Exponential motivic homotopy theory I: exponentiation of coefficient systems (Preliminary version)

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November 15, 2022

Abstract

We construct new six functor formalisms capturing cohomological invariants of varieties with potentials. Starting from any six functor formalism $C(-)$, encoded as a coefficient system, we associate a new six functor formalism $C_{\exp}(-)$; this requires in particular constructing the convolution product symmetric monoidal structure at the ∞-categorical level. We study $C_{\exp}(-)$ and how it relates to $C(-)$. We also define motives in $C_{\exp}(-)$ attached to varieties with potential and study their properties.

Contents

Keywords: motivic homotopy theory, coefficient system, six-functor formalism, exponential motives, varieties with potential, Landau-Ginzburg model

1 Introduction

Exponential cohomological invariants

A scheme with potential (or Landau–Ginzburg model) is a pair (X, a) where X is a scheme and $a \in \mathcal{O}(X)$ is a regular function, which we think of as a morphism $a : X \to \mathbb{A}^1 \simeq \mathbb{G}_a$ to the additive group. Schemes with potential occur naturally in several areas of algebraic geometry: nearby and vanishing cycles, mirror symmetry for Fano varieties [CCG+13], Landau-Ginzburg/Calabi–Yau correspondance for quasihomogeneous potentials [FJR13], Donaldson-Thomas theory [KS11], etc.

Schemes with potential have "exponential" cohomological invariants which generalize classical Weil cohomologies for ordinary schemes. For instance, if X is smooth over a field k of characteristic 0, one can consider the *twisted de Rham cohomology* $H^*_{\mathrm{dR}}(X, \mathcal{E}^a)$ where $\mathcal{E}^a := (\mathcal{O}_X, d - (da \wedge \_))$ is the exponential line bundle with connection associated to the function a; this case gives its name to the general theory. If X is locally of finite type over $\mathbb{C}$, there is the *rapid decay cohomology* $H^*_{\mathrm{rd}}(X^{\mathrm{an}}, \mathbb{Z})$ which is roughly speaking the singular cohomology of a generic fiber of $a(\mathbb{C}) : X^{\mathrm{an}} := X(\mathbb{C}) \to \mathbb{C}$. Over a field k of characteristic $p > 0$, we have the *twisted ℓ-adic cohomology* $H^*_\ell(X_{\bar{k}}, a^*\mathcal{L}_\psi)$ where $\mathcal{L}_\psi \in \mathrm{Sh}_{\mathrm{\acute{e}t}}(\mathbb{G}_{a,k}, \overline{\mathbb{Q}}_\ell)$ is the Artin–Schreier sheaf attached to an additive character $\psi : \mathbb{Z}/p\mathbb{Z} \to \overline{\mathbb{Q}}_\ell^\times$.

When a is the zero function, these cohomology theories recover the standard de Rham, Betti and ℓ-adic cohomology groups of X respectively. In general however, they are genuinely new invariants which cannot be the realisation of classical motives. For instance, twisted de Rham cohomology groups carry an *irregular Hodge filtration* with rational breaks which does not fit into an Hodge structure [ESY17], and the comparison between twisted de Rham cohomology and rapid decay cohomology for varieties over number fields leads to *exponential periods*, which form a subalgebra of $\mathbb{C}$ that contains e and the Euler–Mascheroni constant γ and is (conjecturally) strictly larger than classical periods [KZ01, §4.3] [FJ22]. Finally, twisted ℓ-adic cohomology over finite fields gives rise via taking traces of Frobenius to many interesting exponential sums such as Kloosterman sums [Del77, Exposé 6].

Each of these cohomology theories fits into a sheaf theory: holonomic $\mathcal{D}$-modules in the de Rham case, ℓ-adic sheaves in the ℓ-adic case and enhanced ind-sheaves (in the sense of [DK16, §4]) in the Betti case. Exponential cohomology is obtained as the cohomology of an "exponential sheaf": $\mathscr{E}^a$ in the de Rham case, $a^*\mathscr{L}_\psi$ in the ℓ-adic case and the exponential enhanced (ind-)sheaves of [DK18, §3] in the Betti case[1]. When a is not constant, the exponential sheaf $\mathscr{E}^a$ has *irregular singularities* at infinity, and this is another reason why its cohomology is not of motivic origin in the classical sense.

The emerging theory of exponential motives aims to generalize results and conjectures about classical motives to varieties with potential. The idea of such a theory was first suggested by Kontsevich and Zagier in [KZ01, §4.3], developed in the context of *exponential mixed Hodge structures* by Kontsevich and Soibelman in [KS11, §4] and explored in the context of *exponential Nori motives* by Fresán and Jossen [FJ22]. Fresán and Jossen define, for $k \subset \mathbb{C}$, a neutral $\mathbb{Q}$-linear Tannakian category $M^{\mathrm{exp}}(k,\mathbb{Q})$ with two fiber functors corresponding to twisted de Rham and rapid decay cohomology. The category $M^{\mathrm{exp}}(k,\mathbb{Q})$ contains as a full subcategory the abelian category of Nori motives [FJ22, Theorem 5.1.1]. This construction allows them, among other things, to define a period torsor and to state and study a generalization of the Grothendieck-Kontsevich-Zagier period conjecture for exponential periods.

Outline

In this paper, we define categories of exponential motivic sheaves in the context of motivic homotopy theory after Morel and Voevodsky. Our approach is centered on the six operation formalism, which we approach through the notion of *coefficient system*. A coefficient system is an axiomatisation of the common properties satisfied both by classical sheaf theories and by the stable homotopy category $\mathrm{SH}(-)$ (see Definition 3.1). As we recall in section 3, a coefficient system can be extended to a six operation formalism satisfying all the usual properties.

Given a coefficient system $C(-)$, we construct an associated *exponential coefficient system* $C_{\mathrm{exp}}(-)$. The definition, inspired by [KS11], is very simple. For a scheme S over B, writing $\pi : \mathbb{G}_{\mathrm{a},S} \to S$ for the structure morphism of the additive group over S, we put

$$C_{\mathrm{exp}}(S) := \{M \in C(\mathbb{G}_{\mathrm{a},S}) | \pi_! M = 0\}$$

(or equivalently the Verdier quotient $C(\mathbb{G}_{\mathrm{a},S})/\pi^! C(S)$, see section 4.1) which we equip with the convolution product

$$M * N := \mu_!(M \boxtimes_S N)$$

where $\mu : \mathbb{G}_{\mathrm{a},S} \times \mathbb{G}_{\mathrm{a},S} \to \mathbb{G}_{\mathrm{a},S}$ is the addition map. The inclusion functor $C_{\mathrm{exp}}(S) \to C(\mathbb{G}_{\mathrm{a},S})$ admits a right adjoint

$$\Pi : C(\mathbb{G}_{\mathrm{a},S}) \to C_{\mathrm{exp}}(S), M \mapsto M * \mathbb{E}_0$$

where $\mathbb{E}_0 \in C_{\mathrm{exp}}(S)$ is the unit of the convolution product.

Once we establish that $C_{\mathrm{exp}}(-)$ forms a coefficient system, the general theory provides us with a six operation formalism. In particular, for a (separated finite type) morphism $f : T \to S$ of B-schemes, we have adjunctions

$$\underline{f}^* : C_{\mathrm{exp}}(S) \leftrightarrows C_{\mathrm{exp}}(T) : \underline{f}_* \quad \text{and} \quad \underline{f}_! : C_{\mathrm{exp}}(T) \leftrightarrows C_{\mathrm{exp}}(S) : \underline{f}^!$$

[1] We could not find a precise statement of the comparison between the framework of d'Agnolo-Kashiwara and rapid decay cohomology, but apparently this comparison is known to experts.

(we underline these to distinguish the functoriality in C and $C_{\mathrm{exp}}(-)$) . We describe these functors as well as the internal Hom explicitly in terms of the operations in $C(-)$ and the functor Π.

We can then apply this general construction to motivic homotopy theory. From the coefficient system $\mathrm{SH}(-)$ of *stable motivic homotopy types* we obtain a six operation formalism $\mathrm{SH}_{\mathrm{exp}}(-)$ of *exponential stable motivic homotopy types*. Similarly, from the coefficient system $\mathrm{DM}(-)$ of *motivic sheaves*, we obtain a six operation formalism $\mathrm{DM}_{\mathrm{exp}}(-)$ of *exponential motivic sheaves*.

This simple-looking definition hides a difficulty. We want our six operation formalisms to be valued in symmetric monoidal stable ∞-categories rather than bare triangulated categories; this additional structure exists in all natural examples and is extremely useful to treat questions of descent and equivariance. The difficulty is then to make sense of the convolution product on $C_{\mathrm{exp}}(S)$ as a symmetric monoidal ∞-category. We propose two approaches in this paper. One is to use the idea, originally due to Lurie, of encoding six-functor formalisms using correspondences. This idea has been developped in [GR17c], with an extension to the lax-monoidal setting in [RS21, Appendix A]. This conceptually very satisfying approach relies however on unproven assumptions about $(\infty, 2)$-categories[2]. The other is to do a technically simpler, strictly $(\infty, 1)$-categorical construction relying on symmetric monoidal cartesian fibrations.

The definition of $C_{\mathrm{exp}}(S)$ does not involve, at face value, any S-schemes with potential. If $C = \mathrm{SH}(-)$ is Morel-Voevodsky's stable motivic homotopy category (the "initial six operation formalism" by [DG20]), then $\mathrm{SH}_{\mathrm{exp}}(S)$ is a subcategory of $\mathrm{SH}(\mathbb{G}_{\mathrm{a},S})$: this means $\mathrm{SH}_{\mathrm{exp}}(S)$ is built out of smooth $\mathbb{G}_{\mathrm{a},S}$-schemes, or in other words S-schemes $f : X \to S$ with a *smooth* potential $a : X \to \mathbb{G}_{\mathrm{a}}$. However, with the six operation formalism at hand, we can construct exponential motives for arbitrary S-schemes with potentials. We first construct, for any coefficient system C and any potential $a : X \to \mathbb{G}_{\mathrm{a}}$, an *exponential twist* $\mathbb{E}_a \in C_{\mathrm{exp}}(X)$ which plays the role of the exponential $\mathscr{D}$-module $\mathscr{E}^a$; the object $\mathbb{E}_0$ associated to the zero potential is the unit of the convolution product. We then define the corresponding *exponential (homological) motive* $M_S(X, a) \in C_{\mathrm{exp}}(S)$ as

$$M_S(X, a) := \underline{f}_!(\mathbb{E}_a * \underline{f}^!\mathbb{E}_0)$$

and study basic properties of this construction in section 6.1. This procedure allows us, under some assumptions, to construct generators of the categories $C_{\mathrm{exp}}(X)$. In particular, we study generators of the categories $\mathrm{DM}_{\mathrm{exp}}(X)$ and $\mathrm{SH}_{\mathrm{exp}}(X)$. This also leads us to define *exponential (motivic) cohomology groups* as certain Hom groups in $C_{\mathrm{exp}}(X)$, which we compute in terms of morphism groups in the original coefficient system C.

Note that the idea of adding a coordinate to encode "exponential" phenomena and irregular singularities has been used in several guises in recent advances on irregular Riemann-Hilbert correspondence and irregular Hodge theory, through the aforementioned enhanced sheaves of Tamarkin [Tam18] and d'Agnolo-Kashiwara [DK16] (using sheaves on $X \times \mathbb{R}$ where X is a real or complex manifold), the mixed twistor $\mathscr{D}$-modules of Mochizuki [Moc15] and the irregular mixed Hodge modules of Sabbah-Yu [Sab18] (using sheaves on $X \times \mathbb{C}$).

[2] collected in [GR17c, Chapter 10, 0.4.2]; as far as we know, although there has been progress since then, not everything in that list has been conclusively established

Future work

With this basic formalism in place, we plan to explore other constructions specific to exponential motives. One of them is the Fourier transform. Sheaf-theoretic Fourier transforms exist in various sheaf theories: Fourier-Laplace transform of holonomic $\mathscr{D}$-modules in characteristic 0, Fourier-Deligne transform for ℓ-adic sheaves in positive characteristic [Lau87], enhanced Fourier-Sato transform for enhanced ind-sheaves, etc. However it is well-known that, for classical motivic sheaves, there is no Fourier transform lifting all of these along realisations functors. One motivation for exponential motives is that they admit such a Fourier transform, satisfying all the standard properties of the Fourier-Deligne transform.

Using the Fourier transform, we will construct realisation functors which connect the categories $\mathrm{SH}_{\exp}(-)$ and $\mathrm{DM}_{\exp}(-)$ to classical sheaf theories and recover the exponential cohomology theories (rapid decay, exponential de Rham, twisted ℓ-adic) that motivated the theory in the first place.

Let k be a subfield of $\mathbb{C}$. Then we plan to construct a realisation functor

$$R_{\mathrm{Nori}} : \mathrm{DM}_{\exp,c}(k, \mathbb{Q}) \to D^b(M^{\exp}(k, \mathbb{Q}))$$

where $M^{\exp}(k, \mathbb{Q})$ is the category of exponential Nori motives of [FJ22].

2 Conventions

2.1 Algebraic geometry

Throughout we fix a Noetherian scheme B which serves as our base scheme. All our schemes are of finite type over B, and we denote the category of B-schemes of finite type by $\mathrm{Sch}\, B$.

2.2 Category theory

Generally, we follow the notations of [Lur09, Lur] for ∞-categories and (symmetric) monoidal ∞-categories. In particular, we denote (symmetric) monoidal ∞-categories by symbols like $C^{\otimes}$ when we want to emphasize the monoidal structure, but we sometimes omit the superscript $\otimes$ when it clutters the notation.

We write Fin_* for the (nerve of) the category of pointed finite sets. For $n \in \mathbb{N}$, we have the object $\langle n \rangle = \{0, 1, \dots n\}$ pointed at 0, and we write $\langle n \rangle^{\circ} = \{1, \dots, n\}$; every object in Fin_* is equivalent to one of $\langle n \rangle$.

We denote by $\mathrm{Cat}_{\mathrm{st}}$ (resp. $\mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$) the ∞-category of stable (resp. stable symmetric monoidal) ∞-categories and exact (resp. and symmetric monoidal) functors. We always equip $\mathrm{Cat}_{\mathrm{st}}$ with the cartesian symmetric monoidal structure.

Given a symmetric monoidal ∞-category $C^{\otimes}$, we denote by $\mathrm{Mod}(C^{\otimes})$ the generalised ∞-operad of module objects in $C^{\otimes}$. Informally, the objects of $\mathrm{Mod}(C^{\otimes})$ are pairs (A, M) where A is a commutative algebra object in $C^{\otimes}$ and M is an A-module, and morphisms from (A, M) to (B, N) are pairs $(\phi : A \to B, f : M \to N)$ where ϕ is a morphism of commutative algebras and f is a morphism of A-modules, with N considered as an A-module via ϕ. In particular, we have $\mathrm{Mod}(\mathrm{Cat}_{\mathrm{st}}^{\otimes})$, which admit a forgetful functor (a map of generalised ∞-operads) to $\mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}}^{\otimes})$.

