Six-functor formalism; why solid abelian groups?

Simon Pepin Lehalleur

November 10, 2022

These notes are based on a talk given on Friday 11th in the Intercity seminar on "Condensed Mathematics".

Contents

1 Introduction

Cohomologies want to be sheaf theories
Sheaf theories want to be six-functor formalisms

Algebraic geometry is famous for having (too?) many cohomology theories. Let us list some:

- Coherent cohomology $\mathrm{H}^*_{\mathrm{coh}}(-, \mathcal{O})$ (with the simplest coherent coefficient sheaf possible $\mathcal{O}$ to start with)
- Singular (or Betti) cohomology $\mathrm{H}^*_{\mathrm{B}}(-, \mathbb{Z})$
- ℓ-adic cohomology $\mathrm{H}^*_{\ell}(-, \mathbb{Z}_\ell)$
- Algebraic de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(-/k)$
- Rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(-/K)$

These cohomology theories only make sense (or are well-behaved) in certain contexts:

- Coherent cohomology makes sense for arbitrary schemes over some ring R, and produces R-modules. For simplicity in this talk I will always consider finite type R-schemes with R Noetherian.

- Betti cohomology is defined for finite-type $\mathbb{C}$-schemes, and produces abelian groups.
- ℓ-adic cohomology works best over an algebraically closed field k of characteristic different from the fixed prime ℓ, and produces $\mathbb{Z}_\ell$-modules.
- Algebraic de Rham cohomology works best for finite-type k-schemes with k a field of characteristic 0, and produces k-vector spaces.
- Rigid cohomology is defined for k-varieties with k the characteristic p residue field of a characteristic 0 non-archimedean valued field K, and produces K-vector spaces.

Grothendieck, who contributed in important ways to the development of all these theories, famously observed that Betti, ℓ-adic, de Rham and rigid [1] cohomology have very similar properties such as Künneth formulas and Poincaré duality. In the Betti case, these properties are inherited from algebraic topology, but for the others they have to be established in completely different ways.

Grothendieck collectively named them *Weil cohomologies*[2], because of Weil's original insight that such cohomology theories should exist in positive characteristic and that they could help tackle his conjectures on zeta functions over finite fields. Grothendieck conjectured further that the similarities between all Weil cohomologies, and their relationship with algebraic cycles, should be ultimately explained by the existence of universal cohomological objects called *motives*. Because of this deep similarity, we will concentrate in this talk on the Betti case (the most geometrically intuitive and technically the simplest) except for a few remarks, but almost all results hold with small modifications for all Weil cohomologies.

At first sight, coherent cohomology behaves very differently from Weil cohomologies.

Example 1.1. We have

$$\mathrm{H}^*_B(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Z}) = \mathrm{H}^*_{\mathrm{sing}}(\mathbb{C}, \mathbb{Z}) = \mathrm{H}^0_{\mathrm{sing}}(\mathbb{C}, \mathbb{Z}) = \mathbb{Z}[0]$$

because $\mathbb{C}$ is contractible, while we have

$$\mathrm{H}^*_{\mathrm{coh}}(\mathbb{A}^1_A, \mathcal{O}) = \mathrm{H}^0_{\mathrm{coh}}(\mathbb{A}^1_A) = \mathcal{O}(\mathbb{A}^1_A) = A[t][0]$$

The affine line thus "appears contractible" for Weil cohomologies but not for coherent cohomology. This also illustrates another important difference: Weil cohomology groups of finite type k-schemes are always finitely generated, while coherent cohomology groups are often not finitely generated over R even for finite type R-schemes.

Example 1.2. We have

$$\mathrm{H}^*_B(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Z}) = \mathrm{H}^*_{\mathrm{sing}}(S^2, \mathbb{Z}) = \mathbb{Z}[0] \oplus \mathbb{Z}(1)[2]$$

where $\mathbb{Z}(1) := \mathrm{H}^2_B(\mathbb{P}^1_{\mathbb{C}})$ is a free rank 1 abelian group. The object $\mathbb{Z}(1)$ is called a *Tate twist*. Tate twists are defined similarly as $H^2(\mathbb{P}^1)$ in all Weil cohomologies and play a central role in their study. On the other hand we have $\mathrm{H}^*_{\mathrm{coh}}(\mathbb{P}^1_A, \mathcal{O}) = A[0]$, so coherent cohomology has no Tate twists.

Example 1.3. Let R be a non-reduced finite type $\mathbb{C}$-algebra, such as $R = \mathbb{C}[x]/(x^2)$. Then $\mathrm{Spec}(R)^{\mathrm{an}} = \mathrm{Spec}(R^{\mathrm{an}}_{\mathrm{red}})$ hence

$$\mathrm{H}^*_B(\mathrm{Spec}(R), \mathbb{Z}) \simeq \mathrm{H}^*_B(\mathrm{Spec}(R_{\mathrm{red}}), \mathbb{Z})$$

while

$$\mathrm{H}^*_{\mathrm{coh}}(\mathrm{Spec}(R), \mathcal{O}) = R[0] \neq R_{\mathrm{red}}[0] = \mathrm{H}^*_{\mathrm{coh}}(\mathrm{Spec}(R_{\mathrm{red}}), \mathcal{O}).$$

In general, Weil cohomologies do not see the difference between X and X_{red}, and in fact between X and Y where $f : X \to Y$ is a universal homeomorphism, while coherent cohomology very much does.

[1] which for Grothendieck meant crystalline and was essentially restricted to smooth projective varieties, since rigid cohomology was defined after Grothendieck distanced himself from the mainstream mathematical community.

[2] The name "Weil cohomology" is often used for the restriction of those cohomology theories to smooth projective varieties, but for this talk we are really interested in all varieties.

The main message of this talk is that despite these profound differences, with the right perspective, Weil cohomologies and coherent cohomology actually have a lot in common: they share a very rich functoriality, the *six-functor formalism*. This formalism encodes the fundamental properties common to all these cohomology theories. In particular, it unifies and generalizes Poincaré duality (on the Weil side) and Serre duality (on the coherent side), and also makes the proof of these duality theorems easier and conceptual. The name "six-functor" refers to the six operations

$$f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{Hom}}$$

on derived categories of sheaves; we will explain this notation in detail later.

The Weil/coherent parallel has been understood in some form since the beginnings of the subject: Grothendieck developed parallel duality formalisms first in coherent cohomology [Har66], then (in collaboration with many others) in étale/ℓ-adic cohomology in SGA4 and SGA5, and Verdier developped the Betti case in [Ver95]. However the classical theory of derived categories of quasicoherent sheaves of [Har66] does *not* admit a six-functor formalism, but only a *five*-functor formalism! The basic issue, as we will explain below, is that "coherent cohomology with compact supports" does not make sense classically and there is no coherent $f_!$ functor. This means that the proofs in [Har66] and in subsequent developments of the coherent theory were technical and less natural than in the Betti and ℓ-adic case.

