Applied Linear Algebra

(Course Code: EE 635)

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Time: **1 hour and 15 minutes**

Quiz-3

Total Points: 40

Instructions

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- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Prove or disprove (possibly through some counter-example) the following:

   (a) For the Jordan form representation of a nilpotent matrix, $N \in \mathbb{C}^{n\times n}$, the number of Jordan blocks of size $i \times i$ or greater is equal to $\text{rank}(N^{i-1}) - \text{rank}(N^i)$.

   (b) If $A \in \mathbb{C}^{n\times n}$ has a set of n orthonormal eigenvectors, it must be a Hermitian matrix.

   (c) For $A \in \mathbb{R}^{n\times n}$, suppose $\{w_1, w_2, \ldots, w_n\}$ is a set of n linearly independent eigenvectors and for $w = \sum_{k=1}^{k=n} k w_k$, we have $Aw = \sum_{k=1}^{k=n} (k^2 + k) w_k$. Then the minimal polynomial, $\mu_A(x)$, and characteristic polynomial, $\chi_A(x)$, of A are identical.

   (d) For a linear operator $A : \mathbb{V} \to \mathbb{V}$ on a finite dimensional vector space, a positive integer m, and a vector $v \in \mathbb{V}$, suppose $A^{m-1}v \neq 0$, but $A^m v = 0$. Then $\{v, Av, A^2v, \ldots, A^{m-1}v\}$ must be linearly independent.

[3× 4]

2. (a) The *spectral* norm of a matrix, $A \in \mathbb{R}^{m\times n}$, is defined as $||A||_2 := \max_{x\neq 0} \dfrac{||Ax||_2}{||x||_2}$, where $x \in \mathbb{R}^n$ and the vector norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ result from the conventional inner products on those spaces. If the non-zero singular values of A are given by $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ $(r \leq \min(m, n))$, then prove that $||A||_2 = \sigma_1$.

   (b) The *Frobenius* norm of a matrix $A \in \mathbb{R}^{m\times n}$ is given by $||A||_F := \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$. If the non-zero singular values of A are given by $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ $(r \leq \min(m, n))$, then prove that $||A||_F = (\sigma_1^2 + \sigma_2^2 + \ldots \sigma_r^2)^{\frac{1}{2}}$.

[5+5]

3. Provide an example for each of the following or explain clearly why no such example can be provided, if such is the case.

   (a) Two matrices $A, B \in \mathbb{R}^{7\times7}$ are such that $\chi_A(x) = \chi_B(x)$, $\mu_A(x) = \mu_B(x)$, and A and B have the same eigenvalues with same algebraic and geometric multiplicities for each eigenvalue, but there exists no $T \in \mathbb{R}^{7\times7}$ such that $B = T^{-1}AT$.

   (b) A lower/upper triangular matrix, $A \in \mathbb{C}^{n\times n}$, satisfying $A^HA = AA^H$, which is not a diagonal matrix.

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[[illegible] × 2]

4. (a) For matrices $A, B \in \mathbb{C}^{n\times n}$, show that $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ represent the same operator under different choices of bases. Hence, show that $\chi_{AB}(x) = \chi_{BA}(x)$.

   (b) (*Rayleigh quotient*) Suppose the eigenvalues of a Hermitian matrix, $A \in \mathbb{C}^{n\times n}$, are given by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Show that $\lambda_1 = \max_{||x||_2=1} x^HAx$ and $\lambda_n = \min_{||x||_2=1} x^HAx$.

[5+5]