Applied Linear Algebra

(Course Code: EE 635)

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Time: **1 hour and 15 minutes**

Quiz-2

Total Points: 40

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Instructions

- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Prove or disprove (possibly through some counter-example) the following:

   (a) For two subspaces, $\mathbb{W}_1$ and $\mathbb{W}_2$, of an inner product space, we have $(\mathbb{W}_1 + \mathbb{W}_2)^{\perp} = \mathbb{W}_1^{\perp} \cap \mathbb{W}_2^{\perp}$.

   (b) For an inner product space $\mathbb{V}$, two vectors $v_1, v_2 \in \mathbb{V}$ are equal if and only if $\langle v_1 | v \rangle = \langle v_2 | v \rangle$ for every $v \in \mathbb{V}$.

   (c) If $v_1, v_2 \in \mathbb{V}$ (an inner product space) are orthogonal, then the vectors $v_1 + v_2$ and $v_1 - v_2$ cannot be orthogonal.

   (d) If Ξ_1 and Ξ_2 are non-empty subsets of an inner product space $\mathcal{V}$ such that $\Xi_1 \subseteq \Xi_2$, then $\Xi_1^{\perp} \subseteq \Xi_2^{\perp}$.

   (e) Let $\{v_1, v_2, \ldots, v_m\}$ be a linearly independent set of vectors in a real inner product space $\mathbb{V}$. There exist exactly 2^m orthonormal sets of vectors of the form $\{w_1, w_2, \ldots, w_m\}$ in $\mathbb{V}$ such that span($\{v_1, v_2, \ldots, v_i\}$)=span($\{w_1, w_2, \ldots, w_i\}$) for all $i \in \{1, 2, \ldots, m\}$.

   [2.5× 5]

2. (a) For $v_1, v_2 \in \mathbb{V}$ (an inner product space), show that $\langle v_1 | v_2 \rangle = 0$ if and only if $||v_1|| \leq ||v_1 + \alpha v_2|| \; \forall \alpha \in \mathbb{C}$.

   (b) Suppose $\Pi : \mathbb{V} \rightarrow \mathbb{V}$ is an idempotent linear operator on the finite dimensional vector space $\mathbb{V}$, and satisfies the condition: $||\Pi v|| \leq ||v|| \; \forall v \in \mathbb{V}$. Show that there exists a subspace $\mathbb{U} \subseteq \mathbb{V}$, such that Π is an orthogonal projection from $\mathbb{V}$ onto $\mathbb{U}$.

   [3.5+4]

3. (a) Show that a linear operator, ϕ, on an inner product space, $\mathbb{V}$ (over $\mathbb{R}$ or $\mathbb{C}$), is identically the 'zero' operator if and only if $\langle \phi(v_1) | v_2 \rangle = 0$ for all $v_1, v_2 \in \mathbb{V}$.

(b) Suppose v_1 and v_2 belong to an inner product space, $\mathbb{V}$ (over $\mathbb{C}$), and $\alpha, \beta \in \mathbb{C}$, while $\phi : \mathbb{V} \to \mathbb{V}$ is a linear operator. Show that $\alpha\bar{\beta}\langle\phi(v_1)|v_2\rangle + \bar{\alpha}\beta\langle\phi(v_2)|v_1\rangle = \langle\phi(\alpha v_1 + \beta v_2)|\alpha v_1 + \beta v_2\rangle - |\alpha|^2\langle\phi(v_1)|v_1\rangle - |\beta|^2\langle\phi(v_2)|v_2\rangle$.

(c) Show that for an inner product space, $\mathbb{V}$ over $\mathbb{C}$, a necessary and sufficient condition for a linear operator, ϕ, to be the 'zero' operator is that $\langle\phi(v)|v\rangle = 0$ for all $v \in \mathbb{V}$

(d) Provide an example of a non-zero linear operator, ϕ, on an inner product space, $\mathbb{V}$ over $\mathbb{R}$, such that $\langle\phi(v)|v\rangle = 0$ for all $v \in \mathbb{V}$.

(e) Show that for an inner product space, $\mathbb{V}$ over $\mathbb{R}$, a necessary and sufficient condition for a linear, self-adjoint operator, ϕ, to be the 'zero' operator is that $\langle\phi(v)|v\rangle = 0$ for all $v \in \mathbb{V}$

[2+2+5+2+5]

4. Suppose $\mathcal{P} : \mathbb{V} \to \mathbb{U}$ is an orthogonal projection onto a subspace $\mathbb{U}$ of $\mathbb{V}$, where $\mathbb{V}$ is a finite dimensional inner product space. Prove that $\mathcal{P}$ is self-adjoint. [4]