# Seminar Amsterdam/Nijmegen Spring 2020: Hilbert schemes of points on surfaces and their cohomology

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## Overview

Let $X$ be a smooth quasi-projective algebraic variety over a field $k$. One of the most basic examples of moduli spaces in algebraic geometry is the configuration space $\mathrm{Conf}^n(X)$ of (unordered) $n$ distinct points on $X$, which is simply the quotient of the complement of the union of all the diagonals in $X^n$ by the obvious action by the symmetric group $\Sigma_n$. The geometry of the smooth quasi-projective variety $\mathrm{Conf}^n(X)$ is very interesting but still controlled in some sense by the geometry of $X$.

One defect of $\mathrm{Conf}^n(X)$ is that it is not projective, even when $X$ is. It is natural to ask for compactifications of $\mathrm{Conf}_n(X)$. The first thing to look at is the symmetric powers $\mathsf{Sym}^n(X) := X^n/\Sigma_n$. They are the moduli spaces of effective 0-cycles on $X$, which means that we only record the multiplicity when points come together. When $X = C$ is a curve, $\mathsf{Sym}^n(C)$ is a smooth variety which plays a key role in the study of the geometry of $C$. However $\mathsf{Sym}^n(X)$ is singular as soon $\dim(X) > 1$ and $n > 1$.

In this seminar, we study the next case where $X = S$ is a surface. We will see that in this case, the *Hilbert scheme* $\mathrm{Hilb}^n(S)$, which is the moduli of 0-dimensional subschemes of length $n$ in $S$, is a smooth, irreducible variety which is a resolution of singularities of $\mathsf{Sym}^n(S)$ (and hence a compactification of $\mathrm{Conf}^n(S)$). Note that in dimensions greater than 2 and for general $n$, the Hilbert scheme is much less well behaved and no smooth functorial compactification of $\mathrm{Conf}^n(X)$ is known.

The geometry of $\mathrm{Hilb}^n(S)$ is very interesting and related with the geometry of $S$ in a subtle way. We will focus on a special instance of this general phenomenon: the computation of the cohomology of $\mathrm{Hilb}^n(S)$ in terms of the one of $S$. For concreteness, we assume that $k = \mathbb{C}$ and we consider Betti cohomology cohomology with rational coefficients, but similar considerations apply to $\mathbb{Q}_\ell$-adic cohomology, for instance, and in fact some results will be "motivic" in nature. One of the key ideas will be that the cohomology of individual Hilbert schemes $\mathrm{Hilb}^n(S)$ is complicated, and that it pays to consider them for all $n \geq 0$ simultaneously.

After learning basic facts about the geometry of $\mathrm{Hilb}^n(S)$, the first main result we will study is a formula due to Göttsche for the Betti numbers. It takes the form of a generating series:

$$\sum_{n\geq 0, i\geq 0} b_i(\mathrm{Hilb}^n(S))\, t^i q^n = \prod_{m>0, 0\leq j\leq 4} (1-(-1)^j t^{2m-2+j} q^m)^{(-1)^{j+1} b_j(S)}.$$

From this formula it is possible to extract formulas for the individual $\mathrm{Hilb}^n(S)$, but they are very complicated and one sees right away the advantage of looking at all the Hilbert schemes at once.

The second main result we will study is a construction due to Nakajima (and independently Grojnowski). They construct an action of a certain infinite dimensional graded Lie algebra $\mathfrak{h}_{H^*(S)}$ (which depends on $H^*(S)$ equipped with its intersection form), the Heisenberg Lie algebra, on the bigraded vector space

$$\mathbb{H}(S) := \sum_{n=0}^{\infty} H^*(\mathrm{Hilb}^n(S), \mathbb{Q})$$

which identifies $\mathbb{H}(S)$ with an explicit representation of $\mathfrak{h}_{H^*(S)}$ called the Fock space representation. The Lie algebra $\mathfrak{h}_{H^*(S)}$ is defined by generators and relations, and the proof proceeds by constructing algebraic correspondences between the various $\mathrm{Hilb}^n(S)$ and checking that the induced operators satisfy the required relations in cohomology.

