EE-635

(Applied Linear Algebra)

Assignment-2

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2] Suppose $\exists\ \alpha, \beta, \gamma \in \mathbb{R} \ni \alpha(x+y) + \beta(y+z) + \gamma(z+x) = 0$, $\alpha, \beta, \gamma \neq 0$

Then, $\Rightarrow (\alpha+\gamma)x + (\alpha+\beta)y + (\beta+\gamma)z = 0$

Now, this holds iff $(\alpha+\gamma), (\alpha+\beta), (\beta+\gamma)$ all are equal to 0.

($\because$ given $x, y, z \rightarrow$ LI)

$$\therefore \left.\begin{array}{l} \alpha+\gamma=0 \\ \alpha+\beta=0 \\ \beta+\gamma=0 \end{array}\right\} \text{Adding all} \Rightarrow 2(\alpha+\beta+\gamma)=0$$

$$\Rightarrow \alpha+\beta+\gamma=0 \quad -\star$$

$$\therefore \alpha=\beta=\gamma=0$$

(comparing each of the 3 above eqns with eqn $\star$)

$\because$ This contradicts our assumption of the given vectors being NOT Linearly independent ($\alpha, \beta, \gamma \neq 0$), $\therefore$ assumption is wrong.

$\Rightarrow (x+y), (y+z), (z+x)$ are L.I. $\quad \therefore$ Proved.

LI = Linearly independent

4] $\because$ $\mathbb{C}$ is vector space over $\mathbb{R}$,

$\therefore$ 1 and x LI $\Leftrightarrow$ $\alpha(1) + \beta(x) = 0$ iff $\alpha, \beta = 0$

where $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{C}$.

Suppose NOT! Assume x is a real number!

Then, $a(1) + b(x) = 0$ holds for $a = -bx$ where $a, b \in \mathbb{R}$.

~~going a step further~~ if $x = 0$ then $a = 0$ and $b \neq 0$

if $x \neq 0$ then $x = -a/b$, $b \neq 0$.

$\because$ $\exists$ $a, b \neq 0$ and $a, b \in \mathbb{R}$ satisfying $a(1) + b(x) = 0$,

(both simultaneously $\neq 0$)

$\Rightarrow$ 1 and x are NOT LI.

$\because$ this contradicts ~~our~~ our assumption of $x \in \mathbb{R}$,

$\therefore$ proved that $x \notin \mathbb{R}$ if 1 and x are LI where

$1, x \in \mathbb{R}[\mathbb{C}; \mathbb{R}]$.

Now, if given that x is NOT real, then

$\alpha(1) + \beta(x) = 0 \Rightarrow \alpha = -\beta x$ where $\alpha, \beta \in \mathbb{R}$.

$\because$ RHS is NOT real but LHS is real, this can happen only if

$\alpha, \beta = 0$ ~~$\Rightarrow \alpha = 0$~~ $\Rightarrow$ 1, x are linearly independent!

$\therefore$ Proved.

9] Given, $\mathcal{B}_1 = \{v_1, v_2, \dots, v_{n-1}, v_n\}$ is a basis for fin-dim $\mathbb{V}$;

$\Rightarrow$ if $v \in \mathbb{V}$,

then $v = a_1 v_1 + a_2 v_2 + \dots + a_{n-1} v_{n-1} + a_n v_n$

Now, rewriting $a_2 = a_2 + a_1 - a_1$

$a_3 = a_3 - (a_2 + a_1) + (a_2 + a_1)$ $\left(a_i = a_i - \left(\sum_{1}^{i-1} a_j\right) + \sum_{j=1}^{i-1} a_j \quad \forall\, i \in [2, n] \right)$

$\vdots$

$a_n = a_n \cancel{- a_{n-1} + a_{n-1}} - \left(\sum_{j=1}^{n-1} a_j\right) + \left(\sum_{j=1}^{n-1} a_j\right)$

$$\Rightarrow v = a_1 v_1 + (a_2 + a_1 - a_1) v_2 + \left(a_3 - (a_2 + a_1) + (a_2 + a_1)\right) v_3 + \dots + v_n \left(a_n - \left(\sum_{j=1}^{n-1} a_j\right) + \left(\sum_{j=1}^{n-1} a_j\right)\right)$$

$$= a_1 (v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_2 + a_1)(v_3 - v_4) + \dots + \left(\sum_{j=1}^{n-1} a_j\right)(v_{n-1} - v_n) + \left(a_n + \sum_{j=1}^{n-1} a_j\right) v_n$$

$$= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \left(\sum_{j=1}^{i} a_j\right) + v_n \left(a_n + \sum_{j=1}^{n-1} a_j\right)$$

Thus, any vector $v \in \mathbb{V}$ can be represented as linear combination of $(v_1 - v_2), (v_2 - v_3), \dots, (v_{n-1} - v_n), v_n$.

