Filter Design (Type-2) Assignment

EE338: Digital Signal Processing

Mid-Semester Optional Extension

Sravan K Suresh

(22B3936)

Filter number: **63**

Reviewed by:

Neel Bhavesh Rambhia (22B1298)

Swarup Dasharath Patil (22B3953)

Under the guidance of:

Prof. Vikram M Gadre

Indian Institute of Technology Bombay

February, 2025

EE338 Mid-Semester Take-Home Extension

March 7, 2025

Contents

1 Introduction

In digital signal processing (DSP), filter design is essential for isolating or removing certain frequency components from a signal. Filters are used in many fields, such as audio processing, telecommunications, and biomedical signal analysis. Designing a filter involves taking theoretical specifications and turning them into practical, working solutions. The challenge is to achieve the desired frequency response while sticking to requirements like passband and stopband tolerances, transition bandwidths, and filter types (e.g., IIR or FIR).

The process of designing the filter includes several important steps: first, deriving both the un-normalized and normalized specifications, then using the bilinear transformation to convert the analog filter specifications into a digital filter. Finally, we will check if the filter works as expected by analyzing its frequency response. Since we're restricted from using automatic design tools in MATLAB, we'll take a hands-on, step-by-step approach to the design process, which will help us better understand how IIR filters are built.
The report is organized as follows:
Section 2 discusses the un-normalized and normalized filter specifications.
Section 3 walks through the design steps, including the creation of the analog filter's transfer function and its conversion to a digital filter.
Section 4 shows how we verify the filter by analyzing its frequency response.

2 Specifications

The design problem involves the creation of a multi-bandpass Infinite Impulse Response (IIR) filter to process an analog signal bandlimited to 280 kHz, sampled at a rate of 630 kHz. The filter is designed to extract specific frequency bands while suppressing others, adhering to strict tolerances on the passband and stopband magnitudes. The following subsections outline the un-normalized and normalized filter specifications.

2.1 Un-Normalized Discrete Time Filter Specifications

The un-normalized discrete-time filter specifications are as follows:

- **Sampling Rate:** The analog signal is sampled at a rate of **630 kHz**.
- **Passband Tolerances:** The magnitude response in the passband must lie **between 1 and 0.85** (for IIR filters).
- **Stopband Tolerances:** The magnitude response in the stopband **must not exceed 0.15**.
- **Transition Bands:** For bandpass filters, the transition bands are **5 kHz** on either side of each passband.
- **Passband Frequencies:**
  - Group I: **65 kHz to 95 kHz**

  - Group II: **210 kHz to 240 kHz**

  This is because the Filter number assigned to me is $\mathbf{M = 63}$, thus $\mathbf{Q = 5}$ and $\mathbf{R = 8}$.

- **Stopband Frequencies:** All frequencies outside the specified passbands are considered stopbands.
- **Filter Type:** The passbands are **Equi-ripple** (oscillatory) and the stopbands are **monotonic** (non-oscillatory).
  - The Analog Transfer Function is defined by the two center frequencies and bandwidths as follows:
    * $\Omega_{c_1} = 0.9932$ and $B_1 = 0.1767$ for the first band.
    * $\Omega_{c_2} = 0.9959$ and $B_2 = 0.8159$ for the second band.

2.2 Normalized Digital Filter Specifications

The normalized digital filter specifications are derived by scaling the frequencies with respect to the Nyquist frequency, which is half the sampling rate (315 kHz). The normalized frequencies are calculated as follows:

$$f_{\text{normalized}} = \frac{f_{\text{actual}}}{f_{\text{Nyquist}}} = \frac{f_{\text{actual}}}{315\,\text{kHz}}$$

- **Normalized Passband Frequencies:**
  - Group I:
  $$\frac{65\,\text{kHz} \times 2\pi}{630\,\text{kHz}} \approx 0.6483 \quad \text{to} \quad \frac{95\,\text{kHz} \times 2\pi}{630\,\text{kHz}} \approx 0.9475$$
  - Group II:
  $$\frac{210\,\text{kHz} \times 2\pi}{630\,\text{kHz}} \approx 2.0944 \quad \text{to} \quad \frac{240\,\text{kHz} \times 2\pi}{630\,\text{kHz}} \approx 2.3936$$
- **Normalized Transition Bands:**
  $$\frac{5\,\text{kHz} \times 2\pi}{630\,\text{kHz}} \approx 0.0159$$
- **Normalized Stopband Frequencies:** All frequencies outside the normalized passbands are considered stopbands.

