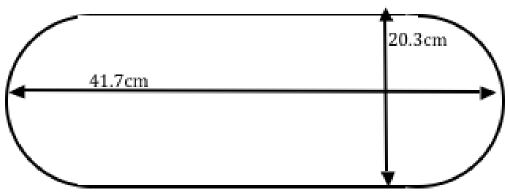
Tired of burnt edges? Fed up with strangely shaped pans?

Do you want a pan that will make you the most even, perfect brownies? Or do you find that 2 batches of brownies just isn't enough? Are you looking to fit the maximum number of batches of brownies in your home oven?

We can find a brownie pan to fit all of your baking needs. We will minimize those crunchy edges that result from corners where heat is concentrated, as well as making sure that you can make the best use of your oven space. Our method is general enough that we can fit a pan to your needs. Is the biggest issue for you burnt edges? We can show you pans that give great edge results. Is your problem that you can't make enough brownies at once? We can show you how to select a pan shape that will absolutely maximize your space usage. And if you are somewhere in the middle? Are you not willing to sacrifice quality, but need brownies for a crowd? Tell us how many pans have to fit, and we can find you a pan shape that gets the best edges? Or if you are a stickler for quality, we can find a pan that is almost as good, but maybe fit a whole other pan per oven rack.

Don't believe us?
We examine a whole range of rectangular pans with rounded corners, to find the best pans in the world for your needs. Depending on your preference, we select the best pan for evenly cooked brownies for the space efficiency that you need, or find the most spece efficient pan possible for a the high quality egdes that you love.

Based on our analysis, we want to sell you a pan today. We chose to mass produce the pan that will fit best in the most ovens: our classic model. It slides to the back of your average home oven, so you can line up 3 pans per shelf. Because our revolutionary pan is long and thin, pans in smaller home oven fit easily, as well as make good use of space. For those of you running bigger brownie-baking operations, don't worry! For any large size of oven, our pan will fit in that oven much better than your old pans–less wasted space in the oven, more brownies for you.



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