Useful Vector Equations and Identities

Dot (Scalar) Product: $\vec{A}\cdot\vec{B} = A\,B\cos\theta$

- Distributive property: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$
- Commutative property: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- Scaling property: $k(\vec{A}\cdot\vec{B}) = (k\vec{A})\cdot\vec{B} = \vec{A}\cdot(k\vec{B})$

Cross (Vector) Product: $\vec{A}\times\vec{B} = A\,B\sin\theta\,\overrightarrow{a_n}$

- Distributive property: $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$
- Not Commutative: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
- Scaling property: $k(\vec{A}\times\vec{B}) = (k\vec{A})\times\vec{B} = \vec{A}\times(k\vec{B})$

Rectangular coordinate system:

Dot Product: $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

Cross Product:
$$\vec{A}\times\vec{B} = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Magnitude of a vector: $A = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Differential Elements:

Length: $\overrightarrow{dl} = dx\,\overrightarrow{a_x} + dy\,\overrightarrow{a_y} + dz\,\overrightarrow{a_z}$

Volume: $dv = dx\,dy\,dz$

Surfaces: $\overrightarrow{ds_x} = dy\,dz\,\overrightarrow{a_x}$ $\quad\overrightarrow{ds_y} = dx\,dz\,\overrightarrow{a_y}$ $\quad\overrightarrow{ds_z} = dx\,dy\,\overrightarrow{a_z}$

Cylindrical Coordinate System:

A point $P(x,y,z)$ in the RCS system is given in the CCS system as $P(\rho,\phi,z)$ where ρ and z are the lengths and ϕ is the angle as measured from the x-axis.

$x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$

$\rho = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}(y/x)$

- **Vector transformations one way**:

$$\begin{bmatrix} \overrightarrow{a_x} \\ \overrightarrow{a_y} \\ \overrightarrow{a_z} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{a_\rho} \\ \overrightarrow{a_\phi} \\ \overrightarrow{a_z} \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_Z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

- **Vector transformations the other way**:

$$\begin{bmatrix} \overrightarrow{a_\rho} \\ \overrightarrow{a_\phi} \\ \overrightarrow{a_z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{a_x} \\ \overrightarrow{a_y} \\ \overrightarrow{a_z} \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_Z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

- **Differential elements**:

Length: $\overrightarrow{dl} = d\rho\,\overrightarrow{a_\rho} + \rho\,d\phi\,\overrightarrow{a_\phi} + dz\,\overrightarrow{a_z}$

Volume: $dv = \rho\,d\rho\,d\phi\,dz$

Surfaces: $\overrightarrow{ds_\rho} = \rho\,d\phi\,dz\,\overrightarrow{a_\rho}$ $\overrightarrow{ds_\phi} = d\rho\,dz\,\overrightarrow{a_\phi}$ $\overrightarrow{ds_z} = \rho\,d\rho\,d\phi\,\overrightarrow{a_z}$

Spherical Coordinate System:

A point $P(x,y,z)$ in the RCS system is given in the SCS system as $P(r,\theta,\phi)$ where r is the length, θ is measured from the positive z-axis and ϕ is the angle as measured from the x-axis.

Note that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$

$r = \sqrt{x^2 + y^2 + z^2}$, $\phi = \tan^{-1}(y/x)$ and $\theta = \cos^{-1}(z/r)$

Vector transformations one way:

$$\begin{bmatrix} \overrightarrow{a_x} \\ \overrightarrow{a_y} \\ \overrightarrow{a_z} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{a_r} \\ \overrightarrow{a_\theta} \\ \overrightarrow{a_\phi} \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_Z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Vector transformations the other way:

$$\begin{bmatrix} \overrightarrow{a_r} \\ \overrightarrow{a_\theta} \\ \overrightarrow{a_\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{a_x} \\ \overrightarrow{a_y} \\ \overrightarrow{a_z} \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Differential elements:

Length: $\overrightarrow{dl} = dr\,\overrightarrow{a_r} + r\,d\theta\,\overrightarrow{a_\theta} + r\sin\theta\,d\phi\,\vec{a}_\phi$