These normalized specifications are used as the basis for the design of the analog and digital filters, ensuring that the filter meets the desired frequency response characteristics.

3 Design Process

The IIR multi-bandpass filter was designed using a Chebyshev analog prototype filter with the bilinear transformation method. I have followed the protocol outlined in the problem statement, avoiding the use of MATLAB's complete filter design commands. The key steps are detailed below.

3.1 Analog Filter Specifications

The design began with the following critical specifications:

- Sampling frequency: $F_s = 630\,\text{kHz}$
- Passbands: Group I (65-95 kHz) and Group II (210-240 kHz)
- Transition bandwidth: 5 kHz on either side of passbands
- Passband tolerance: $\delta_1 = 0.15$ (magnitude between 1 and 0.85)
- Stopband tolerance: $\delta_2 = 0.15$ (magnitude ≤ 0.15)

3.2 Frequency Normalization

Frequencies were normalized to the Nyquist frequency (315 kHz):

$$\omega = \frac{2\pi f}{F_s}$$

Resulting in:

- Group I: $\omega_{p1} = [0.6483, 0.9475]$ rad/sample (MATLAB: `omega_p1`)
- Group II: $\omega_{p2} = [2.0944, 2.3936]$ rad/sample (MATLAB: `omega_p2`)

3.3 Bilinear Transformation and Analog Prototype

The bilinear transformation ($s = \frac{1-z^{-1}}{1+z^{-1}}$) was applied to map digital specifications to the analog domain. Critical parameters were calculated as:

- Transition band edges: $f_{stop_1} = [60, 100]\,\text{kHz}$, $f_{stop_2} = [205, 245]\,\text{kHz}$ (MATLAB: `f_stop1`, `f_stop2`)
- Analog frequency transformation:

$$\Omega = \tan\left(\frac{\omega}{2}\right) \quad \text{(MATLAB: Omega\_p1, Omega\_p2)}$$

- Bandwidths (B) and center frequencies (Ω_0):

$$B_1 = 0.1767, \quad \Omega_{0_1} = 0.9932 \quad \text{(Group I)} \quad \text{(MATLAB: B\_1, Omega0\_1)}$$

$$B_2 = 0.8159, \quad \Omega_{0_2} = 0.9959 \quad \text{(Group II)} \quad \text{(MATLAB: B\_2, Omega0\_2)}$$

3.4 Lowpass Prototype Design

The bandpass-to-lowpass transformation was implemented using:

$$\Omega_L \to \frac{\Omega^2 - \Omega_0^2}{B \cdot \Omega} \quad \text{(MATLAB: } \texttt{LP\_Omega1}, \texttt{LP\_Omega2}\text{)}$$

and the lowpass-to-bandpass transformation was implemented using:

$$s_L \to \frac{s^2 + \Omega_0^2}{B \cdot s} \quad \text{(MATLAB: } \texttt{analog\_bpf1}, \texttt{analog\_bpf2}\text{)}$$

where s_L is the prototype lowpass filter variable. The filter order N for the Chebyshev Filter was determined by solving:

$$N \geq \left\lceil \frac{\cosh\left(\sqrt{D_2/D_1}\right)}{\cosh\left(\Omega_s/\Omega_p\right)} \right\rceil \quad \text{(MATLAB: } \texttt{N1}, \texttt{N2}\text{)}$$

with $D_1 = 0.3841$ and $D_2 = 43.4444$. Both bands required $N = 10$.

3.5 Pole-Zero Plots

The pole-zero plots for the designed filters are shown in Figures 1 and 2. All the poles plotted here are the ones lying in the LHCP and the rest are discarded.

