EE-229 (Signal Processing-I)

Homework-4

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Sol 1] T.P.T $\int_{-\infty}^{\infty} \text{sinc}(t)\,dt = 1$ and $\int_{-\infty}^{\infty} \text{sinc}^2(t)\,dt = 1$ where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

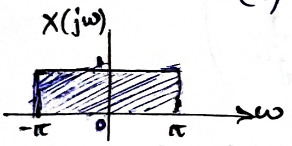
Proof: I use the following properties:

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

(Parseval's theorem)

and

$$\boxed{\frac{\sin(\pi t)}{\pi t} \xrightarrow{FT} X(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}}$$



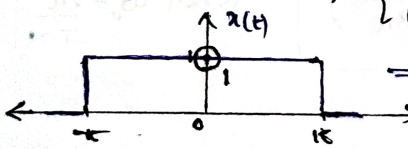
∵ sinc(t) is a real-valued fn, $\text{sinc}^2(t) = |\text{sinc}(t)|^2$. ∴ $\int_{-\infty}^{\infty} |\text{sinc}(t)|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, d\omega$

$$\Rightarrow \int_{-\infty}^{\infty} \text{sinc}^2(t)\,dt = \frac{1}{2\pi}\left(1\cdot\pi - 1\cdot(-\pi)\right)$$

$$= 2\pi/2\pi = 1$$

∴ Proved.

Now, consider $x(t) = \begin{cases} 1, & |t| \le \pi \\ 0, & |t| > \pi \end{cases}$



Now, $X(\omega) = \frac{2\sin(\omega\cdot\pi)}{\omega} \times \frac{\pi}{\pi} = 2\pi\,\text{sinc}(\omega)$.

⇒ clearly, $\boxed{x(0) = 1}$. ~~Ata~~ So, in the process of deriving x(t) back,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\,\text{sinc}(\omega)e^{j\omega t}d\omega$$

$$\Rightarrow x(t) = \int_{-\infty}^{\infty} \text{sinc}(\omega)e^{j\omega t}d\omega,$$

Put t = 0 !!

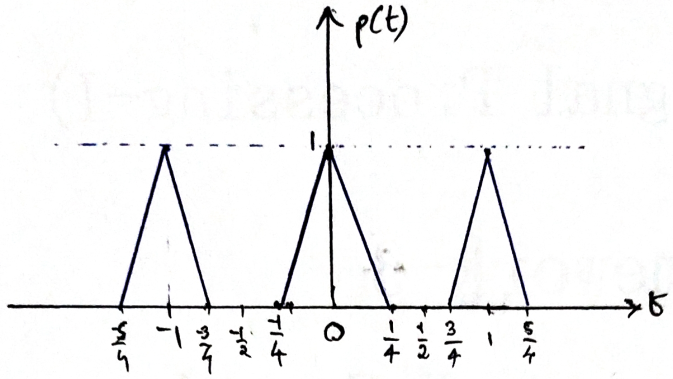
$$\Rightarrow x(0) = \int_{-\infty}^{\infty} \text{sinc}(\omega)e^{0}\,d\omega$$

$$= 1 = \int_{-\infty}^{\infty} \text{sinc}(t)\,dt.$$

∴ proved.

Sol.2] We know that FT of a triangular pulse is $\text{sinc}^2 f_n$, let's anyways derive its

-a]



$$p(t) = \begin{cases} 4t+1 & -\frac{1}{4} \leq t \leq 0 \\ -4t+1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(considering one period)

a) $$a_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-j\omega_0 kt}\, dt = \frac{1}{(1)}\left(\int_{-1/4}^{0} (4t+1)\, e^{-j\omega t} dt + \int_{0}^{1/4} (-4t+1)\, e^{-j\omega t} dt\right)$$ $\left(\because \int_{-1/2}^{-1/4} 0\, dt = 0 \text{ and } \int_{1/4}^{1/2} 0\, dt = 0\right)$

