In[1]:=
```
Clear[i, s, x, y, α, β];
s[α_, β_] :=
 NDSolve[{β * y''[x] + α * Sin[y[x]] y[x] == 0, y[0] == 1, y'[0] == 0}, y, {x, 0, 50}]
```

In[2]:=
```
data1 = Flatten[Table[{α, β, (Part[(y[x] /. s[α, β]) /. {x → 50.0}, 1]) - 0.7},
    {α, -1.0, 12.7, 0.2}, {β, -0.98, 20.5, 0.2}], 1]
```

Out[2]=
```
{{-1., -0.98, 0.225103}, {-1., -0.78, -2.35575}, {-1., -0.58, -4.62111}, {-1., -0.38, -0.0627312},
 {-1., -0.18, -1.09886}, {-1., 0.02, 2.82746}, {-1., 0.22, 0.790605}, {-1., 0.42, 0.484073}, {-1., 0.62, 3.38737},
 ··· 7435 ···, {12.6, 19.02, 0.140052}, {12.6, 19.22, 0.22777}, {12.6, 19.42, 0.281622}, {12.6, 19.62, 0.299994},
 {12.6, 19.82, 0.283418}, {12.6, 20.02, 0.23437}, {12.6, 20.22, 0.156871}, {12.6, 20.42, 0.0560789}}
```

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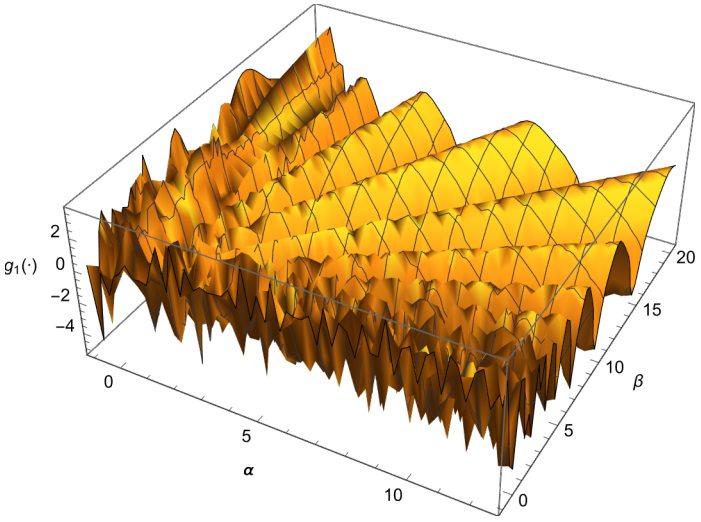
In[3]:=
```
(*ListPlot3D[data0]*)
```

In[4]:=
```
gg = Interpolation[data1]
```

Out[4]=
```
InterpolatingFunction[ Domain: {{-1., 12.6}, {-0.98, 20.4}}
                       Output: scalar ]
```

In[5]:=
```
Plot3D[gg[x1, y1], {x1, -1., 12.6}, {y1, -0.98, 20.4}, AxesLabel → {"α", "β", "g₁(·)"}]
```

Out[5]=



In[6]:=
```
fn = x1
```

Out[6]=
```
x1
```

In[7]:=
```
n = 2; m = 1; K = 3; aw = (K - 1.) / K;
```

In[8]:=
```
ndiv = 30; Δ1 = 5.0; Δ2 = 7.0; t1 = 1.001; t2 = 2.001;
```

In[9]:=
```
xz = {x1 → Δ1, x2 → Δ2};
f = Simplify[(fn /. xz) + (∂x1 fn /. xz) * (x1 - (x1 /. xz)) + (∂x2 fn /. xz) * (x2 - (x2 /. xz))]
```

