EE-635

(Applied Linear Algebra)

Assignment-5

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Assignment - 5

Sol. 4] Given: ① $A = A^T$, $A \in \mathbb{R}^{n \times n}$

② If $u \in W \Rightarrow Au \in W$.

T.P.T: $v \in W^{\perp} \Rightarrow Av \in W^{\perp}$.

Proof:

$\langle v | u \rangle = 0$ and $\because Au \in W$, $\forall u \in W$

$\therefore \langle v | Au \rangle = 0$

$\Rightarrow v^T \cdot Au = 0 \Rightarrow (v^T Au)^T = 0$

$\Rightarrow u^T A^T v = 0$

$\Rightarrow u^T A v = 0$ $\quad (\because A = A^T$ (from ①))

$\Rightarrow \langle u | Av \rangle = 0$ $\quad \forall u \in W$.

$\therefore Av \in W^{\perp}$.

Using $v \in W^{\perp}$, we arrived at $Av \in W^{\perp}$. $\therefore$ proved.

Sol. 6] Given: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$;

T.P.T i) $v \in (\mathcal{C} = \mathrm{im}([B \; AB \dots A^{n-1}B])) \Rightarrow Av \in \mathcal{C}$.

ii) $v \in (\bar{\mathcal{O}} = \ker([C^T \; A^TC^T \dots (A^{n-1})^T C^T]^T)) \Rightarrow Av \in \bar{\mathcal{O}}$.

Proof: Suppose $v \in \mathcal{C}$.

$\Rightarrow v = [B \; AB \dots A^{n-1}B] x$.

Now, consider $Av = [AB \; A^2B \dots A^n B] x$

$\because$ A satisfies its own char. eqn $\chi_A(x) = 0$ (Cayley-Hamilton)

and $\deg(\chi_A(x)) = n$ and it is monic,

$\therefore$ any power of A greater than $n-1$ can be written as a linear combination of $[I, A, A^2, \dots, A^{n-1}]$.

$$\Rightarrow A^n B = \left(\sum_{i=0}^{n-1} \alpha_i A^i\right) B \qquad (\text{note}: A^0 = I)$$

$$\Rightarrow Av = \left[AB \ A^2B \ \ldots \ \sum_{i=0}^{n-1} \alpha_i A^i B\right] x, \quad x \in \mathbb{R}^n.$$

Thus,

Av is expressible as lin. combination of $B, AB, \ldots, A^{n-1}B$

$\therefore Av \in \operatorname{im}[B, AB, \ldots, A^{n-1}B]$.

$\therefore C$ is A-inv. $\qquad \therefore$ proved.

iii) Let $v \in \bar{O}$. $\therefore \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0 \rightarrow (1)$ (taking transpose)

Now consider Av: $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} Av = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} v$

from eqn (1), $Cv = CAv = \ldots = CA^{n-1}v = 0$

$\therefore \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ CA^n v \end{bmatrix}$ but $\because A^n = \sum_{i=0}^{n-1} \alpha_i A^i \Rightarrow CA^n v = C \sum_{i=0}^{n-1} \alpha_i A^i v = 0$

$\therefore CA^n v = 0$

$\therefore \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0$ $\quad \therefore e$ is also invariant $\quad \therefore \bar{O}$ is also invertible.

Sol. 7 Given: $W \subseteq \mathbb{R}^n \ni w \in W \Rightarrow Aw \in W$ and $W = \langle w_1, w_2, \ldots, w_k \rangle$

$\therefore Aw_i \in W \Rightarrow Aw_i = \sum_{j=1}^{k} \alpha_{ij} w_j$

$\because W = [w_1 \ w_2 \ldots w_k]$

$$\Rightarrow AW = [Aw_1 \ Aw_2 \ldots Aw_k] = \left[\sum_{j=1}^{k} \alpha_{1j} w_j \ \ \sum_{j=1}^{k} \alpha_{2j} w_j \ \ldots \ \sum_{j=1}^{k} \alpha_{kj} w_j\right]$$

$$\Rightarrow AW = [w_1 \ w_2 \ldots w_k] \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{k1} \\ \alpha_{12} & \cdot & \alpha_{k2} \\ \vdots & & \vdots \\ \alpha_{1k} & \cdots & \alpha_{kk} \end{bmatrix}$$

T.P.T $\exists\ S \in \mathbb{R}^{k\times k} \ni AW = WS$.

$$S = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{k1} \\ \alpha_{12} & \cdot & \alpha_{k2} \\ \vdots & \ddots & \vdots \\ \alpha_{1k} & \cdots & \alpha_{kk} \end{bmatrix}$$

(from previous eqn)

$\therefore$ we've found and proved that $\exists\ S \ni AW = WS$.

