EE-635

(Applied Linear Algebra)

Assignment-4

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Assignment - 4

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Sol 6] and Sol 7] Consider a linear functional $f: \mathbb{V} \to \mathbb{F}$ such that

$f(\alpha \tilde{v}_1 + \tilde{v}_2) = \alpha f(\tilde{v}_1) + f(\tilde{v}_2) \quad \forall \; \vec{v}_1, \vec{v}_2 \in \mathbb{V}$ and $\alpha \in \mathbb{C}$.

Now,

$f(\cdot)$ can be characterised by its action on some basis of $\mathbb{V}$, say $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$.

Let $f(v_i) = \beta_i \in \mathbb{F}$.

$$\Rightarrow f(v) = f(\Sigma \alpha_i v_i) \quad \text{where } \alpha_i \neq 0 \; \forall i$$

$$= \Sigma \alpha_i f(v_i) = \Sigma \alpha_i \beta_i$$

$$= [\alpha_1 \; \alpha_2 \ldots \alpha_n] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \left\langle [v]_{\mathcal{B}} \middle| [\tilde{v}]_{\mathcal{B}} \right\rangle$$

where $\tilde{v} = \bar{\beta}_1 v_1 + \bar{\beta}_2 v_2 + \ldots + \bar{\beta}_n v_n$ (where $\bar{\beta}_i = \text{conj}(\beta_i)$)

Thus, $\boxed{f(v) = \langle v | \tilde{v} \rangle}$ where $\tilde{v} = \Sigma \overline{f(v_i)} \cdot v_i$

$\therefore$ proved.

Now, consider $u \in Ker(f)$. $\Rightarrow f(u) = 0_{\mathbb{F}}$

$$\Rightarrow \langle u | \tilde{v} \rangle = 0.$$

By inner pdt property, $\overline{\langle \tilde{v} | u \rangle} = 0$.

Taking complex conjugate of this eqn, we get $\overline{\overline{\langle \tilde{v} | u \rangle}} = \bar{0}$

$$\Rightarrow \langle \tilde{v} | u \rangle = 0 \quad \forall u \in Ker(f).$$

$$\Rightarrow \boxed{\tilde{v} \in (Ker(f))^{\perp}}$$ $\therefore$ proved.

Now,

given $P(v) = w$, where $w \in (Ker(f))^{\perp}$.

$$\Rightarrow f(P(v)) = f(w)$$

$$= \langle w | \tilde{v} \rangle.$$

Also, $\because w = P(v) \quad \therefore \langle (v-w) | \tilde{v} \rangle = 0$ as $\tilde{v} \in (Ker(f))^{\perp}$.

$$\Rightarrow \langle v | \tilde{v} \rangle - \langle w | \tilde{v} \rangle = 0 \Rightarrow \langle v | \tilde{v} \rangle = \langle w | \tilde{v} \rangle$$

$$\Rightarrow f(v) = f(w)$$

$$\Rightarrow \boxed{f(v) = f(P(v))}$$ $\therefore$ proved.

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Sol 8] Given, $T: \mathbb{V} \to \mathbb{V}$ (linear)

$v_1 \mapsto \langle Tv_1 | v_2 \rangle$ is a linear functional acting on $v_1 \in \mathbb{V}$

$\because \langle T(\alpha u_1 + u_2) | v_2 \rangle = \langle \alpha T(u_1) + T(u_2) | v_2 \rangle$

$= \alpha \langle Tu_1 | v_2 \rangle + \langle Tu_2 | v_2 \rangle$

From soln 6, we concluded that

$\exists$ a unique $\tilde{v}_2$: $v_1 \mapsto \langle Tv_1 | v_2 \rangle$

is the same as an inner pdt with a unique vector,

$v_1 \mapsto \langle v_1 | \tilde{v}_2 \rangle$

($\because$ every linear functional can be linked with a unique vector so that ~~its~~ is described as an inner pdt with that vector)

$\Rightarrow \langle Tv_1 | v_2 \rangle = \langle v_1 | \tilde{v}_2 \rangle$.

Now, ~~let~~ let $T^* : v_2 \mapsto \tilde{v}_2$

Clearly, then $\langle Tv_1 | v_2 \rangle = \langle v_1 | T^* v_2 \rangle$.

To now show linearity of T^*:

consider $\langle v_1 | \underline{T^*(cw_1 + w_2)} \rangle = \langle Tv_1 | cw_1 + w_2 \rangle$

$= \langle Tv_1 | cw_1 \rangle + \langle Tv_1 | w_2 \rangle$

$= c \langle Tv_1 | w_1 \rangle + \langle Tv_1 | w_2 \rangle$

$= \bar{c} \langle v_1 | T^* w_1 \rangle + \langle v_1 | T^* w_2 \rangle$

$= \langle v_1 | cT^* w_1 \rangle + \langle v_1 | T^* w_2 \rangle$

$= \langle v_1 | \underline{cT^* w_1 + T^* w_2} \rangle$

$\Rightarrow T^*(cw_1 + w_2) = cT^* w_1 + T^* w_2$.

$\therefore$ T^* is LINEAR.

