Heisenberg Lie algebra and Nakajima operators

Reference: Manfred Lehn, Lecture notes on Hilbert schemes

Emma Brakkee

April 21, 2020

Goal

S smooth surface over $\mathbb{C}$

$\mathrm{b}_i(S^{[n]}) = \dim \mathrm{H}^i(S^{[n]}, \mathbb{Q})(n)$ (degree $-2n$ to $2n$)

Göttsche's formula

$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} \mathrm{b}_i(S^{[n]}) t^{i-2n} q^n = \prod_{m=1}^{\infty} \prod_{j=0}^{4} (1 - (-1)^j t^{j-2} q^m)^{-(-1)^j \mathrm{b}_j(S)}$$

Goal: The RHS of Göttsche's formula is the Poincaré series of an irreducible representation of the (infinite-dimensional) *Heisenberg Lie superalgebra*.

Definition

Lie superalgebra: $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a graded linear map $[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for $x, y \in \mathfrak{g}$ homogeneous:

1. $[x, y] = -(-1)^{|x||y|}[y, x]$
2. $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$

Representation of $\mathfrak{g}$ on $\mathbb{Z}/2\mathbb{Z}$-graded vector space V: linear map $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ such that for x,y homogeneous:

1. $x \in \mathfrak{g}_0 \Rightarrow \rho(x)$ homogeneous of deg. 0
   $x \in \mathfrak{g}_1 \Rightarrow \rho(x)$ homogeneous of deg. 1
2. $\rho([x, y]) = \rho(x)\rho(y) - (-1)^{|x||y|}\rho(y)\rho(x)$.

Examples

Let A be the $\mathbb{Q}$-algebra of matrices of the form $\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$, basis:

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 1: (3-dimensional) Heisenberg Lie algebra

$\mathfrak{g} = (A, [\ ,\ ])$ where $[a, b] = ab - ba$.

- $[p, q] = c$, $[q, p] = -c$
- $[p, c] = 0 = [q, c]$

There is a Lie algebra representation of $\mathfrak{g}$ on $M := \mathbb{Q}[x] = S^*(\mathbb{Q}x)$:

$$p \mapsto \frac{\partial}{\partial x}, \ q \mapsto x \cdot (-), \ c \mapsto \mathrm{id}$$

It is irreducible: If $W \subset M$ is invariant and $0 \neq f = \sum_{i=0}^{n} a_i x^i \in W$,

- $W \ni 1 = \frac{1}{n! a_n} p^n f$
- apply $q \Rightarrow$ all powers of x are in W. So $W = M$.

Example 2: Modification, a Lie superalgebra

$\mathfrak{g}' = A = A_0 \oplus A_1 = \langle c \rangle \oplus \langle p, q \rangle$ with $[a, b] = ab - (-1)^{|a||b|} ba$.

- $[p, q] = c = [q, p]$
- $[p, c] = 0 = [q, c] = [c, c]$

There is a representation of $\mathfrak{g}'$ on $M' := \bigwedge^*(\mathbb{Q}x) = \mathbb{Q} \oplus \mathbb{Q}x$:

$$p \mapsto \tfrac{\partial}{\partial x},\ q \mapsto x \wedge (-),\ c \mapsto \mathrm{id}\,.$$

Poincaré series:

1. $\sum_{n \geq 0} \dim_{\mathbb{Q}}(M_n) t^n = \sum_{n \geq 0} t^n = (1 - t)^{-1}$
2. $\sum_{n \geq 0} \dim_{\mathbb{Q}}(M'_n) t^n = 1 + t$

So both $(1 - \varepsilon t)^{-\varepsilon}$, $\varepsilon = \pm 1$. Compare with Göttsche's formula.

