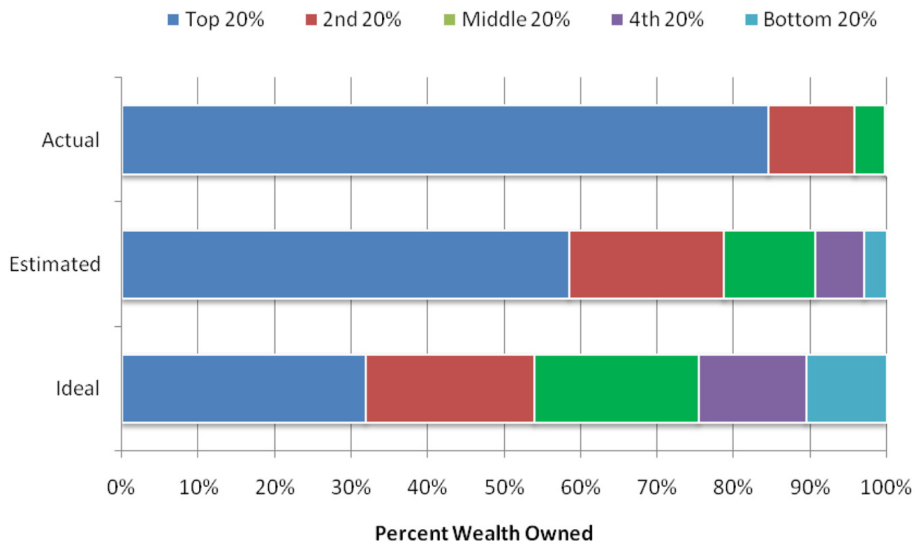
Introduction

An editorial in the August 8, 2017 issue of *Nature Human Behavior* states:

> Economic inequality has become a serious concern for many in the political, public and academic spheres. The gap between rich and poor has been increasing for decades, to the extent that more than 50% of the world's wealth is now held by the top 1% of earners. The geography of inequality can be mapped against poorer health standards, lesser educational attainment, higher crime rates, greater unhappiness and lack of trust in one's fellow citizens.
>
> Research in psychology and economics provides a seeming paradox when compared with these grim statistics of growing inequality in wealth and income. Numerous laboratory-based studies have shown that people reject unfair payment distributions. Participants even reject inequitable outcomes on behalf of unknown others in third-party scenarios. This work suggests that equality should be a powerful and desirable social norm. However, if this is the case, why is wealth inequality so tolerated in real life[1]?

There is evidence showing that it may not be so much that people *tolerate* wealth inequality, but rather that people may not have a clear understanding of the actual distribution of wealth in whatever populations they belong to. In a paper[2] published in 2011, Michael I. Norton and Dan Ariely present evidence supporting this proposition. They report results of a study that shows Americans have misconceptions about the distribution of wealth in their country. Wealth inequality is an issue in many countries, but much of the research on this subject focuses on the USA.

The following three bar graphs, which appear in their paper, summarize average views regarding the actual, perceived, and ideal distributions of wealth in the USA.



The top graph (labeled "Actual") shows that the wealthiest one-fifth (20%) of the US population owns about 84% of the wealth. The next wealthiest fifth owns about 11%. The third wealthiest own 4% and the bottom two fifths own 0.2% and 0.1%, respectively. The shares of the two bottom fifths are actually too small to be visible on the graph.

The next graph shows what Americans estimate the distribution of wealth to be, on average, and the last one shows an "ideal" distribution of wealth, according to the average American. Both of these graphs show a much more equal distribution than the actual one. These three graphs indicate that on average, Americans do not have an accurate understanding of the actual distribution of wealth in the country.

Whatever the facts are, it's not clear that we should expect the average citizen to be able to cite accurate statistics on this matter, without first studying it carefully. We do encounter statements of the form "The top x% owns y% of the wealth" frequently in various media but it can be hard to keep the numbers straight. Furthermore, the mathematics of the distribution of wealth is less intuitive than one might think. For example, roughly 10% of American households actually have *negative* net worth, by some measures. This means that a single family with a positive net worth, say US$100, owns more wealth than the bottom 10% of American families.

The purpose of this document is to help you understand some of the most commonly measures of wealth inequality so that you can read further on the subject.

A Clarification

We will focus on *wealth* inequality, as opposed to *income* inequality here. The measures we cover will be applicable to both, as they are related but distinct issues that are frequently discussed together.

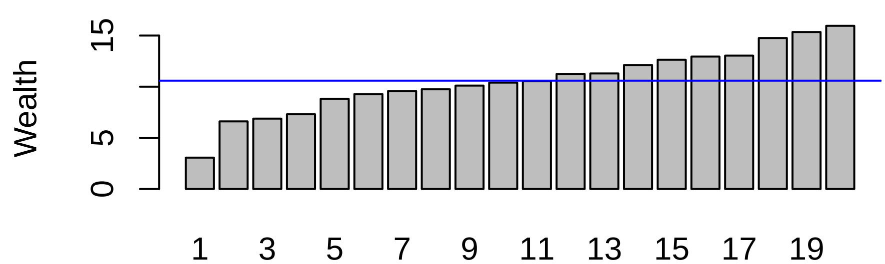
The Hoover Index

The first measure of wealth inequality we consider is called the *Hoover Index* (named after the economist Edgar M. Hoover, Jr.), or sometimes the *Robin Hood Index*. This second name refers to the fictional hero of English legend who stole from the rich and gave to the poor. How much (in percent terms) would Robin Hood have to steal from the rich and give to the poor, in order that everyone ends up with the same wealth? That's essentially what the Hoover/Robin Hood index is.

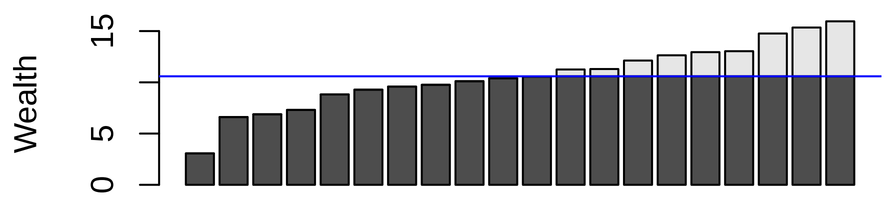
Here's a more formal definition: The *Hoover Index* (also called *Robin Hood Index* of a population is the percentage of total wealth that would have to be redistributed in order for every individual to have the same wealth. We will call this number H.

