Post-Midsem :

Recap: Discrete Random Variables : $\exists$ E such that $|E| \leq |\mathbb{N}|$ and $\mathbb{P}_X(E) = 1$

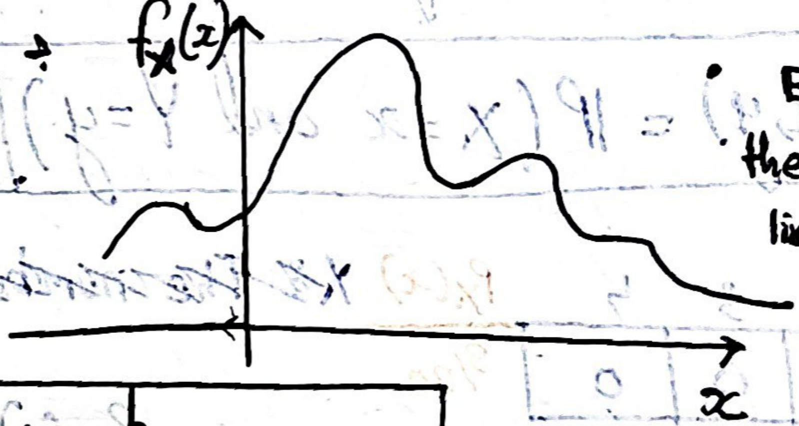
Recall: i) CDF ii) PMF iii) $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$

- CDF defined $\iff$ $\mathbb{P}_X(B)$ $\forall$ $\mathcal{B}(\mathbb{R})$ is defined.

Continous Random Variables :

- Probability Density Function : $f_X(x) \geq 0 \;\; \forall x$.

$\Rightarrow$ $$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$



Then, $$\boxed{\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,dx.}$$

- EVEN Though the PMF had to lie b/w 0 and 1
  ↓
  PDF can take any arbitirary value
  (Ex : $\delta(x)$ satisfies)

- A random variable is continous if it is described by a PDF.

Why is this the density? → $[a, a+\delta]$, δ small.

$\mathbb{P}(a \le X \le a+\delta) = \int_a^{a+\delta} f_X(x)\,dx$. As δ is small $\approx f_X(a)\cdot\delta$

$\rightarrow \boxed{f_X(a) = \dfrac{\mathbb{P}(a \le X \le a+\delta)}{\delta}}$ → Probability per unit something

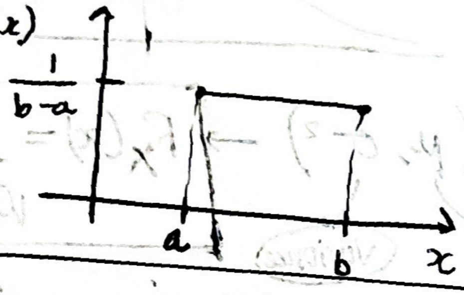
↳ $\boxed{\text{Probability density.}}$ ✓.

If $\delta \to 0$, $\boxed{\mathbb{P}(X=a) \approx 0}$ — The probability of a single value 'a' in a continous Random variable is zero $\forall\ a$.

• Many 'tending to' zeros sum up (integrate) to give something finite.

$\rightarrow \boxed{F_X(x) = \int_{-\infty}^{x} f_X(\tau)\,d\tau}$ $\left[\dfrac{dF_X(x)}{dx} = f_X(x) \text{ where the derivative exists.} \right]$

Example :- $f_X(x)$



$\rightarrow E[X] = \int_a^b \dfrac{x \cdot 1}{b-a}\,dx = \left[\dfrac{x^2}{2}\right]_0^b \cdot \dfrac{1}{b-a} = \dfrac{b^2-a^2}{2(b-a)}$

$\rightarrow E[X] = \left(\dfrac{b+a}{2}\right)$

Definitions :-

i) Expectation :- $E[X]$

$E[X] = \int_{-\infty}^{\infty} x\, \underbrace{f_X(x)\,dx}_{\text{(weight)}}$

a) If $\mathbb{P}(X \ge 0) = 1$, $E[X] \ge 0$.

b) If $a \le X \le b$, $a \le E[X] \le b$.

c) $E[aX+b] = aE[X]+b$.

d) $E[g(X)] = \int_{-\infty}^{\infty} g(x)\cdot f_X(x)\,dx$ [LOTUS].