Clausen and Scholze discovered that condensed mathematics can be used to build theories of quasicoherent sheaves and six-functor formalisms for "analytic geometry" in a new, very general sense, including complex analytic geometry, rigid analytic geometry and much more besides [**CS:analytic**; **CS:complex**]. However, their ideas already bring something new to quasicoherent sheaves in algebraic geometry. In particular, their theory of *solid abelian groups* and solid modules can be used to construct a full quasicoherent six-functor formalism and to complete the analogy between the Weil and coherent six-functor formalisms. This is precisely the goal of the rest of this seminar.

Six-functor formalisms have many applications all over algebraic and arithmetic geometry, which we will not discuss at all. On the coherent side, interesting functors and equivalences between derived categories of schemes are almost always *Fourier-Mukai transforms* whose very definition and basic properties are based on the six-functor formalism [Huy06]. On the Weil side, the original and perhaps still most spectacular application is the proof of the Weil conjectures using ℓ-adic cohomology [**WeilII**; SGA5; Del74].

The Betti six-functor formalism is discussed in depth in [KS90, Chapters 2-3]. Here are other general introductions to six-functor formalisms: [**Scholze**; **Gallauer:six**].

2 From cohomologies to sheaf theories

Let X be a finite type R-scheme with structure morphism $\pi_X : X \to \mathrm{Spec}(R)$. Let $\mathrm{F} \in \mathrm{QCoh}(X)$ be a quasicoherent sheaf. We have

$$\mathrm{H}^i_{\mathrm{coh}}(X, \mathrm{F}) \simeq R^i \pi_{X*} \mathrm{F}$$

where

$$R^i \pi_{X*} : \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(A) \simeq A - \mathrm{Mod}$$

is the i-th derived functor of the pushforward functor

$$\pi_{X*} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(A).$$

It is convenient to package all cohomology degrees together by introducing the derived category $\mathbf{D}_{\mathrm{qcoh}}(X)$ of complexes of Zariski sheaves on X with quasicoherent cohomology up to quasi-isomorphism. Then, for any morphism of schemes $f : X \to Y$, we can consider the derived pushforward functor

$$Rf_* : \mathbf{D}_{\mathrm{qcoh}}(X) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(Y)$$

which provides a "relative version" of cohomology. This functor has a left adjoint, the derived pullback functor

$$Lf^* : \mathbf{D}_{\mathrm{qcoh}}(Y) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(X)$$

which is a relative version of the constant sheaf functor. Besides these two functors, basic sheaf theory also provides a derived tensor product functor

$$- \otimes^L - : \mathbf{D}_{\mathrm{qcoh}}(X) \times \mathbf{D}_{\mathrm{qcoh}}(X) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(X)$$

which makes $\mathbf{D}_{\mathrm{qcoh}}(X)$ into a symmetric monoidal category and a derived internal Hom

$$\underline{\mathrm{RHom}}(-,-) : \mathbf{D}_{\mathrm{qcoh}}(X) \times \mathbf{D}_{\mathrm{qcoh}}(X) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(X)$$

which is related to $\otimes^L$ by the adjunction

$$\mathrm{Hom}(\mathrm{F} \otimes^L \mathrm{G}, \mathrm{H}) \simeq \mathrm{Hom}(\mathrm{F}, \underline{\mathrm{RHom}}(\mathrm{G}, \mathrm{H}))$$

The same pattern, restricted to finite type schemes over $\mathbb{C}$, holds for Betti cohomology. Indeed, complex algebraic varieties are nice[3] topological spaces, which implies that Betti cohomology $\mathrm{H}^*_B(X, \mathbb{Z})$ is isomorphic to the cohomology of the constant sheaf $\mathbb{Z}_X$ on $X^{\mathrm{an}} := X(\mathbb{C})$, see [**Petersen**]. We thus have

$$\mathrm{H}^i_B(X, \mathbb{Z}) = \mathrm{H}^i R\pi_{X*}\mathbb{Z}_X[0]$$

with

$$R\pi_{X*} : \mathbf{D}(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow \mathbf{D}(\mathrm{Spec}(\mathbb{C})^{\mathrm{an}}, \mathbb{Z}) \simeq \mathbf{D}(\mathrm{Ab}).$$

the derived pushforward and $\mathbf{D}(X^{\mathrm{an}}, \mathbb{Z})$ the derived category of sheaves of abelian groups on X^{an}. We also have derived pullbacks, derived tensor products and derived internal Homs in this setting.

Let's abstract the common structure obtained so far:

Definition 2.1. Let S be a base scheme and Sch_S be the category of finite type schemes over S. A *sheaf theory* (or *four functor formalism*)[4] is a functor

$$\mathbf{D}(-) : \mathrm{Sch}_S^{\mathrm{op}} \longrightarrow \mathrm{TriCat}^{\otimes}$$

to the category $\mathrm{TriCat}^{\otimes}$ of symmetric monoidal triangulated categories (and symmetric monoidal functors), such that

- for any $f : X \to Y$ in Sch_S, the functor

  $$f^* := \mathbf{D}(f) : \mathbf{D}(Y) \longrightarrow \mathbf{D}(X)$$

  admits a right adjoint

  $$f_* : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y).$$

- The symmetric monoidal structure on each $\mathbf{D}(X)$ is closed, i.e., there are internal Hom bifunctors $\underline{\mathrm{Hom}}$ such that

  $$\mathrm{Hom}(\mathrm{F} \otimes \mathrm{G}, \mathrm{H}) \simeq \mathrm{Hom}(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})).$$

Remark 2.2. (i) Implicit in the definition is the fact that f^* is symmetric monoidal for any f, i.e., we have natural isomorphisms

$$f^*(\mathrm{F} \otimes \mathrm{G}) \simeq f^*(\mathrm{F}) \otimes f^*(\mathrm{G}).$$

This holds in the general context of sheaves on a locally ringed space (by simple properties of tensor products of modules), so in particular in our coherent and Betti examples.

[3] cohomologically locally connected

[4] Neither name is quite standard terminology, but they are convenient for this talk.

(ii) We have dropped the left/right derived functor decorations from the notation, writing e.g f_* for Rf_*, and will do so consistently in the rest of the talk. All the structure and results that matter fo us happen at the level of derived/triangulated categories, and in some examples of sheaf theories we do not even necessarily have underlying abelian categories. Consequently, the word "sheaf" in what follows should be interpreted as "complex of sheaves", or even better "object in some derived category of sheaves".

(iii) Definition 2.1 is not quite right as it stands: neither our coherent or Betti examples satisfy it! The problem is that pullbacks of (complexes of) sheaves are not strictly speaking functorial, but only functorial up to a natural isomorphism. This can be fixed by interpreting $\mathrm{TriCat}^{\otimes}$ as a 2-category and $\mathbf{D}(-)$ as a 2-functor.