Now, given $\rho(W) = r$,

T.P.T $\underline{A}$ and $\underline{S}$ share atleast r eigenvalues,

$\because \rho(W) = r \Rightarrow \therefore \exists$ atmost $(k-r)$ L I vectors $\ni W\underline{v} = 0$.

$\therefore$ Ker(W) can accomodate atmost $(k-r)$ eigenvectors of S.

$$\Downarrow$$

$k-(k-r)$ eigenvectors of S (at the least) must ~~have~~ be such that $Wv \neq 0$.

Let $v \notin$ Ker(W) be one of those eigenvectors of S with eigenvalue 'λ'.

$\Rightarrow Sv = \lambda v \Rightarrow \underline{W}(Sv) = \lambda Wv$

$\Rightarrow A\underline{Wv} = \lambda \underline{Wv}$

let $Wv = x$.

then $Ax = \lambda x$.

$\therefore$ if v is eigenvector of $S \ni v \notin$ Ker(W), then it is also an eigenvector of A.

Thus, there are atleast 'r' eigenvalues (correspondingly) common to A and S.

$\therefore$ proved.

Sol. 12] Given, $A \in \mathbb{R}^{m\times n} \xrightarrow{SVD} A = U\begin{bmatrix} \Sigma_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} V^T$ and MPP$(A) = A^+$

where $A^+ = V\begin{bmatrix} \Sigma^{-1}_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} U^T$. $\because$ SVD $\therefore$ $\underline{U}$ and $\underline{V}$ are orthogonal $(UU^T = VV^T = I)$ and $\boxed{\Sigma}$ is $\boxed{\text{diag}}$.

$$\Downarrow$$

$\Sigma^T = \Sigma$

$\Sigma^{-1^T} = \Sigma^{-1}$

a) $A^{-1} = A^+$.

Proof: $A^{-1} = \left(U\begin{bmatrix} \Sigma_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} V^T\right)^{-1} = (V^T)^{-1}\left(\begin{bmatrix} \Sigma_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}(U)^{-1}$

$= (V^T)^T\begin{bmatrix} \Sigma^{-1}_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}(U)^T$ ($\because$ given A is invertible)

(NOTE: $\text{diag}^T = \text{diag}$)

$$= V\begin{bmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{bmatrix} U^T = A^+ \quad \therefore \text{ proved.}$$

b) $(A^+)^+ = A$

Proof: Taking ~~transpose~~ inverse of A^+: $(A^+)^{-1} = \left(V\begin{pmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{pmatrix} U^T\right)^{-1}$

$$= (U^T)^{-1} \cdot \begin{pmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{pmatrix}^{-1} \cdot (V)^{-1}$$

$$= (U^T)^T \cdot \begin{pmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{pmatrix}^{-1} \cdot (V)^T = U\begin{pmatrix}\Sigma_{r\times r} & 0 \\ 0 & 0\end{pmatrix} V^T$$

$$= A \quad \therefore \text{ proved.}$$

c) $(A^+)^T = (A^T)^+$

Proof: LHS → $(A^+)^T = \left(V\begin{bmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{bmatrix} U^T\right)^T = (U^T)^T\begin{bmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{bmatrix}(V)^T$

$$= U\begin{bmatrix}\Sigma_{r\times r}^{-1} & 0 \\ 0 & 0\end{bmatrix} V^T$$

$$= \left(V\begin{bmatrix}\Sigma_{r\times r} & 0 \\ 0 & 0\end{bmatrix} U^T\right)^+$$

$$= \left(\left(U\begin{bmatrix}\Sigma_{r\times r} & 0 \\ 0 & 0\end{bmatrix} V^T\right)^T\right)^+ = (A^T)^+ = \text{RHS}$$

$\therefore$ proved.

d) $\because A \in \mathbb{R}^{m\times n} \therefore A^TA \in \mathbb{R}^{n\times n}$ and $AA^T \in \mathbb{R}^{m\times m}$. Also, $VV^T = I_n$ and $UU^T = I_m$.

Furthermore, $\because$ given that A has FCR to prove $A^+ = (A^TA)^{-1}A^T$
and that A has FRR to prove $A^+ = A^T(AA^T)^{-1}$
it allows us to invert A^TA for the first part and AA^T for the second.