Note: If $H = H_0 \oplus H_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, then

$$\begin{aligned} S^* H &= T^* H / \langle x \otimes y - (-1)^{|x||y|} y \otimes x \mid x, y \text{ homogeneous} \rangle \\ &= S^*(H_0) \otimes \textstyle\bigwedge^*(\Pi H_1) \ \ (\Pi \text{ changes parity}) \end{aligned}$$

Heisenberg Lie superalgebra $\mathfrak{h}$

Fix a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Q}$-vector space H with non-degenerate bilinear form $\langle\, , \,\rangle \colon H \times H \to \mathbb{Q}$ such that

i) $\langle \alpha, \beta \rangle = (-1)^{|\alpha||\beta|} \langle \beta, \alpha \rangle$, α, β homogeneous
ii) $H_0 \perp H_1$.

Let $\mathfrak{h} = \mathfrak{h}(H) := H[t, t^{-1}] \oplus \mathbb{Q}c$ with grading

$$\mathfrak{h}_0 = H_0[t, t^{-1}] \oplus \mathbb{Q}c, \;\; \mathfrak{h}_1 = H_1[t, t^{-1}]$$

and Lie superbracket $[\, , \,]$ given by

$$\begin{cases} [c, u] = 0, & u \in H[t, t^{-1}] \\ [\alpha f, \beta g] = \langle \alpha, \beta \rangle \operatorname{res}_t(g \, \mathrm{d}f) \cdot c, & \alpha, \beta \in H; \; f, g \in \mathbb{Q}[t, t^{-1}] \end{cases}$$

Note: $[\alpha t^n, \beta t^m] = n\delta_{n,-m} \langle \alpha, \beta \rangle \cdot c$

Heisenberg Lie superalgebra $\mathfrak{h}$

- $H = H_0 \oplus H_1$, $\langle\, , \rangle \colon H \times H \to \mathbb{Q}$
- $\mathfrak{h} := H[t, t^{-1}] \oplus \mathbb{Q}c$ with
  - $\mathfrak{h}_0 = H_0[t, t^{-1}] \oplus \mathbb{Q}c, \; \mathfrak{h}_1 = H_1[t, t^{-1}]$
  - $\begin{cases} [c, u] = 0, & u \in H[t, t^{-1}] \\ [\alpha t^n, \beta t^m] = n\delta_{n,-m}\langle \alpha, \beta \rangle \cdot c, & \alpha, \beta \in H \end{cases}$

Remark

Let $\{\alpha^i\}$ be a homogeneous basis of H and $\{\beta^i\}$ its dual basis, so $\langle \alpha^i, \beta^j \rangle = \delta_{ij}$. Then

$$(\mathrm{span}(\alpha^i t^n, \beta^i t^{-n}, c), [\, ,])$$

is isomorphic to Example 1 ($\alpha^i \in H_0$) or 2 ($\alpha^i \in H_1$).

Fock space $\mathbb{F}$

Let $\mathfrak{h}_- := t^{-1}H[t^{-1}] \subset \mathfrak{h}$, so this is $\mathbb{Z}/2\mathbb{Z}$-graded. Let

$$\mathbb{F} = \mathbb{F}(H) := S^*\mathfrak{h}_-,$$

a "super-polynomial ring" (two odd elements anticommute) in the variables $\alpha^i t^{-n}$. The *vacuum element* $\mathbb{1}$ is the unit $1 \in \mathbb{F}$.

There is a representation of $\mathfrak{h}$ on $\mathbb{F}$:

- $c \mapsto \mathrm{id}$
- $\alpha^i t^{-m} \mapsto \alpha^i t^{-m} \cdot (-)$, $m > 0$
- $\beta^j t^m \mapsto m\frac{\partial}{\partial \alpha^j t^{-m}}$, $m \geq 0$.

Lemma

This representation is irreducible.