However, this issue is only the first of increasingly complicated categorical "coherence" issues in the study of six-functor formalisms, and triangulated categories quickly prove unfit to the task. For instance, one would like to make sense of gluing statements of the form:

"Given an open cover $\{U_i\}$ of X, the category $D(X)$ is the (higher-categorical) limit of the categories $D(U_{i_1} \cap \ldots U_{i_n})$ along the Čech nerve of the cover."

One also wants, given a group G acting on X, to define the category $\mathbf{D}_G(X)$ of G-equivariant sheaves directly in terms of "the induced action of G on $\mathbf{D}(X)$". None of this cannot be made to work in the 2-category $\mathrm{TriCat}^{\otimes}$.

A very effective way to deal with all those issues in one fell swoop is to work with ∞-categories, and to define a sheaf theory as an (∞-)functor $\mathbf{D}(-) : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\mathrm{st},\otimes}$ to the ∞-category of symmetric monoidal stable ∞-categories satisfying the conditions above.

Since these coherence issues are quite orthogonal to the main point of this talk, we will mostly pretend they do not exist and work with triangulated categories without comments.

At this point we have two sheaf theories: $\mathbf{D}_{\mathrm{coh}}(-)$ on Sch_A and $\mathbf{D}_B(-) := \mathbf{D}((-)^{\mathrm{an}}, \mathbb{Z})$ on $\mathrm{Sch}_{\mathbb{C}}$. In fact, each Weil cohomology mentioned in the introduction can be promoted (with considerable technical difficulties) to a sheaf theory:

- ℓ-adic cohomology to the theory of ℓ-adic sheaves [SGA4]
- Algebraic de Rham cohomology to the theory of holonomic $\mathcal{D}$-modules [HTT08]
- Rigid cohomology to the theory of arithmetic $\mathcal{D}$-modules [AL22]

Remark 2.3.

- The sheaf theories described so far have the special property that when k is an algebraically closed field, we have $\mathbf{D}(k) \simeq R-\mathrm{Mod}$ for some coefficient ring R, reflecting the fact that the corresponding cohomology groups are just graded R-modules. Weil cohomology theories carry interesting additional structures, like the mixed Hodge structure on Betti cohomology and the Galois actions on ℓ-adic cohomology, and it would be good to integrate them as well.

  For Galois actions on ℓ-adic cohomology, this is actually already part of the theory of ℓ-adic sheaves (since $\mathbf{D}(k_{\text{ét}}, \mathbb{Z}_\ell)$ is roughly speaking a derived category of $\mathrm{Gal}(\bar{k}/k)$-representations on $\mathbb{Z}_l$-modules when k is not algebraically closed). For mixed Hodge structures, this requires a lot more work and the theory of *mixed Hodge modules* of Saito [Sai86; Sai17].

- Another very interesting source of sheaf theories in the sense above is *motivic homotopy theory* after Morel-Voevodsky. This leads in particular to sheaf theories $\mathrm{SH}(-)$ of *stable motivic homotopy types* and $\mathrm{DM}(-)$ of *motivic sheaves* which are our best current candidates to realise Grothendieck's motivic project [Ayo07a; Ayo07b; CD19]. The categories $\mathrm{SH}(k)$ and $\mathrm{DM}(k)$ are very complicated even when k is a field; morphism groups in $\mathrm{DM}(k)$ for instance encodes intersection theory of Chow groups and higher Chow groups of smooth k-varieties.

- Variants of Definition 2.1 (and of the discussion of six-functor formalisms that follows) make sense in other geometric contexts: complex analytic geometry, rigid analytic geometry [**Mann**; AGV22]... The Betti sheaf theory can clearly extended to arbitrary topological spaces, and the associated six-functor formalism has good properties for locally compact spaces [KS90]

3 Six-functor formalisms: motivation

Why are sheaf theories, as defined above, not enough? Why do we need to go from four to six functors? Here are two important motivations.

First, given a sheaf theory $\mathbf{D}(-)$ as above, we want to use it to "compute" cohomology in interesting ways. For this, we can use natural transformations which are defined completely canonically from $\mathbf{D}(-)$ using the various adjunctions. Here are the most important ones:

- (Base change) Let

$$\begin{array}{ccc} Y' & \xrightarrow{\tilde{g}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a cartesian commutative square in Sch_S. The corresponding base change map is the composition

$$g^* f_* \to \tilde{f}_* \tilde{f}^* g^* f_* \simeq \tilde{f}_* (g\tilde{f})^* f_* \simeq \tilde{f}_* \tilde{g}^* f^* f_* \to \tilde{f}_* \tilde{g}^*$$

where we have used the (co)units of the pullback/pushforward adjunctions and the functoriality of pullbacks[5]. In particular, when f is the inclusion of a closed point $x \in X$, this gives a natural transformation

$$(g_*(-))_x \longrightarrow (g_x)_*(-).$$

which compares "the fibre of the cohomology with the cohomology of the fibre".

- (Projection) By a similar game with adjunctions starting from the fact that f^* is monoidal, we get a natural transformation

$$f_*\mathrm{F} \otimes \mathrm{G} \longrightarrow f_*(\mathrm{F} \otimes f^*\mathrm{G}).$$

- (Künneth) Given two morphisms $f : Y \to X$ and $h : Z \to X$, there is a natural transformation

$$f_*\mathrm{F} \otimes h_*\mathrm{G} \longrightarrow (f \times_X h)_*(\mathrm{F} \boxtimes_X \mathrm{G})$$

where $\boxtimes_X : \mathbf{D}(Y) \times \mathbf{D}(Z) \to \mathbf{D}(Y \times_X Z)$ is the exterior product

$$\mathrm{F} \boxtimes_X \mathrm{G} := p_1^*\mathrm{F} \otimes p_2^*\mathrm{G}.$$

Exercise 3.1. Construct the projection and Künneth maps using adjunctions and monoidality.

In the coherent setting, these canonical maps tend to be isomorphisms (with Tor-independence assumptions for base change) [Stacks, Tags 08ET, 0FLN].

In the Weil setting, these maps are however far from isomorphisms in general. Such maps are only useful for computations if we can understand when they are isomorphisms. There are two basic facts of the matter: base change along open immersions and the the proper base change theorem.

Proposition 3.2. *(Open base change) Let $\mathbf{D}(-)$ be any one of the examples of sheaf theories above (quasicoherent, Betti, other Weil type, motivic...). Let j be an open immersion and g be any morphism. Then the corresponding base change map is an isomorphism:*

$$j^* g_* \mathrm{F} \xrightarrow{\sim} \tilde{g}_* \tilde{j}^* \mathrm{F}.$$

[5]The definition of the map does not use that the square is cartesian, but usually this is necessary for it to have good properties.