So,

consider RHS $= (A^TA)^{-1}A^T$, where $A^T = (V^T)^T\begin{bmatrix}\Sigma_{r\times r} & 0 \\ 0 & 0\end{bmatrix}^T U^T$

$$= \left(\underbrace{V\begin{bmatrix}\Sigma_{r\times r} & 0 \\ 0 & 0\end{bmatrix}U^T}_{\text{~~let this be X~~}}\ \underbrace{U\begin{bmatrix}\Sigma_{r\times r} & 0 \\ 0 & 0\end{bmatrix}V^T}_{\text{~~and this be Y~~}}\right)^{-1} A^T$$

I_m

Now, $(A^TA) = V\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix} I \begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix} V^T$.

$\because \Sigma$ is diagonal, $\therefore \Sigma\cdot\Sigma = \Sigma^2$ is also diagonal. (Let $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$)

then $\begin{bmatrix}\sigma_1 \; \sigma_2 \cdots \sigma_r & 0_{r\times(n-r)}\\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)}\end{bmatrix}\begin{bmatrix}\sigma_1 \; \sigma_2 \cdots \sigma_r & 0_{r\times n-r}\\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)}\end{bmatrix} = \begin{bmatrix}\sigma_1^2 \; \sigma_2^2 \cdots \sigma_r^2 & 0\\ 0 & 0\end{bmatrix}$

$= \begin{bmatrix}\Sigma_{r\times r}^2 & 0\\ 0 & 0\end{bmatrix}$

$\therefore (A^TA) = V\begin{bmatrix}\Sigma_{r\times r}^2 & 0\\ 0 & 0\end{bmatrix}V^T \Rightarrow (A^TA)^{-1} = (V^T)^{-1}\begin{bmatrix}\Sigma_{r\times r}^2 & 0\\ 0 & 0\end{bmatrix}^{-1}(V)^{-1}$

$\therefore (A^TA)^{-1}A^T = (V^T)^{-1}\begin{bmatrix}\Sigma_{r\times r}^{-2} & 0\\ 0 & 0\end{bmatrix} \underbrace{V^{-1}\cdot V}_{I}\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T$

$\because V^T = V^{-1}$, $\therefore (V^{-1})^{-1}\begin{bmatrix}\Sigma_{r\times r}^{-2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T$

$\Rightarrow (A^TA)^{-1}A^T = V\begin{bmatrix}\Sigma_{r\times r}^{-1} & 0\\ 0 & 0\end{bmatrix}U^T$

$= A^+$ $\therefore$ proved.

Now, T.P.T $A^+ = A^T(AA^T)^{-1}$,

$\therefore$ RHS $= \left(V\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T\right)\left(\left(U\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}\underbrace{V^T\Big)\Big(V}_{I_n}\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T\right)\right)^{-1}$

$AA^T = U\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}I\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}U^T = U\begin{bmatrix}\Sigma_{r\times r}^2 & 0\\ 0 & 0\end{bmatrix}U^T$

$\therefore (AA^T)^{-1} = (U^T)^{-1}\left(\begin{bmatrix}\Sigma_{r\times r}^2 & 0\\ 0 & 0\end{bmatrix}\right)^{-1}(U)^{-1}$

$\therefore A^T(AA^T)^{-1} = V\begin{bmatrix}\Sigma_{r\times r} & 0\\ 0 & 0\end{bmatrix}\underbrace{U^T(U^T)^{-1}}_{I}\begin{bmatrix}\Sigma_{r\times r}^{-2} & 0\\ 0 & 0\end{bmatrix}\underbrace{U^{-1}}_{U^T}$ ($\because U \to$ orthogonal)

$= V\begin{bmatrix}\Sigma_{r\times r}^{-1} & 0\\ 0 & 0\end{bmatrix}U^T$

$= A^+$ $\therefore$ proved.

e) $(A^TA)^+ = A^+(A^T)^+$ and $(AA^T)^+ = (A^T)^+A^+$

Proof: As calculated in part d),

$$A^TA = V\begin{bmatrix}\Sigma^2_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}V^T \text{ and } AA^T = U\begin{bmatrix}\Sigma^2_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}U^T$$

$$\Rightarrow (A^TA)^+ = (V^T)^T\begin{bmatrix}\Sigma^{-2}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}(V)^T \text{ and } (AA^T)^+ = (U^T)^T\begin{bmatrix}\Sigma^{-2}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}(U)^T$$

Now, $\because (A^T)^+ = (A^+)^T$ (proved in part c)

$$\therefore (A^T)^+ = \left(V\begin{bmatrix}\Sigma^{-1}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}U^T\right)^T$$

$$= U\begin{bmatrix}\Sigma^{-1}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}V^T$$

$$\therefore A^+(A^T)^+ = \left(V\begin{bmatrix}\Sigma^{-1}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}\underbrace{U^T\Big)\Big(U}_{I}\begin{bmatrix}\Sigma^{-1}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}V^T\right) = V\begin{bmatrix}\Sigma^{-2}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}V^T$$

$$= (A^TA)^+$$

(as computed above)

$\therefore$ 1st eqn proved.