Example 1

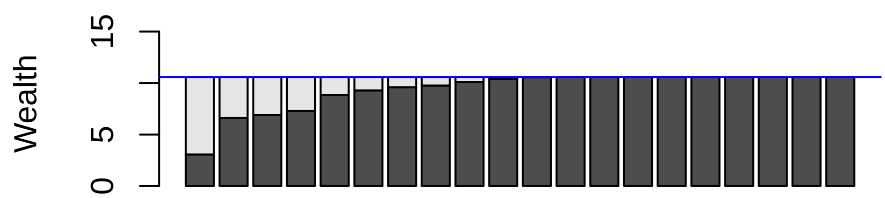
We'll consider an intuitive, visual example involving a population with 20 individuals. Here's a bar graph showing the wealth per person (each person corresponds to a single bar). The people are sorted so that the wealth increases as we move to the right. The blue line indicates the mean wealth per person.



You can imagine slicing the bars that extend above the blue line:



Now glue these pieces to the shorter bars in such a way that every bar reaches exactly to the blue line:



In effect, this redistributes the wealth to achieve 100% equality.

The Hoover Index equals the area of the bars that was sliced off, divided by the total area, times 100%. In the chart above, the Hoover Index equals the light gray area divided by the area of the entire shaded region which here is approximately 11%.

Some Useful Notation

Before we try a numerical example, let's discuss some notation that will make it easier to express the Hoover Index in equation form.

We'll assume we're studying a population of N people, numbered 1, 2, 3, ..., N.

The wealth of person i will be denoted x_i.

We'll frequently refer to the mean or average wealth in the population, and a standard notation for this is $\bar{x}$.

Various sums will also come up often, for example the total wealth is:

$$x_1 + x_2 + x_3 + \cdots + x_N.$$

In words, we could describe this expression as:

The sum of all x_i, where i ranges from 1 to N

In mathematical notation, "the sum of all" is compressed to the symbol $\sum$ (the Greek letter sigma), and the limits on the range are written below and above the $\sum$, like this:

$$\text{The sum of all } x_i\text{, where } i \text{ ranges from 1 to } N \quad \Longrightarrow \quad \sum_{i=1}^{N} x_i$$

That is:

$$x_1 + x_2 + x_3 + \cdots + x_N = \sum_{i=1}^{N} x_i$$

The expression $\sum\limits_i x_i$ is even more compact and is used when it's clear from context what values i takes on. The form $\sum\limits_{i=1}^{N} x_i$ is used when we want to be more explicit.

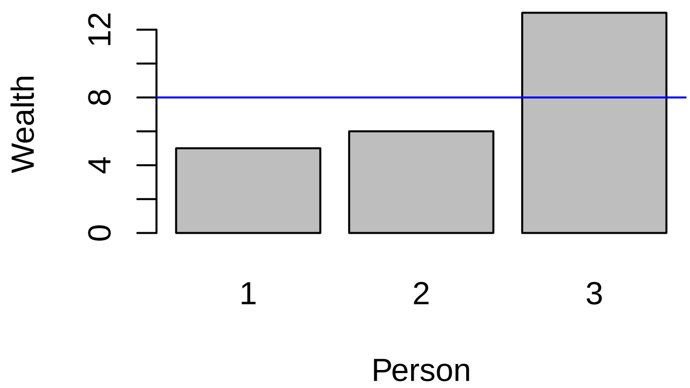
Here's an application: With this notation, the mean of x_1, x_2, x_3, ..., x_N, which is the sum of all the x_i divided by N, can be expressed in this way:

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{1}{N} \sum_{x=1}^{N} x_i$$

Example 3

Let's say our population had three individuals, person 1, person 2, and person 3, with wealth 5, 6, and 13 units, respectively.

The mean wealth per individual is $\bar{x} = \dfrac{5+6+13}{3} = \dfrac{24}{3} = 8$ units. Here's a bar graph showing the wealth of each of the three persons; the blue line marks the average wealth.



To achieve exact equality, where each person owns 8 units of wealth, person 3 would have to give away 5 units of wealth, 3 units to person 1 and 2 units to person 2. That is, $3+2=5$ units of wealth have to be redistributed, and 5 is about 20.8% of $5+6+13=24$, the total amount of wealth in the population. Therefore the Hoover Index for this population is 20.8%.

Note that the amount of wealth that had to be redistributed shows up in two ways:

- The less wealthy in the population (persons 1 and 2) are $3+2=5$ units "short" of the mean altogether.
- The more wealthy in the population (person 3), at 13 units, is $13-8=5$ units above the mean.

Therefore if we add up the total amount by which each person's wealth differs from the average wealth (in absolute value) we should get *twice* the amount of wealth that needs to be redistributed to achieve complete equality. To make the following calculation clearer, let's set A equal to the *amount* of wealth that has to be redistributed to reach full equality (as opposed to the *percentage* of total wealth it's necessary to distribute).

Then the first sentence in the above paragraph states:

$$\sum_{i=1}^{N} |x_i - \bar{x}| = 2A$$

If we divide both sides by 2, we get:

$$\frac{\sum_{i=1}^{N} |x_i - \bar{x}|}{2} = A$$

which is equivalent to:

$$A = \frac{\sum_{i=1}^{N} |x_i - \bar{x}|}{2}$$

Now this is the amount of wealth that needs to be distributed; to get the percentage, we divide by the total wealth $\sum_{x=1}^{N} x_i$ and multiply by 100%:

$$H = \frac{\sum_{i=1}^{N} |x_i - \bar{x}|}{2\sum_{i=1}^{N} x_i} \times 100\%$$

We can use the fact that $\sum_{x=1}^{N} x_i = N \times \bar{x}$, which follows from $\bar{x} = \sum_{x=1}^{N} x_i$ to get this useful alternate form:

$$H = \frac{\sum_{i=1}^{N} |x_i - \bar{x}|}{2N\bar{x}}.$$

As a check, let's apply the formula to our simple example. Recall that the wealth values were 5, 6, and 13 units.

$$
\begin{aligned}
H &= \frac{\sum_i |x_i - \bar{x}|}{2\sum_i x_i} \times 100\% \\
&= \frac{|5-8| + |6-8| + |13-8|}{2 \times (5+6+13)} \times 100\% \\
&= \frac{3+2+5}{48} \times 100\% \\
&= \frac{10}{24} \times 100\% \\
&= 0.208\bar{3} \times 100\% \\
&\approx 20.8\%
\end{aligned}
$$

Problem 1

Suppose a population of five people have wealth values 7, 10.5, 12, 12, and 23. Round your answer to the nearest tenth of a percent. Find the Hoover Index. (Solution is on page xx)

Bounds on the Hoover Index

If everyone in the population has the same amount of wealth, then no wealth needs to be redistributed to achieve full equality. Therefore H can equal 0% in this extreme case. It can never be negative.

