Applied Linear Algebra

(Course Code: EE 635)

Department of Electrical Engineering
Indian Institute of Technology Bombay

Instructor: Dwaipayan Mukherjee
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Time: **1.25 Hours**

Mid-semester Examination

Total Points: 50

Instructions

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- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Prove or disprove (possibly through some counter-example) the following:

   (a) For two subspaces $\mathbb{U}$ and $\mathbb{W}$ of the vector space $\mathbb{V}$, if every vector belonging to $\mathbb{V}$ either belong to $\mathbb{U}$ or to $\mathbb{W}$ (or both), then either $\mathbb{V} = \mathbb{U}$, or $\mathbb{V} = \mathbb{W}$ (or both).

   (b) Every subfield of the complex numbers must contain every rational number.

   (c) Suppose $v_1, v_2 \in \mathbb{V}$ (a vector space). Further, suppose $\mathbb{U}$ and $\mathbb{W}$ are subspaces of $\mathbb{V}$. If $v_1 + \mathbb{U} = v_2 + \mathbb{W}$, then $\mathbb{U} = \mathbb{W}$.

   (d) If $\mathbb{U}_1, \mathbb{U}_2$, and $\mathbb{W}$ are subspaces of the vector space $\mathbb{V}$ such that $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{W}$ and $\mathbb{V} = \mathbb{U}_2 \oplus \mathbb{W}$, then $\mathbb{U}_1 = \mathbb{U}_2$.

[3 × 4]

2. Give an example of each of the following

   (a) A mapping $T : \mathbb{R}^2 \to \mathbb{R}$ such that $T(\alpha v) = \alpha T(v)$ for $\alpha \in \mathbb{R}, v \in \mathbb{R}^2$, but T is not linear.

   (b) A mapping $T : \mathbb{C} \to \mathbb{C}$ such that $T(v_1 + v_2) = T(v_1) + T(v_2)$ for $v_1, v_2 \in \mathbb{C}$, but T is not linear.

[3 × 2]

3. For a linear transformation $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{U})$, show that

   (a) dim(im(φ'))=dim(im(φ)), and

   (b) im(φ')=(ker(φ))^0

[4+5]

4. For vector spaces $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$, of which $\mathbb{U}$, and $\mathbb{V}$ are finite dimensional, consider two linear transformations $\tau : \mathbb{U} \to \mathbb{V}$ and $\psi : \mathbb{V} \to \mathbb{W}$. Suppose the composition of the two transformations is given by $\psi \circ \tau$. Establish that $\dim(\mathrm{Ker}(\psi \circ \tau)) \leq \dim(\mathrm{Ker}(\tau)) + \dim(\mathrm{Ker}(\psi))$. When does equality hold? What can you say about the inequality when τ is surjective? [4+1+2]

5. Let $\mathbb{V} = \mathbb{R}[x]_2$, the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$, that have degree 2 or less, given by $p(x) = c_2x^2 + c_1x + c_0$. Define $f_1(p) = \int_0^1 p(x)dx$, $f_2(p) = \int_0^2 p(x)dx$, and $f_3(p) = \int_0^{-1} p(x)dx$. Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbb{V}'$. Obtain the basis for $\mathbb{V}$ to which $\{f_1, f_2, f_3\}$ is a dual. [3+3]

6. (a) Suppose $\mathbb{U} \subseteq \mathbb{V}$ and $\{v_1 + \mathbb{U}, v_2 + \mathbb{U}, \ldots, v_m + \mathbb{U}\}$ is a basis for $\mathbb{V}/\mathbb{U}$, while $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{U}$. Show that $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{V}$.

   (b) Verify the first isomorphism theorem (you are not allowed to use it/invoke it at any point!) for the case when $\mathbb{V}$ is the space of all polynomials in x over the real field, of degree less than or equal to n, $\mathbb{W} = \mathbb{V}$, and the linear map $T : \mathbb{V} \to \mathbb{W}$ is the derivative map. Then provide an interpretation for what each element in $\mathbb{V}/\mathrm{Ker}(T)$ represents. In other words, you are required to define the induced map corresponding to the derivative map and show that this induced map is linear, injective, has an image identical with the original derivative map, and finally, $\mathbb{V}/\mathrm{Ker}(T) \cong \mathrm{im}(T)$. Thereafter, you need to describe what each element in $\mathbb{V}/\mathrm{Ker}(T)$ represents.

[5+5]
