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EEN442 - Power Systems II (Συστήματα Ισχύος II)

Part 3: Power flow analysis

https://sps.cut.ac.cy/courses/een442/

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Today's learning objectives

After this part of the lecture and additional reading, you should be able to . . .

1. . . . derive models of the main power system components that can be used in power flow studies;
2. . . . derive the power flow equations for a given power network;
3. . . . formulate a standard power flow problem together with the required constraints;
4. . . . explain the functioning of standard numerical methods to solve the power flow problem.

Outline

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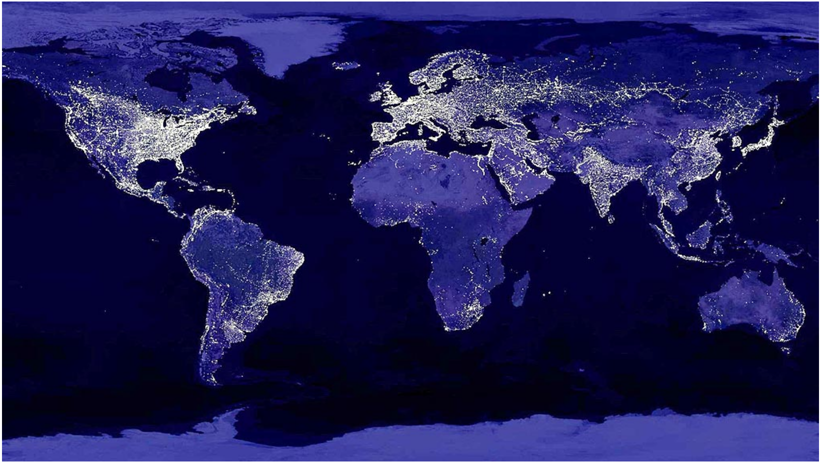
1 From small- to large-scale systems...



1 The importance of power flow analysis

- It is highly important to know the voltages, currents and powers in different parts of a power system
- This is necessary for . . .
  - . . . an adequate design of the different components (generators, lines, transformers,. . .)
  - . . . keeping losses low and ensuring an economic operation of the system, while taking relevant constraints into account (e.g., voltage, line and generation limits)
  - . . . monitoring the (steady-state) stability of the system
- Yet, it is practically impossible to physically measure *all* voltages and currents in the system
- A power flow computation is an efficient tool to obtain the *complete state* of the system, i.e., all complex voltages at all nodes in the system (once the voltages are known, the currents and powers can also be computed)
- Power flow computations are the most used computations in power systems!

1 Power flow computation

- A power flow computation is an efficient tool to obtain the *complete state* of the system, i.e., all complex voltages at all nodes, under balanced steady-state conditions
- Once the voltages are known, the currents and powers can also be computed
- Power flow computations are usually performed using dedicated software
- Useful tool for both analysis of an existing network and of projected network expansions or load growth

→ Power flow computations are the most used computations in power systems!

1 General formulation of power flow problem (1)

- In general, the power flow problem (also called load flow problem) is formulated as a set of nonlinear equations

$$\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathbf{0},$$

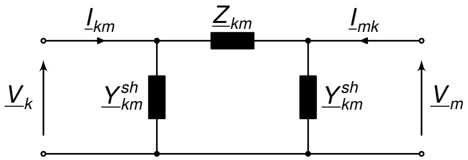
where

  - $\mathbf{f}$ is an n-dimensional nonlinear function
  - $\mathbf{x}$ is an n-dimensional complex vector containing the states of the system. These are the unknown voltage magnitudes and phase angles at the nodes in the system.
  - $\mathbf{u}$ is an input vector with known entries (e.g., voltages at generator nodes with voltage control)
  - $\mathbf{p}$ is a vector that contains the parameters of the network components (e.g., line and transformer impedances)

1 General formulation of power flow problem (2)

- The power flow problem consists in formulating the equations in $\mathbf{f}$ and then solving them with respect to $\mathbf{x}$
- Both aspects are covered in the remainder of this part of the lecture
- A necessary condition for the power flow problem to have a physically meaningful solution is that $\mathbf{f}$ and $\mathbf{x}$ have the same dimension, since then we have the same number of unknowns as equations
- But even then, there might not be a unique solution or even no solution at all!

2.1 Review: Π-model of a transmission line (1)



- To formulate network equations for power flow problem, typically line admittance is used instead of line impedance

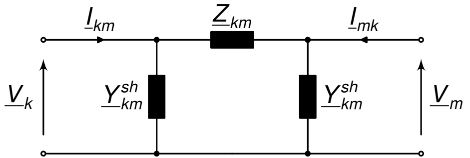
$$\underline{Y}_{km} = \underline{Z}_{km}^{-1} = (R_{km} + jX_{km})^{-1} = G_{km} + jB_{km}$$

with

$$G_{km} = \frac{R_{km}}{R_{km}^2 + X_{km}^2}, \quad B_{km} = -\frac{X_{km}}{R_{km}^2 + X_{km}^2}$$

- In most cases, shunt conductance is very small and therefore neglected
- For standard transmission lines both R_{km} and X_{km} are positive and thus G_{km} is positive and B_{km} is negative

2.1 Review: Π-model of a transmission line (2)



- Complex currents given in terms of complex voltages $\underline{V}_k$ and $\underline{V}_m$ by

$$\underline{I}_{km} = \underline{Y}_{km}\,(\underline{V}_k - \underline{V}_m) + \underline{Y}^{sh}_{mk}\underline{V}_k$$
$$\underline{I}_{mk} = \underline{Y}_{km}\,(\underline{V}_m - \underline{V}_k) + \underline{Y}^{sh}_{mk}\underline{V}_m$$

- Or, equivalently, in matrix form

$$\begin{bmatrix} \underline{I}_{km} \\ \underline{I}_{mk} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{km} + \underline{Y}^{sh}_{km} & -\underline{Y}_{km} \\ -\underline{Y}_{km} & \underline{Y}_{km} + \underline{Y}^{sh}_{km} \end{bmatrix} \begin{bmatrix} \underline{V}_k \\ \underline{V}_m \end{bmatrix}$$

- Note: above matrix is symmetric (i.e., it is identical to its transposed); this reflects the fact that lines and cables are symmetrical elements

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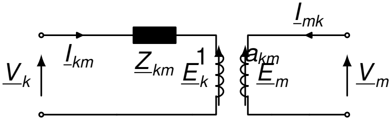


- In-phase transformer without core losses, turns ratio c_{km} and $a_{km} = c_{km}^{-1}$

$$\underline{E}_m = a_{km}\underline{E}_k = \underline{V}_m \qquad \underline{I}_{km} = -a_{km}\underline{I}_{mk}$$

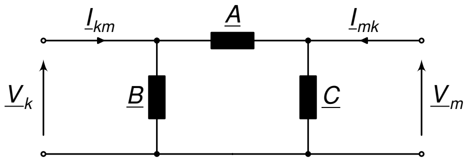
- Using Ohm's law, we have that (with $\underline{Y}_{km} = \underline{Z}_{km}^{-1}$)

$$\underline{I}_{km} = \underline{Y}_{km}(\underline{V}_k - \underline{E}_k) = \underline{Y}_{km}(\underline{V}_k - a_{km}^{-1}\underline{V}_m)$$
$$\underline{I}_{mk} = -a_{km}^{-1}\underline{Y}_{km}(\underline{V}_k - \underline{E}_k) = -a_{km}^{-1}\underline{Y}_{km}(\underline{V}_k - a_{km}^{-1}\underline{V}_m)$$

- Or, equivalently, in matrix form

$$\begin{bmatrix} \underline{I}_{km} \\ \underline{I}_{mk} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{km} & -a_{km}^{-1}\underline{Y}_{km} \\ -a_{km}^{-1}\underline{Y}_{km} & a_{km}^{-2}\underline{Y}_{km} \end{bmatrix} \begin{bmatrix} \underline{V}_k \\ \underline{V}_m \end{bmatrix}$$