There are important applications of Nakajima operators to the determination of the cup product on $H^*(\mathrm{Hilb}^n(S))$ (for a *fixed n*). In particular, Lehn and Sorger gave a very explicit description of this cup product when $S$ has trivial canonical bundle [11]. Unfortunately the proof is very technical and computational in nature, and it makes little sense to present it in this seminar.

Hilbert schemes of surfaces have found many important applications in algebraic geometry (construction of most of the known examples of projective hyperkähler manifolds, study of moduli spaces of sheaves on surfaces), combinatorics and representation theory (Haiman's work on Macdonald polynomials, generalized McKay correspondence), .... Most recently, they have played a central role in new advances on the P=W conjecture on moduli spaces of Higgs bundles; in particular, the work of Shen and Zhang [14], which is the topic of the Intercity seminar this year, has the theorem of Lehn and Sorger mentioned above as central input.

Our main references will be Bertin's lecture notes [1], Lehn's lecture notes [10], Ellingsrud-Göttsche's lecture notes [5] and a paper by de Cataldo and Migliorini [4].

It is perfectly fine to work over the complex numbers all the way through; you are nonetheless welcome to point out which constructions and results work over a more general field (non necessarily algebraically closed or of characteristic 0).

## 1 Talk 1: Hilbert schemes I

We define the Hilbert schemes and the Hilbert-Chow morphism, look at some examples, and prove Fogarty's theorem.

- Define the symmetric powers of a quasi-projective algebraic variety. Show that they are smooth for a curve and singular otherwise. A good reference is [2, §7.1].
- Define the Hilbert functor of a smooth projective variety, and the subfunctors corresponding to a fixed Hilbert polynomial. State Grothendieck's representability theorem (and its immediate extension to quasi-projective schemes). State Hartshorne's connectedness theorem. We will not prove these theorems in general but only in our special case.
- State and prove the formula for the tangent space to the Hilbert scheme at a point [10, Theorem 3.2].
- (Optional) Sketch the proof of the representability theorem in the special case of Hilbert scheme of points as explained in [1, §2.1]. Note that this construction does not show a priori that $\mathrm{Hilb}^n(X)$ is projective when $X$ is, this requires a small separate argument [1, Proposition 2.13].
- Construct the Hilbert-Chow morphism. This is somewhat subtle at the scheme level and discussed in [1, Theorem 2.16]. Again, the method does not show projectivity of the morphism a priori but this can be shown separately, see [1, Remark 2.19]. Show that the Hilbert-Chow morphism is birational.
- Let $C$ be a smooth curve. Prove that $\mathrm{Hilb}^n(C)$ is representable by $\mathsf{Sym}^n(C)$ (and the Hilbert-Chow morphism is an isomorphism).
- Let $X$ be a smooth quasi-projective variety. Prove that we have

  $$\mathrm{Hilb}^2(X) = \mathrm{Bl}_\Delta \mathsf{Sym}^2(X)$$

  in such a way that the Hilbert-Chow morphism is the blow-up map. For this, one can look at the sequence of exercises in [2, 7.3.E.(2)-(5)].
- Discuss the incidence schemes and the "induction diagram". Use it to prove Hartshorne's theorem in the special case of Hilbert schemes of points: if $X$ is connected, then $\mathrm{Hilb}^n(X)$ is connected [10, §3.3] [1, Propositions 2.10, 2.20, 2.21].
- Prove that, if $S$ has an algebraic symplectic form (for instance if $S$ is a K3 surface or an abelian surface), then $\mathrm{Hilb}^n(S)$ has also an algebraic symplectic form induced from the one on $S$ (Beauville) [13, Theorem 1.17]. If you want, you can have a look at the special case of $\mathrm{Hilb}^2(X)$ which is more explicit in [9, §2.1].

