EE-635

(Applied Linear Algebra)

Assignment-3

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Problem Set-3

1] $T = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ $\because$ $3x_1 = (3)x_1 + (0)x_2 + (0)x_3$

$x_1 - x_2 = (1)x_1 + (-1)x_2 + (0)x_3$

$2x_1 + x_2 + x_3 = (2)x_1 + (1)x_2 + (1)x_3$

Checking if T is invertible by RREF and augmented matrix:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{② } R_1 \to R_1/3]{\text{① } R_3 \to R_3 + R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-1)R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$RREF(T) = I$ $\quad$ $T^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$\therefore T^{-1}$ exists.

Now,

$$T^2 = T \cdot T = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 2 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix}$$

$\therefore (T^2 - I) \cdot (T - 3I)$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{\underline{0_{3\times3}}}$$

$\therefore$ Proved.

4] Given, $\phi : \mathbb{F}^5 \to \mathbb{F}^2 \ni Ker(\phi) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 :$
$x_1 = \alpha x_2 \, ; \, x_3 = x_4 = x_5\}$ $(\alpha \in \mathbb{F})$

$\therefore \mathcal{B}(Ker(\phi)) = \left\langle \begin{bmatrix} \alpha \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$ $\therefore dim(Ker(\phi)) = 2.$

L.I

Now, by Rank-Nullity theorem,

$$dim(\mathbb{F}^5) = dim(Ker(\phi)) + dim(im(\phi))$$
$$5 = 2 + dim(im(\phi))$$
$$\Rightarrow dim(im(\phi)) = 3$$

But $\because im(\phi) \subseteq \mathbb{F}^2$ $(\because \phi : \mathbb{F}^5 \to \mathbb{F}^2)$
and $dim(\mathbb{F}^2) = 2.$

$\therefore \nexists$ any such map.

6] Given, $\mathcal{L}_1 : C^1(\mathbb{R}) \to C^0(\mathbb{R})$, $\mathcal{L}_1(f(x)) = \frac{d}{dx}(f(x)) + \alpha f(x)$, $\alpha \in \mathbb{R}$.

$\mathcal{L}_2 : C^2(\mathbb{R}) \to C^0(\mathbb{R})$, $\mathcal{L}_2(f(x)) = \frac{d^2}{dx^2}(g(x)) + \omega^2 g(x)$, $\omega \in \mathbb{R}$.

For arbitrary $f_1 \in \ker(\mathcal{L}_1)$:

$$\frac{d}{dx}(f_1(x)) + \alpha f_1(x) = 0$$

$$\left(\int \frac{y'}{y} = \int -\alpha \Rightarrow \ln y = -\alpha x + C \right.$$
$$\left. \Rightarrow f(x) = e^{-\alpha x} \cdot e^{C} \right)$$

$$\Rightarrow \underset{f(x)}{\frac{f_1'(x)}{}} = -\alpha \Rightarrow f_1(x) = ce^{-\alpha x}, \quad c \in \mathbb{R}.$$

$$\therefore \boxed{\mathcal{B}_f(\ker(\mathcal{L}_1)) = \{e^{-\alpha x}\}} \Rightarrow \boxed{\dim(\mathcal{L}_1) = 1}$$

Now, for any $g_1 \in \ker(\mathcal{L}_2)$:

$$\frac{d^2}{dx^2}(g_1(x)) + \omega^2 g_1(x) = 0$$

$$\Rightarrow g_1''(x) + \omega^2 g_1(x) = 0 \Rightarrow g_1(x) = \alpha \sin(\omega x) + \beta \cos(\omega x), \quad \alpha, \beta \in \mathbb{R}.$$

$$\therefore \boxed{\mathcal{B}_g(\ker(\mathcal{L}_2)) = \{\sin(\omega x), \cos(\omega x)\}} \Rightarrow \boxed{\dim(\mathcal{L}_2) = 2}$$

7] Given, $\mathbb{V}$ and $\mathbb{W}$ → fdvs ; $\phi \in \mathcal{L}(\mathbb{V},\mathbb{W})$ and $\phi' \in \mathcal{L}(\mathbb{W}',\mathbb{V}')$.

T.P.T if ψ is a linear map from ϕ to ϕ' then it is an isomorphism of $\mathcal{L}(\mathbb{V},\mathbb{W})$ onto $\mathcal{L}(\mathbb{W}',\mathbb{V}')$.