Given two ∞-categories C and D, we denote by $\mathrm{Fun}(C, D)$ the corresponding functor ∞-category. If D is the underlying ∞-category of a symmetric monoidal ∞-category $D^{\otimes}$, then the functor ∞-category $\mathrm{Fun}(C, D)$ has a corresponding symmetric monoidal structure $\mathrm{Fun}(C, D)^{\otimes}$ given by the pointwise product (this is the special case [Lur, Remark 2.1.3.4] with $\mathcal{O}^{\otimes}$ the trivial ∞-operad). Explicitly, for all $n \in \mathbb{N}$, we have we have

$$\mathrm{Fun}(C, D)^{\otimes}_{\langle n \rangle} = \mathrm{Fun}(C, D^{\otimes}) \times_{\mathrm{Fun}(C, \mathrm{Fin}_*)} \{\langle n \rangle\}$$

where the $\langle n \rangle$ refers to the corresponding constant functor in $\mathrm{Fun}(C, \mathrm{Fin}_*)$. With respect to this symmetric monoidal structure, there is an equivalence

$$\mathrm{CAlg}(\mathrm{Fun}(C, D)^{\otimes}) \simeq \mathrm{Fun}(C, \mathrm{CAlg}(D^{\otimes})).$$

3 Coefficient systems

The notion of coefficient system provides a convenient framework for discussing theories of (derived) sheaves on schemes in algebraic geometry, with a view towards the formalism of Grothendieck's six operations. It is due to Voevodsky and Ayoub [Del01, Ayo07] (in a 2-categorical setting) and further studied in Cisinski-Déglise [CD19], Drew [Dre18] (in our current infinity-categorical setting) (see also [Kha16, Rob14, Hoy17, Ayo]). For a leisurely introduction we refer to [Gal21]. The term "coefficient system", suggested by [Dre18] after Grothendieck [Gro], is meant to suggest that we are parametrizing the possible coefficients of various sheaf cohomology theories.

As we discuss in Section 3.2, classical sheaf theories (sheaves for the analytic topology, ℓ-adic sheaves, $\mathcal{D}$-modules) provide important examples of coefficient systems. The definition is nevertheless particularly suitable for theories coming from motivic homotopy theory, since it axiomatises the operations and properties which are formally available from the Morel-Voevodsky construction.

3.1 Definitions

Definition 3.1. A *coefficient system (over B)* is a functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$ satisfying the following axioms (where we write $f^* = C(f)$ for a morphism f of B-schemes).

(1) **(Left)** For each smooth morphism $p : Y \to X \in \mathrm{Sch}_B$, the functor $p^* : C(X) \to C(Y)$ admits a left adjoint $p_\sharp$, and we have the following equivalences.

(Smooth base change) For each cartesian square

$$\begin{tikzcd} Y' \arrow[r, "p'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$

in Sch_B, the Beck-Chevalley transformation $p'_\sharp (f')^* \to f^* p_\sharp$ is an equivalence.

(Smooth projection formula) The canonical transformation

$$p_\sharp(p^*(-)\otimes -) \to - \otimes p_\sharp(-)$$

is an equivalence.

(2) **(Right)** For every $X \in \mathrm{Sch}_B$ and every $f : Y \to X$, the following properties hold.

(Internal hom) The symmetric monoidal structure on $C(X)$ is closed.

(Push-forward) The pull-back functor f^* admits a right adjoint $f_* : C(Y) \to C(X)$.

(3) **(Localization)** The ∞-category $C(\emptyset) \simeq 0$ is trivial. And for each closed immersion $i : Z \hookrightarrow X$ in Sch_B with complementary open immersion $j : U \hookrightarrow X$, the square

$$\begin{tikzcd} C(Z) \arrow[r, "i_*"] \arrow[d] & C(X) \arrow[d, "j^*"] \\ C(\emptyset) \simeq 0 \arrow[r] & C(U) \end{tikzcd} \tag{3.2}$$

is Cartesian in $\mathrm{Cat}^{\mathrm{st}}_\infty$.

(4) For each $X \in \mathrm{Sch}_B$, if $p : \mathbb{A}^1_X \to X$ denotes the canonical projection with zero section $s : X \to \mathbb{A}^1_X$, then:

($\mathbb{A}^1$-homotopy) The functor $p^* : C(X) \to C(\mathbb{A}^1_X)$ is fully faithful.

(Tate stability) The composite $p_\sharp s_* : C(X) \to C(X)$ is an equivalence.

Remark 3.3. Let us expand a little on condition (Smooth projection formula). The functor f^*, like any symmetric monoidal functor in $\mathrm{CAlg}(\mathrm{Cat}^{\otimes}_{\mathrm{st}})$, has a canonical upgrade to a morphism $(C(S), C(S)) \to (C(S), C(X))$ in $\mathrm{Mod}(\mathrm{Cat}_{\mathrm{st}})$, where $C(S)$ acts on $C(X)$ via f^* itself. In particular, if we write $m : C(S)\times C(X) \to C(X)$ the action map, we have a commutative square

$$\begin{tikzcd} C(S)\times C(S) \arrow[r, "\mathrm{id}\times f^*"] \arrow[d, "\otimes_{C(X)}"] & C(S)\times C(X) \arrow[d, "m"] \\ C(S) \arrow[r, "f^*"] & C(X) \end{tikzcd}$$

and the canonical map in (Smooth projection formula) is the mate of this square when passing to left adjoints. The condition (Smooth projection formula) is that this mate is an isomorphism. By [Rob14, Proposition 9.4.15], this is equivalent to $f_\sharp$ lifting to a map $(C(S), C(X)) \to (C(S), C(S))$ in $\mathrm{Mod}(\mathrm{Cat}_{\mathrm{st}})$ (more precisely, [Rob14, Proposition 9.4.15] is rather about $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{st}})$, but the proof works in the non-presentable setting as well).

Definition 3.4. A coefficient system C over B is *presentable* if it factors through the subcategory $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{st}})$ of stable presentably symmetric monoidal ∞-categories. In other words, for every $X \in \mathrm{Sch}_B$, the symmetric monoidal ∞-category $C(X)$ is presentably symmetric monoidal, and for every morphism $f : X \to Y$, the functor $f^* : C(Y) \to C(X)$ is cocontinuous (or equivalently is a left adjoint).

Similarly, we say that C is *compactly generated* if it factors through the subcategory $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{st}}_{\omega})$ of compactly generated ∞-categories and compact-preserving cocontinuous functors.

Remark 3.5. For a functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{st}})$ taking values in stable presentable ∞-categories, the adjoint functor theorem allows to reformulate part of the axioms.

1. The existence of f_* is automatic (since f^* is cocontinuous), and so is the existence of internal homs (since the symmetric monoidal product is cocontinuous in each variable). In other words, the axiom (Right) is automatic.
2. The existence of $f_\sharp$ for f smooth is equivalent to f^* being continuous. And the axiom (Smooth projection formula) can also be expressed as $f_\sharp$ lifting to a map in $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{st}})$.

Definition 3.6. (i) Let C_1, C_2 be coefficient systems over B. A *morphism of coefficient systems* is a morphism $\phi : C_1 \to C_2$ in the functor ∞-category $\mathrm{Fun}(\mathrm{Sch}_B^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}}))$ with the following property: for every morphism $f : X \to Y$ in Sch_B, the Beck-Chevalley transformation

$$f_\sharp \phi_X \to \phi_Y f_\sharp : C_1(X) \to C_2(Y)$$

is invertible.

(ii) The ∞-category CoSy_B of coefficient systems over B is defined as the subcategory of $\mathrm{Fun}(\mathrm{Sch}_B^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}}))$ whose objects are coefficient systems and 1-morphisms are defined as above. We also write $\mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$ (resp. $\mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}_\omega}$) for the subcategory of CoSy_B spanned by presentable (resp. compactly generated) coefficient systems and cocontinuous (resp. and compact-preserving) morphisms.

3.2 Examples

Many existing "sheaf theories" form coefficient systems. All those examples of coefficient systems are either small or presentable, with small coefficient systems occuring as subcategories of compact or constructible objects in presentable coefficient systems. For some of these examples, the necessary constructions and the verification of the axioms of coefficient systems are not available in a convenient form in the literature.

3.2.1. Motivic homotopy theory Stable motivic homotopy theory $\mathrm{SH}(-)$, in its ∞-categorical incarnation constructed in [Rob14, §9.1] forms a presentable coefficient system over any base B [Rob14, Theorem 9.4.36]. Even though the definition of coefficient system does not directly refer to motivic homotopy theory, the coefficient system $\mathrm{SH}(-)$ plays a distinguished role because it is the initial object in $\mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$ by [DG20, Theorem 7.14, Remark 7.15].

The arguments of [Rob14] also show that $\mathrm{DA}^{\mathrm{Nis}}(-)$ (the $\mathbb{A}^1$-derived category of Morel), $\mathrm{SH}^{\text{ét}}(-)$ (the étale local version of stable motivic homotopy theory) and $\mathrm{DA}^{\text{ét}}(-)$ (the étale motives of Morel) form coefficient systems over any base B.

3.2.2. Motives Spitzweck [Spi18] constructs a motivic ring spectrum $\mathscr{M} \in \mathrm{SH}(\mathrm{Spec}(\mathbb{Z}))$ that represents Bloch-Levine motivic cohomology and then defines

$$\mathrm{DM}(-) := \mathrm{SH}(-; \mathscr{M}) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{st}})$$

as the functor which associates with the scheme X the ∞-category of modules in $\mathrm{SH}(X)$ over the pullback of $\mathscr{M}$ to X. See [Dre18, Theorem 8.10] for more details on this construction. This functor can be seen as a coefficient system of integral motivic sheaves.

Spitzweck proves that over a field k its compact part is equivalent to Voevodsky's category of geometric motives [Voe00], while with rational coefficients (and for any scheme) one recovers Beilinson motives in the sense of [CD19] (which are themselves equivalent to $\mathrm{DA}^{\text{ét}}(-, \mathbb{Q})$ by [CD19, Theorem 16.2.18]).

3.2.3. Analytic sheaves Let k be a field of characteristic 0 equipped with a complex embedding $\sigma : k \to \mathbb{C}$ and let $\Lambda \in \mathrm{CAlg}(\mathcal{S}p)$ be a commutative ring spectrum. There is a presentable coefficient system $\mathrm{Shv}((-)^{\mathrm{an}}, \Lambda)$ over the base $\mathrm{Spec}(k)$, whose objects are sheaves of Λ-modules for the analytic topology (see for instance [Ayo, Proposition 1.26], where coefficient systems are called "Voevodsky pullback formalisms"); when Λ is a discrete ring, this is equivalent to the unbounded derived category of sheaves of Λ-modules. The subfunctor of algebraically constructible sheaves forms a small coefficient system $\mathrm{Shv}_{\mathrm{ct}}((-)^{\mathrm{an}}, \Lambda)$, and one can take the Ind-completion [Gal21, § 2.4.2] to get a compactly generated coefficient system of ind-constructible sheaves $\mathrm{Ind}(\mathrm{Shv}_{\mathrm{ct}}((-)^{\mathrm{an}}, \Lambda))$. (This is [Ayo, Corollary 1.27], with slightly different notations.) The Betti realisation functor can then be interpreted as the morphism of coefficient systems $\mathrm{SH}(-) \to \mathrm{Shv}((-)^{\mathrm{an}}, \Lambda)$ or $\mathrm{SH}(-) \to \mathrm{Ind}(\mathrm{Shv}_{\mathrm{ct}}((-)^{\mathrm{an}}, \Lambda))$ [Ayo, Theorem 1.28].

3.2.4. Nori motivic sheaves Let k be as in 3.2.3. In [Ayo, §2.3], Ayoub defines a presentable coefficient system $\mathrm{Shv}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(-, \mathbb{Q})$ of "(triangulated) Nori motivic sheaves". For $X \in \mathrm{Sch}_k$, the stable ∞-category $\mathrm{Shv}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(X, \mathbb{Q})$ is equipped with a t-structure whose heart is equivalent to the indization of the abelian category of finite-dimensional representations of Ayoub's motivic Galois group $\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}(k, \mathbb{Q})$ of k, or equivalently by [CGAdS17], to the indization of the abelian category of Nori motives over k. This construction thus provides a reasonable candidate for "Nori motivic sheaves" with an associated six operation formalism. Ayoub suggests as a precise relative version of the motivic t-structure conjecture that the natural morphism of coefficient systems $\mathrm{DA}^{\text{ét}}(-, \mathbb{Q}) \to \mathrm{Shv}_{\mathrm{geo}}^{\mathscr{G}^{\mathrm{cl}}_{\mathrm{mot}}}(-, \mathbb{Q})$ induces an equivalence on constructible objects.

3.2.5. Étale and ℓ-adic sheaves A general construction of an ∞-category $\mathrm{D}_{\mathrm{cons}}(X; \Lambda)$ of constructible étale sheaves on a scheme X with coefficients in a condensed ring Λ is performed in [HRS20]. When applied to $\Lambda \in \{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell\}$, one recovers the classical theory of ℓ-adic constructible sheaves. Thus for example, the functor $\mathrm{D}_{\mathrm{cons}}(-; \mathbb{Q}_\ell) : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$ defined on k-schemes for an algebraically closed field k is a coefficient system, by Lemma 3.10 together with the classical results on ℓ-adic sheaves in [Del77].

3.2.6. Holonomic $\mathscr{D}$-modules There should be a coefficient system $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathscr{D}_{-})$ of holonomic $\mathscr{D}$-modules on schemes over a field k of characteristic zero although we don't have a convenient reference for this claim.

3.2.7. Mixed Hodge modules Similarly, there should be an ∞-categorical lift of Saito's bounded derived category of mixed Hodge modules [Sai90] although we do not know of a reference. An alternative by Drew [Dre18] provides a coefficient system of *motivic* Hodge modules $\mathrm{DH}(-)$ on $\mathrm{Spec}(\mathbb{C})$. It comes with well-behaved realization functors, shares many desirable properties with Saito's theory, and is conjectured to embed fully faithfully into the latter.

3.2.8. Non-examples There are important examples of "sheaf theories" with a rich functoriality but which do not form coefficient systems. Given a base scheme S, we have functors

$$\mathrm{QCoh}(-),\ \mathrm{IndCoh}(-),\ \mathrm{Dmod}(-) : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$$

(quasi-coherent sheaves, ind-coherent sheaves, and general left D-modules) constructed in [GR17a, GR17b]. None of them are coefficient systems. For $\mathrm{QCoh}(-)$ and $\mathrm{IndCoh}(-)$, equipped with the natural pullback operation (which makes sense for morphisms of schemes), one issue is that j^* does not have a left adjoint when j is an open immersion, contradicting (1). For $\mathrm{Dmod}(-)$, the situation is even worse as only three out of six operations (cf. Section 3.4) are defined. A general possible strategy to obtain more functoriality has been proposed by Clausen-Scholze using condensed mathematics [CS19].

3.3 More on axioms

3.3.1. Localization In practice, (Localization) is often the most difficult to verify. In order to establish that it is preserved by certain constructions, we will find it useful to reformulate the axiom as follows.