Now let's consider how large the Hoover Index could be. First, it's clear that The Hoover Index cannot exceed 100%. The only way to redistribute more than 100% is to distribute some of the wealth more than once (i.e., give one person some money, then later take it back and give it to a second person). This is unnecessary, because you could gave given the second person the money in the first place.

Let's imagine a population of five people where one person owns all the wealth. To achieve full equality, we have to divide that one person's wealth into five equal parts, and redistribute four of the five parts. In other words, $4/5 = 80\%$ of the wealth must be distributed.

If the population consisted of 99 people and one person with all the wealth, then we would divide that wealth into 100 equal parts and distribute 99 of those parts to the others who initially had no money. Therefore $H = 99/100 = 99\%$.

We could construct larger and larger examples of this type in which the Hoover Index is arbitrarily close to 100%. Therefore we conclude that for any population,

$$0\% \leq H < 100\%.$$

Problem 2

Suppose everyone in a population has his or her wealth doubled. Will that affect the Hoover Index? What if, instead, each person receives $10,000—will the Hoover Index be affected? Finally, suppose a wealthy person gives a poorer person a small portion of their wealth (not enough to bring the poorer person up to average). Will H be affected? You can use a very small hypothetical population to investigate these questions, for example, three people with wealth amounts $1000, $2000, and $3000.

The Gini Index

The Gini Index, named after the statistician Corrado Gini (1884–1965), is another, slightly more complex measure of inequality in the distribution of goods such as wealth across a population. We'll derive a formula for the Gini Index.

Let's consider a population of five people and hypothetical values for the net worth of each. The numbers will be small (perhaps we'll work in units of $10,000) and chosen to make the computations simple.

Person	Units of Wealth
Aya	10
Ben	19
Carmen	8
Dan	9
Elana	4

Just as with the Hoover Index, the goal is to capture the amount of wealth inequality in the form of a single statistic. We will start with a simple idea, and gradually refine it until we have something useful.

First Attempt

Suppose we choose a pair of people from our population at random and compute the difference in wealth between them. We would like a statistic which estimates this difference (on average) as accurately as possible.

That's exactly what the (arithmetic) mean of the absolute value of the differences in wealth is designed to estimate. Let's try that. I'll call this "AWD1" for "average wealth difference, attempt 1".

Using summation notation, we will compute:

$$\text{AWD1} = \frac{\sum\limits_{\text{pairs } \{i,\, j\}} |x_i - x_j|}{\text{number of pairs}}$$

For each pair of people, we'll subtract one wealth amount from the other, take the absolute value, and finally take the mean of all those absolute values.

For example, for the pair $\{\text{Aya},\ \text{Ben}\}$, we calculate $|10-19| = |-9| = 9$.

We don't need to include a term $\{\text{Ben},\ \text{Aya}\}$, because that is the same pair of people.

$\{\text{Aya},\ \text{Aya}\}$ is not a true "pair" of people (most likely, it would be interpreted as a set with a single person as its one element), so we won't consider such things in the mean.

I'll do the calculation in tabular form. First, note that we can systematically write down all the distinct pairs of people this way:

1. First write all the pairs including the first person in our list (Aya).

2. Second, write all the pairs including the second person in our list, Ben, without repeating the pair $\{\text{Ben}, \text{Aya}\}$.

3. Third, write all the pairs including the third person, without repeating any pairs.

4. And so forth ...

Aya/Ben
Aya/Carmen
Aya/Dan
Aya/Elana
Ben/Carmen
Ben/Dan
Ben/Elana
Carmen/Dan
Carmen/Elana
Dan/Elana

Now for each pair, compute the absolute value of the wealth difference, then take the mean:

Pair	absolute difference in wealth
Aya/Ben	$\lvert 10-19\rvert = 9$
Aya/Carmen	$\lvert 10-8\rvert = 2$
Aya/Dan	$\lvert 10-9\rvert = 1$
Aya/Elana	$\lvert 10-4\rvert = 6$
Ben/Carmen	$\lvert 19-8\rvert = 11$
Ben/Dan	$\lvert 19-9\rvert = 10$
Ben/Elana	$\lvert 19-4\rvert = 15$
Carmen/Dan	$\lvert 8-9\rvert = 1$
Carmen/Elana	$\lvert 8-4\rvert = 4$
Dan/Elana	$\lvert 9-4\rvert = 5$
Sum	64

The sum of the absolute differences is 64, so the mean of the absolute differences is $64/10 = 6.4$.

To sum up, if we choose a pair of people from our population at random (with each pair being equally likely to be chosen), then our best guess for the difference in their wealth is 6.4 units. We could then use this statistic on a larger scale to estimate average wealth difference among citizens of an entire country.

First Refinement

One issue with this statistic is that it is scale dependent. Because different countries use different units of currency, values of $AWD1$ computed for two different countries are likely not to be directly comparable. For example, one US dollar historically has value roughly 100 Japanese yen.

We will have a much more useful statistic if we can adjust the formula so that a particular value has the same meaning regardless of the what units of wealth are being used. This process is called *normalization*.

Here's what we'll do: We'll divide our $AWD1$ by the average value of wealth across the population (which you can check is 10 in this example).

In this case, our initial value of 6.4 becomes 0.64. We could call that AWD2. You could check that if we multiply all our hypothetical wealth values by 5, for example, that would have no effect on our AWD2 statistic.

Here is how we could express AWD2 using the sigma notation:

$$\text{AWD2} = \frac{\sum\limits_{\text{pairs }\{\text{i, j}\}} |x_i - x_j|}{(\text{number of pairs}) \times \bar{x}}$$

Second Refinement

Can we do better? Some statistics have the nice property that all the possible values lie in a finite interval of real numbers. If that's the case, then it's especially convenient if we can arrange that that interval be $[0, 1]$.

Let's look at some extreme hypothetical values of our statistic AWD2.