2.1 Π-equivalent model of in-phase three-phase transformer (1)



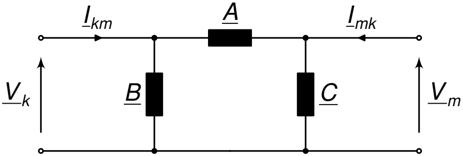
- In-phase transformer model can also be represented by Π-equivalent model
- From the above Π-model we have that

$$\underline{I}_{km} = (\underline{A} + \underline{B})\underline{V}_k - \underline{A}\,\underline{V}_m$$
$$\underline{I}_{mk} = -\underline{A}\,\underline{V}_k + (\underline{A} + \underline{C})\underline{V}_m$$

- Or, equivalently, in matrix form

$$\begin{bmatrix} \underline{I}_{km} \\ \underline{I}_{mk} \end{bmatrix} = \begin{bmatrix} \underline{A} + \underline{B} & -\underline{A} \\ -\underline{A} & \underline{A} + \underline{C} \end{bmatrix} \begin{bmatrix} \underline{V}_k \\ \underline{V}_m \end{bmatrix}$$

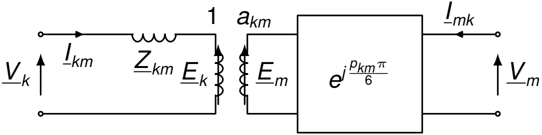
2.1 Π-equivalent model of in-phase three-phase transformer (2)



- Comparing coefficients with transformer model yields

$$\underline{A} = a_{km}^{-1}\underline{Y}_{km} \quad \underline{B} = (1 - a_{km}^{-1})\underline{Y}_{km} \quad \underline{C} = (a_{km}^{-1} - 1)a_{km}^{-1}\underline{Y}_{km}$$

- Note: Π-equivalent model of a real transformer is symmetric, but if $c_{km}^2 \neq 1$ then its diagonal elements differ since then $\underline{B} \neq \underline{C}$



- Complex transformation ratio

$$\underline{a}_{km} = \underline{c}_{km}^{-1} = \left(c_{km} e^{j\frac{p_{km}\pi}{6}}\right)^{-1} = c_{km}^{-1} e^{-j\frac{p_{km}\pi}{6}} = a_{km} e^{j\varphi_{km}}$$

- Then

$$\underline{E}_k = \underline{a}_{km} \underline{V}_m$$
$$\underline{I}_{km} = -\underline{a}_{km}^{*} \underline{I}_{mk}$$

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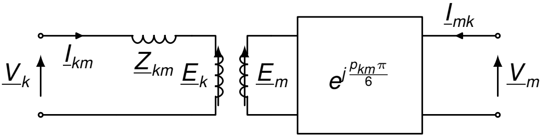
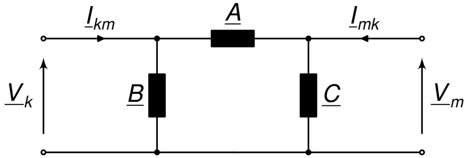


- Using Ohm's law, we have that (with $\underline{Y}_{km} = \underline{Z}_{km}^{-1}$)

$$\underline{I}_{km} = \underline{Y}_{km}(\underline{V}_k - \underline{E}_k) = \underline{Y}_{km}(\underline{V}_k - \underline{a}_{km}^{-1}\underline{V}_m)$$
$$\underline{I}_{mk} = -(\underline{a}_{km}^*)^{-1}\underline{Y}_{km}(\underline{V}_k - \underline{E}_k) = -(\underline{a}_{km}^*)^{-1}\underline{Y}_{km}(\underline{V}_k - \underline{a}_{km}^{-1}\underline{V}_m)$$

- Or, equivalently, in matrix form

$$\begin{bmatrix} \underline{I}_{km} \\ \underline{I}_{mk} \end{bmatrix} = \begin{bmatrix} \underline{Y}_{km} & -\underline{a}_{km}^{-1}\underline{Y}_{km} \\ -(\underline{a}_{km}^*)^{-1}\underline{Y}_{km} & \underline{a}_{km}^{-2}\underline{Y}_{km} \end{bmatrix} \begin{bmatrix} \underline{V}_k \\ \underline{V}_m \end{bmatrix}$$

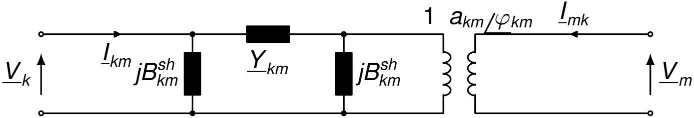


- The matrix

$$\begin{bmatrix} \underline{Y}_{km} & -\underline{a}_{km}^{-1}\underline{Y}_{km} \\ -(\underline{a}_{km}^{*})^{-1}\underline{Y}_{km} & a_{km}^{-2}\underline{Y}_{km} \end{bmatrix}$$

  is not symmetric if $\underline{a}_{km}$ is not real!

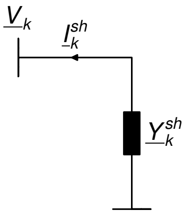
- Hence, it is not possible to determine parameters $\underline{A}$, $\underline{B}$ and $\underline{C}$ of equivalent Π-model since $\underline{a}_{km}^{-1}\underline{Y}_{km} \neq (\underline{a}_{km}^{*})^{-1}\underline{Y}_{km}$
- Phase-shifting transformer model can NOT be modelled by Π-equivalent circuit!



- Above circuit captures models for power lines (with zero shunt conductance), cables and transformers
- Power line or cable $\rightarrow$ set $a_{km} = 1$ and $\varphi_{km} = 0$
- In-phase transformer $\rightarrow$ set $B^{sh}_{km} = 0$ and $\varphi_{km} = 0$
- Phase-shifting transformer $\rightarrow$ set $B^{sh}_{km} = 0$ and $\varphi_{km} \neq 0$
- Complex current

$$
\begin{aligned}
\underline{I}_{km} &= jB^{sh}_{km}\underline{V}_k + \underline{Y}_{km}(\underline{V}_k - a_{km}^{-1}e^{-j\varphi_{km}}\underline{V}_m) \\
&= jB^{sh}_{km}\underline{V}_k + (G_{km} + jB_{km})(\underline{V}_k - a_{km}^{-1}e^{-j\varphi_{km}}\underline{V}_m)
\end{aligned}
$$

2.2 Shunt elements

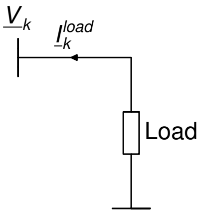


- Shunt element = impedance with one port connected to neutral
- Shunts are, e.g., capacitors or reactors
- Usually, current of a shunt is defined positive when flowing into network (generator convention)

$$\underline{I}_k^{sh} = -\underline{Y}_k^{sh}\underline{V}_k$$

- Then we obtain for complex power

$$\underline{S}_k^{sh} = P_k^{sh} + jQ_k^{sh} = -(\underline{Y}_k^{sh})^*|\underline{V}_k|^2$$

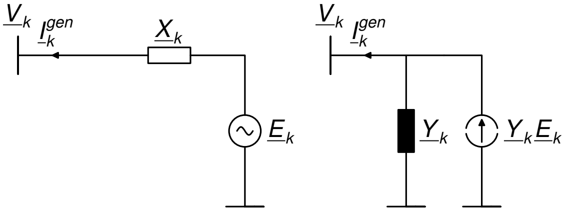


- Loads are fundamental component of power systems
- Accurate modelling of loads very important
- For high-voltage systems, loads usually represent aggregated demand of a complete lower voltage network (as seen from a substation)
- Then, load typically modelled as constant power load

$$\underline{S}_k^{load} = P_k^{load} + jQ_k^{load}$$

where P_k^{load} and Q_k^{load} are fixed values

2.4 Generators - Standard model



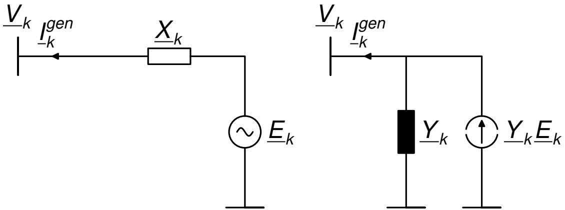
- Generators are usually modelled as voltage sources behind an impedance with variable current injection
- Using the Norton equivalent, we can write the equations as

$$\underline{Y}_k\underline{E}_k - \underline{Y}_k\underline{V}_k = \underline{I}_k^{gen}$$