This is already very useful because of:

Proposition 3.3. *(Zariski separation) Let $\mathbf{D}(-)$ be any one of the examples of sheaf theories above (quasicoherent, Betti, other Weil type, motivic...). Let $\{j_i : U_i \to X\}_{i\in I}$ be an open covering of a scheme X.*

Then the functors $\{j_i^*\}_{i\in I}$ are jointly conservative: a morphism $\alpha : \mathrm{F} \to \mathrm{G}$ in $\mathbf{D}(X)$ is an isomorphism if and only if $j_i^*(\alpha)$ is an isomorphism for all $i \in I$.

Theorem 3.4. *(Proper base change) Let $\mathbf{D}(-)$ be a Weil or motivic sheaf theory. With the notations above, assume that f and h are proper. Then the base change, projection and Künneth natural transformations are all isomorphisms:*

$$f^*g_*\mathrm{F} \xrightarrow{\sim} \tilde{g}_*\tilde{f}^*\mathrm{F}$$

$$f_*\mathrm{F} \otimes \mathrm{G} \xrightarrow{\sim} f_*(\mathrm{F} \otimes f^*\mathrm{G})$$

$$f_*\mathrm{F} \otimes h_*\mathrm{G} \xrightarrow{\sim} (f \times_X h)_*(\mathrm{F} \boxtimes_X \mathrm{G}).$$

Exercise 3.5. Prove the third isomorphism “formally”, using only the first two (Easy version: prove that there is one such isomorphism; hard version: prove that the isomorphism constructed in the previous exercise is an isomorphism)

From this perspective, six-functor formalisms give us control over when these canonical maps are isomorphisms beyond the proper case.

The second motivation is that, as mentioned in the introduction, we want the formalism to encode duality theorems: Serre duality in the coherent case, Poincaré duality in the Betti - and more generally Weil - case.

Let $X/\mathbb{C}$ be a smooth, non-necessarily proper d-dimensional variety and L be a local system of $\mathbb{Q}$-vector spaces (for simplicity). Then Poincaré duality takes the form

$$\mathrm{H}^i_{B,c}(X, L)^\vee \simeq \mathrm{H}^{2d-i}_B(X, L^\vee \otimes \mathbb{Q}(d))$$

where $\mathrm{H}^*_{B,c}$ denotes cohomology with compact supports and $\mathbb{Q}(d)$ is a Tate twist. So to integrate Poincaré duality into the formalism, we need to treat cohomology and cohomology with compact supports on the same footing.

Serre duality seems of a quite different nature, since it is classically only defined for a *proper* smooth scheme X of relative dimension d over R and a vector bundle E over X:

$$\mathrm{H}^{d-i}_{\mathrm{coh}}(X, E)^\vee \simeq \mathrm{H}^i_{\mathrm{coh}}(X, E^\vee \otimes \Omega^n_{X/R})$$

and there is no cohomology with compact supports in sight. As we will see, the work of Clausen-Scholze removes this restriction to proper schemes and makes the parallels between Poincaré and Serre duality much stronger.

4 Six-functor formalism: Betti

The above discusion of Poincaré duality suggests that, in the same way that we have “relativized” cohomology with (derived) pushforwards, we should have for every morphism $f : X \to Y$ of finite type $\mathbb{C}$-schemes an *exceptional pushforward functor*

$$f_! : \mathbf{D}_B(X, \mathbb{Z}) \longrightarrow \mathbf{D}_B(Y, \mathbb{Z})$$

(pronounced “f lower shriek”) such that

$$\mathrm{H}^i_{B,c}(X, \mathbb{Z}) \simeq \mathrm{H}^i \pi_{X!}\mathbb{Z}_X.$$

In this Betti setting, it is possible to write such a functor very explicitly. Let F be a sheaf of abelian groups on X^{an}. We define $f_!\mathrm{F} \in \mathrm{Shv}(Y^{\mathrm{an}}, \mathbb{Z})$ as

$$(f_!\mathrm{F})(U) = \{s \in \mathrm{F}(f^{-1}(U)) | f : \mathrm{Supp}(s) \to U \text{ is proper}\}.$$

This yields a left-exact functor between abelian categories

$$f_! : \mathrm{Shv}(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow \mathrm{Shv}(Y^{\mathrm{an}}, \mathbb{Z})$$

and we define

$$f_! : \mathbf{D}_B(X, \mathbb{Z}) \longrightarrow \mathbf{D}_B(Y, \mathbb{Z})$$

as its right derived functor.

By construction, there is a natural transformation

$$f_! \longrightarrow f_*$$

which is an isomorphism when f is proper. We also have functoriality isomorphisms

$$(f \circ g)_! \simeq f_! \circ g_!.$$

Lemma 4.1. *Let $j : U \to X$ be an open immersion.*

(i) $j_! : \mathrm{Shv}(U^{\mathrm{an}}, \mathbb{Z}) \longrightarrow \mathrm{Shv}(X^{\mathrm{an}}, \mathbb{Z})$ is the "extension by zero" functor, i.e. $j_!F$ is the subsheaf of j_*F with

$$(j_!F)_x \simeq \begin{cases} F_x, & x \in U \\ 0, & x \notin U \end{cases}$$

In particular $j_!$ is exact.

(ii)

$$j_! : \mathbf{D}_B(U, \mathbb{Z}) \longrightarrow \mathbf{D}_B(X, \mathbb{Z})$$

is the left *adjoint of j^*.*

(iii) $j_!$ commutes with arbitrary colimits.

Proof. Let us prove (i). By definition, $j_!F$ is a subsheaf of j_*F and so it suffices to compute the stalks at $x \notin U$. Let $x \in V \subset X$ be an open neighbourhood and $s \in (j_!F)(V)$. Then by definition, the map $\mathrm{supp}(s) \to X$ is proper, so $\mathrm{supp}(s)$ is closed in X. Let $W = (\mathrm{supp}(s))^c$. Then $V \cap W$ is an open neighbourhood of x in and $s_{|W \cap V} = 0$. This shows that $(j_!F)_x = 0$ and concludes the proof. The exactness of $j_!$ can then be checked on stalks.

To prove (ii), since j^* and $j_!$, it suffices to show the adjunction at the level of abelian categories of sheaves (and then derive trivially). This adjunction is then standard [Stacks, Tag 03DH]. □

The explicit definition of $f_!$ in the Betti case is unfortunately not available for other Weil cohomologies. In [SGA4], Grothendieck and Deligne figured out an alternative construction in the étale and ℓ-adic case, which works in many other contexts. Let $f : X \to Y$ be a separated finite type morphism of quasicompact quasiseparated (e.g Noetherian) schemes. Then the Nagata compactification theorem states that f can be factored as $f = \bar{f} \circ j$ with $\bar{f}$ a proper morphism and j an open immersion [Con07]. If we want to define $f_!$ with the properties above in any given sheaf theory, we should have

$$f_! \simeq \bar{f}_! \circ j_! \simeq \bar{f}_* \circ j_!$$

where $j_!$ is the uniquely determined left adjoint of j^* (if it exists). When j_* is fully faithful (which happens in all our examples), then we have $j^* j_* \simeq \mathrm{id}$ and by adjunction we get a natural transformation $j_! \to j_*$, which in general provides a natural transformation

$$f_! \to f_*$$

which is an iso if f is proper.