One possible scenario is that everyone has the same amount of wealth, say 10 units:

Person	Units of Wealth
Aya	10
Ben	10
Carmen	10
Dan	10
Elana	10

Now if we calculate the sum of the wealth differences, we simply get a sum of zeros, which totals to 0. The average wealth equals 10, and $\frac{0}{10} = 0$, so $\text{AWD2} = 0$ in this case.

Note that AWD2 can never be negative, because the absolute value of any real number is at least zero. Therefore the values of AWD2 is always a nonnegative real number.

At the other extreme, it seems logical that the greatest average difference may be achieved if one person has very high wealth and everyone else has zero (thus driving the average wealth down).

Person	Units of Wealth
Aya	10
Ben	0
Carmen	0
Dan	0
Elana	0

The only pairs with different wealth are $\{\text{Aya, Ben}\}$, $\{\text{Aya, Carmen}\}$, $\{\text{Aya, Dan}\}$, and $\{\text{Aya, Elana}\}$.

The sum of all the wealth differences is thus $10+10+10+10=40$, and if we divide by the number of pairs (10) times the average, which is $(10+0+0+0+0)/5=2$, we find that AWD2 $= \dfrac{40}{10\times 2} = 2$.

It is in fact true that the largest value that AWD2 can attain is 2, when a single person has all the wealth and everyone else has zero. Therefore AWD2 always lies in the interval $[0,2]$.

We could therefore properly normalize this statistic by dividing AWD2 by 2. This AWD2 is the actual Gini Index.

To summarize, here is a procedure for calculating the Gini Index for a population:

1. For each pair in the population, find the absolute value of their wealth difference.

2. Find the average of these absolute wealth differences.

3. Divide that average by twice the mean wealth in the population.

In sigma notation:

$$G = \frac{\displaystyle\sum_{\text{pairs } \{i,j\}} |x_i - x_j|}{2(\text{number of pairs})\bar{x}}$$

Example 4

Consider a population of five people, A, B, C, D, and E, with wealth values 1, 4, 1, 3, and 10, respectively.

Let's compute the Gini Index. I'll use the tabular format again, writing out one row per pair:

Pair	Absolute Wealth Difference
A/B	$\|1-4\| = 3$
A/C	$\|1-1\| = 0$
A/D	$\|1-3\| = 2$
A/E	$\|1-10\| = 9$
B/C	$\|4-1\| = 3$
B/D	$\|4-3\| = 1$
B/E	$\|4-10\| = 6$
C/D	$\|1-3\| = 2$
C/E	$\|1-10\| = 9$
D/E	$\|3-10\| = 7$
Sum	42

Now we divide this sum of 42 by $2(\text{number of pairs})\bar{x}$. The number of pairs is once again 10, and the mean of the wealth values is $(1+4+1+3+10)/5 = 3.8$.

Therefore:

$$
\begin{aligned}
G &= \frac{\sum_{\text{pairs } \{i,j\}} |x_i - x_j|}{2(\text{number of pairs})\bar{x}} \\
&= \frac{42}{2(10)3.8} \\
&= 0.55
\end{aligned}
$$

Interpreting the Gini Index

As with the Hoover Index, the Gini Index lies between 0 and 1. A Gini Index of 0 means that everyone has the same amount of wealth. A Gini Index of 1 means a single person has all the wealth and the rest have zero. The Gini Indices of countries lie mostly in the range 0.25 to 0.60.

Comparing the Hoover and Gini Indices

We have now discussed two measures of the inequality of wealth in a population. This raises the question of how the two measures compare to each other. We would hope that they both measure inequality well. If their values are always equal, that would be interesting mathematically, but it would indicate that one is redundant.

Can one index always be calculated exactly from the other index? It turns out the answer to this question is "no".

Consider two populations, A and B. Population A consists of two people, with wealth values 1 and 2 respectively.

Population B consists of three people with wealth values 1, 1, and 2.

Population A has Hoover Index 16.7% and Gini Index 33.3% (to one decimal place).

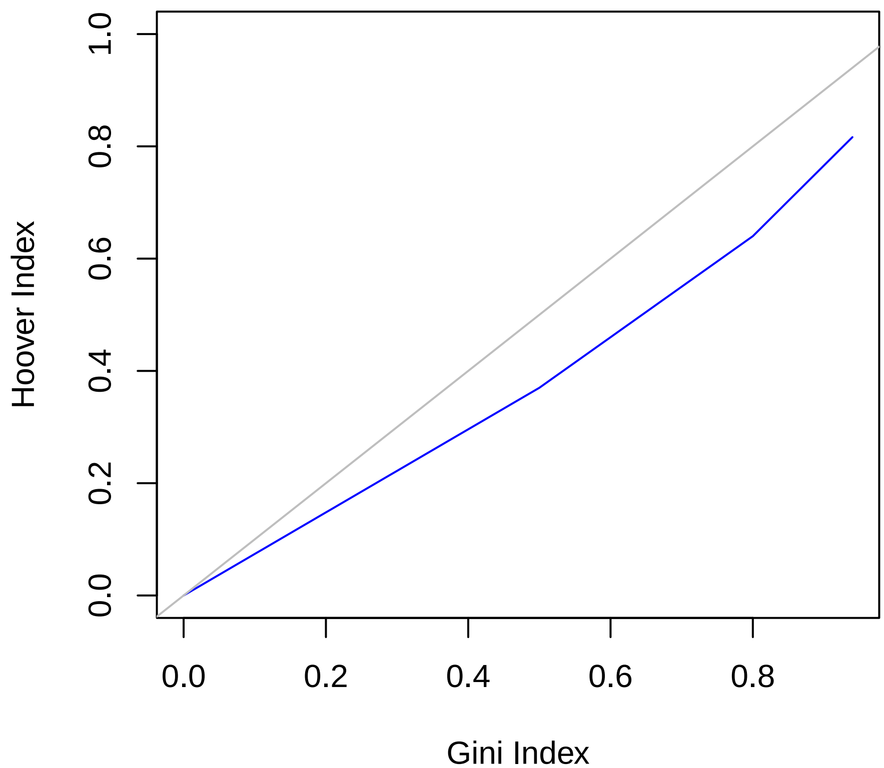
Population B has Hoover Index 16.7% and Gini Index 25%.

Therefore just knowing the Hoover Index of a population does not imply that we know the Gini Index of that population.

