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EEN320 - Power Systems I (Συστήματα Ισχύος Ι)

Part 8: The transmission line

https://sps.cut.ac.cy/courses/een320/

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Today's learning objectives

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After this part of the lecture and additional reading, you should be able to . . .

1. . . . name the main components of an overhead line and describe their functionality;
2. . . . understand how to derive the Π-equivalent circuit of a transmission line;
3. . . . determine the parameters of the Π-equivalent circuit of a transmission line from its concentrated parameters;
4. . . . explain how and under which conditions or assumptions the Π-equivalent circuit can be further simplified.

Outline

1. **Physical relevance of power lines**
2. **Structure of overhead lines**
   - Conductors
   - Support structures
   - Insulators
   - Shield wires
3. **Derivation of lumped inductor and capacitor values**
   - Inductance
   - Capacitance
4. **Overhead line parameters**
   - Concentrated parameters
   - Some brief remarks on cables
5. **Equivalent circuits for power lines**
   - Differential equation of a power line
   - Solution of differential equation of a power line
   - Π-equivalent circuit
   - Model simplifications and their validity

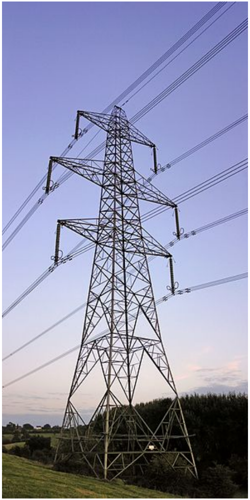
1 Outline

1. **Physical relevance of power lines**
2. Structure of overhead lines
3. Derivation of lumped inductor and capacitor values
4. Overhead line parameters
5. Equivalent circuits for power lines

1 Power lines - Overview

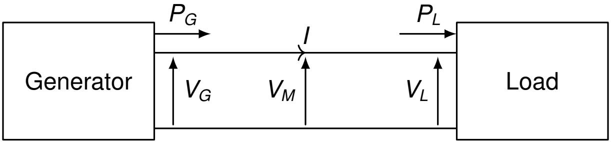
- Task: Transport electricity
- 2 main types:
  - Overhead line (OL)
  - Cable
- OLs and cables possess different structure and operational properties
- At same voltage level, costs for cables approx. 10-20 times than costs of OLs

$\rightarrow$ OLs economically more viable option



Overhead power line in Gloucestershire, England ©Yummifruitbat

1 Recap: Motivation for high voltage transmission (1)



Simplified DC transmission system

- Line is resistive $\rightarrow$ voltage drop across line $\rightarrow$ $V_G > V_M > V_L$
- Average power transmitted over line: $P_{trans} = V_M I$
  (V_M is voltage at middle of line length)
- Denote total line resistance by R $\rightarrow$ line losses given by

$$P_{loss} = RI^2 = R\left(\frac{P_{trans}}{V_M}\right)^2$$

- Ratio of power losses to transmitted power

$$\frac{P_{loss}}{P_{trans}} = R\frac{P_{trans}}{V_M^2}$$

1 Recap: Motivation for high voltage transmission (2)

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- Ratio of power losses to transmitted power

$$\frac{P_{loss}}{P_{trans}} = R\frac{P_{trans}}{V_M^2}$$

→ Power losses inversely proportional to square of operational voltage V_M^2

- Power lines usually operated at high voltage
- However, higher voltage means higher insulation of components

→ Higher costs

1 Costs vs. transmission voltage

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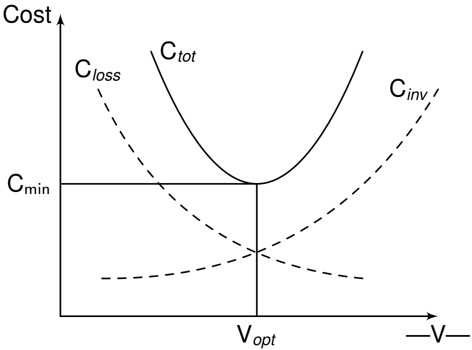


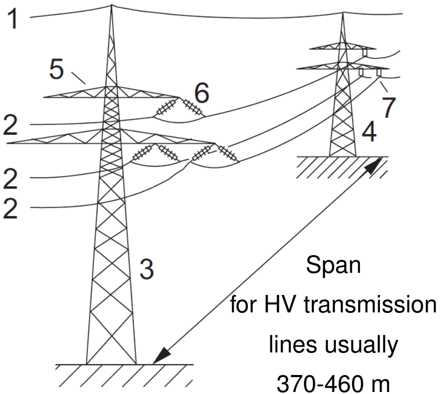
- Total costs $C_{tot}=C_{loss}+C_{inv}$
- Minimum costs C_{min} $\rightarrow$ economically optimal operating voltage

2 Outline

1. Physical relevance of power lines
2. **Structure of overhead lines**
   - Conductors
   - Support structures
   - Insulators
   - Shield wires
3. Derivation of lumped inductor and capacitor values
4. Overhead line parameters
5. Equivalent circuits for power lines

An overhead line consists of

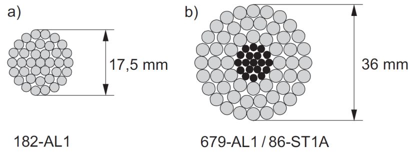
- Conductors (2)
- Support structures
  - Towers (and poles) (3,4)
  - Traverse (5)
- Shield wires (1)
- Insulators (6,7)
  - Strain-type insulator (6)
  - Suspension-type insulator (7)



Source: K. Heuck et al., ”Elektrische Energieversorgung”, Springer, 2013

2.1 Conductors - Overview

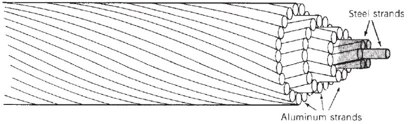
- Aluminium is most common conductor metal
- Copper also used, but less frequently as heavier and more expensive
- Mechanical strain acting on conductors limits span between towers (line sag)
- For short lines, conductors of pure aluminium strands may be used (Figure (a))
- For longer lines, conductors are *reinforced* with central steel strands (Figure (b))



(a) (b)

Source: K. Heuck et al., "Elektrische Energieversorgung", Springer, 2013

Strands are usually twisted to reduce Eddy currents



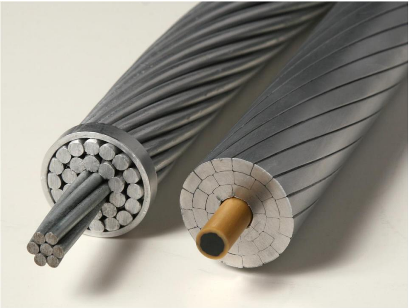
Source: J. Duncan Glover et al., "Power System Analysis & Design", Cengage Learning, 2008

2.1 Conductors - Types

- Common types of conductors
  - Aluminium conductor steel-reinforced (ACSR)
  - All-Aluminium conductor (AAC)
  - All-Aluminium alloy conductor (AAAC)
  - Aluminium conductor composite reinforced (ACCR)
  - Aluminium conductor composite core (ACCC)
- Conductors labeled based on cross section (in mm^2) of aluminium and core strand

  Example: 243-AL1/39-ST1A

  (code after AL and ST denotes finishing properties of AL and ST)