- However, generators are typically controlled such that they inject fixed amount of active power and keep voltage amplitude at terminals constant. So, in power flow calculations they are modelled as

$$P_k^{gen} = P_k^{set}, \quad |\underline{V}_k| = V_k^{set}$$

2.4 Generators - Standard model

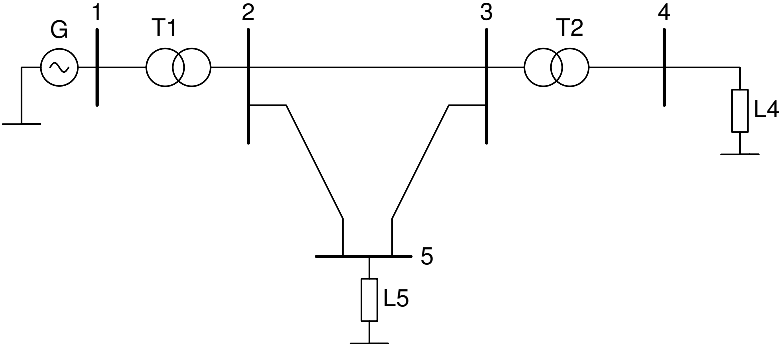


Main constraints are

- **Active power limitations**: active power setpoint has to be within limits of plant
- **Reactive power limitations**: voltage magnitude mainly controlled via reactive power generation; yet, reactive power capability of generator limited by several factors, including active power generation and bus voltage

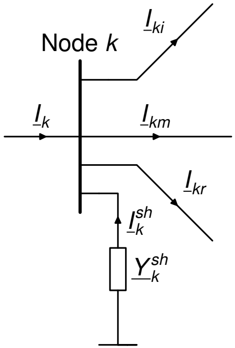
$\rightarrow$ Not possible to control active power and voltage magnitude outside certain boundaries

3 Nodal formulation of network equations - Overview



- Next, we derive the basic network equations from Kirchhoff's Current Law (KCL)
- We then put these equations in a form that is suitable for formulating the power flow equations
- As part of this procedure, we also discuss the *admittance matrix* of a power system

3.1 Net complex current injection at network buses (1)



- Consider a network with $N \geq 2$ nodes
- From KCL

$$\underline{I}_k + \underline{I}_k^{sh} = \sum_{m \in \mathcal{N}_k} \underline{I}_{km} \quad \text{for } k = 1, \ldots, N$$

- $\underline{I}_k$ is net current injection from generators and loads
- $\underline{I}_k^{sh}$ is current injection from shunts
- $\mathcal{N}_k$ is set of nodes adjacent to node k (e.g., in figure on left $\mathcal{N}_k$ contains m, i, r)

- Recall complex current from node k to m in multi-purpose circuit

$$\underline{I}_{km} = jB^{sh}_{km}\underline{V}_k + \underline{Y}_{km}(\underline{V}_k - a^{-1}_{km}e^{-j\varphi_{km}}\underline{V}_m)$$

- Recall complex shunt current

$$\underline{I}^{sh}_k = -\underline{Y}^{sh}_k\underline{V}_k$$

- Hence, we obtain

$$\underline{I}_k = \left(\underline{Y}^{sh}_k + \sum_{m\in\mathcal{N}_k}\left(jB^{sh}_{km} + \underline{Y}_{km}\right)\right)\underline{V}_k - \sum_{m\in\mathcal{N}_k}\underline{Y}_{km}a^{-1}_{km}e^{-j\varphi_{km}}\underline{V}_m$$

for $k = 1, \ldots, N$

3.2 The admittance matrix

- Expression for $\underline{I}_k$, $k = 1, \ldots, N$ can be written in matrix form

$$\underline{\mathbf{I}} = \underline{\mathbf{Y}}\,\underline{\mathbf{V}},$$

where
  - $\underline{\mathbf{I}}$ is vector with current injections $\underline{I}_k$, $k = 1, \ldots, N$
  - $\underline{\mathbf{V}}$ is vector with nodal voltages $\underline{V}_k = V_k e^{j\theta_k}$, $k = 1, \ldots, N$
  - $\underline{\mathbf{Y}} = \mathbf{G} + j\mathbf{B}$ is *nodal admittance matrix* with elements

$$\boxed{\begin{aligned} \underline{\mathcal{Y}}_{kk} &= \mathcal{G}_{kk} + j\mathcal{B}_{kk} = \underline{Y}_k^{sh} + \sum_{m \in \mathcal{N}_k} \left(B_{km}^{sh} + \underline{Y}_{km} \right) \\ \underline{\mathcal{Y}}_{km} &= \mathcal{G}_{km} + j\mathcal{B}_{km} = -\underline{Y}_{km} a_{km}^{-1} e^{-j\varphi_{km}} = -(G_{km} + jB_{km}) a_{km}^{-1} e^{-j\varphi_{km}} \end{aligned}}$$

- Admittance matrix is compact representation of network interconnections
- Synchronous generators are modeled with their Norton equivalent

3.2 Forming the nodal admittance matrix

Main steps:

1. We derive the per-phase per-unit equivalent circuit
2. We add the per-unit series impedance of transformer and transmission lines.
3. We transform each voltage source in series with an impedance to an equivalent current source in parallel with that impedance using Norton's theorem.
4. We expressed impedance values as admittance.
5. We use Kirchhoff's current law to formulate the system equations.

Note: Current sources always flow into a node, hence power flow for generators will be positive, power flow for motors and loads will be negative.

3.2 Nodal current expressed via admittance matrix

- Now, k-th component of nodal current vector $\underline{\mathbf{I}}$ can be written as

$$
\begin{aligned}
\underline{I}_k &= \underline{\mathcal{Y}}_{kk}\underline{V}_k + \sum_{m \in \mathcal{N}_k} \underline{\mathcal{Y}}_{km}\underline{V}_m \\
&= (\mathcal{G}_{kk} + j\mathcal{B}_{kk}) V_k e^{j\theta_k} + \sum_{m \in \mathcal{N}_k} (\mathcal{G}_{km} + j\mathcal{B}_{km}) V_m e^{j\theta_m}
\end{aligned}
$$

- Note: In presence of transformers, admittance matrix is NOT necessarily symmetric
- Note: In presence of transformers, the series admittance $\underline{Y}_{km}$ between nodes k and m and the (k, m)-th entry $\underline{\mathcal{Y}}_{km}$ of admittance matrix do NOT have the same values!

