EE 325
Probability and Random Processes

Mid-semester Examination
September 21, 2023
1:30 – 3:30 pm

No clarifications. If you think a question is wrong/incomplete, make suitable assumptions. Clearly state and justify these assumptions.

Name:

Roll Number:

	Points Awarded		Points Awarded
Question 1		Question 5	
Question 2		Question 6	
Question 3		Question 7	
Question 4		Question 8	
		Total	

1. (a) Define countable and uncountable sets. (2 points)

Solution: A *countable set* is a set that has either a finite number of elements or can be put into a one-to-one correspondence (bijection) with the set of natural numbers ($\mathbb{N}$). Countable sets can be finite or infinite, but their elements can be enumerated or listed in a systematic way. An *uncountable set* is a set that is not countable, meaning it has too many elements to be put into a one-to-one correspondence with the set of natural numbers ($\mathbb{N}$). Uncountable sets have a higher cardinality than countable sets and are typically associated with the real numbers ($\mathbb{R}$).

(b) Is the set of polynomials of degree k with integer coefficients countable? Give reasons for your answer. (3 points)

Solution: Yes the set of polynomials of degree k with integer coefficients are countable. Let P_k be the set of all k-th degree polynomials with integer coefficients. Observe that for some polynomial $p \in P_k$, p is defined uniquely by its $k+1$ coefficients. These coefficients can be taken from a $(k+1)$-tuple of $\mathbb{Z}^{k+1}$, which is a countable set. Thus P_k is countable.

(c) A number is said to be an algebraic number if it is a root of a polynomial equation with integer coefficients. For example, 3 is an algebraic number since it is a root of the polynomial $x^3 - 27$. However, it is known that π is not an algebraic number. Is the set of algebraic numbers countable? Give reasons for your answer. (5 points)

Solution: A complex number z is said to be algebraic if it satisfies some polynomial equation of positive degree,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

The algebraic numbers are countable. For $p \in P_k$, let $R_p := \{z \in \mathbb{C} | p(z) = 0\}$ be the set of all roots of p. It is intuitively obvious to the most casual of observers that R_p is finite. (*A polynomial of degree k has at most k roots*). Then the set of all roots of polynomials of degree k can be defined as $W_k = \bigcup_{p \in P_k} R_p$. This is a countable union of countable sets, and so W_k is countable. Let $\mathcal{A}$ be the set of all algebraic numbers, defined by $\mathcal{A} = \bigcup_{k \in \mathbb{N}} W_k$. This is (yet again) a countable union of countable sets.

2. I choose an integer uniformly at random from the range $[1, 1000000]$.

(a) Determine the probability that the chosen number is divisible by 2. (2 points)

Solution: The basic formula of probability says that,

$$\texttt{The probability of an event} = \frac{\texttt{Number of favourable outcome}}{\texttt{Total number of outcome}}$$

Number of even numbers in [1, 1000000] = 50000 and the number of outcome = 1000000. Hence, The probability that the chosen number is divisible by 2 = $\frac{50000}{1000000} = \frac{1}{2}$

(b) Determine the probability that the chosen number is divisible by 5. (2 points)

Solution: Similarly, the number of integers divisible by 5 in [1, 1000000] = 20000 and the number of outcome = 1000000. Hence, the probability that the chosen number is divisible by 5 = $\frac{1}{5}$.

(c) Determine the probability that the chosen number is divisible by both 2 and 5. (2 points)

Solution: Similarly, the number of integers divisible by both 2 and 5 or by 10 in [1, 1000000] = 10000 and the number of outcome = 1000000. Hence, the probability that the chosen number is divisible by both 2 and 5 = $\frac{1}{10}$.

(d) Determine the probability that the chosen number is divisible by at least one number in $\{2, 5\}$. (2 points)

Solution: We can use the inclusion-exclusion principle to find the probability that the chosen number is divisible by at least one number in $\{2, 5\}$. The required probability = Probability that the chosen number is divisible by 2 + Probability that the chosen number is divisible by 5 - the probability that the chosen number is divisible by both 2 and 5 = $\frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{6}{10}$

(e) Determine the probability that the chosen number is divisible by exactly one number in $\{2, 5\}$. (2 points)

Solution: The probability that the chosen number is divisible by at least one number in $\{2, 5\}$ = the probability that the chosen number is divisible by exactly one number in $\{2, 5\}$ + the probability that the chosen number is divisible by both 2 and 5. Hence, the probability that the chosen number is divisible by exactly one number in $\{2, 5\} = \frac{6}{10} - \frac{1}{10} = \frac{1}{2}$

3. Let $A_1, A_2, A_3, \cdots$ be a sequence of sets satisfying the property $A_k \subseteq A_{k+1}, \forall k$. Let B be another set satisfying the property that B is independent of all A_ks. Consider the set $A = \cup_k A_k$. Prove that B is independent of A. (10 points)

