Qn 1

(a) Define a Sigma-algebra **(2 points)**

(b) Let A and B belong to a sigma-algebra $\mathcal{F}$. Use the definition above to show that $A \cap B^c \in \mathcal{F}$. **(3 points)**

(c) Let F be a sigma-algebra of subsets of Ω and suppose $B \in \mathcal{F}$. Show that $G = A \cap B : A \in \mathcal{F}$ is a sigma-algebra of subsets of B. **(5 points)**

Solution.

(a) A sigma-algebra, denoted by $\mathcal{F}$, on a set Ω is a collection of subsets of Ω that satisfies the following three properties:

- The empty set $\emptyset$ is in $\mathcal{F}$.
- If A is in $\mathcal{F}$, then its complement A^c is also in $\mathcal{F}$.
- If $A_1, A_2, A_3, \ldots$ is a countable sequence of sets in $\mathcal{F}$, then their union $\bigcup_{i=1}^{\infty} A_i$ is also in $\mathcal{F}$.

(b) We need to prove $A \cap B^c \in \mathcal{F}$

- As $A \in \mathcal{F}$, then by (property 2), A^c is also in $\mathcal{F}$. $B \in \mathcal{F}$ implies, $A^c \cup B \in \mathcal{F}$ by (property 3).
- As $A^c \cup B \in \mathcal{F}$, then $(A^c \cup B)^c \in \mathcal{F}$ by (property 2).
- By DeMorgan's laws, $(A^c \cup B)^c = A \cap B^c$.
- $(A \cap B^c) \in \mathcal{F}$

(c) Let F be a sigma-algebra of subsets of Ω, and suppose $B \in F$. We want to show that $G = \{A \cap B : A \in F\}$ is a sigma-algebra of subsets of B.

1. **Contains the Empty Set:**

$$\emptyset = \emptyset \cap B$$

Since $\emptyset \in F$ (property 1 of sigma-algebra), $\emptyset \cap B$ is in G.

2. **Closed under Complement:** Let E be in G, so $E = A \cap B$ for some A in F. Then, the complement of E is given by:

$$E^c = (A \cap B)^c = A^c \cup B^c$$

Since $A^c \in F$ (property 2 of sigma-algebra) and $B^c \in F$ (as $B \in F$), E^c is in G.

3. **Closed under Countable Unions:** Let $E_1, E_2, \ldots$ be a countable sequence in G. So, $E_i = A_i \cap B$ for each i, where $A_i \in F$.

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_i \cap B) = (\bigcup_{i=1}^{\infty} A_i) \cap B$$

Since $\bigcup_{i=1}^{\infty} A_i \in F$ (property 3 of sigma-algebra), $\bigcup_{i=1}^{\infty} E_i$ is in G.

Therefore, G is a sigma-algebra of subsets of B.

2(a): Suppose, $i < j$ and $m < n$. If $j < m$, then A_{ij} and A_{mn} are determined by distinct independent rolls, and are therefore independent. For the case $j = m$ we have that,

$$\begin{aligned}\mathbb{P}(A_{ij} \cap A_{jn}) &= \mathbb{P}(i^{th},\ j^{th} \text{ and } n^{th} \text{ rolls show same number}),\\ &= \sum_{r=1}^{6} \frac{1}{6} \mathbb{P}(j^{th} \text{ and } n^{th} \text{show } r | i^{th} \text{shows } r) = \frac{1}{36} = \mathbb{P}(A_{ij})\mathbb{P}(A_{jn}),\end{aligned}$$

[2 Marks]

as required. However, if $i \neq j \neq k$,

$$\mathbb{P}(A_{ij} \cap A_{jk} \cap A_{ik}) = \frac{1}{36} \neq \frac{1}{216} = \mathbb{P}(A_{ij})\mathbb{P}(A_{jk})\mathbb{P}(A_{ik}).$$

[3 Marks]

2(b):

$$\mathbb{P}(1^{\text{st}} \text{ shows } r \text{ and sum is } 7) = \frac{1}{36} = \mathbb{P}(1^{\text{st}} \text{ shows } r)\mathbb{P}(\text{sum is } 7).$$

[5 Marks]

Q3. A dart is thrown by a player at a circular target of a radius of three units. Assume that the dart is guaranteed to hit the circular target. Further, assume that the probability that the dart lands in a region of the target is proportional to the area of that region.