Now,

$$(A^T)^+A^+ = \left(U\begin{bmatrix}\Sigma^{-1}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}\underbrace{V^T\Big)\Big(V}_{I}\begin{bmatrix}\Sigma^{-1}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}U^T\right) = U\begin{bmatrix}\Sigma^{-2}_{\ell\times\ell} & 0\\ 0 & 0\end{bmatrix}U^T$$

$$= (AA^T)^+$$

(as calculated above)

Thus, proved.

Sol. 13] a) Consider the characteristic polynomials to prove it:

$$\chi_A(x) = \det(A - xI)$$
$$\chi_{A^H}(y) = \det(A^H - yI)$$

Now,

$$\chi_{A^H}(y) = \overline{\overline{\det(A^H - yI)}} = \overline{\det(\overline{A^H} - y^* I)}$$
$$= \overline{\det(A^T - y^* I)}$$
$$= \overline{\det(A^T - y^* I^T)}$$
$$= \overline{\chi_{A^T}(y^*)}$$

But $\because$ A and A^T have same char poly, ($\because$ det does NOT change on Transposition)

$$\therefore \chi_{A^H}(y) = \overline{\chi_A(y^*)}$$

$\therefore$ for $y \in \mathbb{C}$: $\chi_A(y) = 0 \Leftrightarrow \chi_{A^H}(y^*) = 0$.

$\therefore$ proved

b) As given, $Au = \lambda u$ and $A^H v = \mu v \Rightarrow \lambda \neq \mu^*$, $v \neq 0$, $u \neq 0$.

$$\Rightarrow (Au)^H = (\lambda u)^H \text{ and } (A^H v)^H = (\mu v)^H$$
$$\Rightarrow u^H A^H = \lambda^* u^H \text{ and } v^H A = \mu^* v^H$$
$$\Rightarrow v^H u^H A^H = \lambda^* v^H u^H \text{ and } v^H A u = \mu^* v^H u$$

$$\Rightarrow v^H A u = \lambda(v^H u) \text{ and } v^H A u = \mu^*(v^H u)$$

Subtracting these two eqns:

$$0 = (\lambda - \mu^*)(v^H u)$$

$\because \lambda \neq \mu^*$, $\Rightarrow v^H u = 0$.

$\Rightarrow \langle u | v \rangle = 0$. $\Rightarrow$ u and v are ORTHOGONAL.

$\therefore$ proved.

c) Any matrix $\in \mathbb{R}^{n\times n}$ can be also treated as it belonging to $\mathbb{C}^{n\times n}$.

So, let $J \in \mathbb{C}^{n\times n}$ be its Jordan cannonical form of A.

$\because$ given $A.M(\lambda) = 1$,

$\therefore$ we have $J_1 = \begin{bmatrix} \lambda & \text{—} \\ | & * \end{bmatrix}$ for some $V = [v_1\ v_2 \ldots v_n]$

where $Av_1 = \lambda v_1$

Clearly, $v_1 \rightarrow$ Right eigenvector (A).

For left eigenvectors, we have

$$\underbrace{\begin{bmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_n^* \end{bmatrix}}_{W} A = \begin{bmatrix} \lambda & 0 \\ 0 & Q \end{bmatrix} \underbrace{\begin{bmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_n^* \end{bmatrix}}_{W}$$

w_1^* → "left-eigenvector [A]"

Let,

for a moment assume the contrary :- w_1 and v_1 to be orthogonal,

ie : $\langle v_1 | w_1 \rangle = 0$.

$\Rightarrow w_1^* v_1 = 0$.

$\therefore$

$$WAv_1 = \begin{bmatrix} \lambda & 0 \\ 0 & Q \end{bmatrix} Wv_1 \Rightarrow \lambda \begin{bmatrix} 0 \\ w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 \\ w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix}$$

Comparing the last $(n-1)$ terms of vectors on both sides, we get

$$\lambda \begin{bmatrix} w_1^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix} = Q \begin{bmatrix} w_2^* v_1 \\ \vdots \\ w_n^* v_1 \end{bmatrix}$$

$\because v_1 \neq 0$; $\therefore v_1$ CANNOT be orthogonal to w_i for all $i = 1, 2, \ldots, n$.