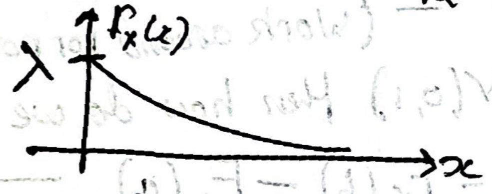
(ii) → Variance :- $Var(X) = \int_{-\infty}^{\infty} (x-E[X])^2 f_X(x)\,dx = E[X^2]-(E[X])^2$

In our Ex, $Var(X) = E[X^2]-(E[X])^2 = \int_a^b \dfrac{x^2\,dx}{b-a} - \left(\dfrac{a+b}{2}\right)^2$

$= \dfrac{b^3-a^3}{3(b-a)} - \left(\dfrac{a^2+2ab+b^2}{4}\right) = \dfrac{a^2+ab+b^2}{3} - \dfrac{a^2+2ab+b^2}{4} = \dfrac{a^2+b^2-2ab}{12} = \dfrac{(a-b)^2}{12}$

• Exponential Random Variable :-

$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & o/w. \end{cases}$



$\Rightarrow E[X] = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx = \dfrac{1}{\lambda}$, $F_X(x) = \int_0^{x} \lambda e^{-\lambda x'}\,dx' = 1-e^{-\lambda x}$ $\forall\ x \ge 0$

• Duration of phone calls is modelled as an Exponential Random Variable — Memory lessness

$\mathbb{P}(X > k+a \mid X > k) = \mathbb{P}(X > a)$

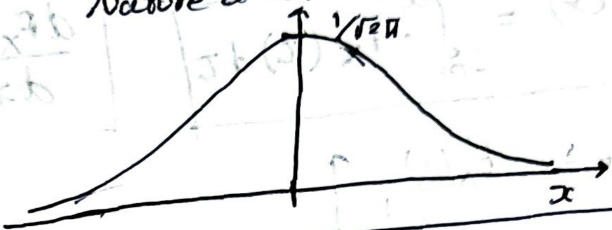
P-TO

$$\text{LHS} = \frac{\mathbb{P}(X > k+a \text{ and } X > k)}{\mathbb{P}(X>k)} = \frac{\mathbb{P}(X>k+a)}{\mathbb{P}(X>k)} = \frac{e^{-\lambda(a+k)}}{e^{-\lambda k}}$$

$$= e^{-\lambda a} = \mathbb{P}(X > a).$$

- $E[X^2] = \int_0^\infty x^2 \cdot \lambda \cdot e^{-\lambda x}\, dx = \frac{2}{\lambda^2} \Rightarrow Var(X) = \frac{1}{\lambda^2}$.

Normal / Gaussian Random variable — Appear in Nature a lot.



- CLT → central limit theorem
- nice analytical properties.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \forall x.$$

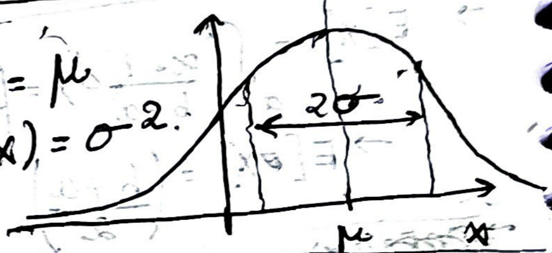
$$\mathcal{N}(0,1) \longrightarrow X \sim \mathcal{N}(\mu, \sigma^2) \longrightarrow \boxed{F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

mean, variance (for $\mathcal{N}(0,1)$); mean, variance (for μ, σ^2)

- The Gaussian Random variables are used to model noise.

$E[X] = \mu$

$Var(X) = \sigma^2$.



So a communication system, $X \sim$ transmitted signal, $Y \sim$ recieved signal.