Lemma 3.7. *Let $C \in \mathscr{P}(B)$ be a functor satisfying (Push-forward) and (Smooth base change). Then the following are equivalent:*

(i) C satisfies (Localization).

(ii) For each closed immersion i with complementary open immersion j, the functor C satisfies all of:

(1) $C(\emptyset) \simeq 0$;

(2) i_* is fully faithful;

(3) the pair (i^*, j^*) is jointly conservative.

Proof. This is well-known, see for example [Dre18, Remark 5.9.(3)] or [CD19, § I.2.3]. □

Remark 3.8. If C is a coefficient system and i, j as in Lemma 3.7 then one automatically has the following additional properties:

(1) i_* admits a right adjoint $i^!$.

(2) $j^* i_* \simeq 0$.

(3) $j_*, j_\sharp$ are fully faithful.

(4) There are cofiber sequences for all M:

$$j_\sharp j^* M \to M \to i_* i^* M, \qquad i_* i^! M \to M \to j_* j^* M$$

The functors i^* and j^* thus exhibit $C(X)$ as a *recollement* of $C(Z)$ and $C(U)$ in the sense of [Lur, Appendix A.8]

3.3.2. Triangulated coefficient systems The notion of coefficient system is an ∞-categorical version of the closed symmetric monoidal stable homotopy 2-functors of [Ayo07, Definitions 1.4.1, 2.3.1, and 2.3.50]. We prefer to use a slightly different terminology for this latter notion than in loc.cit. to emphasize the parallel with coefficient systems.

Let us write $\mathrm{Tri}^{\otimes}$ for the $(2,1)$-category whose objects are symmetric monoidal triangulated categories, morphisms are monoidal exact functors and 2-morphisms are invertible monoidal exact natural transformations.

Definition 3.9. A *triangulated coefficient system (over B)* is a 2-functor $D : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Tri}^{\otimes}$ satisfying the triangulated analogues of the properties in Definition 3.1. To be precise, these are (1), (2), (4) of Definition 3.1, together with (ii) of Lemma 3.7. A *morphism of triangulated coefficient systems* is a natural transformation between triangulated coefficient systems which satisfies the triangulated analogue of Definition 3.6.

Triangulated coefficient systems have the advantage that they can be manipulated without the heavy machinery of ∞-category theory. Moreover, as a rule of thumb, statements about (symmetric monoidal) stable ∞-categories which involve checking properties rather than constructing new structure often reduce immediately to statements about their (symmetric monoidal) triangulated homotopy categories. Here are some instances of this rule of thumb which we will use.

Lemma 3.10. *Let $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$ be a functor.*

1. *By passing to the homotopy categories, one gets a functor $\mathrm{h}C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Tri}^{\otimes}$.*
2. *The functor C is a coefficient system if and only if $\mathrm{h}C$ is a triangulated coefficient system.*
3. *Let $C \to C'$ be a natural transformation of functors $\mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$. Then it is a morphism of coefficient systems if and only if the induced natural transformation $\mathrm{h}C \to \mathrm{h}C'$ is a morphism of triangulated coefficient systems.*

Proof. See [Dre18, Remark 5.10, Proposition 5.11, Corollary 5.13] and [DG20, Remark 7.6]. □

3.3.3. Shifted coefficient system Fix a scheme $X \in \mathrm{Sch}_B$. We denote by $C_X : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$ the coefficient system shifted by X that is informally defined by

$$C_X(Y) = C(Y \times_B X).$$

Given a morphism $f : Y \to X$, evaluation at f defines a morphism of coefficient systems $f^* : C_X \to C_Y$. Assume now that f is a proper morphism (resp. smooth morphism). Then the right (resp. left) adjoints define a natural transformation of functors $\mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_{\mathrm{st}}$

$$f_* : C_Y \to C_X \qquad (\text{resp. } f_\sharp : C_Y \to C_X).$$

This follows from proper (resp. smooth) base change (for which see Section 3.4.3 below) and [Lur, Corollary 4.7.4.18.(3)].

In fact, when viewing C_Y as a C_X-module, the latter being a commutative algebra object in $\mathrm{Fun}(\mathrm{Sch}_B^{\mathrm{op}}, \mathrm{Cat}_{\mathrm{st}})$, Remark 3.3 yields that f_* (resp. $f_\sharp$) may be upgraded to a morphism of C_X-modules.

3.4 Six operations

In [Ayo07] it is shown that a triangulated coefficient system affords the formalism of the six operations on quasi-projective schemes, and with slightly stronger assumptions, [CD19] extends this to all (separated finite type) schemes. Part of this formalism has been lifted to ∞-categories, see for example [LZ12, Rob14, Kha16, Dre18, AGV22]. We recall some of the properties we will need in the sequel. A more complete survey can be found in [Gal21]. Fix a coefficient system $C \in \mathrm{CoSy}_B$.

3.4.1. Exceptional functoriality An important fact is that there are exceptional functors associated with any separated (finite type) morphism $f : X \to Y$ of B-schemes:

$$f_! : C(X) \to C(Y), \qquad\qquad f^! : C(Y) \to C(X),$$

the first being left adjoint to the second. When f is proper, there is an equivalence $f_! \simeq f_*$, and when f is an open immersion, there is an equivalence $f_! \simeq f_\sharp$. These two properties essentially determine $f_!$ for general f.

3.4.2. Linearity The functor $f_!$ is 'linear' in the sense that the canonical morphism

$$f_!(M \otimes f^* N) \xrightarrow{\sim} f_! M \otimes N$$

is an equivalence for any $M \in C(X)$, $N \in C(Y)$. And equally, for a Cartesian square in Sch_B

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

the canonical morphism

$$f'_!(g')^* \xrightarrow{\sim} g^* f_!$$

is an equivalence. By adjunction, there is also an equivalence

$$f^! g_* \xrightarrow{\sim} g'_*(f')^!.$$

3.4.3. Smooth and proper base change An important consequence of the previous two properties is *proper base change*: If in the Cartesian square above f (and therefore f') is proper then the canonical morphism

$$g^* f_* \xrightarrow{\sim} f'_*(g')^*$$

is an equivalence. The same conclusion holds if instead g (and therefore g') is smooth. This is essentially equivalent to the axiom (Smooth base change).

3.4.4. Relative purity Given a locally free $\mathcal{O}_X$-module $\mathcal{V}$ viewed as a vector bundle $p : V \to X$ with zero section $s : X \hookrightarrow V$, one denotes by

$$\mathrm{Th}(\mathcal{V}) := p_\sharp s_* : C(X) \xrightarrow{\sim} C(X)$$

the associated Thom equivalence. Thom equivalences behave well under base change in X and direct sums of vector bundles. The inverse is denoted by $\mathrm{Th}^{-1}(\mathcal{V}) = s^! p^*$.

If $V = \mathbb{A}^1_X$ is the free rank 1 bundle then this equivalence is the subject of the axiom (Tate stability) and twice desuspended typically denoted by (1). More generally, one has 'Tate twists' (n) for arbitrary $n \in \mathbb{Z}$.

For $f : X \to Y$ *smooth* and separated, there are equivalences of functors

$$f^! \simeq \mathrm{Th}(\Omega_f) f^*$$

and

$$f_! \simeq f_\sharp \mathrm{Th}^{-1}(\Omega_f).$$

3.4.5. Exterior products and Künneth formula Given two schemes X_i, $i = 1, 2$, denote by $p_i : X_1 \times_B X_2 \to X_i$ the canonical projection. We denote the external product by

$$M_1 \boxtimes M_2 = p_1^* M_1 \otimes p_2^* M_2$$

for $M_i \in C(X_i)$. If $f_i : X_i \to Y_i$ are B-morphisms, $i = 1, 2$, then we have an equivalence in $C(Y_1 \times_B Y_2)$:

$$(f_1 \times f_2)_!(M_1 \boxtimes M_2) \xrightarrow{\sim} (f_1)_! M_1 \boxtimes (f_2)_! M_2.$$

In fact, the exterior products also provide an alternative way to encode the symmetric monoidal structure on the categories $C(X)$ for varying X. To see this, we use the following lemma:

Lemma 3.11. *[Lur, Theorem 2.4.3.18] Let C be an ∞-category with finite coproducts and $D^\otimes$ be a symmetric monoidal ∞-category. There is a canonical equivalence of ∞-category*

$$\mathrm{Fun}^{\mathrm{lax}}(C^{\amalg}, D^\otimes) \simeq \mathrm{Fun}(C, \mathrm{CAlg}(D^\otimes))$$

where $\mathrm{Fun}^{\mathrm{lax}}(-,-)$ denotes the category of symmetric lax-monoidal functors between two symmetric monoidal categories.

Lemma 3.11 and the interplay between Cartesian and coCartesian monoidal structures on opposite ∞-categories imply that a coefficient system C can be equivalently described as a lax-monoidal functor

$$C^{\boxtimes} : (\mathrm{Sch}_B^{\times})^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\mathrm{st}}.$$

and that the whole ∞-category cosy_B is equivalent to a (non-full) subcategory of $\mathrm{Fun}^{\mathrm{lax}}((\mathrm{Sch}_B^{\times})^{\mathrm{op}}, \mathrm{Cat}_{\infty}^{\mathrm{st}})$.

3.5 Correspondences and the six operation formalism

An insight due to Lurie, and independently to Hörmann, is that the six operation formalism can be very compactly encoded in terms of extensions of a presheaf in $\mathscr{P}(S)$ to categories of correspondences. To make this idea precise, one cannot work only with triangulated categories; it is necessary to use some kind of enhancement. In Hörmann's formulation, the enhancement is provided by fibered multiderivators [Hï8]. In Lurie's formulation, which we adopt, the enhancement is provided by $(\infty,1)$- and $(\infty,2)$-categories.

Lurie's idea has been developed extensively by Gaitsgory and Rozenblyum in [GR17a], as well as by Liu-Zheng in [LZ12] and most recently by [Man22]. We summarize the approach from [GR17a], completed by [RS20, Appendix A]. This is to date the most complete approach,

We denote by $\mathrm{Corr}(\mathrm{Sch}_B)$ the $(\infty,1)$-category of correspondences defined in [GR17a, §7.0.1.5]. By construction, its objects are the objects of Sch_B, its 1-morphisms are correspondences of the form $X \xleftarrow{g} Z \xrightarrow{f} Y$ and a 2-morphism $\alpha : (X \xleftarrow{g} Z \xrightarrow{f} Y) \to (X \xleftarrow{g'} Z' \xrightarrow{f'} Y)$ is given by a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, "p"', "\sim"] & \\ X & & Y \\ & Z' \arrow[ul, "g'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

The composition of correspondences is given by fibre product. We also write $\mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}$ for the wide subcategory of $\mathrm{Corr}(\mathrm{Sch}_B)$ where we restrict to those correspondences as above with f separated; this category naturally occurs because we only consider exceptional push-forwards $f_!$ when f is separated.

We denote by $\mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}^{\mathrm{prop},2-op}$ the $(\infty,2)$-category of correspondences defined in [GR17a, §7.1.2.5], where all denotes the class of all morphisms, sep the class of separated morphisms, and prop the class of proper morphisms. The category $\mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}^{\mathrm{prop},2-op}$ has the same objects and 1-morphisms as $\mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}$, and a 2-morphism $\alpha : (X \xleftarrow{g} Z \xrightarrow{f} Y) \to (X \xleftarrow{g'} Z' \xrightarrow{f'} Y)$

is given by a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & Y \\ & Z' \arrow[ul, "g'"] \arrow[uu, "p"] \arrow[ur, "f'"'] & \end{tikzcd}$$

with p a proper morphism (note the direction of the arrow; this is what the superscript 2-op indicates).

Remark 3.12. In fact, because Sch_B is an ordinary $(1,1)$-category, $\mathrm{Corr}(\mathrm{Sch}_B)$ (resp. $\mathrm{Corr}(\mathrm{Sch}_B)^{\mathrm{prop},2-\mathrm{op}}_{\mathrm{all},\mathrm{sep}}$) is a $(2,1)$-category (resp. a $(2,2)$-category) and the above description of objects, 1-morphisms and 2-morphisms together with a little more data about compositions describes it completely, with no additional higher coherences. We will not use this fact and thus do not prove it. For the application to coefficient systems (Theorem ?? below), it is really necessary to consider $\mathrm{Corr}(\mathrm{Sch}_B)$ (resp. $\mathrm{Corr}(\mathrm{Sch}_B)^{\mathrm{prop},2-\mathrm{op}}_{\mathrm{all},\mathrm{sep}}$) as an $(\infty,1)$-category (resp. $(\infty,2)$-category. Moreover, we also need to consider symmetric monoidal structures on $\mathrm{Corr}(\mathrm{Sch}_B)^{(\mathrm{prop},2-\mathrm{op})}_{\mathrm{all},\mathrm{sep}}$, and symmetric monoidal $(2,1)$ (or worse $(2,2)$)-categories are quite cumbersome to work with explicitely.

The construction of categories of correspondences is functorial [GR17a, Chapter IX, §2.1.2]. We do not need to spell out the whole formalism, but let us observe that $\mathrm{Sch}_B^{\mathrm{op}}$ and $\mathrm{Sch}_{B,\mathrm{sep}}$ are themselves equivalent to categories of "correspondences with one leg", namely $\mathrm{Sch}_B^{\mathrm{op}} \simeq \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{isom}}$ and $\mathrm{Sch}_{B,\mathrm{sep}} \simeq \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{isom},\mathrm{sep}}$ (where isom denotes the class of isomorphisms). The functoriality of correspondences then provides a functor

$$\iota^* : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}$$

which sends a scheme to itself and a morphism $f : X \to Y$ to the correspondence $(Y \xleftarrow{f} X = X)$, and

$$\iota_! : \mathrm{Sch}_{B,\mathrm{sep}} \to \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}$$

which sends a scheme to itself and a separated morphism $f : X \to Y$ to the correspondence $(X = X \xrightarrow{f} Y)$.

We denote by $\iota^{2,*}$ (resp. $\iota_{2,!}$) the composite $(\infty,2)$-functor

$$\mathrm{Sch}_{op}\, B \xrightarrow{\iota} \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}} \to \mathrm{Corr}(\mathrm{Sch}_B)^{\mathrm{prop},2-\mathrm{op}}_{\mathrm{all},\mathrm{sep}}$$

(resp.

$$\mathrm{Sch}_{sep}\, B \xrightarrow{\iota} \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}} \to \mathrm{Corr}(\mathrm{Sch}_B)^{\mathrm{prop},2-\mathrm{op}}_{\mathrm{all},\mathrm{sep}})$$

By [GR17a, Chapter IX, §2.1.3], the cartesian symmetric monoidal structure of Sch_B induces a structure of symmetric monoidal $(\infty,2)$-category on $\mathrm{Corr}(\mathrm{Sch}_B)^{\mathrm{prop},2-op}_{\mathrm{all},\mathrm{sep}}$ (resp. a structure of symmetric monoidal $(\infty,1)$-category on $\mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}}$). Since the functor "Corr : Trpl $\to$ 2 – Cat" (in the notation of loc.cit.) is symmetric monoidal, and remembering than $\mathrm{Sch}_{op}\,B \simeq \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{isom}}$ and $\mathrm{Sch}_{sep}\,B \simeq \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{isom},\mathrm{sep}}$, we conclude that the functors $\iota^{2,*}$ and $\iota_{2,!}$ (resp. ι^* and $\iota_!$) are symmetric monoidal.

