In[1]:=
```
fn = -3.0 * x3 * x1^2 + 7.5 * x2^2 - 5.0 * x1^2;
```

In[2]:=
```
g1 = Sqrt[x1] - 4. * x2^2 - 2 * x3;
g2 = 6.0 * x3^2 - 1.2 * x1^2 + 3.0 * x3 * x2;
```

In[4]:=
```
Minimize[{fn, g1 ≥ 0, g2 ≥ 0, 1.899` - x1 ≥ 0, x1 ≥ 0.001`,
  1.699` - x2 ≥ 0, x2 ≥ 0.001`, 0.899` - x3 ≥ 0, x3 ≥ 0.001`}, {x1, x2, x3}]
```

Out[4]=
```
{-10.9835, {x1 → 1.28349, x2 → 0.0550422, x3 → 0.560397}}
```

In[5]:=
```
SeedRandom[4];
```

In[6]:=
```
(* Initialization *)
```

In[7]:=
```
n = 3; m = 2; K = 3; Ñ1 = 10; Ñ2 = 10; Ñ3 = 10; pos = 1;
```

In[8]:=
```
Pos = pos;
```

In[9]:=
```
τ = 0.95; γ1 = 0.98;
```

In[10]:=
```
(* Box constraints. *)
```

In[11]:=
```
a1 = 0.001; a2 = 0.001; a3 = 0.001;
```

In[12]:=
```
Δ1 = 1.9; Δ2 = 1.4; Δ3 = 0.9; q1 = 0.59`; qq2 = 1.0; TOL = 1.0 × 10^-7;
```

In[13]:=
```
bi_ := q1 * Δi
```

In[14]:=
```
{b1, b2, b3}
```

Out[14]=
```
{1.121, 0.826, 0.531}
```

In[15]:=
```
μ1,1 = a1
μ2,1 = a2
μ3,1 = a3
```

Out[15]=
```
0.001
```

Out[16]=
```
0.001
```

Out[17]=
```
0.001
```

In[18]:=
$$\mu_{i\_,k\_} := \mu_{i,k-1} + \frac{b_i - a_i}{(K-1)}$$

In[19]:=
```
{{μ1,1, μ1,2, μ1,3}, {μ2,1, μ2,2, μ2,3}, {μ3,1, μ3,2, μ3,3}}
```

Out[19]=
```
{{0.001, 0.561, 1.121}, {0.001, 0.4135, 0.826}, {0.001, 0.266, 0.531}}
```

In[20]:=
$$\delta_{i\_,k\_} := \frac{\mu_{i,k+1} - \mu_{i,k}}{\tilde{N}_i}$$

In[21]:=
$$\{\{\delta_{1,1}, \delta_{1,2}, \delta_{1,3}\}, \{\delta_{2,1}, \delta_{2,2}, \delta_{2,3}\}, \{\delta_{3,1}, \delta_{3,2}, \delta_{3,3}\}\}$$

Out[21]=
$$\{\{0.056, 0.056, 0.056\}, \{0.04125, 0.04125, 0.04125\}, \{0.0265, 0.0265, 0.0265\}\}$$

In[22]:=
$$\mathbf{cx_{i\_} := x_i \to (b_i)}$$

In[23]:=
$$\mathbf{cl_{i\_} := \{b_i - x_i \geq 0,\ x_i \geq a_i\}}$$

In[24]:=
$$\mathbf{xz = Flatten[Table[\{cx_i\}, \{i, 1, 3\}]]}$$

Out[24]=
$$\{x_1 \to 1.121, x_2 \to 0.826, x_3 \to 0.531\}$$

In[25]:=
$$\mathbf{f = Simplify\left[(fn /. xz) + \sum_{jj=1}^{n} \left(\partial_{x_{jj}} fn /. xz\right) * (x_{jj} - (x_{jj} /. xz))\right]}$$