## 2 Talk 2: Hilbert schemes II

We study in more details the geometry of Hilbert schemes and the Hilbert-Chow morphism.

- Prove Fogarty's theorem stating that $\mathrm{Hilb}^2(S)$ is smooth if $S$ is a smooth surface. There are two variants of the proof: one uses some homological algebra of regular local rings [1, Theorem 3.1] (note that he also has a different argument for connectedness, which we will revisit in the next talk), the other uses the Hirzebruch-Riemann-Roch theorem [10, Theorem 3.3].
- Show that $\mathrm{Hilb}^n(X)$ is smooth for any smooth quasi-projective variety when $n \leq 3$. Show that $\mathrm{Hilb}^4(\mathbb{A}^3)$ is singular [10, Corollary 3.4, Remark 3.5]. The situation gets even worse: for any smooth quasi-projective variety $X$ of dimension $d \geq 3$ and any $n$ large enough, $\mathrm{Hilb}^n(X)$ is reducible, with some irreducible components of dimension greater than $nd$ (Iarrobino). Concretely, it means that most 0-dimensional subschemes in $X$ cannot be obtained as the limit of a family of reduced subschemes.
- Introduce the punctual Hilbert schemes, which are the fibers of the Hilbert-Chow map over a 0-cycle of the form $n \cdot x$ for $x \in S$ closed point. Explain why the punctual Hilbert scheme of any smooth surface $S$ at any closed point is (non-canonically) isomorphism to the punctual Hilbert scheme of $\mathbb{A}^2$ at 0 (under the assumption $k = \bar{k}$). Define curvilinear 0-dimensional subschemes and prove that curvilinear points are open in the punctual Hilbert scheme [1, §2.3.2].
- State and prove Briançon's theorem following [10, Theorem 3.10]. A nice interpretation of this theorem is that the curvilinear locus is dense in the punctual Hilbert scheme. Note that the connectedness of the punctual Hilbert scheme together with the easy fact that $\mathsf{Sym}^n(X)$ is connected whenever $X$ is connected gives another proof of the connectedness of $\mathrm{Hilb}^n(S)$ established in the previous talk. Along the way, you can also prove that the boundary of the Hilbert scheme is irreducible as the argument is very similar [10, Proposition 3.9].

## 3 Talk 3: Göttsche's formula I

We first look at a direct computation for $S = \mathbb{A}^2$ and $S = \mathbb{P}^2$. Along the way we get more details about the geometry of $\mathrm{Hilb}^n(\mathbb{A}^2)$.

- State Göttsche's formula on Betti numbers above. State its corollary for the generating function of the Euler characteristics of the $\mathrm{Hilb}^n$. Explain why, in the particular case of a K3 surface, this last generating function is a modular form.
- We start by looking at a computation for the Betti numbers in the cases $S = \mathbb{P}^2$ and $S = \mathbb{A}^2$, by following the argument of Ellingsrud and Stromme. The method also gives some interesting information on the punctual Hilbert scheme. State the main theorem [6, Theorem 1.1]. State [6, Corollary 1.3].
- Define cellular decompositions and state (and sketch the proof of) [6, Proposition 1.5]. State [6, Theorem 1.6].
- Define the torus action on $\mathrm{Hilb}^n(\mathbb{P}^2)$ (and the similar action on $\mathrm{Hilb}^n(\mathbb{A}^2)$) and explain its geometric consequences [6, §2]. In particular, note that this shows that the punctual Hilbert scheme at a fixed point of the torus action is cellular; since we have seen in the previous talk that all punctual Hilbert schemes of smooth surfaces are isomorphic, this shows that those are always cellular as well. This is only one of the many uses of this torus action; a lot of recent results on $\mathrm{Hilb}^n(\mathbb{A}^2)$ use it in a crucial way.
- Explain why how to use the torus action to construct a nice open cover of $\mathrm{Hilb}^n(\mathbb{A}^2)$ and prove directly that $\mathrm{Hilb}^n(\mathbb{A}^2)$ is smooth without appealing to Fogarty's theorem [1, §3.2.1].
- The rest of the argument in [6] is a computation, with most of the effort devoted to computing the weights of the 2-dimensional torus action on the tangent spaces to the fixed points. Rather than doing it in detail, try to convince us that it is possible in principle, and explain how to conclude (for instance, prove [6, Proposition 4.1], which is enough for $S = \mathbb{A}^2$).