$\rightarrow Y = aX + Z$, a constant, $Z \sim \mathcal{N}(\mu, \sigma^2)$

Let's say, $X \sim \mathcal{N}(\mu, \sigma^2)$, $Y = aX + b$

$E[X] = a\mu + b$, $Var(Y) = a^2\sigma^2$.

- what is non-trivial / surprising? → $Y = \mathcal{N}(a\mu + b, a^2\sigma^2)$ — Proof Later

→ CDF is painful to obtain in the closed form. : $\boxed{F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x'-\mu)^2}{2\sigma^2}}\, dx'}$

- CDF tables are available for $\mathcal{N}(0,1)$ in any standard book. — (work around for not being in open form).

Lets say $X \sim \mathcal{N}(0,1)$ then how do we compute $\mathbb{P}(a \le X \le b)$?

→ $\mathbb{P}(a \le X \le b) = F_X(b) - F_X(a)$ → use table.

Now lets say $X \sim \mathcal{N}(\mu, \sigma^2)$ → then $\boxed{Y = \frac{X-\mu}{\sigma}} \Rightarrow Y \sim \mathcal{N}(0,1)$

$$\mathbb{P}(a \le X \le b) = \mathbb{P}\left(\frac{a-\mu}{\sigma} \le Y \le \frac{b-\mu}{\sigma}\right) = F_Y\left(\frac{b-\mu}{\sigma}\right) - F_Y\left(\frac{a-\mu}{\sigma}\right)$$

P.T.O

Conditional PDF's :

$f_{X|A}(x)\cdot\delta = \mathbb{P}(x \le X \le x+\delta \mid A)$ or $\boxed{\mathbb{P}(a \le X \le b \mid A) = \int_a^b f_{X|A}(x')\cdot dx'}$

Example : $A = \{a \le X \le b\}$

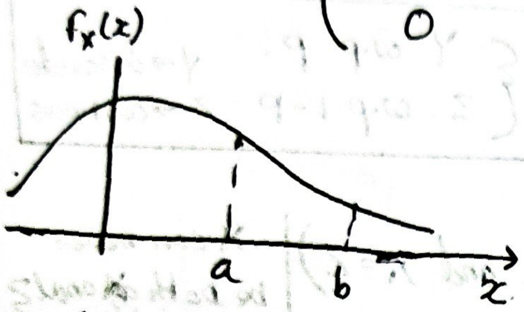
So, $f_{X|A}(x)\cdot\delta = \mathbb{P}(x \le X \le x+\delta \mid a \le X \le b) = \dfrac{\mathbb{P}(x \le X \le x+\delta \text{ and } a \le X \le b)}{\mathbb{P}(a \le X \le b)}$

→ If $x < a$ or $x > b$ $f_{X|A}(x) = 0$ since Numerator $= 0$.

→ If $a \le x \le b$ → $f_{X|A}(x)\cdot\delta = \dfrac{\mathbb{P}(x \le X \le x+\delta)}{\mathbb{P}(a \le X \le b)} = \dfrac{f_X(x)\cdot\delta}{\mathbb{P}(A)}$

So $f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(A)} & \text{for } a \le x \le b \\ 0 & \text{o/w.} \end{cases}$

$\boxed{\text{only works for } A \text{ of a specific Type.}}$ ✓✓



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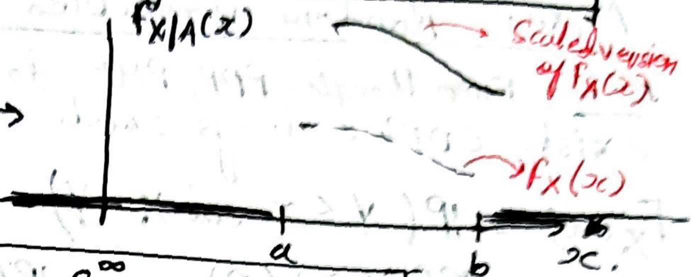
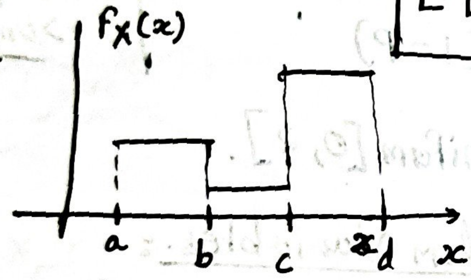