However, the two indices are strongly related, and, for example, a high Gini Index implies a high Hoover Index. Edward Allen has determined a formula[3] for the Hoover Index in terms of the Gini Index which is accurate to 5% provided $0 \leq G \leq 0.95$. Here is that formula:

$$H(G) = \left\{ \begin{array}{ll} 0.74G & \text{if } G \leq 0.5 \\ 0.37 + 0.90(G - 0.50) & \text{if } 0.5 \leq G \leq 0.8 \\ 0.64 + 1.26(G - 0.80) & \text{if } 0.8 \leq G \leq 0.95 \end{array} \right\}$$

The blue graph gives Allen's estimate of the Hoover Index from the Gini Index. According to this estimate, the Gini Index tends to be slightly larger than the Hoover Index for $0 \leq G \leq 0.95$.



An Alternate Formula for G

There is one awkward aspect of the expression for the Gini Index which we have derived: It involves a sum over all wealth differences between *pairs* in the population. This makes sense conceptually, as we are interested in how far apart in wealth different members of the population are. However, it raises the issue of how, if we want to compute the Gini Index for a large population, we are going to enumerate all these pairs.

A more commonly seen formula for G sidesteps this issue at the cost of some redundancy. To illustrate, let's look back to a previous example, where we had to form all distinct pairs using the group Aya, Ben, Carmen, Dan, and Elana.

The following table shows the distinct pairs we formed:

	Aya	Ben	Carmen	Dan	Elana
Aya	$\times$	•	•	•	•
Ben	$\times$	$\times$	•	•	•
Carmen	$\times$	$\times$	$\times$	•	•
Dan	$\times$	$\times$	$\times$	$\times$	•
Elana	$\times$	$\times$	$\times$	$\times$	$\times$

The bullets indicate the genuinely distinct pairs of two different people. The $\times$'s indicate pairs that either were already counted or really consisted of a single person. There were ten pairs/bullets. The Gini Index is the sum of the 10 absolute wealth differences divide by $2 \times (\text{number of pairs})\bar{x}$.

Now if we dispense with this step of enumerating only the "genuine" pairs of two distinct people, include the "pairs" corresponding to boxes with $\times$ symbols in them, it turns out that we will just get *twice* the sum $\sum_{\text{pairs } \{i,j\}} |x_i - x_j|$.

That's because in the the diagonal corresponding to the "pairs" $\{\text{Aya}, \text{Aya}\}$, $\{\text{Ben}, \text{Ben}\}$, $\ldots$, $\{\text{Elana}, \text{Elana}\}$, all the wealth differences are zero.

The pairs below this diagonal are duplicates of the pairs above the diagonal; the people are just listed in reverse order.

Therefore we could simply add up the absolute wealth differences over the entire grid, as indicated below, then divide the sum by 2 to compensate for the double-counting.

	Aya	Ben	Carmen	Dan	Elana
Aya	•	•	•	•	•
Ben	•	•	•	•	•
Carmen	•	•	•	•	•
Dan	•	•	•	•	•
Elana	•	•	•	•	•

If we use x_1, x_2, x_3, x_4, and x_5 to represent the wealth values for Aya, Ben, Carmen, Dan, and Elana, then this new sum of absolute wealth differences could be represented by:

$$\sum_{1 \le i,\ j \le 5} |x_i - x_j|$$

This just means we add up all the $|x_i - x_j|$ for all possible combinations of i and j where i and j are between 1 and 5. Note that there are no other restrictions: $i = 1$ and $j = 1$ are included, as are $i = 1$, $j = 2$ and $i = 2$, $j = 1$. There are thus $5 \times 5 = 25$ different absolute differences to add up.

We now have the following formula for the Gini Index, which is equivalent to the first:

$$\begin{aligned} G &= \frac{\sum\limits_{\text{pairs } \{i,\ j\}} |x_i - x_j|}{2(\text{number of pairs})\bar{x}} \\ &= \frac{\sum\limits_{1 \le i,\ j \le n} |x_i - x_j|}{4(\text{number of pairs})\bar{x}} \end{aligned}$$

Now the denominator still refers to the number of pairs of people. Here we'll use a useful counting formula: Given n people, the number of distinct pairs (unordered) of those people is $\frac{n(n-1)}{2}$. We can therefore substitute this into the denominator and simplify a bit more:

$$G = \frac{\sum\limits_{1\ \leq i,\ j\ \leq n} |x_i - x_j|}{4(\text{number of pairs})\bar{x}}$$

$$= \frac{\sum\limits_{1\ \leq i,\ j\ \leq n} |x_i - x_j|}{4(\frac{n(n-1)}{2})\bar{x}}$$

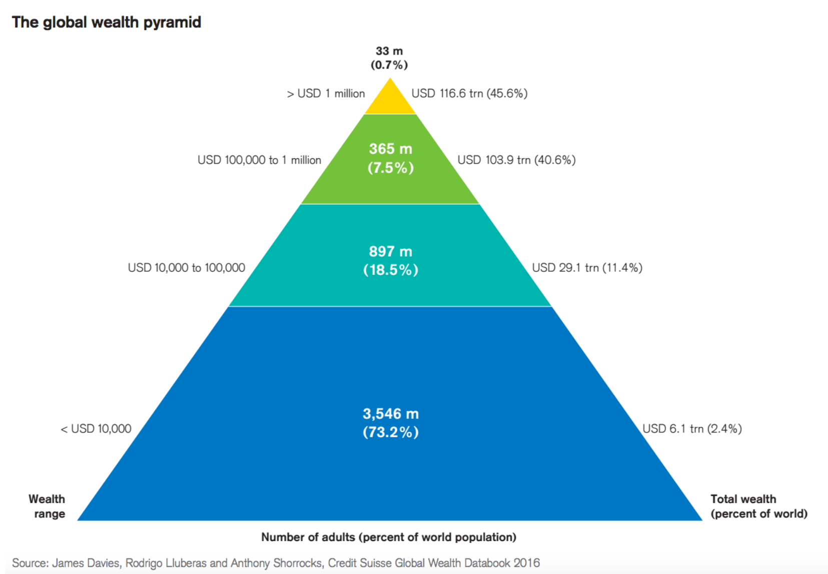
$$= \frac{\sum\limits_{1\ \leq i,\ j\ \leq n} |x_i - x_j|}{2n(n-1)\bar{x}}$$

[This is clearly far too notation-heavy for Math 106; perhaps the formula could be described verbally.]