Volume: $dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

Surfaces: $\overrightarrow{ds_r} = r^2\sin\theta\,d\theta\,d\phi\,\overrightarrow{a_r}$ $\overrightarrow{ds_\theta} = r\sin\theta\,dr\,d\phi\,\overrightarrow{a_\theta}$ $\overrightarrow{ds_\phi} = r\,dr\,d\theta\,\overrightarrow{a_\phi}$

Gradient of a scalar is a vector:

RCS: $$\nabla f = \frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z$$

CCS: $$\nabla f = \frac{\partial f}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\vec{a}_\phi + \frac{\partial f}{\partial z}\vec{a}_z$$

SCS: $$\nabla f = \frac{\partial f}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{a}_\theta + \frac{1}{r\,\sin\theta}\frac{\partial f}{\partial \phi}\vec{a}_\phi$$

The Divergence Theorem:

$$\int_V \nabla\cdot\vec{F}\,dv = \oint_S \vec{F}\cdot\vec{ds}$$

where S is the surface that completely encloses the volume V.

Definition of divergence: $$\nabla\cdot\vec{F} = \underbrace{\lim}_{\nabla v\to 0}\frac{1}{\Delta v}\oint_s \vec{F}\cdot\vec{ds}$$

where s is the surface that completely encloses the differential volume Δv. Divergence of a vector is a scalar quantity. It is a measure of net outward flow of a field through a closed surface. If the net outward flow through a closed surface is zero, the field is said to be **continuous** or **solenoidal**. It can be computed at any point in space as follows:

RCS: $$\nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

CCS: $$\nabla\cdot\vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

SCS: $$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial\left(r^2 F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, F_\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

Stokes' Theorem:

$$\int_S (\nabla \times \vec{F}) \cdot \overrightarrow{ds} = \oint_C \vec{F} \cdot \overrightarrow{d\ell}$$

where S is the surface that is completely enclosed by the contour C.

Definition of Curl: $$\nabla \times \vec{F} = \underbrace{\lim}_{\Delta s \to 0} \frac{1}{\Delta s} \oint_c \vec{F} \cdot \overrightarrow{d\ell}$$

where c is the contour that completely encloses the differential surface Δs.

Curl is a measure of net circulation of a vector field. If the net circulation around a closed path is zero, the field is said to be **conservative** or **irrotational**.

Note that the curl of a vector field is also a vector quantity.

Curl can be computed as follows:

RCS: $$\nabla \times \vec{F} = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

CCS: $$\nabla \times \vec{F} = \frac{1}{\rho} \begin{vmatrix} \overrightarrow{a_\rho} & \rho \overrightarrow{a_\phi} & \overrightarrow{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

SCS: $$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \overrightarrow{a_r} & r \overrightarrow{a_\theta} & r \sin\theta \overrightarrow{a_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix}$$

The Laplacian of a scalar quantity is also a scalar quantity.

It is second order differential operator.

Definition: $\nabla^2 f = \nabla \cdot \nabla f$

Laplace's equation: $\nabla^2 f = 0$

Poisson's equation: $\nabla^2 f \neq 0$

It can be computed as follows:

RCS: $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

CCS: $$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial f}{\partial \rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

SCS: $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial f}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Some Useful Vector Identities:

$$\nabla \times \nabla f = 0 \quad \text{(a)}$$

$$\nabla \cdot \left[\nabla \times \vec{F}\right] = 0 \quad \text{(b)}$$

$$\nabla^2 f = \nabla \cdot \nabla f \quad \text{(c)}$$

$$\nabla^2 \vec{F} = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla \times \nabla \times \vec{F} \quad \text{(d)}$$

$$\nabla(f + g) = \nabla f + \nabla g \quad \text{(e)}$$

$$\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G} \quad \text{(f)}$$

$$\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G} \quad \text{(g)}$$

$$\nabla (f\, g) = g\, \nabla f + f\, \nabla g \quad \text{(h)}$$

$$\nabla \cdot (f\, \vec{A}) = f\, \nabla \cdot \vec{A} + \vec{A} \cdot \nabla f \quad \text{(i)}$$

$$\nabla \times (f\, \vec{A}) = f\, \nabla \times \vec{A} + \nabla f \times \vec{A} \quad \text{(j)}$$

$$\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot \left(\nabla \times \vec{B}\right) \quad \text{(k)}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \quad (\ell)$$