Conditional Expectation :

$$\boxed{E[X|A] = \int_{-\infty}^{\infty} x\cdot f_{X|A}(x)\cdot dx}$$

Example :



$A = \{\frac{a+b}{2} \le x \le b\}$ compute $E[X|A]$

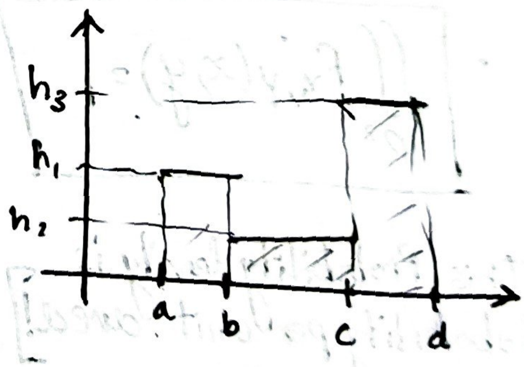
Total probability Theorem :

Let $A_1, A_2, \ldots A_n$ form a partition of Ω, i.e, $A_i \cap A_j = \phi \ \forall i \ne j$

$\bigcup_{i=1}^{\infty} A_i = \Omega$. Then $f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\cdot\mathbb{P}(A_i)$

Total Expectation Theorem :

$E[X] = \sum_{i=1}^{n} E[X|A_i]\cdot P(A_i)$



Ex: $E[X] = \int_a^b x h_1 dx + \int_b^c x h_2 dx + \int_c^d x h_3 dx$

$= h_1\dfrac{(b^2-a^2)}{2} + h_2\dfrac{(c^2-b^2)}{2} + h_3\dfrac{(d^2-c^2)}{2}.$

$=$

$A_1 = \{a \le X \le b\}$	$\mathbb{P}(A_1) = h_1(b-a)$	$E[X\mid A_1] = a+b/2$
$A_2 = \{b \le X \le c\}$	$\mathbb{P}(A_2) = h_2(c-b)$	$E[X\mid A_2] = b+c/2$
$A_3 = \{c \le X \le d\}$	$\mathbb{P}(A_3) = h_3(d-c)$	$E[X\mid A_3] = c+d/2$

$$E[X] = \sum_{i=1}^{3} E[X|A_i] \cdot \mathbb{P}(A_i) = \left(\frac{b^2-a^2}{2}\right)h_1 + \left(\frac{c^2-b^2}{2}\right)h_2 + \left(\frac{d^2-c^2}{2}\right)h_3$$

Example →

$$X = \begin{cases} \text{Uniform } [0,2] & \text{w.p } 1/2 \\ 1 & \text{o/w.} \end{cases}$$

You can think of this as a two step process with deciding the type of R.V X. is by flipping a coin.

$\mathbb{P}(X=1) = 1/2$ → But for it to be a continous R.V., $\mathbb{P}(X=1)=0$ which is not the case here and hence this ain't a continous R.V.

→ Thus X is neither a continous nor discrete R.V and we can't define it's PDF or PMF.

Mixed Random Variables:

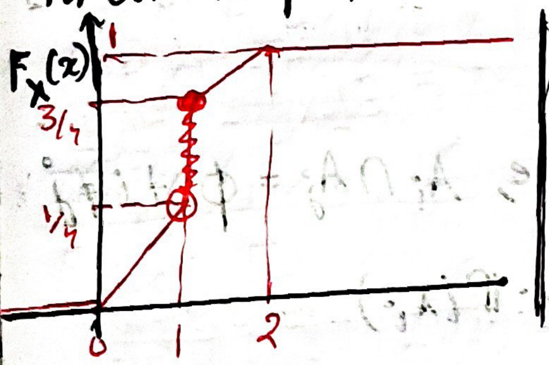
$$X = \begin{cases} Y & \text{w.p } p. & Y \to \text{discrete} \\ Z & \text{w.p } 1-p & Z \to \text{continous.} \end{cases}$$

