III Minimal regular models

Setup S (integral) Dedekind scheme

$C/_\eta$ smooth projective connected curve

By applying resolution of singularities to any proper flat model of C (see next chapter) we get $X/_S$ proper flat with X regular connected. and $X_\eta \simeq C$.

Goal: Modify X to make it "minimal".

The resulting theory is very similar to the study of minimal models of smooth projective surfaces over an alg. closed field (see [Hartshorne, V]). The two theories can be developped in parallel, and the fibered case is somewhat easier because vertical divisors play a distinguished role.

def 1 | A *fibered surface* is a proper flat morphism $f : X \longrightarrow S$ with $\begin{cases} X \text{ 2-dim. noetherian} \\ S \text{ Dedekind scheme.} \end{cases}$

· Fibers of such a morphism are proper curves over general fields, which can be very singular. (For X regular, they are at least l.c.i).

rmk X curve. (over field)

$$\begin{array}{ccccc} X \text{ reduced} & \not\Longleftrightarrow & X \text{ l.c.i} & \Longrightarrow & \omega_{X/k} \text{ exists and is invertible} \\ \checkmark \Downarrow & (S1) & \Downarrow \text{[Liu 8.2.18]} & & \\ X \text{ has no embedded points} & \Longleftrightarrow & X \text{ Cohen-Macaulay} & \Longrightarrow & \omega_{X/k} \text{ exists.} \end{array}$$

1) Degree of divisors on singular curves:

· In this section, X is a proper curve on a field k (not necc. irred or reduced).

· Recall that if A is a noetherian 1-dim ring and $f \in A$ is not a zero-divisor, then the length $\operatorname{len}_A(A/_f)$ is finite, and that
$$\operatorname{len}_A(A/_{fg}) = \operatorname{len}_A(A/_f) + \operatorname{len}_A(A/_g) \quad [\text{Liu, Lemma 7.1.26}].$$

def 2 | Let $x \in X^{(0)}$ and $f \in \mathcal{O}_{X,x}$ non-zerodivisor. The *multiplicity of f at x is* is $\operatorname{mult}_x(f) := \operatorname{len}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/_f) < \infty$.

· Because of additivity, we can extend mult_x to the total ring of fractions.

def 3 | Let D be a Cartier divisor on X. The <u>multiplicity of D at x</u> is $\mathrm{mult}_x(D) := \mathrm{mult}_x(f_x) - \mathrm{mult}_x(g_x)$ for $f_x/g_x \in \mathrm{Frac}(\mathcal{O}_{X,x})$ local representative of D.

· The associated Weil divisor is then $\sum_{x \in X^{(0)}} \mathrm{mult}_x(D) \cdot [x] \in Z^1(X)$.

def 4 | The <u>(total) degree of D</u> is
$$\deg(D) := \sum_{x \in X^{(0)}} \mathrm{mult}_x(D) \cdot [\kappa(x) : k] \qquad (⚠ \text{ depends on } k)$$

· We have
$$\begin{cases} \deg(D_1 + D_2) = \deg(D_1) + \deg(D_2) \\ \deg(D) = \dim_k H^0(X, \mathcal{O}_D) \text{ if } D \text{ is effective.} \end{cases}$$

· The basic fact of life, as in the case of smooth curves, is:

thm 5 | (Riemann-Roch)
$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X).$$

The proof is the same as in the smooth case, by reduction to the effective case. [Liu, 7.3.17].

cor 6 | Let $f \in \kappa(X)$. Then $\deg(\mathrm{div}(f)) = 0$. (proof: $\mathcal{O}(\mathrm{div} f) \simeq \mathcal{O}$ via f)

def 7 | Let $\mathcal{L}$ be a line bundle on X. Its <u>(total) degree</u> is
$$\deg(\mathcal{L}) := \chi(\mathcal{L}) - \chi(\mathcal{O}_X).$$

def 8 | The <u>arithmetic genus of X</u> is $p_a(X) := 1 - \chi(\mathcal{O}_X)$.

· To go further in the study of RR, want to apply Serre duality.

prop | (RR + duality)
X proper CM curve
$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \omega_{X/k}(-D)) = \deg D + 1 - p_a(X).$$

· def | X proper l.c.i curve. A Cartier divisor $K_{X/k}$ with $\mathcal{O}(K_{X/k}) \simeq \omega_{X/k}$ (which is invertible in this case) is called a <u>canonical divisor</u>.

prop 10 [Liu, 7.3.31] | X proper l.c.i.
· $\deg(\omega_{X/k}) = 2(p_a - 1)$.
· $\dim_k H^0(X, \omega_{X/k}) = p_a$ if X is geom. red. and geom. connected.

