Néron models:

. We revert to our standard notations: S (connected) Dedekind, η generic point, etc.

1) Generalities

def: Let $X/_\eta$ be a smooth scheme. The Néron model $N(X)$ of X is a smooth model of X such that, for all smooth S-schemes Z, the restriction map
$$\mathrm{Hom}_S(Z, N(X)) \longrightarrow \mathrm{Hom}_\eta(Z_\eta, X)$$
is a bijection.

. The Néron model does not exist in general, but if it does it is unique up to unique isomorphism.

. lemma: If $X/_\eta$ is a group scheme which admits a Néron model, then $N(X)$ is a group scheme over S.

proof: . Using the universal property, we see that $N(X) \times_S N(X)$ is a Néron model of $X \times_\eta X$. Then, the structure morphisms $X \times_\eta X \xrightarrow{m} X$, $\eta \xrightarrow{e} X$ and $i : X \to X$ extend to maps $N(X) \times_S N(X) \xrightarrow{\tilde{m}} N(X)$, $S \xrightarrow{\tilde{e}} N(X)$ and $\tilde{i} : N(X) \to N(X)$ satisfying the axioms of group schemes. $\square$

. Néron models can be constructed locally on S, so that one can often assume that S is a trait.

. Given a Néron model $\mathcal{N}$, we can consider its fiber $\mathcal{N}_\sigma$ at a closed point. It is a smooth commutative group scheme. Let us briefly describe the general structure of such an object $G/_{\sigma = \mathrm{Spec}(k)}$.

- G has an identity component $G^\circ \hookrightarrow G$: a connected smooth commutative group scheme.

of finite type [G connected + group $\Rightarrow$ G geom. connected ; can pass to

rmk: A connected locally of finite type scheme X over a field is not necc. of finite type. There are easy non-separated examples but also separated examples (use blow-ups). However this cannot happen if $X = G$ is a group scheme:

lemma: $G/_k$ locally of finite type connected grp scheme $\Rightarrow$ G of finite type.

proof: G geom. connected $\Rightarrow$ WLOG $k = \bar{k}$. G_{red} is smooth $\underset{G \text{ connected}}{\Rightarrow}$ G is irreducible. Let U be a affine non-empty open. Then the map $U \times U \xrightarrow{m} G$ is faithfully flat: flat because m is, surjective because for any $g \in G(k)$, $U \cap g \cdot U^{-1} \neq \emptyset$. Since $U \times U$ is quasi-compact, G is as well. $\square$

- $\pi_0(G/k) := G/G^\circ$ is an étale group scheme, of finite type iff G is.
- The structure of G° over an imperfect field can be really complicated.
- <u>Assume k perfect</u>.

 A theorem of Chevalley says that there exists a unique exact sequence

$0 \to L \to G^\circ \to B \to 0$ with * L smooth <u>affine</u> commutative alg. group.
* B abelian variety.

Moreover L has a unique decomposition has $U \times T$ with U unipotent ($\overset{k \text{ perfect}}{\Longleftrightarrow}$ U is successive extension of copies of $\mathbb{G}_a$) and a torus T ($T_{\bar{k}} \simeq \mathbb{G}_m^n$).

2) Main existence theorems

· <u>thm</u>: A/η abelian variety. Then A admits a Néron model, which is moreover quasi-projective (hence separated of finite type).

· <u>rmk</u>: - there are other group schemes besides abelian varieties, for instance a group like $\mathbb{G}_m$ admits a Néron model which is not of finite type ($0 \to \mathbb{G}_{m,S} \to N(\mathbb{G}_m)_S \to \mathbb{Z} \to 0$)

<u>idea of the proof</u>: S = trait = spectrum of a DVR for simplicity.

<u>Steps</u>: 0) Start with any proper flat model $\mathcal{A}_0$.

1) Construct a <u>smoothening</u> (a certain weak form of desingularization) $\mathcal{A}_1$ of $\mathcal{A}_0$.
2) Take the smooth locus $\mathcal{A}_2$ of $\mathcal{A}_1$ and prove it is a <u>weak Néron model</u>.
3) Remove the irrelevant irreducible special components to get $\mathcal{A}_3$.
4) Construct a <u>birational group law</u> on $\mathcal{A}_3$ and extend it to an actual group law to get $\mathcal{A}_4$.
5) Prove that $\mathcal{A}_4 \simeq N(A)$.

0) · Can start with any projective embedding $A \hookrightarrow \mathbb{P}^N_\eta$ and close it up in $\mathbb{P}^N_S \rightsquigarrow \mathcal{A}_0$ proper flat.

1) <u>def</u>: Let X/S be of finite type with X_η/η smooth. A <u>smoothening of X</u> is a proper morphism $X' \xrightarrow{f} X$ with f_η isomorphism and which satisfies:
$\forall\, S' \to S$ étale morphism, the canonical map
$(X')^{sm}(S') \longrightarrow X(S')$ is bijective.

· <u>rmk</u>: A resolution of singularities of X is always a smoothening [BLR, 3.1/2]

· Néron and Raynaud proved that smoothenings always exists and can be obtained by a sequence of blow-ups [BLR, 3.1/3].

2) Let $\mathcal{A}_2$ be the smooth locus of $\mathcal{A}_1$. By construction, it is an instance of the following definition.

<u>def:</u> Let X_η/η be a smooth finite type scheme. A <u>weak Néron model</u> X of X_η is a smooth finite type S-scheme such that, for all $S' \to S$ étale, the natural map $X(S') \longrightarrow X_\eta(S'_\eta)$ is a bijection.