Note: Even though PDF, PMF do not exist. CDF's always exist.

$$F_X(x) = \mathbb{P}(Y \le x \text{ and } X = Y) + \mathbb{P}(Z \le x \text{ and } X = Z)$$
$$= \mathbb{P}(Y \le x)(p) + \mathbb{P}(Z \le x)(1-p)$$
$$= F_Y(x)\cdot(p) + F_Z(x)\cdot(1-p)$$

X can never be both Y and Z
→ disjoint
→ union = sum

• For our example, $p = 1/2$ and $Z \sim \text{Uniform}[0,2]$.



• $$\boxed{\iint_{\mathbb{R}^2} f_{X,Y}(x,y) = 1}$$

[• This Probability density is Probability per unit area!]

Joint Random Variables:

$f_{X,Y}(x,y)$ ← joint PDF

$$\mathbb{P}((x,y) \in B) = \iint_{\substack{x,y \\ (x,y)\in B}} f_{X,Y}(x,y)\,dx\,dy$$

Inaccurate Math

$\mathbb{P}(x \le X \le x+\delta \text{ and } y \le Y \le y+\delta)$

Assume δ is small.

Then $\int_x^{x+\delta}\int_y^{y+\delta} f_{X,Y}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}$

So $$\boxed{\frac{\mathbb{P}(x \le X \le x+\delta \text{ and } y \le Y \le y+\delta)}{\delta^2} = f_{X,Y}(x,y)}$$

Conditioning on Random Variables :-

$p_{X|Y}(x|y) = \mathbb{P}(X=x \mid Y=y)$ ← definition. $\mathbb{P}(Y=y)>0$

$= \dfrac{\mathbb{P}(X=x \text{ and } Y=y)}{\mathbb{P}(Y=y)} = \dfrac{\mathbb{P}_{XY}(x,y)}{\mathbb{P}_Y(y)}$.

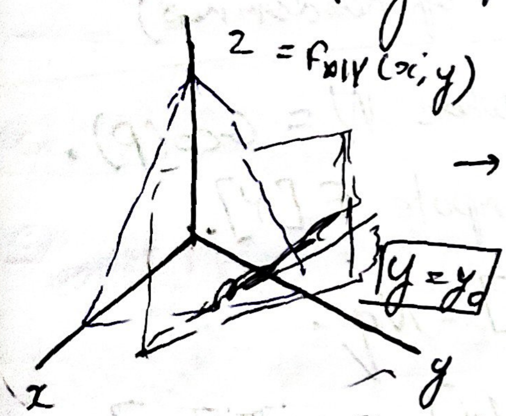
$\boxed{f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}}$ → How do we interpret this ?

Since $\mathbb{P}(Y=y)=0$ for continous r.v's

$y \le Y \le y+\varepsilon \quad x \le X \le x+\varepsilon$

$\dfrac{\mathbb{P}(x \le X \le x+\delta \text{ and } y \le Y < y+\varepsilon)}{\mathbb{P}(y \le Y \le y+\varepsilon)} \approx \dfrac{f_{XY}(x,y)\,\delta\varepsilon}{f_Y(y)\,\varepsilon}$ | "δ" small, "ε" small

$= \dfrac{f_{XY}(x,y)\,\delta}{f_Y(y)} \triangleq f_{X|Y}(x|y)\,\delta$.



The line of intersection of the two gives us the PDF of the conditional.

↳ It has the right slope, we will have to scale it to get the PDF (normalization)

$|y.\ X,Y \quad Z = X|Y$

Multiplication Rule :- $\boxed{p_{X|Y}(x,y) \cdot p_Y(y) = p_{XY}(x,y) = p_{Y|X}(y,x) \cdot p_X(x)}$

$f_{XY}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y) = f_{Y|X}(y|x) \cdot f_X(x)$.

Total Probability Law :-

$p_X(x) = \sum_y p_{X,Y}(x,y) = \sum_y p_{X|Y}(x,y) \cdot p_Y(y)$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty} f_{X|Y}(x,y) \cdot f_Y(y)\,dy$.