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Standard round-wire ACSR (left) and ACCC with trapezoidal wires (right)

ACCR and ACCC use carbon and glas fiber core $\rightarrow$ up to 10 times lower thermal expansion coefficient than steel $\rightarrow$ can use more aluminium $\rightarrow$ reduced line losses

2.1 Conductors - Bundles

- Each phase of a three-phase transmission line consists of one or more conductors
- More than one conductor/phase $\rightarrow$ bundled conductor
- Advantages:
  - Smaller series resistance
  - Reduced electric field strength at conductor surface $\rightarrow$ reduced Corona effect
- Transmission line may also consist of several three-phase systems in parallel

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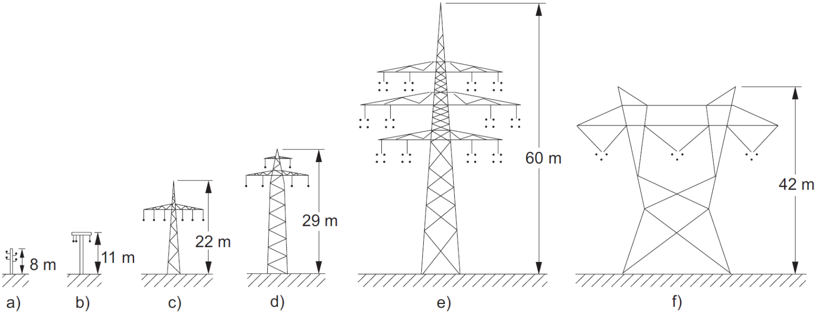
Triple-circuit 400 kV overhead line with 4 conductors per phase

2.2 Support structures - Towers and poles

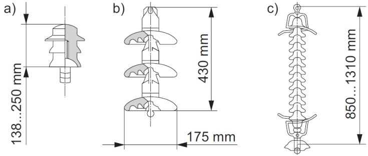


Source: K. Heuck et al., ”Elektrische Energieversorgung”, Springer, 2013

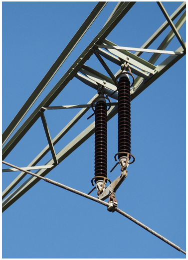
- Large variety of support structures
- Poles made of wood or concrete [a) and b)] used for voltages $\leq$ 110 kV
- Self-supporting lattice steel towers [c) - f)] used for voltages $\geq$ 110 kV

2.3 Insulators

- Need to insulate ”live” conductors from tower
- Pin-type insulators (for lower voltages < 60 kV); material: porcelain; Figure (a)
- Suspension-type insulators (for voltages > 60 kV)
  - Suspension disc insulator; material: glas; Figure (b)
  - Long-rod insulator; material: porcelain; Figure (c)
    (Strain-type insulator some times also used)
- To prevent sparkovers, insulators need to be sufficiently long (approx. 1.5 cm/kV) and possess appropriate shape to minimise leakage currents



Source: K. Heuck et al., ”Elektrische Energieversorgung”, Springer, 2013



110 kV double long-rod suspension string
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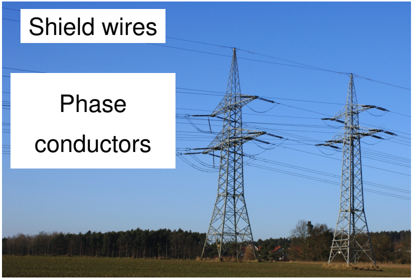
2.4 Shield wires

- Shield wires located above phase conductors to provide protection against lightning
- Shield wires are grounded to tower

→ They also serves as parallel path with Earth for fault currents

- Predominantly used above 110 kV
- Much smaller cross section than phase conductors
- Modern shield wires contain optical fibres for communication/control
- Usually, 1-2 shield wires used



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3 Outline

1. Physical relevance of power lines
2. Structure of overhead lines
3. **Derivation of lumped inductor and capacitor values**
   - Inductance
   - Capacitance
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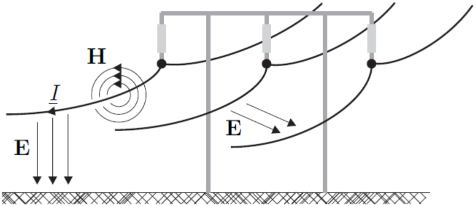
Magnetic and electric fields of conducting power line

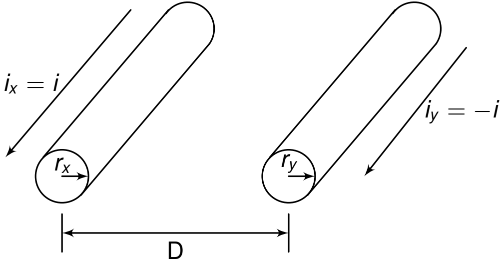


E. . . electric field

H. . . magnetic field

- Due to the alternating current, there is an alternating magnetic field in each line affecting neighbouring lines.
- Similarly, there is an electric field between lines and from lines to the ground.
- The parameters of the line model are dictated by the line material (ohmic losses) as well as the magnetic and electric fields.

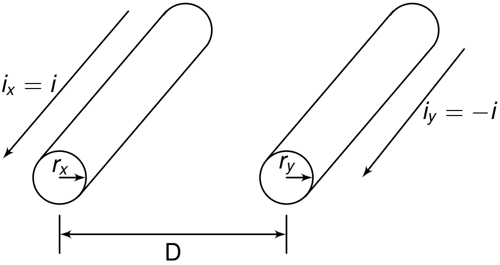
3.1 Inductance of a single-phase two-wire line (1)



- r_x, r_y: radius of cylindrical conductors
- D: spacing between conductors
- i: current flowing in conductors
- **Assumptions**: Conductors are of infinitely length, non-magnetic ($\mu = \mu_0$)[1] and have uniform current density (skin-effect neglected)

[1] μ_0 is vacuum permeability constant

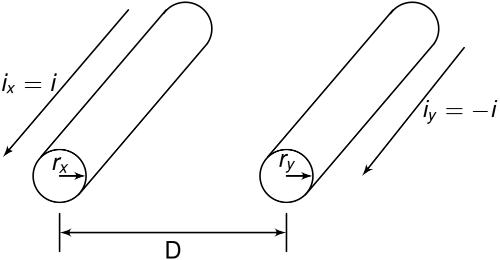
3.1 Inductance of a single-phase two-wire line (2)



- Inductance of one conductor $k = \{x, y\}$

$$L'_k = \frac{\mu_0}{2\pi} \ln\left(\frac{D}{r'_k}\right) = 2 \cdot 10^{-7} \ln\left(\frac{D}{r'_k}\right) \text{ [H/m]} \qquad r'_k = r_k e^{-\frac{1}{4}} \approx 0.778r$$

$$\mu_0 = 4\pi \cdot 10^{-7}$$



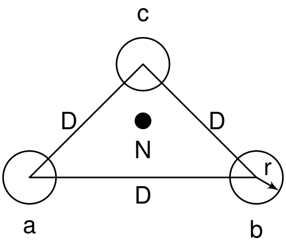
- Total inductance of single-phase two-wire line

$$L' = L'_x + L'_y = 2 \cdot 10^{-7} \left(\ln\left(\frac{D}{r'_x}\right) + \ln\left(\frac{D}{r'_x}\right) \right)$$

$$= 2 \cdot 10^{-7} \ln\left(\frac{D^2}{r'_x r'_y}\right) = 4 \cdot 10^{-7} \ln\left(\frac{D}{\sqrt{r'_x r'_y}}\right) \text{ [H/m]}$$