[Question: Does this have $n(n-1)$ in the denominator instead of n^2 in order for it to be an unbiased estimator for G? This formula with $n(n-1)$ seems to agree with the Gini function in DescTools.]

The Lorenz Curve

Based in Switzerland, Credit Suisse is a large investment bank and financial services company which was founded in 1856. Credit Suisse one of several institutions considered by the Swiss National Bank to be a “systemically important bank” in that its failure could have a severe impact on the world economy. Regulators require systemically important banks to meet stricter standards than other banks to prevent failure. It is therefore in the interest of Credit Suisse to have a clear understanding of the global economy. After the “Great Recession” of the late 2000’s, Credit Suisse began publishing annual Global Wealth Reports and Databooks. The following chart appeared in the 2016 Global Wealth Report:



(Image source: `https://www.statista.com/chart/11857/the-global-pyramid-of-wealth/`, accessed 12/19/2022)

This chart conveys information about how wealth is distributed worldwide. In particular, it describes what percentage of adults have wealth within various ranges. For example, according to the numbers along the left side and the middle of the triangle, 73.2% of adults worldwide have wealth less than US$10,000 as of 2016. About 0.7% of adults have wealth greater than $1 million.

The numbers on the right side tell us, for example, that the bottom 73.2% of the adult population (in terms of wealth) own 2.4% of the world's wealth, while the top 0.7% of adults own 45.6% of the world's wealth.

While this chart provides a vivid display of quite a bit of information, it does raise some questions about how we could analyze wealth distributions more deeply. For example:

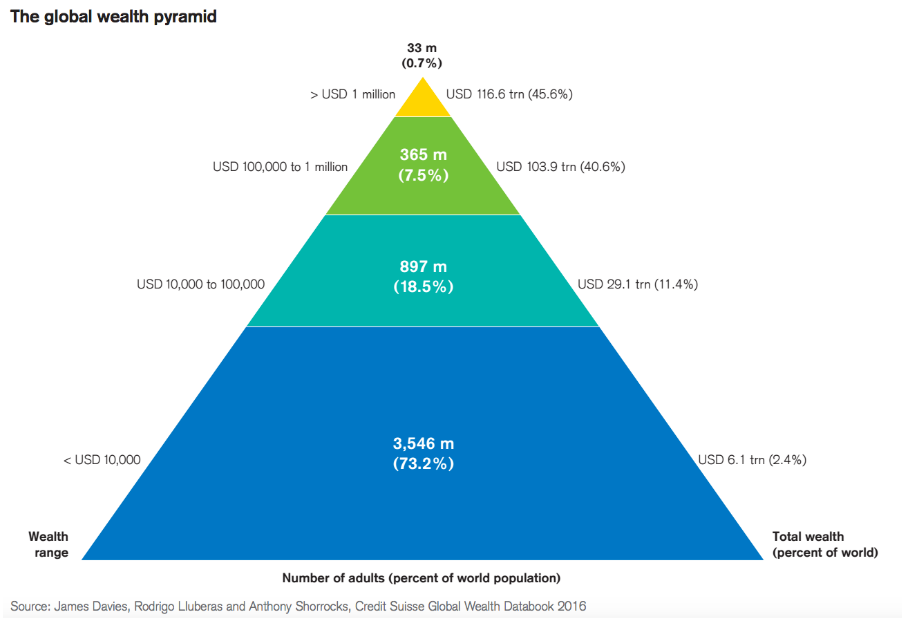
1. The areas of the four colored regions correspond to the percentage of adults in a particular wealth range. For example, the area of the blue region is about 73.2% of the area of the entire triangle. The area of the yellow peak is about 0.7% of the area of the triangle. Clearly the yellow area is much smaller than the blue area, but by how much? That is difficult to judge. People have difficulty making accurate comparisons of two areas; they tend to do better when comparing lengths to one another. The blue area at the bottom is over 100 times as large as the yellow area at the top—is that about what you would have guessed without being given their areas? Perhaps there are better ways to communicate the same information?

2. The wealth ranges (0 to $10,000, $10,000–$100,000, etc.) are somewhat arbitrary, and based on US dollars. Perhaps this chart should be called "*A* Global Wealth Pyramid" rather than "*The* Global Wealth Pyramid".

3. What if we wanted to make similar charts for individual countries which don't use US dollars? Is there a step-by-step procedure we could use which doesn't depend on the unit of currency?

Defining the Lorenz Curve

Let's go back to the Global Wealth Pyramid:



We'll extract the percentages and put them in a table:

Group	Percentage of Population	Percentage of Wealth
1	73.2	2.4
2	18.5	11.4
3	7.5	40.6
4	0.7	45.6

The American economist Max Lorenz (1876–1959) devised a way to transform this table into a more useful format by creating "cumulative" versions of the percentages of population and wealth at each level. The Lorenz Curve gives a very detailed description of the distribution of wealth in a population. In fact, both the Hoover and Gini indices, as well as other measures of wealth inequality can be derived from it.

Here's a description of the idea in words:

From row 1: The bottom 73.2% of the population owns 2.4% of the wealth.

From row 2: The bottom $73.2\% + 18.5\% = 91.7\%$ of the population owns $2.4\% + 11.4\% = 13.8\%$ of the wealth.

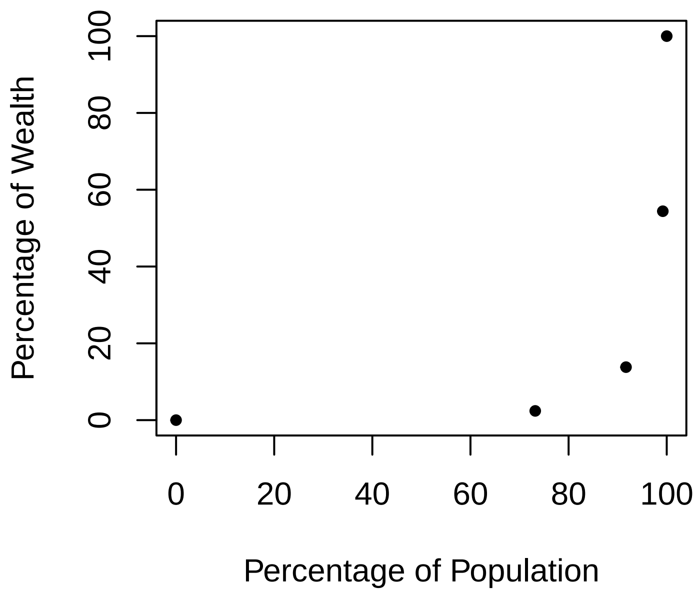
From row 3: The bottom $73.2\% + 18.5\% + 7.5\% = 99.2\%$ of the population owns $2.4\% + 11.4\% + 40.6\% = 54.4\%$ of the wealth.