→ **~~Combinational Expectional~~ Conditional Expectation** :- $E[X|Y=y] = \sum_x x \cdot p_{X|Y}(x|y)$

→ $E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x,y)\,dx$

P.T.O

• Tower rule, Total expectation theorem, law of iterated expectation.

Claim: $E[X] = E[E[X|Y]]$.

$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\cdot f_Y(y)\,dx\,dy$

$= \int_{-\infty}^{\infty} f_Y(y)\left[\int_{-\infty}^{\infty} x f_{X|Y}(x,y)\,dx\right] dy = E[E[X|Y]]$

$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \cdot \int f_{X,Y}(x,y)\,dy\,dx.$

Ex: ~~coin~~
coin, $P(\text{head}) = Q \sim \text{uni}[0,1]$. Toss the coin, $P(\text{heads})$

$X = \begin{cases} 1 & \text{if coin lands on heads} \\ 0 & \text{o/w.} \end{cases}$ $\quad E[X \mid Q = q] = q.$

(Two layers of Randomness)

• We toss a coin with bias 'q', N times; where $N = \text{Geo}(p)$. Y is the total no. of heads obtained. Compute $E[Y]$.

$Y \mid N = n \sim \text{Bin}(n, q)$. $\rightarrow E[Y|N] = Nq$.

$\therefore E[Y] = E[E[Y|N]] = E[Nq] = q\,E[N] = \frac{q}{p}$

Independence of Random variables:

Recall: For events A and B, independence is defined as
$P(A \cap B) = P(A)\cdot P(B)$.

If discrete: $P_{X,Y}(x,y) = P_X(x)\cdot P_Y(y) \ \forall\, x, y$.

If continuous: $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) \quad \forall\, x, y$.

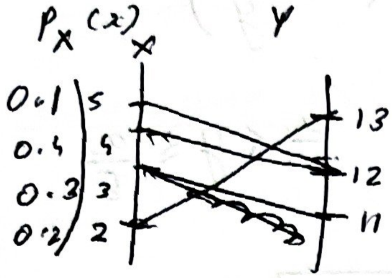
• Let $X \perp\!\!\!\perp Y$, Then $E[XY] = E[X]\cdot E[Y]$.

Proof: $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\cdot f_Y(y)\,dx\,dy$.

$= \left(\int_{-\infty}^{\infty} x f_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} y f_Y(y)\,dy\right) = E[X]\cdot E[Y]$. Hence Proved

Recall: $E[X+Y] = E[X] + E[Y]$.

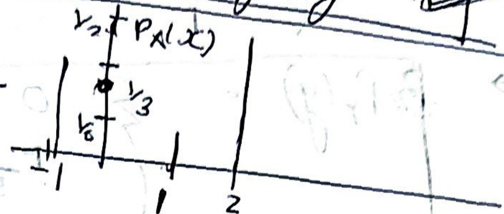
Derived Distributions: $Y = g(X)$, $f_X(x)$, $P_X(x)$ given. ~~Ne to compt~~
Need to compute $f_Y(y)$ or $P_Y(y)$.



$$\begin{bmatrix} P_Y(13) = 0.2 \\ P_Y(12) = 0.5 \\ P_Y(11) = 0.3 \end{bmatrix}$$

$$\boxed{P_Y(y) = \sum_{x: g(x)=y} P_X(x)}$$

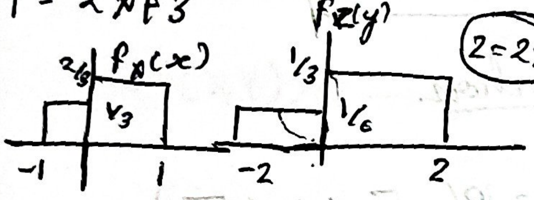
EX: $\boxed{Y = 2X + 3}$



$R_Y = \{1, 5, 7\}$.