- Identical conductors ($r_x = r_y$): $L' = 4 \cdot 10^{-7} \ln\left(\frac{D}{r'}\right)$ [H/m]

3.1 Inductance of a three-phase three-wire line



- Assumptions: balanced phase currents, equidistant spacing D, identical conductor radii r
- Line-neutral inductances of three-phase three-wire line

$$L' = L'_a = L'_b = L'_c = 2 \cdot 10^{-7} \ln\left(\frac{D}{r'}\right) \text{ [H/m]}$$

- This is half the inductance of a single-phase two-wire line!
- Inductances balanced $\rightarrow$ can use single-phase equivalent circuit for network calculations

3.1 Inductance of a three-phase three-wire line - Transposition of conductors

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- In practice, conductors rarely spaced in equidistant manner

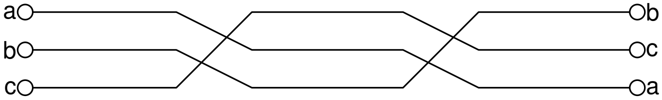
$\rightarrow$ Inductances become unbalanced ($L_a \neq L_b \neq L_c$) $\rightarrow$ this causes unbalanced voltage drops even if currents are balanced!

- Practical remedy: restore balance by exchanging conductor positions along line (e.g. at substations)
- This is called *transposition*
- For transposed line with equivalent spacing $D_{eq} = \sqrt[3]{D_{12} D_{23} D_{31}}$

$$L' = L'_a = L'_b = L'_c = 2 \cdot 10^{-7} \ln\left(\frac{D_{eq}}{D_S}\right) \text{ [H/m]}$$

$D_S \ldots$ Geometric Mean Radius (GMR) for stranded conductors

$D_S = r'$ for solid cylindrical conductors



3.1 Example: Determine inductance of a three-phase three-wire line

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Task. A completely transposed 50-Hz three-phase line has flat horizontal phase spacing with 10m between adjacent conductors. The geometric mean radius (GMR) of the conductors is 0.0159m. The line length is $\ell = 200$km. Determine the inductance in H and the inductive reactance in Ω.

Solution. We have that

$$D_{eq} = \sqrt[3]{10 \cdot 10 \cdot 20} = 12.6\ m$$

Hence,

$$L' = 2 \cdot 10^{-7} \ln \left(\frac{D_{eq}}{D_S} \right) = 2 \cdot 10^{-7} \ln \left(\frac{12.6}{0.0159} \right) = 1.335\mu \text{ H/km}$$

and

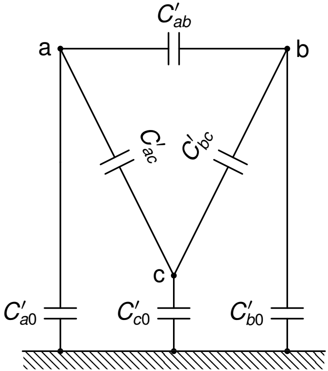
$$L = L' \cdot \ell = 1.335 \cdot 10^{-6} \cdot 200 = 0.267 \text{ H},$$

as well as

$$X = L\omega = 0.267 \cdot 2 \cdot \pi \cdot 50 = 83.88\ \Omega$$

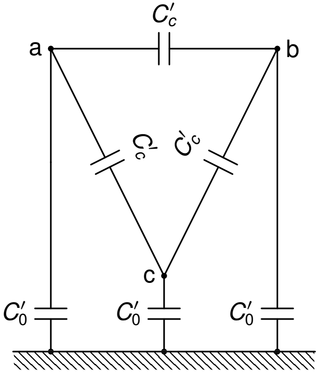
3.2 Capacitance - Coupling and earth capacitances

- Line capacitance can be obtained in similar fashion to inductances
- Need to consider interaction of electric fields between conductors *and* between individual conductors and earth
- This can be modelled via coupling and earth capacitances

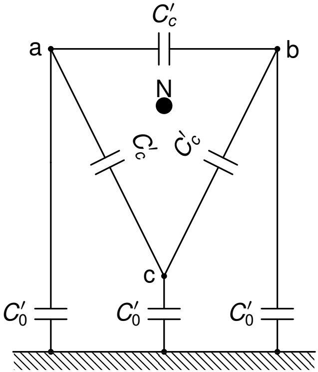


3.2 Capacitance - Balanced three-phase three-wire line (1)

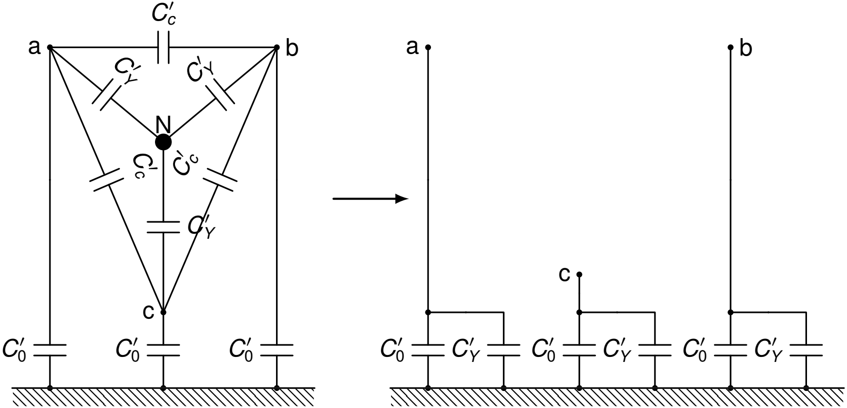
- Assume balanced line (e.g. via transposition)
- Then, $C'_0 = C'_{a0} = C'_{b0} = C'_{c0}$ and $C'_c = C'_{ab} = C'_{ac} = C'_{bc}$
- Coupling conductors C'_c form Δ-connection



- Assume balanced line (e.g. via transposition)
- Then, $C'_0 = C'_{a0} = C'_{b0} = C'_{c0}$ and $C'_c = C'_{ab} = C'_{ac} = C'_{bc}$
- Coupling conductors C'_c form Δ-connection
- Introduce *fictitious* neutral point N



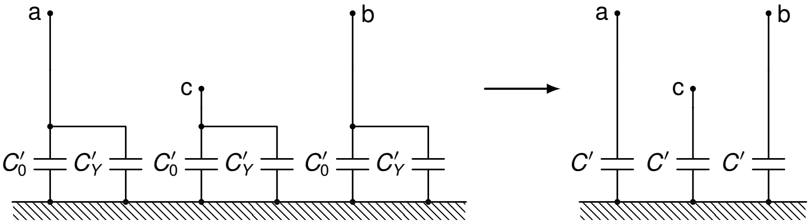
- Balanced conditions $\rightarrow$ sum of currents at N equal zero $\rightarrow$ N has same potential as ground
- Parallel connection of coupling and earth capacitances $C'_Y = 3C'_c$