Out[25]=
$$5.16979 - 14.7815\, x_1 + 12.39\, x_2 - 3.76992\, x_3$$

In[26]:=
$$\mathbf{Cons = Flatten[Table[cl_i, \{i, 1, 3\}]]}$$

Out[26]=
$$\{1.121 - x_1 \geq 0, x_1 \geq 0.001, 0.826 - x_2 \geq 0, x_2 \geq 0.001, 0.531 - x_3 \geq 0, x_3 \geq 0.001\}$$

In[27]:=
$$\mathbf{x0 = 0;}$$

In[28]:=
$$\mathbf{Constraints = Cons}$$

Out[28]=
$$\{1.121 - x_1 \geq 0, x_1 \geq 0.001, 0.826 - x_2 \geq 0, x_2 \geq 0.001, 0.531 - x_3 \geq 0, x_3 \geq 0.001\}$$

In[29]:=
$$\mathbf{xy_{i\_,k\_} := \mu_{i,k} \leq x_i \leq \mu_{i,k+1}}$$

In[30]:=
$$\mathbf{Z_{i,k} = Table\left[\left\{xy_{i,k}\right\}, \{i, 1, n\}, \{k, 1, K-1\}\right]}$$

Out[30]=
$$\{\{\{0.001 \leq x_1 \leq 0.561\}, \{0.561 \leq x_1 \leq 1.121\}\},$$
$$\{\{0.001 \leq x_2 \leq 0.4135\}, \{0.4135 \leq x_2 \leq 0.826\}\}, \{\{0.001 \leq x_3 \leq 0.266\}, \{0.266 \leq x_3 \leq 0.531\}\}\}$$

In[31]:=
```
Clear[i, k, k1, k2, k3];
GenD := Table[Table[Table[dat[k1, k2, k3] =
      Table[{x1, x2, x3, -g_pos, f}, {x1, μ1,k1, μ1,k1+1, δ1,k1}, {x2, μ2,k2, μ2,k2+1, δ2,k2},
       {x3, μ3,k3, μ3,k3+1, δ3,k3}], {k1, 1, K - 1}], {k2, 1, K - 1}], {k3, 1, K - 1}];
```

In[32]:=
```
Clear[i, k, k1, k2, k3]; GenD;
```

In[33]:=
```
Gen2 := Table[Table[Table[data[k1, k2, k3] = Flatten[dat[k1, k2, k3], 2], {k1, 1, K - 1}],
    {k2, 1, K - 1}], {k3, 1, K - 1}]
```

In[34]:=
```
Clear[i, k, k1, k2, k3]; Gen2;
```

In[35]:=
```
scut = {};  cnt = 0;
```

In[36]:=
```
InitialSolution = Minimize[fn, Constraints, {x1, x2, x3}]
```

Out[36]=
$$\{-8.28503, \{x_1 \to 1.121, x_2 \to 0.000999991, x_3 \to 0.531\}\}$$

In[37]:=
```
xz = Part[InitialSolution, 2];
f = Simplify[(fn /. xz) + Sum[(∂xjj fn /. xz) * (xjj - (xjj /. xz)), {jj, 1, n}]]
```

Out[38]=
$$10.2869 - 14.7815\, x_1 + 0.0149999\, x_2 - 3.76992\, x_3$$

In[39]:=
```
Values = {Table[(gjj /. Part[InitialSolution, 2]), {jj, 1, m}]};
xx = -Max[-Values]; x0 = xx;
```

In[41]:=
```
gCut = ExpandAll[
  xx + Sum[(∂xjj gPos /. Part[InitialSolution, 2]) * (xjj - (xjj /. Part[InitialSolution, 2])), {jj, 1, n}]]
```