Proof :

Let $\dim(\mathbb{V}) = m$ and $\dim(\mathbb{W}) = n$

$\Rightarrow \dim(\mathbb{V}') = \dim(\mathbb{V}) = m$, $\dim(\mathbb{W}') = \dim(\mathbb{W}) = n$

and $\dim(\mathcal{L}(\mathbb{V},\mathbb{W})) = mn$.

$\dim(\mathcal{L}(\mathbb{W}',\mathbb{V}')) = mn$.

Let $\mathcal{B}_1, \mathcal{B}_2$ be ordered basis of $\mathcal{L}(\mathbb{V},\mathbb{W})$ and $\mathcal{L}(\mathbb{W}',\mathbb{V}')$ respectively.

Let $\psi : \mathcal{L}(\mathbb{V},\mathbb{W}) \to \mathcal{L}(\mathbb{W}',\mathbb{V}')$ and

$\psi(\phi_i) = \varphi'_i \ \forall i$ where $\phi_i \in \mathcal{B}_1$ and $\phi'_i \in \mathcal{B}_2$.

Now,

let $\varphi_0 \in \mathcal{L}(\mathbb{V},\mathbb{W}) \ni \psi(\varphi_0) = 0 \in \mathcal{L}(\mathbb{W}',\mathbb{V}')$.

$\Rightarrow \exists\, a_i \ni \psi(\sum_i a_i\varphi_i) = 0$ $\quad (\because \{\varphi_i\} = \mathcal{B}_1)$

$\Rightarrow \sum_i a_i \psi(\varphi_i) = 0 \Rightarrow \sum_i a_i(\varphi'_i) = 0$

Now, $\because \{\varphi'_i\} = \mathcal{B}_2 \Rightarrow a_i = 0 \ \forall i \Rightarrow \varphi_0 = 0 \in \mathcal{L}(\mathbb{V},\mathbb{W})$

Thus, the map is one-to-one (1)

Now, consider arbitrary $\varphi' \in \mathcal{L}(\mathbb{W}',\mathbb{V}')$.

$\therefore \varphi' = \sum_i \beta_i\varphi'_i = \sum_i \beta_i \psi(\varphi_i) = \sum_i \psi(\beta_i\varphi_i)$

$= \cancel{\sum_i \psi}\ \psi(\phi)$ with $\varphi = \sum_i \beta_i \phi_i$.

$\therefore \forall \varphi' \in \mathcal{L}(\mathbb{W}',\mathbb{V}'), \exists\, \varphi \in \mathcal{L}(\mathbb{V},\mathbb{W}) \ni \psi(\varphi) = \varphi'$.

$\therefore \psi$ is onto — (2)

Using (1) and (2),

it is proved that the linear map ψ is an isomorphism of $\mathcal{L}(\mathbb{V},\mathbb{W})$ onto $\mathcal{L}(\mathbb{W}',\mathbb{V}')$.

8] Given, U and W are subspaces of V.

T.P.T $(U \cap W)^\circ = U^\circ + W^\circ$.

Proof:

Let $g \in U^\circ + W^\circ \Rightarrow \exists g_1 \in U^\circ$ and $g_2 \in W^\circ$ such that $g = g_1 + g_2$.

Consider any arbitrary $v \in U \cap W$:

$$g(v) = g_1(v) + g_2(v) = 0 + 0 = 0$$

$(\because v \in U)$ $(\because v \in W)$

$\Rightarrow g \in (U \cap W)^\circ \therefore U^\circ + W^\circ \subseteq (U \cap W)^\circ$.

$\Rightarrow \dim(U^\circ + W^\circ) \leq \dim((U \cap W)^\circ)$ — ①

$\Rightarrow$ ~~Also~~, $\dim(U^\circ + W^\circ) = \dim(U^\circ) + \dim(W^\circ) - \dim(U^\circ \cap W^\circ)$

noting that $\dim(U^\circ) + \dim(U) = \dim(V)$

$\dim(W^\circ) + \dim(W) = \dim(V)$

$\therefore \dim(U^\circ + W^\circ) = 2\dim(V) - \dim(U) - \dim(W) - \underline{\dim(U^\circ \cap W^\circ)}$. — ②

Consider $f \in (U + W)^\circ$

$\Rightarrow$ every vector $v = u_1 + w_1$ is killed by f.