$\therefore \Rightarrow \lambda$ is an eigenvalue of Q.

But $\chi_A(x) = \chi_{J_2}(x)$

$= (x-\lambda)\, \chi_Q(x)$

Now, $\because \lambda \to$ eigenvalue (Q),

$\therefore \chi_Q(x) = (x-\lambda)\, q(x)$ for some $q(x) \in \mathbb{R}[x]$.

$\therefore \chi_A(x) = (x-\lambda)^2 q(x)$

→ leads to $AM(\lambda) = 2$. ?! $(\neq 1)$

CONTRADICTION!

$\therefore \langle v_1 | w_1 \rangle \neq 0$.

$\therefore$ proved.

Sol. 20] a) $a = \{ p(x) \in \mathbb{C}[x] : p[x] = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x \}$

Let ~~$p, q \in a$~~ $p(x), q(x) \in a$. where $q(x) = \sum_{i=1}^{n} b_i x^i$

Clearly, $p+q = (a_n + b_n) x^n + \dots + (a_1 + b_1) x \in a$

Now,

consider $g(x) = c_m x^k + \dots + c_1 x + c_0 \in \mathbb{C}[x]$

$pg = c_n x^k (p(x)) + \dots + c_0 (p(x))$

Here, each term of $g(x)$ gets multiplied by powers of x $\because$ $p(x)$ has zero constant coeff. Thus, $pg(x)$ also has all of its terms with non-zero powers of x.

$\therefore$ $p(x) g(x) \in a$.

$\therefore$ a is an ideal.

b) $b = \{ p(x) \mid p(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots \}$

If $p(x), g(x) \in b$, clearly $(p+q)(x) \in a$,

Now,

consider $g(x) = a_0 + a_1 x + \dots + a_k x^k \in \mathbb{C}[x]$

$pg(x) = a_0^2 + \underline{a_0 a_1 x} + (a_0 a_2 + a_0 a_2) x^2 + \dots$

$\hookrightarrow$ odd-powered term exists in $pg(x)$!

$\therefore$ $p(x) g(x) \notin a$.

$\therefore$ NOT an ideal.

c) $c = \{0, 2, 4\} \subseteq \mathbb{Z}_6$.

Now,
$$\left.\begin{aligned} 0+2 &= 2 \bmod 6 = 2 \in c \\ 2+4 &= 6 \bmod 6 = 0 \in c \\ 0+4 &= 4 \bmod 6 = 4 \in c \end{aligned}\right\} \Rightarrow p, q \in a \Rightarrow (p+q)_{\mathbb{Z}_6} \in a.$$

Now,

for $p \in \{0, 1, 2, 3, 4, 5\}$, $2p \in \underbrace{\{0, 2, 4\}}_{\in c}$ and $4p \in \underbrace{\{0, 2\}}_{\in c}$

and $0p = 0 \in c$

$\therefore$ $\{0, 2, 4\} \subseteq \mathbb{Z}_6$ is an ideal.

d) $d := \{ p(x) \in \mathbb{Z}[x] \mid p(x)$ has even coeff $\}$

Let $p(x) = \sum_{i=0}^{n} a_i x^i \in d$ and $q(x) = \sum_{j=0}^{m} a_j x^j \in d$.

$\because$ even + even = even $\Rightarrow (p+q)(x) \in d$. ($\because a_i + a_j$ = even $\forall i, j$)

Let $g(x) = c_n x^n + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$

then $p(x)g(x)$ will have all coeff even irrespective of whether the coeff in $g(x)$ are even or odd. ($\because$ even $\times$ odd = even, even $\times$ even = even)

$\Rightarrow$ every coeff of $pg(x)$ is even. $\Rightarrow pg \in d$.

$\therefore$ <u>d is an ideal</u>.

e) $\mathbb{R} \subseteq \mathbb{R}[x]$

Consider $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}[x]$.

but $\alpha x \notin \mathbb{R}$.

$\therefore$ <u>NOT an ideal</u>.

Sol. 22) Given $\langle v, Av, \ldots, A^{n-1}v \rangle = \mathbb{C}^n$

$\Rightarrow \mathcal{B} = \{v, Av, \ldots, A^{n-1}v\}$ is a basis for $\mathbb{C}^n$

(set of n vectors spanning $\mathbb{C}^n$ must be LI, $\therefore$ a basis)

$$[A]_{\mathcal{B}} = \left[[Av]_{\mathcal{B}} \; [A\,Av]_{\mathcal{B}} \cdots\cdots [A\,A^{n-1}v]_{\mathcal{B}} \right]$$

Since we can have an ordered basis $\{v_1, v_2, \ldots, v_n\}$ where $v_i = A^{i-1}v$.