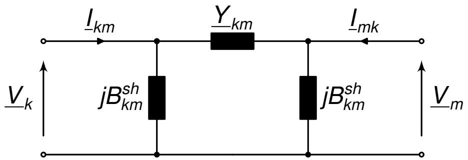
3.2 Some comments on admittance matrix

- For practical large power systems, admittance matrix is usually sparse
- Sparsity typically increases with network size
- This sparsity can be used effectively to design efficient numerical algorithms to perform power flow computations and other calculations in power systems
- Example: a power system with 1000 buses and 1500 branches (=lines and transformers) usually has a degree of sparsity greater than 99%, i.e., less than 1% of entries of admittance matrix have nonzero values

4 Outline

4 Power flows - Overview

- We have discussed simplified versions of the power flow equations for an individual transmission line (e.g., for lossless impedances)
- Next, we derive expressions for the active and reactive power flows in different network elements (transmission lines and transformers)
- All discussed components possess *linear* behaviour with respect to the relation between voltages and currents
- Yet, in power systems one is usually interested in powers rather than currents
- The power flow equations are *nonlinear* algebraic equations
- This makes computing their solution challenging

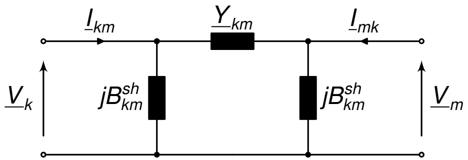


- Assume shunt conductance of line is small $\rightarrow \underline{Y}^{sh}_{km} = jB^{sh}_{km}$
- Complex current from node k to m

$$\underline{I}_{km} = \underline{Y}_{km}(\underline{V}_k - \underline{V}_m) + jB^{sh}_{km}\underline{V}_k$$

- Complex power from node k to m

$$\begin{aligned}\underline{S}_{km} &= \underline{V}_k\,\underline{I}^*_{km} \\ &= \underline{V}_k\left(\underline{Y}^*_{km}(\underline{V}^*_k - \underline{V}^*_m) - jB^{sh}_{km}\underline{V}^*_k\right) \\ &= \underline{Y}^*_{km}V_k e^{j\theta_k}(V_k e^{-j\theta_k} - V_m e^{-j\theta_m}) - jB^{sh}_{km}V_k^2\end{aligned}$$



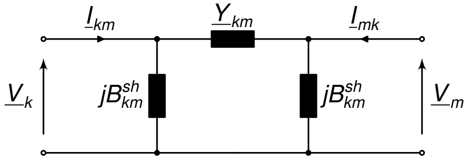
- Define short-hand $\theta_{km} = \theta_k - \theta_m$
- Split $\underline{S}_{km} = P_{km} + jQ_{km}$ into real and imaginary part
- Active power flow

$$P_{km} = \Re(\underline{S}_{km}) = V_k^2 G_{km} - V_k V_m \left(G_{km}\cos(\theta_{km}) + B_{km}\sin(\theta_{km})\right)$$

- Reactive power flow

$$Q_{km} = \Im(\underline{S}_{km}) = -V_k^2 (B_{km} + B^{sh}_{km}) + V_k V_m \left(B_{km}\cos(\theta_{km}) - G_{km}\sin(\theta_{km})\right)$$

4.1 Active and reactive power flows on transmission line from node *m* to node *k*



- Power flows from *m* to *k* can be obtained in same way
- Note that

$$\sin(\theta_{mk}) = \sin(-\theta_{km}) = -\sin(\theta_{km}) \quad \cos(\theta_{mk}) = \cos(-\theta_{km}) = \cos(\theta_{km})$$

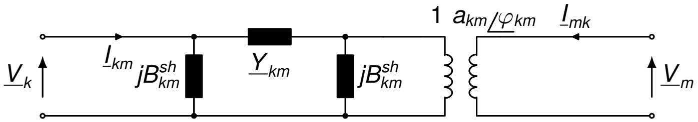
- Active power flow

$$P_{mk} = V_m^2 G_{km} - V_k V_m G_{km} \cos(\theta_{km}) + V_k V_m B_{km} \sin(\theta_{km})$$

- Reactive power flow

$$Q_{mk} = -V_m^2 (B_{km} + B_{km}^{sh}) + V_k V_m B_{km} \cos(\theta_{km}) + V_k V_m G_{km} \sin(\theta_{km})$$

4.2 Complex power flow in multi-purpose two-port from node k to m



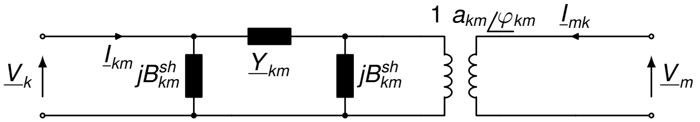
- Complex current from node k to m

$$\underline{I}_{km} = jB_{km}^{sh}\underline{V}_k + \underline{Y}_{km}(\underline{V}_k - a_{km}^{-1}e^{-j\varphi_{km}}\underline{V}_m)$$

- Complex power from node k to m

$$\begin{aligned}
\underline{S}_{km} &= \underline{V}_k\, \underline{I}_{km}^* \\
&= \underline{V}_k \left(-jB_{km}^{sh}\underline{V}_k^* + \underline{Y}_{km}^*(\underline{V}_k^* - a_{km}^{-1}e^{j\varphi_{km}}\underline{V}_m^*)\right) \\
&= \underline{Y}_{km}^* V_k e^{j\theta_k}\left(V_k e^{-j\theta_k} - a_{km}^{-1}e^{j\varphi_{km}}V_m e^{-j\theta_m}\right) - jB_{km}^{sh}V_k^2 \\
&= \underline{Y}_{km}^*\left(V_k^2 - \frac{V_k V_m}{a_{km}}e^{j(\theta_k - \theta_m + \varphi_{km})}\right) - jB_{km}^{sh}V_k^2
\end{aligned}$$

4.2 Active and reactive power flows in multi-purpose two-port from node k to node m



- Active power flow

$$P_{km} = V_k^2 G_{km} - \frac{V_k V_m}{a_{km}} \left(G_{km} \cos(\theta_{km} + \varphi_{km}) + B_{km} \sin(\theta_{km} + \varphi_{km}) \right)$$

- Reactive power flow

$$Q_{km} = -V_k^2 (B_{km} + B^{sh}_{km}) + \frac{V_k V_m}{a_{km}} \left(B_{km} \cos(\theta_{km} + \varphi_{km}) - G_{km} \sin(\theta_{km} + \varphi_{km}) \right)$$

- For transmission line set $a_{km} = 1$, $\varphi_{km} = 0$
- For in-phase transformer set $B^{sh}_{km} = \varphi_{km} = 0$
- For phase-shifting transfomer set $B^{sh}_{km} = 0$

5 Outline

5 Power flows expressed via admittance matrix

- Using admittance matrix, complex power injection at node k is

$$\underline{S}_k = \underline{V}_k \underline{I}_k^* = V_k e^{j\theta_k} \left((\mathcal{G}_{kk} + j\mathcal{B}_{kk}) V_k e^{j\theta_k} + \sum_{m \in \mathcal{N}_k} (\mathcal{G}_{km} + j\mathcal{B}_{km}) V_m e^{j\theta_m} \right)^*$$

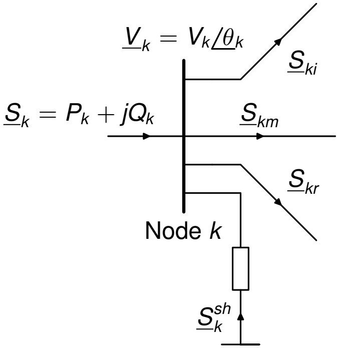
$$= V_k e^{j\theta_k} \left((\mathcal{G}_{kk} - j\mathcal{B}_{kk}) V_k e^{-j\theta_k} + \sum_{m \in \mathcal{N}_k} (\mathcal{G}_{km} - j\mathcal{B}_{km}) V_m e^{-j\theta_m} \right)$$

- Identifying real and imaginary part of above expression, yields active and reactive power flows

$$P_k = \underbrace{\mathcal{G}_{kk} V_k^2 + V_k \sum_{m \in \mathcal{N}_k} V_m \left(\mathcal{G}_{km} \cos(\theta_{km}) + \mathcal{B}_{km} \sin(\theta_{km}) \right)}_{= f_k(\theta_1, \dots, \theta_N, V_1, \dots, V_N)}$$

$$Q_K = \underbrace{-\mathcal{B}_{kk} V_k^2 + V_k \sum_{m \in \mathcal{N}_k} V_m \left(\mathcal{G}_{km} \sin(\theta_{km}) - \mathcal{B}_{km} \cos(\theta_{km}) \right)}_{= g_k(\theta_1, \dots, \theta_N, V_1, \dots, V_N)}$$