$\Rightarrow f(v) = f(u_1 + w_1) = \underset{=0}{f(u_1)} + \underset{=0}{f(w_1)} = 0$

$\Rightarrow f \in U^\circ$ and $f \in W^\circ$

$\Rightarrow f \in U^\circ \cap W^\circ \quad \therefore (U + W)^\circ \subseteq U^\circ \cap W^\circ$.

$\Rightarrow \dim((U+W)^\circ) \leq \dim(U^\circ \cap W^\circ)$.

Thus, eqn ② becomes

$\dim(U^\circ + W^\circ) \geq 2\dim(V) - \dim(U) - \dim(W) - \underline{\dim((U+W)^\circ)}$

$\because \dim(U) + \dim(W) = \dim(U+W) + \dim(U \cap W)$

$\Rightarrow \dim(U^\circ + W^\circ) \geq 2\dim(V) - \dim(U+W) - \dim(U \cap W) - \dim((U+W)^\circ)$

$\because \dim(U+W) + \dim((U+W)^\circ) = \dim(V)$

$\Rightarrow \dim(U^\circ + W^\circ) \geq \dim(V) - \dim(U \cap W)$.

Now,

$\because \dim(V) = \dim(U \cap W) + \dim((U \cap W)^\circ)$

$\Rightarrow \dim(U^\circ + W^\circ) \geq \dim((U \cap W)^\circ)$. — ③

From eqn ① and ③, $\boxed{U^\circ + W^\circ = (U \cap W)^\circ}$ $\therefore$ proved.

12] a) Given $\Lambda : V \to V''$

Consider $v_1, v_2 \in V$. $\Rightarrow \Lambda(\alpha v_1 + v_2)\varphi$ $(\alpha \in \mathbb{F})$

$= \varphi(\alpha v_1 + v_2)$ (by defn of Λ)

$= \varphi(\alpha v_1) + \varphi(v_2)$

$= \alpha\varphi(v_1) + \varphi(v_2) = \alpha\Lambda(v_1) + \Lambda(v_2)$.

$\therefore$ Λ is a linear map.

b) Let $v \in V$ and $\varphi \in V'$.

Then, $((\underline{\underline{\tau'' \circ \Lambda}})(v))(\varphi) = (\tau''(\Lambda v)(\varphi))$

$= (\Lambda v)(\tau'\varphi)$ (by defn of dual)

$= (\tau'\varphi)(v)$ (by defn of Λ)

$= \varphi(\tau v)$ (by defn of dual)

$= (\Lambda(\tau v))(\varphi)$

$= ((\underline{\underline{\Lambda \circ \tau}})(v))(\varphi)$. $\therefore$ $\underline{\underline{\tau'' \circ \Lambda = \Lambda \circ \tau}}$

$\therefore$ Proved.

c) T.P.T if $V \to$ fdvs $\Rightarrow \Lambda$ is isomorphism of V onto V''.

Proof: If $v \neq 0$, then $(\Lambda v)\varphi = \varphi(v)$
$\neq 0$

$\{\because$ if $v \in \text{Null}(\Lambda) \Rightarrow \varphi(v) = 0 \ \forall \varphi \in V'$ but $v \neq 0\}$

Thus Λ is non-singular.

$\Rightarrow \Lambda$ is invertible.

Also, $\underline{\dim(V'')} = \dim(V') = \underline{\dim(V)}$.

$\therefore$ Λ is isomorphism $\therefore$ Proved.

14] Let $\varphi \in \mathcal{L}(V_1 \times \dots \times V_m, W)$

Let S denote a map from $\mathcal{L}(V_1 \times \dots \times V_m) \to \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$

such that

$$S(\varphi) = (\varphi_1, \varphi_2, \dots, \varphi_m), \text{ where } \varphi_i \in \mathcal{L}(V_i; W)$$

$v = (0, 0, \dots, v_i, 0, \dots, 0) \in V_1 \times V_2 \times \dots \times V_m$ $\forall i$

$\Rightarrow \varphi_i(v_i) = \varphi(v) = w \in W$

Thus, due to linearity of φ, φ_i's are also linear maps.