Thus, $Av_i = v_{i+1} \Rightarrow [Av_i]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ $(i+1)^{th}$ posn

for $i = 1, 2, \ldots, n-1$.

From Cayley-Hamilton Theorem

$$\chi_A(A) = 0.$$

$$\Rightarrow A^n = -\sum_{i=0}^{n-1} \alpha_i A^i$$

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$$\Rightarrow [Av_n]_B = \begin{bmatrix} -\alpha_0 \\ -\alpha_1 \\ \vdots \\ -\alpha_{n-1} \end{bmatrix} \Rightarrow [A]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & & & -\alpha_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

$\because \{v, Av, \ldots, A^{n-1}v\}$ is LI, $\nexists \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} : \sum \alpha_i A^i v = 0$ unless $\alpha_i = 0 \ \forall i$.

Minimal polynomial of A wrt v must have deg $\geq n$.
$\because$ Also $\mu(x)$ multiple of this polynomial, $\deg(\mu_A(x)) = n$.
and we know that

$\mu_A(x) \mid \chi_A(x)$ $\quad\therefore\quad$ $\underline{\mu_A(x) = \chi_A(x)}$.

Soln 24] For $\mathbb{Z}[x]$ to be a PID, every ideal generated must be a principle ideal (ideal generated by a single element)

Consider the ideal $a = \langle 2, x^2 \rangle$

$\therefore f(x) = 2p(x) + x^2 q(x)$, where $f(x) \in a$ and $p(x), q(x) \in \mathbb{Z}[x]$.

If $\mathbb{Z}[x]$ is a PID then $a = \langle 2, x^2 \rangle = \langle h(x) \rangle$.

So $\quad 2 = h(x) f_1(x) \quad$ — ①

$\quad x^2 = h(x) f_2(x) \quad$ — ② $\quad (f_1(x), f_2(x) \in \mathbb{Z}[x])$

from ① it is clear that $h(x) f_1(x)$ has to a const. poly :

$\therefore h(x) = \{\pm 1, \pm 2\}$.

Suppose $a = \langle 2, x^2 \rangle = \langle 1 \rangle$.

$\exists\ r_1(x)$ and $r_2(x)$ such that $[r_1, r_2 \in \mathbb{Z}[x]]$

$$2r_1(x) + x^2 r_2(x) = 1$$

The term $x^2 r_2(x)$ has no constant term, thus the only constant term available is that of $r_1(x)$.

Suppose $r_1(0)$ is the CONSTANT term then by comparison of the polynomial coefficients we have

$2r_1(0) = 1$ this is possible only if $r_1(0) = 1/3$

$\therefore r_1(x) \notin \mathbb{Z}[x]$

$\therefore \langle 2, x^2 \rangle \neq \langle 1 \rangle$

Suppose $\langle 2, x^2 \rangle = \langle 2 \rangle$ $\quad$ (ie : $h(x) = \pm 2$)

then $\exists\, r(x) \ni x^2 = 3\, r(x)$

which again implies $r(x) \notin \mathbb{Z}[x]$

$\therefore \langle 2, x^2 \rangle$ is NOT a principle Ideal.

$\Rightarrow \mathbb{Z}[x]$ is NOT PID.

~~S~~ol. 27] Given two diagonalisable matrices $A, B \in \mathbb{R}^{n \times n}$.

T.P.T $AB = BA \iff \exists\, S \in \mathbb{R}^{n \times n} \ni S^{-1}AS$ and $S^{-1}BS$ are diagonal.

Proof :-

($\Leftarrow$) Assume $S^{-1}AS = \Omega_1$ and $S^{-1}BS = \Omega_2$.

Now, $(S^{-1}AS)(S^{-1}BS) = \Omega_1 \Omega_2$

Noting that diagonal matrices are commutable,

$$S^{-1}ABS = \underline{\Omega_2 \Omega_1}$$

Now,

$$\Omega_2 \cdot \Omega_1 = (S^{-1}BS)(S^{-1}AS)$$

$$= S^{-1}BAS.$$

$\Rightarrow S^{-1}ABS = S^{-1}BAS$

$\Downarrow$

(pre-multiply by S and post-multiply by S^{-1}), $\underline{AB = BA}$.

Now, ($\Rightarrow$) Supposing $AB = BA$:

"Commuting matrices have a common set of eigenvectors",

and this property will help us show existence of S.