Out[10]=
```
0. + x1
```

In[11]:=
```
Off[Join::heads]; Off[Set::write];
lbnd = 0.0;
```

In[13]:=
```
Cons = {Δ1 - x1 ≥ 0, Δ2 - x2 ≥ 0, x1 ≥ t1, x2 ≥ t2}; x0 = 0;
```

In[14]:=
```
Constraints = Cons
```

Out[14]=
```
{5. - x1 ≥ 0, 7. - x2 ≥ 0, x1 ≥ 1.001, x2 ≥ 2.001}
```

In[15]:=
```
δ1 = Δ1/2; δ2 = Δ2/2; η1 = 40; η2 = 40; pos = 1; mfo = 20.0; ialg = 1;
σ = 0.98; γ1 = 1.0;
τ = 0.95;
β = 1.0; Pos = pos = 1; scut = Constraints;
```

In[18]:=
```
lbnd := Which[i == 1, t1, i == 2, t2]
```

In[19]:=
```
For[i = 0, i ≤ 1, {For[j = 0, j ≤ 1, { lw[i, j] = lbnd + (j - 1) * δi;
    ur[i, j] = lbnd + (j) * δi; Print[lw[i, j], " @ ", ur[i, j]];}, j++]}, i++]
```

```
1.001 @ 3.501
3.501 @ 6.001
2.001 @ 5.501
5.501 @ 9.001
```

In[20]:=
```
Cons = {Δ1 - x1 ≥ 0, Δ2 - x2 ≥ 0, x1 ≥ t1, x2 ≥ t2}; x0 = 0;
```

In[21]:=
```
Constraints = Cons
```

Out[21]=
```
{5. - x1 ≥ 0, 7. - x2 ≥ 0, x1 ≥ 1.001, x2 ≥ 2.001}
```

In[22]:=
```
scut = {};
```

In[23]:=

In[24]:=
```
cnt = 0; For[j1 = 0, j1 ≤ 1, {For[j2 = 0, j2 ≤ 1, {cnt = cnt + 1; data[j1, j2] = {};
    data[j1, j2] = Flatten[
      Table[{x1, x2, -gg[x1, x2], f}, {x1, lw[1, j1], ur[1, j1], (ur[1, j1] - lw[1, j1])/η1},
       {x2, lw[2, j2], ur[2, j2], (ur[2, j2] - lw[2, j2])/η2}], 1];}, j2++];}, j1++]
```

In[25]:=
```
data[1, 2]
```

Out[25]=
```
{{1.001, 5.501, 0.283837, 1.001}, {1.001, 5.5885, 0.161042, 1.001}, {1.001, 5.676, 0.0491547, 1.001},
 {1.001, 5.7635, -0.0498808, 1.001}, {1.001, 5.851, -0.133893, 1.001}, {1.001, 5.9385, -0.201139, 1.001},
 {1.001, 6.026, -0.251403, 1.001}, {1.001, 6.1135, -0.283277, 1.001}, {1.001, 6.201, -0.298052, 1.001},
 ··· 1664 ···, {3.501, 8.3885, 1.00611, 3.501}, {3.501, 8.476, 0.875603, 3.501},
 {3.501, 8.5635, 0.748353, 3.501}, {3.501, 8.651, 0.624004, 3.501}, {3.501, 8.7385, 0.501752, 3.501},
 {3.501, 8.826, 0.383393, 3.501}, {3.501, 8.9135, 0.269359, 3.501}, {3.501, 9.001, 0.161777, 3.501}}
```

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In[26]:=
```
Print[cnt];
```

```
4
```

In[27]:=
```
ialg = 1;
```

In[28]:=
```
InitialSolution = Minimize[f, Constraints, {x1, x2}]
```

Out[28]=
$$\{1.001, \{x_1 \to 1.001, x_2 \to 2.001\}\}$$

In[29]:=
```
xz = Part[InitialSolution, 2];
f = (fn /. xz) + (∂x1 fn /. xz) * (x1 - (x1 /. xz)) +
    (∂x2 fn /. xz) * (x2 - (x2 /. xz)) + (∂x3 fn /. xz) * (x3 - (x3 /. xz));
```