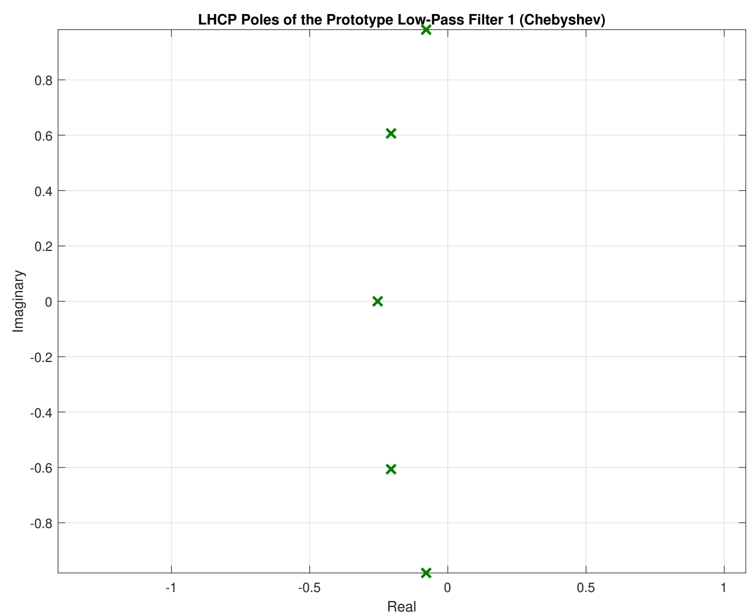


Figure 1: Pole-Zero Plot of the First Bandpass Filter

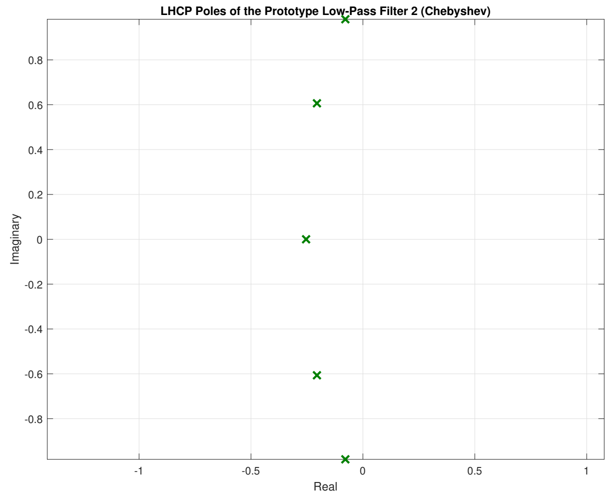


Figure 2: Pole-Zero Plot of the Second Bandpass Filter

3.6 Pole Locations and Transfer Functions for Chebyshev Filter

The 5th-order Chebyshev prototype poles were calculated as:

$$\sigma_k = -\sinh(\eta) \cdot \sin\left(\frac{(2k-1)\pi}{2N}\right), \quad \omega_k = \cosh(\eta) \cdot \cos\left(\frac{(2k-1)\pi}{2N}\right)$$

where $\eta = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)$ and $\epsilon = \sqrt{D_1}$

Left-half-plane poles were selected for stability (Table 1). The analog transfer function for Group I was derived as:

$$H_{\text{Analog,LPF}}(s_L) = \frac{1}{(s_L - s_{L_1})(s_L - s_{L_2})\cdots(s_L - s_{L_5})}$$

where $\Omega_c = 0.9932$ (Group I) and $\Omega_c = 0.9959$ (Group II).

Table 1: Selected Left-Half-Plane Poles (Group I and Group II)

Group I

Pole #	Value (rad/s)
1	$-0.1657 + 1.0464i$
2	$-0.4810 + 0.9440i$
3	$-0.7491 + 0.7491i$
4	$-0.9440 + 0.4810i$
5	$-1.0464 + 0.1657i$

Group II

Pole #	Value (rad/s)
1	$-0.0785 + 0.9812i$
2	$-0.2054 + 0.6064i$
3	$-0.2539 + 0.0000i$
4	$-0.2054 + -0.6064i$
5	$-0.0785 + -0.9812i$

3.7 Digital Filter Realization

The bilinear transformation was applied to both bandpass filters:

$$H_{\text{Digital}}(z) = H_{\text{Analog}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{(MATLAB: \texttt{digital\_bpf1}, \texttt{digital\_bpf2})}$$

Resulting in 20th-order IIR filters for each band. The final combined transfer function was obtained by parallel combination:

$$H_{\text{Total}}(z) = H_1(z) + H_2(z) \quad \text{(MATLAB: \texttt{num\_discrete\_total}, \texttt{den\_discrete\_total})}$$

