EE-229 (Signal Processing-I)

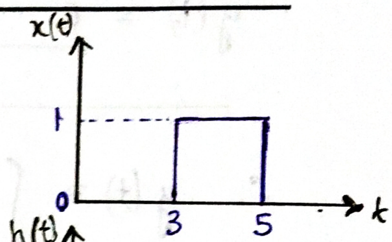
Homework-2

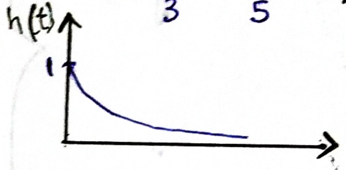
Name: Sravan K Suresh

Roll No: 22B3936

Sol. 1] $x(t) = u(t-3) - u(t-5) = \begin{cases} 1, & 3 \leq t < 5 \\ 0, & \text{elsewhere} \end{cases}$

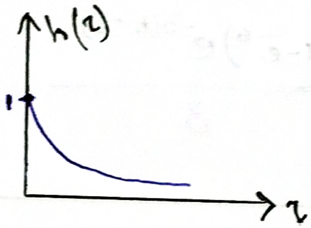
$h(t) = e^{-3t} u(t)$

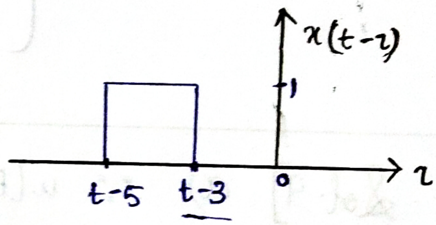




$x(t-\tau) = \begin{cases} 1, & t-5 \leq \tau < t-3 \\ 0, & \text{elsewhere} \end{cases}$

$h(\tau) = e^{-3\tau} u(\tau)$





$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

$= \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$ (we will 'slide' $x(t-\tau)$ to convolve)

Now, for $t-3 < 0 \Rightarrow t < 3$, $\because$ $x(t-\tau)$ will have NO overlap with $h(\tau)$,

$\therefore$ $\underline{y(t) = 0 \quad \forall\, t < 3}$

For $(t-3) \geq 0$ and $(t-5) < 0 \Rightarrow t \in [3, 5)$, there will be overlapping of samples and the limits of integration will be from $\tau = 0$ to $\tau = t-3$.

$\therefore$ $y(t) = \int_{0}^{t-3} x(t-\tau) h(\tau) d\tau = \int_{0}^{t-3} (1) e^{-3\tau} d\tau$

$y(t) = \left[-\frac{e^{-3\tau}}{3} \right]_{0}^{t-3} = -\frac{\left(e^{-3(t-3)} - e^{-3(0)}\right)}{3}$

$y(t) = \frac{1 - e^{-3(t-3)}}{3}, \quad \forall\, t \in [3, 5)$

For $(t-3) < \infty$ and $(t-5) \geq 0 \Rightarrow t \in [5, \infty)$,
the overlap of samples can be computed by applying limits of integration from $\tau = t-5$ to $\tau = t-3$.

$$\therefore \; y(t) = \int_{t-5}^{t-3} x(t-\tau)\, h(\tau)\, d\tau = \int_{t-5}^{t-3} (1)\, e^{-3\tau}\, d\tau$$

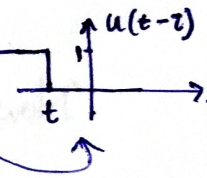
$$= \left[-\frac{e^{-3\tau}}{3} \right]_{t-5}^{t-3} = \frac{e^{-3(t-5)} - e^{-3(t-3)}}{3}$$

$$y(t) = \frac{e^{-3(t-5)}\left(1 - e^{-6}\right)}{3}, \quad \text{if } 5 \leq t < \infty$$

$$\therefore \; y(t) = \begin{cases} 0 & -\infty < t < 3 \\ \dfrac{1 - e^{-3(t-3)}}{3} & 3 \leq t < 5 \\ \dfrac{(1-e^{-6})\, e^{-3(t-5)}}{3} & 5 \leq t < \infty \end{cases}$$

Sol. 9] a) $\because \; u(t) = \begin{cases} 0 & \forall\, t < 0 \\ 1 & \forall\, t \geq 0 \end{cases}$

The convolution integral simplifies to $r(t) = \int_0^t u(\tau) \cdot u(t-\tau)\, d\tau$ due to the limited range of integration.