$$= \left[(4t+1)\frac{e^{-j\omega t}}{(-j\omega)} - \int 4 \cdot \frac{e^{-j\omega t}}{(-j\omega)} dt\right]_{-1/4}^{0} + \left[(-4t+1)\frac{e^{-j\omega t}}{(-j\omega)} - \int (-4)\frac{e^{-j\omega t}}{(-j\omega)} dt\right]_{0}^{1/4}$$

$$= \left(\frac{-(1+4t)e^{-j\omega t}}{j\omega} + \frac{4e^{-j\omega t}}{\omega^2}\right)_{-1/4}^{0} + \left(\frac{(4t-1)e^{-j\omega t}}{j\omega} - \frac{4e^{-j\omega t}}{\omega^2}\right)_{0}^{1/4}$$

$$= \left(\frac{-1}{j\omega} + \frac{4}{\omega^2} - \left(0 + \frac{4e^{+j\omega/4}}{\omega^2}\right)\right) + \left(0 - \frac{4e^{-j\omega/4}}{\omega^2} - \left(\frac{-4e^{-j\omega(0)}}{\omega^2} - \frac{e^{0}}{j\omega}\right)\right)$$

$$= \left(\cancel{\frac{-1}{j\omega}} + \frac{4}{\omega^2} - \frac{4e^{j\omega/4}}{\omega^2}\right) + \left(\frac{-4e^{-j\omega/4}}{\omega^2} + \frac{4}{\omega^2} + \cancel{\frac{1}{j\omega}}\right)$$

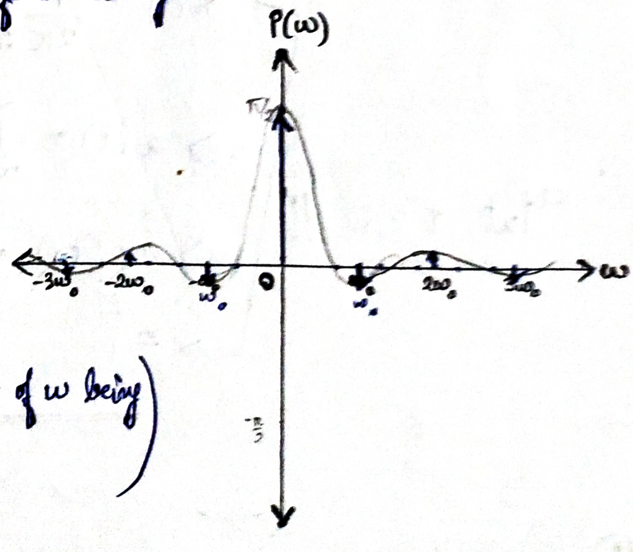
$$= 8/\omega^2 - \overset{4}{\cancel{8}}/\omega^2 \left(\frac{e^{j\omega/4} + e^{-j\omega/4}}{2}\right) 2 = 8/\omega^2 \left(1 - \cos\frac{\omega}{4}\right)\Big|_{\omega = k\omega_0}$$

$$\Rightarrow \underline{\underline{a_k}} = \frac{8}{\omega^2} \cdot 2\sin^2\frac{\omega}{8} = \frac{1}{4}\left(\frac{\sin(\omega/8)}{(\omega/8)}\right)^2 = \boxed{\frac{1}{4}\,\text{sinc}^2\left(\frac{k\omega_0}{8\pi}\right)}$$ where $\omega_0 = \frac{2\pi}{T = 1}$

b) Now that we've got Fourier coefficients, life is easy:

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\, \delta(\omega - k\omega_0)}$$

$$\therefore \underset{P(\omega)}{\cancel{X(j\omega)}} = \frac{\pi}{2}\sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k\omega_0}{8\pi}\right)\delta(\omega - k\omega_0)$$



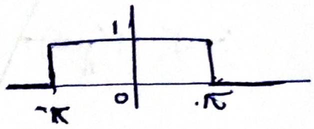
($\{a_k\}$ scaled by 2π and sampled at values of ω being integral multiples of ω_0 $(= 2\pi/1)$)

2.c) Let $X(j\omega)$ be the Fourier Transform of the aperiodic signal $x(t)$.