Thus, these vectors span $\mathbb{V}$.

Now,

to prove ~~ino~~ linear independence,

Let $b_1(v_1-v_2)+b_2(v_2-v_3)+\dots+b_{n-1}(v_{n-1}-v_n)+b_n v_n=0$
where $b_1, b_2, \dots b_{n-1}, b_n \neq 0$ (none of them are zero).

$\Rightarrow b_1 v_1+(b_2-b_1)v_2+(b_3-b_2)v_3+\dots+(b_n-b_{n-1})v_n=0.$

$\because$ given $\{v_1, v_2, \dots, v_n\}$ is a basis $\Rightarrow$ they are L.I.

$\Rightarrow b_1=0,\ b_2-b_1=0,\ b_3-b_2=0, \dots, b_n-b_{n-1}=0.$

$\Rightarrow b_2=0 \Rightarrow b_3=0 \dots b_n=0$

$\Rightarrow b_1=b_2=b_3=\dots=b_n=0.$

$\because$ This CONTRADICTS our assumption ($b_i \neq 0\ \forall i \in [1,n]$.

$\Updownarrow$

$(v_1-v_2), \dots (v_{n-1}-v_n), v_n$ are not LI)

$\Leftrightarrow$

Thus, proved that
$(v_1-v_2), (v_2-v_3), \dots, (v_{n-1}-v_n), v_n$ are Lin. independent.
$\because$ we had just proved that they also span V,
$\therefore \{(v_1-v_2), (v_2-v_3), \dots, (v_{n-1}-v_n), v_n\}$ is a BASIS of V.

Proved.

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Page No.:

Date:

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11] Let $B_1 \to$ Basis of $W_1 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \Rightarrow \dim(W_1) = \boxed{3}$

$B_2 \to$ Basis of $W_2 = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \Rightarrow \dim(W_2) = \boxed{3}$

Now, $W_1 \cap W_2$ will have any matrix in it as the form $\begin{bmatrix} a & -a \\ -a & b \end{bmatrix}$

$\therefore B_{1\cap 2} \to$ Basis of $W_1 \cap W_2 = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \Rightarrow \dim(W_1 \cap W_2) = \boxed{2}$

Now,

$\because$ W_1 and W_2 both are subspaces of the same vectorspace $\mathbb{F}^{2\times 2}$,

$$\therefore \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$= 3 + 3 - 2$$

$$= \boxed{4}$$

Given

$= \boxed{4}$

12] Given $A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$ and T.P.T $\rho(C) \leq \min(\rho(A), \rho(B))$

($\rho(A) = \text{rank}(A)$)

$\because \rho(C) =$ maximum number of linearly independent rows of C.

and as matrix multiplication of A and B ~~is~~ means each row of C is a linear combination of rows of A using coeff given by corresponding columns of B.

Assuming $\rho(A) \leq \rho(B)$ (the other case is similar), we have $\rho(C) \leq \rho(A)$ going by the explanation aforementioned.

In case, if

$\rho(B) \leq \rho(A)$ then we would have obtained $\rho(C) \leq \rho(B)$.

$\therefore$ in either case,

we have $\rho(C) \leq \min(\rho(A), \rho(B))$. $\therefore$ Proved.

14] $\because$ Given that W consists all elements 'p' all of which $\in \mathbb{F}[x]_3$
$\therefore W \subset \mathbb{F}[x]_3$. —— (a)

Now,
for this subset W to be a subspace, it must satisfy
i) Closure under addition
ii) Closure under scalar multiplication.

Consider $p_1(x) (\in W) = a_0 + a_1 x + a_2 x^2 + (a_0 + a_1 - a_2) x^3$ $a_0, a_1, a_2 \in \mathbb{F}$
$p_2(x) (\in W) = b_0 + b_1 x + b_2 x^2 + (b_0 + b_1 - b_2) x^3$ $b_0, b_1, b_2 \in \mathbb{F}$

then

$$p_1(x) + p_2(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_0+b_0+a_1+b_1-(a_2+b_2))x^3$$

Let $a_0 + b_0 = c_0$, $b_1 + a_1 = c_1$, $a_2 + b_2 = c_2$.

