Uncertainty in Intensity

Abdel Elabd
Scalable Quantum Research Lab
University of Washington

$I = \kappa \Omega^2 \Longrightarrow \mathrm{Var}[I] = \kappa^2 \mathrm{Var}[\Omega^2]$

$$\begin{aligned}
\sigma_I^2 &= \kappa^2 \Bigg[E[\Omega^4] - E^2[\Omega^2] \Bigg] \\
&= \kappa^2 \Bigg[E[\Omega^4] - E^2[\Omega^2] \Bigg] \\
&\text{Substitute } \sigma_\Omega^2 = E[\Omega^2] - E^2[\Omega] \Longrightarrow E[\Omega^2] = \sigma_\Omega^2 + E^2[\Omega] \\
&= \kappa^2 \Bigg[E[\Omega^4] - \Bigg(\sigma_\Omega^2 + E^2[\Omega] \Bigg)^2 \Bigg] \\
&= \kappa^2 \Bigg[E[\Omega^4] - \Bigg(\sigma_\Omega^4 + 2\sigma_\Omega^2 E^2[\Omega] + E^4[\Omega] \Bigg) \Bigg] \\
\sigma_I^2 &= \kappa^2 \Bigg[E[\Omega^4] - \Bigg(\sigma_\Omega^4 + 2\sigma_\Omega^2 \bar{\Omega}^2 + \bar{\Omega}^4 \Bigg) \Bigg] \qquad (1)
\end{aligned}$$

Let's put a pin in that and compute $E[\Omega^4]$:

The pdf of Ω is given by $p(\Omega) = \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \exp\left(-\frac{(\Omega - \bar{\Omega})^2}{2\sigma_\Omega^2}\right)$. Therefore:

$$\begin{aligned}
E[\Omega^4] &= \int_{-\infty}^{\infty} \Omega^4 p(\Omega) d\Omega \\
&= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} \Omega^4 \exp\left(-\frac{(\Omega - \bar{\Omega})^2}{2\sigma_\Omega^2}\right) d\Omega
\end{aligned}$$

Substitute $x = \Omega - \bar{\Omega} \Longrightarrow dx = d\Omega, \Omega^4 = (x + \bar{\Omega})^4$:

$$\begin{aligned}
E[\Omega^4] &= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} (x + \bar{\Omega})^4 \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} (x^2 + 2x\bar{\Omega} + \bar{\Omega}^2)^2 \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} (x^2 + 2x\bar{\Omega} + \bar{\Omega}^2)(x^2 + 2x\bar{\Omega} + \bar{\Omega}^2) \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} (x^4 + 2x^3\bar{\Omega} + x^2\bar{\Omega}^2 + 2x^3\bar{\Omega} + 4x^2\bar{\Omega}^2 + 2x\bar{\Omega}^3 + x^2\bar{\Omega}^2 + 2x\bar{\Omega}^3 + \bar{\Omega}^4) \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} (x^4 + 4x^3\bar{\Omega} + 6x^2\bar{\Omega}^2 + 4x\bar{\Omega}^3 + \bar{\Omega}^4) \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx
\end{aligned}$$

The odd moments vanish, so:

$$E[\Omega^4] = \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} (x^4 + 6x^2\bar{\Omega}^2 + \bar{\Omega}^4) \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} x^4 \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx + \frac{6\bar{\Omega}^2}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx$$
$$+ \frac{\bar{\Omega}^4}{\sqrt{2\pi\sigma_\Omega^2}} \int_{-\infty}^{\infty} \bar{\exp}\left(-\frac{x^2}{2\sigma_\Omega^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_\Omega^2}} \left(3\sigma_\Omega^5\sqrt{2\pi}\right) + + \frac{6\bar{\Omega}^2}{\sqrt{2\pi\sigma_\Omega^2}} \left(\sigma_\Omega^3\sqrt{2\pi}\right) + \frac{\bar{\Omega}^4}{\sqrt{2\pi\sigma_\Omega^2}} \left(\sigma_\Omega\sqrt{2\pi}\right)$$

$$E[\Omega^4] = 3\sigma_\Omega^4 + 6\bar{\Omega}^2\sigma_\Omega^2 + \bar{\Omega}^4 \tag{2}$$

Plugging Eqn. (2) into Eqn. (1):

$$\sigma_I^2 = \kappa^2 \left[\left(3\sigma_\Omega^4 + 6\bar{\Omega}^2\sigma_\Omega^2 + \bar{\Omega}^4\right) - \left(\sigma_\Omega^4 + 2\sigma_\Omega^2\bar{\Omega}^2 + \bar{\Omega}^4\right) \right]$$

$$\sigma_I^2 = \kappa^2 \left[2\sigma_\Omega^4 + 4\bar{\Omega}^2\sigma_\Omega^2 \right] \tag{3}$$

There's our answer.