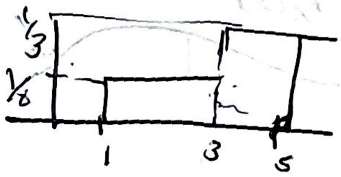
→ $P_Y(1) = \frac{1}{3}$, $P_Y(5) = 1/6$ $P_Y(7) = 1/2$.

$Y = 2X + 3$



$Z = 2X$

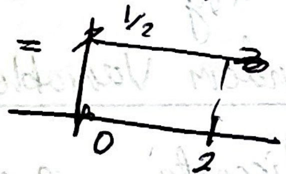
→ $Y = 2X + 3$



Recipie: Compute $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y) = \int_{x: g(x) \le y} f_X(x)\,dx$.

→ $f_Y(y) = \frac{dF_Y(y)}{dy}$ at points of continuity.

→ Example $Y = X^3$, $f_X(x) =$



→ $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y) = \int_{-\infty}^{y^{1/3}} f_X(x)\,dx$.

I] $y \le 0$ → $P(Y \le y) = 0$.

II] $y > 0$ → $P(Y \le y) = \int_0^{y^{1/3}} f_X(x)\,dx = \int_0^{y^{1/3}} \frac{1}{2}\,dx = \frac{y^{1/3}}{2}$.

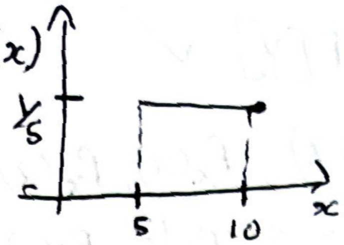
III] $y > 8$ → $P(Y \le y) = 1$

$$\boxed{\frac{dF_Y(y)}{dy} = \begin{cases} 0 & y \le 0 \\ \frac{1}{6} y^{-2/3} & 0 < y \le 8 \\ 0 & y > 8 \end{cases}}$$

- CDF —
- CCDF — complementary — $\overline{F_X}(x) = \mathbb{P}(X > x) = 1 - F_X(x)$

P-10

Ex: $Y = \dfrac{a}{X}$



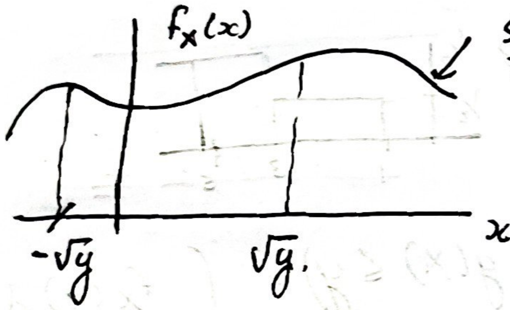
$F_Y(y) = \mathbb{P}(Y \le y)$
$= \mathbb{P}\left(\frac{a}{X} \le y\right)$
$= \mathbb{P}\left(X \ge \frac{a}{y}\right)$

Decreasing Function.

$$F_Y(y) = \begin{cases} 0 & y \le a/10 \\ \displaystyle\int_{a/y}^{10} \frac{dx}{5} = \frac{1}{5}\left(10 - \frac{a}{y}\right) = 2 - \frac{a}{5y} & \frac{a}{10} < y \le \frac{a}{5} \\ 1 & y > a/5. \end{cases}$$

$\rightarrow$
$$\boxed{f_Y(y) = \begin{cases} 0 & y \le a/10 \\ +\dfrac{a}{5y^2} & \dfrac{a}{10} < y \le a/5 \\ 0 & y > a/5. \end{cases}}$$

Ex:



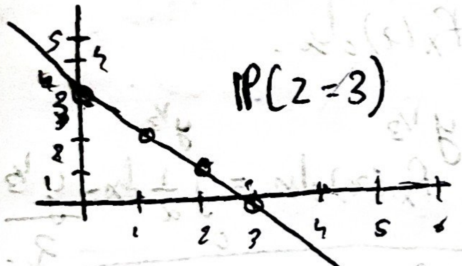
Some general function

$\mathbb{P}(Y \le y) = \mathbb{P}(-\sqrt{y} \le x \le \sqrt{y})$
$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$.