Then $p_1(x) + p_2(x) = c_0 + c_1 x + c_2 x^2 + (c_0 + c_1 - c_2) x^3$, $c_0, c_1, c_2 \in \mathbb{F}$

$\in W$

So $\because p_1(x) \in W$ and $p_2(x) \in W \Rightarrow p_1(x) + p_2(x) \in W$ $\therefore$ closed under addition.
$\hookrightarrow$ (b)

Now, consider $\alpha \in \mathbb{F}$, then

$$\alpha p_1(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + (\alpha a_0 + \alpha a_1 - \alpha a_2) x^3$$
$$= \beta_0 + \beta_1 x + \beta_2 x^2 + (\beta_0 + \beta_1 - \beta_2) x^3 \in W.$$

where $\beta_0 = \alpha a_0$, $\beta_1 = \alpha a_1$, $\beta_2 = \alpha a_2$.

So $\because p_1(x) \in W \Rightarrow \alpha p_1(x) \in W$ $\therefore$ closed under scalar multiplication
$\hookrightarrow$ (c)

From (a), (b) and (c), it is proved that W is indeed a subspace of $\mathbb{F}[x]_3$.

Now, considering an arbitrary element $p \in W$:

$$p(x) = a_0 + a_1 x + a_2 x^2 + (a_0 + a_1 - a_2) x^3, \quad a_0, a_1, a_2 \in \mathbb{F},$$
$$= a_0(1+x^3) + a_1(x+x^3) + a_2(x^2 - x^3).$$

Thus, $p(x)$ can be written as a linear combination of $(1+x^3)$, $(x+x^3)$, (x^2-x^3). Also, consider

$$A(1+x^3)+B(x+x^3)+C(x^2-x^3)=0$$

$$\Rightarrow A+Bx+Cx^2+(A+B-C)x^3=0.$$

For LHS to be identically zero $\forall x \in \mathbb{F}$, $A=0, B=0, C=0$.

$\therefore$ $(1+x^3)$, $(x+x^3)$, (x^2-x^3) are linearly independent.

$\therefore B(W)=\{(1+x^3),(x+x^3),(x^2-x^3)\} \Rightarrow \boxed{\dim(W)=3}$

Now, given $p_1(x) = 1+x+x^2+x^3$

$$= (1)+(1)(x)+(1)(x^2)+(1+1-1)(x^3)$$

$$= a_0+a_1x+a_2x^2+(a_0+a_1-a_2)x^3 \in W.$$

where $a_0=1$, $a_1=1$, $a_2=1$. $\therefore p_1(x) \in W$.

Also,

$$p_2(x) = 1+x^2 = (1)+(0)x+(1)x^2+(1+0-1)x^3$$

$$= a_0+a_1x+a_2x^2+(a_0+a_1-a_2)x^3 \in W$$

where $a_0=1, a_1=0, a_2=1$ $\therefore p_2(x) \in W$.

Consider

$$\alpha p_1(x)+\beta p_2(x)=0$$

$$\Rightarrow \alpha(1+x+x^2+x^3)+\beta(1+x^2)=0$$

$$\Rightarrow (\alpha+\beta)+\alpha x+(\alpha+\beta)x^2+\alpha x^3=0$$

For LHS to be identically zero $\forall x \in \mathbb{F}$, $\alpha=0 \Rightarrow \beta=0$; $\alpha,\beta \in \mathbb{F}$.

$\therefore p_1(x), p_2(x)$ are linearly independent.

$\therefore S=\{p_1,p_2\}$ is a linearly independent set of W.

17] Given $\exists$ subspaces W_i, $i \in \{1, 2, \ldots, m\} \ni V = \bigoplus_{i=1}^{m} W_i$ where V is a f.d. v.s.

By definition,

$$V = \bigoplus_{i=1}^{m} W_i \Rightarrow V = \sum_{i=1}^{m} W_i \text{ and } \bigcap_{i \in \{1,2,\ldots,m\}} W_i = \{0\}.$$

① ② (Proved at the end)

Now, $\because \dim\left(\sum_{i=1}^{m} W_i\right) = \sum_{i=1}^{m} \dim(W_i) - \dim\left(\bigcap_{i \in \{1,2,\ldots,m\}} W_i\right)$ — ③

(extended property $\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$)

Substituting ① & ② in eqn ③,

$$\Rightarrow \boxed{\dim(V) = \sum_{i=1}^{m} \dim(W_i)} \cancel{= 0}$$

$\therefore$ Proved.

(The implication $V = \bigoplus_{i=1}^{m} W_i \Rightarrow V = \sum_{i=1}^{m} W_i$ and $\bigcap_{i=\{1,2,\ldots m\}} W_i = \{0\}$ is proved via principle of mathematical inductn.