- Balanced conditions $\rightarrow$ sum of currents at N equal zero $\rightarrow$ N has same potential as ground
- Parallel connection of coupling and earth capacitances $C' = C'_Y + C'_0$

$$C' = \frac{2\pi\varepsilon_0}{\ln\left(\frac{D_{eq}}{r}\right)} \text{ [F/m]} \qquad \varepsilon_0 \ldots \text{vacuum permittivity}$$

- Typical value for overhead lines $C' \approx 10$ nF/km



Note: similar calculations applicable to conductor bundles

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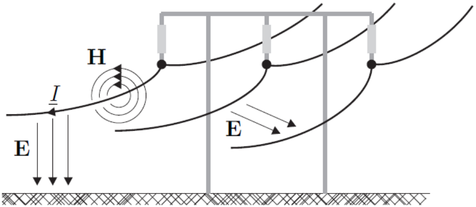
   - Concentrated parameters
   - Some brief remarks on cables
5. Equivalent circuits for power lines

Magnetic and electric fields of conducting power line

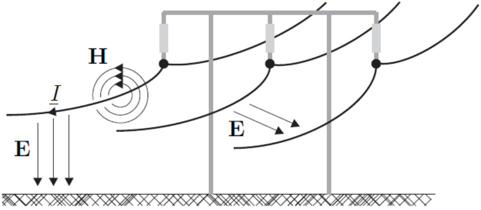


E. . . electric field

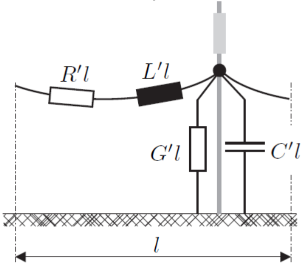
H. . . magnetic field

- Each power line has characteristic line parameters
- Parameters dependent on line geometry and material
- Parameters often indicated in [unit]/km and by giving the line length ℓ

Magnetic and electric fields of conducting power line



Power line model with concentrated parameters



- Line resistance R' [Ω/km] ↔ Ohmic resistance of conductor
- Line inductance L' [H/km] ↔ Magnetic field of conductor
- Capacitance C' [F/km] ↔ Electric field of conductor
- Shunt conductance G' [S/km] ↔ Leakage currents at insulators

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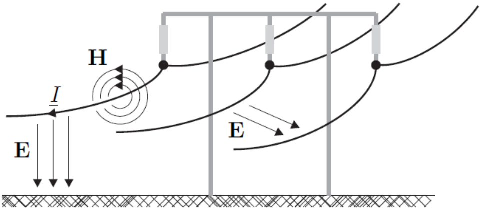
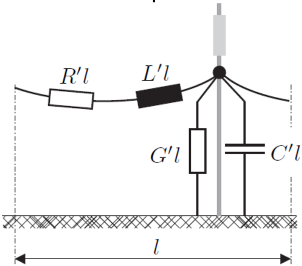
Magnetic and electric fields of conducting power line



Power line model with concentrated parameters



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- For performing circuit analysis involving power lines (e.g. to determine the network conditions or design) we need to know the concentrated parameters of the lines
- Usually, concentrated parameters indicated by manufacturer

Please see the course book for a detailed derivation.

4.1 Resistance - AC vs. DC resistance

- Real conductors are not lossless!

→ This can be accounted for by including a series resistance in the conductor model

- For DC current, resistance of conductor can easily be determined from its diameter, length and specific conductivity
- For AC current, in addition the *skin effect* needs to be considered when determining the resistance of a conductor
- Skin-effect: current not distributed homogeneously over conductor diameter, but concentrated towards conductor boundaries

→ Current density increases towards conductor boundaries

→ Effective diameter of conductor is reduced and, hence, ohmic resistance is increased compared to DC resistance (typically by a few percent)

4.1 Resistance - Calculation

- For steel-reinforced aluminium conductors (ACSR), AC resistance is approximately same as DC resistance
- Reason: Skin-effect $\rightarrow$ reduced AC current in steel strands $\rightarrow$ increase in AC resistance by skin-effect comparable to higher DC current in steel strands
- Conductor losses result in heat dissipation $\rightarrow$ maximum conductor current limited, as long-term high temperatures ($> 80^\circ$) decrease mechanical strength of conductor material $\rightarrow$ line sags
- Line resistance operating at temperature of ϑ° can be calculated via

$$
\begin{aligned}
R' &= R'_{20}(1 + \alpha(\vartheta - 20^\circ\mathrm{C}))\ [\mathrm{R/m}] \\
R'_{20} &= \frac{\rho_{20}}{A} \text{ resistance of conductor at } 20^\circ\mathrm{C} \\
\rho_{20} &\ldots \text{specific resistance of of conductor material at } 20^\circ\mathrm{C} \\
A &\ldots \text{effective conductor area}
\end{aligned}
$$

- For practical conductors, resistance values obtained via measurements

- Also, losses due to insulator leakage currents and corona
- Corona: high value of electric field strength at conductor surface causes air to become electrically ionised and to conduct
- Corona losses dependent on meteorological conditions (rain; humidity) and conductor surface irregularities
- For overhead lines, conductance G' can only be estimated from measurements, while it can be determined experimentally for cables
- Usually, conductance is very small and therefore most often neglected in power system studies

4.2 Cables vs. overhead lines

- Cables mostly used at low voltage levels ($<$110 kV)
- Often installed underground
- Physical characteristics of cables fundamentally different from overhead transmission lines (OHLs)!
- Main reasons:
  - Distance between conductors as well as between conductors and earth much smaller in cables than in OHLs
  - Conductors in cables typically surrounded by other metallic materials, e.g. skin
  - Insulation material of OHLs is air, while in cables materials such as paper, oil or SF_6 are used
- Consequences:
  - Inductance of OHLs usually higher as that of cables
  - Capacitance of cables usually much higher as that of OHLs

4.2 Cables vs. overhead lines - Typical parameters

- Typical values for parameters of OHLs at 50 Hz

Rated voltage in kV	230	345	500	765
R' [Ω/km]	0.050	0.037	0.028	0.012
$X'_L = \omega L'$ [Ω/km]	0.407	0.306	0.271	0.274
$Y'_C = \omega C'$ [μS/km]	2.764	3.765	4.333	4.148

- Typical values for parameters of cables at 50 Hz

Rated voltage in kV	115	230	500
R' [Ω/km]	0.059	0.028	0.013
$X'_L = \omega L'$ [Ω/km]	0.252	0.282	0.205
$Y'_C = \omega C'$ [μS/km]	192.0	204.7	80.4

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   - Π-equivalent circuit
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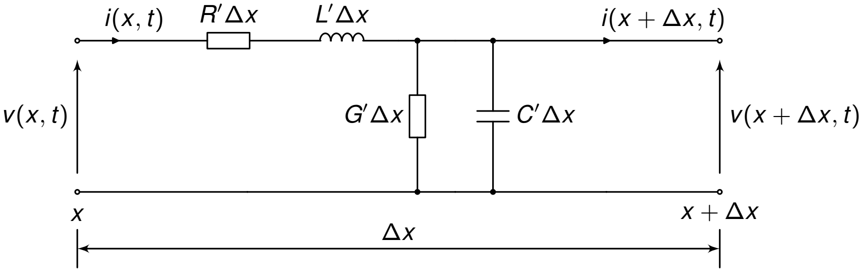
5 Why do we need power line models?