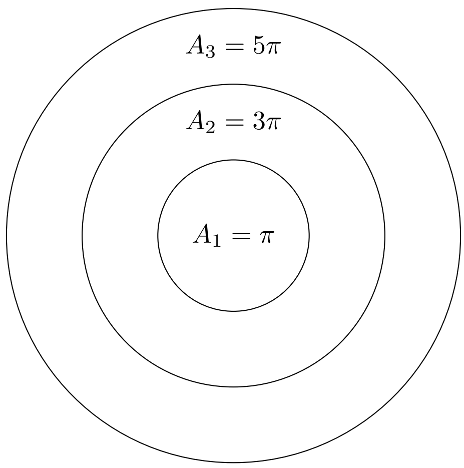
(a) The target is partitioned into three concentric annuli A_1, A_2, and A_3, where

$$A_k = \{(x, y) : k - 1 \leq \sqrt{x^2 + y^2} < k\}$$

The player scores an amount k if and only if the dart hits A_k. Let X be the random variable denoting the score. Characterize and plot the CDF of X.

(b) Let us revise the scoring method so that the player gets an amount equal to the distance between the hitting point and the center of the target. Let Y be the random variable denoting the score. Characterize and plot the CDF of Y.

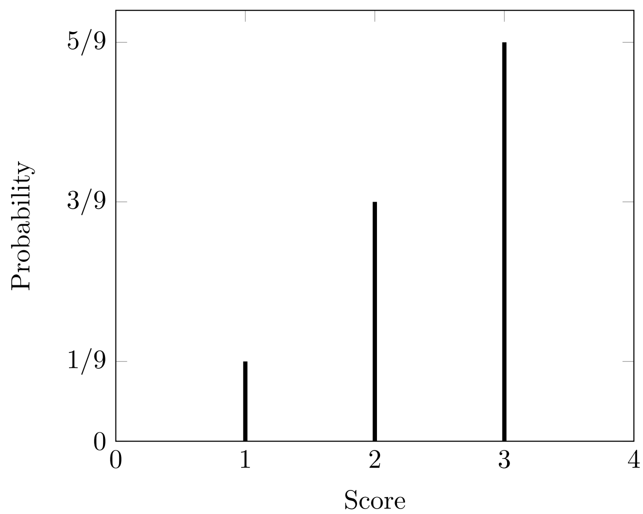
Ans a)



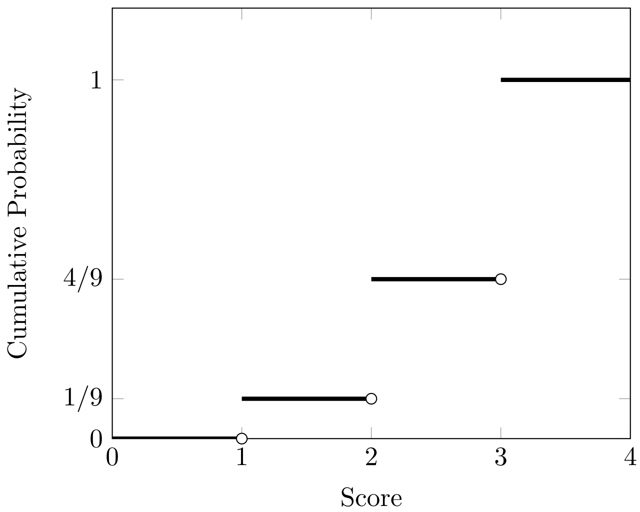
Since the probability of the dart landing in a region is proportional to the target, we have

$$\begin{aligned}
P(\text{Dart in } A_k) &\propto \text{Area of } A_k \\
\implies P(\text{Dart in } A_1) &\propto \text{Area of } A_1 = \pi * 1^2 = \pi \\
P(\text{Dart in } A_2) &\propto \text{Area of } A_2 = \pi * 2^2 - \pi * 1^2 = 3 * \pi \\
P(\text{Dart in } A_3) &\propto \text{Area of } A_3 = \pi * 3^2 - \pi * 2^2 = 5 * \pi
\end{aligned}$$

