EE 229 **Spring 2023-24**

HW 1

Assigned: 10/08/23 Due: 17/08/23

You are encouraged to discuss these problems with others, but you need to write up the actual solutions alone. Start early, and come to office hours (TBA) with any doubts. Your only have to submit your solutions to **the questions marked** [†]. *Drop off your submission in the dropbox labeled EE 229 in the EE office by 5.30 pm on the due date.*

Preliminaries

A signal $x(t)$ is said to be *even* if $x(t) = x(-t)$ for all t, and *odd* if $x(t) = -x(-t)$ for all t. An arbitrary signal $x(t)$ can be decomposed as a sum of an even and an odd signal as follows:

$$x(t) = x_e(t) + x_o(t),$$

where

$$x_e(t) = \frac{x(t) + x(-t)}{2},$$
$$x_o(t) = \frac{x(t) - x(-t)}{2}.$$

x_e is referred to as the *even part* of x, and x_o is referred to as its *odd part*.

Note: Is the above decomposition of a signal as a sum of an even and an odd signal unique?

Note: While the above definitions are given for continuous-time signals, it is easy to write out the corresponding versions for discrete-time signals.

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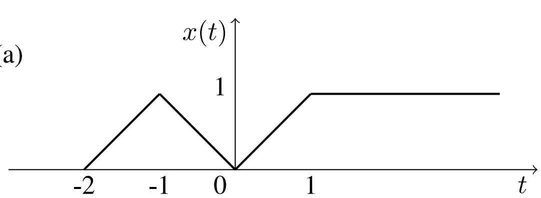
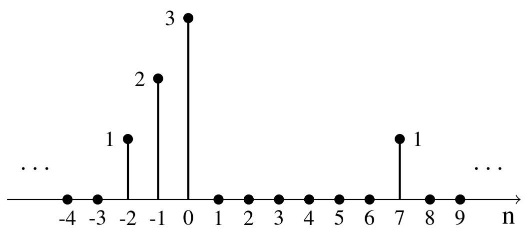
1. [†] Carefully sketch the following signals. Mark all the critical points.
   (a) $g(t) = tu(-t-1) - u(-t-1)$
   (b) $h(t) = e^{-tu(t)}, \quad -1 \leq t \leq 1$
   (c) $m(t) = \left(\dfrac{\sin\left[\frac{\pi}{2}(t-2)\right]}{t^2+4} \right) \delta(1-t)$
   (d) $p[n] = (\frac{1}{2})^n u[n-3]$
2. Determine and sketch the even and odd parts of the signals depicted in figures below. Label your sketches carefully.

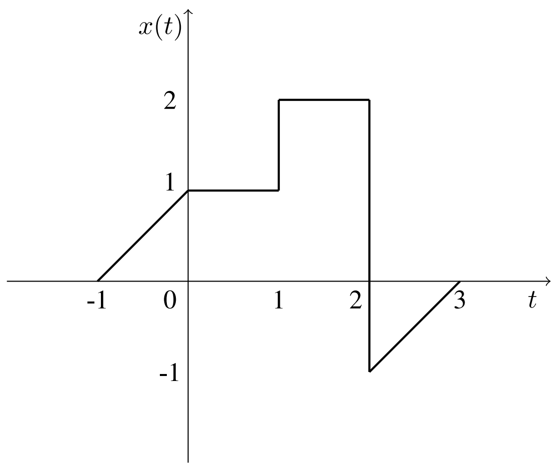
(a)



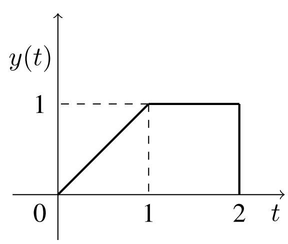
(b)



3. For the continuous time signal $x(t)$ shown below, sketch and label carefully each of the following signals: (a) $x(t-1)$, (b) $x(2-t)$, (c) $x(2t+1)$, (d) $[x(t)+x(-t)]u(t)$.



4. [†] Consider the signal $y(t) = (1/5)x(-2t-3)$ shown below. Determine and carefully sketch the original signal $x(t)$.