In[31]:=
```
Values = {(gg[x1, x2] /. Part[InitialSolution, 2])}
xx = -Max[-Values, x0];
gCut =
 ExpandAll[xx + (1/0.0001 ((gg[x1, x2] /. Part[InitialSolution, 2]) - (gg[x1 - 0.0001, x2] /. Part[
          InitialSolution, 2])) * (x1 - (x1 /. Part[InitialSolution, 2])) +
     1/0.0001 ((gg[x1, x2] /. Part[InitialSolution, 2]) - (gg[x1, x2 - 0.0001] /.
          Part[InitialSolution, 2])) * (x2 - (x2 /. Part[InitialSolution, 2])))]
Cons = Join[Cons, {gCut ≥ 0}];
```

Out[31]=
$$\{-5.02533\}$$

Out[33]=
$$5.2299 + 12.5065\, x_1 - 11.3814\, x_2$$

In[35]:=
```
Off[InterpolatingFunction::dmval]; cnv = {};
```

In[36]:=
```
For[ii = 0, ii ≤ 12, {
ialg = ialg + 1; (*ψ1=Part[Part[xz,1],2];
   ψ2=Part[Part[xz,2],2];
   ur[1,j1]=ψ1;
   ur[2,j2]=ψ2;*)
pos = Pos; cnt = 0; For[j1 = 0, j1 ≤ 1,
    {For[j2 = 0, j2 ≤ 1, {cnt = cnt + 1; data[j1, j2] = Flatten[Table[{x1, x2, -gg[x1, x2], f},
```

```
            {x1, lw[1, j1], ur[1, j1], (ur[1, j1] - lw[1, j1]) / η1}, {x2, lw[2, j2],
             ur[2, j2], (ur[2, j2] - lw[2, j2]) / η2}], 1]; }, j2++];}, j1++];
Print[cnt];
cnt = 0; For[j1 = 0, j1 ≤ 1, {For[j2 = 0, j2 ≤ 1, {
 cnt = cnt + 1;
fo[j1, j2] = Table[Part[Part[data[j1, j2], i], 4], {i, 1, Length[data[j1, j2]]}];
ev[j1, j2] = Table[{Abs[τ * Min[fo[j1, j2]] - Part[Part[data[j1, j2], i], 4]]},
          {i, 1, Length[data[j1, j2]]}];
eo3[j1, j2] = Part[Flatten[Position[ev[j1, j2], Min[ev[j1, j2]]]], 1];
kdat[j1, j2] = Part[data[j1, j2], eo3[j1, j2]];
kxo[j1, j2] = Part[kdat[j1, j2], 3];
kx1[j1, j2] = Part[kdat[j1, j2], 1];
kx2[j1, j2] = Part[kdat[j1, j2], 2];
rr = {x1 → kx1[j1, j2], x2 → kx2[j1, j2]};
Θ1 = 1/0.0001 ((gg[x1, x2] /. rr) - (gg[x1 - 0.0001, x2] /. rr));
Θ2 = 1/0.0001 ((gg[x1, x2] /. rr) - (gg[x1, x2 - 0.0001] /. rr));
CuttingHyperplane2[cnt] =
Simplify[(0.98 * kxo[j1, j2] +
((Θ1 * (x1 - kx1[j1, j2])) + Θ2 * (x2 - kx2[j1, j2]))) /
            Max[kxo[j1, j2], kx1[j1, j2], kx2[j1, j2]]];
}, j2++];}, j1++];
γ = Max[Flatten[Table[mxq[j1, j2], {j1, 1, 2}, {j2, 1, 2}]]];
pi_[a_] := Coefficient[a, xi, 1];