Also, total probability should be 1, which gives $P(\text{Dart in } A_1) = \frac{1}{9}, P(\text{Dart in } A_2) = \frac{3}{9}, P(\text{Dart in } A_3) = \frac{5}{9}$. Score is equal to the annuli that the dart landed on, hence $P(X = 1) = \frac{1}{9}, P(X = 2) = \frac{3}{9}, P(X = 3) = \frac{5}{9}$
PMF of the above function is:



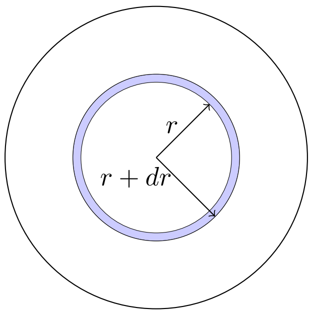
The CDF would be



Marking Scheme:

- 3 points for PDF calculation
  - 3 points for correct calculation of probabilities/PDF/CDF
  - 2 points if there is minor calculation mistake or in defining probability
  - 2 points if proportionality constant not calculated
  - 0 points otherwise
- 2 points for CDF plot
  - 2 points for correct CDF plot
  - 1 point if PDF is plot instead
  - 1 point is Score > 3 is missing
  - 1 point if labels on the CDF graph is incorrect
  - 1 point if lines/shape is incorrect

Ans b) Now , the score is defined to be the distance between the centre and the dart. The probability to receive a score r is the same as landing in a thin circle of radius $'r'$, as shown in the shaded region.



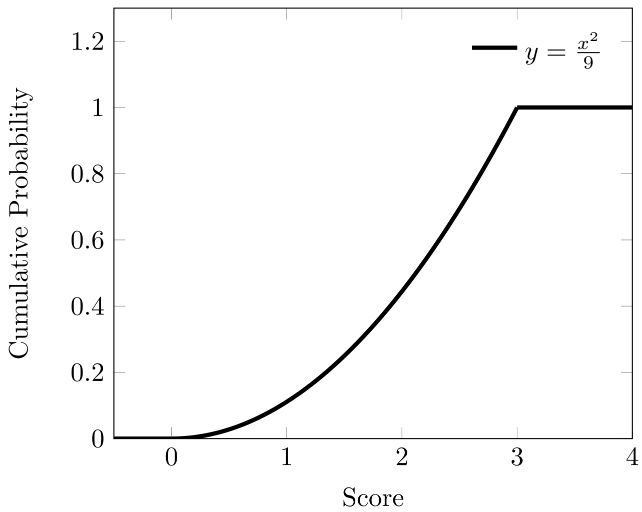
This implies $P(Y = r) \propto 2\pi r = 2\pi rk$. Since the score is now continuous, we have

$$\int_0^3 P(Y = r)dr = 1 \implies \pi(3^2)k = 1 \implies k = \frac{1}{9\pi}$$

The PDF of the score is $P(Y = r) = \frac{2r}{9}$. The CDF of the score would be

$$P(Y \leq r) = \int_0^r (P(Y = x)dx = \frac{r^2}{9} \text{ for } 0 \leq r \leq 3$$
$$P(Y \leq r) = 1 \text{ for } r \geq 3$$
$$P(Y \leq r) = 0 \text{ otherwise}$$

One could directly argue this CDF from the area of the circle of radius r and the area of the circle of radius 3. The plot of the CDf is:



Marking Scheme:

