University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 12: (Co)limits of infinity-categories and Joyal extension

1 (Transitivity property of pushout squares) Let C be an ∞-category. Suppose given a map $\sigma : \Delta^2 \times \Delta^1 \to C$. We can depict this as a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z'. \end{tikzcd}$$

This is slightly misleading: what is not depicted here? Nevertheless, for every subset A of $\{x, y, z, x', y', z'\}$, write $X(A)$ for the full ∞-subcategory of $X = \Delta^2 \times \Delta^1$ spanned by A and $\sigma(A)$ for the restriction of σ to $X(A)$. We assume that the left square is a pushout in C. The goal of the exercise is to show that the right square is a pushout iff the outer square is a pushout, following [HTT, Lemma 4.4.2.1].

- Show that the natural map $C_{\sigma(y,z,y',z')/} \to C_{\sigma(z,y',z')/}$ is a trivial fibration. (Hint: this is where the assumption is used.)
- Show that the map $C_{\sigma(y,z,x',y',z')/} \to C_{\sigma(z,x',z')/}$ is the composite of $C_{\sigma(z,x',y',z')/} \to C_{\sigma(z,x',z')/}$ with a pullback of the map from the previous question. Deduce that it is also a trivial fibration.
- Show that the map $X(y, x', y') \to X(y, z, x', y', z')$ is left anodyne. Deduce that $C_{\sigma(y,z,x',y',z')/} \to C_{\sigma(y,x',y')/}$ is a trivial fibration.
- Recall from exercise sheet 11, exercise 3 that if $F : C \to D$ is a trivial fibration of ∞-categories, then if D has an initial object, so does C. Prove that the converse is true as well: if C has an initial object c, then $F(c)$ is an initial object of D.
- Prove that the right square is a pushout iff the outer square is a pushout.

2 Let $F : X \to K$ be a left or right fibration of simplicial sets. Assume that that K is a Kan complex.

- Prove that X is an ∞-groupoid (hint: left/right fibrations are inner fibrations, and are conservative), hence (by a result we will see first thing next lecture) a Kan complex.
- Deduce from the Joyal lifting theorem that F admits the right lifting property with respect to horn inclusions $\Lambda^n_k \to \Delta^n$ with $n \geqslant 2$ and $0 \leqslant k \leqslant n$.
- Show that F also has the right lifting property with respect to $\Lambda^1_k \to \Delta^1$ with $k = 0, 1$. (Hint: left/right fibrations are isofibrations).
- Deduce that F is a Kan fibration.

3 Let C be an ∞-category which admits an initial object $\varnothing$ and a terminal object $*$. Prove that C is a pointed ∞-category iff there exists a morphism $* \to \varnothing$.