- Being able to describe the behaviour of a power systems by a mathematical model is a fundamental prerequisite for network planning and operation
- We will derive a model of a power line that is valid under *stationary* (or steady-state) conditions
- Note: A real power system is never exactly in steady-state due to continuous variations of load and generation
- However, under normal conditions this variations are of small magnitude compared to overall power flows in network
- Also, normal load patterns change over fairly long period (several tens of minutes)

$\rightarrow$ Steady-state model suitable for describing nominal network operating conditions

5.1 Differential equation of a power line - Line section

Section of length Δx of homogeneous power line



- Line parameters R', L', G' and C' are not lumped, but (uniformly) distributed along length of line; Δx denotes a small distance
- Propagation of current $i(t,x)$ and voltage $v(t,x)$ across that line segment is not instantaneous
- Propagation can be described by a partial-differential equation (i.e. propagation depends on time t and location x)

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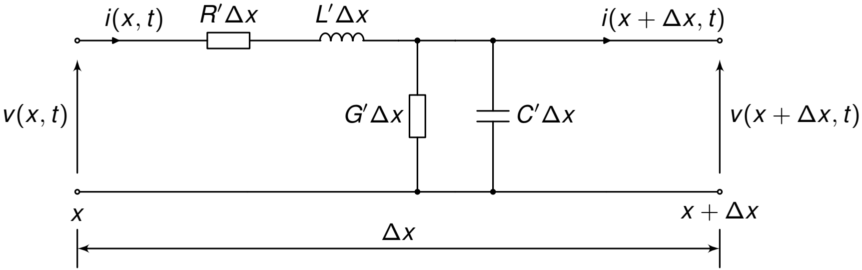
Section of length Δx of homogeneous power line

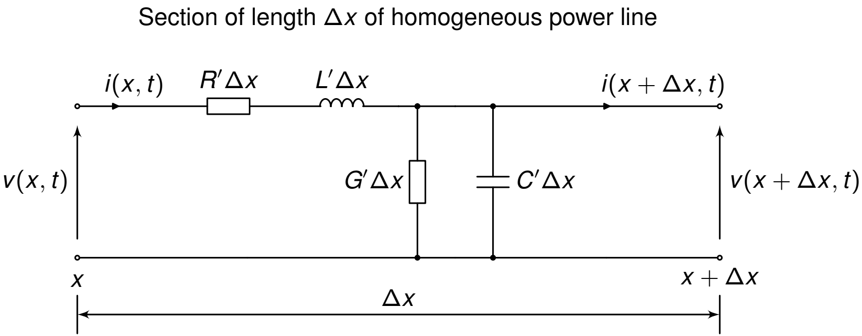


- From KVL

$$v(x+\Delta x,t) = v(x,t) - R'\Delta x i(x,t) - L'\Delta x \frac{\partial i(x,t)}{\partial t}$$

- From KCL

$$i(x+\Delta x,t) = i(x,t) - G'\Delta x v(x+\Delta x,t) - C'\Delta x \frac{\partial v(x+\Delta x,t)}{\partial t}$$



- For infinitesimally small section length $\Delta x \to 0$, previous equations are equivalent to

$$\frac{\partial v}{\partial x} = -\left(R' + L'\frac{\partial}{\partial t}\right) i$$

$$\frac{\partial i}{\partial x} = -\left(G' + C'\frac{\partial}{\partial t}\right) v$$

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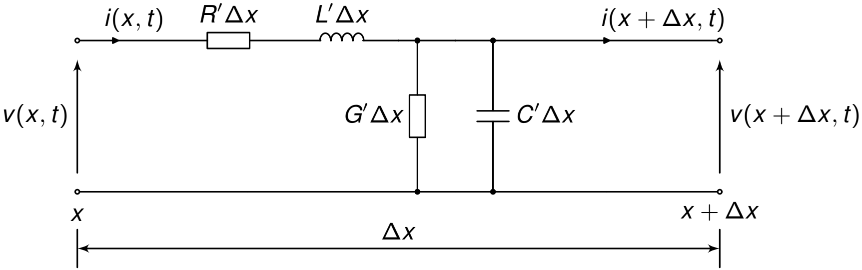


- Decouple equations by differentiating first wrt x and second wrt t and insert resulting expressions in equations (derived by Maxwell around 1860)

$$\boxed{\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= R'G'v + \left(R'C' + L'G'\right)\frac{\partial v}{\partial t} + L'C'\frac{\partial^2 v}{\partial t^2} \\ \frac{\partial^2 i}{\partial x^2} &= R'G'i + \left(R'C' + L'G'\right)\frac{\partial i}{\partial t} + L'C'\frac{\partial^2 i}{\partial t^2} \end{aligned}}$$

5.2 Introducing phasors to solve telegrapher's equations

- In power systems, we are mostly interested in solving telegrapher's equations for special case of *sinusoidal excitation*
- For that case, voltage $v(x,t)$ and current $i(x,t)$ can be represented as phasors with complex amplitudes $\underline{V}$ and $\underline{I}$ and frequency $\omega = 2\pi f$:

$$u(x,t) = \Re\left(\underline{V}(x)e^{j\omega t}\right)$$
$$i(x,t) = \Re\left(\underline{I}(x)e^{j\omega t}\right)$$

- By using phasors, telegrapher's equations reduce to two linear first-order differential equations

$$\frac{d\underline{V}}{dx} = -(R' + j\omega L')\underline{I}$$
$$\frac{d\underline{I}}{dx} = -(G' + j\omega C')\underline{V}$$

- Eliminating $\underline{I}(x)$ leaves us with a linear homogeneous second-order differential equation, which is called *wave equation*

$$\boxed{\frac{d^2\underline{V}}{dx^2} = (R' + j\omega L')(G' + j\omega C')\underline{V}}$$

- Note: introduction of phasors transforms partial differential equation in ordinary differential equation (i.e. in one variable)

5.2 Solution of the wave equation - Voltage

- The solution of the wave equation can be computed as

$$\boxed{\underline{V}(x) = \underline{V}^{+} e^{-\underline{\gamma} x} + \underline{V}^{-} e^{\underline{\gamma} x}}$$

- $\underline{V}^{+}$ and $\underline{V}^{-}$ are integration constants
- $\underline{\gamma}$ is called *propagation constant*

$$\boxed{\underline{\gamma} = \sqrt{(R' + j\omega L')(G' + j\omega C')}}$$

5.2 Solution of the wave equation - Interpretation

- Writing the solution of the wave equation as a function of time, we obtain

$$v(x,t) = \Re\left(\underbrace{\underline{V}^{+} e^{-\underline{\gamma} x} e^{j\omega t}}_{\text{forward travelling wave}} + \underbrace{\underline{V}^{-} e^{\underline{\gamma} x} e^{j\omega t}}_{\text{backward travelling wave}} \right)$$

- Forward travelling (voltage) wave moves in positive x-direction
- Backward travelling (voltage) wave moves in negative x-direction (also called *reflected* wave)
- Complex propagation constant $\underline{\gamma}$ can be split in real and imaginary part

$$\underline{\gamma} = \alpha + j\beta$$

- α describes damping of (voltage) wave and is measured in Nepers per unit length [2]
- β describes phase of (voltage) wave at distance x from origin and is measured in radians per unit length

[2] Np=Neper is a logarithmic unit to measure physical field quantities.