Then, for $y(t) = x(t)\cdot p(t)$, $Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\, P(j(\omega-\theta))\, d\theta$.

$$\therefore\ Y(\omega) = \frac{1}{2\pi}\cdot\frac{\pi}{2}\int_{-\infty}^{+\infty} X(\theta)\cdot\sum_{-\infty}^{\infty} \text{sinc}^2\left(\frac{k\omega_0}{8\pi}\right)\delta\left((\omega-\theta)-k\omega_0\right)d\theta\ .$$

2 d) For $x(t) = \text{sinc}(t)$, $X(j\omega) = \begin{cases} 1, & |\omega| \le \pi \\ 0, & |\omega| > \pi \end{cases}$

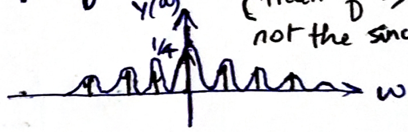


$\because$ $P(\omega)$ is a train of impulses ($\cancel{\delta(\omega)\pm}\ \delta(\omega - k\omega_0),\ k\in\mathbb{Z}$)

$\therefore$ when it is convolved with $X(j\omega)$,

it reproduces itself! $\therefore$ only change is the scaling by $1/2\pi$.

$$\therefore\ \boxed{Y(\omega) = \frac{1}{4}\sum_{k=-\infty}^{+\infty} \text{sinc}^2\left(\frac{k\omega_0}{8\pi}\right)\delta(\omega - k\omega_0)}$$



(train of $\uparrow$, not the sinc²)

Sol. 4]

LTI system: $x(t) * h(t) = y(t)$ and $X(j\omega)\cdot H(j\omega) = Y(j\omega)$.

Also, $x(t) = e^{-at}u(t) \xrightarrow{FT} X(j\omega) = \frac{1}{a+j\omega}$ $(Re(a) > 0)$.

$\therefore$ a) $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$. Now, $X(j\omega) = \frac{1}{1+j\omega} + \frac{1}{3+j\omega}$

and $Y(j\omega) = \frac{2}{1+j\omega} + \frac{(-2)}{4+j\omega}$

$$\Rightarrow H(j\omega) = \frac{3(3+j\omega)}{(2+j\omega)(4+j\omega)}\ .$$

b) Decomposing $H(j\omega)$ into partial fractions, $3\times\frac{(2+j\omega+4+j\omega)}{(2+j\omega)(4+j\omega)}\times\frac{1}{2}$

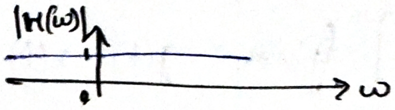
$$= \frac{3}{2}\left(\frac{1}{4+j\omega} + \frac{1}{2+j\omega}\right).$$

$$\therefore\ h(t) = \frac{3}{2}\left(e^{-4t} + e^{-2t}\right)u(t)$$

c) $\because \frac{Y(j\omega)}{X(j\omega)} = \frac{(9+3j\omega)}{8+6j\omega-\omega^2} \Rightarrow 8Y + 6j\omega\cdot Y - \omega^2\cdot Y = 9\cdot X + 3j\omega\cdot X$ $\xrightarrow{\text{Inv. FT}}$

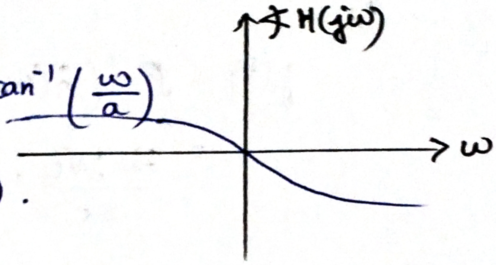
$$\Rightarrow 8y(t) + 6\frac{d}{dt}y(t) + \frac{d^2}{dt^2}y(t) = 9x(t) + 3\frac{d}{dt}x(t)\ .$$

Sol. 5] a) $|H(j\omega)| = \dfrac{|a-j\omega|}{|a+j\omega|} = \dfrac{\sqrt{a^2+\omega^2}}{\sqrt{a^2+\omega^2}} = 1$ $\therefore$