As explained in section 3.4.5, the data of a coefficient system C is equivalent to a functor

$$C^{\boxtimes} : (\mathrm{Sch}_B^{\times})^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\mathrm{st}}$$

satisfying certain properties. We can now state the main theorem about this approach to the six operation formalism for coefficient systems, due to Gaitsgory-Rozenblyum [GR17a] with some additional arguments by Richarz-Scholbach [RS21].

Theorem 3.13. *Let $C(-)$ be a coefficient system. There is an extension*

$$C_!^{\boxtimes,*} : \mathrm{Corr}(\mathrm{Sch}_B)^{\mathrm{prop},2-op}_{\mathrm{all},\mathrm{sep}} \to \mathrm{Cat}_{\mathrm{st}}$$

of the functor $C^{\boxtimes}$ along ι as a lax-monoidal $(\infty,2)$-functor, which is the unique lax-monoidal extension of $C^{\boxtimes}$ satisfying base change.

Proof. This is proven in [GR17a] in the special case where $C^{\boxtimes}$ is a strict monoidal functor. Building on [GR17a], this is then proven in the special case of $\mathrm{DM}(-,\mathbb{Q})$ in [RS21, Lemma A.7]; the arguments there apply to any coefficient system. □

Corollary 3.14. *Let $C^{\otimes}$ be a symmetric monoidal coefficient system, considered as a lax-monoidal functor $C^{\boxtimes} : (\mathrm{Sch}_{\times}\,B)^{\mathrm{op}} \to \mathrm{Cat}_{\mathrm{st}}$. There is an extension*

$$C_!^{\boxtimes,*} : \mathrm{Corr}(\mathrm{Sch}_B)_{\mathrm{all},\mathrm{sep}} \to \mathrm{Cat}_{\mathrm{st}}$$

of $C^{\boxtimes}$ along ι as a lax-monoidal ∞-functor

Remark 3.15.

- Unfortunately, the proof of Theorem 3.13, and thus of Corollary 3.14, relies on unproven statements about $(\infty,2)$-categories (collected in [GR17c, Chapter 10, 0.4.2]). These statements are widely expected to hold, but are not as far as we not proven.
- For many purposes, Corollary 3.14 is enough, and we don't need to explicitly consider $(\infty,2)$-categories; The only additional structure encoded by Theorem 3.13 which is not present in Corollary 3.14 seems to be the natural transformation

$$f_! \to f_*$$

and its (higher) compatibilities with the rest of the operations. However, at least in the approach of [GR17a], the proof of Corollary 3.14 really passes through Theorem 3.13 and thus through $(\infty,2)$-categories.

4 Exponentiation as a triangulated coefficient system

The main construction of this paper associates to a (resp. triangulated) coefficient system C another (resp. triangulated) coefficient system $C_{\exp}$, the *exponentiation* of C. In the present section we perform the construction at the triangulated level, while in Section 5 we will work at the level of ∞-categories. Many results about the latter construction will then be formal consequences of the results in this section. We proceed in this way to keep most of the discussion elementary and accessible to readers who don't want to wade into technicalities about symmetric monoidal ∞-categories.

So, throughout this section we fix a triangulated coefficient system C over B. The construction of $C_{\exp}$ depends on a choice of smooth commutative B-group scheme A satisfying certain conditions. The most important case, which motivates the construction and in which those conditions are always satisfied, is the additive group $A = \mathbb{G}_{a,B}$. We encourage the reader to always keep this case in mind.

Convention 4.1. We fix a commutative B-group scheme $(A, \mu, 0)$ where $\mu : A \times_B A \to A$ and $0 : B \to A$ denote the multiplication and unit. We write $\pi : A \to B$ for the structure map. For $X \in \mathrm{Sch}_B$, we write $\pi_X : X_A := X \times_B A \to X$, and often also denote π_X as π when there is no confusion. We make the following assumptions on A:

1. $\pi : A \to B$ is smooth, and
2. the functor $\pi_X^* : C(X) \to C(X_A)$ is fully faithful for all $X \in \mathrm{Sch}$.

These assumptions are satisfied for $A = \mathbb{G}_{a,B}$ by the axiom ($\mathbb{A}^1$-homotopy) of coefficient systems.

4.1 Definition through vanishing (co)homology

As explained in the introduction, we would like to define the exponentiation as the Verdier quotient

$$C_{\exp}(X) = C(X_A)/\pi^* C(X).$$

For technical reasons, it is very useful to realize the resulting category as a full subcategory of $C(X_A)$. We thus choose the following as our definition.

Definition 4.2. Let $X \in \mathrm{Sch}_B$. We define $C_{\exp}(X) \subseteq C(X_A)$ as the full subcategory given as the kernel of the exact functor $\pi_{X!} : C(X_A) \to C(X)$.

In this subsection, in a series of remarks , we clarify the relationship between this definition and the Verdier quotient above, and provide several other equivalent characterisations of $C_{\exp}(X)$. These equivalent constructions will not be used in the rest of the paper, and the reader happy with Definition 4.2 can jump directly to section 4.2.

Remark 4.3. The functor $\pi^* : C(X) \to C(X_A)$ has a left adjoint $\pi_\sharp$ (since π is smooth) and a right adjoint π_*. By assumption, it is also fully faithful so that we obtain a recollement

$$
\begin{tikzcd}
C(X) \arrow[r, "\pi^*" description] & C(X_A) \arrow[l, bend right, "\pi_\sharp"'] \arrow[l, bend left, "\pi_*"] \arrow[r, "Q" description] & C(X_A)/\pi^*C(X) \arrow[l, bend right, "Q_\lambda"'] \arrow[l, bend left, "Q_\rho"]
\end{tikzcd}
$$

in which

- Q_λ is fully faithful and induces an equivalence

$$C(X_A)/\pi^*C(X) \xrightarrow{\sim} \ker(\pi_\sharp) = {}^\perp(\pi^*C(X))$$

  onto the left orthogonal complement, and
- Q_ρ is fully faithful and induces an equivalence

$$C(X_A)/\pi^*C(X) \xrightarrow{\sim} \ker(\pi_*) = (\pi^*C(X))^\perp$$

  onto the right orthogonal complement.

Remark 4.4. The ordinary and exceptional pullback functors $\pi^*, \pi^! : C(X) \to C(X_A)$ are closely related. Indeed, we have (Section 3.4.4)

$$\pi^! \simeq \mathrm{Th}(\Omega_\pi) \circ \pi^*$$

where $\mathrm{Th}(\Omega_\pi)$ denotes the Thom equivalence with respect to the relative cotangent bundle $\Omega_\pi = \Omega_{X_A/X}$ of π_X. Since A is a smooth group scheme, it follows that this relative cotangent bundle is in fact the pullback of the conormal sheaf to the embedding $0 : B \to A$. In particular we have

$$\pi^! \simeq \mathrm{Th}(\Omega_\pi) \circ \pi^* \simeq \pi^* \circ \mathrm{Th}\big((0^*\Omega_{A/B})\,|_X\big)$$

and it follows that the images of $\pi^!$ and π^* coincide. (In the special case of $A = \mathbb{G}_a$, this Thom equivalence can be identified with a Tate twist and shift: $\mathrm{Th}(\Omega_{\mathbb{G}_a}) \simeq (-)(1)[2]$.) The recollement of Remark 4.3 is thus 'isomorphic' to a twisted version:

$$
\begin{tikzcd}
C(X) \arrow[r, "\pi^!" description] & C(X_A) \arrow[l, bend right, "\pi_!"'] \arrow[l, bend left, "\pi_*\circ\mathrm{Th}^{-1}(\Omega_\pi)"] \arrow[r, "Q" description] & C(X_A)/\pi^!C(X) \arrow[l, bend right, "Q_\lambda"'] \arrow[l, bend left, "Q_\rho"]
\end{tikzcd}
$$

and of course, we then also have

$$C(X_A)/\pi^*C(X) \underset{\sim}{\xrightarrow{Q_\lambda}} \ker(\pi_!).$$

Remark 4.5. While $\ker(\pi_*)$ and $\ker(\pi_!) = \ker(\pi_\sharp)$ are equivalent, they are *not* equal as subcategories of $C(X_A)$. In our construction of exponentiation we have chosen to focus on the realization of $C_{\exp}(X)$ as $\ker(\pi_!)$. The reason is that coefficient systems encode the ordinary pullback functors (or, $*$-functoriality) and while for $f : Y \to X$, the induced functor

$$f_A^* : C(X_A) \to C(Y_A)$$

restricts to the respective kernels of $\pi_!$, the same is not true for the kernels of π_*, see Lemma 4.8.

We note that this is in contrast to [KS11], where Kontsevich and Soibelman define exponential mixed Hodge structures in a similar way (with $X = B = \mathrm{Spec}(\mathbb{C})$), but take the kernel of π_*.

Remark 4.6. When viewing the colocalization $Q_\lambda \circ Q$ as a functor

$$Q_\lambda \circ Q : C(X_A) \to \ker(\pi_!),$$

we will, below, denote it by Π. It is right adjoint to the canonical inclusion and thus a *coreflector*, cf. ??. It may also be expressed explicitly as the convolution product with a certain object in $\ker(\pi_!)$, cf. Corollary 4.32.

Remark 4.7. Suppose C affords a Verdier duality, that is, for each $X \in \mathrm{Sch}$, there is an anti-involution

$$\mathbb{D}_X : C(X) \xrightarrow{\sim} C(X)^{\mathrm{op}}$$

which exchanges the $*$-functoriality and the !-functoriality:

$$\mathbb{D}_Y f^* \simeq f^! \mathbb{D}_X$$

for every $f : Y \to X$. Then $\mathbb{D}_X$ sends $\ker(\pi_!)$ to $\ker(\pi_*)$ and restricts to an antiequivalence

$$\ker(\pi_!) \xrightarrow{\sim} \ker(\pi_*)^{\mathrm{op}}.$$

4.2 Four functor formalism

In this section, we construct the non-monoidal part of the triangulated coefficient system C_{exp}. We will use (often without mentioning) the basic properties of coefficient systems described in Section 3.4.

Lemma 4.8. *Let $f : X \to Y$ be a morphism in Sch_B. We have $f_A^* C_{\mathrm{exp}}(Y) \subset C_{\mathrm{exp}}(X)$. Hence C_{exp} defines a subfunctor of C_A (cf. Section 3.3.3).*

Proof. This follows from the base change isomorphism: $\pi_{X!} f_A^* \simeq f_A^* \pi_{Y!}$. □

Our first goal is to verify the subset of the axioms of a triangulated coefficient system for C_{exp} which do not involve the monoidal structure. We start with some easy observations.

Proposition 4.9.
1. *Let $f : X \to Y$ be a smooth morphism in Sch_B. The functor $(f_A)_\sharp$ restricts to a functor $(f_A)_\sharp : C_{\mathrm{exp}}(X) \to C_{\mathrm{exp}}(Y)$ left adjoint to $f_A^* : C_{\mathrm{exp}}(Y) \to C_{\mathrm{exp}}(X)$.*
2. *Let $f : X \to Y$ be a proper morphism in Sch_B. The functor $(f_A)_*$ restricts to a functor $(f_A)_* : C_{\mathrm{exp}}(X) \to C_{\mathrm{exp}}(Y)$ right adjoint to $f_A^* : C_{\mathrm{exp}}(Y) \to C_{\mathrm{exp}}(X)$.*
3. *The functor C_{exp} satisfies the (triangulated version of) the axioms (Smooth base change) and ($\mathbb{A}^1$-homotopy).*

Proof. For a smooth morphism $f : X \to Y$, we have a natural isomorphism of left adjoints $\pi_{Y\sharp}(f_A)_\sharp = f_\sharp \pi_{X\sharp}$. Together with the fact that $\pi_\sharp$ and $\pi_!$ differ by a Thom equivalence, this shows that $(f_A)_\sharp$ restricts to a functor $C_{\exp}(X) \to C_{\exp}(Y)$. It is a left adjoint to f_A^* and satisfies the Beck-Chevalley condition for cartesian squares, since those properties are satisfied at the level of C_A. This proves the first statement together with (Smooth base change). For a proper morphism $f : X \to Y$, we have $f_! \simeq f_*$ and the same argument shows that $(f_A)_*$ restricts to a right adjoint of f_A^*.

Finally, for ($\mathbb{A}^1$-homotopy) we notice that it is equivalent to the counit $(p_A)_\sharp (p_A)^* \to \mathrm{id}$ being invertible hence this follows as well from ($\mathbb{A}^1$-homotopy) for C_A. □

We turn to the construction of the other functors. Unlike the pullbacks and the $\sharp$-pushforwards, the pushforward $(f_A)_*$ does not restrict directly from C_A in general and an additional step is needed. We need a right adjoint (or coreflector) of the inclusion $C_{\exp} \hookrightarrow C_A$.

Remark 4.10. Consider the endomorphism $L = \pi^* \pi_\sharp$ of $C_A(X)$ together with the morphism $\eta : \mathrm{id} \to \pi^* \pi_\sharp$ given by the unit of the adjunction. By our Convention 4.1, L is an exact localization functor and therefore there is a functorial fiber of η, describing the associated colocalization [Kra10, § 4.11]. (In the notation of Remark 4.3, this colocalization is given by $Q_\lambda Q : C_A(X) \to C_A(X)$.) The image of this colocalization is precisely $\ker(L) = C_{\exp}(X)$, see [Kra10, Proposition 4.11.1] if necessary.

Definition 4.11. We denote by $\Pi : C_A(X) \to C_{\exp}(X)$ the coreflector that is characterized by the distinguished triangle for every $M \in C_A(X)$:

$$\Pi(M) \to M \to \pi^* \pi_\sharp(M) \tag{4.12}$$

We will often abuse notation (as just done here) and identify Π with the colocalization $C_A(X) \xrightarrow{\Pi} C_{\exp}(X) \hookrightarrow C_A(X)$.

Remark 4.13. The morphism π is smooth, so by relative purity we have a natural isomorphism $\pi^* \pi_\sharp \simeq \pi^! \pi_!$ (the Thom and inverse Thom equivalences compensate).

Lemma 4.14. *The endofunctor Π commutes with pullbacks, $\sharp$-pushforwards and $!$-pushforwards in C_A in the following sense. Let $f : X \to Y$ be a morphism in Sch_B.*

1. *The natural transformation $(f_A)^* \Pi \to \Pi (f_A)^*$ is an isomorphism.*
2. *Assume that f is smooth. The natural transformation $(f_A)_\sharp \Pi \to \Pi (f_A)_\sharp$ is an isomorphism.*
3. *Assume that f is finite type separated. The natural transformation $(f_A)_! \Pi \to \Pi (f_A)_!$ is an isomorphism.*

Proof. The functors π^*, $\pi_\sharp$ between C and C_A as well as the unit $\mathrm{id} \to \pi^* \pi_\sharp$ commute with arbitrary pullbacks, $\sharp$-pushforwards and $!$-pushforwards (using base change and the fact that π is smooth in the latter case). Hence all claims follow from the defining distinguished triangle (4.12). □

Proposition 4.15. *The functor $C_{\exp}$ satisfies the axioms (Push-forward), (Tate stability) and condition (ii) of Lemma 3.7.*

Proof. For any f in Sch_B, the composite $\Pi(f_A)_*$ is a right adjoint to $(f_A)^*$. This proves (Push-forward).