(Proof as in Example 1)

Weights

Add a $\mathbb{Z}$-grading on $\mathfrak{h}$, the *weight*:

$$\mathrm{wt}(\alpha t^n) := -n, \ \mathrm{wt}(c) := 0.$$

This induces a grading $\mathbb{F} = \bigoplus_{n \geq 0} \mathbb{F}^n$. Poincaré series:

- There is an isomorphism of graded vector spaces

$$\begin{aligned}
\bigoplus_{n \geq 0} \mathbb{F}^n q^n &= S^*\Big(\mathfrak{h}_- = \bigoplus_{m>0} H t^{-m} = \bigoplus_{m>0} H q^m\Big) \\
&= S^*\Big(\bigoplus_{m>0} H_0 q^m \oplus \bigoplus_{m>0} H_1 q^m\Big) \\
&\cong \bigotimes_{m>0} \underbrace{S^*(H_0 q^m)}_{(\text{Ex. } 1)^{\otimes \dim H_0}} \otimes \bigotimes_{m>0} \underbrace{\textstyle\bigwedge^*(H_1 q^m)}_{(\text{Ex. } 2)^{\otimes \dim H_1}}
\end{aligned}$$

- Taking dimensions:

$$\sum_{n \geq 0} \dim(\mathbb{F}^n) q^n = \prod_{m>0} (1 - q^m)^{-\dim H_0} (1 + q^m)^{\dim H_1} \qquad (1)$$

Assume the $\mathbb{Z}/2\mathbb{Z}$-grading on H comes from $\mathbb{Z}$-grading $H = \bigoplus H_i$. Get a bigrading on $\mathbb{F}$:

$$\mathbb{F} = \bigoplus_{n,i} \mathbb{F}^{n,i},$$

bidegree of αt^{-n} is $(n, |\alpha|)$ (α homogeneous). Extension of (1):

$$\sum_{n \geq 0} \sum_{i} \dim(\mathbb{F}^{n,i}) q^n p^i = \prod_{m>0} \prod_{j} (1 - (-1)^j p^j q^m)^{-(-1)^j \dim H^j} \quad (2)$$

Take $H = \mathrm{H}^*(S, \mathbb{Q})(1)$ with $\langle \alpha, \beta \rangle = -\int_S \alpha \smile \beta$.

(2) + Göttsche: $\mathbb{F} = \mathbb{F}(H)$ has the same Poincaré series as

$$\mathbb{H} := \bigoplus_{n,i} \mathbb{H}^{n,i}, \quad \mathbb{H}^{n,i} = \mathrm{H}^i(S^{[n]}, \mathbb{Q})(n).$$

It follows that $\mathbb{F} \cong \mathbb{H}$ as graded vector spaces.

Nakajima: representation of $\mathfrak{h}$ on $\mathbb{H}$ such that $\mathbb{F} \cong \mathbb{H}$ as $\mathfrak{h}$-modules.

Nakajima operators $\mathbb{H} \to \mathbb{H}$

For $\alpha \in H = \mathrm{H}^*(S, \mathbb{Q})(1)$ and $n \geq 0$, we will construct

1. "Creation operator" $\alpha_{-n} \colon \mathbb{H}^{\ell} \to \mathbb{H}^{\ell+n}$
2. "Annihilation operator" $\alpha_{n} \colon \mathbb{H}^{\ell+n} \to \mathbb{H}^{\ell}$

For $n, \ell \geq 0$, let $Z = Z^{\ell,\ell+n}$ be the incidence variety

$$\{(\xi', x, \xi) \mid \xi \subset \xi',\ \rho(\xi') = \rho(\xi) + nx\} \subset S^{[\ell+n]} \times S \times S^{[\ell]}$$

where ρ is the Hilbert–Chow morphism.

- $\dim Z = 2\ell + n + 1$ (possibly irreducible components of lower dimension)
- $[Z] \in \mathrm{H}_{2(2\ell+n+1)}(S^{[\ell+n]} \times S \times S^{[\ell]})$ (Borel–Moore homology)

Nakajima operators $\mathbb{H} \to \mathbb{H}$

- $\alpha \in H = \mathrm{H}^*(S, \mathbb{Q})(1)$ and $n \geq 0$
- $Z \subset S^{[\ell+n]} \times S \times S^{[\ell]}$, p_i projection to i-th factor.

"Creation operator" α_{-n}

$\alpha_{-n} \colon \mathbb{H}^{\ell} \to \mathbb{H}^{\ell+n}$ sends $y \in \mathbb{H}^{\ell} = \mathrm{H}^*(S^{[\ell]})(\ell)$ to

$$\mathrm{PD}^{-1}\, p_{1,*}\big([Z] \frown (p_2^*\alpha \smile p_3^* y)\big)$$

where PD means Poincaré duality.