Remark 4.2. One also need to show (in a given context) that this definition is independent of the choice of compactification, and that it is suitably functorial in f. The necessary properties of $\mathbf{D}(-)$ can be axiomatised, but since this is orthogonal to our point we omit it.

The exceptional pushforward gives us generalisations of the isomorphisms of Theorem 3.4:

- (Base change isomorphism) Let
$$\begin{array}{ccc} Y' & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{f} & X \end{array}$$
be a cartesian square in Sch_S. There is a natural isomorphism
$$f^* g_! \xrightarrow{\sim} \tilde{g}_! \tilde{f}^*$$
In particular, when f is the inclusion of a closed point $x \in X$, this gives a natural isomorphism
$$(g_!(-))_x \xrightarrow{\sim} (g_x)_!(-).$$

- (Projection formula) There is a natural isomorphism
$$f_!\mathrm{F} \otimes \mathrm{G} \xrightarrow{\sim} f_!(\mathrm{F} \otimes f^*\mathrm{G}).$$

- (Künneth isomorphism) Given two morphisms $f : Y \to X$ and $g : Z \to X$, there is a natural isomorphism
$$f_!\mathrm{F} \otimes g_!\mathrm{G} \longrightarrow (f \times_X g)_!(\mathrm{F} \boxtimes_X \mathrm{G}).$$

Now let's see how Poincaré duality fits into the picture.

Lemma 4.3. *Let $f : Y \to X$ be a proper morphism. Then the functor $f_* : \mathbf{D}_B(Y, \mathbb{Z}) \to \mathbf{D}_B(X, \mathbb{Z})$ preserves direct sums.*

Proof. By proper base change, this reduces to the case where X is a point, so we have to show this property for sheaf cohomology on a compact Hausdorff topological space Y. A priori we have to prove this for arbitrary complexes of sheaves, but it is possible to reduce to the case of a single sheaf, so let's only treat that case for simplicity.

Since sheaf cohomology commutes with finite sums, it suffices to show that $H^*(Y, -)$ commutes with filtered colimits of sheaves. For $H^0(Y, -)$ it follows directly from the fact that the support of sections are closed in Y hence compact. It follows from the H^0 case that filtered colimits of soft sheaves on Y are soft. Then the general case follow from the H^0 and the existence of resolutions by soft sheaves. □

Since $j_!$ is a left adjoint by Lemma 4.1, it also commutes with direct sums. We conclude that for any separated finite type morphism $f : Y \to X$, $f_!$ commutes with direct sums (this can also be proved directly by the same argument as Lemma 4.3. The triangulated categories $\mathbf{D}_B(X, \mathbb{Z})$ and $\mathbf{D}_B(Y, \mathbb{Z})$ admit arbitrary direct sums; this is an advantage of working with unbounded derived categories and arbitrary sheaves. Now a functor between nice enough such "large" triangulated categories which commutes with direct sums admits a right adjoint: this is the adjoint functor theorem of [Nee01, Theorem 8.4.4]. This also holds at the level of stable ∞-categories, by [Lur09, Corollary 5.5.2.9 (1)] combined with [Lur17, Proposition 1.4.4.1 (2)].

Hence $f_!$ admits a right adjoint
$$f^! : \mathbf{D}_B(X, \mathbb{Z}) \to \mathbf{D}_B(Y, \mathbb{Z})$$
(pronounced "f upper shriek"). This defining adjunction $f_!$ can be upgraded formally to a sheaf isomorphism as follows.

Proposition 4.4. *(Formal local Poincaré-Verdier duality) Let $f : Y \to X$ be a separated finite type morphism. There is a canonical isomorphism*

$$\underline{\mathrm{Hom}}(f_!\mathrm{F}, \mathrm{G}) \simeq f_*\underline{\mathrm{Hom}}(\mathrm{F}, f^!\mathrm{G}).$$

In particular, when $\mathrm{G} = \mathbb{Z}_X$ *is constant, we get*

$$(f_!\mathrm{F})^\vee := \underline{\mathrm{Hom}}(f_!\mathrm{F}, \mathbb{Z}_X) \simeq f_*\underline{\mathrm{Hom}}(\mathrm{F}, f^!\mathbb{Z}_X).$$

and when furthermore $\mathrm{F} = \mathbb{Z}_Y$ *is constant, we get*

$$(f_!\mathbb{Z}_Y)^\vee \simeq f_*f^!\mathbb{Z}_X.$$

Proof. Using the various adjunctions and the projection formula, we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}(\mathrm{H}, \underline{\mathrm{Hom}}(f_!\mathrm{F}, \mathrm{G})) &\simeq \mathrm{Hom}(\mathrm{H} \otimes f_!\mathrm{F}, \mathrm{G}) \\ &\simeq \mathrm{Hom}(f_!(f^*\mathrm{H} \otimes \mathrm{F}), \mathrm{G}) \\ &\simeq \mathrm{Hom}(f^*\mathrm{H} \otimes \mathrm{F}, f^!\mathrm{G}) \\ &\simeq \mathrm{Hom}(f^*\mathrm{H}, \underline{\mathrm{Hom}}(\mathrm{F}, f^!\mathrm{G})) \\ &\simeq \mathrm{Hom}(\mathrm{H}, f_*\underline{\mathrm{Hom}}(\mathrm{F}, f^!\mathrm{G})) \end{aligned}$$

which imply the result by the Yoneda lemma. □

If $f = \pi_X$ is the structure morphism of a separated finite-type $\mathbb{C}$-scheme X, we get the following formula (say with $\mathbb{Q}$-coefficients for simplicity):

$$H^*_c(X, \mathbb{Q})^\vee \simeq H^*(X, \pi^!_X\mathbb{Q}).$$

So to recover Poincaré duality, and generalize it to a family of smooth varieties and other coefficient sheaves, it suffices to compute the functor $f^!$ for a smooth morphism.

