EE702: Self-Study-1

Representation of Motion using Quaternions

Defn: $q \triangleq [q(1) \quad q(2) \quad q(3) \quad q(4)]^T = [v \ s]^T$

Quaternion

scalar part

vector part

where $v \in \mathbb{R}^3$, $s \in \mathbb{R}$

Equivalent defn: $q \triangleq i\,q(1) + j\,q(2) + k\,q(3) + q(4)$

where i, j, k are complex quantities forming a non-abelian group,

i.e:

$$i \otimes i = -1$$

$$j \otimes j = -1$$

$$k \otimes k = -1$$

$$i \otimes j = -j \otimes i = k$$

$$j \otimes k = -k \otimes j = i$$

$$k \otimes i = -i \otimes k = j$$

$$i \otimes j \otimes k = -1.$$

[1] The symbol $\otimes$ is used to denote the quaternion multiplication. Also, a quaternion q is interchangeably used as a vector or an algebraic quantity depending on its use in matrix operations or quaternion operations, respectively.

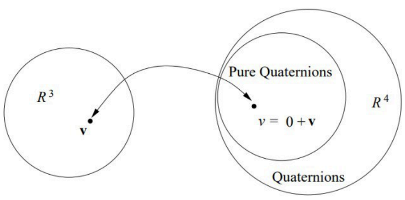
3 Quaternion Rotation Operator

How can a quaternion, which lives in $\mathbb{R}^4$, operate on a vector, which lives in $\mathbb{R}^3$? First, we note that a vector $\boldsymbol{v} \in \mathbb{R}^3$ is a *pure quaternion* whose real part is zero. Let us consider a unit quaternion $q = q_0 + \boldsymbol{q}$ only. That $q_0^2 + \|\boldsymbol{q}\|^2 = 1$ implies that there must exist some angle θ such that

$$\cos^2\theta = q_0^2,$$
$$\sin^2\theta = \|\boldsymbol{q}\|^2.$$

In fact, there exists a unique $\theta \in [0, \pi]$ such that $\cos\theta = q_0$ and $\sin\theta = \|\boldsymbol{q}\|$. The unit quaternion can now be written in terms of the angle θ and the unit vector $\boldsymbol{u} = \boldsymbol{q}/\|\boldsymbol{q}\|$:

$$q = \cos\theta + \boldsymbol{u}\sin\theta.$$



Using the unit quaternion q we define an operator on vectors $\boldsymbol{v} \in \mathbb{R}^3$:

$$\begin{aligned} L_q(\boldsymbol{v}) &= q\boldsymbol{v}q^* \\ &= (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q}\cdot\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times\boldsymbol{v}). \end{aligned} \tag{3}$$

If $q_1 = [v_1 \ s_1]^T$, $q_2 = [v_2 \ s_2]^T$,

Addition: $q_1 + q_2 = [v_1 + v_2 \, , \ s_1 + s_2]^T$

Multiplication: $q_1 \otimes q_2 = [v \ s]^T$

where $v = v_1 \times v_2 + s_1 v_2 + s_2 v_1$

$s = s_1 s_2 - v_1 v_2^T$

Norm: $N(q) = \|q\| = +(s^2 + vv^T)^{1/2}$

Inverse: $q^{-1} = \dfrac{q^*}{\|q\|^2} = \dfrac{[-v \ s]}{\|q\|^2}$

complex conjugate

A point $\mathbf{p} = [x \ y \ z]^T$ in 3-D space can be regarded as a quaternion having a zero scalar component. We may call it a vector quaternion $p = [\mathbf{p} \ 0]^T$. Vector quaternions are, thus, a way of embedding 3-D Cartesian information in a 4-D space that provides a common representation for both translational and rotational information (see [25]). If $\mathbf{n} = [n_1 \ n_2 \ n_3]$ represents the axis of rotation passing through the origin and θ is the angle of rotation, then the unit quaternion (i.e., the norm is unity) representing the rotation is expressed as

$$q = [\mathbf{n}\sin(\theta/2) \quad \cos(\theta/2)]^T. \tag{1}$$

If a and b are vector quaternions representing the position vectors of a point in two different co-ordinate frames, then $b = q \otimes a \otimes q^* + t$ is the affine transformation between the two coordinate frames, with q and t $(= [\mathbf{t} \ 0]^T)$ representing the rotation and the translation between the two frames, respectively.

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