- 3 points for PDF calculation
  - 3 points for correct calculation of probabilities/PDF/CDF
  - 2 points if there is minor calculation mistake but CDF $\propto r^2$
  - 2 points if proportionality constant not calculated
  - 0 points otherwise
- 2 points for CDF plot
  - 2 points for correct CDF plot
  - 2 points for the CDF plotted of incorrect PDF calculated ($\frac{x^3}{27}$ or linear)
  - 0 points for plotting CDF/PDF of previous question plotted
  - 1.5 point if shape is incorrect
  - 1 point is Score >3 is missing
  - 1 point if labels on the CDF graph is incorrect

4. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $P(\{\omega_1\}) = P(\{\omega_2\}) = P(\{\omega_3\}) = 1/3$. Define random variables X, Y, and Z on Ω as follows:

$$X(\omega_1) = 1, X(\omega_2) = 2, X(\omega_3) = 3,$$
$$Y(\omega_1) = 2, Y(\omega_2) = 3, Y(\omega_3) = 1,$$
$$Z(\omega_1) = 2, Z(\omega_2) = 2, Z(\omega_3) = 1.$$

(a) Show that X and Y have the same probability mass functions. (2 points)
(b) Find the probability mass function of $U = X \times Y$. (2 points)
(c) Find $p_{Y|Z}$. (3 points)
(d) Find $p_{Z|Y}$. (3 points)

Solution

(a) Clearly $p_X(i) = p_Y(i) = \frac{1}{3}$ for $i = 1, 2, 3$, and other values are 0.
(b) $(XY)(\omega_1) = 2, (XY)(\omega_2) = 6, (XY)(\omega_3) = 3$, and therefore $p_{XY}(i) = \frac{1}{3}$ for $i = 2, 3, 6$, and other values are 0.
(c)

$$p_{Y|Z}(2 \mid 2) = \frac{P(Y = 2, Z = 2)}{P(Z = 2)} = \frac{P(\omega_1)}{P(\omega_1 \cup \omega_2)} = \frac{1}{2},$$

and similarly $p_{Y|Z}(3 \mid 2) = \frac{1}{2}, p_{Y|Z}(1 \mid 1) = 1$, and other values are 0 .
(d) Likewise $p_{Z|Y}(2 \mid 2) = p_{Z|Y}(2 \mid 3) = p_{Z|Y}(1 \mid 1) = 1$, and other values are 0 .

Marking scheme:

2 points each for reaching the final answer in parts (a) and (b). In part B, no partial marks awarded if the sample space of U is misinterpreted. For parts (c) and (d) partial marks are awarded only if steps are detailed out correctly.

EE325 Endsem Solutions

Autumn 2023

1 Question 5

1.1 Part a

Given $X \geq_{st} Y$ and Theorem 1, we always have random variables X' and Y' satisfying $F_X(x) = F_{X'}(x)$ and $F_Y(y) = F_{Y'}(y)$ **on the same sample space** such that $\mathbb{P}(X' \geq Y') = 1$. (1 mark)

Also $\mathbb{E}[X] = \mathbb{E}[X']$ and $\mathbb{E}[Y] = \mathbb{E}[Y']$. (1 mark)

The random variable $X' - Y'$ takes non-negative values with probability 1. Hence $\mathbb{E}[X' - Y'] \geq 0$ or $\mathbb{E}[X] = \mathbb{E}[X'] \geq \mathbb{E}[Y'] = \mathbb{E}[Y]$ (3 marks)

Alternatively

One can write $\mathbb{E}[X] = \int_0^\infty (1 - F_X(a))da - \int_{-\infty}^0 F_X(a)da$, also $\mathbb{E}[Y] = \int_0^\infty (1 - F_Y(a))da - \int_{-\infty}^0 F_Y(a)da$ if the expectations exist. (4 marks)

Given $X \geq_{st} Y$ we have $F_X(a) \leq F_Y(a)\ \forall a$. Combing the two we have $\mathbb{E}[X] \geq \mathbb{E}[Y]$. (1 mark)

Using integral of CCDF is the expected value is only true for positive random variables - partial marks (3 marks)