5 Variables in power flow problem



- Consider a network with $N \geq 2$ nodes
- To each node k, $k = 1, \ldots, N$, there are 4 main variables associated
  - V_k voltage magnitude
  - θ_k voltage phase angle
  - P_k net active power (algebraic sum of generation and load)
  - Q_k net reactive power (algebraic sum of generation and load)
- We may also associate additional *operational constraints* to a node k (e.g., generation or voltage limits)

5.1 Basic bus types - Two main types

- Main variables of power flow problem: V_k, θ_k, P_k and Q_k
- Usually some of these variables are known (i.e., fixed) and some are unknown (i.e., need to be calculated via power flow computation)
- Depending on which variables are known and which are unknown, we can distinguish two main bus types
  1) **PQ bus**: P_k and Q_k are known; V_k and θ_k are calculated
  2) **PV bus**: P_k and V_k are known; Q_k and θ_k are calculated

5.1 Basic bus types - Usage

- **PQ bus** usually used to represent load buses without voltage control

  Justification: active and reactive power demand of load bus is often known (at least with certain accuracy)

- **PV bus** bus typically used to represent generator buses with voltage control

  Justification: Synchronous machine usually equipped with automatic voltage regulator (AVR) that adjusts excitation voltage such that terminal voltage magnitude (or other voltage magnitude close to generator) is kept at set value

- **PV bus** also used to represent synchronous compensators

  Justification: Synchronous compensators (also: synchronous condensers) are synchronous machines that do not generate any active power (besides internal losses) and that are used for reactive power and voltage control

- In practical power systems, majority of buses are PQ buses (typically over 80%)

- In addition to PQ and PV buses a third bus type is needed: the **Vθ bus**

1) Active power losses are unknown in advance $\rightarrow$ can not specify all active power injections P_k at all buses *before* solving power flow equations

$$\sum_{k=1}^{N} P_k = \text{active power losses} = f_k(\theta_1, \ldots, \theta_N, V_1, \ldots, V_N) = ???$$

2) Voltage phase angles θ_k, θ_m only appear through differences $\theta_{km} = \theta_k - \theta_m$ in power flow equations

- We can add arbitrary constant *c* to all phase angles in network without changing electric state and power flows in network

$\rightarrow$ Need to take phase angle at one bus as *reference phase angle*

- At slack bus *i* active power balance equation replaced by

$$\theta_i = 0 \quad \text{0: arbitrary value; any other constant would work as well}$$

5.1 Basic bus types - How to choose the slack bus?

- At slack bus i, active power injection takes value

$$P_i = -\sum_{k=1,k\neq i}^{N} P_k + p,$$

where all P_k in the sum are known and p is a variable that is determined at the end of power flow computation to satisfy above active power balance

$\rightarrow$ Need to select a bus at which a generator is connected as slack bus (= *slack generator*)

5.1 Basic bus types - Slack bus: reactive power and data

- What about reactive power losses at slack bus?
  - Not possible to specify reactive power injections at all buses
  - Q_k not specified at PV buses (= for reactive power each PV bus acts as slack bus)

  → No problem, as long as there is at least one PV bus

- What data is needed at slack bus?
  - Need to specify either V_i or Q_i
  - As generator is connected at slack bus, it is natural to specify voltage magnitude V_i

  → Slack bus = $\mathbf{V}\theta$ bus

5.1 Basic bus types - Summary

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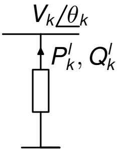
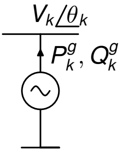
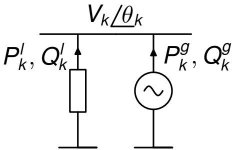
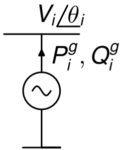
Bus configuration

Load bus	Generator bus		Load & generator bus		Slack bus i
					
Bus type					
PQ bus	PQ bus	**PV bus**	PQ bus	**PV bus**	Vθ bus
Power balance equations					
$P_k^l = f_k(\ldots)$	$P_k^g = f_k(\ldots)$		$P_k^g + P_k^l = f_k(\ldots)$		$\theta_i = 0$
$Q_k^l = g_k(\ldots)$	$Q_k^g = g_k(\ldots)$	$V_k = V_k^r$	$Q_k^g + Q_k^l = g_k(\ldots)$	$V_k = V_k^r$	$V_i = V_i^r$
Unknowns					
V_k, θ_k	V_k, θ_k	θ_k	V_k, θ_k	θ_k	-

Note: With chosen convention in this module $P_k^l \leq 0$, $P_k^g \geq 0$; $Q_k < 0 \rightarrow$ ind. react. pow.

5.2 Inequality constraints

- Formulation of power flow problem typically involves a set of inequality constraints to impose operating limits on certain variables
- Examples: voltage mangnitude (PQ buses) and reactive power (PV buses)
- Mathematically these constraints can be formulated as

$$Q_k^{\text{min}} \leq Q_k \leq Q_k^{\text{max}}$$
$$V_k^{\text{min}} \leq V_k \leq V_k^{\text{max}}$$

- If bus limit is violated, then bus status has to be changed to enforce equality constraint at limiting value
- This is usually done by changing the bus type
- Example: reactive power constraint violated at a PV bus $\rightarrow$ convert that bus into PQ bus (then Q is specified and V becomes a variable)
- Other constraints: line power flows, active power generation, phase shifter angles, . . .

5.3 Problem solvability

- When defining the bus types in the power flow problem, it is important to ensure that the resulting set of power balance equations contains the same number of equations as unknowns
- This is usually a necessary (though not always sufficient) condition for solvability
- Consider a system with $N \geq 2$ buses and suppose that N_{PV} are PV buses, N_{PQ} are PQ buses and 1 is the Vθ bus
- State of system is fully specified when all voltage magnitudes and all phase angles at all N buses are known $\rightarrow$ need to know $2N$ variables
- Voltage magnitudes at N_{PV} PV buses and at Vθ bus are given + phase angle at Vθ bus is given

5.3 Problem solvability

- Hence, unknowns are phase angles and voltage magnitudes at N_{PQ} PQ buses and phase angles at N_{PV} are PV buses, which gives a total of

$$N_{\text{unknown}} = 2N_{PQ} + N_{PV}$$

- From the PV buses we get N_{PV} active power balance equations
- From the PQ buses we get N_{PQ} active power balance equations and N_{PQ} reactive power balance equations $\rightarrow$ total number of equations is

$$N_{\text{equations}} = 2N_{PQ} + N_{PV}$$

- Hence, $N_{\text{unknown}} = N_{\text{equations}}$ and solvability condition is fulfilled

6 Outline

6 Numerical methods for solving the power flow problem

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- In practical cases, power balance equations can NOT be solved analytically
- Instead, numerical methods have to be used
- There exist a variety of free (partially open source) and commercial software packages to perform this task
- Some of the most common software tools: Matpower, DigSilent PowerFactory, PowerWorld, PSSE (Siemens)
- In the following, we discuss an iterative numerical method that is often used to solve the power flow problem: the Newton-Raphson method
- We also discuss some standard assumptions employed to simplify and speed-up the computations (Pθ-QV decoupling; DC power flow)

- To get a basic understanding of the method, we first take a look at the one-dimensional nonlinear equation

$$f(x) = 0,$$

where x is the unknown and $f(x)$ is a continuously differentiable scalar function

- Examples: $f(x) = \sin(x-3)$, $f(x) = x^2 + 9$, $f(x) = e^x$
- Objective: Given a starting value $x^{(0)}$, find appropriate root x^*, i.e., an x^* that satisfies $f(x^*) = 0$
- Newton-Raphson method provides an *iterative algorithm* to tackle this problem
- Let ν denote the iteration counter
- Define a tolerance $\epsilon > 0$ that determines the required minimal accuracy for the solution

6.1 Newton-Raphson method - Algorithm for one-dimensional case

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1) Set $\nu = 0$ and choose an appropriate starting value $x^{(0)}$;

2) Compute $f\left(x^{(\nu)}\right)$;

3) Compare $f\left(x^{(\nu)}\right)$ with tolerance ϵ:

If $|f\left(x^{(\nu)}\right)| \leq \epsilon$, then $x = x^{(\nu)}$ is admissible solution to $f(x) = 0$

If $|f\left(x^{(\nu)}\right)| > \epsilon$, then proceed to step 4;

4) Linearise $f(x)$ at current solution point $x^{(\nu)}$, i.e.,

$$f\left(x^{(\nu)} + \Delta x^{(\nu)}\right) \approx f\left(x^{(\nu)}\right) + f'\left(x^{(\nu)}\right)\Delta x^{(\nu)}$$

where $f'\left(x^{(\nu)}\right)$ denotes the first derivative of f evaluated at $x^{(\nu)}$;

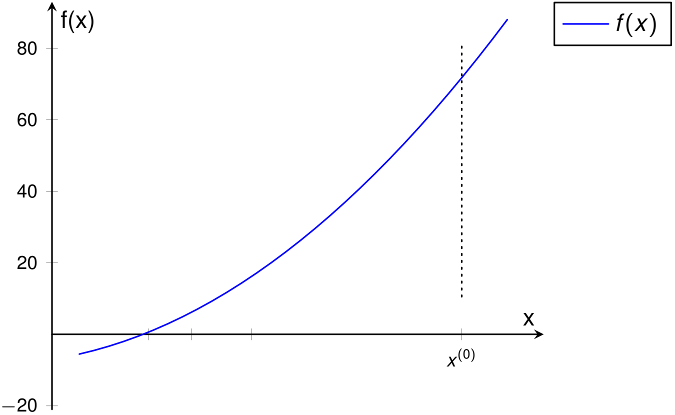
5) Solve $f\left(x^{(\nu)}\right) + f'\left(x^{(\nu)}\right)\Delta x^{(\nu)} = 0$ for $\Delta x^{(\nu)}$ and compute the update for x as

$$x^{(\nu+1)} = x^{(\nu)} + \Delta x^{(\nu)} = x^{(\nu)} - \frac{f\left(x^{(\nu)}\right)}{f'\left(x^{(\nu)}\right)};$$

6) Update iteration counter $\nu + 1 \to \nu$ and go to step 2.

6.1 Newton-Raphson method - Example

- Find the root of $f(x) = 3x^2 + 4x - 7 = 0$
- Initial guess $x^{(0)} = 4.5$
- Estimate after 3 iterations: $x^{(3)} = 1.06$ (exact solution $x^* = 1$)



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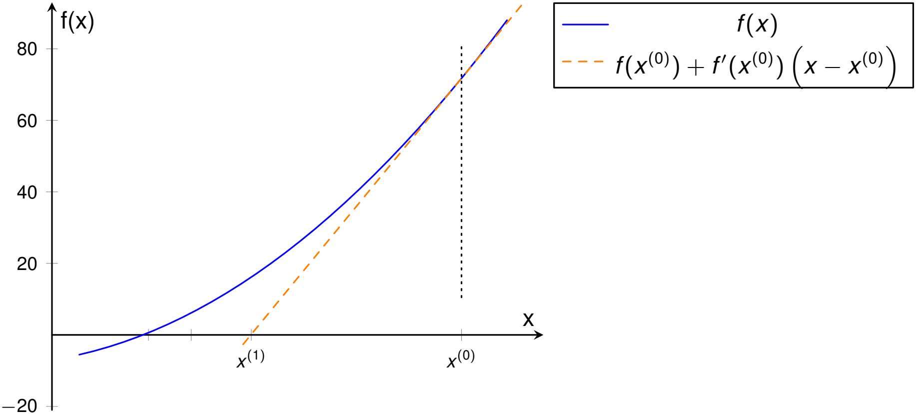

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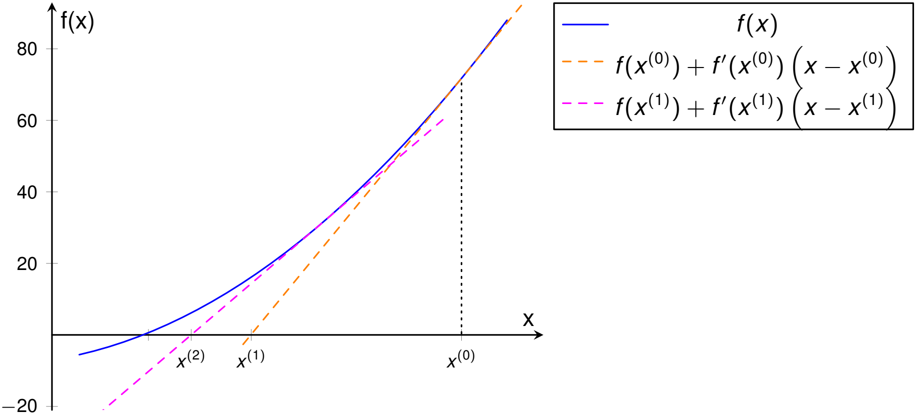

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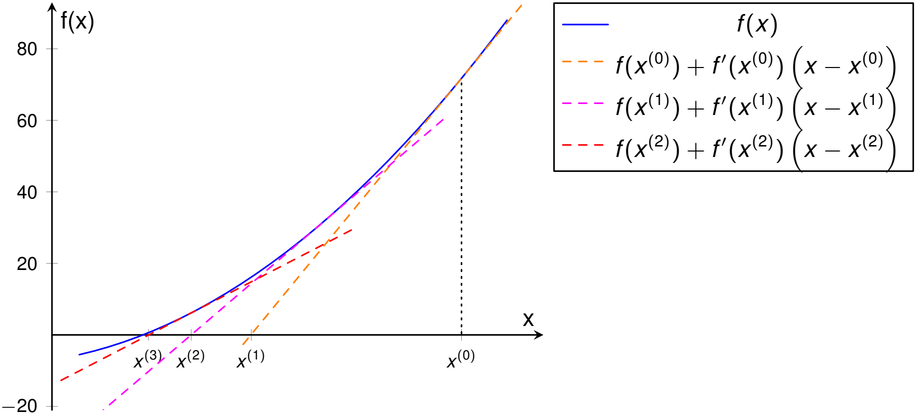

- Convergence rate (locally, i.e., for $x^{(\nu)}$ close enough to actual root x^*)

$$\left|x^* - x^{(\nu+1)}\right| = \frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right| \cdot \left|x^* - x^{(\nu)}\right|^2$$

$\rightarrow$ Locally, Newton-Raphson algorithm converges quadratically (this is fast)

- Dishonest Newton-Raphson algorithm
  - Use *constant* derivative $f'\left(x^{(\nu)}\right) = f'(x^{(0)})$ in each iteration
  - Advantage: no need to recalculate f' for each iteration
  - Disadvantage: number of iterations required for convergence usually higher
  - Overall: often lower computational burden $\rightarrow$ faster convergence
  - May be useful simplification if only limited accuracy of solution required

- Now, we come back to the multi-dimensional case and consider the n-dimensional system of nonlinear equations

$$\mathbf{f}(\mathbf{x}) = \mathbf{0},$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$$

- We assume that all $f_k : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions
- In principle, the same Newton-Raphson algorithm as in the one-dimensional case can be used to compute the roots $\mathbf{x}^*$ of $\mathbf{f}$
- Main modification: replace f' with *Jacobian matrix* $\mathbf{J}$ of $\mathbf{f}$

6.1 Newton-Raphson method - Jacobian matrix

- Jacobian matrix of **f** defined as

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

- Linearisation of **f** at $\mathbf{x}^{(\nu)}$ approximated by first-order *Taylor expansion*

$$\mathbf{f}\left(\mathbf{x}^{(\nu)} + \Delta\mathbf{x}^{(\nu)}\right) \approx \mathbf{f}\left(\mathbf{x}^{(\nu)}\right) + \mathbf{J}\left(\mathbf{x}^{(\nu)}\right)\Delta\mathbf{x}^{(\nu)}$$