3.8 Filter Coefficients

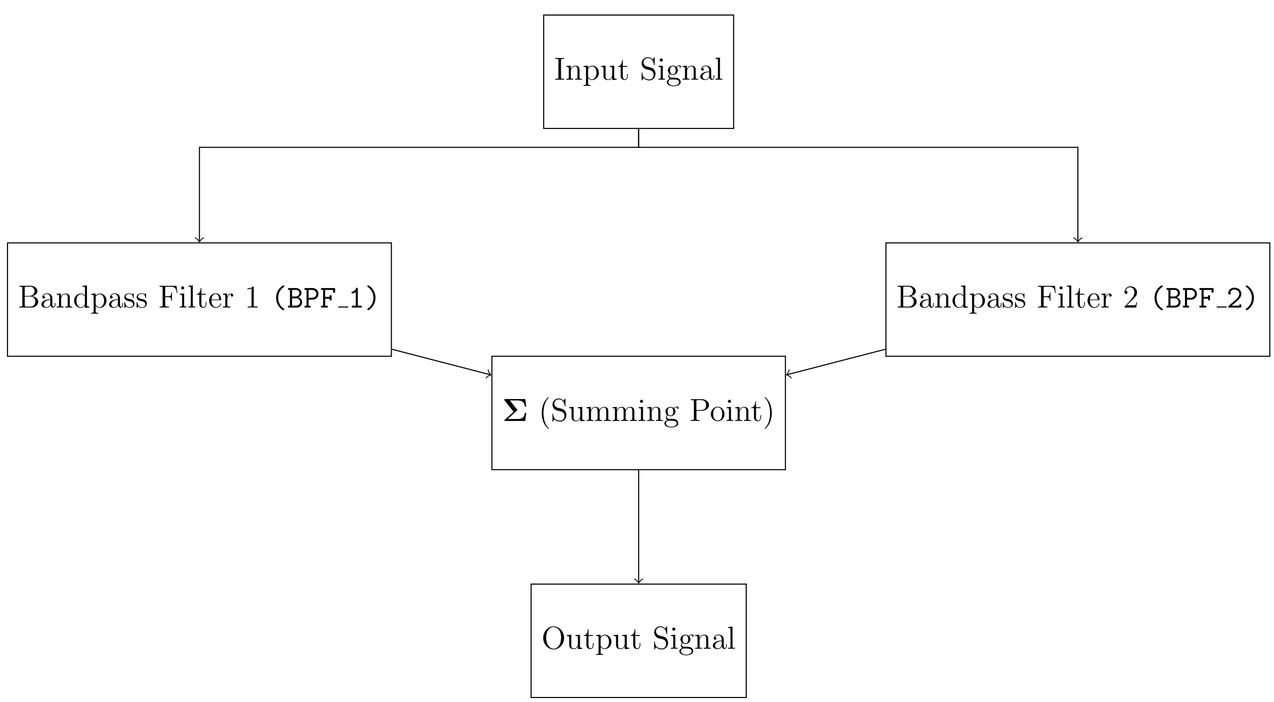
The normalized coefficients after bilinear transformation are shown below (first 5 terms shown for brevity). Full coefficient tables are provided in Appendix A.

- **Group I Numerator:** $b_1 = [-8.62 \times 10^{-6}, 0, 4.31 \times 10^{-5}, ..., 8.62 \times 10^{-6}]$
- **Group I Denominator:** $a_1 = [1, -7.15, 25.66, ..., 1.28]$
- **Group II Numerator:** $b_1 = [-8.62 \times 10^{-6}, 0, 4.31 \times 10^{-5}, ..., 8.62 \times 10^{-6}]$
- **Group II Denominator:** $a_1 = [1, -7.15, 25.66, ..., 1.28]$

3.9 Configuration of Combined Filter for Multi-bandpass

The magnitude functions of the filters add up, resulting in the overall transfer function:

$$H = H_1 + H_2$$



3.10 Final Transfer Function

The final combined transfer function after bilinear transformation is given by:

$$H(z) = \frac{N(z)}{D(z)}$$

where:

$$N(z) = -1.724882428 \times 10^{-5} z^0 - 6.604152056 \times 10^{-6} z^1 + \cdots + 2.205299714 \times 10^{-5} z^{20}$$

$$D(z) = 1z^0 - 0.765750981 z^1 + 1.465347667 z^2 - \cdots + 1.634617886 z^{20}$$

The full numerator and denominator coefficients are provided in Appendix A.

4 Verification

The designed IIR multi-bandpass filter was verified by analyzing its frequency response and performance across the specified passbands and stopbands. The magnitude and phase responses were computed using MATLAB's `freqz` function, and the results are presented below.