For Linearity (S), consider $S(\varphi_1) + \alpha S(\varphi_2)$

$$= (\psi_{11}, \psi_{12}, \ldots, \psi_{1m}) + (\alpha\psi_{21}, \alpha\psi_{22}, \ldots, \alpha\psi_{2m})$$

Now for any $v_i \in V_i$,

consider $\psi_{1i}(v_i) + \alpha\psi_{2i}(v_i)$

$$= \varphi_1(v) + \alpha\varphi_2(v) \quad [\because v = (0, 0, \ldots, v_i, \ldots, 0)]$$

But $\because \varphi_1(v) + \alpha\varphi_2(v) = (\varphi_1 + \alpha\varphi_2)(v)$

$$\therefore S(\varphi_1 + \alpha\varphi_2) = (\psi_{11}, \psi_{12}, \ldots, \psi_{1m}) + (\alpha\psi_{21}, \alpha\psi_{22}, \ldots, \alpha\psi_{2m})$$

$$= S(\varphi_1) + \alpha S(\varphi_2) \quad \therefore S \text{ is also linear. } (1)$$

For Surjectivity (S):

consider $(\psi_1, \psi_2, \ldots, \psi_m) \in \mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}(V_m, W)$.

$\exists\, \varphi \in \mathcal{L}(V_1 \times \ldots \times V_m, W) \ni \forall\, v = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m),\ \vec{v}_i \in V_i$

and $v \in V_1 \times \ldots \times V_m$

$$\varphi(v) \to (\psi_1(v_1), \ldots, \psi_m(v_m))$$

Thus, $\exists\, \varphi \in \mathcal{L}(V_1 \times \ldots \times V_m, W)\ \forall\, (\psi_1, \psi_2, \ldots, \psi_m)$

$\ni S(\varphi) = (\psi_1, \psi_2, \ldots, \psi_m)$.

$\Rightarrow$ S is onto. (2)

For injectivity (S):

let $S(\varphi) = (0, 0, \ldots, 0) \in \mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}(V_m, W)$

$\Rightarrow \psi_i(v_i) = 0 \quad \forall\, i$

But $\because \psi_i(v_i) = \varphi(v)$

if $v = (0, \ldots, v_i, 0, \ldots, 0) \Rightarrow \varphi(v) = 0 \quad \forall\, v \in V_1 \times \ldots \times V_m$.

(Because $\psi_i(v_i) = 0\ \forall\, i$!!)

$\Rightarrow \varphi$ has to be nothing but an identically zero functional.

ie: $\varphi = 0_{\mathcal{L}(V_1, \ldots, V_m, W)}$

$\Rightarrow$ S is one-to-one (3)

From (1), (2) and (3)

S is a linear Bijection.

$\Rightarrow$

$\mathcal{L}(V_1 \times V_2 \times \ldots \times V_m, W)$ is isomorphic with $\mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}(V_m, W)$

16] Given $\mathbb{U}$ and $\mathbb{V}$ are subspaces of $\mathbb{W}$,

T.P.T $w_1 + \mathbb{U} = w_2 + \mathbb{V} \Rightarrow \mathbb{U} = \mathbb{V} \quad \forall w_1, w_2 \in \mathbb{W}$.

Proof: Let $\vec{u}, \tilde{u} \in \mathbb{U}$ and $\vec{v}, \tilde{v} \in \mathbb{V}$.

$$\Rightarrow w_1 + \vec{u} = w_2 + \vec{v}$$

$$\Rightarrow (w_1 + \vec{u}) - w_2 = \vec{v} \in \mathbb{V}.$$

$$\Rightarrow (w_1 + \vec{u}) + \mathbb{V} = w_2 + \mathbb{V}$$

$$= w_1 + \mathbb{U} \quad \text{(given)}$$

$$\Rightarrow \cancel{w_1} + \vec{u} + \tilde{v} = \cancel{w_1} + \tilde{u}$$

$$\Rightarrow \tilde{v} = \tilde{u} - \vec{u} \in \mathbb{U}$$

$$\Rightarrow \mathbb{V} \subseteq \mathbb{U} \quad \text{— (1)}$$

$$(w_2 + \vec{v}) - w_1 = \vec{u} \in \mathbb{U}$$

$$(w_2 + \vec{v}) + \mathbb{U} = w_1 + \mathbb{U} = w_2 + \mathbb{V}$$

$$\Rightarrow \cancel{w_2} + \vec{v} + \tilde{u} = \cancel{w_2} + \tilde{v}$$

$$\Rightarrow \tilde{u} = \tilde{v} - \vec{v} \in \mathbb{V}$$

$$\Rightarrow \mathbb{U} \subseteq \mathbb{V}. \quad \text{— (2)}$$

from (1) and (2) $\Rightarrow \boxed{\mathbb{U} = \mathbb{V}}$ $\therefore$ proved.