- Hence, correction vector $\Delta\mathbf{x}(\nu)$ is solution to

$$\mathbf{f}\left(\mathbf{x}^{(\nu)}\right) + \mathbf{J}\left(\mathbf{x}^{(\nu)}\right)\Delta\mathbf{x}^{(\nu)} = \mathbf{0}$$

- The above *n*-dimensional system of equations is *linear* and usually solved via Gauss elimination (LU factorisation)

6.1 Newton-Raphson method - Algorithm for multi-dimensional case

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1) Set $\nu = 0$ and choose an appropriate starting value $\mathbf{x}^{(0)}$;

2) Compute $\mathbf{f}\left(\mathbf{x}^{(\nu)}\right)$;

3) Compare $\mathbf{f}\left(\mathbf{x}^{(\nu)}\right)$ with tolerance ϵ:

If $|f_k\left(\mathbf{x}^{(\nu)}\right)| \leq \epsilon$ for all $k = 1, \ldots, n$, then $\mathbf{x} = \mathbf{x}^{(\nu)}$ is admissible solution to $\mathbf{f}(\mathbf{x}) = 0$

Otherwise proceed to step 4;

4) Compute Jacobian matrix $\mathbf{J}\left(\mathbf{x}^{(\nu)}\right)$;

5) Update solution

$$\Delta\mathbf{x}^{(\nu)} = -\mathbf{J}^{-1}\left(\mathbf{x}^{(\nu)}\right)\mathbf{f}\left(\mathbf{x}^{(\nu)}\right)$$
$$\mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} + \Delta\mathbf{x}^{(\nu)};$$

6) Update iteration counter $\nu + 1 \rightarrow \nu$ and go to step 2.

6.2 Newton-Raphson method applied to power flow problem (1)

- We now apply the Newton-Raphson method to the power flow problem
- We assume that there are N buses and that bus 1 is the slack bus (Vθ bus)
- Therefore, we define the state vector $\mathbf{x}$ of phase angles and voltage magnitudes

$$\mathbf{x} = \begin{bmatrix} \theta \\ \mathbf{V} \end{bmatrix}$$

- Initial value $\mathbf{x}^{(0)}$ (unless more precise estimate available)
  - At all buses: set voltage phase angle to 0° (=phase angle imposed at slack bus)
  - At all PQ buses: set voltage magnitude to 1pu

6.2 Newton-Raphson method applied to power flow problem (2)

- Next, we order the nonlinear function $\mathbf{f}$ such that the first components correspond to the active power flows and the last ones to the reactive power flows

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \Delta\mathbf{P}(\mathbf{x}) \\ \Delta\mathbf{Q}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \mathbf{P}(\mathbf{x}) - \mathbf{P}^{s} \\ \mathbf{Q}(\mathbf{x}) - \mathbf{Q}^{s} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_2(\mathbf{x}) - \mathbf{P}_2^{s} \\ \vdots \\ \mathbf{P}_N(\mathbf{x}) - \mathbf{P}_N^{s} \\ \mathbf{Q}_2(\mathbf{x}) - \mathbf{Q}_2^{s} \\ \vdots \\ \mathbf{Q}_n(\mathbf{x}) - \mathbf{Q}_n^{s} \end{bmatrix}$$

- $P_k(\mathbf{x})$ and $Q_k(\mathbf{x})$ are the active, respectively reactive, power flow equations at node k
- P_k^s and Q_k^s denote the known active, respectively reactive, power setpoints at node k
- If there are N_{PQ} PQ buses, then $n = N_{PQ} + 1$

- The load flow problem can now be written as

$$\mathbf{f} = \begin{bmatrix} \Delta\mathbf{P}(\mathbf{x}) \\ \Delta\mathbf{Q}(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$

- The functions $\Delta\mathbf{P}(\mathbf{x})$ and $\Delta\mathbf{Q}(\mathbf{x})$ are called *active and reactive (power) mismatches*

- Updates for $\mathbf{x}$ at iteration step $\nu + 1$ are calculated from

$$\mathbf{J}\left(\mathbf{x}^{(\nu)}\right)\begin{bmatrix} \theta^{(\nu)} \\ \mathbf{V}^{(\nu)} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{P}(\mathbf{x}^{(\nu)}) \\ \Delta\mathbf{Q}(\mathbf{x}^{(\nu)}) \end{bmatrix} = \mathbf{0}$$

- Jacobian of $\mathbf{f}$ evaluated at $\mathbf{x}^{(\nu)}$ is given by

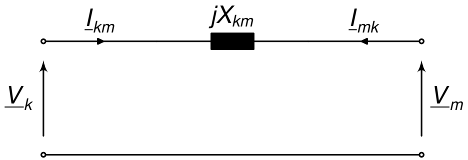
$$\mathbf{J}\left(x^{(\nu)}\right) = \begin{bmatrix} \frac{\partial\Delta\mathbf{P}(\mathbf{x}^{(\nu)})}{\partial\theta} & \frac{\partial\Delta\mathbf{P}(\mathbf{x}^{(\nu)})}{\partial\mathbf{V}} \\ \frac{\partial\Delta\mathbf{Q}(\mathbf{x}^{(\nu)})}{\partial\theta} & \frac{\partial\Delta\mathbf{Q}(\mathbf{x}^{(\nu)})}{\partial\mathbf{V}} \end{bmatrix} = \begin{bmatrix} \frac{\partial\mathbf{P}(\mathbf{x}^{(\nu)})}{\partial\theta} & \frac{\partial\mathbf{P}(\mathbf{x}^{(\nu)})}{\partial\mathbf{V}} \\ \frac{\partial\mathbf{Q}(\mathbf{x}^{(\nu)})}{\partial\theta} & \frac{\partial\mathbf{Q}(\mathbf{x}^{(\nu)})}{\partial\mathbf{V}} \end{bmatrix},$$

  where last equality follows since $\mathbf{P}^s$ and $\mathbf{Q}^s$ are constant vectors

6.3 $P\theta$ and QV decoupling

- Standard power flow problem involves variables at each network node k
  - Voltage amplitude V_k
  - Voltage angle θ_k
  - Net active power P_k
  - Net reactive power Q_k
- Transmission systems exhibit a strong coupling between P and θ as well as V and Q
- This property can be used to simplify and speed-up power flow computations

6.3 Power flows on lossless transmission line



- We consider a Π-model of a transmission line with zero resistance and zero shunt admittance
- For this scenario we already know that the active and reactive power flows are given by

$$P_{km} = \frac{V_k V_m \sin(\theta_{km})}{X_{km}}$$

$$Q_{km} = \frac{V_k^2 - V_k V_m \cos(\theta_{km})}{X_{km}}$$

- The *sensitivities* between power flows P_{km}, Q_{km} and states θ_k, V_k are given by

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{V_k V_m \cos(\theta_{km})}{X_{km}}, \qquad \frac{\partial P_{km}}{\partial V_k} = \frac{V_m \sin(\theta_{km})}{X_{km}}$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = \frac{V_k V_m \sin(\theta_{km})}{X_{km}}, \qquad \frac{\partial Q_{km}}{\partial V_k} = \frac{2V_k - V_m \cos(\theta_{km})}{X_{km}}$$

- For $\theta_{km} = 0$, we observe ideal *decoupling conditions*, i.e.,

$$\left.\frac{\partial P_{km}}{\partial \theta_k}\right|_{\theta_{km=0}} = \frac{V_k V_m}{X_{km}}, \qquad \left.\frac{\partial P_{km}}{\partial V_k}\right|_{\theta_{km=0}} = 0$$

$$\left.\frac{\partial Q_{km}}{\partial \theta_k}\right|_{\theta_{km=0}} = 0, \qquad \left.\frac{\partial Q_{km}}{\partial V_k}\right|_{\theta_{km=0}} = \frac{2V_k - V_m}{X_{km}}$$

$\rightarrow$ For small voltage angles: $P\theta$ and QV decoupling

- Strong coupling between active power and voltage angle, as well as between reactive power and voltage magnitudes
- Weak coupling between active power and voltage magnitudes , as well as between reactive power and voltage angle

6.3 Sensitivities of power flows in transmission systems (1)

- In HV transmission lines usually line inductance dominating component
- In usual transmission system operating conditions $\theta_{km} \ll 90^\circ$

$\rightarrow$ $P\theta$ and QV decoupling true in usual transmission system operation

- Note: for large voltage angles decoupling does NOT hold!
- Example: in neighbourhood of 90° $\sin(\theta_{km}) \approx 0, \quad \cos\theta_{km} \approx 1$

  $\rightarrow$ Strong coupling between Q and V, as well as between P and θ_{km}

6.3 (Fast) Decoupled power flow computation (1)

- Neglecting $P - V$ and $Q - V$ couplings allows to simplify Newton-Raphson scheme

- With decoupling assumption, power flow Jacobian becomes

$$\mathbf{J}(x)_{\text{dec}} = \begin{bmatrix} \frac{\partial \mathbf{P}(\mathbf{x}^{(\nu)})}{\partial \theta} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{Q}(\mathbf{x}^{(\nu)})}{\partial \mathbf{V}} \end{bmatrix}$$

$\rightarrow$ No coupling between updates of voltage magnitudes and phase angles!