4.1 Magnitude Response Analysis

The magnitude response of the filter was evaluated to ensure compliance with the design specifications. The following plots illustrate the filter's performance:

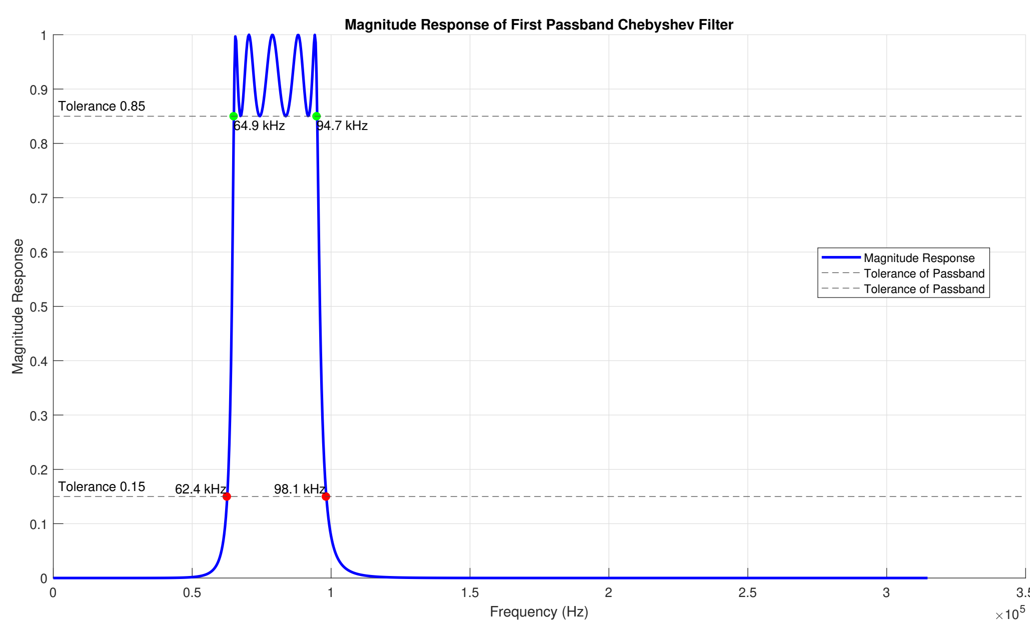


Figure 3: Magnitude Response in the First Passband (65-95 kHz)

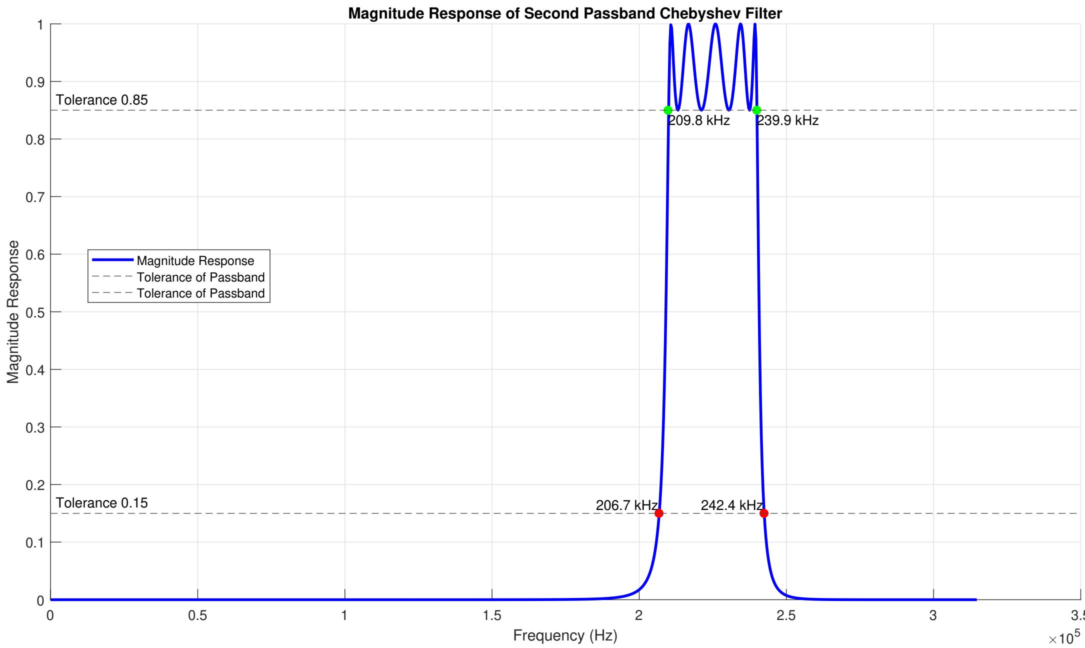


Figure 4: Magnitude Response in the Second Passband (210-240 kHz)