5. [†] A system is said to be *invertible* if distinct input signals always result in distinct output signals. In other words, for an invertible system, the input signal corresponding to any given output signal can be uniquely determined.

   Identify whether each of the following systems is invertible. (x denotes the input signal, and y the output signal.) Justify your answers.

   (a) $y[n] = \sum_{k=-\infty}^{n} x[k]$
   (b) $y[n] = x[n]x[n-2]$
   (c) $y[n] = nx[n]$
   (d) $y[n] = x[2n]$

6. [†] Check if the following systems are linear, time-invariant and causal. (x denotes the input signal, and y the output signal.) Justify your answers.

   (a) $y(t) = \sin(x(t) - x(0))$
   (b) $y(t) = x(\sin(t))$
   (c) $y(t) = \max_{s \in [t-1,t]}\{x(s)\}$

(d) $y(t) = x(t/3)$

7. [†] This is a two-part question on 'system construction.'

   (a) Construct a discrete-time system that has memory, is causal, stable and shift-invariant, but not linear.

   (b) Construct a continuous-time system that has memory, is causal, linear and unstable, but not shift-invariant.

8. [†] Let $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$ and $h[n] = 2\delta[n+1] + 2\delta[n-1]$. Compute and plot each of the following convolutions.

   (a) $y_1[n] = x[n] \star h[n]$

   (b) $y_2[n] = x[n+2] \star h[n]$

   (c) $y_3[n] = x[n] \star h[n+2]$

9. [†] Let

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{elsewhere} \end{cases}$$

and

$$h[n] = \begin{cases} 1, & 0 \le n \le N \\ 0, & \text{elsewhere} \end{cases}$$

where $N \le 9$ is an integer. Determine the value of N, given that $y[n] = x[n] \star h[n]$, $y[4] = 5$ and $y[14] = 0$.

EE 229 **Autumn 2023**

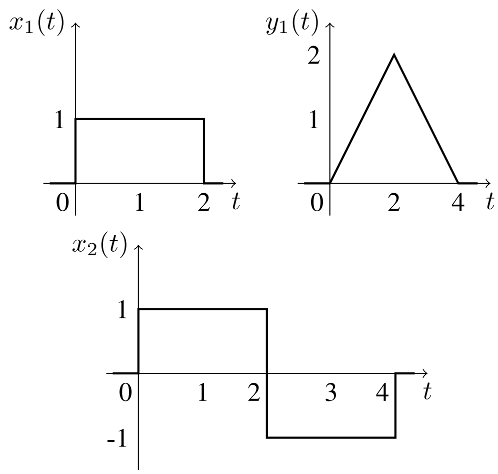
HW 2

Assigned: 24/08/23 Due: 31/08/23

You are encouraged to discuss these problems with others, but you need to write up the actual solutions alone. Start early, and come to office hours with any doubts. Your only have to submit your solutions to **the questions marked** [†]. *Drop off your submission in the dropbox labeled EE229 in the EE office by 5.30 pm on the due date.*

1. [†] Let $x(t) = u(t-3) - u(t-5)$, $h(t) = e^{-3t}u(t)$. Compute $(x \star h)(t)$

2. Consider an LTI system whose response to the signal $x_1(t)$ in figure below is the signal $y_1(t)$ illustrated below. Determine and sketch carefully the response of the system to the input $x_2(t)$ shown below.



3. The goal of this exercise is to verify the sifting property of the unit impulse in continuous time. Consider the impulse approximation
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & t \in [0, \Delta] \\ 0 & \text{elsewhere} \end{cases}.$$
For $x(t) = a + bt + ct^2$, compute $\hat{x}(1) := \int_{-\infty}^{\infty} x(s)\delta_\Delta(1-s)ds$.

   Compute now the error between $x(1)$ and $\hat{x}(1)$. What happens to the error as $\Delta \downarrow 0$?

4. The discrete time signals x and y have finite support. Specifically, $x[n]$ is zero outside $n_1 \le n \le n_2$, $y[n]$ is zero outside $n_3 \le n \le n_4$.

   Prove that $(x \star y)[n]$ is zero outside $n_1 + n_3 \le n \le n_2 + n_4$.