The fourth row essentially says the bottom 99.9% of the population owns 100% of the wealth (the first percentage should in theory be 100% rather than 99.9%; the discrepancy is due to rounding error). We could even add a zeroth row, which states that the bottom 0% of the population owns 0% of the wealth, which will give us one additional row. Notice we don't need a column for Group number anymore:

The bottom ____ %	Own ____% of the wealth
0	0
73.2	2.4
91.7	13.8
99.2	54.4
100	100

We could now think of each row as representing an ordered pair, and plot the resulting points. The points are:

$$(0,0),\ (73.2,2.4),\ (91.7,13.8),\ (99.2,54.4),\ (100,100)$$



The Lorenz Curve for Canada

Next, we'll look at some data from a different population—the residents of Canada. We'll use statistics drawn from the World Bank's Poverty and Equity database (`datacatalog.worldbank.org/search/dataset/0038020/Poverty-and-Equity-Database`)

Let's separate the population of Canada into two groups: The median (individual) net worth is that value such that 50% of individuals in Canada have net worth less than that amount, and 50% have more than that amount.

The median value divides the population into two halves, a wealthier half, and the poorer half.

According to the database, the poorer half of the population of Canada owns about 28.2% of the total wealth (so the wealthier half owns about 100% minus 28.2% or 71.8% of the wealth.) The poorer half of the population has than half the wealth because they're poorer, of course! The fact that their share of the wealth is only 28.2% rather than 50% indicates there is *some* wealth inequality in Canada.

In fact, this gives us one point on the graph of the Lorenz Curve for Canada. Let's call the function L. We have found that:

$$L(0.50) = 0.282.$$

Equivalently, the point $(0.50, 0.282)$ lies on the graph of L.

By dividing up the population of Canada into more groups (by wealth), we can find more points on the Lorenz Curve. Here are some further statistics from the World Bank database (2016 and 2017 numbers):

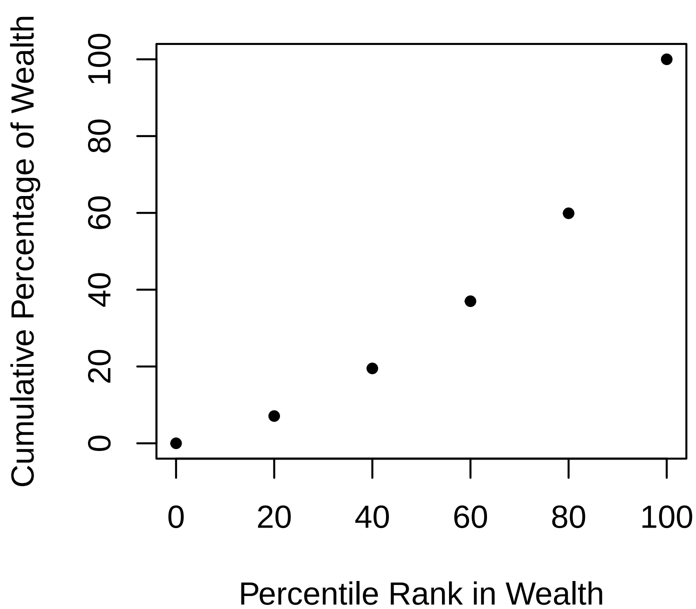
The bottom 20% own 7.1%

The bottom 40% own 19.5%

The bottom 60% own 37.0%

The bottom 80% own 59.9%

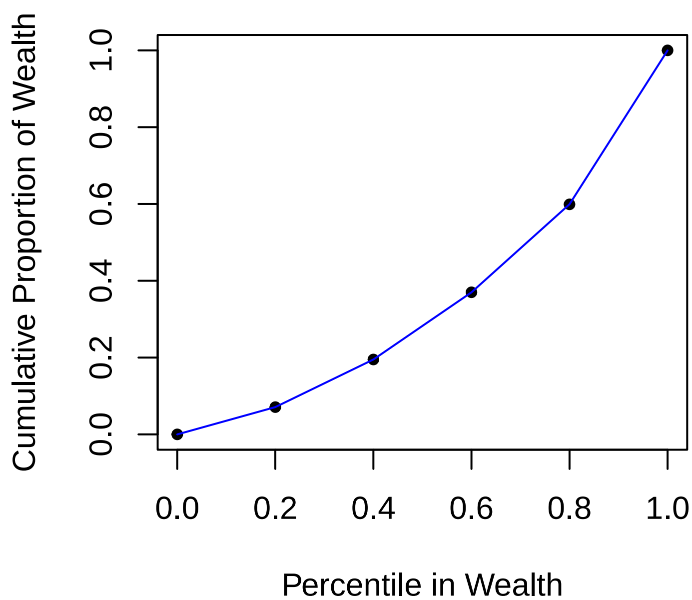
We also know that the bottom 100% (which is the entire population of Canada) owns 100% = 1.00 = 1 of the wealth, and the bottom 0% own 0% of the wealth (assuming no negative wealth here). This gives us six points on the graph of L, plotted below:



I've labeled the horizontal axis using the term "percentile". The 20th percentile in wealth is that wealth value x such that 20% of the population owns less than x dollars in wealth.

With more and more data, we can fill in the graph of L so that it approximates a continuous, unbroken curve. In principle, this process of filling in more points cannot go on forever, because the population of Canada is finite—there are a bit fewer than 40 million residents currently, which means even if we had complete wealth information on every citizen, there could not be more than 40 million points on the graph! However, it's far easier to work with a function defined by a continuous curve than a graph with 40 million points.

In fact, it's likely that with just the six points plotted above, if we draw a continuous curve through them the result will be close enough to the "true" graph of L to be useful.



The Lorenz Curve from Scratch

Let's consider a hypothetical population of eight people:

Name	Wealth
Nicole	11
Deshawn	14
Reid	7
Saniya	11
Evan	5
Ari	10
April	3
Cameron	19

The next steps will be much easier if we sort the table by wealth; I'll rewrite the rows so that the wealth values are in descending order. I'll also add a Totals row at the bottom.