p[i_] := If[Part[ans, i] < 0, Rescale[Part[ans, i], {0, 10^4}, {0, Δi}], Part[ans, i]];
For[cnt = 1, cnt ≤ 3, {
aa1 = CuttingHyperplane2[1];
aa2 = CuttingHyperplane2[cnt];
a1 = {p1[aa1], p2[aa1]};
a2 = {p1[aa2] * RandomReal[{0.95, 0.99}], p2[aa2] * RandomReal[{0.98, 0.996}]};
b1 = Coefficient[Coefficient[aa1, x1, 0], x2, 0];
b2 = Coefficient[Coefficient[aa2, x1, 0], x2, 0];
      ax1 = (a1) / (a1.a2);
      ax2 = (a2) / (a1.a2);
bx1 = (b1) / (b1 * b2);
      bx2 = (b2) / (b1 * b2);
ω = ArcCos[Mod[ax1.ax2, 1]];
a1 = ax1; a2 = ax2; b1 = bx1; b2 = bx2;
x1 = ((b1 - b2 * Cos[ω]) / Sin[ω]^2) * {{Part[a1, 1]}, {Part[a1, 2]}} +
        ((b2 - b1 * Cos[ω]) / Sin[ω]^2) * {{Part[a2, 1]}, {Part[a2, 2]}};
A = {a1, a2};
P = IdentityMatrix[2];
G = A.Transpose[P];
Off[RowReduce::luc]; Z = RowReduce[G];
```

```
F = {Part[Z, 1], Part[Z, 2]}; ab = {Part[Part[F, 1], 2], Part[Part[F, 2], 2]};
ς = -1;
sv = P.(-ab);
cx = (sv * ς);
ans = Flatten[x1 + cx];
pnt[cnt] = {p[1], p[2]};
}, cnt++];
cnt = 0; ca = {};
For[ cnt = 1, cnt ≤ 3, {
rh = {x1 → Part[pnt[cnt], 1], x2 → Part[pnt[cnt], 2]};
pdat = {x1, x2, gg[x1, x2]} /. rh; (*y1=Part[pdat,1]/.rh;
       y2=Part[pdat,2]/.rh;
       Print["yvals=",y1," * ",y2]*);
      (*Clear[i];
      For[i=0,i<2,{zi=If[yi<ti,ti,yi];
         zi=If[yi>Δi,Δi,yi]; (*zi=If[yi≥ti && yi≤Δi,yi,yi]*);
         (* zi=Rescale[zi,{-20,10^4},{ti,Δi}]*);
        Print[" *** xval=",zi];},i++] ;
      Print["xvals=",z1," * ",z2];*);
      γ = (x1 + x2) /. rh(*{x1→z1,x2→z2}*);
      x1 = 1.0 * (x1 - 0.5 * (x1 - γ)) /. rh(*{x1→z1,x2→z2}*);
      x2 = 1.0 * (x2 - 0.5 * (x2 - γ)) /. rh(*{x1→z1,x2→z2}*);
      rh2 = {x1 → x1, x2 → x2}(*{x1→z1,x2→z2}*);
      pdat = {x1, x2, gg[x1, x2]} /. rh2; Print["xvals=", x1, " * ", x2];
ca = Join[ca, {pdat}];
pxo = Part[pdat, 3];
px1 = Part[pdat, 1];
px2 = Part[pdat, 2];
r3 = {x1 → px1, x2 → px2};
θ1 = 1/0.0001 ((gg[x1, x2] /. r3) - (gg[x1 - 0.0001, x2] /. r3));
θ2 = 1/0.0001 ((gg[x1, x2] /. r3) - (gg[x1, x2 - 0.0001] /. r3));
Cut1[cnt] = Simplify[(γ1 * pxo + (θ1 * (x1 - px1) + θ2 * (x2 - px2))) /
