Exercise sheet 6: lifting calculus

1 Show that a morphism which is both inner anodyne and an inner fibration is an isomorphism. Hint: lift it against itself.

2 Show that an inner anodyne morphism of simplicial sets induces a bijection on vertices. Hint: this is true for horn inclusions; show the morphisms which induce a bijection on vertices is saturated.

3 The goal of this exercise is to show that the fundamental category functor $\tau : \mathrm{sSet} \to \mathrm{Cat}$ commutes with finite products (in exercise 5.5, we saw that it does not commute with infinite-products).

- Show that τ commutes with final objects, and deduce that it is enough to treat the case of the product of two objects.
- Show that if $f : X \to Y$ is an inner anodyne map, then the induced functor $\tau f : \tau X \to \tau Y$ between their fundamental categories is an equivalence of (1-)categories. Hint: use that $N\tau X \to N\tau Y$ is an inner fibration (as is the nerve of any functor between 1-categories). In fact, more is true: inner anodyne maps are always categorical equivalences. However at this point we have not even defined categorical equivalences between general simplicial sets!
- Let X be a simplicial set. Show that there exists an inner anodyne morphism $X \to X'$ with X' an ∞-category. Hint: factor the morphism $X \to \Delta^0$ using the appropriate weak factorisation system).
- Let X, Y be two simplicial sets. Show that the canonical map $\tau(X \times Y) \to \tau(X) \times \tau(Y)$ is an isomorphism of categories. Hint: first, show that it is enough to prove it is an equivalence of categories (what are the objects?). Then use the two previous questions, together with the fact that $\tau(X) = h(X)$ for an ∞-category and that h does commute with products.

4 Let C be a small category. Show that the pushout-product of two monomorphisms in $\mathrm{PSh}(C)$ is a monomorphism. Hint: reduce to the case of Set, and use the explicit description of pushouts in Set.

5 This exercise introduces a "trick" to formulate uniqueness of liftings in lifting problems.

- Let C be a category with pushouts. Given a morphism $f : A \to B$, let $f^\vee : B \coprod_A B \to B$ be the "fold map", i.e. the unique morphism such that the composition with each of the canonical maps $B \to B \coprod_A B$ is f which is provided by the universal property of the pushout. Show that for $g : X \to Y$, we have that g has the right lifting property with respect to both f and $f^\vee$ if and only if g has the right lifting property with respect to f and the lifts are always unique.
- Deduce from the small object argument that for any set of morphisms in sSet, we have $(\overline{S \cup S^\vee}, (S \cup S^\vee)^{\boxslash})$ is an orthogonal factorisation system.

- Show that if S is the set of inner horn inclusions, the resulting factorisation for the morphism $X \to \Delta^0$ with $X \in \mathrm{sSet}$, say $X \to Y \to \Delta^0$, is precisely $X \to N\tau X \to \Delta^0$ with τX the fundamental category of X (Hint: show that Y is the nerve of a category using the Grothendieck-Segal caracterisation of nerves; then show that the map $X \to Y$ is the universal map from X to the nerve of a category using the uniqueness of liftings). This actually gives an alternative proof of the existence of fundamental categories.