1.2 Part b

Let $P(X = a) = \frac{e^{-\lambda}\lambda^a}{a!}$ be denoted by $f_X^\lambda(a)$ and $F_X^\lambda(a) = \sum_{i=0}^{a} f_X^\lambda(i) = \sum_{i=0}^{a} \frac{e^{-\lambda}\lambda^i}{i!}$

Given $\lambda \geq \mu$, one can show $\exists i \in \mathbb{N} \cup \{0\}$ such that $f_X^\lambda(k) \leq f_Y^\mu(k)\ \forall k \leq i$ and $f_X^\lambda(k) > f_Y^\mu(k)\ \forall k > i$. (2 marks)

We clearly have $F_X^\lambda(a) \leq F_Y^\mu(a)\ \forall\ a \leq i$. Now assume $F_X^\lambda(b) > F_Y^\mu(b)$ for some $b > i$. Then we have

$$1 = \sum_{i=0}^{\infty} f_X^\lambda(i) = F_X^\lambda(b) + \sum_{i=b+1}^{\infty} f_X^\lambda(i) > F_Y^\mu(b) + \sum_{i=b+1}^{\infty} f_Y^\mu(i) = \sum_{i=0}^{\infty} f_Y^\mu(i) = 1$$

which is a contradiction, hence $F_X^\lambda(a) \leq F_Y^\mu(a)\ \forall a$ and thus $X \geq_{st} Y$ (3 marks)

Any valid way that shows $F_X^\lambda(a) \leq F_Y^\mu(a)\ \forall a$ will be awarded full marks (5 marks)

6. Let $X_1, X_2, X_3, ...$ be a sequence of independent and identically distributed random variables with a moment generating function $M_X(s)$. Let N be a random variable independent of $X_1, X_2, ...$ with a moment generating function $M_N(s)$. Let $S = X_1 + X_2 + ... + X_N$. Compute the moment generating function of S in terms of $M_X(s)$ and $M_N(s)$.

Solution: By the definition of a moment generating function we have

$$M_X(s) = \mathbb{E}(e^{X \cdot s})$$

$$M_N(s) = \mathbb{E}(e^{N \cdot s})$$

$$M_S(s) = \mathbb{E}(e^{S \cdot s}) \qquad \textbf{(2 marks)}$$

S depends on N conditionally which gives

$$\mathbb{E}(e^{S \cdot s}) = \mathbb{E}(\mathbb{E}(e^{S \cdot s}|N)) = \sum_n \mathbb{E}(e^{S \cdot s}|N = n)\mathbb{P}(N = n) \qquad \textbf{(2 marks)}$$

This gives us

$$M_S(s) = \sum_n \mathbb{E}(e^{S \cdot s}|N = n)\mathbb{P}(N = n)$$

$$= \sum_n \mathbb{E}(e^{(X_1 + X_2 + ... + X_n) \cdot s})\mathbb{P}(N = n)$$

$$= \sum_n \mathbb{E}(e^{X_1 \cdot s})...\mathbb{E}(e^{X_n \cdot s})\mathbb{P}(N = n) \qquad \textbf{(2 marks)}$$

$$= \sum_n \big(M_X(s)\big)^n \mathbb{P}(N = n)$$

$$= \mathbb{E}(M_X(s)^N)$$

Now $M_X(s)^N = e^{N \cdot \ln M_X(s)}$ which gives

$$\mathbb{E}(M_X(s)^N) = \mathbb{E}(e^{N \cdot \ln M_X(s)}) \qquad \textbf{(2 marks)}$$

$$= M_N(\ln M_X(s)) \qquad \textbf{(2 marks)}$$

Q7

- **Part a)** To show

$$|\mathbb{E}[XY]| \le \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

Consider the linear combination $Z = aX + bY$ where $a, b \in \mathbb{R}$. The second moment of Z is

$$\begin{aligned} \mathbb{E}[Z^2] &= \mathbb{E}[(aX + bY)^2] \\ &= \mathbb{E}[a^2X^2 + b^2Y^2 + 2abXY] \\ &= a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] + 2ab\mathbb{E}[XY] \end{aligned}$$