5. Consider the continuous time system defined by the LCCDE
$$\frac{d\,y(t)}{dt} + y(t) = x(t).$$
   (a) Under the initial condition $y(0) = 1$, show that the system is neither linear nor time-invariant.
   (b) Under the initial condition $y(0) = 0$, show that the system is linear but not time-invariant.

6. Consider a continuous time system described by the LCCDE

$$\sum_{k=1}^{N} a_k \frac{d^k\, y(t)}{dt^k} = \sum_{k=1}^{M} b_k \frac{d^k\, x(t)}{dt^k}$$

and the *initial rest* condition. The input signals x_1 and x_2 satisfy the property that $x_i(t) = 0\ \forall t < t_i$. Let y_1 and y_2 denote the corresponding output signals.

   (a) Show that the output signal corresponding to the input $\alpha x_1(t) + \beta x_2(t)$ is $\alpha y_1(t) + \beta y_2(t)$.

   (b) Show that the output signal corresponding to the input signal $x_1(t-\kappa)$ is $y_1(t-\kappa)$.

   Note: This almost verifies that the system under consideration is LTI. Why almost?

7. Consider a system defined by the linear constant coefficient difference equation

$$\sum_{k=0}^{N} y[n-k] = \sum_{k=0}^{M} x[n-k],$$

along with intitial condition of initial rest:

$$x[n] = 0\ \forall\, n < n_0 \Rightarrow y[n] = 0\ \forall\, n < n_0.$$

Show that over the space of signals of the form $\{x:\ x[n] = 0\ \forall\, n < n_0\}$, the system is causal.

8. Argue that $\delta(at) = \frac{1}{|a|}\delta(t)$.

9. [†] Perform the following convolutions where $\star$ indicates convolution.

   (a) For $u(t)$ a unit step function, find $r(t) = u(t) \star u(t)$.

   (b) Find $x(t) \star h(t)$, where $h(t) = (-e^{-t} + 2e^{-2t})u(t)$ and $x(t) = 10e^{-3t}u(t)$.

   (c) Find the output $y(t)$ of an LTI system with impulse response $h(t) = 2e^{-2t}u(t)$ when excited with an input $x(t)$ given by

$$x(t) = \begin{cases} 1, & 2 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}.$$

10. [†] Given that $f(t) \star g(t) = y(t)$, where $\star$ denotes convolution,

   (a) Find $f(t-T_1) \star g(t-T_2)$, for some finite-valued real numbers T_1 and T_2.

   (b) Use the result of (a) and the fact that $u(t) \star u(t) = r(t)$, to find $(u(t+1) - u(t-2)) \star (u(t-3) - u(t-4))$. Verify the result graphically.

11. [†] Given $y(t) = f(t) \star g(t)$, derive a general formula to compute $f(ct) \star g(ct), c \neq 0$. Hence, if $f(t) = u(t+1) - u(t-2)$ and $g(t) = r(t)(u(t) - u(t-1))$, find $f(2t) \star g(2t)$.

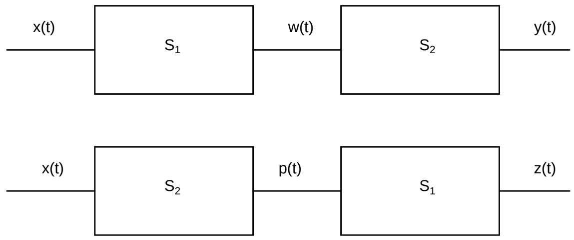
12. Given below are the impulse response of some systems. Determine whether the systems are (a) Stable (b) Causal.

   (a) $h(t) = e^{-(t+2)}u(t)$.

   (b) $h(t) = e^{-|t|}$.

   (c) $h(t) = \delta(t) + \delta(t-3)$.

13. [†] Let $x(t) = e^{-2t}u(t)$. The system S_1 is described by $y(t) = x(2t)$ and the system S_2 has an impulse response $h(t) = e^{-t}u(t)$. Find the output for the following two cascaded connections. Are the outputs expected to be the same in both cases?



14. [†] Determine whether each of the following statements concerning LTI systems is true or false. Justify your answers.

(a) If $h(t)$ is the impulse response of an LTI system, and h(t) is periodic and nonzero, the system is unstable.