Thus, T^* can be uniquely determined by studying the effect of the functional $\langle Tv | v_2 \rangle$ on a basis set for $v \in \mathbb{V}$.

Sol 10] a) To analyse $(R+T)^*$, we begin by considering $R+T$:

$$\langle (R+T) v_1 | v_2 \rangle \quad (\because \text{ inner pdt is NOT bilinear})$$
$$= \langle R v_1 | v_2 \rangle + \langle T v_1 | v_2 \rangle$$
$$= \langle v_1 | R^* v_2 \rangle + \langle v_1 | T^* v_2 \rangle$$
$$= \langle v_1 | (R^* + T^*) v_2 \rangle \quad — ①$$

But then

$\langle (R+T) v_1 | v_2 \rangle$ is also equal to $\langle v_1 | (R+T)^* v_2 \rangle$ — ②

from ① & ②,

$$\boxed{(R+T)^* = R^* + T^*} \quad \therefore \text{ proved,}$$

b) Consider $\langle (\alpha T) v_1 | v_2 \rangle = \langle v_1 | (\alpha T)^* v_2 \rangle$

$$= \alpha \langle T v_1 | v_2 \rangle$$
$$= \alpha \langle v_1 | T^* v_2 \rangle$$
$$= \langle v_1 | \bar{\alpha} T^* v_2 \rangle$$

$$\therefore \boxed{\bar{\alpha} T^* = (\alpha T)^*} \quad \therefore \text{ proved.}$$

c) Consider $\langle (TR) v_1 | v_2 \rangle = \langle v_1 | (TR)^* v_2 \rangle$

$$= \langle R v_1 | T^* v_2 \rangle$$
$$= \langle v_1 | R^* T^* v_2 \rangle$$

$$\boxed{R^* T^* = (TR)^*} \quad \therefore \text{ proved}$$

*** d) Consider $\langle T^* v_1 | v_2 \rangle = \langle v_1 | (T^*)^* v_2 \rangle$

$$= \overline{\langle v_2 | T^* v_1 \rangle} = \overline{(\langle T v_2 | v_1 \rangle)}$$

$$= \langle v_1 | T v_2 \rangle$$

$$\boxed{T = (T^*)^*} \quad \therefore \text{ proved}$$

*** e) We know few things: $TT^{-1} = I = T^{-1}T$, $T^* T^{*-1} = I = T^{*-1} T^*$

To study $(T^{-1})^*$ on RHS, I begin with:

$$\langle T^{-1} v_1 | v_2 \rangle = \langle v_1 | (T^{-1})^* v_2 \rangle \quad — ①$$

Now, I need to cook up something that leaves me with $(T^*)^{-1}$ at 2^{nd} argument.

So... $\langle T^{-1}v_1 | v_2 \rangle = \langle T^{-1}v_1 | \underline{T^*} (T^*)^{-1} v_2 \rangle$!!

Now...

!!

$\Rightarrow \quad = \langle (T) T^{-1} v_1 | (T^*)^{-1} v_2 \rangle$

$= \langle v_1 | (T^*)^{-1} v_2 \rangle$ ——— ②

$\therefore \boxed{(T^*)^{-1} = (T^{-1})^*}$ $\therefore$ proved

(from ① & ②)

f) T.P.T $\text{im}(T^*) = (\text{Ker}(T))^{\perp}$:

~~for~~ Let $u \in (\text{Im}(T^*))^{\perp}$

$\Rightarrow \langle u | T^*(v) \rangle = 0 \quad \forall v \in V \quad (\because T^*(v) \in \text{Im}(T^*) \ \forall v \in V)$

$\Rightarrow \langle Tu | v \rangle = 0 \quad \forall v \in V$

for this to hold for every $v \in V$, $T(u)$ must be 0 as substituting $v = T(u)$ leads to $\langle Tu | Tu \rangle = 0 \Rightarrow \|Tu\|^2 = 0$

$\Rightarrow T(u) = 0$.

Thus, $u \in \text{Ker}(T)$. $\Rightarrow (\text{Im}(T^*))^{\perp} \subseteq \text{Ker}(T)$. —①

Now,

let $v \in \text{Ker}(T)$.

$\Rightarrow \langle T(v) | \tilde{v} \rangle = 0 \quad \forall \tilde{v} \in V$.

$\Rightarrow \langle v | T^* \tilde{v} \rangle = 0 \quad \forall \tilde{v} \in V$.

$\because T^*(\tilde{v}) \in \text{Im}(T^*) \ \forall \tilde{v} \in V$,

$\therefore v \in (\text{Im}(T^*))^{\perp}$.

$\Rightarrow \text{Ker}(T) \subseteq (\text{Im}(T^*))^{\perp}$ —②

from ① and ②,

$(\text{Im}(T^*))^{\perp} = \text{Ker}(T)$

$\Rightarrow \boxed{\text{Im}(T^*) = (\text{Ker}(T))^{\perp}}$

Sol. 11] a) Suppose T is self-adjoint, ie: $T = T^*$.