$\because$ $V = W_1 \oplus W_2 \Rightarrow V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$, (case $i=2$)

then let it be true for case $i = m-1$,

ie; $V = \bigoplus_{i=1}^{m-1} W_i \Rightarrow V = \sum_{i=1}^{m-1} W_i$ and $\bigcap_{i=1}^{m-1} W_i = \{0\}$.

then

$$V = \bigoplus_{i=1}^{m} W_i = \bigoplus_{i=1}^{m-1} W_i \oplus W_m \Rightarrow V = \sum_{i=1}^{m-1} W_i + W_m$$

and $\bigcap_{i=1}^{m-1} W_i \cap W_m = \{0\}$

$$\Rightarrow V = \sum_{i=1}^{m} W_i \text{ and } \bigcap_{i=\{1,2,\ldots m\}} W_i = \{0\}.$$

$\therefore$ Proved.)

22] Given : $\phi : \mathbb{R}^2 \to \mathbb{R}^2 \ni \phi(x_1, x_2) = (-x_2, x_1)$

$\phi(\epsilon_1) = \phi(1,0) = (0,1)$

$\phi(\epsilon_2) = \phi(0,1) = (-1,0)$. $\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow [\phi]_M = [\epsilon_1 \ \epsilon_2]$

Matrix of linear map ϕ

$\therefore [\phi]_M = A = \begin{array}{cc} \epsilon_1 & \epsilon_2 \\ \end{array} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\therefore \phi^2 = \phi \cdot \phi = A \cdot A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= -I$

$\therefore \underline{\phi^2 = -I}$.

Now, given

$B = \{(1,2), (-1,1)\}$ then $\cancel{[\phi]_B = [}$ to find $[\phi]_B$:

$\phi[1,2] = (-2,1)$ $\therefore$ row-reducing $\left[\begin{array}{cc|cc} 1 & -1 & -2 & -1 \\ 2 & 1 & 1 & -1 \end{array}\right]$

$\phi(-1,1) = (-1,-1)$

$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & -1 & -2 & -1 \\ 0 & 3 & 5 & 1 \end{array}\right] \xrightarrow{R_2/3} \left[\begin{array}{cc|cc} 1 & -1 & -2 & -1 \\ 0 & 1 & 5/3 & 1/3 \end{array}\right]$

24] Given, $\mathbb{V} = \mathbb{R}[x]_n$; $\mathcal{B} = \{f_0, f_1, \ldots, f_n\}$, $\mathcal{B}' = \{g_0, g_1, \ldots, g_n\}$

$f_i = x^i$ $\qquad$ $g_i = (x+\alpha)^i$

Differential operator: $D: \mathbb{V} \to \mathbb{V}$. $\qquad$ $\alpha \to$ scalar

Consider Basis $\mathcal{B}$:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \quad -(1)$$

then

$$D(p(x)) = a_1 + 2a_2 x + \cdots + (n-1)a_{n-1} x^{n-2} + a_n \cdot n \cdot x^{n-1}.$$

$$\therefore [D_{\mathcal{B}}(p(x))] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & 3 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Now, consider Basis $\mathcal{B}'$:

$$p(x) = b_0 + b_1(x+\alpha) + b_2(x+\alpha)^2 + \cdots + b_n(x+\alpha)^n \quad -(2)$$

then

$$D_{\mathcal{B}'}(p(x)) = b_1 + 2b_2(x+\alpha) + \cdots + nb_n (x+\alpha)^{n-1} + 0(x+\alpha)^n$$

$$\therefore [D_{\mathcal{B}'}(p(x))] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & 3 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Now,

for obtaining these bases using basis transformation:

observe:

$$x + \alpha = (1)(x) + (1)(\alpha)$$

$$(x+\alpha)^2 = (1)(x^2) + (2\alpha)(x) + (\alpha^2) 1$$

$$(x+\alpha)^i = {}^iC_0 \alpha^i + {}^iC_1 \alpha^{i-1} x + {}^iC_2 \alpha^{i-2} x^2 + \cdots + {}^iC_i \alpha^{i-i} x^i$$

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$$\therefore \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\text{in } \mathcal{B}} = \underbrace{\begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^n \\ 0 & 1 & 2\alpha & 3\alpha^2 & \cdots & n\alpha^{n-1} \\ 0 & 0 & 1 & 6\alpha & \cdots & n(n-1)\alpha^{n-2} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}}_{T^{-1} \to \text{ Transformation matrix}} \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\text{in } \mathcal{B}'}$$