By definition $f(a, b) = \mathbb{E}[Z^2] \ge 0 \ \forall a, b \in \mathbb{R}$. Thus we have

$$a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] + 2ab\mathbb{E}[XY] \ge 0. \qquad (\textbf{2 marks})$$

Substituting $\frac{b}{a} = -\frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}$ we have

$$\begin{aligned} \mathbb{E}[X^2] + \left(\frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}\right)^2 \mathbb{E}[Y^2] - 2\frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}\mathbb{E}[XY] \ge 0 \\ \implies \mathbb{E}[X^2] - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}\mathbb{E}[XY] \ge 0 \end{aligned}$$

Thus we can conclude from above that

$$\begin{aligned} &\mathbb{E}[X^2]\mathbb{E}[Y^2] \ge (\mathbb{E}[XY])^2 \\ \implies &\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]} \ge |\mathbb{E}[XY]| \end{aligned} \qquad (\textbf{3 marks})$$

- **Part b)** To show

$$|\rho(X, Y)| \le 1$$

By definition we have

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

where $\mathrm{Cov}(X, Y)$ is the covariance given as $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ and σ_X, σ_Y are the standard deviation of X and Y resp. given as $\sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]}$ and $\sqrt{\mathbb{E}[(Y - \mathbb{E}[Y])^2]}$. Consider random variables $\bar{X} = X - \mathbb{E}[X]$ and $\bar{Y} = Y - \mathbb{E}[Y]$ then we have

$$\rho(X, Y) = \frac{\mathbb{E}[\bar{X}\bar{Y}]}{\sqrt{\mathbb{E}[\bar{X}^2]\mathbb{E}[\bar{Y}^2]}}. \qquad (\textbf{2 marks})$$

From Part 1, we have that

$$|\mathbb{E}[\bar{X}\bar{Y}]| \le \sqrt{\mathbb{E}[\bar{X}^2]\mathbb{E}[\bar{Y}^2]}. \qquad (1)$$

Using the definition and (1) we get

$$|\rho(X, Y)| \le 1 \qquad (\textbf{3 marks})$$

8. Let $U_1, U_2, \ldots$ be a sequence of independent random variables, with each being uniformly distributed over the interval $[0, 1]$, and let $X_n = \min\{U_1, \ldots, U_n\}$ for $n \geq 1$.

(a) Does $\{X_n\}$ converge in distribution? If yes, to what? (2 marks)

(b) Does $\{X_n\}$ converge in probability? If yes, to what? (2 marks)

(c) Does $\{X_n\}$ converge in the mean-squared sense? If yes, to what? (3 marks)

(d) Does $\{X_n\}$ converge almost surely? If yes, to what? (3 marks)

Solution

(a) $X_n \xrightarrow{d} 0$:

Note that

$$F_{U_n}(u) = \begin{cases} 0 & \text{if } u < 0, \\ 1 & \text{if } u \geq 1, \\ 1 - u & \text{if } 0 \leq u \leq 1. \end{cases}$$

Also,

$$\begin{aligned} F_{X_n}(x) = P(X_n \leq x) &= 1 - P(X_n > x) = 1 - P(U_1 > x, U_2 > x, \ldots, U_n > x) \\ &= 1 - P(U_1 > x)P(U_2 > x) \ldots P(U_n > x) \quad \text{(since } U_i\text{'s are independent)} \\ &= 1 - (1 - F_{U_1}(x))(1 - F_{U_2}(x)) \ldots (1 - F_{U_n}(x)) \\ &= 1 - (1 - x)^n. \end{aligned}$$

Therefore, we conclude

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Therefore, $X_n \xrightarrow{d} 0$. **(2 marks)**

(b) $X_n \xrightarrow{p} 0$:

Note that as we found in part (a)

$$F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - (1 - x)^n & \text{if } 0 \leq y \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