Equivalently, α_{-n} is the "correspondence" given by $p_{13,*}([Z] \frown p_2^*\alpha) \in \mathrm{H}_*(S^{[\ell+n]} \times S^{[\ell]})$.

If α is homogeneous, then $\alpha_{-n} \colon \mathbb{H} \to \mathbb{H}$ is homogeneous of bidegree $(n, |\alpha|)$.

Nakajima operators $\mathbb{H} \to \mathbb{H}$

- $\alpha \in H = \mathrm{H}^*(S, \mathbb{Q})(1)$ and $n \geq 0$
- $Z \subset S^{[\ell+n]} \times S \times S^{[\ell]}$, p_i projection to i-th factor.

"Creation operator" α_{-n}

$\alpha_{-n} \colon \mathbb{H}^\ell \to \mathbb{H}^{\ell+n}$ sends $y \in \mathbb{H}^\ell = \mathrm{H}^*(S^{[\ell]})(\ell)$ to

$$\mathrm{PD}^{-1}\, p_{1,*}\big([Z] \frown (p_2^*\alpha \smile p_3^* y)\big)$$

Intuition: If α is the class of $A \subset S$ and y the class of $Y \subset S^{[\ell]}$, then $\alpha_{-n}(y)$ is the class of

$$\left\{ W' \in S^{[n+\ell]} \;\middle|\; \begin{array}{l} \exists W \in Y \text{ s.t. } W \subset W', \\ \rho(W') = \rho(W) + nx \text{ for some } x \in A \end{array} \right\}$$

So α_{-n} "adds n-folds of points in A"

Nakajima operators $\mathbb{H} \to \mathbb{H}$

- $\alpha \in H = \mathrm{H}^*(S, \mathbb{Q})(1)$ and $n \geq 0$
- $Z \subset S^{[\ell+n]} \times S \times S^{[\ell]}$, p_i projection to i-th factor.

"Creation operator" α_{-n}

$\alpha_{-n} \colon \mathbb{H}^\ell \to \mathbb{H}^{\ell+n}$ sends $y \in \mathbb{H}^\ell = \mathrm{H}^*(S^{[\ell]})(\ell)$ to

$$\mathrm{PD}^{-1}\, p_{1,*}\big([Z] \frown (p_2^*\alpha \smile p_3^* y)\big)$$

"Annihilation operator" α_n

$\alpha_n \colon \mathbb{H}^{\ell+n} \to \mathbb{H}^\ell$ sends $y \in \mathbb{H}^{\ell+n} = \mathrm{H}^*(S^{[\ell+n]})(\ell+n)$ to

$$(-1)^n\, \mathrm{PD}^{-1}\, p_{3,*}\big([Z] \frown (p_2^*\alpha \smile p_1^* y)\big)$$

- α_{-n} "adds n-folds of points", α_n "subtracts n-folds of points"
- α_n is adjoint to α_{-n} w.r.t. $\langle x, y\rangle_{S^{[\ell]}} := (-1)^\ell \int_{S^{[\ell]}} x \smile y$.

Nakajima relations

Theorem (Nakajima)

Let $\alpha, \beta \in H = \mathrm{H}^*(S)(1)$ and $n, m \in \mathbb{Z}$. Then

$$[\alpha_n, \beta_m] = n\delta_{n,-m}\langle \alpha, \beta \rangle \, \mathrm{id}_{\mathbb{H}} \, .$$

(Here $[\,,\,]$ is the supercommutator and $\langle \alpha, \beta \rangle = -\int_S \alpha \smile \beta$).

Corollary 1

The assignment

$$\mathfrak{h} \ni c \mapsto \mathrm{id}_{\mathbb{H}}, \quad \alpha t^n \mapsto \alpha_n$$

defines a representation of $\mathfrak{h}$ on $\mathbb{H}$.

Corollary 2

$\mathbb{F} \cong \mathbb{H}$ as representations of $\mathfrak{h}$.