Solution: Clearly A_k's are monotonically increasing sets. $A = \bigcup_k A_k$. Hence by continuity of probability $\mathbb{P}(A) = \lim_{k\to\infty} \mathbb{P}(A_k)$. We can also see that $B \cap A_k$'s are monotonically increasing sets.

$$\begin{aligned} B \cap A &= \cup_k (B \cap A_k) \\ \implies \mathbb{P}(B \cap A) &= \lim_{k\to\infty} \mathbb{P}(B \cap A_k) \end{aligned}$$

Now, $B \perp A_k$ gives,

$$\begin{aligned} \mathbb{P}(B \cap A_k) &= \mathbb{P}(B)\mathbb{P}(A_k) \\ \implies \lim_{k\to\infty} \mathbb{P}(B \cap A_k) &= \lim_{k\to\infty} \mathbb{P}(B)\mathbb{P}(A_k) \\ \implies \mathbb{P}(B \cap A) &= \mathbb{P}(B)\mathbb{P}(A) \end{aligned}$$

Hence, B is independent of A.

4. Consider the experiment of rolling a fair 6-faced dice once. Give examples of events that satisfy the following cases. For each part, clearly define the events and compute all relevant probabilities.

(a) $\mathbb{P}(A_1 \cap B_1) > \mathbb{P}(A_1)\mathbb{P}(B_1)$ (2 points)

Solution: $A_1 = B_1 =$ Event of getting a odd number

$$\mathbb{P}(A_1 \cap B_1) = \frac{1}{2} > \frac{1}{4} = \mathbb{P}(A_1)\mathbb{P}(B_1)$$

(b) $\mathbb{P}(A_2 \cap B_2) < \mathbb{P}(A_2)\mathbb{P}(B_2)$ (2 points)

Solution: $A_2 =$ Event of getting a odd number, $B_2 =$ Event of getting a even number

$$\mathbb{P}(A_2 \cap B_2) = 0 < \frac{1}{4} = \mathbb{P}(A_2)\mathbb{P}(B_2)$$

(c) $\mathbb{P}(A_3 \cap B_3|C_3) > \mathbb{P}(A_3)\mathbb{P}(B_3)$ (3 points)

Solution: $A_3 =$ Event of getting 1 as the output, $B_3 =$ Event of getting a number ≤ 2, $C_3 =$ Event of getting a number ≥ 3

$$\mathbb{P}(A_3 \cap B_3|C_3) = 0 < \frac{1}{6} = \mathbb{P}(A_3 \cap B_3)$$

(d) $\mathbb{P}(A_4 \cap B_4|C_4) < \mathbb{P}(A_4)\mathbb{P}(B_4)$ (3 points)

Solution: $A_4 =$ Event of getting 1 as the output, $B_4 =$ Event of getting a number ≤ 2, $C_4 =$ Event of getting a number ≤ 3.

$$\mathbb{P}(A_4 \cap B_4|C_4) = \frac{1}{3} > \frac{1}{6} = \mathbb{P}(A_4 \cap B_4)$$

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5. (a) Prove that $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$. (5 points)

Solution:

$$\begin{aligned} \text{var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] \geq 0 \\ \implies \mathbb{E}[X^2 - 2 \cdot X \cdot \mathbb{E}(X) + \mathbb{E}^2[X]] \geq 0 \\ \implies \mathbb{E}[X^2] - 2 \cdot \mathbb{E}[X] \cdot \mathbb{E}(X) + \mathbb{E}^2[X] \geq 0 \\ \implies \mathbb{E}[X^2] \geq (\mathbb{E}[X])^2 \end{aligned}$$

Didn't get this

(b) Give an example of a random variable with finite expectation and infinite variance. Compute the expectation and variance of this random variable to show that it has these properties. (5 points)

Solution: An example is a random variable X having a student's t-distribution with $\nu = 2$ degrees of freedom. It's mean is $\mathbb{E}[X] = 0$ for $\nu > 1$, but its second moment $\mathbb{E}[X^2] = \infty \implies \text{var}[X] = 0$ for $1 < \nu \leq 2$. $\mathbb{E}[X^2] = \frac{1}{\sqrt{\nu} B(\frac{1}{2}, \frac{\nu}{2})} \int_{-\infty}^{\infty} x^2 (1 + \frac{x^2}{\nu})^{-\frac{\nu+1}{2}} dx = \frac{1}{\sqrt{\nu} B(\frac{1}{2}, \frac{\nu}{2})} \cdot \frac{\nu}{\nu - 2}$. Hence, $\mathbb{E}[X^2] = \infty$ for $\nu = 2$.

6. We roll a fair six-faced dice over and over. What is the expected number of rolls until the first pair of consecutive sixes appears? (10 points)

Solution: First, lets look at how long it will take to get our first six. There are six, equally likely, outcomes, and so we'll need to roll, on average, six times to get a six. Let's define E_1 to be the expected number of rolls to get the first six. There are two things that could happen when we roll the die:

- We roll a six (1/6 chance), and this takes one roll.
- We miss the six (5/6 chance), which burns a roll, and puts us back exactly where we started.