Now, $\measuredangle H(j\omega) = \measuredangle\left(\dfrac{a-j\omega}{a+j\omega}\right) = \tan^{-1}\left(\dfrac{-\omega}{a}\right) - \tan^{-1}\left(\dfrac{\omega}{a}\right)$

$= -2\tan^{-1}(\omega/a)$.



b) for $a=1$, $|H(j\omega)|=1$, $\measuredangle H(j\omega) = -2\tan^{-1}(\omega)$ $\Rightarrow \cancel{H(j\omega) = \dfrac{1-j\omega}{1+j\omega}}$.

$\therefore H(\omega) = |H(\omega)| e^{\phi(\omega)}$

$= e^{j\phi(\omega)} = e^{j(-2\tan^{-1}(\omega))}$.

Now, for $x(t) = \cos(t/\sqrt{3}) + \cos(t) + \cos(\sqrt{3}t)$

$= \dfrac{e^{jt/\sqrt{3}} + e^{-jt/\sqrt{3}}}{2} + \dfrac{e^{jt}+e^{-jt}}{2} + \dfrac{e^{j\sqrt{3}t} + e^{-j\sqrt{3}t}}{2}$

$\Rightarrow X(j\omega) = \cancel{2}\pi\left[\dfrac{\delta(\omega-1/\sqrt{3})+\delta(\omega+1/\sqrt{3})}{\cancel{2}} + \dfrac{\delta(\omega-1)+\delta(\omega+1)}{\cancel{2}} + \dfrac{\delta(\omega-\sqrt{3})+\delta(\omega+\sqrt{3})}{\cancel{2}}\right]$

$\therefore Y(j\omega) = X(j\omega)\cdot H(j\omega) = \pi e^{-2j\tan^{-1}\omega}\big(\delta(\omega-1/\sqrt{3}) + \delta(\omega+1/\sqrt{3}) + \delta(\omega-1) + \delta(\omega+1) + \delta(\omega-\sqrt{3}) + \delta(\omega+\sqrt{3})\big)$

$\therefore y(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} Y(j\omega)\cdot e^{j\omega t}\,d\omega$. But $\because \displaystyle\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$.

$\therefore y(t) = \dfrac{1}{2}\Big(e^{j(t/\sqrt{3} - 2\tan^{-1}(1/\sqrt{3}))} + e^{j(-t/\sqrt{3} - 2\tan^{-1}(-1/\sqrt{3}))} + e^{j(t/1 - 2\tan^{-1}1)} + e^{j(t/-1 - 2\tan^{-1}(-1))} + e^{j(\sqrt{3}t - 2\tan^{-1}(\sqrt{3}))} + e^{j(-\sqrt{3}t - 2\tan^{-1}(-\sqrt{3}))}\Big)$

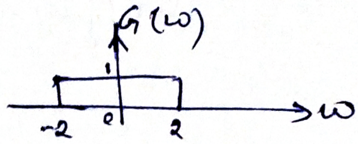
$= \dfrac{1}{2}\Big(2\cos\left(t/\sqrt{3} - 2\pi/6\right) + 2\cos\left(t/1 - \pi/4\times 2\right) + 2\cos\left(\sqrt{3}t - 2\pi/3\right)\Big)$

$y(t) = \cos\left(\dfrac{t}{\sqrt{3}} - \dfrac{\pi}{3}\right) + \cos\left(t - \dfrac{\pi}{2}\right) + \cos\left(\sqrt{3}t - \dfrac{2\pi}{3}\right)$.