In particular, note that X_n is a continuous random variable. To show $X_n \xrightarrow{p} 0$, we need to show that

$$\lim_{n\to\infty} P(|X_n| \geq \epsilon) = 0, \quad \text{for all } \epsilon > 0.$$

Since $X_n \geq 0$, it suffices to show that

$$\lim_{n\to\infty} P(X_n \geq \epsilon) = 0, \quad \text{for all } \epsilon > 0.$$

For $\epsilon \in (0, 1)$, we have

$$P(X_n \geq \epsilon) = 1 - P(X_n < \epsilon) = 1 - P(X_n \leq \epsilon) \quad \text{(since } X_n \text{ is a continuous random variable)}$$
$$= 1 - F_{X_n}(\epsilon) = (1-\epsilon)^n.$$

Therefore,

$$\lim_{n\to\infty} P(|X_n| \geq \epsilon) = \lim_{n\to\infty} (1-\epsilon)^n = 0, \quad \text{for all } \epsilon \in (0, 1].$$

(2 marks)

(c) $X_n \xrightarrow{L^2} 0$:

By differentiating $F_{X_n}(x)$, we obtain

$$f_{X_n}(x) = \begin{cases} n(1-x)^{n-1} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E[|X_n|^2] = \int_0^1 nx^2(1-x)^{n-1}\,dx &\leq \int_0^1 nx(1-x)^{n-1}\,dx \\ &= -x(1-x)^n|_0^1 + \int_0^1 (1-x)^n\,dy \quad \text{(integration by parts)} \\ &= \frac{1}{n+1}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} E[|X_n|^2] = 0.$$

(3 marks)

(d) $X_n \xrightarrow{a.s.} 0$:

We will prove

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) < \infty,$$

which implies $X_n \xrightarrow{a.s.} 0$. By our discussion in part (b),

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) = \sum_{n=1}^{\infty} (1-\epsilon)^n = \frac{1-\epsilon}{\epsilon} < \infty \quad \text{(geometric series).}$$

(3 marks)

Marking scheme:

Partial marks($\approx$ 1 mark) are awarded if the final answer or the definitions of convergence are correct.

Problem 9. Let $\{X_i\}$ and $\{Y_i\}$ be independent Bernoulli processes with parameters p and q respectively.

(a) Consider the process $Z_i = \min\{X_i, Y_i\}$. What kind of process is $\{Z_i\}$? (3 points)

Solution.
Let's consider the probability distribution of Z_i.
Given that $\{X_i\}$ and $\{Y_i\}$ are independent processes:

$$\begin{aligned}\mathbb{P}(Z_i = 1) &= \mathbb{P}(\min\{X_i, Y_i\} = 1) \\ &= \mathbb{P}(X_i = 1 \text{ and } Y_i = 1) \\ &= \mathbb{P}(X_i = 1) \times \mathbb{P}(Y_i = 1) \\ &= p \times q\end{aligned}$$

[+1]

$$\begin{aligned}\mathbb{P}(Z_i = 0) &= \mathbb{P}(\min\{X_i, Y_i\} = 0) \\ &= \mathbb{P}(X_i = 0 \text{ or } Y_i = 0) \\ &= 1 - \mathbb{P}(Z_i = 1) \\ &= 1 - p \times q\end{aligned}$$

[+1]

Therefore, the probability distribution of the process $\{Z_i\}$ is:

$$\begin{aligned}\mathbb{P}(Z_i = 1) &= pq \\ \mathbb{P}(Z_i = 0) &= 1 - pq\end{aligned}$$

$\{Z_i\}$ is also a Bernoulli process but with a parameter equal to the product of the component parameters p and q. [+1]

□

(b) Is $\{Z_i\}$ strict sense stationary (SSS)? (3 points)

Solution. Consider time instants i and j, such that $j = i + n,\ \forall\ n \leq N$:

$$\begin{aligned}Z_i &= \min\{X_i, Y_i\} \\ Z_j &= \min\{X_j, Y_j\}\end{aligned}$$

[+1]