$\forall\, t < 0$, $u(\tau)$ and $u(t-\tau)$ will be zero (as shown in plots)
$\because$ they're defined only for $\tau \geq 0$. $\therefore \; r(t) = 0 \;\; \forall\, t < 0$.

When $t \geq 0$, both $u(\tau)$ and $u(t-\tau)$ will be equal to 1 for $0 \leq \tau \leq t$. $\therefore \; r(t) = \int_0^t 1 \times 1 \cdot d\tau = t$

$$\therefore \; r(t) = \begin{cases} 0 & \forall\, t < 0 \\ t & \forall\, t \geq 0 \end{cases}$$

b)

$$x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$

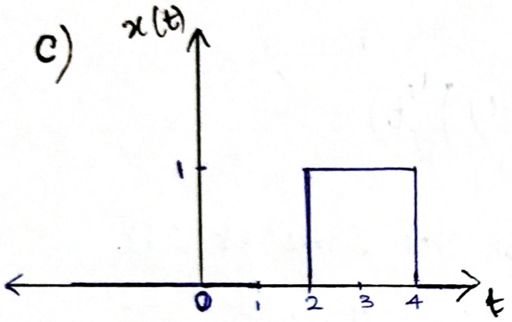
$\because$ the fns are multiplied by $u(\tau)$ and $u(t-\tau)$, the convolution integral reduces to :

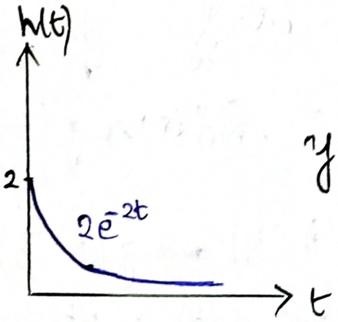
$$(x*h)(t) = \int_0^t \left(10e^{-3(t-\tau)}\cancel{u(t-\tau)}\right)\cdot\left(-e^{-\tau}+2e^{-2\tau}\right)\cancel{u(\tau)}\,d\tau$$

$$= 10e^{-3t}\int_0^t \left(2e^{\tau}-e^{2\tau}\right)d\tau = 10e^{-3t}\left[2e^{\tau}-\frac{e^{2\tau}}{2}\right]_0^t$$

$$(x*h)(t) = 10e^{-3t}\left(2e^{t}-\frac{e^{2t}}{2}-\frac{3}{2}\right).$$

c)





$$y(t) = (x*h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

for $t-2<0 \Rightarrow t<2$, $\because h(\tau<0)=0$, $\therefore$ No overlap.

$\therefore\ y(t) = 0 \quad , \quad t<2.$

for $t-2\geq 0$ and $t-4<0 \Rightarrow t\in[2,4)$,

$$y(t) = \int_0^{t-2} 1\times 2e^{-2\tau}\,d\tau = \left[\frac{2}{(-2)}e^{-2\tau}\right]_0^{t-2} = 1-e^{4-2t}, \quad 2\leq t<4.$$

for $t-4\geq 0 \Rightarrow t\geq 4$,

$$y(t) = \int_{t-4}^{t-2} 2e^{-2\tau}\,d\tau = \left[\frac{2}{(-2)}e^{-2\tau}\right]_{t-4}^{t-2} = e^{8-2t}-e^{4-2t}, \quad t\geq 4.$$

$$\therefore\ y(t) = \begin{cases} 0 & , \ t<2. \\ 1-e^{4-2t} & , \ 2\leq t<4. \\ e^{4-2t}(e^{4}-1) & , \ t\geq 4. \end{cases}$$