Sol 6] Given, $g(t) = x(t)\cos(t)$ and $FT(g(t)) : G(\omega) = \begin{cases} 1 & |\omega| \le 2 \\ 0 & |\omega| > 2 \end{cases}$

a)

$$g(t) = \mathcal{F}^{-1}(G(\omega))$$



$$= \frac{\sin \overset{2}{\cancel{W}}t}{\pi t} = \frac{2\sin t\cos t}{\pi t}$$

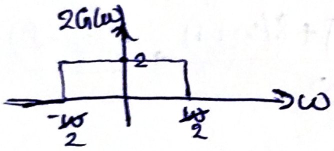
$$\therefore x(t) = \frac{g(t)}{\cos(t)} = \boxed{\frac{2\sin t}{\pi t}}$$

b) Modulation property of CTFT:

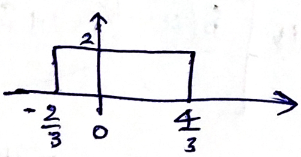
If $x(t) \xrightarrow{FT} X(\omega)$ then $x(t)\cos(\omega_0 t) \xrightarrow{FT} \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$.

$\therefore$ if $g(t) = x_1(t)\cos(2t/3)$ then $G(\omega) = \frac{1}{2}\left[X_1\left(\omega - \frac{2}{3}\right) + X_1\left(\omega + \frac{2}{3}\right)\right]$

$$\Rightarrow 2G(\omega) = X_1(\omega - 2/3) + X_1(\omega + 2/3).$$



$\therefore X_1(\omega) =$



$$= \begin{cases} 2 & , \omega \in \left[-\frac{2}{3}, \frac{4}{3}\right] \\ 0 & , \text{elsewhere} \end{cases}$$

Sol. 7] Considering Fourier transform of the entire eqn:

$$\Rightarrow j\omega Y(j\omega) + 10Y(j\omega) = X(j\omega)Z(j\omega) - X(j\omega) \quad (\because x * y \leftrightarrow X(\cdot)Y(\cdot))$$

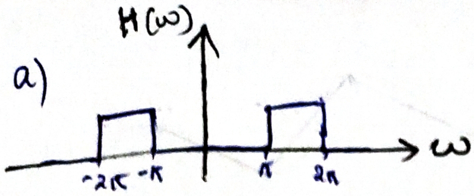
a) $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{Z(j\omega) - 1}{(10 + j\omega)}$, Now, $Z(j\omega) = \dfrac{1}{1+j\omega} + 3$

$$\therefore H(j\omega) = \frac{3 + 2j\omega}{(1+j\omega)(10+j\omega)} = \frac{1}{9}\cdot\frac{1}{1+j\omega} + \frac{17}{9}\cdot\frac{1}{10+j\omega}$$

b) $h(t) = \mathcal{F}^{-1}(H(j\omega))$

$$= \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t).$$

Sol. 8] a)



$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-2\pi}^{\pi} e^{j\omega t}\, d\omega + \int_{\pi}^{2\pi} e^{j\omega t}\, d\omega \right)$$

$$= \frac{1}{2\pi}\left(\int_{2\pi}^{\pi} e^{-j\tilde{\omega} t}(-d\tilde{\omega}) + \int_{\pi}^{2\pi} e^{j\omega t}\, d\omega \right) = \frac{1}{2\pi}\left(\int_{\pi}^{2\pi} e^{-j\omega t}\, d\omega + \int_{\pi}^{2\pi} e^{j\omega t}\, d\omega \right)$$

$$= \frac{1}{2\pi}\left(\int_{\pi}^{2\pi} 2\cos(\omega t)\, d\omega \right) = \frac{1}{\cancel{2}\pi} \left. \frac{\cancel{2}\sin \omega t}{t} \right|_{\pi}^{2\pi}$$

$$= \frac{1}{\pi t}\left(\sin(2\pi t) - \sin(\pi t) \right) = \underline{\underline{2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)}} \, .$$

b) NO!