19] Given, fdvs $\mathbb{U} \subseteq \mathbb{V}$; T.P.T $\exists\, \mathbb{W} \subseteq \mathbb{V} \ni \dim(\mathbb{W}) = \dim(\mathbb{V}/\mathbb{U})$ and $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$. Also, T.P.T $\mathbb{V}$ is isomorphic to $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.

Proof: Consider a basis $\{v_1 + \mathbb{U}, \ldots, v_m + \mathbb{U}\}$ of $\mathbb{V}/\mathbb{U}$.

Claim: $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

Proof: Suppose $\sum_{i=1}^{m} a_i v_i = 0$ for some $a_i \neq 0$

$$\Rightarrow \sum_{i=1}^{m} a_i v_i + \mathbb{U} = 0 + \mathbb{U} \quad \text{for some } a_i \neq 0$$

$$\Rightarrow \sum_{i=1}^{m} a_i (v_i + \mathbb{U}) = 0_{\mathbb{V}/\mathbb{U}} \quad \text{for some } a_i \neq 0.$$

We just considered $\{v_i + \mathbb{U}\}$ to be $\mathcal{B}(\mathbb{V}/\mathbb{U})$,

$\therefore a_i = 0 \quad \forall i \in \{1, \ldots, m\}$.

$\therefore$ Contradiction!

$\Rightarrow \{v_1, \ldots, v_m\}$ is linearly independent.

Now, Suppose $\sum_i a_i v_i \neq 0$ and $\sum_i a_i v_i \in \mathbb{U}$ for some $a_i \neq 0$.

$$\Rightarrow \sum_i a_i v_i + \mathbb{U} = 0 + \mathbb{U} = 0_{\mathbb{V}/\mathbb{U}}$$

$\Rightarrow$ Contradiction!

Thus, unless $\sum_i a_i v_i = 0$, $\sum_i a_i v_i \notin \mathbb{U}$. $\therefore v_i \notin \mathbb{U} \; \forall i$.

$\therefore$ by the principle of basis extension, upon appending v_i to the set $\{u_1, u_2, \ldots, u_n\}$, the resulting set will be linearly independent.

Further, $\dim(\mathbb{V}/\mathbb{U}) = \dim(\mathbb{V}) - \dim(\mathbb{U})$

$$\Rightarrow \dim(\mathbb{V}) = \dim(\mathbb{V}/\mathbb{U}) + \dim(\mathbb{U})$$
$$= m + n$$

$\therefore \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ is surely a basis for $\mathbb{V}$.

Now,

consider $v \in \mathbb{V}$: $v = \sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j = u + w$

where $u \in \mathbb{U}$ and ~~$w \in \mathbb{W}$~~

$w \in \langle v_1, v_2, \ldots, v_m \rangle$. Let $\langle v_1, v_2, \ldots, v_m \rangle := \mathbb{W}$.

$\because v = u + w \Rightarrow \mathbb{V} = \mathbb{U} \oplus \mathbb{W}$.

Consider a linear mapping $\phi : \mathbb{V} \to \mathbb{U} \times \mathbb{V}/\mathbb{U}$ given by

$$\phi(v) = (\underbrace{v - \textstyle\sum_j b_j v_j}_{(\sum_i^n a_i u_i = u)}, v + \mathbb{U})$$

To prove surjection: consider any $(\tilde{u}, \tilde{v} + \mathbb{U}) \in \mathbb{U} \times \mathbb{V}/\mathbb{U}$ $(\tilde{u} \in \mathbb{U}, \tilde{v} \in \mathbb{V})$

$\therefore$ $\tilde{u} = \sum_{i=1}^{n} r_i u_i$ and $\tilde{v} = \cancel{\tilde{u}} + \tilde{w}$ where $\tilde{w} = \sum_{j=1}^{m} \tilde{b}_j v_j$.