Let $\{\lambda_i\}_{i=1}^{n}$ be eigenvalues of A and corresponding LI eigenvectors $\longrightarrow v_1, v_2, \ldots, v_n$.

$\because A$ is diagonalisable,

$\therefore$ ~~$\exists P \ni$~~ $A = PDP^{-1}$

where $P = [[v_1][v_2] \cdots [v_n]]$ and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Similarly, let $\{\mu_i\}_{i=1}^{n} \rightarrow$ eigenvalues of B and corresponding LI eigenvectors $\rightarrow u_1, u_2, \ldots, u_n$.

$\because$ B is also diagonalisable, $\therefore B = QDQ^{-1}$

where $Q = [[u_1][u_2] \cdots [u_n]]$ and $D = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$

Now, $A = PDP^{-1} \Rightarrow P^{-1}A = DP^{-1} \Rightarrow P^{-1}AP = D \longrightarrow$ diagonal ✓

$\therefore S = P \quad (S^{-1}AS = D)$

~~Now, consider $S^{-1}BS = (P)(QDQ^{-1})(P)$~~

NOTE: S is clearly invertible as the column vectors composing it are LI ($\because$ they're eigenvectors).

Now,

consider $S^{-1}BS = (P^{-1})(QDQ^{-1})(P)$.

$\because$ A and B share same eigenvectors, $\therefore P = Q$.

$\therefore S^{-1}BS = (P^{-1}P)\, D\, (P^{-1}P)$

$= D \longrightarrow$ diagonal ✓

$\therefore$ proved.

Sol. 28] Given $A = \begin{bmatrix} -\pi/2 & \pi/2 \\ \pi/2 & -\pi/2 \end{bmatrix}$

let λ be eigenvalue of A. Then, by Cayley-Hamilton Theorem,

$\det(\lambda I - A) = 0 \Rightarrow \det\begin{pmatrix} \lambda+\pi/2 & -\pi/2 \\ -\pi/2 & \lambda+\pi/2 \end{pmatrix} = 0$

$\Rightarrow \left(\lambda + \frac{\pi}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2 = 0$

$\Rightarrow \lambda^2 + \frac{2\pi\lambda}{2} + \left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2 = 0$

$\Rightarrow \lambda(\lambda+\pi) = 0$

$\Rightarrow \lambda = 0, \lambda = -\pi$ (eigenvalues)

To find eigenvectors corresponding to these eigenvalues so as to diagonalise A,

for $\lambda = 0$: $\begin{bmatrix} +\pi/2 & -\pi/2 \\ -\pi/2 & +\pi/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \quad \therefore v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(\lambda I - A)X = O_{2\times 2}$

for $\lambda = -\pi$: $\begin{bmatrix} -\pi+\pi/2 & -\pi/2 \\ -\pi/2 & -\pi+\pi/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -x_2$

$\therefore V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now, we can write $A = P^{-1} D . P^{-1}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (0) & 0 \\ 0 & (-\pi) \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$\therefore \cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \ldots$

noting that $A^2 = (P^{-1} D P^{-1})(P^{+1} D P^{-1}) = P^{+1} D^2 P^{-1}$ and that similarly $A^n = (P^{+1} D P)^{-1} \ldots (P^{+1} D P)^{-1} = P^{+1} D^n P^{-1}$ (easily proved inductively)

$$\therefore \cos(A) = P^{+1} . \cos(D) . P^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}\begin{bmatrix} \cos(0) & 0 \\ 0 & \cos(-\pi) \end{bmatrix}\begin{bmatrix} 1 & +1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}\begin{pmatrix} 1 & +1 \\ +1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & +2 \\ +2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}$$

Sol.29] Given, $A = -A^T$,

T.P.T

$$(e^A)(e^{A^T}) = I$$

Proof:

Let $P = e^A$.

Now, $PP^T = e^A e^{A^T}$

$= e^{(-A^T)} e^{A^T}$ (from given)

Now, we know that

$e^X . e^Y = e^{X+Y}$ only if $XY = YX$.

(Proved in Soln 32] the immediate next soln)

Clearly, $\because (-A^T)(A^T) = (A^T)(-A^T)$ (which is $= -(A^T)^2$)

$\therefore \quad PP^T = e^{-A^T + A^T}$

$= e^{0_{n\times n}}$

$= I_{n\times n}$ $\quad \therefore$ proved.

Sol. 32] Given $AB = BA$.