We certainly have $C_{\exp}(\emptyset) = 0$. Let i be a closed immersion with complementary open immersion j. The pair (i_A^*, j_A^*) is conservative at the level of C_A, so it is also conservative for the subfunctor $C_{\exp}$. Moreover, the right adjoint to i_A^* at the level of $C_{\exp}$ is given by $(i_A)_*$, by Proposition 4.9.2. Therefore, (ii) of Lemma 3.7 for $C_{\exp}$ follows from the corresponding axiom for C_A.

We now prove (Tate stability). Let $X \in \mathrm{Sch}$. By definition of the $\sharp$ and $*$-pushforwards, we have to show that the functor $(q_A)_\sharp\Pi(i_A)_* : C_{\exp}(X) \to C_{\exp}(X)$ is an equivalence. Again by Proposition 4.9.2, $(i_A)_*$ preserves $C_{\exp}$, so we have $(q_A)_\sharp\Pi(i_A)_* \simeq (q_A)_\sharp(i_A)_*$ which is an equivalence by (Tate stability) for C_A. □

At this point, combining Proposition 4.9 and Proposition 4.15, we have shown that the functor $C_{\exp}$ is a (non-monoidal) stable homotopy 2-functor in the sense of [Ayo07, Definitions 1.4.1]. By [Ayo07, Scholie 1.4.2], we have a "four functor formalism" for $C_{\exp}$. To distinguish notationally from the operations in C, we adopt the following.

Notation. Let f be a morphism in Sch. We write

(a) $\underline{f}^* = C_{\exp}(f)^*$ and

(b) $\underline{f}_* = C_{\exp}(f)_*$.

Assume f is separated. We write

(c) $\underline{f}_! = C_{\exp}(f)_!$ and

(c) $\underline{f}^! = C_{\exp}(f)^!$.

Let us identify the resulting operations.

Proposition 4.16. *Let f be a morphism in* Sch*. Then:*

(a) $\underline{f}^* = f_A^*$ *and*

(b) $\underline{f}_* = \Pi(f_A)_*$.

Assume f is separated. We write

(c) $\underline{f}_! = (f_A)_!$,

(c) $\underline{f}^! = \Pi f_A^!$.

Moreover,

(c) if f is smooth then $\underline{f}_\sharp = (f_A)_\sharp$,

(d) if f is smooth and separated then $\underline{f}^! = f_A^!$,

(e) if f is proper then $\underline{f}_* = (f_A)_*$ *and*

(f) *if E is a vector bundle then $C_{\exp}(\mathrm{Th})(E) = \mathrm{Th}(E_A)$.*

Finally, we always have $\Pi(f_A)_*\Pi \simeq \Pi(f_A)_*$, and $\Pi(f_A)^!\Pi \simeq \Pi f_A^!$ for f separated.

Proof. The identification $C_{\exp}(f)^* = (f_A)^*$ is true by definition of $C_{\exp}$ and Lemma 4.8. The claim about $f_\sharp$ if f is smooth (resp. f_* if f is proper) follows from Lemma 4.9. This implies the claim about Thom equivalences. For f smooth, relative purity then shows $C_{\exp}(f)^! = f_A^!$.

For general f choose a compactification $f = \overline{f}j$ with $\overline{f}$ proper and j open immersion. Then $C_{\exp}(f)_! = C_{\exp}(\overline{f})_* C_{\exp}(j)_\sharp$ which by the previous paragraph equals $(\overline{f}_A)_*(j_A)_\sharp = (f_A)_!$. Passing to right adjoints this also gives $C_{\exp}(f)^! = \Pi f_A^!$.

The final statement follows by uniqueness of right adjoints. □

4.3 Convolution product

We now come to the monoidal structure.

Definition 4.17. Let $X \in \mathrm{Sch}_B$. The *convolution product* on $C_A(X)$ is the bifunctor

$$* : C_A(X) \times C_A(X) \to C_A(X), (M, N) \mapsto \mu_!(M \boxtimes_X N).$$

In the study of the convolution product, it is useful to introduce some additional morphisms related to the group A.

Definition 4.18. The two *shear isomorphisms* are the two morphisms $\delta_1, \delta_2 : A \times_B A \simeq A \times_B A$ of B-schemes uniquely characterised by

$$\mu\delta_1 = p_1,\ p_2\delta_1 = p_2 \text{ and } p_1\delta_2 = p_1,\ \mu\delta_2 = p_2$$

i.e. given informally by the formulas $\delta_1(x, y) = (x - y, y)$ and $\delta_2(x, y) = (x, y - x)$. We also write $d = p_1\delta_1$, i.e $d(x, y) = x - y$.

Lemma 4.19. *Let $X \in \mathrm{Sch}_B$. Let $M \in C_A(X)$ and $N \in C(X)$. There is an isomorphism*

$$0_*N * M \simeq \pi^*N \otimes M$$

which is natural in M and N.

Proof. We have

$$0_*N * M := \mu_!(0_*N \boxtimes M) \simeq \mu_!(0 \times \mathrm{id})_*(N \boxtimes M) \simeq N \boxtimes M = \pi^*N \otimes M$$

where the second isomorphism uses the closed projection formula and base change, and the second uses $\mu(0 \times \mathrm{id}) = \mathrm{id}$. □

Corollary 4.20. *The object $0_*\mathbb{1}$ is a unit for the convolution product on $C_A(X)$: there are isomorphisms*

$$0_*\mathbb{1} * M \simeq M \simeq M * 0_*\mathbb{1},$$

natural in $M \in C_A(X)$.

Proof. This follows directly from Lemma 4.19. ◻

Lemma 4.21. *Let $X \in \mathrm{Sch}_B$ and $M, N \in C_A(X)$. Then we have an isomorphism*

$$\pi_!(M * N) \simeq \pi_! M \otimes \pi_! N$$

in $C(X)$. Consequently, the convolution product restricts to a bifunctor

$$* : C_{\exp}(X) \times C_{\exp}(X) \to C_{\exp}(X).$$

Proof. We have

$$\pi_!(M * N) := \pi_! \mu_!(M \boxtimes_X N) \simeq (\pi \times \pi)_!(M \boxtimes_X N) \simeq \pi_! M \otimes \pi_! N.$$

by the observation that $\pi\mu = \pi \times \pi$ and the Künneth formula for !-pushforwards (which is a formal consequence of base change and the projection formula and thus holds in any coefficient system [JY21, Lemma 2.2.3]) . The claim follows since $C_{\exp}(X) = \ker(\pi_!)$. ◻

Remark 4.22. We will prove below that this convolution product on $C_{\exp}(X)$ underlies a symmetric monoidal structure. In particular, the unit is described in Corollary 4.29. The symmetry constraint is in particular the obvious one, coming from the isomorphism $A^2 \simeq A^2$ exchanging the two factors.

Lemma 4.23. *Let $M \in C(X)$ and $N \in C_A(X)$. We have a canonical and functorial isomorphism*

$$\pi^* M * N \simeq \pi^*(M \otimes \pi_! N).$$

In particular, convolution with a constant object always produces a constant object.

Proof. The isomorphism is obtained as follows.

$$\begin{aligned}
\pi^* M * N &\simeq \mu_!(\pi^* M \boxtimes_X N) \\
&\simeq \mu_! \delta_{2!} \delta_2^*(\pi^* M \boxtimes_X N) \\
&\simeq p_{2!}(p_1^* \pi^* M \otimes \delta_2^* p_2^* N) \\
&\simeq (p_2)_!(p_2^* \pi^* M \otimes \delta_2^* p_2^* N) \\
&\simeq \pi^* M \otimes p_{2!} \delta_2^* p_2^* N \\
&\simeq \pi^* M \otimes \mu_! \delta_{2!} \delta_2^* p_2^* N \\
&\simeq \pi^* M \otimes p_{1!} p_2^* N \\
&\simeq \pi^* M \otimes \pi^* \pi_! N \\
&\simeq \pi^*(M \otimes \pi_! N)
\end{aligned}$$

where we have used the projection formula, proper base change, the identities of Definition 4.18 and the fact that, for any isomorphism g, we have $\mathrm{id} \simeq g_! g^*$. ◻

Lemma 4.24. *Let $M, N \in C_A(X)$. The natural transformation $\Pi \to \mathrm{id}$ induces isomorphisms*

$$\Pi M * N \simeq \Pi(M * N) \simeq \Pi M * \Pi N.$$

Proof. Let us write $\theta : \Pi \to \mathrm{id}$ for the natural transformation of the coreflector. By Lemma 4.21, we have $\Pi M * N \in C_{\exp}(X)$, hence by the universal property of the coreflector Π the morphism $\theta_M * \mathrm{id} : \Pi M * N \to M * N$ factors uniquely through $\Pi(M * N)$, providing a natural morphism $\alpha : \Pi M * N \to \Pi(M * N)$. Moreover, since $\Pi M * N \in C_{\exp}(X)$, we also see that the morphism $\theta_{\Pi M * N} : \Pi(\Pi M * N) \to \Pi(M * N)$ is an isomorphism.

By naturality, the morphism $\alpha * \theta_{\Pi M * N} : \Pi(\Pi M * N) \to \Pi(M * N)$ coincides with $\Pi(\theta_M * \mathrm{id})$, so it is an isomorphism iff the cofiber of $\theta_M * \mathrm{id}$, which is nothing else than $\mathrm{Cofib}(\theta_M) * N$, is in $\pi^* C(X)$. Since $\mathrm{Cofib}(\theta_M)$ is in $\pi^* C(X)$, this follows from Lemma 4.23.

This proves the first isomorphism, and the second follows by two applications of the first. ◻

We can also describe the interaction between Π and the internal Hom in $C_A(-)$.

Lemma 4.25. *Let $M, N \in C_A(X)$. The natural transformation $\Pi \to \mathrm{id}$ induces a commutative square of isomorphisms*

$$\begin{tikzcd}
\Pi \underline{\mathrm{Hom}}(M, \Pi N) \arrow[r, "\sim"] \arrow[d, "\sim"] & \Pi \underline{\mathrm{Hom}}(M, N) \arrow[d, "\sim"] \\
\Pi \underline{\mathrm{Hom}}(\Pi M, \Pi N) \arrow[r, "\sim"] & \Pi \underline{\mathrm{Hom}}(\Pi M, N)
\end{tikzcd}$$

Proof. This is a formal consequence of the tensor/internal Hom adjunction and of Lemma 4.24. ◻

We can also relate the internal Homs in $C_{\exp}$ and C_A.

Notation. Let $X \in \mathrm{Sch}$ and $N, P \in C_{\exp}(X)$. We denote the internal hom of N and P in $C_{\exp}(X)$ by $\underline{\mathrm{Hom}}_{\exp}(N, P)$.

Lemma 4.26. *Let $X \in \mathrm{Sch}$ and $N, P \in C_{\exp}(X)$. We have a canonical isomorphism*

$$\underline{\mathrm{Hom}}_{\exp}(N, P) \simeq \Pi (p_1)_* \underline{\mathrm{Hom}}_{C_{A^2}(X)}(p_2^* N, \mu^! P)$$

in $C(X_A)$.

Proof. This follows formally from the definition of the convolution product and adjunctions. ◻

Recall from the introduction that one motivation of exponentiation is to introduce analogues of the exponential $\mathscr{D}$-modules $\mathscr{E}^a$ attached to a function $a : X \to \mathbb{G}_a$. In the context of a general coefficient system, we call the resulting objects exponential twists.

Definition 4.27. Let $(X, a) \in \mathrm{Sch}_A$. Write $z(a) : X \to A \times_B X$ for the closed immersion of the graph of a, and $u(a)$ for the open immersion of the open complement. The *exponential twist functor* associated to (X, a) is

$$\mathbb{E}_a(-) : C(X) \to C_{\exp}(X), \qquad M \mapsto \Pi z(a)_* M$$

We write $\mathbb{E}_a := \mathbb{E}_a(\mathbb{1})$.

Lemma 4.28. *Let $X \in \mathrm{Sch}_B$, let $M \in C_A(X)$, and $N \in C(X)$. Then we have an isomorphism*

$$M * \mathbb{E}_0(N) \simeq \Pi(M \otimes \pi^* N),$$

natural in M and N.

Proof. This is the composite

$$\begin{aligned} M * \Pi 0_* N &\simeq \Pi(M * 0_* N) && \text{Lemma 4.24} \\ &\simeq \Pi(M \otimes \pi^* N) && \text{Lemma 4.19} \end{aligned}$$

□

Corollary 4.29. *The object $\mathbb{E}_0 = \Pi 0_* \mathbb{1} \in C_{\exp}(X)$ is a unit for the convolution product:*

$$\mathbb{E}_0 * M \simeq M \simeq M * \mathbb{E}_0$$

naturally in $M \in C_{\exp}(X)$.

Proof. This follows directly from Lemma 4.28. □

Lemma 4.30. *Let $f : (X, af) \to (Y, a)$ be a morphism of A-schemes. Then the following diagram in Sch_B is Cartesian.*

$$\begin{tikzcd} X \arrow[r, "\mathrm{id} \times af"] \arrow[d, "f"] & X \times A \arrow[d, "f \times A"] \\ Y \arrow[r, "\mathrm{id} \times a"] & Y \times A \end{tikzcd}$$

Proof. Let P denote the pullback with structure maps $\gamma : P \to X \times A$ and $\delta : P \to Y$. Since the square in the statement clearly commutes we obtain a map $\alpha : X \to P$, and we define $\beta : P \to X$ as the composite $p_1 \circ \gamma$, where p_1 denotes the projection on the first factor.

By construction we have $\beta \circ \alpha = \mathrm{id}$ and we need to prove $\alpha \circ \beta = \mathrm{id}$. As γ is a closed immersion it suffices to show instead $\gamma \circ \alpha \circ \beta = \gamma : P \to X \times A$. We treat both factors separately:

$$\begin{aligned} p_1 \circ \gamma \circ \alpha \circ \beta &= p_1 \circ (\mathrm{id} \times af) \circ \beta \\ &= \beta \\ &= p_1 \circ \gamma \end{aligned}$$

and

$$\begin{aligned}
p_2 \circ \gamma \circ \alpha \circ \beta &= p_2 \circ (f \times A) \circ \gamma \circ \alpha \circ \beta \\
&= p_2 \circ (\mathrm{id} \times a) \circ \delta \circ \alpha \circ \beta \\
&= a \circ f \circ \beta \\
&= a \circ f \circ p_1 \circ \gamma \\
&= a \circ p_1 \circ (f \times A) \circ \gamma \\
&= a \circ p_1 \circ (\mathrm{id} \times a) \circ \delta \\
&= a \circ \delta \\
&= p_2 \circ (\mathrm{id} \times a) \circ \delta \\
&= p_2 \circ (f \times A) \circ \gamma \\
&= p_2 \circ \gamma
\end{aligned}$$

This completes the proof. □

Lemma 4.31. *(a) Let $f : (X, af) \to (Y, a)$ be a morphism in Sch_A. There is an equivalence of functors $C(Y) \to C_{\exp}(X)$*

$$f^* \circ \mathbb{E}_a(-) \simeq \mathbb{E}_{af}(f^* -).$$

(b) There is an equivalence of functors

$$\mathbb{E}_a(-) \simeq u(a)_* u(a)^* \pi^!(-)[-1].$$

In particular, we have

$$\mathbb{E}_0 \simeq u_* u^* \pi^! \mathbb{1}[-1].$$

Proof. Claim (a) follows directly from the definition together with Lemma 4.14 and Lemma 4.30.