(b) The inverse of a causal LTI system is always causal.

(c) If $|h[n]| < K$ for each n, where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

(d) If a discrete-time LTI system has a impulse response $h[n]$ of finite duration, the system is stable.

(e) If an LTI system is causal, it is stable.

(f) The cascade of a noncausal LTI system with a causal one is necessarily noncausal.

(g) A continuous-time LTI system is stable if and only if its step response $s(t)$ is absoultely integrable, that is,

$$\int_{-\infty}^{+\infty} |s(t)|dt < \infty$$

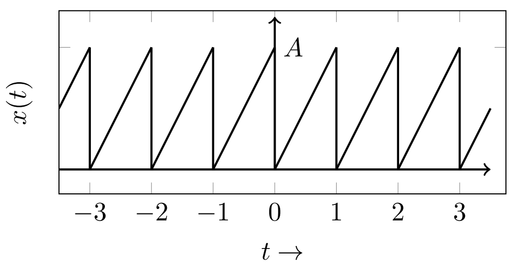
(h) A discrete-time LTI system is causal if and only if its step response $s[n]$ is zero for $n < 0$.

EE 229 **Spring 2023**

HW 3

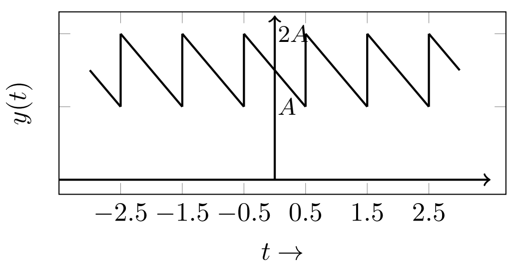
Assigned: 14/09/23 Due: No need to submit

1. [†] A periodic signal $x(t)$ is given below.



   (a) Determine its Fourier Series coefficients. Sketch its magnitude and phase spectrum.

   (b) Using the results in part (a) above and without doing elaborate integrations, determine the coefficients of the Fourier series of the periodic signal $y(t)$ shown below. Sketch the magnitude and phase spectrum.



2. A 2π periodic signal $x(t)$ is specified over one period as

$$x(t) = \begin{cases} \dfrac{t}{A} & 0 \le t < A \\ 1 & A \le t < \pi \\ 0 & \pi \le t < 2\pi \end{cases}$$

   Represent the function as a Fourier series.

3. The Fourier series coefficients of a periodic signal $x(t)$ is given by

$$d_k = \begin{cases} jk, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$$

   The fundamental period of the signal is T_0 = 4. Determine the signal $x(t)$.

4. Suppose we are given the following information about signal $x(t)$:

   i) $x(t)$ is a real signal

   ii) $x(t)$ is periodic with period T=6 and has Fourier coefficients a_k

   iii) a_k=0 for $k = 0$ and $k > 2$

iv) $x(t) = -x(t-3)$

v) $\frac{1}{6}\int_{-3}^{3}|x(t)|^2 dt = \frac{1}{2}$

vi) a_1 is a positive real number

Show that $x(t) = A\cos(Bt + C)$ and determine the value of constants A, B and C.

5. [†] Consider the following three discrete-time signals with fundamental period of 6:

$$x[n] = 1 + \cos\left(\frac{2\pi n}{6}\right), \quad y[n] = \sin\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right), \quad z[n] = x[n]y[n]$$

(a) Find the Fourier series coefficients of $x[n]$
(b) Find the Fourier series coefficients of $y[n]$
(c) Find the Fourier series coefficients of $z[n]$

6. [†] Consider a causal continuous-time LTI system whose input $x(t)$ and output $y(t)$ are related by the following differential equation:

$$\frac{d}{dt}y(t) + 4y(t) = x(t)$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs.

(a) $x(t) = \cos(2\pi t)$
(b) $x(t) = \sin(4\pi t) + \cos(6\pi t + \pi/4)$

7. [†] Consider the periodic impulse train in continuous time $x(t) = \sum_{k=-\infty}^{\infty}\delta(t - kT)$. Plot the approximations obtained by truncating the Fourier series of $x(\cdot)$. Specifically, plot $\sum_{k=-N}^{N} a_k e^{j(\frac{2\pi}{T})kt}$ for $N = 5$, 10, 100, where $\{a_k\}$ denotes the Fourier series of x. Interpret your plots.