- Clearly, if X is not irreducible, then the total degree is a rather weak invariant. However:

<u>prop 11</u>
a) X is projective.

b) Let D be a Cartier divisor on X. Then

$$D \text{ ample} \iff \forall\, Y \text{ irred. comp of } X,\ \deg(D_{|Y}) > 0.$$

<u>idea of proof</u> [Liu, exc. 4.1.16, 7.5.3, 7.5.4]

- The order of proof is: (a) for normal curves) $\Rightarrow$ b) $\Rightarrow$ a).

- a) for normal curves is proved by patching embeddings of affine opens: If $C = \bigcup U_i$ affine open cover and $U_i \hookrightarrow Y_i$ with Y_i/k projective, then the natural map $\bigcap U_i \to \prod Y_i$ extends to C by normality and valuative criterion and gives a projective embedding.

- b): By Serre's criterion, enough to show that for all $\mathcal{F} \in Coh(X)$ and $n >> 0$, $H^1(X, \mathcal{F} \otimes \mathcal{O}(nD)) = 0$.

- Let $\pi : X' \to X$ be the normalization. Then $\deg(\pi^* \mathcal{O}(nD)) = \deg(\mathcal{O}(nD))$

Hence (by the usual argument, since X' is projective) $\pi^*\mathcal{O}(nD)$ is ample / (to be precise, need to do this for each irred. comp. of X')

We have a short exact sequence

$$(+)\ 0 \to \pi_* \pi^! \mathcal{F} \to \mathcal{F} \to \underset{\substack{\uparrow \\ \text{skyscraper sheaf}}}{\mathcal{L}} \to 0 \quad \text{with } \pi^! \mathcal{F} := \mathcal{H}om_{\mathcal{O}_X}(\pi_* \mathcal{O}_{X'}, \mathcal{F})$$

equipped with its natural coherent $\mathcal{O}_{X'}$-module structure (π finite)

and
$$H^1(X, (\pi_* \pi^! \mathcal{F}) \otimes \mathcal{O}(nD)) \underset{\text{projection formula}}{\simeq} H^1(X, \pi_*(\pi^! \mathcal{F} \otimes \pi^* \mathcal{O}(nD)))$$

$$\underset{\pi \text{ finite}}{\simeq} H^1(X', \pi^! \mathcal{F} \otimes \pi^* \mathcal{O}(nD))$$

$$\underset{\pi^*\mathcal{O}(nD) \text{ ample}}{\simeq} 0 \quad \text{for } n >> 0$$

- The result then follows from the LES of (+).

- a): it is then enough to construct an effective Cartier divisor which meets every irreducible component of X. But, given any locally noetherian scheme X and any non-associated point x, there is an effective Cartier divisor on X with support containing x. $\square$

- Application to fibered surfaces (surprisingly not used in the sequel: remember that equations are evil!)

thm 12 (Lichtenbaum [Liu, 8.3.16])
Let $f : X \to S$ be a regular fibered surface (i.e. X regular)
Then f is projective (in the sense of EGA, not Hartshorne)

proof: Let $\pi : Y \to T$ be a proper morphism of noetherian schemes, and $\mathcal{L}$ be a line on Y. We need the following classical fact about ampleness:

> If for all $t \in T$, $\mathcal{L}_t$ is ample on Y_t, then π is projective (in EGA-sense).

- We can assume X connected $\Rightarrow X_\eta$ connected.

- Let $x \in X_\eta^{(0)}$ be a closed point. Then $D_0 = \overline{\{x\}}$ is a Weil = Cartier divisor on X. Since $\mathcal{O}(D_0)_\eta$ is ample by X connected curve, there exists $U \subseteq S$ non-empty open with $(s \in U \Rightarrow (D_0)_s$ ample$)$

Let $S \setminus U = \{s_1, \dots, s_n\}$.

lemma 13 There exists an effective divisor D_i which meets all irreducible components of X_{s_i}.

(see [Liu, 8.3.35 a)] for the proof)

- Then $D := D_0 + D_1 + \dots + D_n$ is a Cartier divisor such that $\forall s \in S$, $\mathcal{O}(D)_s$ is ample. $\square$

rmk: for $f : X \to S$ smooth on S "nice" (e.g quasi-excellent), f has finitely many singular fibers and we can dispense with D_0.

rmk:
- Smooth proper surfaces over a field are projective. [Badescu, Thm 1.2.8].
- There exist normal proper non-projective surfaces [Schröer].