X_i, Y_i, X_j and Y_j are independent of each other, hence, Z_i and Z_j are independent of each other. Also, X_i and X_j share the parameter p and, Y_i and Y_j share the parameter q. [+1]

Therefore,

$$\text{PMF}(Z_i) \equiv \text{PMF}(Z_j)$$

Thus, $\{Z_i\}$ is N^{th} order stationary. Since the above argument holds for all $N \in \mathbb{N}$, $\{Z_i\}$ is strict sense stationary. [+1]

□

(c) For the process $\{X_i\}$, what is the probability that the k^{th} success occurs before the j^{th} failure? (4 points)

Solution.

We want at least k successes to have occurred before the j^{th} failure occurs, which implies at most $(j-1)$ failures can occur before the k^{th} success.

Consider the case where the k^{th} success occurs after exactly n failures. The corresponding probability $\mathbb{P}_{n,k}$ is obtained by multiplying the probability of observing exactly $k-1$ successes and n failures by the probability of getting the k^{th} success immediately after:

$$\begin{aligned}\mathbb{P}_{n,k} &= \left(\binom{k-1+n}{k-1} \cdot p^{k-1} \cdot (1-p)^n\right) \times p \\ &= \binom{k-1+n}{k-1} \cdot p^k \cdot (1-p)^n\end{aligned}$$

[+2]

The required quantity is then observed by summing $\mathbb{P}_{n,k}$ for the failures n varying from 0 to $(j-1)$:

$$\sum_{n=0}^{j-1} \mathbb{P}_{n,k} = \sum_{n=0}^{j-1} \binom{k-1+n}{k-1} \cdot p^k \cdot (1-p)^n$$

[+2]

□

EE 325
Probability and Random Processes

Endsem Q.10

- A random walk is a discrete-time random process with $X_0 = 0$, and the update rule,

$$X_{n+1} = X_n + U_n, \quad n \in \mathbb{N} \cup \{0\},$$

where the U_n's are independent and identically distributed random variables with mean μ and variance σ^2.

1. Prove that $X_t - X_s = \sum_{i=s}^{t} U_i$. (3 Points)

Solution: The random walk has the "independent increment property". To see this, note that for $s, t \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} X_t &= X_t - X_0 \\ X_t &= (X_t - X_s) + (X_s - X_0) \\ \implies X_t - X_s &= (X_t - X_0) - (X_s - X_0) \\ X_t - X_s &= \sum_{i=s}^{t} U_i \end{aligned}$$

which is a function of $U_s, \cdots, U_{t-1}$, while $X_s - X_0$ is a function of $U_0, \cdots, U_{s-1}$. Since U is an i.i.d. RV, the independent increment property holds.

2. Compute $\mu_X(t)$. (2 Points)

Solution: $\mathbb{E}[U_n] = \mu$. Hence, $\mu_X(t) = \mathbb{E}[X_t] = t\mu$.

3. Compute $R_X(t, s)$. (3 Points)

Solution:

$$\begin{aligned} R_X(t,s) &= \mathbb{E}[X_t - X_s] \quad \text{(assume WLOG that } t > s) \\ &= \mathbb{E}[(X_t - X_0)(X_s - X_0)] \\ &= \mathbb{E}[(X_s - X_0)(X_t - X_s + Xs - X0)] \\ &= \mathbb{E}[(Xs - X0)(Xt - Xs)] + \mathbb{E}[(X_s - X_0)^2] \\ &= \mathbb{E}[X_s - X_0]E[X_t - X_s] + E[(X_s - X_0)^2] \\ &= st\mu^2 + s\sigma^2. \end{aligned}$$

4. Is this process WSS? (2 Points)

Solution: For WSS,

(a) $\mu_X(t) = \mu_X, \forall t \in \mathbb{R}$

(b) $R_X(t, s) = R_X(t - s), \forall t, s \in \mathbb{R}$.

It is immediate that if $\mu \neq 0$ or $\sigma \neq 0$, then the random walk above is not stationary.