8. [†] Consider a continuous-time LTI system $\mathcal{S}$ whose frequency response is

$$H(\omega) = \begin{cases} 1 & |\omega| \geq 250 \\ 0 & \text{otherwise} \end{cases}.$$

The input to this system is a signal $x(t)$ with fundamental period $T = \pi/7$, and Fourier series coefficients a_k. If the output signal $y(t) = x(t)$, for which values of k is a_k guaranteed to be zero?

9. [†] Consider a discrete-time LTI system with impulse response

$$h[n] = \begin{cases} 1 & 0 \leq n \leq 2 \\ -1 & -2 \leq n \leq -1 \\ 0 & \text{otherwise} \end{cases}.$$

Given that the input to this system is

$$x[n] = \sum_{k=-\infty}^{\infty}\delta[n - 4k],$$

determine the Fourier series coefficients of the output $y[n]$.

10. Consider two discrete-time signals $x[n]$ and $y[n]$ that are both periodic with period N. Suppose that

$$x[n] \xrightarrow{\text{DTFS}} a_k, \quad y[n] \xrightarrow{\text{DTFS}} b_k.$$

Exploit the duality principle to show that the property

$$x[n]y[n] \xrightarrow{\text{DTFS}} \sum_{l \in <N>} a_l b_{k-l}$$

implies the property

$$\frac{1}{N}\sum_{l \in <N>} x[l]y[n-l] \xrightarrow{\text{DTFS}} a_k b_k.$$

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HW 4

Assigned: 16/10/23 Due: 23/10/23

You are encouraged to discuss these problems with others, but you need to write up the actual solutions alone. Your only have to submit your solutions to **the questions marked** [†]. *Drop off your submission in the dropbox labeled EE229 in the EE office by 5.30 pm on the due date.*

1. [†] Prove that $\int_{-\infty}^{\infty} \mathrm{sinc}(t)dt = \int_{-\infty}^{\infty} \mathrm{sinc}^2(t)dt = 1$. (Recall that $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.)

2. [†] Let $p(t)$ denote the periodic triangular pulse train as shown in Fig. 2.

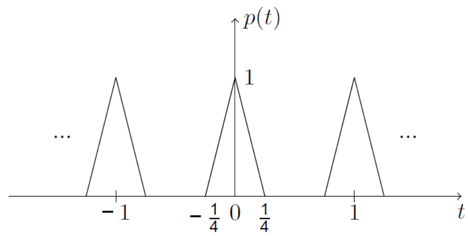

Figure 1: $p(t)$

   (a) Compute the Fourier series coefficients p_k corresponding to $p(t)$
   (b) Compute and sketch the Fourier transform $P(\omega)$ corresponding to $p(t)$
   (c) Let $x(t)$ be an aperiodic signal. Define $y(t) = x(t)p(t)$. Obtain an expression for $Y(\omega)$.
   (d) Sketch $Y(\omega)$ when $x(t) = \mathrm{sinc}(t)$.

3. Let $X(\omega)$ be the Fourier transform of the signal $x(t)$ shown in Figure 4. Do the following computations without explicitly evaluating $X(\omega)$.

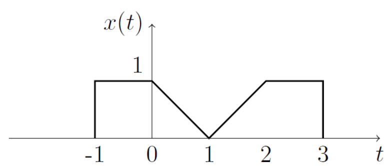

Figure 2: $x(t)$ for Problem 7

   (a) $X(0)$
   (b) $\int_{-\infty}^{\infty} X(\omega)d\omega$
   (c) $\int_{-\infty}^{\infty} X(\omega)e^{j\omega}d\omega$
   (d) $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$
   (e) Sketch the inverse Fourier transform of $\mathrm{Real}(X(\omega))$. (Here, $\mathrm{Real}(a)$ denotes the real part of a.)

4. [†] Consider an LTI system whose response to the input $x(t) = [e^{-t} + e^{-3t}]u(t)$ is $y(t) = (2e^{-t} - 2e^{-4t})u(t)$.