          (γ1 * If[Max[pxo, px1, px2] > 1, Max[pxo, px1, px2], 1])];
      Label[Hi];}, cnt++];
rhs2 =
     Expand[FindFit[ca, 0.98 * pxo + ((ϕ1 * (x1 - px1) + ϕ2 * (x2 - px2))), {ϕ1, ϕ2}, {x1, x2}]];
aa1 = Simplify[0.98 * pxo + ((ϕ1 * (x1 - px1) + ϕ2 * (x2 - px2))) /. rhs2];
    a1 = {p1[aa1], p2[aa1]};
    GCH = Simplify[ Chop[ (1.0 * aa1) / (√(a1.a1)) ]];
    (*Print["GCH=",GCH];*)
Constr1 = {};
    For[cnt = 1, cnt ≤ 3, {Constr1 = Join[Constr1, {CuttingHyperplane2[cnt] ≥ 0}];}, cnt++];
Constr2 = {}; For[cnt = 1, cnt ≤ 3, {Constr2 = Join[Constr2, {Cut1[cnt] ≥ 0}];}, cnt++];
```

```
ConstrT = Join[Constr1, Constr2];
Cons = Join[Cons, {GCH ≥ 0}];
temp = Join[Cons, Constr2];
   Constraints = Join[Constraints, temp];
   Print["**1**", xz, f];
InitialSolution = Minimize[f, Constraints, {x1, x2}]; Print[InitialSolution];
xz = Part[InitialSolution, 2];
   f = Simplify[(fn /. xz) + (∂x1 fn /. xz) * (x1 - (x1 /. xz)) + (∂x2 fn /. xz) * (x2 - (x2 /. xz))];
   Print["**2**", xz, f];
Values = {(gg[x1, x2] /. Part[InitialSolution, 2])};
xx = If[ialg > 2, -Max[-Values], -Max[-Values]];
   Print[ialg, " ** xx=", xx]; x0 = xx; cnv = Join[cnv, {{ialg, xx}}];
   Print["***   ", {Values}, xx]; Print[xx]; If[Abs[xx] < 1.0 × 10^-10, {Print[xx];
     Break[];}];
Pos = 1; Print["Pos=", Pos, "   ", xx];
gCut =
    ExpandAll[xx + (1/0.0001 ((gg[x1, x2] /. Part[InitialSolution, 2]) - (gg[x1 - 0.0001, x2] /.
            Part[InitialSolution, 2])) * (x1 - (x1 /. Part[InitialSolution, 2])) +
        1/0.0001 ((gg[x1, x2] /. Part[InitialSolution, 2]) - (gg[x1, x2 - 0.0001] /.
            Part[InitialSolution, 2])) * (x2 - (x2 /. Part[InitialSolution, 2])))];
   Print["gcut=", gCut ≥ 0];
Cons = Join[Cons, {gCut ≥ 0}];
}, ii++];
4
xvals=5.0482 * 2.5226
xvals=0.518823 * 0.969665
xvals=97.5014 * 195.16
**1** {x1 → 1.001, x2 → 2.001} 0. + x1
{3.52688, {x1 → 3.52688, x2 → 2.001}}
**2** {x1 → 3.52688, x2 → 2.001} 0. + x1
2 ** xx=-3.20288
***   {{-3.20288}}-3.20288
-3.20288
Pos=1   -3.20288
gcut=-30.6073 + 17.8055 x1 - 17.6879 x2 ≥ 0
4
xvals=5.03585 * 2.51642
xvals=0.519388 * 0.970077
```