Hence, the expected number of rolls is 1/6 of a chance of taking one roll plus 5/6 chances of taking that roll and being back where we started.

DOUBT: How to derive the expression for the expected no. of rolls until the first 'n' consecutive sixes appear?

$$E_1 = 1/6 \cdot 1 + 5/6 \cdot (1 + E_1)$$

No. of rolls to get a 'six'

Which gives $E_1 = 6$. Now we know that the expected number of rolls to get the first six is 6, from there, we can either roll another six, or, more likely, we don't roll the second six and are right back at the beginning again.

$$E_2 = 6 + 1/6 \cdot 1 + 5/6 \cdot (1 + E_2)$$

Which gives $E_2 = 42$.

7. Two random variables V and W are said to be independent iff $p_{V,W}(v, w) = p_V(v) p_W(w)$ for all v and w. Let V and W be independent Bernoulli random variables with parameters p and q respectively. Let $X = V + W$ and $Y = V - W$.

(a) Characterize $p_X(x)$. (2 points)

Solution: RV V can take values 0 and 1 w.p. p and $1 - p$. Similarly, W can take values 0 and 1 w.p. q and $1 - q$. Hence, RV $X = V + W$ can take values $\{0, 1, 2\}$ with the probability as shown below.

$$p_X(0) = p_{V,W}(0, 0) = p_V(0) p_W(0) = (1 - p)(1 - q)$$

$$p_X(1) = p_{V,W}(0, 1) + p_{V,W}(1, 0) = p_V(0) p_W(1) + p_V(1) p_W(0) = p(1 - q) + q(1 - p)$$

$$p_X(2) = p_{V,W}(1, 1) = p_V(0) p_W(0) = pq$$

X attains value 0 otherwise.

(b) Characterize $p_Y(y)$. (2 points)

Solution: Similarly RV $Y = V - W$ can take values $\{0, 1, -1\}$ with the probability as shown below.

$$p_Y(0) = p_{V,W}(0,0) + p_{V,W}(1,1) = p_V(0)p_W(0) + p_V(1)p_W(1) = (1-p)(1-q) + pq$$

$$p_Y(1) = p_{V,W}(1,0) = p_V(1)p_W(0) = p(1-q)$$

$$p_X(-1) = p_{V,W}(0,1) = p_V(0)p_W(1) = q(1-p)$$

Y attains value 0 otherwise.

(c) Characterize $p_{X,Y}(x,y)$. (4 points)

Solution: Only non-zero values possible for $p_{X,Y}(x,y)$ are $p_{X,Y}(0,0), p_{X,Y}(2,0), p_{X,Y}(1,1)$ and $p_{X,Y}(1,-1)$ and the probabilities are,

$$p_{X,Y}(0,0) = p_{V,W}(0,0) = p_V(0)p_W(0) = (1-p)(1-q)$$

$$p_{X,Y}(2,0) = p_{V,W}(1,1) = p_V(1)p_W(1) = pq$$

$$p_{X,Y}(1,1) = p_{V,W}(1,0) = p_V(1)p_W(0) = p(1-q)$$

$$p_{X,Y}(1,-1) = p_{V,W}(0,1) = p_V(0)p_W(1) = q(1-p)$$

$p_{X,Y}(x,y)$ attains value 0 otherwise.

(d) Are X and Y independent? Give reasons for your answer. (2 points)

Solution: X and Y are not independent as $p_{X,Y}(1,-1) \neq p_X(1)p_Y(-1)$

8. Let X and Y be independent geometric random variables with parameters p and q respectively.

Didn't understand this

(a) Let $Z = \min\{X, Y\}$. Characterize the PMF of Z. (5 points)

Solution: Given $Z = \min\{X, Y\}$

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\min\{X,Y\} \leq z) \\ &= 1 - \mathbb{P}(X > z, X > z) \\ &= 1 - [1 - F_X(z)][1 - F_Y(z)] \\ &= 1 - (1-p)^z(1-q)^z \\ &= 1 - (1 - p - q + pq)^z \end{aligned}$$

Using the fact that the CDF of a geometric RV X is $\mathrm{F}_X(k) = 1 - (1-p)^k$. Hence, we can say $Z \sim \text{Geometric}(p + q - pq)$

(b) Compute the expected value of Z. (2 points)

Solution: The mean of the geometric RV $Z \sim \text{Geometric}(p + q - pq)$ is $\frac{1}{p+q-pq}$

(c) Compute the variance of Z. (3 points)

Solution: The variance of the geometric RV $Z \sim \text{Geometric}(p + q - pq)$ is $\frac{1-p-q+pq}{(p+q-pq)^2}$