Sol. 10] a) $\because (f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\,d\tau$

$\therefore f(t-T_1) * g(t-T_2)(t) = \int_{-\infty}^{\infty} f(\tau - T_1)\cdot g(t-\tau-T_2)\,d\tau$

now, substitute $u = \tau - T_1 \Rightarrow \tau = u + T_1$ and $d\tau = du$:

$\therefore \left(f(t-T_1) * g(t-T_2)\right)(t) = \int_{-\infty}^{\infty} f(u)\cdot g(t-T_2-(u+T_1))\,du$

$= \int_{-\infty}^{\infty} f(u)\cdot g(t-(T_1+T_2)-u)\,du$

let $T' = T_1 + T_2$, then

$$\left(f(t-T_1) * g(t-T_2)\right)(t) = \left(f(t) * g(t-T')\right)(t)$$

$\Rightarrow$ The convolution of $f(t-T_1)$ and $g(t-T_2)$ is the same as the convolution of $f(t)$ and $g(t)$ shifted by $T' = \underline{T_1+T_2}$.

$\therefore \boxed{f(t-T_1) * g(t-T_2) = f(t) * g(t-T_1-T_2)}$

b) $(u(t+1) - u(t-2)) * (u(t-3) - u(t-4))$

$= u(t+1) * \{u(t-3) - u(t-4)\} - u(t-2) * \{u(t-3) - u(t-4)\}$ [Distributive property]

$= u(t) * u(t-2) - u(t) * u(t-3) - u(t) * u(t-5) + u(t) u(t-6)$

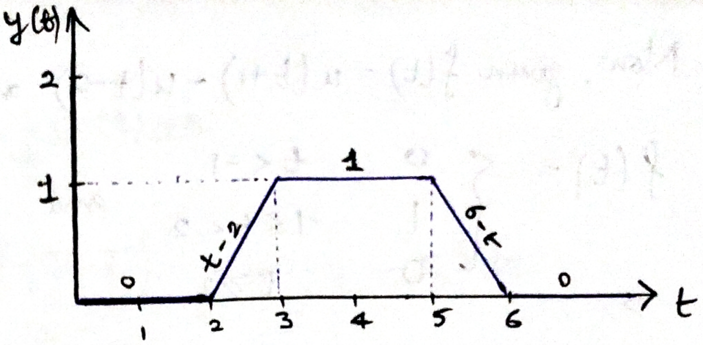
[result proved above]

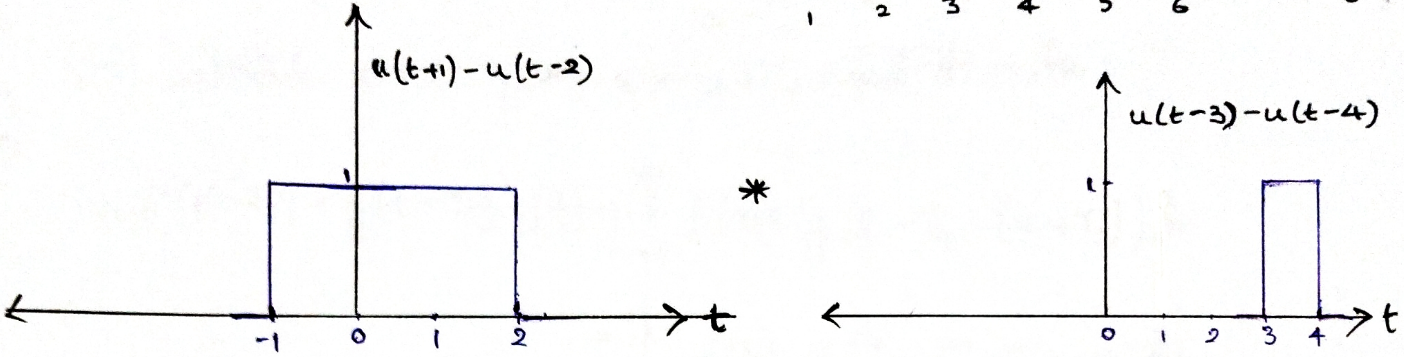
$= r(t-2) - r(t-3) - r(t-5) + r(t-6)$ (using $u(t) * u(t) = r(t)$)