   (a) Determine the frequency response of this system.

(b) Determine the impulse response.

(c) Find the differential equation relating the input to the output.

5. [†] The goal of this problem is to understand the effect of non-linear phase response. Consider the LTI system with frequency response

$$H(\omega) = \frac{a - j\omega}{a + j\omega}.$$

Here, $a > 0$.

(a) Sketch the magnitude and phase response of the system.

(b) Setting $a = 1$, determine the output of the system to the input

$$\cos(t/\sqrt{3}) + \cos(t) + \cos(\sqrt{3}t).$$

6. [†] Suppose $g(t) = x(t)\cos(t)$ and the Fourier transform of $g(t)$ is

$$G(\omega) = \begin{cases} 1 & |\omega| \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Determine $x(t)$

(b) Specify the Fourier transform of $x_1(t)$ such that $g(t) = x_1(t)\cos(2t/3)$

7. [†] The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$y'(t) + 10y(t) = \int_{-\infty}^{\infty} x(s)z(t-s)ds - x(t),$$

where $z(t) = e^{-t}u(t) + 3\delta(t)$.

(a) Find the frequency response of this system.

(b) Find the impulse response of this system.

8. [†] Consider the ideal (continuous-time) bandpass filter with passband $[\pi, 2\pi]$. Specifically, this filter has frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \in [\pi, 2\pi] \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Obtain the impulse response corresponding to this system.

(b) Is this system BIBO stable? Give a clear justification for your answer.

Based on your solution, can you comment on the stability of the ideal low pass filter?

9. Signals $x_1(t)$ and $x_2(t)$ are bandlimited with bandwidths $\bar{\omega}_1$ and $\bar{\omega}_2$, respectively, i.e., for $i = 1, 2$,

$$X_i(\omega) = 0 \text{ when } |\omega| > \bar{\omega}_i$$

Define $z(t) = x_1(t)x_2(t)$.
For what values of T can $z(t)$ be perfectly reconstructed from its samples $\{z(kT), k \in \mathbb{Z}\}$?

10. [†] The bandlimited signal $x(t)$ is sampled with a sampling interval of T. Using these samples, the first-order hold reconstruction $x_{FOH}(t)$ is obtained by linearly interpolating between the samples. Formally, for $t \in [kT, (k+1)T]$,

$$x_{FOH}(t) = x(kT) + \frac{t - kT}{T}\left(x\big((k+1)T\big) - x(kT)\right).$$

Express the spectrum of $x_{FOH}(t)$ in terms of the spectrum of $x(t)$.

11. Consider the bandlimited signal x, which has been sampled with a sampling interval of T. Let $x_{ZOH}(t)$ and $x_{FOH}(t)$ denote, respectively, the zero order hold and the first order hold reconstruction obtained using these samples.

    Obtain the frequency response of the LSI system which generates $x_{FOH}(t)$ as the output corresponding to the input $x_{ZOH}(t)$.

12. *(CHALLENGING)* Consider a signal $x(t)$ that is bandlimited in $[-\pi, \pi]$. Your pointy-haired boss samples this signal at $x(n/2)$, $n \in \mathbb{Z}$. He claims that $\phi(t) = sinc(2t)$ is the *unique interpolation* signal such that

$$x(t) = \sum_{n \in \mathcal{Z}} x(n/2)\phi(t - n/2).$$

    Is your boss correct? If not, provide another choice for $\phi(\cdot)$.

    Note: Problem designed by Prof. Animesh Kumar.

13. *(CHALLENGING)* Consider a class of signals $\mathcal{S}$ whose spectrum has finite support, such that for $x(t) \in \mathcal{S}$,

$$X(\omega) \neq 0 \text{ for } |\omega| \in (\pi, 2\pi)$$

    and $X(\omega) = 0$ otherwise.

    What is the minimal sampling rate required for perfect reconstruction of this class of signals? Come up with a corresponding reconstruction approach.

    Note: Problem designed by Prof. Animesh Kumar.

