University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 7: lifting calculus and categorical equivalences

1 Construct, for any four simplicial sets X, Y, Z, W, a morphism

$$\mathrm{comp}_3 : \mathrm{Fun}(X,Y) \times \mathrm{Fun}(Y,Z) \times \mathrm{Fun}(Z,W) \to \mathrm{Fun}(X,W)$$

which is induced by composition on 0-simplices by the method of Lemma III.28. Show that it can be written in two different ways using the maps comp of Lemma 28. Deduce from this that the composition in the homotopy category of ∞-categories hCat_∞ from Definition III.33 is associative. Show that this composition is as well unital.

2 Let C, D be ∞-categories. Show that a functor $F : C \to D$ is an equivalence of ∞-categories if and only if, for every simplicial set K, the induced map

$$F_* : \mathrm{Fun}(K, C) \to \mathrm{Fun}(K, D)$$

is a categorical equivalence (note the order of the factors!). Hint: Yoneda lemma in hCat_∞.

3 Let C, D be ∞-categories and $F : C \to D$ be a functor. Show that F sends $\mathrm{Core}(C)$ to $\mathrm{Core}(D)$. Let $\mathrm{Core}(F) : \mathrm{Core}(C) \to \mathrm{Core}(D)$ be the induced functor. Show that, if F is an equivalence, then so is $\mathrm{Core}(F)$.

4 Consider a pushout diagram of simplicial sets

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where $X \to X'$ is a monomorphism and $X \to Y$ is a categorical equivalence. Show that $X' \to Y'$ is a categorical equivalence.

5 Show that the retract of a categorical equivalence of simplicial sets is a categorical equivalence.

6 The goal of this exercise is to show one of the key steps which we omitted in the proof of Proposition II.19.

- Let $0 < j < n$. Show that there are unique morphisms of simplicial sets $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$ which are determined on vertices by

$$s(y) = \begin{cases} (0,y) \text{ if } y < j, \\ (1,y) \text{ if } y = j, \\ (2,y) \text{ if } y > j \end{cases} \quad \text{and } r(x,y) = \begin{cases} y \text{ if } x = 0 \text{ and } y < j, \\ y \text{ if } x = 2 \text{ and } y > j, \\ j \text{ otherwise} \end{cases}$$

- Check that $rs = \mathrm{id}$.
- Show that $s(\Lambda^n_j) \subset \Delta^2 \times \Lambda^n_j$, so that in particular $s(\Lambda^n_j) \subset (\Delta^2 \times \Lambda^n_j) \cup (\Lambda^2_1 \times \Delta^n)$ (Hint: show first that $p_2 s = \mathrm{id}$ where $p_2 : \Delta^2 \times \Delta^n \to \Delta^n$ is the second projection.)
- Show that $r(\Delta^2 \times \Lambda^n_j) \subset \Lambda^n_j$ and $r(\Delta^n \times \Lambda^2_1) \subset \Lambda^n_j$. (Hint: it's "just" a computation!)
- Conclude that the inner horn inclusion $\Lambda^n_j \subset \Delta^n$ is a retract of the pushout-product $(\Lambda^2_1 \subset \Delta^2) \boxtimes (\Lambda^n_j \subset \Delta^n)$.