Out[41]=
$$0.52939 + 0.472245\, x_1 - 0.00799993\, x_2 - 2\, x_3$$

In[42]:=
```
Conz = Cons;
```

In[43]:=
```
Cons = Join[Cons, {gCut ≥ 0}]
```

Out[43]=
$$\{1.121 - x_1 \geq 0,\ x_1 \geq 0.001,\ 0.826 - x_2 \geq 0,\ x_2 \geq 0.001,\ 0.531 - x_3 \geq 0,\ x_3 \geq 0.001,\ 0.52939 + 0.472245\, x_1 - 0.00799993\, x_2 - 2\, x_3 \geq 0\}$$

In[44]:=
```
(*ψi_:=qq2*Part[Part[xz,i],2]*) ψi_ := q1 * Δi; bi_ := ψi;
```

In[45]:=
```
{ψ1, ψ2, ψ3}
```

Out[45]=
$$\{1.121, 0.826, 0.531\}$$

In[46]:=
```
ialg = 0;
```

In[47]:=
```
Off[Set::write]; Off[Join::heads];
```

In[48]:=
```
For[ip = 0, ip ≤ 28, {
ialg = ialg + 1;
  Print["Upper bound=", {b1, b2, b3}];
  Clear[i, k, k1, k2, k3];
  GenD;
  Clear[i, k, k1, k2, k3]; Gen2;
  Clear[ii, k1, k2, k3];
g2 := Flatten[
    Table[Part[Part[data[k1, k2, k3], i], n + 2], {i, 1, Length[data[k1, k2, k3]]}], 4];
fodat := Flatten[Table[
     Table[Table[fo[k1, k2, k3] = g2, {k1, 1, K - 1}], {k2, 1, K - 1}], {k3, 1, K - 1}], 1];
evdat := Flatten[Table[Table[Table[ev[k1, k2, k3] =
         Table[Abs[τ * Min[fo[k1, k2, k3]] - Part[Part[data[k1, k2, k3], ii], n + 2]], {ii, 1,
           Length[data[k1, k2, k3]]}], {k1, 1, K - 1}], {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
epos := Flatten[Table[Table[Table[
       eo3[k1, k2, k3] = Part[Flatten[Position[ev[k1, k2, k3], Min[ev[k1, k2, k3]]]], 1],
```

```
        {k1, 1, K - 1}], {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
kdata[k1_, k2_, k3_] := Module[{tt}, Part[data[k1, k2, k3], Part[eo3[k1, k2, k3]]]];
kkdat := Flatten[Table[Table[Table[kdat[k1, k2, k3] = kdata[k1, k2, k3], {k1, 1, K - 1}],
       {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
  fodat;
mxqdat;
evdat;
epos;
kkdat;
kx[0, k1_, k2_, k3_] := Part[kdat[k1, k2, k3], n + 1];
  kx[1, k1_, k2_, k3_] := Part[kdat[k1, k2, k3], 1];
kx[2, k1_, k2_, k3_] := Part[kdat[k1, k2, k3], 2];
kx[3, k1_, k2_, k3_] := Part[kdat[k1, k2, k3], 3];
Clear[kkcon];
  kkcon[k1_, k2_, k3_] := Module[{tt},
    {kx[0, k1, k2, k3], kx[1, k1, k2, k3], kx[2, k1, k2, k3], kx[3, k1, k2, k3]}];
kconst := Flatten[Table[
     Table[Table[kkcon[k1, k2, k3], {k1, 1, K - 1}], {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
kconst;
px_{i_,k1_,k2_,k3_} := x_i → kx[i, k1, k2, k3];
rr_{k1_,k2_,k3_} := {px_{1,k1,k2,k3}, px_{2,k1,k2,k3}, px_{3,k1,k2,k3}};
θ_{i_,k1_,k2_,k3_} := (∂_{x_i} g_pos) /. rr_{k1,k2,k3};