BIBO stability $\equiv \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

But $\int_{-\infty}^{\infty} |2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)|\, dt = 2\int_{0}^{\infty} |2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)|\, dt$

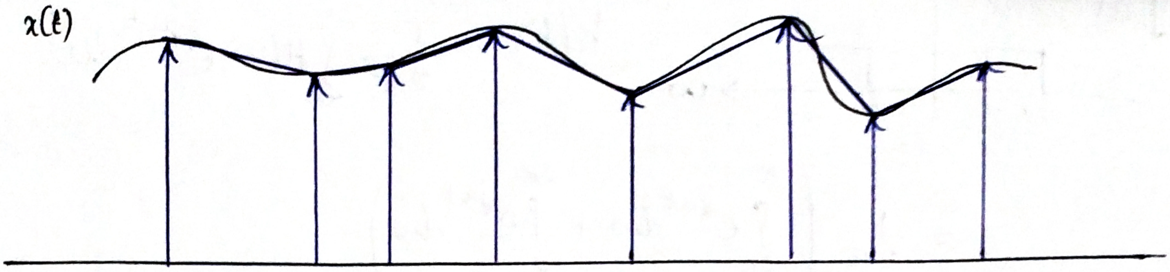
$\because$ |sinc| $\rightarrow$ unstable $\because$ it is an ideal filter,

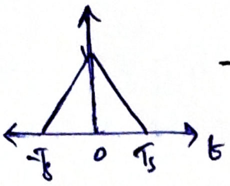
$\therefore 2\int_{0}^{\infty} |2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)|\, dt \geq \infty$

$\therefore$ Unstable. $\therefore$ proved.

Yes, $\because$ every ideal ~~f~~ low-pass filter is unstable.

Sol. 10] $x(t)$





$y(t)$

$$x_{FOH}(t) = \underbrace{\left[\sum x(kT_s)\,\delta(t-kT_s)\right]}_{\frac{1}{T_s}\cdot\tilde{x}(t)} * y(t)$$

$$\frac{1}{T_s}\tilde{x}(W) = \frac{1}{T_s}\sum X(\omega-k\omega_s)$$

Now, $$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-T_s}^{0}\left(\frac{t}{T_s}+1\right)e^{-j\omega t}dt + \int_{0}^{T_s}\left(\frac{-t}{T_s}+1\right)e^{-j\omega t}dt$$

$$= \int_{-T_s}^{T_s} e^{-j\omega t}dt + \frac{1}{T_s}\left(\int_{-T_s}^{0} te^{-j\omega t}dt + \int_{0}^{T_s} -te^{-j\omega t}dt\right)$$

$$= \frac{e^{-j\omega t}}{(-j\omega)}\Bigg|_{-T_s}^{T_s} + \frac{1}{T_s}\left(\int_{0}^{T_s} -te^{j\omega t}dt + \int_{0}^{T_s} -te^{-j\omega t}dt\right)$$

$$= \frac{-2j\sin(\omega T_s)}{-j\omega} + \frac{1}{T_s}\left(\int_{0}^{T_s} -t\times 2\cos(\omega t)dt\right)$$

$$= \frac{2\sin(\omega T_s)}{\omega} + \frac{1}{T_s}\left(-2\left(\frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}\right)\right)\Bigg|_{0}^{T_s}$$

$$= \cancel{\frac{2\sin(\omega T_s)}{\omega}} + \frac{(-2)}{\cancel{T_s}}\left(\cancel{\frac{T_s\sin(\omega T_s)}{\omega}} + \frac{\cos(\omega T_s)}{\omega^2} - \frac{1}{\omega^2}\right)$$

$$= \frac{+2}{\omega^2 T_s}\left(1-\cos(\omega T_s)\right) = \frac{4\sin^2\left(\frac{\omega T_s}{2}\right)}{\omega^2 T_s} = T_s\frac{\sin^2\left(\frac{\omega T_s}{2\pi}\times\pi\right)}{\left(\frac{\omega T_s}{2\pi}\times\pi\right)^2}$$

$$= T_s\,\mathrm{sinc}^2\left(\frac{\omega T_s}{2\pi}\right).$$

$$\therefore X_{FOH}(\omega) = \frac{1}{\cancel{T_s}}\sum X(\omega-k\omega_s)\times \cancel{T_s}\,\mathrm{sinc}^2\left(\frac{\omega T_s}{2\pi}\right)$$

$$= \mathrm{sinc}^2\left(\frac{\omega T_s}{2\pi}\right)\cdot\underbrace{\sum X(\omega-k\omega_s)}_{\text{Spectrum of } x(t)}.$$