University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 10: joins of categories and simplicial sets

1 Let A, B, C be three categories. Prove that functors $F : C \to A \star B$ are in bijective correspondence with triples of functors $(\pi : C \to [1], f_0 : C^{\{0\}} \to A, f_1 : C^{\{1\}} \to B)$ with $C^{\{j\}} \subset C$ is the subcategory of C consisting of objects which π send to j and morphisms which π send to id_j.

2 Prove the description of the functors $\iota_! \dashv \iota^* \dashv \iota_*$ claimed as Lemma 14 in the notes.

3 Let X, Y, Z be three simplicial sets. Prove the associativity of the join by constructing an isomorphism

$$\alpha : (X \star Y) \star Z \simeq X \star (Y \star Z).$$

Hint: give a "symmetrical" description of both sides in terms of cuts of $[n]$ at two places.

4 Let X, Y be two simplicial sets. Construct an isomorphism

$$(X \star Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \star X^{\mathrm{op}}.$$

5 Let X, Y be Kan complexes. Is $X \star Y$ a Kan complex in general?

6 Let A, B be topological spaces. We define the join $A \star B$ to be the iterated pushout

$$A \star B := A \coprod_{A \times \{0\} \times B} (A \times [0,1] \times B) \coprod_{A \times \{1\} B} B.$$

Prove that, unlike the join of simplicial sets, this is a symmetrical construction: there exists an homeomorphism $A \star B \simeq B \star A$. One can show that, if X and Y are simplicial sets and one of X, Y has finitely many non-degenerate simplices, then there is an homeomorphism $|X \star Y| \simeq |X| \star |Y|$ (`https://kerodon.net/tag/017R`). Try to prove this for $X = \Delta^m$ and $Y = \Delta^n$.