Then $TT^* = T(T) = T^2$

and $T^*T = (T)T = T^2$

$\Rightarrow \boxed{T^*T = TT^*}$

($\Leftarrow$) Now suppose $T^*T = TT^*$ and $T = T^2$

Now, $\langle Tv | v \rangle = \langle T^2 v | v \rangle$

$= \langle Tv | T^* v \rangle$

$\Rightarrow \langle v | T^*(v) \rangle = \langle v | T^{*2} v \rangle \quad \therefore \boxed{T^* = T^{*2}}$

~~Also,~~

$\because$ given $T^*T = TT^*$ $\therefore$ $\cancel{\langle T^*T v | v \rangle = \langle TT^* v | v \rangle}$

$\cancel{\langle Tv | (T^*)^* v \rangle = \langle T^* v | (T)^* v \rangle}$

$\cancel{\Rightarrow \langle v | T^*(T) v \rangle = \langle v | TT^* v \rangle}$

Consider $\|(TT^* - T^*)v\|^2$

$= \langle (TT^* - T^*) v | (TT^* - T^*) v \rangle$

$= \langle v | (T^*T - T)(TT^* - T^*) v \rangle$

$= \langle v | (TT^* - T)(TT^* - T^*) v \rangle$

$= \langle v | (TT^*TT^* + TT^* - TT^*T^* - TTT^*) v \rangle$

$= \langle v | (T(TT^*)T^* + TT^* - T(T^{*2}) - (T^2)T^*) v \rangle$

$= \langle v | (T^2 T^{*2} + TT^* - T(T^*) - (T)T^*) v \rangle$

$= \langle v | (T^2 T^{*2} - TT^*) v \rangle$

$= \langle v | ((T)(T^*) - TT^*) v \rangle = \langle v | 0 v \rangle = 0.$

$\Rightarrow \|(TT^* - T^*) v\|^2 = 0 \Rightarrow TT^* = T^*$. —(1)

Similarly, considering $\|(T^*T - T) v\|^2$

$= \langle (T^*T - T) v | (T^*T - T) v \rangle$

$= \langle v | (T^*T - T^*)(T^*T - T) v \rangle$

$= \langle v | (T^*(T^*T)T + T^*T - T^*(T)^2 - (T^*)^2 T) v \rangle$

$= \langle v | (T^{*2} T^2 - T^*T) v \rangle = \langle v | 0 v \rangle$

$= 0$.

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$\therefore T^*T = T$ ②.

From ① and ②, we conclude $\boxed{T = T^*}$ $\left(\because \underline{T^*T = TT^*}_{\text{given}}\right)$

b) Suppose $ST = TS$ and $T = T^*$, $S = S^*$.

T.P.T $ST = (ST)^*$

Proof:

$$\langle STv | v\rangle = \langle TSv | v\rangle$$
$$\Rightarrow \langle Tv | S^*v\rangle = \langle Sv | T^*v\rangle$$
$$\Rightarrow \langle v | \underline{T^*S^*} v\rangle = \langle v | \underline{S^*T^*} v\rangle$$
$$\langle v | (ST)^* v\rangle = \langle v | ST\, v\rangle$$
$$\Rightarrow \boxed{(ST)^* = ST} \quad \therefore ST \text{ is also self-adjoint.}$$

Now,

Suppose $ST = (ST)^*$ and $T = T^*$, $S = S^*$.

T.P.T $ST = TS$.

Proof: $ST = (ST)^*$
$$= (T^*)(S^*)$$
$$= (T)(S) \quad \therefore \boxed{ST = TS}$$

$\therefore S$ and T commute.

c) $V \to$ fd ips T.P.T $T = T^* \Leftrightarrow \langle Tv | v\rangle \in \mathbb{R}\ \forall v \in V$.

Suppose $T = T^*$.

then $\langle Tv | v\rangle = \overline{\langle v | Tv\rangle}$
$$= \langle v | \underline{T^*} v\rangle$$
$$= \langle v | (T) v\rangle \quad (\because T = T^*)$$
$$\Rightarrow \langle Tv | v\rangle = \underline{\langle v | Tv\rangle}$$

$\therefore \langle v | Tv\rangle$ is Real.

$\Rightarrow \langle Tv | v\rangle$ is real for all $v \in V$.

Now,

Suppose $\langle Tv | v\rangle = \overline{\langle Tv | v\rangle}$

$\langle v | T^*v\rangle = \langle v | Tv\rangle \quad \forall v \in V$.

$$\rightarrow T^* = T$$

$\therefore$ proved.

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Sol. 14] T.P.T S is a basis for V, it is enough to prove that S is a linearly independent set because the basis $W = \{w_1, \ldots, w_n\}$ has specified $\dim(V) = n$ and clearly there are 'n' vectors in S.