- Hence, state updates in Newton-Raphson method can be written as two uncoupled equations

$$\frac{\partial \mathbf{P}}{\partial \theta} \Delta\theta^{(\nu)} + \Delta\mathbf{P}(\theta^{(\nu)}, \mathbf{V}^{(\nu)}) = \mathbf{0}$$
$$\frac{\partial \mathbf{Q}}{\partial \mathbf{V}} \Delta\mathbf{V}^{(\nu)} + \Delta\mathbf{Q}(\theta^{\nu+1}, \mathbf{V}^{(\nu)}) = \mathbf{0}$$

6.3 (Fast) Decoupled power flow computation (2)

- Now, two systems of linear equations have to be solved, while overall number of equations remains the same
- But: usually number of operations to solve a system of linear equations increases more than linearly with the system's dimension

$\rightarrow$ Need less operations to solve decoupled power flow than to solve original coupled power flow

- Convergence of decoupled power flow typically slower than that of original power flow
- Yet, gain in faster updates still yields faster overall solution time of decoupled load flow (if system not too heavily loaded)
- An important note regarding accuracy of the solution
  - No approximations have been made in functions $\mathbf{P}(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$
  - Only way to compute update has been simplified

  $\rightarrow$ If decoupled power flow converges, then it converges to a correct solution of the power balance equations

6.4 DC power flow equations - Motivation

- Thus far, we have used exact expressions of power flow equations
- Yet, as power flow equations are solved very frequently in power system operation and planning it is preferable to also have a set of equations that can be solved very fast
- Such equations can be obtained by approximating the exact power flow equations
- The DC power flow equations are the most common approximation of this type

DC power flow equations are simplified equations obtained after

1. Neglecting reactive power flows in all branches
2. Neglecting active power losses in all branches
3. Assuming all voltage magnitudes equal to 1pu
4. Angle difference between buses is small

6.4 DC power flow equations - Approximate equations (1)

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- Hence, we assume
  1. $V_k = 1$ pu, $k = 1, \ldots, N$
  2. $G_{km} = 0$ for all $k = 1, \ldots, N$, $m = 1, \ldots, N$
  3. $B_{km} = -\frac{1}{X_{km}}$ for all $k = 1, \ldots, N$, $m = 1, \ldots, N$
  4. $a_{km} = 1$ (transformer ratio influences mainly reactive power flows)
- Linearisation of active power flow from node k to node m

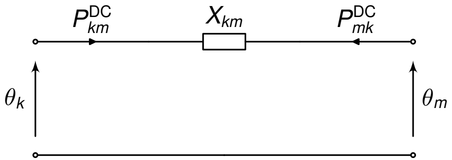
$$
\begin{aligned}
P_{km}^{\text{DC}} &= \frac{\partial P_{km}}{\partial \theta_k}\theta_k + \frac{\partial P_{km}}{\partial \theta_m}\theta_m \\
&= \frac{\cos(\theta_{km})}{X_{km}}\theta_k + \frac{-\cos(\theta_{km})}{X_{km}}\theta_m \\
&= \frac{\cos(\theta_{km})}{X_{km}}\theta_{km} \\
&\approx \frac{1}{X_{km}}\theta_{km} \quad \text{(approximation valid for small phase angle differences)}
\end{aligned}
$$

6.4 DC power flow equations - Approximate equations (2)

- DC power flow equation from node k to node m

$$P_{km}^{\mathrm{DC}} = \frac{\theta_{km}}{X_{km}}$$

- DC power flow equation analogous to Ohm's law applied to a resistor
  - P_{km}^{DC} is DC current
  - θ_k and θ_m are DC voltages at resistor terminals
  - X_{km} is resistance



6.4 DC power flow equations - Approximate equations (3)

- DC power flow at node k

$$P_k^{\mathrm{DC}} = \sum_{m \in \mathcal{N}_k} \frac{\theta_{km}}{X_{km}}$$

- Active power losses neglected

$$\sum_{k=1}^{N} P_k^{\mathrm{DC}} = 0 \quad \Leftrightarrow \quad P_1 = -\sum_{k=2}^{N} P_k^{\mathrm{DC}}$$

$\rightarrow$ No slack bus needed to compensate for unknown active power losses (but still need angle reference bus)

- DC power flow equations can be extended to include phase shifters (not done here)

6.4 DC power flow equations - Matrix form

- DC power flow can be written in matrix form as follows

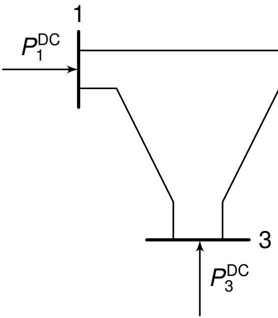
$$\mathbf{P}^{\mathrm{DC}} = \mathbf{A}'\theta,$$

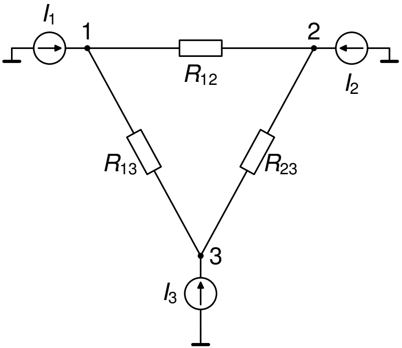
  where
  - $\mathbf{P}$ is vector of net active power injections
  - θ is vector of voltage angles
  - $\mathbf{A}'$ is nodal admittance matrix with elements

$$\mathcal{A}_{km} = -X_{km}^{-1}$$
$$\mathcal{A}_{kk} = \sum_{m \in \mathcal{N}_k} X_{km}^{-1}$$

- Matrix $\mathbf{A}'$ is singular $\rightarrow$ no unique solution for θ
- To make system of equations solvable, need to (arbitrarily) chose one bus as angle reference and remove row and column associated with that bus from $\mathbf{A}'$; we shall call that reduced matrix $\mathbf{A}$

6.4 Circuit interpretation of DC power flow model





- Linearised model $\mathbf{P}^{DC} = \mathbf{A}'\theta$ can be interpreted as network of resistors fed by DC current sources
- Then $\mathbf{P}^{DC}$ are the nodal DC current injections, θ the nodal DC voltages and $\mathbf{A}'$ is the nodal conductance matrix
- Angle reference bus = DC voltage reference bus = Bus 1

6.4 Summary

- Power flow computations are a fundamental tool in power system operation, analysis and planning
- Formulating the power flow problem requires
  - Identifying the bus type of each node in the network (PQ, PV, Vθ)
  - Setting up the power balance equations
  - Determining necessary constraints
- Power flow equations can usually only be solved numerically
- There are many software tools available to do this
- The Newton-Raphson method is a standard numerical method to solve the power flow equations
- Computation of power flow equations can also be simplified via
  - Pθ and QV decoupling
  - DC power flow