Now, $$e^A e^B = \sum_{n=0}^{\infty} \frac{A^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{B^n}{n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A^m}{m!} \cdot \frac{B^n}{n!}.$$

Let $l = m+n \Rightarrow n = l-m$.

$$\therefore e^A e^B = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{A^m B^{l-m}}{m!(l-m)!} \times \frac{l!}{l!}$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} \frac{l!}{m!(l-m)!} A^m B^{(l-m)}$$

$$= \sum_{l=0}^{\infty} \frac{(A+B)^l}{l!} = e^{A+B}$$

(works because A and B commute, $\therefore$ we can write $(A+B)^2 = A^2 + \cancel{AB} \cancel{+BA}\ 2AB + B^2$ and $\therefore$ binomial expansion works).

For counterexample to assertion if $AB \neq BA$,

let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. $AB = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

Clearly, $AB \neq BA$.

Now, $$e^A = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix} \begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix}$$

$$e^B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^1 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$e^{AB} = \begin{pmatrix} \frac{1-\sqrt{7}i}{2} & \frac{1+\sqrt{7}i}{2} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} e^{\left(\frac{1}{2}+\frac{i\sqrt{7}}{2}\right)} & 0 \\ 0 & e^{\left(\frac{1}{2}-\frac{i\sqrt{7}}{2}\right)} \end{pmatrix} \begin{pmatrix} \frac{i\sqrt{7}}{14} & \frac{1}{4}-\frac{i\sqrt{7}}{28} \\ -\frac{i\sqrt{7}}{14} & \frac{1}{4}+\frac{i\sqrt{7}}{28} \end{pmatrix}$$

Clearly, $e^A \cdot e^B \neq e^{(A+B)}$.

Sol. 35] Given, $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 3 & 6 & 8 \end{bmatrix}$

Consider a basis for $\mathbb{R}^3$ as $\{e_1, e_2, e_3\}$; $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$Ae_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A^2e_1 = \begin{bmatrix} 22 \\ 31 \\ 39 \end{bmatrix}$, $A^3e_1 = \begin{bmatrix} 310 \\ 441 \\ 564 \end{bmatrix}$

$\therefore \alpha_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 22 \\ 31 \\ 39 \end{bmatrix} = \begin{bmatrix} 310 \\ 441 \\ 564 \end{bmatrix}$

$\alpha_0 = 5$, $\alpha_1 = 19$, $\alpha_2 = 13$;

$\therefore \mu_{e_1} = s^3 - 13s^2 - 19s - 5$

$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $Ae_2 = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$, $A^2e_2 = \begin{bmatrix} 45 \\ 64 \\ 81 \end{bmatrix}$, $A^3e_2 = \begin{bmatrix} 642 \\ 913 \\ 1167 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \beta_0 + \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \beta_1 + \begin{bmatrix} 45 \\ 64 \\ 81 \end{bmatrix} \beta_2 = \begin{bmatrix} 642 \\ 913 \\ 1167 \end{bmatrix}$

$\Rightarrow \beta_0 = 5$, $\beta_1 = 19$, $\beta_2 = 13$.

$\therefore \mu_{e_2} = s^3 - 13s^2 - 19s - 5$.

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $Ae_3 = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$, $A^2e_3 = \begin{bmatrix} 66 \\ 94 \\ 121 \end{bmatrix}$, $A^3e_3 = \begin{bmatrix} 953 \\ 1355 \\ 1730 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r_0 + \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} r_1 + \begin{bmatrix} 66 \\ 94 \\ 121 \end{bmatrix} r_2 = \begin{bmatrix} 953 \\ 1355 \\ 1730 \end{bmatrix}$

once again, $r_0 = 5$, $r_1 = 19$, $r_2 = 13$.

$\therefore \mu_{e_3} = s^3 - 13s^2 - 19s - 5$.

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Now, eigenvalues are roots of both $\chi(s)$ and $\mu(s)$

Solving $\mu(s) = 0$, we immediately see by inspection that $s = -1$ is a root.

$$
\begin{array}{r|l|l}
s+1 & s^3 - 13s^2 - 19s - 5 & s^2 - 14s - 5 \\
 & -s^3 - s^2 & \\
 & -14s^2 - 19s - 5 & \\
 & +14s^2 + 14s & \\
 & 0 - 5s - 5 & \\
 & +5s + 5 & \\
 & 0 &
\end{array}
$$

$s^2 - 14s - 5 \xrightarrow{\text{gives}} s = 7 \pm 3\sqrt{6}$.

$\frac{14 \pm \sqrt{196 + 20}}{2}$

Thus the Eigenvalues are $\boxed{\lambda_1 = -1}$, $\boxed{\lambda_2 = 7 + 3\sqrt{6}}$ $\boxed{\lambda_3 = 7 - 3\sqrt{6}}$.