CuttingHyperplane2[k1_, k2_, k3_] :=
Simplify[γ1 * kx[0, k1, k2, k3] + Sum[(θ_{ii,k1,k2,k3} * (x_1 - kx[ii, k1, k2, k3])), {ii, 1, 3}]];
Cut2 := Flatten[Table[Table[Table[{CuttingHyperplane2[k1, k2, k3]}, {k1, 1, K - 1}],
       {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
Cut2;
p_{i_}[a_] := Coefficient[a, x_i, 1];
p[i_] := If[Part[ans, i] < 0, Rescale[Part[ans, i], {-90, 100}, {0, Δ_i}], Part[ans, i]];
  (*{0,Δ_i}*);
  u := RandomReal[{0.5, 0.98}];
pnt[k1_, k2_, k3_] := Module[{tt},
aa1 = CuttingHyperplane2[1, 1, 1];
aa2 = CuttingHyperplane2[k1, k2, k3];
xa1 = {p_1[aa1], p_2[aa1], p_3[aa1]};
xa2 = {p_1[aa2], p_2[aa2], p_3[aa2]};
xb_1 = Coefficient[Coefficient[Coefficient[aa1, x_1, 0], x_2, 0], x_3, 0];
xb_2 = Coefficient[Coefficient[Coefficient[aa2, x_1, 0], x_2, 0], x_3, 0];
    ax1 = (xa1) / xa1.xa2;
    ax2 = (xa2) / xa1.xa2;
bx1 = (xb_1) / (xb_1 * xb_2);
    bx2 = (xb_2) / (xb_1 * xb_2);
ω = ArcCos[Mod[ax1.ax2, 1]];
a1 = ax1; a2 = ax2; xb_1 = bx1; xb_2 = bx2;
x_1 = ((xb_1 - xb_2 * Cos[ω]) / Sin[ω]^2) * {{Part[a1, 1]}, {Part[a1, 2]}, {Part[a1, 3]}} +
```

```
       ((xb2 - xb1 * Cos[ω]) / Sin[ω]^2) * {{Part[a2, 1]}, {Part[a2, 2]}, {Part[a2, 3]}};
A = {a1, a2};
P = IdentityMatrix[3];
G = A.P;
     u := RandomReal[{0.6, 0.98}];
Off[RowReduce::luc]; Z = RowReduce[G];
F1 = Transpose[Z]; F = Transpose[Part[F1, 3]];
     ℱ = Flatten[Join[-F, IdentityMatrix[1]]];
     ς = 1 * u;
cx = ς * (P.ℱ);
ans = Flatten[x1 + cx];
{p[1], p[2], p[3]}];
Table[Table[Table[pq[k1, k2, k3] = pnt[k1, k2, k3], {k1, 1, K - 1}], {k2, 1, K - 1}],
   {k3, 1, K - 1}];
px_{i_,k1_,k2_,k3_} := x_i → Part[pq[k1, k2, k3], i];
rh_{k1_,k2_,k3_} := {px_{1,k1,k2,k3}, px_{2,k1,k2,k3}, px_{3,k1,k2,k3}};
Flatten[
   Table[Table[Table[{rh_{k1,k2,k3}}, {k1, 1, K - 1}], {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
GenDat2[k1_, k2_, k3_] := Module[{tt},
     x1 = 1.0 * (x1) /. rh_{k1,k2,k3};
     x2 = 1.0 * (x2) /. rh_{k1,k2,k3};
     x3 = 1.0 * (x3) /. rh_{k1,k2,k3};
     rh2 = {x1 → x1, x2 → x2, x3 → x3};
     {x1, x2, x3, g_pos} /. rh2];
Flatten[Table[Table[Table[{pdat[k1, k2, k3] = GenDat2[k1, k2, k3]; }, {k1, 1, K - 1}],
      {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
Clear[ca];
  ca = {};
  Flatten[Table[Table[Table[{ca = Join[ca, {pdat[k1, k2, k3]}]}], {k1, 1, K - 1}],