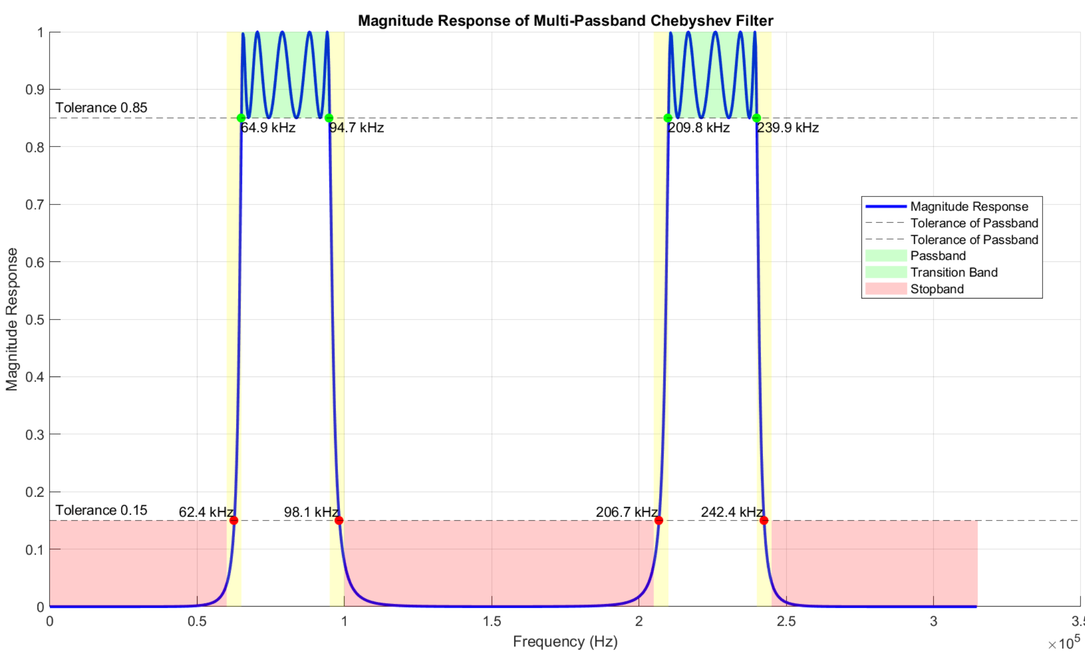


Figure 5: Magnitude Response Across Both Passbands

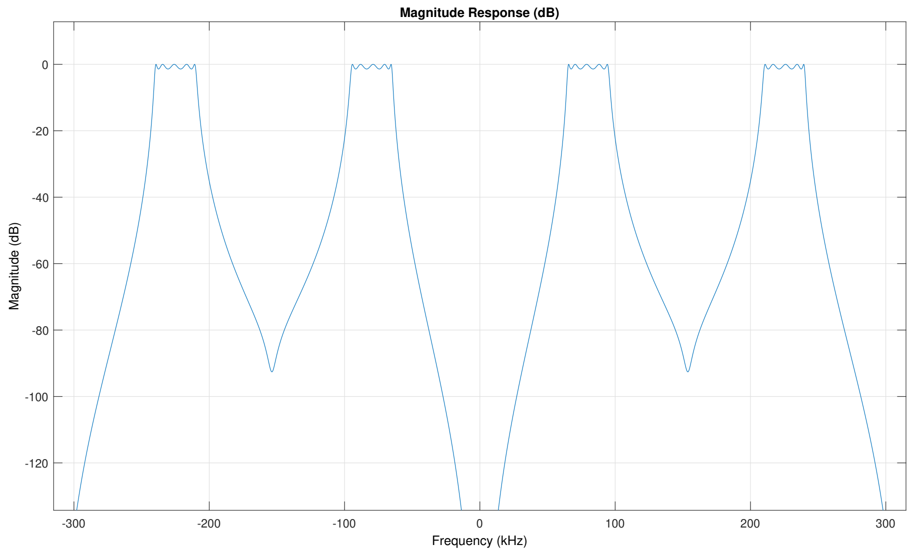


Figure 6: Magnitude Response in Decibels (dB) Across the Entire Frequency Range

4.1.1 Observations

- **First Passband (65-95 kHz):** As shown in Figure 3, the filter exhibits an oscillatory magnitude response within the passband, with the magnitude lying between 1 and 0.85, as specified. The transition bands are sharp, with a width of approximately 5 kHz on either side.
- **Second Passband (210-240 kHz):** Figure 4 demonstrates that the filter meets the design requirements in the second passband, with the magnitude response adhering to the specified tolerances.
- **Multi-Passband Response:** Figure 5 shows the combined response of both passbands, confirming that the filter successfully isolates the desired frequency bands while suppressing others.
- **Decibel Response:** The dB-scale plot in Figure 6 provides a detailed view of the stopband attenuation, which is consistently below $-15\,\text{dB}$ (corresponding to a magnitude of 0.15), as required.