By definition and Remark 4.13, we have $\Pi z(a)_* = \mathrm{Fib}(z(a)_* \to \pi^! \pi_! z(a)_*)$. It therefore suffices to prove that $u(a)_* u(a)^* \pi^!(-)$ is a cofiber of the same map. Under the identifications $\pi_! z(a)_* \simeq (\pi z(a))_! = (\mathrm{id})_! \simeq \mathrm{id}$ and $z(a)^! \pi^! \simeq \mathrm{id}$, this map is given by

$$z(a)_* z(a)^! \pi^! \to \pi^!,$$

whose cofiber is as claimed, by (Localization) for the coefficient system C. This proves claim (b). □

Corollary 4.32. *We have a natural isomorphism of functors $\Pi \cong u_* u^* \pi^! \mathbb{1}[-1] * (-) : C_A \to C_{\exp}$.*

Corollary 4.33. *Let $a : X \to A$ be a potential. Then we have in $C_{\exp}(X)$:*

$$\mathbb{E}_a \simeq a^* \mathbb{E}_{\mathrm{id}}$$

Remark 4.34. This partly explains the special role that the object $\mathbb{E}_{\mathrm{id}} \in C_{\exp}(A)$ plays in the theory. It is called the *exponential kernel*.

Lemma 4.35. *Let $(X,a),(X',a') \in \mathrm{Sch}_A$. Then there is a canonical isomorphism in $C_{\exp}(X \times X')$:*

$$\mathbb{E}_a \boxtimes \mathbb{E}_{a'} \simeq \mathbb{E}_{a \boxplus a'}$$

Proof. We have a sequence of equivalences:

$$\begin{aligned}
\mathbb{E}_a \boxtimes \mathbb{E}_{a'} &= \pi_1^* \Pi z(a)_* \mathbb{1} * \pi_2^* \Pi z(a')_* \mathbb{1} \\
&\simeq \Pi \left(\pi_1^* z(a)_* \mathbb{1} * \pi_2^* z a'_* \mathbb{1} \right) \\
&\simeq \Pi \left(z(a \circ \pi_1)_* \mathbb{1} * z(a' \circ \pi_2)_* \mathbb{1} \right) \\
&= \mathbb{E}_{a \circ \pi_1} * \mathbb{E}_{a' \circ \pi_2}
\end{aligned}$$

so the claim follows from the next Lemma 4.36. □

Lemma 4.36. *Let $a, a' : X \to A$ be two potentials. Then there is a canonical isomorphism in $C_{\exp}(X)$:*

$$\mathbb{E}_a * \mathbb{E}_{a'} \simeq \mathbb{E}_{a+a'}$$

Proof.

$$\begin{aligned}
\mathbb{E}_a * \mathbb{E}_{a'} &= z(a)_* \mathbb{1} * \mathbb{E}_{a'} \\
&\simeq \mu_! \left(p_1^* z(a)_* \mathbb{1} \otimes p_2^* \mathbb{E}_{a'} \right) \\
&\simeq \mu_! \left((z(a) \times A)_* \mathbb{1} \otimes p_2^* \mathbb{E}_{a'} \right) \\
&\simeq \mu_! (z(a) \times A)_* (z(a) \times A)^* p_2^* \mathbb{E}_{a'} \\
&\simeq (\mu \circ (z(a) \times A))_! \Pi z(a')_* \mathbb{1} \\
&\simeq \Pi (\mu \circ (z(a) \times A)_! z(a')_* \mathbb{1} \\
&\simeq \Pi (\mu \circ (z(a) \times A \circ z(a'))_! \mathbb{1} \\
&\simeq \Pi z(a + a')_* \mathbb{1} \\
&= \mathbb{E}_{a+a'}
\end{aligned}$$

□

For a fixed $X \in \mathrm{Sch}_B$, the association $(X,a) \mapsto \mathbb{E}_a$ also defines a map $\mathbb{E}_- : \mathrm{Hom}_B(X,A) \to C_{\exp}(X)$. The domain of this map has a canonical group structure (denoted +) induced by the group structure on A.

Corollary 4.37. *For each $a : X \to A$, the object $\mathbb{E}_a$ is $\otimes$-invertible, and the map $\mathbb{E}_-$ defines a group homomorphism*

$$\mathbb{E}_- : \mathrm{Hom}_B(X,A) \to \mathrm{Pic}(C_{\exp}(X)).$$

Proof. We already remarked that the unit $0 : X \to A$ is taken to the unit $\mathbb{E}_0$ for the convolution product. For $a, a' : X \to A$ we have by Lemma 4.35 and Corollary 4.33:

$$\mathbb{E}_a \otimes \mathbb{E}_{a'} \simeq \Delta^* (\mathbb{E}_a \boxtimes \mathbb{E}_{a'}) \simeq \Delta^* \mathbb{E}_{a \boxplus a'} \simeq \mathbb{E}_{(a \boxplus a') \circ \Delta} = \mathbb{E}_{a+a'}$$

Since $a + (-a) = 0$, this shows both claims. □

5 Exponentiation as a coefficient system

In this section we embark on the actual construction of the functor $(-)_{\mathrm{exp}}$ defined on coefficient systems. Throughout we fix a coefficient system $C \in \mathrm{CoSy}_B$ and we adopt the same convention 4.1 on the group scheme A as in the previous section.

The underlying ∞-category $C_{\mathrm{exp}}(X)$ is easy to define.

Convention 5.1. Fix $X \in \mathrm{Sch}$ and denote by $C_{\mathrm{exp}}(X) \subseteq C_A(X)$ the full sub-∞-category spanned by the kernel of $\pi_! : C(X_A) \to C(X)$.

The whole issue is to equip the ∞-category $C_{\mathrm{exp}}(X)$ with a symmetric monoidal structure lifting the convolution product on $\mathrm{h}C_{\mathrm{exp}}(X)$ given by

$$M * N := \mu_!(M \boxtimes_X N).$$

(which makes sense by Lemma 4.21) and to lift the pullback functors f^* to symmetric monoidal functors.

5.1 Convolution product via correspondences

We first explain an approach based on the results of section 3.5. We do not provide complete details as we will rather base our construction on the second approach.

Let $X \in \mathrm{Sch}_{B,\mathrm{sep}}$. The commutative X-group scheme A_X, considered as a separated commutative monoid scheme, induces $A_X \in \mathrm{CAlg}(\mathrm{Sch}^{\times}_{X,\mathrm{sep}})$. Via the symmetric monoidal functor

$$\iota_! : \mathrm{Sch}^{\times}_{X,\mathrm{sep}} \to \mathrm{Corr}(\mathrm{Sch}_X)_{\mathrm{all},\mathrm{sep}}$$

we get $\iota_! A \in \mathrm{CAlg}(\mathrm{Corr}(\mathrm{Sch}_B))_{\mathrm{all},\mathrm{sep}})$. The commutative algebra structure on $\iota_! A$ is given by the correspondence

$$\iota_! A \times \iota_! A \xleftarrow{\sim} \iota_! A^2 \xrightarrow{\mu} \iota_! A$$

The symmetric lax-monoidal functor

$$C_!^{\boxtimes,*} : \mathrm{Corr}(\mathrm{Sch}_X)_{\mathrm{all},\mathrm{sep}} \to \mathrm{Cat}_{\mathrm{st}}$$

of Corollary 3.14 then yields $C(A) \in \mathrm{CAlg}(()\mathrm{Cat}_{\mathrm{st}})$, i.e. a symmetric monoidal structure on the stable ∞-category $C(A_X) = C_A(X)$. The functor $C_!^{\boxtimes,*}$ then sends the correspondence above to

$$C(A_X) \times C(A_X) \xrightarrow{\boxtimes_X} C(A_X^2) \xrightarrow{\mu_!} C(A_X)$$

so precisely to the convolution product Definition 4.17. This construction produces thus a ∞-categorical lift of Definition 4.17.

This construction works for a fixed X; to obtain a compatible symmetric monoidal structures on $C_A(-)$, with symmetric monoidal lifts of the pullbacks f_A^*, it is then necessary to revisit it by working with correspondences indexed by $X \in \mathrm{Sch}_B$. We do not present this extension because we are going to see this type of construction in the context of the second construction in section 5.3 below.

It still remains to check that this convolution product on $C_A(-)$ restricts to $C_{\exp}(-)$ and induces symmetric monoidal structure there as well; the only complication there is that the unit objects are not the same. We will see the necessary arguments for this in the context of the second construction in section 5.5 below.

5.2 Exterior convolution product

We now start our second, more elementary, approach to the convolution product. The first step is to construct the ∞-categorical counterpart of the exterior product $M \boxtimes_X N$, functorially in X.

Convention 5.2. We denote by

$$\begin{tikzcd} \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}} \arrow[d, "t"] \\ \mathrm{Sch}^{\mathrm{op}} \end{tikzcd}$$

the coCartesian fibration associated to the functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending a scheme X to $\mathrm{Sch}_X^{\mathrm{op}}$ and a morphism $f : X' \to X$ to the pullback $- \times_X X'$ along f. Alternatively, $\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}} = \mathrm{Fun}(\Delta^1, \mathrm{Sch})^{\mathrm{op}}$ is the opposite of the arrow category of Sch and the structure map of the coCartesian fibration is evaluation at 1.

Construction 5.3. Consider the composite

$$\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}} \xrightarrow{s} \mathrm{Sch}^{\mathrm{op}} \xrightarrow{\tilde{C}^{\otimes}} \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

of the 'source' functor and the opposite of the given coefficient system: $\tilde{C}(X) = C(X)^{\mathrm{op}}$ with the opposite symmetric monoidal structure [DG20, Example 2.7]. By symmetric monoidal (un)straightening [DG20, Appendix A], this composite classifies a coCartesian fibration:

$$\begin{tikzcd} \tilde{\mathscr{C}}^{\boxtimes} \arrow[d, "p"] \\ \left(\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}}\right)^{\amalg} \end{tikzcd}$$

Remark 5.4. Informally, the ∞-category $\tilde{\mathscr{C}}$ may be described as follows. Objects are pairs $(a : Y \to X, M)$ of morphisms a in Sch and an object $M \in C(Y)$ on the source. A morphism from $(a : Y \to X, M)$ to $(a' : Y' \to X', M')$ is a morphism of arrows

$$\begin{tikzcd} Y \arrow[r, "a"] & X \\ Y' \arrow[u, "f"] \arrow[r, "a'"] & X' \arrow[u] \end{tikzcd}$$

in Sch together with a morphism $M' \to f^*M$ in $C(Y')$. The tensor product is given by the external product:

$$(a, M) \boxtimes (a', M') = (a \times a' : Y \times Y' \to X \times X', M \boxtimes M')$$

Convention 5.5. Recall the commutative nonunital ∞–operad $\mathrm{Comm_{nu}}^{\otimes} \subseteq \mathrm{Comm}^{\otimes}$ [Lur, § 5.4.4]. For any ∞–operad $\mathcal{O}^{\otimes} \to \mathrm{Comm}^{\otimes}$ we will systematically use the notation $\mathcal{O}^{\otimes}_{\mathrm{nu}}$ to denote the fiber product $\mathcal{O}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathrm{Comm_{nu}}^{\otimes}$.

Convention 5.6. Fix a scheme X and consider the non-full subcategory of $\mathrm{Sch}_{X \times A}$ spanned by *smooth* $a : Y \to X \times A = X_A$ and whose morphisms are *smooth*. We denote it by Sm^A_X.

Composition with $\pi : X \times A \to X$ defines a functor

$$\mathrm{Sm}^A_X \to \mathrm{Sch}_X$$

which we integrate to a morphism of coCartesian fibrations $\pi_* : \int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}} \to \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}}$ over $\mathrm{Sch}^{\mathrm{op}}$. The domain admits a nonunital symmetric monoidal structure with *convolution product*:

$$Y * Y' = \left(Y \times Y' \xrightarrow{a \times a'} (X \times A) \times (X \times A) \xrightarrow{\simeq} (X \times X) \times (A \times A) \xrightarrow{\mu} (X \times X) \times A \right)$$

and we see that the functor π_* underlies a nonunital symmetric monoidal structure:

$$\begin{tikzcd}
\left(\int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}}\right)^{\circledast} \arrow[rr, "\pi_*"] \arrow[dr] & & \left(\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}}\right)^{\amalg}_{\mathrm{nu}} \arrow[dl, "t"] \\
& (\mathrm{Sch}^{\mathrm{op}})^{\amalg}_{\mathrm{nu}} &
\end{tikzcd} \tag{5.7}$$

We let $\tilde{\mathscr{C}}^{\circledast\boxtimes}_{/A}$ denote the fiber product

$$\tilde{\mathscr{C}}^{\boxtimes} \times_{\left(\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{op}}\right)^{\amalg}} \left(\int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}} \right)^{\circledast}.$$

Remark 5.8. Informally, the ∞–category $\tilde{\mathscr{C}}_{/A}$ may be described as follows. The objects are pairs (a, M) where $a : Y \to X \times A$ is smooth and $M \in C(Y)$. A morphism $(a, M) \to (a', M')$ is a commutative square

$$\begin{tikzcd}
Y \arrow[r, "a"] & X \times A \\
Y' \arrow[u, "f"] \arrow[r, "a'"] & X' \times A \arrow[u, "g \times A"']
\end{tikzcd}$$

where f is smooth, together with a morphism $M' \to f^*M$ in $C(Y')$. The tensor product is given by

$$(Y, M) \circledast \boxtimes (Y', M') = (Y \circledast Y', M \boxtimes M')$$

where we suppressed the structure morphisms a, a' of Y, Y', respectively.

Construction 5.9. The coCartesian fibration $\tilde{\mathscr{C}}_{/A}^{\circledast\boxtimes} \to (\mathrm{Sch}^{\mathrm{op}})^{\amalg}_{\mathrm{nu}}$ is classified by a functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{nu}}(\mathrm{Cat}_\infty)$$

which we may compose with the symmetric monoidal equivalence which takes an ∞-category to its opposite. The resulting functor classifies another coCartesian fibration, called the *dual coCartesian fibration*, which we denote by $\mathscr{C}_{/A}^{\circledast\boxtimes} \to (\mathrm{Sch}^{\mathrm{op}})^{\amalg}_{\mathrm{nu}}$.