$\sum_{i=1}^{n} \tilde{a}_i u_i$

let $v = \tilde{u} + \tilde{w}$.

$$\therefore \varphi(v) = = \left(v - \sum_{j=1}^{m} \tilde{b}_j v_j, v + \mathbb{U}\right) = \left(\sum_{i=1}^{n} r_i u_i, v + \mathbb{U}\right) = (\tilde{u}, \underline{v} + \mathbb{U})$$

Now, $v - \tilde{v} = (\tilde{u} + \cancel{\tilde{w}}) - \left(\sum_{i=1}^{n} \tilde{a}_i u_i + \cancel{\tilde{w}}\right) = \sum_{i=1}^{n} (r_i - \tilde{a}_i) u_i \in \mathbb{U}$.

$\Rightarrow v + \mathbb{U} = \tilde{v} + \mathbb{U}$. $\therefore \varphi(v) = (\tilde{u}, \tilde{v} + \mathbb{U})$.

$\therefore \exists \, v : \varphi(v) = (\tilde{u}, \tilde{v} + \mathbb{U}) \; \forall \, (\tilde{u}, \tilde{v} + \mathbb{U}) \in \mathbb{U} \times \mathbb{V}/\mathbb{U}$.

Thus, ϕ is surjective. —— 1

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Now, for proving injectivity, consider $\varphi(v) = (0, 0+\mathbb{U}) = 0_{\mathbb{U}\times\mathbb{V}/\mathbb{U}}$ for some $v \in \mathbb{V}$.

$\Rightarrow$ If $v = \sum_{i=1}^{n} \bar{a}_i u_i + \sum_{j=1}^{m} \bar{b}_j v_j$, then $\varphi(v) = (\sum_i \bar{a}_i u_i, v+\mathbb{U}) = (0, 0+\mathbb{U})$

But $\{u_1, u_2, \dots, u_n\}$ is a linearly independent set. ~~$\therefore \sum \bar{a}_i u_i = 0$~~

$\therefore \quad \sum_i \bar{a}_i u_i = 0 \Rightarrow \bar{a}_i = 0 \quad \forall i$.

Thus, v reduces to $\sum_{i=1}^{m} \bar{b}_j v_j$

Further, $v + \mathbb{U} = 0 + \mathbb{U} \Rightarrow v \in \mathbb{U}$.

$\Rightarrow \sum_{i=1}^{m} b_j v_j \in \mathbb{U}$.

But,

$\sum_{i=1}^{m} b_j v_j \in \langle v_1, \dots, v_m \rangle$ and $\mathbb{U} \cap \langle v_1, \dots, v_m \rangle = \{0\}$

$(\because \mathbb{V} = \mathbb{U} \oplus \langle v_1, v_2, \dots, v_m \rangle)$

So, $\sum b_j v_j = 0 \Rightarrow b_j = 0 \; \forall j$ because $\{v_1, \dots, v_m\}$ is a lin. indep. set.

$\Rightarrow v = 0_{\mathbb{V}} \in \mathbb{V}$.

Thus, the $Ker(\varphi) = 0_{\mathbb{V}}$. $\therefore \varphi$ is injective. — (2)

From (1) and (2), $\mathbb{V}$ is isomorphic to $\mathbb{U} \times \mathbb{V}/\mathbb{U}$.

22] Consider $f(x) \in \mathbb{R}[x]$ and suppose, $deg(f) = n$.

Then, $f \in \mathbb{R}[x]_n$. Now, consider the restriction of φ, say φ_n in $\mathcal{L}[\mathbb{R}[x]_n, \mathbb{R}[x]_n]$

$\ni \varphi_n(f) = \varphi(f)$ for any $f \in \mathbb{R}[x]_n$ $(\because deg(\varphi_n(f)) \leq deg(f)$, $\varphi_n(f) \in \mathbb{R}[x]_n)$

Now, φ_n is a linear mapping on a fdvs $\mathbb{R}[x]_n$ and is also injective, $\because$ given φ is injective.

Thus, φ_n must also be surjective.

But this means that for any $f \in \mathbb{R}[x]$, since $\exists\, n \in \mathbb{Z}_+ \ni deg(f) \leq n$,

$\therefore f \in \mathbb{R}[x]_n$ for some n.

Consequently, $\exists\, g \in \mathbb{R}[x]_n : \varphi_n(g) = f$.

$\Rightarrow \varphi(g) = f \quad (\because \varphi_n = \varphi \text{ for } g \in \mathbb{R}[x]_n)$

$\Rightarrow \exists\, g \in \mathbb{R}[x] : \varphi(g) = f$ and $\therefore \varphi$ is a surjection.