     {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
Θ_i_ := (∂_{x_i} g_pos) /. {x1 → px1, x2 → px2, x3 → px3};
CutGen[k1_, k2_, k3_] := Module[{tt}, pdat[k1, k2, k3] = GenDat2[k1, k2, k3];
     pxo = Part[pdat[k1, k2, k3], 4];
α = 1.0; ν = 1.0; μ2 = 1.0;
     px1 = α * Part[pdat[k1, k2, k3], 1];
px2 = ν * Part[pdat[k1, k2, k3], 2];
px3 = μ2 * Part[pdat[k1, k2, k3], 3];
     {a1, a2, a3} = {Θ1, Θ2, Θ3};
     Simplify[γ1 * pxo + (a1 * (x1 - px1) + a2 * (x2 - px2) + a3 * (x3 - px3))]];
Cuts =
   Flatten[Table[Table[Table[Cut1[k1, k2, k3] = CutGen[k1, k2, k3] ≥ 0, {k1, 1, K - 1}],
      {k2, 1, K - 1}], {k3, 1, K - 1}], 2];
rhs2 = Expand[FindFit[ca, 0.98 * pxo + ((ϕ1 * (x1 - px1) + ϕ2 * (x2 - px2) + ϕ3 * (x3 - px3))),
     {ϕ1, ϕ2, ϕ3}, {x1, x2, x3}]];
aa1 = Simplify[0.98 * pxo + ((ϕ1 * (x1 - px1) + ϕ2 * (x2 - px2) + ϕ3 * (x3 - px3))) /. rhs2];
  a1 = {p1[aa1], p2[aa1], p3[aa1]};
```

```
  Print[" **vu_3* ", px1, " * ", px2, " * ", px3, " * ", pxo];
  GCH = ExpandAll[ (1.0 * aa1) / (√(a1.a1)) ];
  Cons = Join[Cons, {GCH ≥ 0}]; Print[" *** GCH = ", GCH];
  temp = Join[Cons, Cuts];
  Constraints = temp; (*Join[Cons,temp];*)
InitialSolution = Minimize[f, Constraints, {x1, x2, x3}];
Print[InitialSolution];
xz = Part[InitialSolution, 2];
  f = Simplify[ (fn /. xz) + Sum[ (∂xjj fn /. xz) * (xjj - (xjj /. xz)), {jj, 1, n} ] ];
Values = Flatten[Table[(gjj /. xz), {jj, 1, m}]];
xx = (*-Max[-Values,x0];*)If[ialg ≥ 2, -Max[-Values, x0], -Max[-Values]];
  Print[ialg, " ** xx=", xx]; If[ialg ≥ 2 && Abs[xx] < TOL && Abs[xx - x0] < TOL, Break[];];
  x0 = xx;
  Print["***   ", {Values}, xx]; Print[xx]; If[Abs[xx] < TOL, {Print[xx]; Break[];}];
Pos = Part[Flatten[Position[Values, x0]], 1];
  Print["Pos=", Pos, "  ", xx];
gCut = ExpandAll[ xx + Sum[ (∂xjj gPos /. xz) * (xjj - (xjj /. xz)), {jj, 1, n} ] ];
  Print["*** gcut=", gCut ≥ 0];
Cons = Join[Cons, {gCut ≥ 0}]; }, ip++]
```

```
Upper bound={1.121, 0.826, 0.531}

 **vu_3* 13.4298 * 0. * 0.737488 * 2.18969

 *** GCH =  0.913255 + 0.0686995 x1 - 0.997637 x3

{-8.27894, {x1 → 1.121, x2 → 0.001, x3 → 0.529384}}

1 ** xx=-2.22045×10^-16

***   {{-2.22045×10^-16, 0.175106}}-2.22045×10^-16

-2.22045×10^-16

-2.22045×10^-16
```

In[49]:=
```
q1
```

Out[49]=
```
0.59
```

In[50]:=
```
Minimize[{fn, g1 ≥ 0, g2 ≥ 0, Conz}, {x1, x2, x3}]
```

Out[50]=
```
{-8.27894, {x1 → 1.121, x2 → 0.00100001, x3 → 0.529384}}
```