<u>MAP</u> :

$$\begin{tikzcd}
{[p(x)]_{\mathcal{B}}} \arrow[r, "D_{\mathcal{B}}"] \arrow[d, bend right, "T"'] & {[p'(x)]_{\mathcal{B}}} \arrow[d, bend left, "T"] \\
{[p(x)]_{\mathcal{B}'}} \arrow[u, "T^{-1}"] \arrow[r, "D_{\mathcal{B}'}"'] & {[p'(x)]_{\mathcal{B}'}} \arrow[u, "T^{-1}"]
\end{tikzcd}$$

Going along the dotted path, we obtain

$$\boxed{D_{\mathcal{B}'} = T^{-1} D_{\mathcal{B}} T}$$

Also, $D_{\mathcal{B}}$ and $D_{\mathcal{B}'}$ are same (as calculated above) -

25] Now the vectorspace is $\mathbb{R}[x]_3$. $\therefore$ let $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$.

$$\hat{D} : V \to V \ni \hat{D}\left(f(x)\right) = 3f(x) + (3-x)\frac{d}{dx}\left(f(x)\right).$$

$$\mathrm{Ker}\left(\hat{D}(f(x))\right) = \left\{ f(x) \,\middle|\, 3f(x) + (3-x)\frac{d}{dx}\left(f(x)\right) = 0 \right\}$$

$$\hat{D}\left(f(x)\right) = \left(3c_0 + 3c_1 x + 3c_2 x^2 + 3c_3 x^3\right) + (3-x)\left(c_1 + 2c_2 x + 3c_3 x^2\right)$$

$$= \left(3c_0 + 3c_1\right) + \left(3c_1 + 6c_2 - c_1\right)x + \left(3c_2 + 9c_3 - 2c_2\right)x^2 + \left(\cancel{3c_3} - \cancel{3c_3}\right)x^3$$

for $\hat{D}\left(f(x)\right) = 0$,

$$3\left(c_0 + c_1\right) = 0$$

$$2\left(c_1 + 3c_2\right) = 0$$

$$\left(c_2 + 9c_3\right) = 0$$

Page No.: YOUVA
Date:

for solving, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

RREF

$R_1 \to R_1 - R_2 + 3R_3,\ R_2 \to R_2 - 3R_3 \implies \begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & -27 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$c_0, c_1, c_2 \to$ Pivotal variables $\quad c_3 \to$ free variable.

$\therefore \boxed{c_0 = -27c_3},\ \boxed{c_1 = 27c_1},\ \boxed{c_2 = -9c_3}$

$\therefore f(x) = c_3(-27 + 27x - 9x^2 + x^3) \to$ Basis for $Ker(\hat{D}(p(x)))$.

To find $img(\hat{D}(p(x)))$: $3c_0 + c_1(3+2x) + c_2(6x+x^2) + c_3(9x^2)$

the set $\to \{3, (3+2x), (6x+x^2), 9x^2\}$ is a generating set for $im(\hat{D})$ but not LI because

$3 \times 3 + (-3)(2x+3) + (1)(x^2+6x) + (-1/9)(9x^2) = 0$.

$\therefore$ considering the set $\{3, 3+2x, 6x+x^2\} \longrightarrow$ LI set.

if $\exists f(x) \ni \hat{D}(f(x)) = g(x)$ then $g \in Im(\hat{D})$

$g(x) = \alpha(3) + \beta(3+2x) + \gamma(6x+x^2)$

$\Rightarrow 7 + 8x = (3\alpha + 3\beta) + (2\beta + 6\gamma)x + \cancel{\gamma(x^2)}\ (\gamma)\ x^2$

Comparing, we get $\gamma = 0$, $2\beta + \cancel{\gamma 6} = 8$, $3(\alpha+\beta) = 7$

$\Rightarrow \beta = 4 \quad \Rightarrow \alpha = -5/3$

$\therefore g(x) \in Im(\hat{D})$.

To find all sol^n of $\hat{D}(f(x)) = g(x)$:

$$(3c_0 + 3c_1) + (2c_1 + 6c_2)x + (c_2 + 9c_3)x^2 = 7 + 8x$$

$$\Rightarrow 3(c_0 + c_1) = 7, \quad 2(c_1) + 6(c_2) = 8, \quad c_2 + 9c_3 = 0$$

$$\begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} \Rightarrow c_3 \rightarrow \text{free var.}$$

$$c_2 = -9c_3, \quad c_1 = 4 + 27c_3, \quad c_0 = \frac{-5}{3} - 27c_3$$

$$\therefore f(x) = c_3(x^3 - 9x^2 + 27x - 27) + 4x - 5/3$$

$\hookrightarrow$ Basis for $\operatorname{im}(\hat{D}(p(x)))$.