4.2 Conclusion

The verification process confirms that the designed IIR multi-bandpass filter meets all specified requirements, including:

- Passband magnitude response between 1 and 0.85.
- Stopband magnitude response ≤ 0.15.
- Transition bands of 5 kHz on either side of the passbands.

The filter's performance is further validated by the sharp transition bands and consistent stopband attenuation, as demonstrated in the provided plots.

5 Design Comparison (Butterworth vs. Chebyshev)

In this section, we compare the Butterworth and Chebyshev filter designs based on their characteristics, performance, and design parameters. The key differences and similarities are summarized in Table 2.

Table 2: Comparison of Butterworth and Chebyshev Filter Designs

Parameter	**Butterworth**	**Chebyshev**
Passband Behavior	Flat	Equiripple (oscillatory)
Stopband Behavior	Monotonic	Rippled
Filter Order	$N = 10$	$N = 5$
Central Frequency (First Band)	$\Omega_{c_1} = 1.0595$	$\Omega_{c_1} = 0.9932$
Central Frequency (Second Band)	$\Omega_{c_2} = 1.0627$	$\Omega_{c_2} = 0.9959$
Passband Tolerance	$0.85 \le \lvert H(\omega)\rvert \le 1$	$0.85 \le \lvert H(\omega)\rvert \le 1$
Stopband Tolerance	$\lvert H(\omega)\rvert \le 0.15$	$\lvert H(\omega)\rvert \le 0.15$
Pole-Zero Map	Poles lie on a unit circle in the s-plane.	Poles lie on an ellipse inscribed within the unit circle.
Advantages	Flat passband Simple design	Lower filter order Sharper roll-off
Disadvantages	Higher filter order Slower roll-off	Passband ripple More complex design

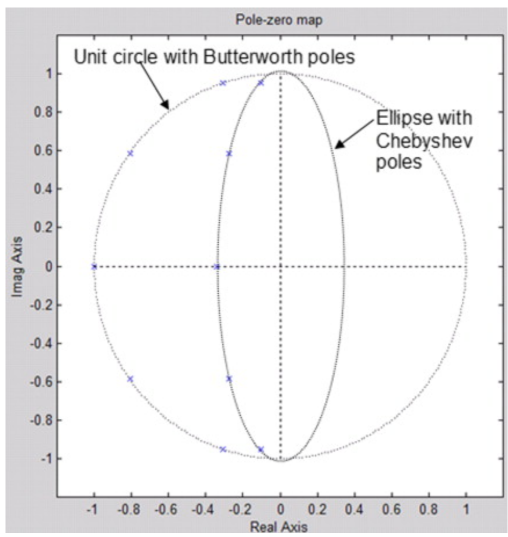


Figure 7: Comparison of the Pole-Zero Maps

5.1 Key Observations

- **Passband Behavior**: The Butterworth filter has a flat passband, which is ideal for applications requiring minimal distortion in the passband. In contrast, the Chebyshev filter has an equiripple passband, which introduces oscillations but achieves a sharper roll-off.
- **Filter Order**: The Chebyshev filter achieves the same specifications with a significantly lower filter order ($N = 5$) compared to the Butterworth filter ($N = 10$). This reduces computational complexity and implementation cost.
- **Central Frequency**: The central frequencies for both bands shifted slightly in the Chebyshev design:

  - First band: From $\Omega_{c_1} = 1.0595$ (Butterworth) to $\Omega_{c_1} = 0.9932$ (Chebyshev).
  - Second band: From $\Omega_{c_2} = 1.0627$ (Butterworth) to $\Omega_{c_2} = 0.9959$ (Chebyshev).
- **Design Trade-offs**: The Chebyshev filter offers a sharper transition band and lower order at the cost of passband ripple. The Butterworth filter, while having a flat passband, requires a higher order to achieve similar stopband attenuation.