We have already seen in Lemma 4.1 that $j^! \simeq j^*$ when j is an open immersion. More generally, one can show that $f^! \simeq f^*$ when f is étale (in the Betti setting, this can be deduced from the fact that étale morphisms are local isomorphisms for the analytic topology). This is generalized substantially in the following:

Theorem 4.5. *(Local Poincaré-Verdier duality for smooth morphisms) Let $f : Y \to X$ be a separated finite type morphism.*

(i) There is a canonical natural transformation

$$f^!\mathbb{Z}_X \otimes f^*(-) \to f^!(-).$$

(ii) If f is smooth of relative dimension d, then this is a natural isomorphism

$$f^!\mathbb{Z}_X \otimes f^*(-) \simeq f^!(-)$$

and we have

$$f^!\mathbb{Z}_X \simeq \mathbb{Z}_Y(d)[2d].$$

Proof. We only provide a sketch. The construction of the morphism in (i) is, as in previous cases of such constructions, entirely formal. Let us now assume that f is smooth of relative dimension d and write

$$\alpha_f : f^!\mathbb{Z}_X \otimes f^*(-) \to f^!(-)$$

Using Zariski separation (Lemma 3.3), it is enough to show that α is an isomorphism after applying $j^*\alpha$ for open immersions $j : U \to Y$ in an open covering of Y. Because the natural transformation α

is so canonical and we have $j^* \simeq j^!$ for open immersions, one can check that $j^*\alpha$ is isomorphic to α_g with $g : U \to f(U)$ the induced morphism. Using the local structure theorem for smooth morphisms [Stacks, p. 054L], we can assume that $g = \pi \circ e$ with $e : Y \to \mathbb{A}^d_X$ étale and $\pi : \mathbb{A}^d_X \to X$ the standard projection. After some more formal diagram chases, this reduces the proof to the case where $f = e$ is étale and $f = \pi$ is the projection.

The reduction above works for "any sheaf theory" satisfying some axioms, in particular for all Weil and motivic sheaf theories. Now we use some arguments specific to the Betti case, which have to be replaced by other arguments in other cases.

First, Étale morphisms are local isomorphisms in the analytic topology, and in the Betti setting the separation argument above works in the analytic topology, so we almost immediately get $e^!\mathbb{Z}_X \simeq \mathbb{Z}_Y$ and $\alpha_e : e^* \simeq e^!$.

In the case of the projection π, there is still some work to do, but it boils down at the end to the Künneth formula and one single computation, namely

$$H^*_c(\mathbb{C}, \mathbb{Z}) \simeq \mathbb{Z}(-1)[-2].$$

See [KS90, §3.1-3] for details. For other Weil/motivic sheaf theories, there is still quite a lot of work to do at this point... □

As usual, in the Betti setting the Tate twist $\mathbb{Z}_Y(d)$ is inconsequential, but it matters when keeping track of additional structures (Hodge structure, Galois representations, etc.).

Remark 4.6. If we extend the Betti six-functor formalism to all locally compact topological spaces, then for $f : Y \to X$ topological submersion of real fiber dimension d, we have instead

$$f^!\mathbb{Z}_X \simeq \mathrm{or}_{Y/X}[d]$$

with $\mathrm{or}_{Y/X}$ the relative orientation sheaf [KS90, pp. 3.3.2–3]. This thus recovers and generalises Poincaré duality for non-orientable manifolds.

The six-functor formalism also encodes a lot of classical algebraic topology.

Proposition 4.7. *Let $X/\mathbb{C}$ be a finite type $\mathbb{C}$-scheme with structure morphism $\pi : X \to \mathrm{Spec}(C)$. Then we have natural isomorphisms*

$$\begin{aligned} H^*(X^{\mathrm{an}}, \mathbb{Z}) &\simeq H^*(\pi_*\pi^*\mathbb{Z}) \\ H^*_c(X^{\mathrm{an}}, \mathbb{Z}) &\simeq H^*(\pi_!\pi^*\mathbb{Z}) \\ H_*(X^{\mathrm{an}}, \mathbb{Z}) &\simeq H^{-*}(\pi_!\pi^!\mathbb{Z}) \\ H^{\mathrm{BM}}_*(X^{\mathrm{an}}, \mathbb{Z}) &\simeq H^{-*}(\pi_*\pi^!\mathbb{Z}) \end{aligned}$$

where H_* (resp. $H^{\mathrm{BM}}_*(X^{\mathrm{an}}, \mathbb{Z})$) denotes singular homology (resp. Borel-Moore homology).

All the usual structures and relations between these various (co)homology groups (functoriality, cap and cup products, Künneth...) can then be deduced formally from properties of the six functors.

This formulation of Poincaré duality also shows that the object $\pi^!_X\mathbb{Z}$ is important even when X is not smooth. It is called the *dualizing complex* on X, and the functor

$$\mathbb{D}_X := \underline{\mathrm{Hom}}(-, \pi^!_X\mathbb{Z}) : \mathbf{D}(X)^{\mathrm{op}} \longrightarrow \mathbf{D}(X)$$

is called *Verdier duality.* It turns out that, in the subcategory $\mathbf{D}_c(X) \subset \mathbf{D}(X)$ of *constructible sheaves*, which contains in particular all sheaves "obtained from geometry by finitely many operations", Verdier duality is a duality, i.e. an autoequivalence with $\mathbb{D}_X \circ \mathbb{D}_X \simeq \mathrm{id}_{D(X)}$. Moreover, on constructible sheaves, it exchanges the left and right adjoints: for $f : Y \to X$, we have

$$f^* \circ \mathbb{D}_Y \simeq \mathbb{D}_X \circ f^! \quad \text{and} \quad f_! \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ f_*.$$

Verdier duality thus reveals a deep symmetry of all Weil (and motivic) sheaf theories, and is very important in further developments, for instance in the study of nearby cycles and perverse sheaves [SGA7; BBD82].

Remark 4.8. Another application of exceptional functors is the *localisation property.* Let $\mathbf{D}(-)$ be a Weil or motivic sheaf theory. Let $i : Z \to X$ be a closed immersion and $j : U \to X$ be the complementary open immersion. Then there are distinguished triangles of functors:

$$j_!j^! \longrightarrow \mathrm{id}_{\mathbf{D}(X)} \longrightarrow i_*i^* \xrightarrow{+}$$

$$i_!i^! \longrightarrow \mathrm{id}_{\mathbf{D}(X)} \longrightarrow j_*j^* \xrightarrow{+}$$

This implies that (i^*, j^*) is jointly conservative, and also (exercise) that for any X the restriction functor $\mathbf{D}(X) \to \mathbf{D}(X_{\mathrm{red}})$ is an equivalence. Example 1.3 shows that this last property does not hold for $\mathbf{D}_{\mathrm{qcoh}}(-)$ and thus that localisation also does not hold.

At this point, we know quite a bit about the Betti six-functor formalism, but we have not defined formally what a six-functor formalism is! Until recently, no such definition was available, and the name "six-functor formalism" was used informally. The situation has changed with the advent of ∞-category theory and a crucial idea of Jacob Lurie of using higher categories of correspondences to encode all the relationships between the six functors. This idea has been developpped by Gaitsgory-Rozenblyum [GR17], Liu-Zheng [LZ12] and Mann []. Since this is not the main object of the talk, we refer to [**Scholze**] for a fuller discussion.

5 Six-functor formalism: coherent pathologies

Can we imitate the constructions from the Betti setting for the (quasi-)coherent sheaf theory? There is a simple but fatal obstruction: when $j : U \to X$ is an open immersion, the pullback functor

$$j^* : \mathbf{D}_{\mathrm{qcoh}}(X) \to \mathbf{D}_{\mathrm{qcoh}}(U)$$

does not admit a left adjoint. Indeed, it does not even commute with arbitrary limits.