5.2 Solution of the wave equation - Current

- By differentiating $\underline{V}(x)$ we obtain

$$\frac{d\underline{V}}{dx} = -\underline{\gamma}\underline{V}^{+} e^{-\underline{\gamma} x} + \underline{\gamma}\underline{V}^{-} e^{\underline{\gamma} x}$$

and, hence,

$$\underline{I}(x) = \frac{1}{-(R' + j\omega L')}\frac{d\underline{V}}{dx} = \frac{-\underline{\gamma}\underline{V}^{+} e^{-\underline{\gamma} x} + \underline{\gamma}\underline{V}^{-} e^{\underline{\gamma} x}}{-(R' + j\omega L')}$$

$$= \sqrt{\frac{G' + j\omega C'}{R' + j\omega L'}}\left(\underline{V}^{+} e^{-\underline{\gamma} x} - \underline{V}^{-} e^{\underline{\gamma} x}\right)$$

- Define *surge impedance* (also called characteristic impedance)

$$\boxed{\underline{Z}_w = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}}$$

$$\Rightarrow \quad \boxed{\underline{I}(x) = \frac{1}{\underline{Z}_w}\left(\underline{V}^{+} e^{-\underline{\gamma} x} - \underline{V}^{-} e^{\underline{\gamma} x}\right)}$$

- Boundary conditions at beginning of line ($x = 0$)

$$\underline{V}(0) = \underline{V}_1 \qquad \underline{I}(0) = \underline{I}_1$$

- Inserting these values in solutions for $\underline{V}(x)$ and $\underline{I}(x)$ at $x = 0$ yields

$$\underline{V}_1 = \underline{V}^+ + \underline{V}^-$$
$$\underline{I}_1 = \frac{\underline{V}^+ - \underline{V}^-}{\underline{Z}_W}$$

- Solving for $\underline{V}^+$ and $\underline{V}^-$, we obtain

$$\underline{V}^+ = \frac{\underline{V}_1 + \underline{Z}_W \underline{I}_1}{2}$$
$$\underline{V}^- = \frac{\underline{V}_1 - \underline{Z}_W \underline{I}_1}{2}$$

- Substituting expressions for $\underline{V}^+$ and $\underline{V}^-$ in equations for $\underline{V}(x)$ and $\underline{I}(x)$ yields

$$\underline{V}(x) = \left(\frac{\underline{V}_1 + \underline{Z}_W \underline{I}_1}{2}\right) e^{-\underline{\gamma} x} + \left(\frac{\underline{V}_1 - \underline{Z}_W \underline{I}_1}{2}\right) e^{\underline{\gamma} x}$$
$$= \left(\frac{e^{\underline{\gamma} x} + e^{-\underline{\gamma} x}}{2}\right) \underline{V}_1 - \underline{Z}_W \underline{I}_1 \left(\frac{e^{\underline{\gamma} x} - e^{-\underline{\gamma} x}}{2}\right)$$

$$\underline{I}(x) = \left(\frac{\underline{V}_1 + \underline{Z}_W \underline{I}_1}{2\underline{Z}_W}\right) e^{-\underline{\gamma} x} - \left(\frac{\underline{V}_1 - \underline{Z}_W \underline{I}_1}{2\underline{Z}_W}\right) e^{\underline{\gamma} x}$$
$$= -\frac{\underline{V}_1}{\underline{Z}_W} \left(\frac{e^{\underline{\gamma} x} - e^{-\underline{\gamma} x}}{2}\right) + \left(\frac{e^{\underline{\gamma} x} + e^{-\underline{\gamma} x}}{2}\right) \underline{I}_1$$

- We can recognise the hyperbolic functions cosh and sinh

$$\cosh(\underline{\gamma} x) = \frac{e^{\underline{\gamma} x} + e^{-\underline{\gamma} x}}{2}, \quad \sinh(\underline{\gamma} x) = \frac{e^{\underline{\gamma} x} - e^{-\underline{\gamma} x}}{2}$$

- Using cosh and sinh gives compact expressions, we obtain equations for propagation of voltage and current from beginning of line

$$\boxed{\begin{aligned} \underline{V}(x) &= \cosh(\underline{\gamma}x)\underline{V}_1 - \underline{Z}_W \sinh(\underline{\gamma}x)\underline{I}_1 \\ \underline{I}(x) &= -\frac{\underline{V}_1}{\underline{Z}_W}\sinh(\underline{\gamma}x) + \cosh(\underline{\gamma}x)\underline{I}_1 \end{aligned}}$$

- In same way, we can define boundary conditions at end of line ($x = \ell$)

$$\underline{V}(\ell) = \underline{V}_2 \qquad \underline{I}(\ell) = \underline{I}_2$$

and obtain equations for propagation of voltage and current from end of line

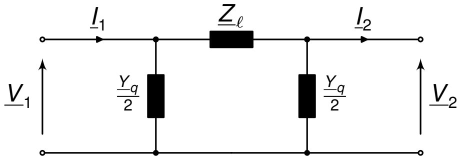
$$\boxed{\begin{aligned} \underline{V}(x) &= \cosh(\underline{\gamma}(\ell - x))\underline{V}_2 + \underline{Z}_W \sinh(\underline{\gamma}(\ell - x))\underline{I}_2 \\ \underline{I}(x) &= \frac{\underline{V}_2}{\underline{Z}_W}\sinh(\underline{\gamma}(\ell - x)) + \cosh(\underline{\gamma}(\ell - x))\underline{I}_2 \end{aligned}}$$

5.3 Simplified models of a power line

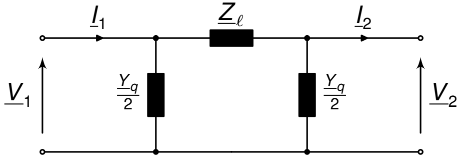
- In practice, we often don't need to use the (rather complicated) wave equation to describe phenomena in power systems
- Reason: Usually, we are interested in the voltage drop across a line or the reactive power flow, but not in the exact trajectory of the voltages and currents along the line

→ Then, we may use simplified models for a power line without compromising the accuracy of our calculations too much

- We will discuss such models in the following
- In particular, we will derive the Π-equivalent circuit of a transmission line



- Π-model contains lumped line parameters
- For model derivation, it is convenient to distribute shunt impedance $\underline{Y}_q$ equally on both sides of quadrupole
- We will derive this model from the wave equation

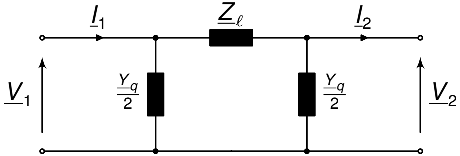


- KCL and KVL yield

$$\begin{bmatrix} \underline{V}_1 \\ \underline{I}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + \underline{Z}_\ell \frac{\underline{Y}_q}{2} & \underline{Z}_\ell \\ \frac{\underline{Y}_q}{2} \left(2 + \underline{Z}_\ell \frac{\underline{Y}_q}{2}\right) & 1 + \underline{Z}_\ell \frac{\underline{Y}_q}{2} \end{bmatrix}}_{=A_1} \begin{bmatrix} \underline{V}_2 \\ \underline{I}_2 \end{bmatrix}$$