Now,

Assuming the contrary (S is NOT LI),

$\exists \{\alpha_i\}_{i=1,2,\ldots n}$ not all zero, such that

$$\sum_{i=1}^{n} \alpha_i s_i = 0$$

Now, $\sum_{i=1}^{n} \alpha_i (s_i - w_i) = 0 - \sum_{i=1}^{n} \alpha_i w_i$

$$\Rightarrow \|\sum \alpha_i (s_i - w_i)\|^2 = \|-\sum_i \alpha_i w_i\|^2$$

$$= \|(\sum \alpha_i w_i)\|^2$$

$$= \langle \sum \alpha_i w_i | \sum_i \alpha_i w_i \rangle$$

$$= \sum_i \alpha_i \overline{\alpha_i} \langle w_i | w_i \rangle$$

($\because \langle w_i | w_j \rangle = 0 \ \forall i \neq j$ due to orthonormality)

$$\Rightarrow \|\sum_i \alpha_i (s_i - w_i)\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \quad \text{—} \ (1)$$

But then, on the other hand,

$$\|\sum \alpha_i (s_i - w_i)\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i (s_i - w_i) \Big| \sum_{i=1}^{n} \alpha_i (s_i - w_i) \right\rangle$$

$$\leq \left| \left\langle \sum_i \alpha_i (s_i - w_i) \Big| \sum_i \alpha_i (s_i - w_i) \right\rangle \right|$$

$$\leq \sum_j \sum_i |\alpha_i| |\overline{\alpha_j}| \left| \langle (s_i - w_i) | (s_j - w_j) \rangle \right|$$

(Δ-inequality) and (inner pdt properties)

$$\leq \sum_j \sum_i |\alpha_i| |\alpha_j| \, \underbrace{\|s_i - w_i\|}_{< 1/\sqrt{n} \ \forall i} \cdot \underbrace{\|s_j - w_j\|}_{< 1/\sqrt{n} \ \forall j}$$

(Cauchy-Schwartz inequality)

$$< \frac{1}{n} \sum_j \sum_i |\alpha_i| |\alpha_j| = \left(\sum_j |\alpha_j| \left(|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \right) \right)$$

from (1)

$$\sum_{i=1}^{n} (|\alpha_i|^2) < \frac{1}{n} \left(\sum_{i=1}^{n} |\alpha_i| \right)^2 \quad \text{—} \ (2)$$

Now consider $\begin{bmatrix} |\alpha_1| \\ |\alpha_2| \\ \vdots \\ |\alpha_n| \end{bmatrix} \in \mathbb{R}^n$ and $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$ with $\langle v_1 | v_2 \rangle = v_2^T v_1$ being the inner product.

then, $$\sum |\alpha_i| = \left\langle \begin{bmatrix} |\alpha_1| \\ \vdots \\ |\alpha_n| \end{bmatrix} \middle| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\rangle$$

$$\leq \left(\sqrt{|\alpha_1|^2 + |\alpha_2|^2 + \dots + |\alpha_n|^2}\right) \cdot \left(\sqrt{1^2 + 1^2 + \dots + 1^2}\right)$$

(Again, Cauchy-Schwartz inequality)

$$\leq \left(\sqrt{\sum_{i=1}^{n} |\alpha_i|^2}\right) \cdot \sqrt{n}$$

$$\Rightarrow \left(\sum |\alpha_i|\right)^2 \leq n \sum |\alpha_i|^2$$

dumping this in RHS of inequation ②,

we get
$$\sum (|\alpha_i|)^2 < \frac{1}{\cancel{n}} \cdot \left(\cancel{n} \sum (|\alpha_i|^2)\right)$$ which is absurd.

$\therefore$ CONTRADICTION.

$\therefore$ S is a linearly independent set. $\therefore$ proved.

Sol. 18] Given: $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$, $u_1(\cdot,\cdot) : \mathbb{U}_1 \times \mathbb{U}_1 \to \mathbb{C}$

$u_2(\cdot,\cdot) : \mathbb{U}_2 \times \mathbb{U}_2 \to \mathbb{C}$, $\mathbb{U}_1^{\perp} = \mathbb{U}_2$,

$v(p,q) = u_1(p,q) = u_2(p,q)$ where $p, q \in \mathbb{U}_1 \cap \mathbb{U}_2$.

T.P.T

there exists a unique inner product $v(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$.

Proof:

Suppose $\langle \cdot | \cdot \rangle_{\mathbb{V}} \to$ inner product in $\mathbb{V}$ as required.

$\because \mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$ (direct sum)

$\therefore \forall f \in \mathbb{V}$, $f = v_1 + v_2$, $v_1 \in \mathbb{U}_1$ and $v_2 \in \mathbb{U}_2$

$\therefore \langle f | \tilde{f} \rangle_{\mathbb{V}}$ (where $f, \tilde{f} \in \mathbb{V}$)

$= \langle v_1 + v_2 | \tilde{v}_1 + \tilde{v}_2 \rangle_{\mathbb{V}}$ where $v_1, \tilde{v}_1 \in \mathbb{U}_1$

$v_2, \tilde{v}_2 \in \mathbb{U}_2$

$$\Rightarrow \langle f|\tilde{f}\rangle_{\mathbb{V}} = \langle v_1|\tilde{v}_1+\tilde{v}_2\rangle_{\mathbb{V}} + \langle v_2|\tilde{v}_1+\tilde{v}_2\rangle_{\mathbb{V}}$$
$$= \langle v_1|\tilde{v}_1\rangle_{\mathbb{V}} + \langle v_1|\tilde{v}_2\rangle_{\mathbb{V}} + \langle v_2|\tilde{v}_1\rangle_{\mathbb{V}} + \langle v_2|\tilde{v}_2\rangle_{\mathbb{V}}$$