ramp fn

$$= \begin{cases} 0 & , t \in (-\infty, 2) \\ t-2 & ; t \in [2,3) \\ (t-2)-(t-3) & , t \in [3,5] \\ (t-2)-(t-3)-(t-5) & , t \in [5,6) \\ (t-2)-(t-3)-(t-5)+(t-6) & , t \in [6,\infty) \end{cases}$$

$$= \begin{cases} 0 & , t < 2 \\ t-2 & , 2 \le t < 3 \\ 1 & , 3 \le t < 5 \\ 6-t & , 5 \le t < 6 \\ 0 & , t \geq 6 \end{cases}$$





As plotted above, the convolution of two rectangular pulses gives rise to a trapezoidal plot. If we imagine sliding the 1-unit ~~1-width~~ long bar ~~on~~ (shown on the plot on right side) through the rectangular plot on the left, then it sweeps as much area as in the left plot itself. This is ~~so~~ exactly equal to the area under the convolved plot (= 3 sq. units). Hence, the ~~g~~ result has been graphically verified.

Sol 11] Given, $y(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$

Now, $(f(ct) * g(ct))(t) = \int_{-\infty}^{\infty} f(c\tau)\, g(ct - c\tau) d\tau$

(NOTE: $c\tau$ and $ct - c\tau$ are new variables of integration, following the same rule of as t and $t-\tau$ in the original convolution)

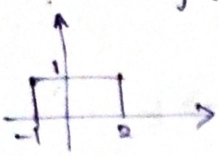
$\therefore (f(ct) * g(ct))(t) = \int_{-\infty}^{\infty} f(u)\, g\left(c\left(t - \frac{u}{c}\right)\right) \frac{du}{c}$ $\boxed{ct = u}$ $\boxed{c\,d\tau = du}$

$$\therefore \boxed{(f(ct) * g(ct))(t) = \frac{1}{c} \int_{-\infty}^{\infty} f(u)\, g\left(c\left(t - \frac{u}{c}\right)\right) du}$$

→ This gives convolution of scaled fns $f(ct)$ and $g(ct)$ at time t, given a general scaling factor 'c'.

Now, given $f(t) = u(t+1) - u(t-2)$ and $g(t) = r(t)\,(u(t) - u(t-1))$

$$f(t) = \begin{cases} 0 & t<-1 \\ 1 & -1\le t<2 \\ 0 & t\geq 2 \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 0 & t<0 \\ t & 0\le t<1 \\ 0 & t\geq 1 \end{cases}$$



$$\therefore \left(f(t) * g(t)\right)(t) = \begin{cases} 0 & , t+1<0 \\ \int_0^{t+1} 1\cdot t\,dt & , 0\le t+1<\tfrac{1}{2} \\ 0 & , 1+t \geq \tfrac{1}{2} \end{cases} = \begin{cases} 0 & , t<-1 \\ \dfrac{(t+1)^2}{2} & , -1\le t<-\tfrac{1}{2} \\ 0 & -\tfrac{1}{2}\le t \end{cases}$$

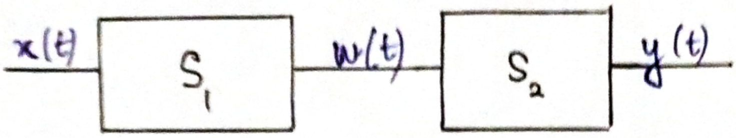
Using the derived formula,

we compute
$$\left(f(2t) * g(2t)\right)(t) = \boxed{\begin{cases} 0 & , t<0\cdot 5 \\ (t+\tfrac{1}{2})^2 & , 0\cdot 5\le t<0 \\ 0 & , t\geq 0 \end{cases}}\ .$$

Sol. 13] Given, $x(t) = e^{-2t}u(t)$, $h(t) = e^{-t}u(t)$.