Name	Wealth
Cameron	19
Deshawn	14
Saniya	11
Nicole	11
Ari	10
Reid	7
Evan	5
April	3
Totals	80

For the next step, we want to divide the sorted table into a number of (approximately) equally sized groups. The more groups, the more points on the Lorenz

Curve we'll get. We only have eight in our population, so I'll chose to divide them into four groups of two people.

Name	Wealth
Cameron	19
Deshawn	14
Saniya	11
Nicole	11
Ari	10
Reid	7
Evan	5
April	3
Totals	80

Cameron and Deshawn make up the wealthiest group. It contains two out of the population of eight people, so this group consists of the top 25% in terms of wealth.

The next wealthiest 25% consists of Saniya and Nicole. Then comes the group of Reid and Ari, and finally, Evan and April make up the bottom 25%.

Now let's pool the total wealth within each of the four groups. For example, the total wealth of the top group is $19+14=33$ units of wealth. Here are the results:

Name	Wealth	Group Wealth
Cameron	19	
Deshawn	14	33
Saniya	11	
Nicole	11	22
Ari	10	
Reid	7	17
Evan	5	
April	3	8
Totals	80	80

The Lorenz Curve is a "cumulative" function, so we'll add a new cumulative column. It gives the total wealth of a group plus the wealth of every group below it.

Name	Wealth	Group Wealth	Cumulative Group Wealth
Cameron	19		
Deshawn	14	33	80
Saniya	11		
Nicole	11	22	47
Ari	10		
Reid	7	17	25
Evan	5		
April	3	8	8
Totals	80	80	

Finally, we convert the Cumulative Group Wealth to percentages by dividing the

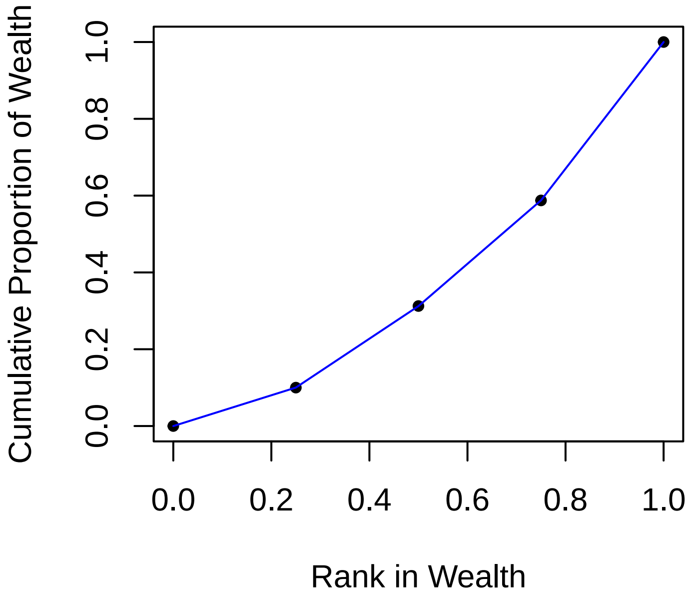
Cumulative Group Wealth Column numbers by 80 (the total wealth), then multiplying by 100% (and rounding a bit if necessary):

Name	Wealth	Group Wealth	Cumulative Group Wealth	Cumulative Group Wealth (%)
Cameron	19			
Deshawn	14	33	80	100%
Saniya	11			
Nicole	11	22	47	58.75%
Ari	10			
Reid	7	17	25	31.25%
Evan	5			
April	3	8	8	10%
Totals	80	80		

This table now gives us four values of the Lorenz Curve Function. It says that the bottom 25% (or 0.25 in decimal form) own 10% of the wealth, the bottom $25\% + 25\% = 50\%$ (or 0.50) own 31.25% of the wealth, the bottom 75% (or 0.75) own 58.75% of the wealth, and the bottom 100% own 100% of the wealth. This last statement is always true of course; similarly, the bottom 0% own 0% of the wealth. Here's a summary, with all values rounded to two decimal places:

x	y
0.00	0.00
0.25	0.10
0.50	0.31
0.75	0.59
1.00	1.00

Here's a plot of those five points:



Note that both the horizontal and vertical coordinates always lie between 0 and 1. This makes it easy to compare Lorenz curves between multiple countries at a glance.

Solutions to Problems

Problem 1

Suppose a population of five people have wealth values 7, 10.5, 12, 12, and 23. Round your answer to the nearest tenth of a percent. Find the Hoover Index.

Solution:

We use the notation $x_1 = 7$, $x_2 = 10.5$, $x_3 = 12$, $x_4 = 12$, and $x_5 = 23$.

The average of these wealth amounts is:

$$\bar{x} = \frac{7+10.5+12+12+23}{5} == 12.9$$

Then:

$$\begin{aligned}
H &= \frac{\sum_{i=1}^{5}}{2\sum_{x=1}^{5} x_i} \times 100\% \\
&= \frac{|7-12.9|+|10.5-12.9|+|12-12.9|+|12-12.9|+|23-12.9|}{2\times 5\times 12.9} \times 100\% \\
&= \frac{5.9+2.4+0.9+0.9+10.1}{129} \times 100\% \\
&= 0.1566 \times 100\% \\
&= 15.7\%
\end{aligned}$$

Problem 2

Suppose everyone in a population has his or her wealth doubled. Will that affect the Hoover Index? What if, instead, each person receives $10,000—will the Hoover Index be affected? Finally, suppose a wealthy person gives a poorer person a small portion of their wealth (not enough to bring the poorer person up to average). Will H be affected? You can use a very small hypothetical population to investigate these questions, for example, three people with wealth amounts \$1000, \$2000, and \$3000.

Answers: No, yes (unless they all were equally wealthy at the beginning), yes. These are three properties that "good" measures of inequality should have.

Notes

[1]Understanding attitudes to inequality. *Nat Hum Behav* 1, 0166 (2017). https://doi.org/10.1038/s41562-017-0166

[2]Norton, M. I., & Ariely, D. (2011). Building a Better America—One Wealth Quintile at a Time. *Perspectives on Psychological Science*, 6(1), 9–12. https://doi.org/10.1177/1745691610393524"

[3]Edward Allen, "Relation Between Two Income Inequality Measures: The Gini coefficient and the Robin Hood Index," WSEAS Transactions on Business and Economics, vol. 19, pp. 760–770, 2022