$(\because$ given $\mathbb{U}_1^{\perp} = \mathbb{U}_2)$, $\therefore \longrightarrow 0 \quad 0$

$$\therefore \langle f|\tilde{f}\rangle_{\mathbb{V}} = \langle v_1|\tilde{v}_1\rangle_{\mathbb{V}} + \langle v_2|\tilde{v}_2\rangle_{\mathbb{V}}$$

Now, to prove uniqueness, check for $f = v_1 \in \mathbb{U}_1$ and $\tilde{f} = \tilde{v}_1 \in \mathbb{U}_1$,
we have $\langle f|\tilde{f}\rangle_{\mathbb{V}} = \langle v_1|\tilde{v}_1\rangle_{\mathbb{V}}$
$= u_1(v_1, \tilde{v}_1)$.
Similarly for $f = v_2 \in \mathbb{U}_2$ and $\tilde{f} = \tilde{v}_2 \in \mathbb{U}_2$,
$\langle f|\tilde{f}\rangle_{\mathbb{V}} = \langle v_2|\tilde{v}_2\rangle_{\mathbb{V}}$
$= u_2(v_2, \tilde{v}_2)$.
$\therefore$ since any vector f or $\tilde{f}$ in $\mathbb{V}$ can be uniquely written as
$f = v_1 + v_2$, $\tilde{f} = \tilde{v}_1 + \tilde{v}_2$.
$$\therefore \langle f|\tilde{f}\rangle_{\mathbb{V}} = u_1(v_1, \tilde{v}_1) + u_2(v_2, \tilde{v}_2)$$

$$\Rightarrow v(f, \tilde{f}) = u_1(v_1, \tilde{v}_1) + u_2(v_2, \tilde{v}_2).$$

$\therefore \langle \cdot|\cdot\rangle_{\mathbb{V}} = v(\cdot,\cdot)$ exists uniquely.

To verify that this is indeed an inner product, checking
i) $v(f,f) \geq 0$ and 0 only for $f=0$:
if $f \neq 0$, then $v_1 + v_2 \neq 0$,
then $u_1(v_1, \tilde{v}_1) + u_2(v_2, \tilde{v}_2) \neq 0$
$\because u_1(\cdot,\cdot)$, $u_2(\cdot,\cdot)$ are defined inner prods.
$\therefore$ verified.

ii) $v(f_a+f_b, f) = u_1(f_{a_1}+f_{b_1}, f_1) + u_2(f_{a_2}+f_{b_2}, f_2)$

where $f_a = \underset{(\in \mathbb{U}_1)}{f_{a_1}} + \underset{(\in \mathbb{U}_2)}{f_{a_2}}$ and $f_b = \underset{(\in \mathbb{U}_1)}{f_{b_1}} + \underset{(\in \mathbb{U}_2)}{f_{b_2}}$ (defined inner prods)

and $f = \underset{(\in \mathbb{U}_1)}{f_1} + \underset{(\in \mathbb{U}_2)}{f_2}$ $\quad \because u_1(\cdot,\cdot), u_2(\cdot,\cdot)$ allow additivity,
so does $v(\cdot,\cdot)$. $\therefore$ checked.

iii) $v(\alpha f, \tilde{f}) = u_1(\alpha f_1, \tilde{f}_1) + u_2(\alpha f_2, \tilde{f}_2)$
where
$f_1, \tilde{f}_1 \in W_1$ and $f_2, \tilde{f}_2 \in W_2$, $\alpha \in \mathbb{C}$ and
$f = f_1 + f_2$, $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$.

$$= \alpha\, u_1(f_1, \tilde{f}_1) + \alpha\, u_2(f_2, \tilde{f}_2)$$
$$= \alpha\left(u_1(f_1, \tilde{f}_1) + u_2(f_2, \tilde{f}_2)\right)$$
$$= \alpha \cdot v(f, \tilde{f})$$
$\therefore$ verified.

iv) $v(f, \tilde{f}) = u_1(f_1, \tilde{f}_1) + u_2(f_2, \tilde{f}_2)$
now,
$$v(\tilde{f}, f) = u_1(\tilde{f}_1, f_1) + u_2(\tilde{f}_2, f_2)$$
$$= \overline{u_1(f_1, \tilde{f}_1)} + \overline{u_2(f_2, \tilde{f}_2)} = \overline{u_1(f_1, \tilde{f}_1) + u_2(f_2, \tilde{f}_2)}$$
$$= \overline{v(f, \tilde{f})} \quad \therefore \text{ verified.}$$
$\therefore \langle \cdot | \cdot \rangle_V = v(\cdot, \cdot)$ is a legitimate inner pdt of V.

Sol. 21] Given : $v_1, v_2 \in$ complex ips ;
T.P.T ~~$v_1 \perp v_2$~~ $\Rightarrow \langle v_1 | v_2 \rangle = 0 \iff \|\alpha v_1 + \beta v_2\|^2 = \|\alpha v_1\|^2 + \|\beta v_2\|^2$
$\forall \alpha, \beta \in \mathbb{C}$.