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HW 5

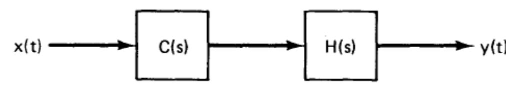
Assigned: 8/11/2023 Due: no need to submit

1. Consider the LSI causal system with the system function

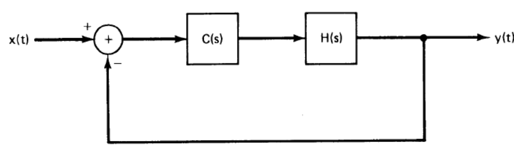
$$H(s) = \frac{1}{(s+1)(s-2)}.$$

This system is clearly unstable, and the goal of this question is to devise a method for its stabilization.

(a) Consider the series compensation scheme as shown below. Give an example of the compensating element $C(s)$ that would stabilize the overall system. Why would this *not* be a particularly useful solution in practice?



(b) Suppose we use the feedback setup as shown below. Can this system be stabilized with a constant gain compensating element, i.e., $C(s) = K$?



(c) How about using a proportional plus derivative controller, i.e., $C(s) = K(s + a)$? Prove that the system can indeed be stabilized in this manner.

2. Compute the discrete time Fourier transform for each of the following signals:

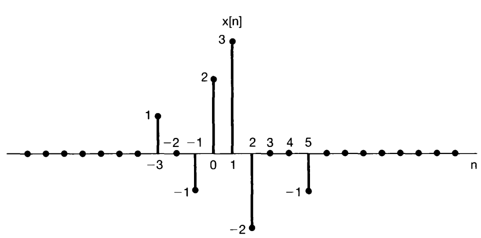
(a) $\sin(n)$

(b) $\cos(\pi n/3 + \pi/4)$

(c) $2^{-|n|}$

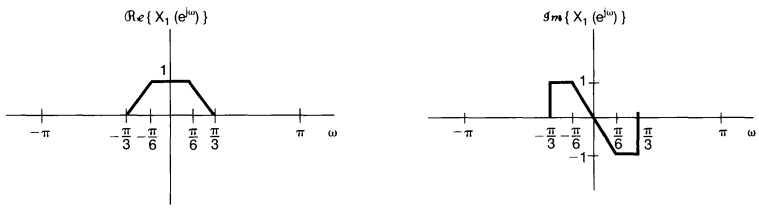
(d) $\delta[n-1] + \delta[n+1]$

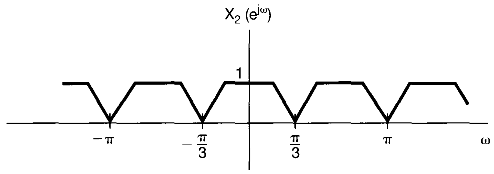
3. Consider the signal depicted below. Let the Fourier transform of this signal be written in rectangular form as $X(\omega) = A(\omega) + jB(\omega)$. Sketch the time domain signal whose transform is $Y(\omega) = B(\omega) + A(\omega)e^{j\omega}$.



4. Consider the LSI system with impulse response $h[n] = 2^{-n}\cos(\pi n/2)u[n]$. Determine the frequency response of this system.

Now, consider the input signal $x[n] = \cos(\pi n/2)$. Detemine the corresponding output signal.

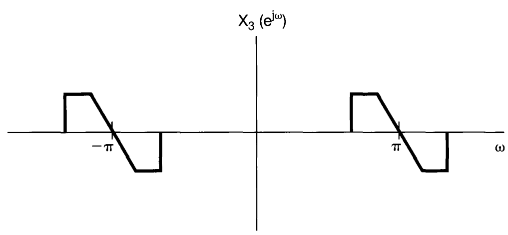




5. The DTFT of the signal $x_1[n]$ is shown below.

    (a) Consider the signal $x_2[n]$ whose spectrum is shown below. Express $x_2[n]$ in terms of $x_1[n]$.

    (b) Consider the signal $x_3[n]$ with spectrum as shown below. Express $x_3[n]$ in terms of $x_1[n]$.



    (c) Compute

$$\alpha := \frac{\sum_{-\infty}^{\infty} n x_1[n]}{\sum_{-\infty}^{\infty} x_1[n]}$$

    without inverting the spectrum of $x_1[n]$.