Remark 5.10. According to [BGN18], the ∞-category $\mathscr{C}_{/A}^{\circledast\boxtimes}$ may informally be described as follows. Objects are pairs (a, M) as in Remark 5.8 and the tensor product also remains unchanged. However, a morphism from (a, M) to (a', M') is a triple (g, f, α) where $g : X' \to X$ is a morphism in Sch, $f : Y \times_X X' \to Y'$ is a smooth morphism over X'_A, and $\alpha : (g')^* M \to f^* M'$ is a morphism in $C(Y \times_X X')$, where we denote by $g' : Y \times_X X' \to Y$ the base change of g.

5.3 Shifted coefficient system and $\sharp$-convolution

The second step of the construction is to construct an ∞-categorical version of the variant of the convolution product given by the formula

$$(M, N) \mapsto \mu_\sharp(M \boxtimes_X N).$$

Construction 5.11. The coCartesian fibration $t \circ \pi_* : \int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{op}}$ admits a section

$$\Delta \times A : \mathrm{Sch}^{\mathrm{op}} \to \int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}}$$

that sends a scheme X to the identity morphism $X_A \to X_A$. The pullback of $\mathscr{C}_{/A}$ along Δ_A is a coCartesian fibration $\mathscr{C}_A \to \mathrm{Sch}^{\mathrm{op}}$ that is classified by the shifted 'coefficient system' (ignoring the symmetric monoidal structure)

$$C_A = C((-)_A) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

cf. Section 3.3.3.

Lemma 5.12. *(a) Let $X \in \mathrm{Sch}$. The inclusion $C(X_A) \hookrightarrow (\mathscr{C}_{/A})_X$ admits a left adjoint. We denote by $L_X : (\mathscr{C}_{/A})_X \to (\mathscr{C}_{/A})_X$ the corresponding localization functor.*

(b) The collection of localization functors $(L_X)_{X \in \mathrm{Sch}^{\mathrm{op}}}$ is compatible with the $(\mathrm{Sch}^{\mathrm{op}})^{\amalg}_{\mathrm{nu}}$-monoidal structure in the sense of [Lur, Definition 2.2.1.6].

Remark 5.13. For the first part, let $M' \in C(X_A)$ and $(a : Y \to X_A, M) \in (\mathscr{C}_{/A})_X$. As follows from Remark 5.10, a morphism $(a, M) \to (\Delta_A)(M')$ corresponds to a morphism $\alpha : M \to a^* M'$ in $C(Y)$. As a is smooth, this is equivalent to giving a morphism $a_\sharp M \to M'$ in $C(X_A)$. To make this argument rigorous we will proceed as in [DG20, Lemma 3.18.(b)].

Similarly, the second part essentially follows from (Smooth base change) and (Smooth projection formula), as in [DG20, Lemma 3.18.(c)].

Proof of Lemma 5.12. Consider the composite

$$\Delta^1 \times \int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}} \xrightarrow{\mathrm{ev}} \mathrm{Sch}^{\mathrm{op}} \xrightarrow{\tilde{C}} \mathrm{Cat}_\infty$$

and the corresponding morphism of coCartesian fibrations

$$\begin{tikzcd} \tilde{\mathscr{C}}_{/A} \arrow[dr, "p"'] & & \tilde{\mathscr{D}}_{/A} \arrow[ll, "\Pi^*"'] \arrow[dl, "q"] \\ & \int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sm}^A)^{\mathrm{op}} & \end{tikzcd}$$

which, over a fixed $a : Y \to X_A \in \mathrm{Sm}^A_X$ is classified by the symmetric monoidal functor $a^* : C(X_A)^{\mathrm{op}} \to C(Y)^{\mathrm{op}}$. Since a is smooth, the functor a^* admits a right adjoint, and it follows [Lur, Corollary 7.3.2.7] that Π^* admits a relative right adjoint. We denote the relative right adjoint by $\Pi_\sharp$.

Let

$$\begin{tikzcd} \tilde{\mathscr{C}}_{/A} \arrow[dr, "p_1"'] & & \tilde{\mathscr{D}}_{/A} \arrow[ll, "\Pi^*"'] \arrow[dl, "q_1"] \\ & \mathrm{Sch}^{\mathrm{op}} & \end{tikzcd}$$

be the diagram obtained by composing p and q with the coCartesian fibration $t \circ \pi_*$. The fiber over $X \in \mathrm{Sch}$ is easily seen to be the functor

$$(\tilde{\mathscr{C}}_{/A})_X \xleftarrow{\Pi^*} (\mathrm{Sm}^A_X)^{\mathrm{op}} \times C(X_A)^{\mathrm{op}}$$

that sends $(a : Y \to X_A, M)$ to (a, a^*M). Now, id_{X_A} is a final object of Sm^A_X, that is, the inclusion $\mathrm{id}_{X_A} : \Delta^0 \hookrightarrow (\mathrm{Sm}^A_X)^{\mathrm{op}}$ is a left adjoint. It follows that the composite

$$\Pi^* \circ \mathrm{id}_{X_A} : C(X_A)^{\mathrm{op}} \to (\tilde{\mathscr{C}}_{/A})_X$$

is a left adjoint, with right adjoint (using [Lur, Proposition 7.3.2.5])

$$\Pi_{X,\sharp} : (\tilde{\mathscr{C}}_{/A})_X \xrightarrow{\Pi_\sharp} (\mathrm{Sm}^A_X)^{\mathrm{op}} \times C(X_A)^{\mathrm{op}} \to C(X_A)^{\mathrm{op}}$$

where the last functor is the canonical projection onto the second factor. This proves the first part.

We now turn to the second part. Let $f : X \to \prod_i X_i$ be a morphism in Sch and let $g_i : (a_i, M_i) \to (a'_i, M'_i)$ be $\Pi_{X_i,\sharp}$-equivalences in $(\tilde{\mathscr{C}}_{/A})_{X_i}$. By definition, we need to show that $\otimes_f\{g_i\}$ is a $\Pi_{X,\sharp}$-equivalence. It follows from (Smooth base change) that the functor $\Pi_\sharp$ preserves $\mathrm{Sch}^{\mathrm{op}}$-coCartesian edges thus we may assume $f = \mathrm{id}_{\prod_i X_i}$. Moreover, by induction and symmetry we reduce to the case of two factors $X_1 \times X_2$ and $g_2 = \mathrm{id}_{(a_2, M_2)}$. Identify g_1 with a smooth morphism $g : Y'_1 \to Y_1$ over $(X_1)_A$ together with a morphism $\alpha : M'_1 \to g^*M_1$ in $C(Y'_1)$. We need to show that the induced morphism

$$\mu_\sharp(a'_1 \times a_2)_\sharp(M'_1 \boxtimes M_2) \to \mu_\sharp(a_1 \times a_2)_\sharp(M_1 \boxtimes M_2)$$

is an equivalence. This is true before applying $\mu_\sharp$ since the morphism identifies, using (Smooth base change) and (Smooth projection formula), with

$$\Pi_{X_1,\sharp} g_1 \boxtimes \mathrm{id} : (a_1')_\sharp M_1' \boxtimes (a_2)_\sharp M_2 \to (a_1)_\sharp M_1 \boxtimes (a_2)_\sharp M_2$$

which is an equivalence by assumption. □

From [Lur, Proposition 2.2.1.9] we deduce the following statement.[3]

Corollary 5.14. *The shifted 'coefficient system' underlies a functor $C_A^{*\sharp} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{nu}}(\mathrm{Cat}_\infty)$ whose associated coCartesian fibration fits into a $(\mathrm{Sch}^{\mathrm{op}})^{\amalg}_{\mathrm{nu}}$-monoidal functor*

$$\begin{tikzcd} \mathscr{C}_{/A}^{\otimes\boxtimes} \arrow[rr] \arrow[dr] & & \mathscr{C}_A^{\circledast_\sharp} \arrow[dl] \\ & (\mathrm{Sch}^{\mathrm{op}})^{\amalg}_{\mathrm{nu}} & \end{tikzcd}$$

which is left adjoint to the inclusion. □

Remark 5.15. Informally, the ∞-category $\mathscr{C}_A$ may be described as follows. An object is a pair (X, M) where $X \in \mathrm{Sch}$ and $M \in C(X_A)$. A morphism $(X, M) \to (X', M')$ consists of a morphism $f : X' \to X$ in Sch together with a morphism $\alpha : f^*M \to M'$ in $C(X')$. The $\sharp$-convolution product is given by

$$(X, M) \circledast_\sharp (X', M') = (X \times X', \mu_\sharp(M \boxtimes M')).$$

5.4 Twisting the $\sharp$-convolution

The morphism μ is smooth, hence by relative purity (section 3.4.4) we have

$$\mu_! = \mu_\sharp \,\mathrm{Th}(-\Omega_\mu).$$

Moreover μ is a group scheme, which implies that Ω_μ is pulled back from B, so that we have

$$\mu_! = \mathrm{Th}(-\mathscr{E})\mu_\sharp$$

with $\mathscr{E} = 0^*\Omega_{A/B}$. This suggests to define the convolution from the $\sharp$-convolution product of the previous section by twisting compatibly with inverse Thom equivalences.

Construction 5.16. Consider the following diagram of solid arrows in the category of simplicial sets:

$$\begin{tikzcd} \Delta^0 \arrow[r, "C_A^{*\sharp}"] \arrow[d, "0"'] & \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{CAlg}_{\mathrm{nu}}(\mathrm{Cat}_\infty)) \arrow[d] \\ J \arrow[r, "\mathrm{Th}(\mathscr{E})"'] \arrow[ur, dotted, "\phi"] & \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Cat}_\infty) \end{tikzcd}$$

where:

[3]Note there is a typo in the statement of said result: The target of the functor in part (2) should be $\mathcal{O}^\otimes$ instead of $\mathrm{N}(\mathrm{Fin}_*)$.

- J is the nerve of the category with two objects $0, 1$ and a unique isomorphism between them,
- the bottom horizontal arrow sends both objects to C_A, the edge $0 \to 1$ to the Thom equivalence $\mathrm{Th}(\mathscr{E})$ (see Lemma 5.17 below) for the conormal sheaf $\mathscr{E} = 0^*\Omega_{A/B}$ to the unit morphism $0 : B \to A$, as in Section 3.4.4, and the edge $1 \to 0$ to the inverse Thom equivalence,
- the top horizontal arrow is provided by Corollary 5.14,
- the right vertical arrow is a Joyal fibration representing the forgetful functor.

Since the left vertical arrow is a trivial cofibration for the Joyal model structure there exists a lift ϕ as indicated by the dotted arrow in the diagram. Evaluating ϕ at 1 yields a new functor $C_A^* : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{nu}}(\mathrm{Cat}_\infty)$.

Lemma 5.17. *Let $p : V \to B$ be a smooth morphism with a section $s : B \to V$, denote by p_X and s_X the pullbacks to $X \to B$ for each X. The family of Thom equivalences $\mathrm{Th}(p_X, s_X) := (p_X)_\sharp (s_X)_* : C(X) \xrightarrow{\sim} C(X)$ underlies an autoequivalence of the functor $C : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.*

Proof. Compare Section 3.3.3. Let us view the morphism p as a functor $\Delta^{1,\mathrm{op}} \to \mathrm{Sch}^{\mathrm{op}}$ and consider then the composite

$$\Delta^{1,\mathrm{op}} \times \mathrm{Sch}^{\mathrm{op}} \xrightarrow{p\times\mathrm{id}} \mathrm{Sch}^{\mathrm{op}} \times \mathrm{Sch}^{\mathrm{op}} \xrightarrow{\times} \mathrm{Sch}^{\mathrm{op}} \xrightarrow{C} \mathrm{Cat}_\infty.$$

By adjunction, it corresponds to a functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Fun}(\Delta^{1,\mathrm{op}}, \mathrm{Cat}_\infty)$ which, we claim, factors through $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^{1,\mathrm{op}}, \mathrm{Cat}_\infty)$ in the sense of [Lur, Definition 4.7.4.16]. Indeed, that amounts to (Smooth base change) for the smooth morphisms $p_X : V_X \to X$. By [Lur, Corollary 4.7.4.18.(3)], passing to the left adjoints $(p_X)_\sharp$ results in another functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat}_\infty)$, thus a morphism $p_\sharp : C_V \to C$ in $\mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Cat}_\infty)$.

The argument for the existence of $s_* : C \to C_V$ is the same, using proper base change instead of (Smooth base change). Combining the two, we obtain a morphism $p_\sharp s_* : C \to C$ in $\mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ which is pointwise the Thom equivalence $\mathrm{Th}(p_X, s_X)$, thus an equivalence. □

By construction, the underlying functor C_A of C_A^* remains unchanged but the nonunital symmetric monoidal structure can informally be described as follows.

Lemma 5.18. *For $X \in \mathrm{Sch}$ and $M, N \in C(X_A)$, we have*

$$M * N \simeq \mu_!(M \boxtimes N).$$

Proof. To see why that is true let us first describe the tensor product on $C_A^{*\sharp}(X)$. Consider the following diagram with (hopefully) obvious notation

$$
\begin{tikzcd}
X_A \arrow[r, "\Delta"] & X_A^2 \\
X_{A^2} \arrow[u, "\mu"] \arrow[r, "\Delta"] \arrow[d, "p_1"'] \arrow[dr, "p_2"' near start] & X_{A^2}^2 \arrow[u, "\mu"'] \arrow[d, "q_2"] \arrow[dl, "q_1" near end] \\
X_A & X_A
\end{tikzcd}
$$

By construction, we then have in $C_A^{*_\sharp}(X)$ (with $\otimes$ denoting the tensor product in C):

$$
\begin{aligned}
M_1 \underset{\sharp}{*} M_2 &\simeq \Delta^* \mu_\sharp (q_1^* M_1 \otimes q_2^* M_2) \\
&\simeq \mu_\sharp \Delta^* (q_1^* M_1 \otimes q_2^* M_2) \\
&\simeq \mu_\sharp (p_1^* M_1 \otimes p_2^* M_2)
\end{aligned}
$$

It follows that in $C_A^*(X)$ we have

$$
\begin{aligned}
M_1 * M_2 &\simeq \mathrm{Th}(\mathscr{E}_{X_A}) \mu_\sharp \left(p_1^* \mathrm{Th}^{-1}(\mathscr{E}_{X_A}) M_1 \otimes p_2^* \mathrm{Th}^{-1}(\mathscr{E}_{X_A}) M_2 \right) \\
&\simeq \mathrm{Th}(\mathscr{E}_{X_A}) \mu_\sharp \left(\mathrm{Th}^{-1}(\mathscr{E}_{X_{A^2}}) p_1^* M_1 \otimes \mathrm{Th}^{-1}(\mathscr{E}_{X_{A^2}}) p_2^* M_2 \right) \\
&\simeq \mathrm{Th}(\mathscr{E}_{X_A}) \mu_\sharp \mathrm{Th}^{-2}(\mathscr{E}_{X_{A^2}}) \left(p_1^* M_1 \otimes p_2^* M_2 \right) \\
&\simeq \mathrm{Th}^{-1}(\mathscr{E}_{X_A}) \mu_\sharp \left(p_1^* M_1 \otimes p_2^* M_2 \right) \\
&\simeq \mu_! \left(p_1^* M_1 \otimes p_2^* M_2 \right)
\end{aligned}
$$

as claimed. □

5.5 Exponentiated coefficient system

Recall the following notation.