Proof : Suppose $v_1 \perp v_2$. $\Rightarrow \langle v_1 | v_2 \rangle = 0$.
Then $\|\alpha v_1 + \beta v_2\|^2 = \langle \alpha v_1 + \beta v_2 | \alpha v_1 + \beta v_2 \rangle$
$$= \langle \alpha v_1 | \alpha v_1 + \beta v_2 \rangle + \langle \beta v_2 | \alpha v_1 + \beta v_2 \rangle$$
$$= \alpha\bar{\alpha}\langle v_1 | v_1 \rangle + \alpha\bar{\beta}\underbrace{\cancel{\langle v_1 | v_2 \rangle}}_{0} + \beta\bar{\alpha}\underbrace{\cancel{\langle v_2 | v_1 \rangle}}_{0} + \beta\bar{\beta}\langle v_2 | v_2 \rangle$$
$$= |\alpha|^2 \|v_1\|^2 + |\beta|^2 \|v_2\|^2$$
$$= \|\alpha v_1\|^2 + \|\beta v_2\|^2.$$

Now, $(\Leftarrow)$ Suppose
$$\|\alpha v_1 + \beta v_2\|^2 = \|\alpha v_1\|^2 + \|\beta v_2\|^2$$
As calculated above, $\downarrow$
$$\cancel{\alpha\bar{\alpha}\langle v_1 | v_1 \rangle} + \alpha\bar{\beta}\langle v_1 | v_2 \rangle + \beta\bar{\alpha}\langle v_2 | v_1 \rangle + \cancel{\beta\bar{\beta}\langle v_2 | v_2 \rangle} = \cancel{\|\alpha v_1\|^2} + \cancel{\|\beta v_2\|^2}$$

$$\Rightarrow \alpha\bar{\beta}\langle v_1 | v_2 \rangle + \beta\bar{\alpha}\langle v_2 | v_1 \rangle = 0.$$

Now, $\beta\bar{\alpha}\langle v_2|v_1\rangle = \overline{\alpha\bar{\beta}\langle v_1|v_2\rangle}$ $\therefore$ $\alpha\bar{\beta}\langle v_1|v_2\rangle + \overline{\alpha\bar{\beta}\langle v_1|v_2\rangle} = 0$.

$\Rightarrow \mathrm{Re}\left(\alpha\bar{\beta}\langle v_1|v_2\rangle\right) = 0$.

Let $\alpha\bar{\beta} = a+ib$, $\langle v_1|v_2\rangle = c+id$, $a,b,c,d \in \mathbb{R}$.

$\Rightarrow \mathrm{Re}\left((a+ib)(c+id)\right) = 0$

$\Rightarrow ac - bd = 0$. $\forall\ a, b \in \mathbb{R}$

If $a=0, b\neq 0 \Rightarrow d=0$

If $b=0, a\neq 0 \Rightarrow c=0$

$\therefore c+id = 0 \Rightarrow \langle v_1|v_2\rangle = 0$

$\Rightarrow v_1 \perp v_2$. $\therefore$ Proved.

Sol. 24]

Let $f \in W$ and $g \in W^{\perp}$. $\Rightarrow \langle f|g\rangle = 0$

$\hookrightarrow = \sum_{i=1}^{n} a_i x^i$

$\Rightarrow \langle (a_0x^0 + a_1x^1 + a_2x^2 + \dots + a_nx^n \,|\, g\rangle = 0$.

$\Rightarrow \int_a^b \cancel{f(x)} (a_0x^0 + a_1x^1 + a_2x^2 + \dots +) g(x)\,dx = 0 \quad \forall \{a_i\} \in \mathbb{R}$

$\Rightarrow \int_a^b a_0 g(x)dx + \int_a^b a_1 x g(x)\,dx + \dots + \int_a^b a_n x^n g(x)dx = 0 \quad \forall \{a_i\} \in \mathbb{R}$.

For a moment, let $a_0 = a_1 = a_2 = \dots = a_{n-1} = 0$, $a_n = 1$.
then we get

$\int_a^b x^n g(x)\,dx = 0 \quad \forall n \in \mathbb{N} \quad (\because f \in \mathbb{R}[x] \in W)$

$\because$ f could have been any polynomial of any degree,

$\therefore$

$\int_a^b g(x)dx = \int_a^b x g(x)\,dx = \int_a^b x^2 g(x)dx = \dots = \int_a^b x^n g(x)dx = 0$.

$\Rightarrow g(x) = 0_f \quad \forall x \quad \therefore \boxed{W^{\perp} = \{0_f\}}$.