- Check that [6, Theorem 1.1] implies Göttsche's formula for $S = \mathbb{A}^2$ (the computation for $S = \mathbb{P}^2$ is also possible but a little more involved).

- Sketch Nakajima's construction of $\mathrm{Hilb}^n(\mathbb{A}^2)$ as a quiver variety [1, §3.2.4] [13, Section 1.2].

# 4 Talk 4: Göttsche's formula II

We prove Macdonald's formula, and set the stage for the next talk by recalling some intersection theory.

- State and prove Macdonald's formula for the generating function of the Betti numbers of the symmetric powers of any CW complex [12, Formula 8.5] (in the case $G = \Sigma_n$; Macdonald talks about "compact polyhedra" but the proof applies much more generally). The paper of Macdonald requires a little bit of representation theory of the symmetric group and symmetric functions but is otherwise short and readable.

- Recall the definition of Chow groups of algebraic varieties over a field. The next talk will rely quite a bit on intersection theory for Chow groups of smooth varieties. You should recall some background material on intersection theory from [8] and explain the refined Gysin formalism of [3, §3-4].

# 5 Talk 5: Göttsche's formula III

We combine Macdonald's formula with the geometry of the Hilbert-Chow map to prove Göttsche's formula in general.

- State [4, Theorem 7.1.1] (you could omit the Hodge structure part and the Chow motives part and just talk about Chow groups and cohomology). Explain why the statement on Chow groups implies the one on cohomology (hint: the cycle class map sends algebraic correspondences to cohomological correspondences). Explain why the cohomological part of this theorem, together with Macdonald's formula, implies Göttsche's formula (this is a simple computation with generating functions).

- Recall the notion of semismall maps. Define relevant strata (those for which the inequality defining semismall maps is an equality).

- Deduce the dimension of the fibers of the Hilbert-Chow morphism from Briancon's theorem [1, Corollary 3.5]. Deduce that the Hilbert-Chow morphism is semi-small.

- State [4, Theorem 4.0.4], and explain how, together with the semismallness of the Hilbert-Chow morphism and Briançon's theorem, it implies [4, Theorem 7.1.1]. Another nice remark to make is that the cohomological version of [4, Theorem 4.0.4] implies the degeneracy of the Leray spectral sequence for the semismall morphism. If you like perverse sheaves, you can explain why [4, Theorem 4.0.4] is a cycle-theoretic refinement of the Decomposition theorem for perverse sheaves in a special case (this is closely related with the degeneracy of the Leray spectral sequence).

- Prove [4, Theorem 4.0.4].

# 6 Talk 6: Heisenberg Lie algebra and Nakajima operators

We construct the action of a certain Lie algebra on the cohomology of the Hilbert schemes.

- Discuss the toy examples of infinite dimensional Lie algebras, their natural irreducible modules and Poincaré series in [10, Example 4.2].

- Define the Heisenberg (super)Lie algebra and its natural irreducible representation [10, §4.3]. One can also look at [13, §8.1].

- Define the Nakajima operators on cohomology and state the main relations [10, Theorem 4.4]. Explain [10, Corollary 4.5]. Explain the remark on the Leray spectral sequence which shows that Göttsche's formula actually follows from [10, Theorem 4.4].

- Prove the relations, modulo the determination of the precise constant $(-1)^{n-1}n$. For this, one can look at [10, §4.4], but also at [5, §6].

- If you have time, sketch the proof by Ellingsrud and Stromme of the determination of the constant [10, Theorem 4.6] [7]. The argument uses the same induction scheme that was used in Talk 2 to prove Briancon's theorem.

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