Models of curves and abelian varieties

I Introduction

1) Generalities

R Dedekind ring (normal noetherian integral domain of dimension $\leqslant 1$)

$K = \mathrm{Frac}(R)$

$\eta = \mathrm{Spec}(K) \longrightarrow S = \mathrm{Spec}(R) \hookleftarrow \sigma$ closed point ($\neq \eta$)

ex
- fields
- discrete valuation rings : $k[t]_{(t)}$, $\mathbb{Z}_{(p)}$, $k[[t]]$, $\mathbb{Z}_p$ $\cdots$
- rings of integers of a number field ; $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}\left[\frac{1+\sqrt{2}}{2}\right]$, $\cdots$
- rings of functions on a regular affine curve : $k[t]$, $\cdots$
- $\cdots$

(can also look at non-affine, non-connected Dedekind schemes)

def X scheme over η. A <u>model</u> of X is a pair $(\mathfrak{X}, i)$ with $\mathfrak{X}$ S-scheme and $i : \mathfrak{X} \times_S \eta \xrightarrow{\sim} X$.
A <u>morphism of models</u> $(\mathfrak{X}, i) \longrightarrow (\mathfrak{X}', i')$ is

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\equiv"] & \mathfrak{X}' \arrow[dl] \\ & S & \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} \mathfrak{X} \times_S \eta \arrow[rr] \arrow[dr, "i"'] & \arrow[d, phantom, "\equiv"] & \mathfrak{X}' \times_S \eta \arrow[dl, "i'"] \\ & X & \end{tikzcd}$$

rmk The morphism i is often uniquely determined and we often omit it from the notation.

Goal: Given a class of varieties, want to understand models with specified properties.

ex: Can ask for $\mathfrak{X}/S$
- faithfully flat (almost always)
- of finite type over S (if X is)
- reduced, integral, normal, regular (if X is)
- separated, proper (if X is)
- smooth (if X is)
- group scheme (extending a group structure on X)
- $\cdots$

rmk: Let $X \hookrightarrow \mathbb{P}^n_\eta$ be projective, defined by homogeneous equations $F_1, \dots, F_m$. We get a flat projective model of X by taking the Zariski closure of X in $\mathbb{P}^n_S$

$$\Updownarrow$$

"chasing denominators in $F_1, \dots, F_m$"

Naive construction, provides raw material for more sophisticated constructions.

def: (for this intro)
A curve over η is a smooth projective geometrically connected scheme over η, of dimension ≤ 1.

def: An abelian variety over η is a smooth projective (geometrically) connected group scheme over η. ($\Rightarrow$ commutative!)

common ground: elliptic curves $\begin{cases} \text{curves of genus } 1 \text{ with } E(K) \neq \emptyset \\ \text{abelian varieties of dim. } 1 \end{cases}$

2) Four fundamental results

- Existence of minimal regular models of curves.
- Existence of Néron models of abelian varieties
- Stable reduction theorem for curves.
- Semi-abelian reduction theorem for abelian varieties.

Minimal regular models

The ideal model of a curve C would be a smooth projective one.
When this exists, we say that C has good reduction.
If $g(C) \geq 1$, the model is then unique up to unique iso.

thm (Lipman '78; Lichtenbaum–Shafarevich ~'66–'68)
Let C be a curve over η. Then C has projective regular models. Moreover, if $g(C) \geq 1$:
i) there exists one such model $\mathcal{C}^{reg}$ which is minimal: for any such model $\mathcal{C}$, any birational map $\mathcal{C} \dashrightarrow \mathcal{C}^{reg}$ is a morphism.
ii) there exists one such model $\mathcal{C}^{nc}$ whose reduced special fibers are normal crossings divisors, and minimal for this property.

- In particular, $\mathcal{C}^{reg}$ (resp. $\mathcal{C}^{nc}$) is a terminal object in the category of projective regular models of C (resp ...), hence unique up to a unique iso; C has good red $\iff \mathcal{C}^{reg}$ smooth
$\iff \mathcal{C}^{nc}$ smooth

Néron models

The ideal model of an abelian variety would be an abelian scheme, i.e, a smooth projective group scheme over S with connected fibers. When this exists, we say that the abelian variety has good reduction, and the abelian scheme model is unique up to a unique iso.

def Let X/η be a smooth scheme. A Néron model of X is a smooth model $\mathcal{N}$ of X such that, for all smooth S-schemes $\mathcal{Z}$, the restriction map
$$\mathrm{Hom}_S(\mathcal{Z}, \mathcal{N}) \longrightarrow \mathrm{Hom}_\eta(\mathcal{Z}_\eta, X)$$
is a bijection.

- In particular, $\mathcal{N}$ is terminal in the category of smooth models of X.
- If X is a group scheme over η, then $\mathcal{N}$ is a group scheme over S in a compatible way.

<u>thm</u> (Néron '64)
Let A be an abelian variety over η. Then A admits a Néron model, which is quasi-projective.

- The structure of the $\begin{cases} \text{minimal regular model of a curve} \\ \text{Néron model of an abelian variety} \end{cases}$

can be quite complicated. The other two major results tell us that, if we are ready to extend K, the situation improves a lot.

Stable reduction

<u>def</u>
- Let k be an algebraically closed field.
- A <u>semi-stable curve</u> over k is a reduced finite type k-scheme C of dimension 1, such that
  C has only nodal singularities: if $x \in C$ is singular, then $\widehat{\mathcal{O}}_{C,x} \simeq k[[u,v]]/(uv)$.
- We say that C is <u>stable</u> if C is moreover proper, connected, of arithmetic genus ≥ 2, and if any irred. component $\simeq \mathbb{P}^1$ meets other components in ≥ 3 pts.
- A morphism $\mathcal{C} \to T$ is a <u>stable</u> (resp. <u>semistable</u>) <u>curve</u> if it is proper flat and its geometric fibers are stable (resp. semistable) curves.

<u>rmk</u> Stable curves have other equivalent, more conceptual definitions.

rmk | The total space of a (semi-)stable curve over a regular base like S may not be regular but tends to have "mild" singularities. (basis of de Jong's res. of sing by alterations!)

thm | (Stable reduction; Deligne-Mumford '69)
Let C be a curve over η, of genus ≥ 2.
There exists a finite separable extension L/K such that C_L admits a stable model over the normal closure S_L of S in L.

· The stable model is unique if it exists, while there can be many different semistable models. In fact:

prop | C admits a stable model
$\Updownarrow$
$\mathcal{C}^{reg}$ semistable
$\Updownarrow$
$\mathcal{C}^{nc}$ semistable.

Semi-abelian reduction

def | A <u>semiabelian variety</u> over a field k is a smooth k-group scheme G which is an extension of an abelian variety by a torus:

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1$$

($\Rightarrow$ commutative)

Recall also that a finite type group scheme G over a field has an identity component G° (connected comp. of e_G)

thm (Semiabelian reduction; Grothendieck '67-'68)
Let A be an abelian variety over η.
There exists a finite separable extension L/K such that, if $\mathcal{N}$ is the Néron model of A_L (over S_L), then for every $\sigma \in S$,
$\mathcal{N}_\sigma^\circ$ is a semiabelian variety.

- Note that $\mathcal{N}_\sigma \neq \mathcal{N}_\sigma^\circ$ even when A already has semiabelian reduction.

rmk: This course will focus on the geometry of models, rather than on their numerous applications to arithmetic geometry, diophantine geometry and moduli theory.
Any serious treatment of any of these applications would require another course and another lecturer!
Nevertheless you are encouraged to force me to learn some of these and explain them to you.