$\Rightarrow (W^{\perp})^{\perp} = V$ ($\because$ every fn is $\perp$ to zero fn)

But $W \subseteq V \Rightarrow W \subset (W^{\perp})^{\perp}$ and $W = (W^{\perp})^{\perp}$

$\boxed{\text{Sol. 25}}$ Given: $\mathcal{B}(\mathbb{R}[x]_3) = \{1, x, x^2, x^3\}$ $\forall x \in [0,1]$,

$$\langle f | g \rangle = \int_0^1 f(t) g(t)\, dt$$

If w_{k+1} is the $(k+1)^{th}$ element of the req orthonormal basis, then

$$w_{k+1} = \left(v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1} | w_i \rangle}{\|w_i\|^2} w_i \right) \text{ divided by its norm.}$$

$\therefore \tilde{w}_1 = v_1 \Rightarrow \tilde{w}_1 = w_1$

$\Rightarrow \underline{w_1 = 1}$, $\tilde{w}_2 = x - \left(\int_0^1 x\,dx\right) \cdot \frac{1}{(1)^2} = x - \frac{1}{2}$

$\Rightarrow \tilde{w}_2 = x - \frac{1}{2}$. $\Rightarrow \|\tilde{w}_2\|^2 = \langle w_2 | w_2 \rangle = \int_0^1 (x - \frac{1}{2})^2 dx$

$\cancel{w_3 = x^2 =}$ $= \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1 = \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3\right]$

$= 1/12$ $\therefore \|\tilde{w}_2\| = 1/2\sqrt{3}$.

$\therefore w_2 = 2\sqrt{3}\left(x - \frac{1}{2}\right) = \underline{\sqrt{3}(2x-1)}$.

$\tilde{w}_3 = x^2 - \int_0^1 x^2 \cdot 1\, dx \times \frac{1}{1^2} - \int_0^1 x^2 \cdot \sqrt{3}(2x-1)\, dx \times \frac{\sqrt{3}(2x-1)}{1^2}$

$= x^2 - \frac{1}{3} - \left(\frac{2\sqrt{3}}{4} - \frac{\sqrt{3}}{3}\right)\sqrt{3}(2x-1) = x^2 - \frac{1}{3} - \frac{3}{2}(2x-1) + (2x-1)$.

$= x^2 - x + \frac{1}{6}$. $\Rightarrow \|\tilde{w}_3\|^2 = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$

$= \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 - \frac{x}{3} + \frac{x^2}{3}\right) dx$

$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{2}{4} + \frac{1}{9} - \frac{1}{6} = \frac{36 + 60 + 5 - 90 + 20 - 30}{180} = \frac{1}{180}$

$\Rightarrow \|\tilde{w}_3\| = 1/6\sqrt{5}$. $\therefore \underline{w_3 = \sqrt{5}(6x^2 - 6x + 1)}$

$\tilde{w}_4 = x^3 - \int_0^1 x^3 \cdot 1\, dx \times 1 - \int_0^1 x^3 \cdot \sqrt{3}(2x-1)\, dx \times \sqrt{3}(2x-1) - \int_0^1 x^3 \sqrt{5}(6x^2 - 6x + 1)\, dx \times \sqrt{5}(6x^2 - 6x + 1)$

$= x^3 - \frac{1}{4} - \sqrt{3}\left(\frac{2}{5} - \frac{1}{4}\right)\sqrt{3}(2x-1) - \sqrt{5}\left(1 - \frac{6}{5} + \frac{1}{4}\right)\sqrt{5}(6x^2 - 6x + 1)$

$$\tilde{w}_4 = x^4 x^3 - \frac{3x^2}{2} - \frac{3}{5}x - \frac{1}{20}$$

$$\Rightarrow \|\tilde{w}_4\|^2 = \int_0^1 \left(x^6 + \frac{69x^4}{20} \cancel{+ \frac{1089x^2}{4} + \frac{289}{4}} - \frac{19x^3}{10} + \frac{51x^2}{100} - \frac{3x}{50} + \frac{1}{400}\right) dx$$

$$= \left(\frac{x^7}{7} - \frac{x^6}{2} + \frac{69x^5}{100} - \frac{19x^4}{40} + \frac{17x^3}{100} - \frac{3x^2}{100} + \frac{x}{400}\right)\Bigg|_0^1$$

$$= \frac{400 - 1400 + 69\times 28 - 19\times 70 + 17\times 28 - 3\times 28 + 7}{2800}$$

$$= \frac{1}{2800} \qquad \therefore \|\tilde{w}_4\| = \frac{1}{20\sqrt{7}}$$

$$\therefore w_4 = \sqrt{7}\left(20x^3 - 30x^2 - 12x - 1\right).$$

∴ Required orthonormal basis:

$$\boxed{\left\{1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1), \sqrt{7}(20x^3-30x^2-12x-1)\right\}}$$

Sol.28] a) Given, $V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}$ with $n \leq m$ and $\{x_i\}_{i=1,2,\dots,m} \in \mathbb{R}$.

T.P.T V must have FCR.

Proof:

Assume the contrary: let its column rank have deficiency,
$\Rightarrow \exists\ \alpha_0, \alpha_1, \dots, \alpha_n$ not all zero such that
$\alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1} = 0$ is satisfied for
m distinct values of x $(x_1, x_2, \dots, x_m)$.
But this eqn being an $(n-1)^{th}$ degree polynomial,
it can have atmost $(n-1)$ distinct roots only.
Thus, $m \leq n-1. \Rightarrow m < n$.
But this is a contradiction to given description of
matrix V, $n \leq m$. Thus, our assumption is wrong.

∴ V must have Full-column-rank.

∴ proved.

b) $\because$ given $(m-1)$ as the degree of the unique polynomial

$\therefore$ comparing with solution a), $m-1 = n-1 \Rightarrow m = n$.