Example 5.1. Let $X = \mathbb{A}^1_k = \mathrm{Spec}(k[T])$ and $U = \mathbb{G}^1_{m,k} = \mathrm{Spec}(k[T]_{(T)})$. Then j^* is the trivially derived functor of the localisation functor:

$$\mathrm{QCoh}(X) = k[T] - \mathrm{Mod} \to k[T]_{(T)} - \mathrm{Mod} = \mathrm{QCoh}(U),\ M \mapsto M \otimes_{k[T]} k[T]_{(T)}.$$

Localisation commutes with finite limits, but not with infinite products

$$(\prod_I k[T]) \otimes k[T]_{(T)} = (\prod_I k[T])_{(T)} \neq (\prod_I k[T]_{(T)})$$

or with cofiltered limits

$$(\lim_n k[T]/(T^n)) \otimes k[T]_{(T)} = k[[T]] \otimes k[T]_{(T)} = k((T)) \neq 0 = \lim_n((k[T]/(T^n) \otimes k[T]_{(T)}).$$

This means that there is no hope of defining a functor $f_!$ with the same formal properties as in the Betti case.

This is however not at all the end of the story, even before involving condensed mathematics. It turns out that the analogue of the functor $f^!$ in $D_{\mathrm{qcoh}}(-)$ *does* exists for a finite type separated morphism between (reasonable) Noetherian schemes, even though $f_!$ does not! Moreover, when $f : Y \to X$ is a smooth morphism of relative dimension d, it satisfies

$$f^!(-) \simeq \Omega^d_{Y/X}[d] \otimes f^*(-)$$

which is closely analoguous to Theorem 4.5. This then implies a vast generalisation of Serre duality.

These results were established by Grothendieck in [] with some important restrictions (only for coherent sheaves), building on the previous work of Serre on coherent duality for projective varieties over a field. However the proofs there were very complicated, the basic issue being that the definition of $f^!$ was ad hoc and not local.

Then work of Deligne, Verdier, Lipman, Neeman, Iyengar, Nayak, Sastry and others steadily improved the situation and made the results stronger and the proofs clearer, but still less transparent than in the Betti case. See [**Neeman-survey**] for an overview of the state of the art pre-Clausen-Scholze.

6 Six-functor formalism with solid modules

We can now sketch how condensed mathematics and solid abelian groups solve the issues in the previous section, and brings the coherent situation much more in line with the Betti case.

Given what we have done so far in the seminar, we can first try to upgrade quasicoherent sheaves to *condensed quasicoherent sheaves*, as follows. The category Cond(Ab) of condensed abelian groups is a symmetric monoidal category, and the fully faithful functor $\underline{\ } : \mathrm{Ab} \to \mathrm{Cond}(\mathrm{Ab})$ which takes an abelian group and returns the corresponding discrete condensed abelian group is symmetric monoidal. This induces a functor $\mathrm{Ring} \to \mathrm{Cond}(\mathrm{Ring})$. Given an ordinary ring R, we define $\mathrm{Cond}(R)$ to be the (abelian) category of condended abelian groups equipped with an $\underline{R}$-module structure. As with usual R-modules over a commutative ring, $\mathrm{Cond}(R)$ gets a symmetric monoidal structure $\otimes_R$ from the symmetric monoidal structure on Cond(Ab).

Let R be a ring and $X = \mathrm{Spec}(R)$. Then $\mathrm{QCoh}(X) \simeq R - \mathrm{Mod}$. So it is tempting to define

$$\mathrm{QCoh}(X) := \mathrm{Cond}(R)$$

and

$$\mathbf{D}^{\mathrm{cond}}_{\mathrm{qcoh}}(X) := D(\mathrm{Cond}(R))$$

and to extend this to other schemes by gluing. This can be done; for this it is very useful to consider $\mathbf{D}^{\mathrm{cond}}_{\mathrm{qcoh}}(X)$ as an ∞-category, so that the ∞-categorical formalism handles the gluing gracefully.

However this does not solve our basic issue: the resulting functor $j^* : \mathbf{D}^{\mathrm{cond}}_{\mathrm{qcoh}}(X) \to \mathbf{D}^{\mathrm{cond}}_{\mathrm{qcoh}}(U)$ for an open immersion $j : U \to X$ still does not commute with colimits. The problem is that the tensor products on Cond(Ab) and on $\mathrm{Cond}(R)$ are essentially algebraic, obtained from the naive tensor product on presheaves of abelian groups/R-modules on **CHaus** by sheaffification, and so do not commute with limits.

This is where solid abelian groups enter the picture. Since this is the topic of the next few talks, I will be very brief here. The idea is that there is a miraculous abelian subcategory $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$ with many good properties such that the inclusion $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$ has a left adjoint

$$(-)^{\blacksquare} : \mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Solid}.$$

The objects $\prod_I \mathbb{Z}$ for any set I are in Solid, are projective in Solid and form a system of projective generators of Solid. There is then a symmetric monoidal structure $\otimes^{\blacksquare}$, the solid tensor product, on Solid defined simply as

$$A \otimes^{\blacksquare} B := (A \otimes B)^{\blacksquare}.$$

As in the condensed case above, given a (discrete) ring R, we can define a category $\mathrm{Solid}(R)$ of solid R-modules, which is projectively generated by the objects $\prod_I R$ for all sets I, and admits a solid tensor product $\otimes^{\blacksquare}_R$.

Warning 6.1. The category $\mathrm{Solid}(R)$ is *not* the same as the subcategory $\mathrm{Mod}_{\underline{R}}(\mathrm{Solid})$ of $\underline{R}$-modules in Solid! Rather, they are both full subcategories of $\mathrm{Cond}(R)$, but

$$\mathrm{Solid}(R) \subsetneq \mathrm{Mod}_{\underline{R}}(\mathrm{Solid}).$$

The category $\mathrm{Mod}_{\underline{R}}(\mathrm{Solid})$ is generated by the objects $(\prod_I \mathbb{Z}) \otimes^{\blacksquare} R$, which are not in $\mathrm{Solid}(R)$ if I is infinite.

Once again these constructions can be derived and globalized, leading to ∞-categories $\mathbf{D}(\mathcal{O}_{X,\blacksquare})$ of *solid quasicoherent sheaves* on schemes. It is for these categories, which contain $\mathbf{D}_{\mathrm{qcoh}}(X)$ as full subcategories of "discrete" objects, that Clausen-Scholze develop a six operation formalism.

The key observation is now this. Let $j : U \to X$ be an open immersion, say $X = \mathrm{Spec}(R)$ and $U = \mathrm{Spec}(R[1/f])$. The corresponding functor

$$j^* : \mathbf{D}(\mathcal{O}_{X,\blacksquare}) \to \mathbf{D}(\mathcal{O}_{U,\blacksquare})$$

corresponds at the level of abelian categories of solid modules to the functor

$$- \otimes_R^{\blacksquare} R[1/f] : \mathrm{Solid}(R) \to \mathrm{Solid}(R[1/f])$$

Because $R[1/f]$ is flat over R, this tensor product is still exact. Moreover, at the level of projective generators, it satisfies

$$(\prod_I R) \otimes_R^{\blacksquare} R[1/f] \simeq \prod_I R[1/f].$$

So j^* commutes with at least some infinite products! Because those are projective generators, Clausen-Scholze manage to bootstrap this to show that j^* commutes with limits (with j any open immersion).