Nakajima relations

Corollary 2

$\mathbb{F} \cong \mathbb{H}$ *as representations of* $\mathfrak{h}$.

Proof.

Let $\mathbb{1}_{\mathbb{H}} := 1 \in \mathrm{H}^0(S^{[0]}) = \mathrm{H}^0(\operatorname{Spec}\mathbb{C})$ be the unit of $\mathbb{H}$.

The morphism of algebras $\mathbb{F} \to \mathbb{H}$ determined by $\mathbb{1} \mapsto \mathbb{1}_{\mathbb{H}}$ defines a morphism of Lie superalgebra representations, injective since $\mathbb{F}$ is irreducible. Hence,

$$\dim \mathbb{F}^{n,i} \leq \dim \mathbb{H}^{n,i}.$$

(2) + Göttsche: these dimensions are the same. So this morphism is also surjective. □

Remark

In fact, you don't need Göttsche's formula – see [Lehn, Prop. 4.5]

Proof of Nakajima relations

Theorem (Nakajima)

Let $\alpha, \beta \in H = \mathrm{H}^*(S)(1)$ *and* $n, m \in \mathbb{Z}$. *Then*

$$[\alpha_n, \beta_m] = n\delta_{n,-m}\langle \alpha, \beta \rangle \, \mathrm{id}_{\mathbb{H}} \, .$$

(Here $[\,,\,]$ *is the supercommutator and* $\langle \alpha, \beta \rangle = -\int_S \alpha \smile \beta$*).*

We prove the relations in the case $n = -m$:

$$\begin{aligned} \alpha_n \beta_{-n} - (-1)^{|\alpha||\beta|} \beta_{-n} \alpha_n &= n \langle \alpha, \beta \rangle \, \mathrm{id}_{\mathbb{H}} \\ &= \Big(-n \int_S \alpha \smile \beta \Big) \, \mathrm{id}_{\mathbb{H}} \end{aligned}$$

(the other cases are similar and easier – see e.g. Ellingsrud–Göttsche).

Proof of Nakajima relations

$$\begin{tikzcd}[column sep=small]
 & S^{[\ell]} \times S \times S^{[\ell+n]} \times S \times S^{[\ell]} \arrow[dl, "s_{123}"'] \arrow[d, "s_{1245}"] \arrow[dr, "s_{345}"] & \\
Z^{\ell,\ell+n} \subset S^{[\ell]} \times S \times S^{[\ell+n]} & S^{[\ell]} \times S \times S \times S^{[\ell]} & S^{[\ell+n]} \times S \times S^{[\ell]} \supset Z^{\ell,\ell+n}
\end{tikzcd}$$

Let $w_+ = s_{1245,*}(s_{123}^*[Z^{\ell,\ell+n}].s_{345}^*[Z^{\ell,\ell+n}]) \in \mathrm{H}_*(S^{[\ell]} \times S \times S \times S^{[\ell]})$. Let r_i be the projection from $S^{[\ell]} \times S \times S \times S^{[\ell]}$ to the i-th factor. Then $\alpha_n \beta_{-n}$ is the "correspondence" given by

$$r_{14,*}(w_+ \frown (r_2^*\alpha \smile r_3^*\beta))$$

Note: w_+ is supported on $W_+ = s_{1245}(s_{123}^{-1}(Z^{\ell,\ell+n}) \cap s_{345}^{-1}(Z^{\ell,\ell+n}))$ which has underlying set

$$\left\{ (\xi, x, y, \zeta) \;\middle|\; \begin{array}{l} \exists \eta \text{ s.t. } \xi \subset \eta \supset \zeta, \\ \rho(\eta) = \rho(\xi) + nx = \rho(\zeta) + ny \end{array} \right\}$$