5.2 Formulae Changes

The following formulae underwent changes during the transition from Butterworth to Chebyshev design:

- **Filter Order Calculation**:
  - Butterworth:
  $$N = \left\lceil \frac{\log_{10}\left(\frac{D_2}{D_1}\right)}{2\log_{10}(\Omega_s)} \right\rceil$$
  - Chebyshev:
  $$N = \left\lceil \frac{\cosh^{-1}\left(\sqrt{\frac{D_2}{D_1}}\right)}{\cosh^{-1}(\Omega_s)} \right\rceil$$
- **Pole Calculation**:
  - Butterworth: Poles are uniformly distributed on a circle in the s-plane.
  - Chebyshev: Poles are distributed on an ellipse in the s-plane, given by:
  $$\sigma_k = -\sinh(\eta)\cdot\sin\left(\frac{(2k-1)\pi}{2N}\right), \quad \omega_k = \cosh(\eta)\cdot\cos\left(\frac{(2k-1)\pi}{2N}\right)$$
  where $\eta = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)$ and $\epsilon = \sqrt{D_1}$.
- **DC Gain Adjustment**:
  - Butterworth: No DC gain adjustment required.
  - Chebyshev: DC gain adjusted based on filter order:
  $$H(0) = \begin{cases} \frac{1}{\sqrt{1+\epsilon^2}} & \text{if } N \text{ is even,} \\ 1 & \text{if } N \text{ is odd.} \end{cases}$$

5.3 Conclusion

The Chebyshev filter design offers a more efficient implementation with a lower filter order and sharper roll-off, making it suitable for applications where passband ripple is acceptable. On the other hand, the Butterworth filter is preferred for applications requiring a flat passband, despite its higher order and slower roll-off. The choice between the two depends on the specific requirements of the application.

A Filter Coefficients

A.1 Group I Filter Coefficients

Table 3: Numerator Coefficients for Group I (b_1)

Index	Value
0	$-8.624412139 \times 10^{-6}$
1	0
2	$4.31220607 \times 10^{-5}$
3	0
4	$-8.624412139 \times 10^{-5}$
5	0
6	$8.624412139 \times 10^{-5}$
7	0
8	$-4.31220607 \times 10^{-5}$
9	0
10	$8.624412139 \times 10^{-6}$

Table 4: Denominator Coefficients for Group I (a_1)

Index	Value
0	1
1	-7.153135408
2	25.65743473
3	-59.07016123
4	95.89971082
5	-114.1221923
6	100.7357103
7	-65.17844606
8	29.73844518
9	-8.708780027
10	1.278521758

A.2 Group II Filter Coefficients

Table 5: Numerator Coefficients for Group II (b_2)

Index	Value
0	$-8.624412139 \times 10^{-6}$
1	0
2	$4.31220607 \times 10^{-5}$
3	0
4	$-8.624412139 \times 10^{-5}$
5	0
6	$8.624412139 \times 10^{-5}$
7	0
8	$-4.31220607 \times 10^{-5}$
9	0
10	$8.624412139 \times 10^{-6}$

Table 6: Denominator Coefficients for Group II (a_2)

Index	Value
0	1
1	6.387384427
2	21.49773865
3	47.40104879
4	75.17887207
5	88.73139022
6	78.96775271
7	52.30037473
8	24.91599901
9	7.776495585
10	1.278521758

A.3 Combined Filter Coefficients

Table 7: Numerator Coefficients for Combined Filter (b_{total})

Index	Value
0	$-1.724882428 \times 10^{-5}$
1	$6.604152056 \times 10^{-6}$
2	-0.0003204415283
3	$6.761847469 \times 10^{-5}$
4	0.0003854877987
5	-0.0002181739125
6	0.001933056053
7	$-4.348708539 \times 10^{-5}$
8	-0.003496088507
9	0.0006691472608
10	-0.0004252653711
11	-0.0006227594749
12	0.003924432361
13	$-3.598942025 \times 10^{-5}$
14	-0.001780636885
15	0.0002559441792
16	-0.0005864455605
17	$-7.086376844 \times 10^{-5}$
18	0.0003610974663
19	$-8.040405255 \times 10^{-6}$
20	$2.205299714 \times 10^{-5}$

Table 8: Denominator Coefficients for Combined Filter (a_{total})

Index	Value
0	1
1	-0.765750981
2	1.465347667
3	-1.561448897
4	6.285460608
5	-4.292706743
6	6.58982776
7	-6.13782591
8	14.6658825
9	-8.427265051
10	10.93701161
11	-8.836453666
12	16.1297337
13	-7.086655686
14	7.966697814
15	-5.458944668
16	8.385501375
17	-2.191961255
18	2.153106637
19	-1.191945943
20	1.634617886