Applied Linear Algebra

(Course Code: EE 635)

Department of Electrical Engineering
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Time: **3.00 Hours**

End-semester Examination

Total Points: 90

Instructions

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- Standard symbols and notations have their usual meanings.
- Answer all questions. You may attempt the questions in any order.
- If some information is missing, make suitable assumptions and state them.

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1. Prove or disprove (possibly through some counter-example) the following:

   (a) Components of the eigenvector for a matrix $A \in \mathbb{C}^{n\times n}$ vary continuously with the entries of A.

   (b) If (λ_k, v_k) is an eigenvalue-eigenvector pair for $A \in \mathbb{R}^{n\times n}$, then $\dfrac{v_k}{\lambda_k - \lambda} = (A - \lambda I)^{-1} v_k$, where λ is not an eigenvalue of A.

   (c) The non-zero matrix $A = uv^T$ for $u, v \in \mathbb{R}^n$ is diagonalizable if and only if $v^T u \neq 0$.

   (d) For $A \in \mathbb{R}^{n\times n}$, $\det(e^A) = e^{\text{trace}(A)}$.

   (e) If $A \in \mathbb{C}^{n\times n}$ is a diagonalizable matrix with characteristic polynomial $\chi_A(x) = (x-1)^{k_1}(x+1)^{k_2}x^{k_3}$, then the rank of A can be increased by adding or subtracting the identity matrix to it.

   (f) Suppose $A \in \mathbb{R}^{n\times n}$ has a characteristic polynomial given by $\chi_A(x) = (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2}$ and $\text{rank}(A - \lambda_1 I) = n - k_1$. Then A must be diagonalizable.

[3 × 6]

2. A common way of encoding messages is to associate distinct positive integers with each letter of the alphabet. For instance, a very basic encoding would be to choose $A \to 1, B \to 2, C \to 3, \ldots, Z \to 26$. However, this can be easily cracked if it falls into the hands of an adversary. To add a level of security, the messenger and receiver both agree upon a predetermined invertible matrix, say $A \in \mathbb{R}^{n\times n}$. Thereafter, the sender splits up the message (without spaces!) into chunks of n consecutive letters in the message leading to vectors in $\mathbb{R}^n$. If the total number of letters is not a multiple of n, the remaining spaces are filled by a place-holder number such as 27. Thereafter,

the matrix acts on each of the vectors and yields transformed vectors as the output, which are then sequentially transmitted to the receiver. The receiver then decodes this message by passing each of these transformed vectors it received through A^{-1}. Then the message is read by mapping the numbers to their corresponding letters.

Example: Suppose the message is 'I SEE' and the matrix is $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ (with $n = 2$). Then the sender, using an encoder machine, first converts the message into $\{9, 19, 5, 5\}$, splits it up into two vectors $v_1 = [9\ 19]^T$, $v_2 = [5\ 5]^T$, operates on v_1 and v_2 using A to get $y_1 = Av_1 = [-10\ 37]^T$ and $y_2 = Av_2 = [0\ 15]^T$, and then transmits y_1 and y_2 sequentially to the receiver. The receiver then recovers $v_1 = A^{-1}y_1$ and $v_2 = A^{-1}y_2$ using a decoder, which then reads the message by replacing the numbers with suitable letters.

Suppose it is known that an adversary uses this form of coding with $n = 3$ but the matrix is unknown to you. However, your spies have managed to steal a model of this encoder machine, used by the sender, and you can now experiment with this machine (without ripping it open!) to find out what this matrix is. You decide to send the message 'LET ME TRY' (without spaces, of course, and using 27 as the place-holder), and obtain a series of numbers at the output.

(a) Determine the matrix $A \in \mathbb{R}^{3\times 3}$ if the encoder gives an output sequence $\{17, -8, 37, 18, -7, 38, 43, -9, 70\}$ for the chosen message. Give an example of a message (not necessarily meaningful words), using which you will certainly not be able to determine the matrix A.

(b) Suppose your friend playfully decides to send you a message using this encoder, which you read as the sequence of numbers given by $\{21, 0, 30, 16, -1, 28, 23, 4, 28, 19, 0, 20, 19, 7, 24, 20, 1, 21\}$ at the output of the encoder. What is your friend trying to tell you? [9 + 9]

3. For each of the following short answers, show appropriate calculations/steps/justifications.

   (a) If u, v, w are orthonormal vectors in an inner product space, $\mathbb{V}$, then what is the value of $||u+v||^2 + ||v+w||^2 + ||w+u||^2$?

   (b) For $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \in \mathbb{Z}_5$, what are the eigenvalues and their corresponding algebraic multiplicities?

   (c) Which of the following matrices, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, is/are diagonalizable over $\mathbb{Z}_2$?

   (d) For a diagonalizable matrix, $A \in \mathbb{C}^{9\times 9}$, with characteristic polynomial given by $\chi(x) = (x-1)^3(x+1)^2(x+2)^4$, what is the rank of $A^T + I$?

   (e) What is the value of t so that the set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ t \end{bmatrix} \right\}$ is linearly dependent in $\mathbb{R}^3$?

(f) Suppose A is a self-adjoint operator on a finite dimensional inner product space over $\mathbb{C}$ and has only 2 and 3 (multiplicities unknown) as its eigenvalues. What is the minimal polynomial of A?

[3×6]

4. (*Competition vs cooperation*) Consider evolution of variables in continuous time for this problem

(a) Consider two species, say $\mathcal{P}$ and $\mathcal{Q}$, in an environment competing for the same resources. Thus, each species' population increases in proportion to its current population and decreases in proportion to that of its competitor. Suppose the constants of proportionality for both $\mathcal{P}$ and $\mathcal{Q}$ are 0.02 (for increase) and -0.01 (for decrease), respectively, while their initial populations are 10,000 and 20,000, respectively. Obtain the expressions for the population evolution of the two species. Does any of the two species go extinct within some finite time? If so, which one?

(b) Suppose two symbiotic species, say $\mathcal{R}$ and $\mathcal{S}$, inhabit the same environment so that each species' population increases in proportion to the current population of its symbiotic neighbour, while it decreases in proportion to its own current population. Further, suppose the two constants of proportionality (for both increase and decrease, respectively) are 0.01 and -0.01 for both the species. If the initial populations of $\mathcal{R}$ and $\mathcal{S}$ are 10,000 and 20,000, respectively, obtain the expressions for the population evolution of the two species. What can you say about the long run trend of either population?

[9+9]

5. (a) For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, establish that $\text{rank}(AB) = \text{rank}(B) - \dim(\text{Ker}(A) \cap \text{Im}(B))$.

(b) For the above matrices, show that $\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$.

(c) Establish that $\text{Im}(C^T C) = \text{Im}(C^T)$ and $\text{Ker}(C^T C) = \text{Ker}(C)$ for $C \in \mathbb{R}^{m\times n}$.

[6×3]