Proof of Nakajima relations

$$\begin{tikzcd}
 & S^{[\ell]} \times S \times S^{[\ell+n]} \times S \times S^{[\ell]} \arrow[dl, bend right, "s_{123}"'] \arrow[dr, bend left, "s_{345}"] \arrow[dd, "s_{1245}"] & \\
Z^{\ell,\ell+n} \subset S^{[\ell]} \times S \times S^{[\ell+n]} & & S^{[\ell+n]} \times S \times S^{[\ell]} \supset Z^{\ell,\ell+n} \\
 & S^{[\ell]} \times S \times S \times S^{[\ell]} & \\
Z^{\ell,\ell-n} \subset S^{[\ell]} \times S \times S^{[\ell-n]} & & S^{[\ell-n]} \times S \times S^{[\ell]} \supset Z^{\ell-\ell+n} \\
 & S^{[\ell]} \times S \times S^{[\ell-n]} \times S \times S^{[\ell]} \arrow[ul, bend left, "t_{123}"] \arrow[ur, bend right, "t_{345}"'] \arrow[uu, "t_{1245}"] &
\end{tikzcd}$$

Similarly, $\beta_{-n}\alpha_n$ is defined by $w_- = t_{1245,*}(t_{123}^*[Z^{\ell,\ell-n}].t_{345}^*[Z^{\ell,\ell-n}])$, supported on a scheme W_- with underlying set

$$\left\{ (\xi, x, y, \zeta) \;\middle|\; \begin{array}{l} \exists \eta \text{ s.t. } \xi \supset \eta \subset \zeta, \\ \rho(\eta) = \rho(\xi) - nx = \rho(\zeta) - ny \end{array} \right\}$$

Proof of Nakajima relations

Steps of proof:

- W_+ and W_- have dimension $2\ell + 2$
- W_+ has two irreducible components of this dimension:
  1. $W' :=$ closure of image of
  $$S^{[\ell-n]} \times B^n \times B^n \dashrightarrow S^{[\ell]} \times S \times S \times S^{[\ell]}$$
  $$(\eta, \eta', \eta'') \mapsto (\eta \cup \eta', \rho(\eta'), \rho(\eta''), \eta \cup \eta'')$$
  where B^n is the n-th Briançon variety.
  2. $\Delta = \{(\xi, x, x, \xi) \mid \xi \in S^{[\ell]}, x \in S\}$

  Hence, $[W_+] = a_+[W'] + N[\Delta]$ for some a_+ and N.
- W_- has one irreducible component of this dimension, namely W'. So $[W_-] = a_-[W']$ for some a_-.
- $a_+ = 1 = a_-$.

Want to show: $\alpha_n \beta_{-n} - (-1)^{|\alpha||\beta|} \beta_{-n} \alpha_n = \big(-n \int_S \alpha \smile \beta \big) \, \mathrm{id}_{\mathbb{H}}$

$\alpha_n \beta_{-n} - (-1)^{|\alpha||\beta|} \beta_{-n} \alpha_n$ is the "correspondence" given by

$$(-1)^n r_{14,*}\Big(w_+ \frown (r_2^*\alpha \smile r_3^*\beta) - (-1)^{|\alpha||\beta|} w_- \frown (r_3^*\beta \smile r_2^*\alpha) \Big)$$
$$= (-1)^n r_{14,*}\Big([W_+] \frown (r_2^*\alpha \smile r_3^*\beta) - [W_-] \frown (r_2^*\alpha \smile r_3^*\beta) \Big)$$
$$= (-1)^n r_{14,*}\Big(N[\Delta] \frown (r_2^*\alpha \smile r_3^*\beta) \Big)$$

Recall: $\Delta = \{(\xi, x, x, \xi) \in S^{[\ell]} \times S \times S \times S^{[\ell]}\}$. It follows that

$$(-1)^n r_{14,*}\Big(N[\Delta] \frown (r_2^*\alpha \smile r_3^*\beta) \Big) = (-1)^n N\Big(\int_S \alpha \smile \beta \Big) [\Delta_{S^{[\ell]}}]$$

$$\alpha_n \beta_{-n} - (-1)^{|\alpha||\beta|} \beta_{-n} \alpha_n = (-1)^n N\Big(\int_S \alpha \smile \beta \Big) \, \mathrm{id}_{\mathbb{H}} \, .$$

Lemma (Ellingsrud–Strømme)

$N = (-1)^{n-1} n.$