. The next step is to strengthen this mapping property to rational maps:

<u>prop:</u> $\mathcal{A}_2$ satisfies the following: for any smooth S-scheme Z, every rational map $Z_\eta \dashrightarrow \mathcal{A}_{2,\eta}$ extends to an S-rational map $Z \dashrightarrow \mathcal{A}_2$.

<u>rmk:</u> these two steps can be applied to get weak Néron models for any smooth variety. Those are important in the theory of motivic integration [Nicaise].

3) Now we start using the fact that A is a group scheme.
This implies that $\Omega^g_{A/k}$ is globally free, generated by an invariant differential ω. By multiplication by a suitable element in R, we can arrange that ω extends to a section ω of $\Omega^g_{\mathcal{A}_2/S}$, which does not vanish on the whole of $\mathcal{A}_{2,6}$. Now put $\mathcal{A}_3 := \mathcal{A}_2 \setminus \bigcup_{\substack{E \subset \mathcal{A}_{2,6} \\ \omega_{|E}=0}} E$.

4) Using the mapping property for rational map, one can show that the multiplication map on A extends to $\mathcal{A}_3$:

<u>thm:</u> The morphism $m_\eta : A \times_\eta A \to A$ extends to an S-rational map $m : \mathcal{A}_3 \times_S \mathcal{A}_3 \dashrightarrow \mathcal{A}_3$. Moreover, the maps
$\mathcal{A}_3 \times \mathcal{A}_3 \overset{(p_1,\, m)}{\dashrightarrow} \mathcal{A}_3 \times \mathcal{A}_3$ and $\mathcal{A}_3 \times \mathcal{A}_3 \overset{(m,\, p_2)}{\dashrightarrow} \mathcal{A}_3 \times \mathcal{A}_3$
are also S-birational.

This puts you in position to apply a theorem of Weil and Artin on extending birational group laws. I will not give the precise statement. This provides an open immersion $\mathcal{A}_3 \hookrightarrow \mathcal{A}_4$ with $\mathcal{A}_4$ smooth separated S-group scheme model of A.

5) Finally, in the presence of a group scheme structure, the mapping property for rational maps can be upgraded to the true Néron mapping property, because of:

<u>thm</u> (Weil) S normal noetherian, $u : Z \dashrightarrow G$ S-rational map with Z smooth and G smooth separated S-group scheme. If u is defined in codimension ≤ 1, it is defined everywhere.

- For Jacobians of curves, it is possible to say more. A simple case is

<u>thm:</u> $X/_S$ flat projective curve such that : * X is regular
* $X/_S$ has geom integral fibers.

Then $\mathrm{Pic}^{\circ}_{X/S}$ is a Néron model of its generic fiber $\mathrm{Pic}^{\circ}_{X_\eta/\eta} \simeq \mathrm{Jac}(X_\eta)$

(in particular it is connected.)

<u>proof:</u> - We have representability of $\mathrm{Pic}_{X/S}$ by $X \to S$ projective with geom. integral fibers, so the statement makes sense (Pic° is the part of Pic with degree $= 0$). One can then reduce to $S = \mathrm{Spec}(R)$, R DVR and admitting a section. We now prove the Néron mapping property.

Let $T \to S$ be a smooth scheme and $u_\eta : T_\eta \longrightarrow \mathrm{Pic}_{X_\eta/\eta}$.

Since $X/_S$ has a section, u_η corresponds to a line bundle $\mathcal{L}$ on $X_\eta \times_\eta T_\eta$. Because X is regular and $T \to S$ is smooth, $X \times_S T$ is regular and $X_\eta \times_\eta T_\eta$ is a dense open in $X \times_S T$. The line bundle $\mathcal{L}$ corresponds to a Weil divisor W on $X_\eta \times T_\eta$; its closure in $X \times_S T$ corresponds to a line bundle on $X \times_S T$ by regularity $\rightsquigarrow$ u_η extends to a morphism $u : T \longrightarrow \mathrm{Pic}_{X/S}$.

- By constancy of the degree in flat families, u factors through $\mathrm{Pic}^{\circ}_{X/S}$. Since $\mathrm{Pic}^{\circ}_{X/S}$ is separated, u is unique. This finishes the proof. $\square$

- For a curve with reducible fibers, we have seen in the chapter on Picard schemes that the representability and separability of $\mathrm{Pic}_{X/S}$ is subtle.

<u>thm</u> (Raynaud) $X/_S$ proper flat regular curve with geometrically integral <u>generic</u> fibre.
We assume that $X \to S$ admits a section [there are weaker hypotheses possible].
Let $\begin{cases} \mathrm{Pic}^{[0]}_{X/S} & \text{be the part of the Picard functor of line bundles of total degree } 0. \\ \cup\!\!\mathsf{I} & \\ E_{X/S} & \text{be the closure of the unit section} \end{cases}$ ($E_{X/S}$ generated by morphisms $Z \xrightarrow{g} \mathrm{Pic}_{X/S}$ with $Z/_S$ flat and g_η factoring through e)

Then $\begin{cases} N(\mathrm{Jac}(X_\eta)) \simeq \mathrm{Pic}^{[0]}_{X/S} / E_{X/S} \\ \\ N(\mathrm{Jac}(X_\eta))^{\circ} \simeq \mathrm{Pic}^{\circ}_{X/S} \end{cases}$ i.e. this quotient is representable by a separated finite type S-scheme.

<u>rmk:</u> This implies a concrete computation of $\pi_0(N(\mathrm{Jac}_{X_\eta}))$ [BLR, 9.5].

- We now come to elliptic curves.

<u>thm:</u> Let $E/_\eta$ be an elliptic curve. Let $\mathcal{E}/_S$ be the minimal regular model of E. Let $\mathcal{E}^{sm}/_S$ be the S-smooth locus of $\mathcal{E}$. Then $N(E) \simeq \mathcal{E}^{sm}$.