Suppose,

for some polynomial f (non-zero) of degree k,

we have

$$\deg(\varphi(f)) < k. \quad \text{Let } g = \varphi(f).$$

Now,

consider $\varphi_{k-1} \in \mathcal{L}(\mathbb{R}[x]_{k-1}, \mathbb{R}[x]_{k-1}) \ni \varphi_{k-1}(f) = \varphi(f)$ for $f \in \mathbb{R}[x]_{k-1}$.

$\because$ we just saw that φ_{k-1} is a surjection and bijection over $\mathbb{R}[x]_{k-1}$.

Thus, for any $p(x) \in \mathbb{R}[x]_{k-1}$, $\exists$ a unique $k(x) \in \mathbb{R}[x]_{k-1} : \varphi_{k-1}(k(x)) = p(x)$.

Now, by our consideration, $g(x) \in \mathbb{R}[x]_{k-1}$.

$\therefore$ we must have $h(x) \in \mathbb{R}[x]_{k-1} : \varphi_{k-1}(h(x)) = g(x)$.

But since $\varphi_{k-1} = \varphi$ on $\mathbb{R}[x]_{k-1}$,

$$\Rightarrow \varphi(h(x)) = g(x).$$

On the other hand, we have $\varphi(f(x)) = g(x)$.

$\because \varphi$ is injective, we must have $f(x) = h(x)$.

But then $\deg(f(x)) = k$ and $\deg(h(x)) < k$ $\therefore$ Contradiction!

$\therefore \deg(\varphi(f)) = \deg(f) \;\; \forall f \in \mathbb{R}[x]$,

$\therefore$ Proved.

24] a) True.

Consider $v \in V$. Now, $\varphi(\psi(v)) \in \operatorname{im}\varphi \subset \underline{\operatorname{Ker}(\psi)}$.

$\therefore \psi(\varphi(\psi(v))) = 0$

$$\Rightarrow \varphi(\psi(\varphi(\psi(v)))) = \varphi(0) = 0$$

($\because$ for a lin. transf, $\varphi(0) = 0$)

b) True

Suppose $\operatorname{im}(\varphi) = \operatorname{Ker}(\varphi) \Rightarrow \dim(\operatorname{im}(\varphi)) = \dim(\operatorname{Ker}(\varphi))$

But RN theorem says $\dim(V) = \dim(\mathrm{im}(\varphi)) + \dim(\mathrm{Ker}(\varphi))$

$= 2\dim(\mathrm{Ker}(\varphi))$

$\Rightarrow \underbrace{2m-1}_{\text{odd integer}} = \underbrace{2\dim(\mathrm{Ker}(\varphi))}_{\text{even integer}}$ i.e. not possible.

c) True.

For lin-functionals that are not zero,

$\dim(V) = \underbrace{\dim(\mathrm{im}(\varphi))}_{=1} + \dim(\mathrm{Ker}(\varphi))$

$\Rightarrow \dim(\mathrm{Ker}(\varphi)) = \dim(V) - 1$.

1) Suppose $\varphi_1 = \varphi_2 = 0$, then it is trivial that $\varphi_1 = \alpha\varphi_2 \ \forall \alpha \in \mathbb{F}$.

2) Suppose $\varphi_1 \neq 0, \varphi_2 \neq 0$.

(one being zero and the other being non-zero is NOT POSSIBLE as their kernels won't be identical in that case.

Let $\{v_1, v_2, \ldots, v_{n-1}\}$ be a basis for $Ker(\varphi_1) = Ker(\varphi_2)$ where $\dim(V) = n$.

Extending it to get a basis for V as $\{v_1, v_2, \ldots, v_n\}$.

Now, for any $v \in V$, $v = \sum_i \alpha_i v_i$

$\Rightarrow \varphi_1(v) = \alpha_n \varphi_1(v_n)$ and $\varphi_2(v) = \alpha_n \varphi_2(v_n)$,

Since $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$, $\therefore \varphi_1(v_n) \neq 0$ and $\varphi_2(v_n) \neq 0$.

$$\Rightarrow \varphi_1(v) = \frac{\alpha_n \cdot \varphi_1(v_n) \cdot \varphi_2(v_n)}{\varphi_2(v_n)}$$

considering this as α,

$$= \frac{\varphi_1(v_n)}{\varphi_2(v_n)} \cdot (\alpha_n \varphi_2(v_n)) = \boxed{\frac{\varphi_1(v_n)}{\varphi_2(v_n)}} \cdot \varphi_2(v) \Rightarrow \varphi_1(\cdot) = \alpha\, \varphi_2(\cdot)$$

d) TRUE Given, $\varphi_1 \varphi_2 \varphi_3 = I_V$

By definition,

$\varphi_1^{-1} = \varphi_2 \varphi_3$ $\qquad\qquad$ $\varphi_3^{-1} = \varphi_1 \varphi_2$