- From wave equation we obtain with $\underline{V}_1 = \underline{V}(x=0)$ and $\underline{I}_1 = \underline{I}(x=0)$

$$\begin{bmatrix} \underline{V}_1 \\ \underline{I}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cosh(\underline{\gamma}\ell) & \underline{Z}_W \sinh(\underline{\gamma}\ell) \\ \frac{1}{\underline{Z}_W} \sinh(\underline{\gamma}\ell) & \cosh(\underline{\gamma}\ell) \end{bmatrix}}_{=A_2} \begin{bmatrix} \underline{V}_2 \\ \underline{I}_2 \end{bmatrix}$$

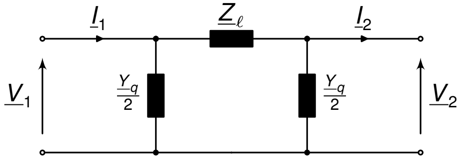


- Comparing coefficients of matrices A_1 and A_2 yields

$$\underline{Z}_\ell = \underline{Z}_W \sinh(\underline{\gamma}\ell)$$

$$\frac{\underline{Y}_q}{2} = \frac{\cosh(\underline{\gamma}\ell) - 1}{\underline{Z}_W \sinh(\underline{\gamma}\ell)} = \frac{1}{\underline{Z}_W} \tanh\left(\frac{\underline{\gamma}\ell}{2}\right)$$

- These parameters correspond to exact relations between currents and voltages according to wave equation for $x = 0$ and $x = \ell$



- For $|\underline{\gamma}\ell| \ll 1$, the expressions for $\underline{Z}_\ell$ and $\underline{Y}_q$ can be simplified

$$\underline{Z}_\ell = \underline{Z}_W \sinh(\underline{\gamma}\ell) \approx \underline{Z}_W \underline{\gamma}\ell = \underline{Z}'\ell$$

$$\frac{\underline{Y}_q}{2} = \frac{1}{\underline{Z}_W} \tanh\left(\frac{(\underline{\gamma}\ell)}{2}\right) \approx \frac{1}{\underline{Z}_W}\frac{\underline{\gamma}\ell}{2} = \frac{\underline{Y}'\ell}{2}$$

→ *Concentrated* elements $\underline{Z}_\ell$ and $\underline{Y}_q$ can be computed from *distributed* parameters R', L', G' and C' if $|\underline{\gamma}\ell| \ll 1$

$$\boxed{\begin{aligned} \underline{Z}_\ell &= \underline{Z}'\ell = (R' + jX')\ell \\ \frac{\underline{Y}_q}{2} &= \frac{\underline{Y}'}{2}\ell = \frac{(G' + jB')}{2}\ell \end{aligned}}$$

5.3 Π-equivalent circuit of homogeneous power line - Validity of model

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- Accuracy of assumption $|\underline{\gamma}\ell| \ll 1$ is crucial for validity of simplified equivalent Π-model
- The larger $|\underline{\gamma}\ell|$, the worse the model with concentrated parameters $\underline{Z}_\ell$ and $\underline{Y}_q$ represents evolution of current and voltage along the line

→ Whenever you use a simplified Π-model to represent a power line, be aware that the model accuracy reduces with increasing line length!

- Rule of thumb:
  - Max. length for overhead line $\approx$ 300 km
  - Max. length for cable $\approx$ 100 km
- Therefore, long power lines are often split into several shorter sections in power flow calculations and each section is represented by individual (simplified) Π-model

5.3 Example: Π-equivalent circuit of homogeneous power line (1)

Task. Consider a power line with the following characteristics

$$R' = 0.05\ \Omega/\text{km}, \quad L' = 1.25\ \text{mH/km}, \quad G' = 0\mu\ \text{S/km}, \quad C' = 10\ \text{nF/km}.$$

Suppose that the line length is 200 km and that the line is operated with a frequency of 50 Hz.

1) Calculate the series impedance $\underline{Z}_\ell^E$ and the shunt admittance $\underline{Y}_q^E$ for the exact Π-equivalent circuit.

2) If $|\underline{\gamma}\ell| \ll 1$, then calculate the simplified series impedance $\underline{Z}_\ell$ and the shunt admittance $\underline{Y}_q$ of the Π-equivalent circuit for that case.

Solution. 1) Surge impedance of line with $G' = 0$ and $\omega = 2\pi f = 314.16$ rad/s

$$\underline{Z}_W = \sqrt{\frac{R' + j\omega L'}{j\omega C'}}$$

$$= \sqrt{\frac{0.05 + j314.16 \cdot 1.25 \cdot 10^{-3}}{j314.16 \cdot 10 \cdot 10^{-9}}} = 354.27 - j22.463 = 354.98\underline{/-3.63^\circ}\ \Omega$$

Propagation constant of line with $G' = 0$ and $\omega = 2\pi f = 314.16$rad/s

$$\underline{\gamma} = \sqrt{(R' + j\omega L')(j\omega C')}$$
$$= \sqrt{(0.05 + j314.16 \cdot 1.25 \cdot 10^{-3})(j314.16 \cdot 10 \cdot 10^{-9})}$$
$$= 0.0001 \text{ Np/km} + j0.0011 \text{ [rad/km]}$$

Np=Neper (logarithmic unit to measure physical field quantities)

5.3 Example: Π-equivalent circuit of homogeneous power line (3)

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Series impedance of Π-equivalent circuit

$$\begin{aligned}\underline{Z}_\ell^E &= \underline{Z}_W \sinh(\underline{\gamma}\ell) \\ &= 354.98\underline{/-3.63^\circ} \cdot \sinh(0.0011\underline{/84.81^\circ} \cdot 200) \\ &= 11.818 + j76.889 = 77.79\underline{/81.27^\circ}\ \Omega\end{aligned}$$

Shunt admittance of Π-equivalent circuit

$$\begin{aligned}\underline{Y}_q^E &= \frac{2}{\underline{Z}_W} \tanh\left(\frac{\underline{\gamma}\ell}{2}\right) \\ &= \frac{2}{354.98\underline{/-3.63^\circ}} \cdot \tanh\left(\frac{0.0011\underline{/84.81^\circ} \cdot 200}{2}\right) \\ &= 1.754 \cdot 10^{-5} + j6.246 \cdot 10^{-4} = 6.248 \cdot 10^{-4}\underline{/88.40^\circ}\ \mathrm{S}\end{aligned}$$

→ Exact Π-equivalent circuit can have shunt conductance even if[3] $G' = 0$!

[3] Physical explanation: We could model the considered line equivalently by two Π-equivalent circuits in series. Then, we would see that there are active power losses in the circuit. Thus, the single Π-equivalent circuit has to have an ohmic component.

2) We have that

$$|\underline{\gamma}\ell| = 0.0011 \cdot 200 = 0.22$$

This value is reasonably smaller than 1

Thus, using the simplified equations valid for $|\underline{\gamma}\ell| \ll 1$, we obtain

$$\begin{aligned} \underline{Z}_\ell &= (R' + j\omega L')\ell \\ &= (0.05 + j314.16 \cdot 1.25 \cdot 10^{-3}) \cdot 200 \\ &= 10 + j78.54 = 79.17\underline{/82.75^\circ}\ \Omega \end{aligned}$$

and

$$\begin{aligned} \underline{Y}_q &= jB'\ell = j\omega C'\ell \\ &= j314.16 \cdot 10 \cdot 10^{-9} \cdot 200 = j6.283 \cdot 10^{-4}\ \text{S} \end{aligned}$$

$\rightarrow$ Simplified Π-equivalent circuit has NO shunt conductance whenever $G' = 0$!