$\therefore$ V is now a square Vandermonde Matrix and being FCR implies full row rank $\Rightarrow$ V is invertible.

Thus,

$\exists$ a unique soln $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}$ for $V \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$

given by $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix} = V^{-1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$.

From given expression, $l(x)$ is a polynomial of degree $(m-1)$

$\therefore$ we only need to prove $l(x_i) = y_i \ \forall \ i \in \{1, 2, \ldots, m\}$.

$$\because l(x) = \sum_{i=1}^{m} \left(y_i \frac{\prod_{j \neq i}^{m} (x - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)} \right) \quad \therefore l(x_i) = y_i \cancel{\frac{\prod_{j \neq i}^{m} (x_i - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)}}$$

$$= y_i.$$

i.e. proved.

c)

Using formula of $l(x)$: ~~y~~

$$y = 0 + 8\frac{(x-0)(x-30)(x-45)(x-60)}{(15-0)(15-30)(15-45)(15-60)} + 15\frac{(x-0)(x-15)(x-45)(x-60)}{(30-0)(30-15)(30-45)(30-60)}$$

$$+ 19\frac{(x-0)(x-15)(x-30)(x-60)}{(45-0)(45-15)(45-30)(45-60)} + 20\frac{(x-0)(x-15)(x-30)(x-45)}{(60-0)(60-15)(60-30)(60-45)}$$

$$\Rightarrow \boxed{y = -8\frac{x(x-30)(x-45)(x-60)}{15 \cdot 15 \cdot 30 \cdot 45} + 15\frac{x(x-15)(x-45)(x-60)}{30 \cdot 15 \cdot 15 \cdot 30} - 19\frac{x(x-15)(x-30)(x-60)}{45 \cdot 30 \cdot 15 \cdot 15} + 20\frac{x(x-15)(x-30)(x-45)}{60 \cdot 45 \cdot 30 \cdot 15}}$$

is the trajectory of the ball in the given vertical plane.

Now,

for landing on ground $\Rightarrow y = l(x) = 0$.

$(x=0, y=0)$ is the coordinate from where the defender kicks the ball,
$\therefore$ not a soln.

So,

$$0 = \frac{1}{15^4}\left(-8\cdot\frac{(x-30)(x-45)(x-60)}{6} + \frac{15(x-15)(x-45)(x-60)}{4}\right.$$
$$\left. - \frac{19(x-15)(x-30)(x-60)}{6} + \frac{20(x-15)(x-30)(x-45)}{24}\right)$$

$$\Rightarrow x^3\left(-4/3 + 15/4 - 19/6 + 5/6\right) + x^2\left(\frac{4}{3}\cdot 135 - \frac{15}{4}\cdot 120 + \frac{19}{6}\cdot 105 - \frac{5}{6}\cdot 90\right)$$

$$+ x(15^2)\left(\frac{-4}{3}(6+8+12) + \frac{15}{4}(3+4+12) - \frac{19}{6}(2+4+8) + \frac{5}{6}(2+3+6)\right)$$

$$+ 15^3\left(\frac{4}{3}\cdot 24 - \frac{15}{4}\cdot 12 + \frac{19}{6}\cdot 8 - \frac{5}{6}\cdot 6\right) = 0$$

$$\Rightarrow x^3 - 150x^2 + 3825x + 297000 = 0$$

$\hookrightarrow \because$ This does NOT have any real +ve root,
this formula isn't helpful in this question.

d) Training the data points over $y = ax^2 + bx + c$,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0^2 \\ 1 & 15 & 15^2 \\ 1 & 30 & 30^2 \\ 1 & 45 & 45^2 \\ 1 & 60 & 60^2 \end{bmatrix}}_{V} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 15 \\ 19 \\ 20 \end{bmatrix}$$

$\because$ V is of FCR, it has a left inverse given by $V^L = (V^T V)^{-1} V^T$

$$\therefore \begin{bmatrix} c \\ b \\ a \end{bmatrix} = V^L \begin{bmatrix} 0 \\ 8 \\ 15 \\ 19 \\ 20 \end{bmatrix} \longrightarrow \text{gives the "best" possible sol}_n.$$

$$V^L = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 15 & 30 & 45 & 60 \\ 0 & 225 & 900 & 2025 & 3600 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 15 & 225 \\ 1 & 30 & 900 \\ 1 & 45 & 2025 \\ 1 & 60 & 3600 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 15 & 30 & 45 & 60 \\ 0 & 225 & 900 & 2025 & 3600 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 150 & 6750 \\ 150 & 6750 & 33750 \\ 6750 & 33750 & 1791250 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 15 & 30 & 45 & 60 \\ 0 & 225 & 900 & 2025 & 3600 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} -0.2286 \\ 0.6636 \\ -0.0054 \end{bmatrix} \quad \therefore y = -0.0054x^2 + 0.6636x - 0.2286$$

$$y = 0 \Rightarrow x = \cancel{0.3453}, \ \underline{122.66}$$

$\therefore$ the ball lands roughly $\boxed{123\,m}$ away from the defender

This solution definitely makes more sense as 123m is a realistic value, unlike the complex ~~ro~~ roots obtained in c). This is probably because the previous solution is obtained from an 'overstrained' system.