$\Rightarrow I_V = \varphi_2 \varphi_3 \varphi_1$ $\qquad\qquad$ $\Rightarrow I_V = \varphi_3 \varphi_1 \varphi_2$

$\Rightarrow \varphi_2^{-1} = \varphi_3 \varphi_1$. $(\because \varphi_2 \varphi_2^{-1} = \varphi_2^{-1} \varphi_2 = I_V)$

e) FALSE.

Suppose $(D - \lambda I)(f(x)) = 0 \Rightarrow f'(x) - \lambda f(x) = 0$.

For $\lambda = 0$, $f(x) = c$ in $Ker(D - \lambda I)$.

This is a non-trivial subspace of $\mathbb{R}[x]$.

For any other $\lambda (\neq 0)$, ~~for~~ let $f'(x) = \lambda f(x)$ for $f(x) \in \mathbb{R}[x]$

$\Rightarrow f(x) = 0$.

$\therefore Ker(D - \lambda I) = 0_{\mathbb{R}[x]}$.

25] Given: $W_1 = \{f(x) \in \mathbb{R}[x] : f(0) = f(10)\}$ and
$W_2 = \{f(x) \in \mathbb{R}[x] : f(0) = f(10) = 0\}$.

T.P.T $\mathbb{R}[x]/W_1$ is isomorphic to $\mathbb{R}[x]/W_2$.

Proof: $\mathbb{R}[x]/W_1 \mapsto$ consists of equivalence classes of polynomials modulo W_1. Two polynomials in $\mathbb{R}[x]$, say $f(x)$ and $g(x)$, (are eq) if $f(x) - g(x) \in W_1$, ie: ~~$(f-g)(0) = (f-g)(10)$~~.

$\mathbb{R}[x]/W_2 \mapsto$ consists of equivalence classes of polynomials modulo W_2. If $f(x), g(x) \in \mathbb{R}[x]$ are equivalent in $\mathbb{R}[x]/W_2 \Rightarrow (f-g)(0) = (f-g)(10) = 0$.

Now, we search for a natural mapping (homomorphism) between these spaces that preserves the structure and operations. Specifically, we need $\varphi : \mathbb{R}[x]/W_1 \to \mathbb{R}[x]/W_2 \ni \varphi(u+W_1) = v + W_2$ where u, v ~~$\in \mathbb{R}[x]$~~ are cosets in $\mathbb{R}[x]/W_1$ and $\mathbb{R}[x]/W_2$ respectively.
Consider the following:

1) If $u + W_1 = 0$ in $\mathbb{R}[x]/W_1$, then $\varphi(u + W_1) = 0 + W_2 = W_2$.
This is because in $\mathbb{R}[x]/W_2$, the zero coset is the coset of the zero polynomial.

2) If $u + W_1 \neq 0$ in $\mathbb{R}[x]/W_1$, then u is a non-zero polynomial that has the same values at 0 and 10 ($\because$ it belongs to W_1).

However, in $\mathbb{R}[x]/W_2$, this would still be a non-zero polynomial, as it does NOT belong to the coset of the zero polynomial.

$\therefore$ we see that φ 'preserves' the zero element (the zero coset) and the non-zero elements in both $\mathbb{R}[x]/W_1$ and $\mathbb{R}[x]/W_2$.

This mapping 'φ' is <u>injective</u> because distinct cosets in $\mathbb{R}[x]/W_1$ map to distinct cosets in $\mathbb{R}[x]/W_2$, and it is <u>surjective</u> (onto) because every coset in $\mathbb{R}[x]/W_2$ is mapped to by some coset in $\mathbb{R}[x]/W_1$.

$\because$ $\varphi : \mathbb{R}[x]/W_1 \rightarrow \mathbb{R}[x]/W_2$ is a bijective mapping that preserves the structure, $\therefore$ $\mathbb{R}[x]/W_1$ is ISOMORPHIC to $\mathbb{R}[x]/W_2$.

$\therefore$ proved.