2) Intersection theory on a regular fibered surface

- Intersection theory in general : try to define intersection pairing $CH^i(X) \times CH^j(X) \longrightarrow CH^{i+j}(X)$ on cycle groups up to rat. equivalence.
- On a surface, only interesting case is 2 divisors.
- <u>Pb 1</u> : without some form of properness, intersections are not invariant under rational equivalence.

  Sol : only allow intersections with at least one divisor proper.
- <u>Pb 2</u> : $CH^2(X)$ is hopelessly complicated and not so interesting for us.

  Sol : only keep track of intersection degrees.

- Write $\mathrm{Div}(X)$ for the group of all Cartier divisors on X,

  $\mathrm{Div}_h(X)$ for the subgroup of <u>horizontal</u> divisors ($f_{|D}$ surjective $\Updownarrow$ finite, for D prime,)

  for $s \in S^{(0)}$, $\mathrm{Div}_s(X)$ for the subgroup of divisors with support in X_s.

  $\mathrm{Div}_v(X) = \sum_{s \in S} \mathrm{Div}_s(X)$ for the subgroup of <u>vertical</u> divisors.

<u>def:</u> Let $D \in \mathrm{Div}(X)$, $E \in \mathrm{Div}_v(X)$.

Write $E = \sum n_\Gamma \cdot [\Gamma]$ with Γ running through the irred. components of closed fibers.

Put $i(D,E) = \sum n_\Gamma \deg_{k(s)}\left(\mathcal{O}(D)_{|\Gamma}\right) \in \mathbb{Z}$.

(this makes sense because $\Gamma/_{k(s)}$ is a proper curve).

<u>thm</u> (i) $i : \mathrm{Div}(X) \times \mathrm{Div}_v(X)$ is a bilinear form.

(ii) $i : \mathrm{Div}_v(X) \times \mathrm{Div}_v(X)$ is symmetric.

(iii) If $D \sim D'$ (i.e $D' - D = \mathrm{div}(F)$ for $F \in K(X)^\times$)

we have $i(D,E) = i(D',E)$.

(iv) If D, E are effective, with no common component, we have :

$$i(D,E) = \sum_{x \in |D| \cap |E|} [K(x) : K(f(x))] \cdot \mathrm{len}_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x} \big/ \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x \right) \geq 0$$

proof: (i): • Linearity in E is by construction.

• Linearity in D follows from additivity of degree.

(iii): $D \sim D' \Rightarrow \mathcal{O}_X(D) \simeq \mathcal{O}_X(D') \Rightarrow i(D,E) = i(D',E)$.

(iv): • First, the condition on supports imply that in a neighborhood of $x \in |D| \cap |E|$, the point x is the only pt of intersection of the supports. This implies

$$m_x \subseteq \sqrt{\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x}$$

Hence $\mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x$ is an artinian ring $\Rightarrow$ length $< \infty$.

• D has no embedded points + condition on supports $\Rightarrow \exists\, D_{|E}$ effective on E with $\mathcal{O}_X(D)_{|E} \underset{(+)}{\simeq} \mathcal{O}_E(D_{|E})$ [Liu, Lemma 7.1.29].

$$\Rightarrow i(D,E) = \deg(\mathcal{O}_E(D_{|E})) = \sum_{x \in |E|} \mathrm{mult}_x(D_{|E}) \cdot [k(x) : k(0)].$$

We have
$$\mathrm{mult}_x(D_{|E}) = \mathrm{len}\left(\mathcal{O}_{E,x} / \mathcal{O}_E(-D_{|E})_x\right)$$
$$\Big\Uparrow \wr \ (+)$$
$$\mathrm{len}\left(\mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x\right) \checkmark$$

(ii): Can reduce to $D = \Gamma_i$, $E = \Gamma_j$ for Γ_i, Γ_j 2 components of a fiber X_s. Then it follows from symmetry in (iv) for $i \neq j$, and it is obvious for $i = j$.

$\square$

notat° | $D \cdot E := i(D,E)$
$E^2 := i(E,E)$

rmk: Compared with case of smooth proj surface [Hartshorne, V §1]:
- no Bertini thm to always reduce to transversality
- still a moving lemma [Liu, 9.1.10-11] but not necessary for theory.