One could then apply the adjoint functor theorem for presentable ∞-categories (which would require additionally to check that j^* is also an accessible functor) to prove that j^* has a left-adjoint

$$j_! : \mathbf{D}(\mathcal{O}_{U,\blacksquare}) \to \mathbf{D}(\mathcal{O}_{X,\blacksquare}).$$

Note that this functor is something really new to the solid context; unlike j^* it does not preserve the subcategories $\mathbf{D}_{\mathrm{qcoh}}(-)$ of discrete quasicoherent sheaves.

Using $j_!$, one can then proceed à la Deligne and define

$$f_! = \bar{f}_* \circ j_! : \mathbf{D}(\mathcal{O}_{Y,\blacksquare}) \to \mathbf{D}(\mathcal{O}_{X,\blacksquare})$$

for any Nagata compactification of a separated finite type morphism $f : Y \to X$, and develop the basic theory as in the Betti case above, in particular showing that $f_!$ commutes with direct sums, so has a right adjoint $f^!$, etc.

Clausen-Scholze do something even more impressive, which I cannot do justice to here: given a separated finite type morphism $f : Y \to X$, they construct a *canonical* compactification of f! Of course such a thing does not exist in the category of schemes, and they have to go to the world of adic spaces (and develop the whole solid theory there as well). But once this is done, the definition $f_! = \bar{f}_* \circ j_!$ becomes canonical. This is in some way parallel to the Betti case where we had a definition of $f_!$ in terms of sections with compact support, without choosing a compactification.

Finally, once the formalism of $f_!$ and $f^!$ is well in place, one can run through the proofs of Poincaré duality in the Betti case above, adapt them directly to solid quasicoherent sheaves, and reduce the proof of Grothendieck-Serre duality to a local computation on $\mathbb{A}^1$. This is precisely what Clausen-Scholze do in [**condensed**].

Moreover, the same reductions to local computations on $\mathbb{A}^1$ can be used to show that the functor $f_!$ always preserves compact objects (in the usual triangulated sense). On the other hand, the functor f_* preserves discrete objects (where it restricts to the usual derived pushforward of quasicoherent sheaves). But then an object in $\mathbf{D}(\mathcal{O}_{X,\blacksquare})$ which is both compact and discrete is bounded and has discrete coherent cohomology sheaves! So this reproves the coherence theorem for derived pushforward by proper morphisms.

References

[AGV22] Joseph Ayoub, Martin Gallauer, and Alberto Vezzani. "The six-functor formalism for rigid analytic motives". In: *Forum Math. Sigma* 10 (2022), Paper No. e61, 182. URL: `https://doi.org/10.1017/fms.2022.55`.

[AL22] Tomoyuki Abe and Christopher Lazda. *A comparison between compactly supported rigid and $\mathscr{D}$-module cohomology.* 2022. URL: `https://arxiv.org/abs/2208.10137`.

[Ayo07a] Joseph Ayoub. "Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I". In: *Astérisque* 314 (2007), x+466 pp. (2008).

[Ayo07b] Joseph Ayoub. "Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II". In: *Astérisque* 315 (2007), vi+364 pp. (2008).

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. “Faisceaux pervers”. In: *Analysis and topology on singular spaces, I (Luminy, 1981)*. Vol. 100. Astérisque. Paris: Soc. Math. France, 1982, pp. 5–171.

[CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives.* Springer Monographs in Mathematics. Springer, Cham, [2019] ©2019, pp. xlii+406. URL: `http://arxiv.org/abs/0912.2110`.

[Con07] Brian Conrad. “Deligne’s notes on Nagata compactifications”. In: *J. Ramanujan Math. Soc.* 22.3 (2007), pp. 205–257.

[Del74] Pierre Deligne. “La conjecture de Weil. I”. In: *Inst. Hautes Études Sci. Publ. Math.* 43 (1974), pp. 273–307.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality.* Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017, xl+533pp. URL: `https://doi.org/10.1090/surv/221.1`.

[Har66] Robin Hartshorne. *Residues and duality.* Lecture Notes in Mathematics, No. 20. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne. Springer-Verlag, Berlin-New York, 1966, pp. vii+423.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory.* Vol. 236. Progress in Mathematics. Translated from the 1995 Japanese edition by Takeuchi. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. xii+407. URL: `https://doi.org/10.1007/978-0-8176-4523-6`.

[Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006, pp. viii+307. URL: `https://doi.org/10.1093/acprof:oso/9780199296866.001.0001`.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds.* Vol. 292. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With a chapter in French by Christian Houzel. Springer-Verlag, Berlin, 1990, pp. x+512. URL: `https://doi.org/10.1007/978-3-662-02661-8`.

[Lur09] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009, pp. xviii+925.

[Lur17] Jacob Lurie. “Higher Algebra”. 2017.

[LZ12] Y. Liu and W. Zheng. “Enhanced six operations and base change theorem for higher Artin stacks”. In: *ArXiv e-prints* (Nov. 2012). arXiv: `1211.5948 [math.AG]`.

[Nee01] Amnon Neeman. *Triangulated categories.* Vol. 148. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2001, pp. viii+449. URL: `https://doi.org/10.1515/9781400837212`.

[Sai17] Morihiko Saito. “A young person’s guide to mixed Hodge modules”. In: *Hodge theory and L^2-analysis.* Vol. 39. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2017, pp. 517–553.

[Sai86] Morihiko Saito. “Mixed Hodge modules”. In: *Proc. Japan Acad. Ser. A Math. Sci.* 62.9 (1986), pp. 360–363. URL: `http://projecteuclid.org/euclid.pja/1195514125`.

[SGA4] *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. Berlin: Springer-Verlag, 1973, pp. vi+640.

[SGA5] *Cohomologie l-adique et fonctions L.* Lecture Notes in Mathematics, Vol. 589. Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie. Berlin: Springer-Verlag, 1977, pp. xii+484.

[SGA7] *Groupes de monodromie en géométrie algébrique. II.* Lecture Notes in Mathematics, Vol. 340. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz. Springer-Verlag, Berlin-New York, 1973, pp. x+438.

[Stacks] The Stacks Project Authors. *Stacks Project.* `http://stacks.math.columbia.edu`. 2015.

[Ver95] Jean-Louis Verdier. "Dualité dans la cohomologie des espaces localement compacts". In: *Séminaire Bourbaki, Vol. 9.* Soc. Math. France, Paris, 1995, Exp. No. 300, 337–349.