Convention 5.19. Fix $X \in \mathrm{Sch}$ and denote by $C_{\exp}(X) \subseteq C_A(X)$ the full sub-∞-category spanned by the kernel of $\pi_! : C(X_A) \to C(X)$.

Lemma 5.20. *The association $X \mapsto C_{\exp}(X)$ underlies a functor $C_{\exp}^* : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{nu}}(\mathrm{Cat}_\infty)$ so that the inclusion $C_{\exp} \subseteq C_A$ is nonunital symmetric monoidal.*

Proof. This follows from [Lur, Proposition 2.2.1.1] once we verify the following two conditions:

1. The sub-∞-category $C_{\exp}(X) \subseteq C_A(X)$ is stable under convolution product.
2. For $f : X' \to X$, the functor $f^* : C_A(X) \to C_A(X')$ takes $C_{\exp}(X)$ to $C_{\exp}(X')$.

Both of these can be tested at the level of homotopy categories. These are Lemmata 4.8 and 4.21. □

Remark 5.21. The proof shows more precisely that $C_{\exp}(X) \subseteq C_A(X)$ is an *ideal* with respect to the convolution product.

Lemma 5.22. *The inclusion $C_{\exp} \subseteq C_A$ admits a global right adjoint Π which is nonunital lax symmetric monoidal.*

Proof. Fix $X \in \mathrm{Sch}$. The inclusion $C_{\exp}(X) \hookrightarrow C_A(X)$ admits a right adjoint by Lemma 5.22. The claim now follows from [Lur, Proposition 2.2.1.1]. □

Remark 5.23. Lemma 5.22 says that $\Pi : C_A \to C_{\exp}$ defines a natural transformation of functors $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which is a right adjoint Π_X to the inclusion at every $X \in \mathrm{Sch}$. Since the inclusion is (nonunital) symmetric monoidal this right adjoint is automatically *lax* symmetric monoidal.

In fact, it admits an explicit description as

$$\Pi \simeq u_* u^* \pi^! \mathbb{1}[-1] * -,$$

see Corollary 4.32.

Lemma 5.24. *The functor $C^*_{\exp}$ underlies a functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ (still denoted $C^*_{\exp}$).*

Proof. By [Lur, Corollary 5.4.4.7], it suffices to prove that $C^*_{\exp}$ factors through $\mathrm{CAlg}_{\mathrm{qu}}(\mathrm{Cat}_\infty)$. In other words, it suffices to check that

1. for $X \in \mathrm{Sch}$, the homotopy category of $C_{\exp}(X)^*$ with the induced tensor structure has a unit, and
2. for $f : X' \to X$, the functor $f^* : C_{\exp}(X) \to C_{\exp}(X')$ at the level of homotopy categories takes units to units.

The first follows from Corollary 4.29, and the second from that together with Lemma 4.31. □

Theorem 5.25. *The functor $C^*_{\exp} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ of Lemma 5.24 is a coefficient system.*

Proof. We noted in Lemma 3.10 that the axioms may be verified at the level of (triangulated) homotopy categories. The statement then follows from ??. □

Remark 5.26. The constructions performed in this section are functorial in the following sense. If $\phi : C \to C'$ is a morphism of coefficient systems then there is an induced morphism $\phi_{\exp} : C_{\exp} \to C'_{\exp}$. This can be proved by carrying the data of ϕ along in each step of the construction, using the theory of *generalized* ∞-operads [Lur, § 2.3.2]. We will leave the construction of a functor

$$(-)_{\exp} : \mathrm{CoSy}_B \to \mathrm{CoSy}_B$$

to the interested reader.

6 Motives of varieties with potential

Recall that a *B-scheme with potential* (X, a) is simply an object of the category Sch_A seen as a pair $X \to B$ and a B-morphism $a : X \to A$. We often identify Sch_B as a full subcategory of Sch_A via the functor $X \mapsto (X, 0)$.

Unlike the exponential cohomology theories described in the introduction, the definition of C_{exp} does not seem directly related to schemes with potential. In this section, we explain how to construct "exponential motives" in C_{exp} associated to Sch_B-schemes with potential. For simplicity, we will express our results at the level of triangulated categories although that is probably not necessary.

6.1 Motives

Let $f : X \to Y$ be a scheme. Recall that the motive associated with X can be defined in terms of the exceptional functors, like so: $\mathrm{M}(X) := f_! f^! \mathbb{1} \in C(Y)$. The motive associated with a variety with potential (X, a) will follow this idea, with a(n exponential) twist:

Definition 6.1. The *motive* of the variety with potential $(f : X \to Y, a : X \to A)$ is defined to be

$$\mathrm{M}(X, a) := \mathrm{M}(X, f, a) := \underline{f}_!(\mathbb{E}_a \otimes \underline{f}^! \mathbb{1}) \in C(Y).$$

More generally, if $g : Z \hookrightarrow X$ is a subscheme then the *relative motive* of (X, Z, a) is defined to be

$$\mathrm{M}(X, Z, a) := \mathrm{Cofib}\left(\mathrm{M}(Z, a \circ g) \to \mathrm{M}(X, a)\right)$$

where the map is given by the composition

$$\begin{aligned}
\underline{f}_!\underline{g}_!(\mathbb{E}_{ag} \otimes \underline{g}^! \underline{f}^! \mathbb{1}) &\simeq \underline{f}_!\underline{g}_!(\underline{g}^* \mathbb{E}_a \otimes \underline{g}^! \underline{f}^! \mathbb{1}) && \text{Lemma 4.31} \\
&\xrightarrow{\sim} \underline{f}_!(\mathbb{E}_a \otimes \underline{g}_! \underline{g}^! \underline{f}^! \mathbb{1}) && \text{projection formula} \\
&\to \underline{f}_!(\mathbb{E}_a \otimes \underline{f}^! \mathbb{1}) && \text{counit of adjunction}
\end{aligned} \tag{6.2}$$

Lemma 6.3. *Let $g : (Z, ag) \to (X, a)$ be a morphism (not necessarily a closed immersion) of varieties with potential. Then the morphism*

$$\mathrm{M}(g) : \mathrm{M}(Z, ag) \to \mathrm{M}(X, a)$$

of (6.2) is functorial in g.

Proof. This is true for each of the morphisms in the composition (6.2). □

Lemma 6.4. *Let (X, f, a) be a variety with potential and assume f is smooth. Then*

$$\mathrm{M}(X, a) \simeq \underline{f}_\sharp \mathbb{E}_a.$$

Proof. Indeed,

$$\underline{f}_\sharp \mathbb{E}_a \simeq \underline{f}_\sharp(\mathbb{E}_a \otimes \underline{f}^* \mathbb{1}) \simeq \underline{f}_!(\mathbb{E}_a \otimes \underline{f}^! \mathbb{1})$$

by relative purity. □

Remark 6.5. The motive of (X, a) is more precisely the *homological* motive of the variety with potential. In fact, there are three additional motives, similarly to the situation for varieties without potential.

cohomology	$f_*(\mathbb{E}_a \otimes f^*\mathbb{1})$
cohomology with compact support	$f_!(\mathbb{E}_a \otimes f^*\mathbb{1})$
homology	$f_!(\mathbb{E}_a \otimes f^!\mathbb{1})$
Borel-Moore homology	$f_*(\mathbb{E}_a \otimes f^!\mathbb{1})$

These exhibit a similar behaviour as in the classical setting: functoriality with respect to proper, étale, ... maps; representing the corresponding theories etc.

For the next result we need some preparations.

Lemma 6.6. *Let $fg = hk$ be a cartesian square with one of f or h smooth. Then the exchange morphism*

$$g^* f^! \to k^! h^*$$

is invertible in any coefficient system.

Proof. Say f is smooth so that $f^! \simeq \mathrm{Th}(\Omega_f) f^*$. Then

$$g^* f^! \simeq g^* \mathrm{Th}(\Omega_f) f^* \simeq \mathrm{Th}(g^*\Omega_f) g^* f^* \simeq \mathrm{Th}(\Omega_k) k^* h^* \simeq k^! h^*$$

which is the exchange morphism [Ay007, Proposition 1.5.19]. □

Lemma 6.7. *Let $f : X \to B$ and $f : X' \to B$ with f smooth. There is a canonical isomorphism $f^!\mathbb{1} \boxtimes (f')^!\mathbb{1} \simeq (f \times f')^!\mathbb{1}$ in any coefficient system.*

Proof. The claim will follow from the observation that the exchange morphism $p^*q^! \to q^! p^*$ is an isomorphism (Lemma 6.6). Indeed, we then have

$$\begin{aligned}
(\mathrm{id}_X \times f')^* f^!\mathbb{1} \otimes (f \times \mathrm{id}_{X'})^*(f')^!\mathbb{1} &\simeq (\mathrm{id}_X \times f')^* f^!\mathbb{1} \otimes (\mathrm{id}_X \times f')^! f^*\mathbb{1} \\
&\simeq (\mathrm{id}_X \times f')^!(f^!\mathbb{1} \otimes f^*\mathbb{1}) \\
&\simeq (\mathrm{id}_X \times f')^! f^!\mathbb{1} \\
&\simeq (f \times f')^!\mathbb{1}.
\end{aligned}$$

□

Proposition 6.8. *Let $(X, a), (X', a') \in \mathrm{Sch}_{Y \times A}$. Then in $C_{\exp}(Y)$ we have a morphism*

$$\mathrm{M}(X, a) \otimes \mathrm{M}(X', a') \to \mathrm{M}(X \times_Y X', a * a')$$

which is invertible if one of X or X' is a smooth Y-scheme.

Proof. This is the following composite:

$$\begin{aligned}\underline{f}_!(\mathbb{E}_a \otimes \underline{f}^! \mathbb{1}) \otimes \underline{f}'_!(\mathbb{E}_{a'} \otimes (\underline{f}')^! \mathbb{1}) &\simeq \underline{f \times f'}_! \Big((\mathbb{E}_a \otimes \underline{f}^! \mathbb{1}) \boxtimes (\mathbb{E}_{a'} \otimes (\underline{f}')^! \mathbb{1}) \Big) \\ &\simeq \underline{(f \times f')}_! \Big(\mathbb{E}_{a * a'} \otimes (\underline{f}^! \mathbb{1} \boxtimes \underline{(f')}^! \mathbb{1}) \Big) \\ &\to \underline{(f \times f')}_! \Big(\mathbb{E}_{a * a'} \otimes \underline{(f \times f')}^! \mathbb{1} \Big)\end{aligned}$$

Here, the first equivalence is linearity, the second is Lemma 4.35, and the third is the canonical isomorphism of Lemma 6.7. □

Convention 6.9. Let $\mathscr{E}$ be a collection $\mathscr{E}_X \subset \mathrm{Pic}(C(X))$ of invertible objects, for each $X \in \mathrm{Sch}_B$, satisfying $g^*\mathscr{E}_Y(n) \subseteq \mathscr{E}_X$ for each $g : X \to Y$, $n \in \mathbb{Z}$. We say that C is *Ind-$\mathscr{E}$-constructible* if for each $Y \in \mathrm{Sch}_B$, the stable ∞-category $C(Y)$ is the smallest localizing subcategory containing the motives $f_\sharp E$ for all $f : X \to Y$ smooth and $E \in \mathscr{E}_X$.

Example 6.10. For example, if $\mathscr{E}_X = \mathbb{1}(\mathbb{Z}) := \{\mathbb{1}(n) \mid n \in \mathbb{Z}\}$ for all X, this is known as of *geometric origin*, see [Gal21, § 2.4.3] for more details on this notion. An important example to have in mind is SH, stable motivic homotopy theory.

More generally, if $\Lambda \subset \mathrm{Pic}(C(B))$ is some class containing all Tate twists and we set $\mathscr{E}_X = f^*\Lambda$ for each $f : X \to B$ then the resulting notion is closely related to Λ-*constructibility* as defined in [Ayo07, § 2.3.10]. (There, Λ is allowed to consist of non-invertible objects as well.)

Proposition 6.11. *If C is Ind-$\mathscr{E}$-constructible then $C_{\exp}$ is Ind-$\mathscr{E}_{\exp}$-constructible, where*

$$\mathscr{E}_{\exp,Y} = \{\mathbb{E}_a(E) \mid E \in \mathscr{E}_Y, a : X \to A\}.^4$$

In particular, if C is of geometric origin then $C_{\exp}(Y)$ has generators

$$\mathrm{M}(X, a)(n), \qquad a : X \to A, n \in \mathbb{Z}.$$

Proof. By assumption, $C_A(Y)$ is generated by motives of the form $b_\sharp E$ where $b : X \to Y \times A$ is smooth and $E \in \mathscr{E}_X$. As $C_{\exp}(Y)$ is a Verdier localization of $C_A(Y)$ with localization functor Π we deduce that the former is generated by objects of the form $\Pi(b_\sharp E)$. By definition of Π (Definition 4.11), we need to identify the fiber of the morphism $b_\sharp E \to \pi^*\pi_\sharp b_\sharp E$. Let $f = \pi \circ b : X \to Y$ and $a = \pi_Y \circ b : X \to A$ be the two components of b. Passing to a small enough Zariski open cover of X we may assume that the vector bundles Ω_f and Ω_b are trivial hence their Thom “spaces” $\mathrm{Th}(\Omega_f)$ and $\mathrm{Th}(\Omega_b)$ are Tate twists (and shifts). It then follows that up to shifts, the object $E' = \mathrm{Th}(\Omega_b)\,\mathrm{Th}(\Omega_f)^{-1}E$ also belongs to $\mathscr{E}_X$. We need to identify the fiber of the morphism

$$b_!(\mathrm{Th}(\Omega_f)E') \to \pi^*\pi_\sharp b_!(\mathrm{Th}(\Omega_f)E') \tag{6.12}$$

in $C_A(Y)$.

Now, $\mathbb{E}_a(E')$ may be written as the fiber of the unit morphism

$$z(a)_*E' \to \pi^*\pi_\sharp z(a)_*E'$$

[4] As seen in the proof, one may restrict oneself to potentials a that factor as a smooth morphism $b : X \to Y \times A$ followed by the canonical projection. Thus if Y is smooth a may be assumed to be smooth too.

from which one easily deduces that the object $f_\sharp \mathbb{E}_a(E')$ in $C_{\exp}(Y)$ is the fiber of the unit morphism

$$(f_A)_\sharp z(a)_* E' \to \pi^* \pi_\sharp (f_A)_\sharp z(a)_* E' \tag{6.13}$$

in $C_A(Y)$. Using purity again in the form of $(f_A)_\sharp = (f_A)_! \operatorname{Th}(\Omega_{f_A}) = (f_A)_! \operatorname{Th}(\pi^*\Omega_f)$ we identify (6.13) with

$$(f_A)_! z(a)_* \operatorname{Th}(\Omega_f) E' \to \pi^* \pi_\sharp (f_A)_! z(a)_* \operatorname{Th}(\Omega_f) E'.$$

Finally, since $f_A \circ z(a) = b$ we get precisely (6.12). □

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