$S_1 \to y(t) = x(2t)$ $\quad$ $S_2 \to$ Impulse response $h(t) = e^{-t}u(t)$.

Case 1:



$$\therefore w(t) = x(2t) = e^{-2(2t)}u(2t) = e^{-4t}u(2t)$$

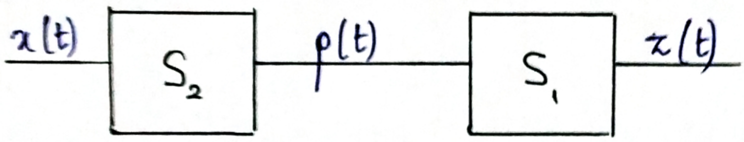
$$y(t) = w(t) * h(t) = \int_{-\infty}^{\infty} w(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-4\tau}\underbrace{u(2\tau)}_{=1 \text{ if } \tau \geq 0}e^{-(t-\tau)}\underbrace{u(t-\tau)}_{=1 \text{ when } \tau \leq t}\,d\tau$$

$$\therefore (w*h)(t) = \int_0^t e^{-4\tau}e^{-(t-\tau)}\,d\tau = \int_0^t e^{-3\tau-t}\,d\tau$$

$$\Rightarrow y(t) = \frac{1}{-3}(e^{-3\tau-t})\Big|_0^t = \boxed{\frac{1}{3}(e^{-t}-e^{-4t})}$$

Case 2:



$$\therefore p(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau}\underbrace{u(2t)}_{=1 \text{ when } \tau \geq 0}e^{-(t-\tau)}\underbrace{u(t-\tau)}_{=1 \text{ when } \tau \leq t}\,d\tau$$

$$\therefore (x*h)(t) = \int_0^t e^{-2\tau}e^{-(t-\tau)}\,d\tau = \int_0^t e^{-\tau-t}\,d\tau$$

$$\Rightarrow p(t) = -e^{-\tau-t}\Big|_{\tau=0}^{\tau=t} = (e^{-t}-e^{-2t}).$$

Now, $z(t) = p(2t) = \boxed{e^{-2t}-e^{-4t}}$ $\quad\therefore \boxed{y(t) \neq z(t)}$

★ The OUTPUTS were expected to be $\boxed{\text{NOT EQUAL}}$ because of 'S_1' being a NON-LTI system in the cascaded connections. Clearly, the result affirms it.

↓

[REASON: $y(t) = x(2t)$ → time being scaled, ∴ NOT time-invariant]

Sol. 14] a) TRUE

If $h(t) \rightarrow$ periodic and non-zero, then $\int_{-\infty}^{\infty} |h(t)| dt \rightarrow \infty$ $\therefore$ UNSTABLE.

b) FALSE

Counterexample: $x[n] = \delta[n-1] \xrightarrow{\text{inverse}} y[n] = \delta[n+1]$

(CAUSAL) (NON-CAUSAL)

c) FALSE

Standard counterexample is the unit fn $x[n] = u[n]$ $\because \sum_{n=-\infty}^{\infty} |x[n]| \rightarrow \infty$.

d) TRUE

Now that $h[n] \rightarrow$ finite duration (say, n_1 to n_2) then $\sum_{k=n_1}^{n_2} |h[k]| < \infty$ (BOUNDED)

e) FALSE

Counterexample: $x[n] = u[n] \rightarrow$ Causal but NOT stable.

f) FALSE

Consider $S_1 : h_1[n] = \delta[n-k]$ and $S_2 : h_2[n] = \delta[n+k]$

causal non-causal

$\Rightarrow y[n] = h_1[n] * h_2[n] = \delta[n]$. causal

g) TRUE

$\delta \leftrightarrow \int_{-\infty}^{\infty} |h(t)|\, t < \infty \Rightarrow$ BOUNDED $\therefore$ Stable.

h) FALSE

Causality has nothing to do with past values being enforced to zero. It may be causal even for non-zero values for $n < 0$, provided it does NOT depend on 'future' values.