$f_Y(y) = \dfrac{f_X(\sqrt{y})}{2\sqrt{y}} + \dfrac{f_X(-\sqrt{y})}{2\sqrt{y}} = \dfrac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$

Sum of Independent Random Variables.

$Z = X + Y$, X, Y, discrete, $P_X(x)$, $P_Y(y)$ given



$\mathbb{P}(Z=3)$

$\mathbb{P}(Z=3) = \sum_i \mathbb{P}(X = i \text{ and } Y = 3-i)$.
$= \sum_i \mathbb{P}(X=i)\,\mathbb{P}(Y=3-i)$ Since $X \perp\!\!\!\perp Y$.

$\mathbb{P}(Z=z) = \underline{\underline{\sum_i \mathbb{P}(X=i) \cdot \mathbb{P}(Y=z-i)}}$

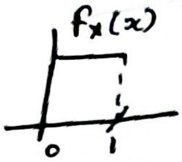
Similarly,
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx.$$

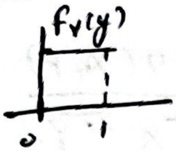
Convolving the two PMFs.

$\cancel{2 \vec{z} \quad x \vec{z} \quad 1+z \quad 0 < x < 1}$
$\cancel{x < z < 1+x}$

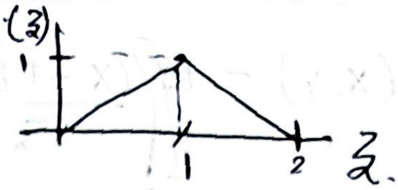
$\boxed{\text{P.T.O}}$

• ***Examples:*** $X, Y \sim \text{uniform}[0,1]$, $X \perp\!\!\!\perp Y$, $Z = X+Y$





~~$f_Z(z) = \int_{-\infty}^{\infty} f(t)\,dx$~~



Covariance and Corelation:

Defⁿ: $\mathrm{cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$

$= E[XY - XE[Y] - E[X]Y + E[X]E[Y]]$

$= E[XY] - E[X]\,E[Y].$

Ans) $\boxed{\mathrm{cov}(X,Y) = E[XY] - E[X]\cdot E[Y]}$

If $\mathrm{cov}(X,Y) < 0 \rightarrow$ the random variables tend to move in opposite directions. — one increases, then the other decreases.

$\mathrm{cov}(X,Y) > 0 \longrightarrow$ the random variables tend to move in the same direction.

$\mathrm{cov}(X,Y) = 0 \longrightarrow$ independent random variables.

Experiment: toss a fair coin 3 times. X – # of heads. Y – # of tails.

	X	Y	XY
HHH	3	0	0
HHT	2	1	2
HTH	2	1	2
THH	2	1	2
HTT	1	2	2
THT	1	2	2
TTH	1	2	2
TTT	0	3	0

$E[X] = 3/2$ $\quad E[Y] = 3/2$

$E[XY] = \frac{3}{2}.$ $\quad \mathrm{cov}(X,Y) = \frac{3}{2} - \left(\frac{9}{4}\right)$

$= -\frac{3}{4}$

• Thus as (# of heads) increases, the (# of tails) decreases}.

$\tilde{X}$ – no. of heads in the first 2 tosses.
$\tilde{Y}$ – no. of tails in the last 2 tosses.

$\tilde{X}$	$\tilde{Y}$	$\tilde{X}\tilde{Y}$
2	0	0
2	1	2
1	1	1
1	0	0
1	2	2
1	1	1
0	1	0
0	2	0

$E[\tilde{X}] = 1$

$E[\tilde{Y}] = 1$

$E[\tilde{X}\tilde{Y}] = 3/4.$

$E[\tilde{X}\tilde{Y}] - E[\tilde{X}]E[\tilde{Y}]$

$= \frac{3}{4} - 1 = -1/4.$

• As the (no. of heads in the first two tosses) increases. The (no. of tails in the last two tosses). decreases.

Ex: Positive covariance

→ $\tilde{X}$ → # of heads in first 2 tosses
$\tilde{Y}$ → # of heads in last 2 tosses.