xvals=96.1542 * 192.464

1 {$x_1 \to 3.52688, x_2 \to 2.001$} 0. + x_1

{3.70676, {$x_1 \to 3.70676, x_2 \to 2.001$}}

2 {$x_1 \to 3.70676, x_2 \to 2.001$} 0. + x_1

3 ** xx=-0.899666

*** {{-0.899666}}-0.899666

-0.899666

Pos=1 -0.899666

gcut=$-5.70882 + 7.30551\, x_1 - 11.1298\, x_2 \geq 0$

4

xvals=5.04326 * 2.52012

xvals=0.517741 * 0.968663

xvals=84.8905 * 169.918

1 {$x_1 \to 3.70676, x_2 \to 2.001$} 0. + x_1

{3.82991, {$x_1 \to 3.82991, x_2 \to 2.001$}}

2 {$x_1 \to 3.82991, x_2 \to 2.001$} 0. + x_1

4 ** xx=-0.227381

*** {{-0.227381}}-0.227381

-0.227381

Pos=1 -0.227381

gcut=$-12.6315 + 3.91532\, x_1 - 1.29496\, x_2 \geq 0$

4

xvals=5.04483 * 2.52091

xvals=0.518747 * 0.969516

xvals=86.855 * 173.85

1 {$x_1 \to 3.82991, x_2 \to 2.001$} 0. + x_1

{3.88799, {$x_1 \to 3.88799, x_2 \to 2.001$}}

2 {$x_1 \to 3.88799, x_2 \to 2.001$} 0. + x_1

5 ** xx=-0.052995

*** {{-0.052995}}-0.052995

-0.052995

Pos=1 -0.052995

gcut=$-19.248 + 2.07562\, x_1 + 5.5597\, x_2 \geq 0$

4

xvals=5.029 * 2.51299

xvals=0.516966 * 0.968005

xvals=84.7096 * 169.556

1 {x$_1$ → 3.88799, x$_2$ → 2.001} 0. + x$_1$

{3.89079, {x$_1$ → 3.89079, x$_2$ → 2.00948}}

2 {x$_1$ → 3.89079, x$_2$ → 2.00948} 0. + x$_1$

6 ** xx=-0.00213572

*** {{-0.00213572}}-0.00213572

-0.00213572

Pos=1 -0.00213572

gcut=-21.1686 + 2.9834 x$_1$ + 4.75678 x$_2$ ≥ 0

4

xvals=5.04459 * 2.52079

xvals=0.518722 * 0.969576

xvals=94.125 * 188.402

1 {x$_1$ → 3.89079, x$_2$ → 2.00948} 0. + x$_1$

{3.89092, {x$_1$ → 3.89092, x$_2$ → 2.00986}}

2 {x$_1$ → 3.89092, x$_2$ → 2.00986} 0. + x$_1$

7 ** xx=-7.22441×10^{-6}

*** $\{\{-7.22441\times10^{-6}\}\}-7.22441\times10^{-6}$

-7.22441×10^{-6}

Pos=1 -7.22441×10^{-6}

gcut=-21.2485 + 3.02305 x$_1$ + 4.71974 x$_2$ ≥ 0

4

xvals=5.02293 * 2.50995

xvals=0.51898 * 0.969644

xvals=93.7643 * 187.68

1 {x$_1$ → 3.89092, x$_2$ → 2.00986} 0. + x$_1$

{3.89092, {x$_1$ → 3.89092, x$_2$ → 2.00986}}

2 {x$_1$ → 3.89092, x$_2$ → 2.00986} 0. + x$_1$

8 ** xx=-9.5115×10^{-9}

*** $\{\{-9.5115\times10^{-9}\}\}-9.5115\times10^{-9}$

-9.5115×10^{-9}

Pos=1 -9.5115×10^{-9}

gcut=-21.2487 + 3.02319 x$_1$ + 4.71961 x$_2$ ≥ 0

4

xvals=5.02652 * 2.51175

xvals=0.51807 * 0.96893

```
xvals=84.4995 * 169.135
```

```
**1** {x1 → 3.89092, x2 → 2.00986} 0. + x1
```

```
{3.89092, {x1 → 3.89092, x2 → 2.00986}}
```

```
**2** {x1 → 3.89092, x2 → 2.00986} 0. + x1
```

```
9 ** xx=-1.24562×10^-11
```

```
***   {{-1.24562×10^-11}}-1.24562×10^-11
```

```
-1.24562×10^-11
```

```
-1.24562×10^-11
```

In[37]:=
```
cnv
```

Out[37]=
```
{{2, -3.20288}, {3, -0.899666}, {4, -0.227381}, {5, -0.052995},
 {6, -0.00213572}, {7, -7.22441×10^-6}, {8, -9.5115×10^-9}, {9, -1.24562×10^-11}}
```

In[38]:=
```
ListPlot[cnv, Joined → True, AxesLabel → {"Iteration Number", "gr(·)"}]
```

Out[38]=