Remaining parameters are very similar to exact values:
$\underline{Z}_\ell \approx \underline{Z}_\ell^E$, $\Im(\underline{Y}_q) \approx \Im(\underline{Y}_q^E)$

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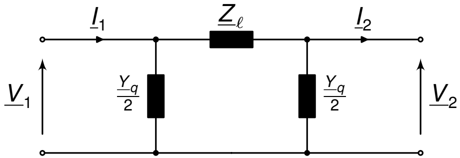


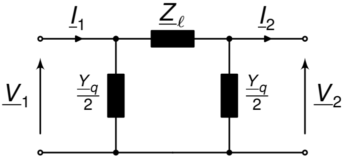
- In practice, G' is small (in particular for voltages > 110kV) and therefore often neglected
- Then, shunt admittance is purely capacitive

$$\underline{Z}_\ell = \underline{Z}'\ell = (R' + jX')\ell \qquad \frac{\underline{Y}_q}{2} = \frac{\underline{Y}'}{2}\ell = \frac{jB'}{2}\ell$$

- Some times, also conductor resistances neglected $\rightarrow R' = 0$; such line model is called *lossless* and its concentrated (or lumped) parameters are given by

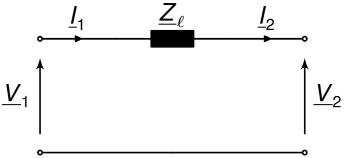
$$\underline{Z}_\ell = \underline{Z}'\ell = jX'\ell \qquad \frac{\underline{Y}_q}{2} = \frac{\underline{Y}'}{2}\ell = \frac{jB'}{2}\ell = \frac{j\omega C'}{2}\ell$$

- For *overhead lines* models can be further simplified
- Typically, overhead lines classified into 3 categories
  - **Short lines (up to 100 km).** Usually, C' and G' very small; model: series impedance $\underline{Z}_\ell = R'\ell + j\omega L'\ell$; shunt admittance $\underline{Y}_q$ is completely neglected
  - **Medium length lines (100 to 300 km).** Use of simplified Π-model with $G' = 0$ without any significant loss of accuracy
  - **Long lines (larger than 300 km).** Significant inaccuracies with concentrated parameter model. Line should either be represented by wave equation or split into several shorter sections



Medium length line model

$$\underline{Y}_q = jB'\ell$$



Short line model

$$\underline{Y}_q = 0$$

5.4 Model simplifications - Comparison: setup

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- We compare results obtained with different models for exemplary 230 kV transmission line with characteristic impedance and propagation constant

  $\underline{Z}_W = 382.2 - j16.5\,\Omega \qquad \underline{\gamma} = \alpha + j\beta = 0.0001$ [Np/km] $+ j0.0011$ [rad/km]

  Np=Neper (logarithmic unit to measure physical field quantities)

- We seek to calculate voltage $\underline{V}_2$ at end of line under *no load* conditions $\rightarrow \underline{I}_2 = 0$
- We assume $|\underline{V}_1| = 1.0$ pu
- We will explore 3 different ways

  1) Using the exact wave equation (Section 4.2)

  2) Using the medium length Π-equivalent circuit (Section 4.3)

  3) Using the short line model (Section 4.4)

5.4 Model simplifications - Comparison: calculation

1) For $\underline{I}_2 = 0$, exact wave equation reduces to (see matrix A_2)

$$\underline{V}_1 = \underline{V}(x=0) = \underline{V}_2 \cosh(\underline{\gamma}\ell)$$

2) Medium length Π-model

$$\underline{Z}_\ell \approx \underline{Z}_W \underline{\gamma}\ell \qquad \frac{\underline{Y}_q}{2} \approx \frac{1}{\underline{Z}_W} \frac{\underline{\gamma}\ell}{2}$$

Hence,

$$\underline{V}_1 = \left(1 + \frac{\underline{Z}_\ell \underline{Y}_q}{2}\right) \underline{V}_2 = \left(1 + \frac{(\underline{\gamma}\ell)^2}{2}\right) \underline{V}_2$$

3) Short line model: there is no voltage drop across series element for zero current $\rightarrow \underline{V}_2 = \underline{V}_1$

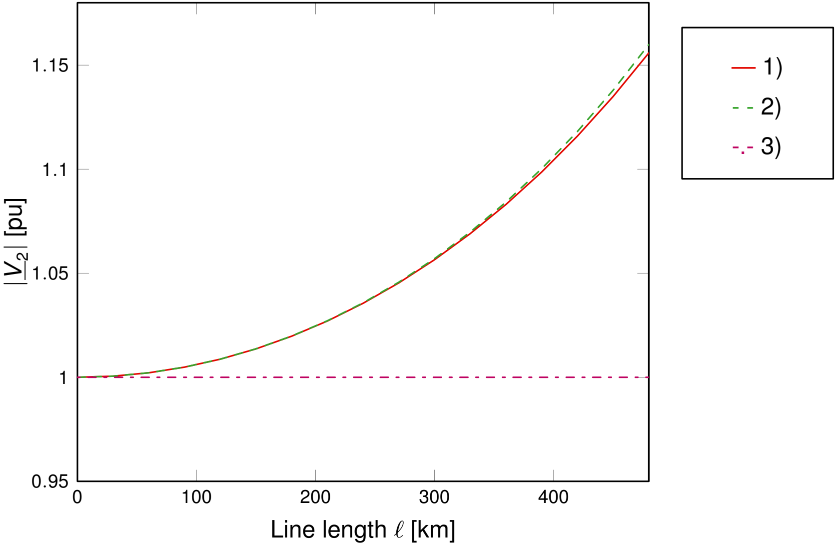
5.4 Model simplifications - Comparison: results

Values for $|\underline{V}_2|$ obtained for different line lengths and different models

Length ℓ in km	$\|\underline{\gamma}\ell\|$	1)	2)	3)
50	0.0552	1.0015	1.0015	1.000
100	0.1105	1.0060	1.0060	1.000
300	0.3314	1.0565	1.0570	1.000
500	0.5523	1.1710	1.1759	1.000

- For short line lengths ($<$ 50 km) all models provide almost identical results
- With increasing line length, results with short line clearly differ from those with other models
- Accuracy of Π-model fairly good up to 300 km, but increasing deviation with increasing length

5.4 Model simplifications - Comparison: Plots



5.4 Summary

- Overhead lines most economic solution for long-distance power transmission
- An overhead line consists of conductors, support structures, shield wires and insulators
- Characteristics of power lines can be represented by set of concentrated parameters R', L', C' and G'
- Exact propagation of voltage and current in a power line can be described by telegrapher's equations (in time-domain), respectively by the wave equation (in phasor-domain)
- For most practical applications, the use of a Π-equivalent circuit suffices to accurately describe the voltage and current relations on a power line
- The validity of the Π-model reduces significantly for long lines ($>$300 km)