Applied Linear Algebra: Problem set-5

Instructor: Dwaipayan Mukherjee*
Indian Institute of Technology Bombay, Mumbai- 400076, India

1. What are all possible eigenvalues of (a) an orthogonal projection matrix, and (b) a square matrix whose kernel is equal to its image?

2. Show that for a nilpotent matrix $A \in \mathbb{F}^{n\times n}$ such that $A^n = 0$, the matrix $A - I$ must be nonsingular.

3. For $A \in \mathbb{C}^{n\times n}$, if every vector in $\mathbb{C}^n$ is an eigenvector of A, show that A must be a scalar multiple of the identity matrix.

4. Suppose for a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\mathbb{U}$ is an A-invariant subspace. Show that $\mathbb{U}^\perp$ must also be A-invariant.

5. For a matrix, $A \in \mathbb{C}^{n\times n}$, show that if $\text{rank}(A) = r$, then A can have at most $r+1$ distinct eigenvalues.

6. For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, show that the following subspaces are A-invariant: (a) $\mathcal{C} = \text{im}([B\ AB\ \dots\ A^{n-1}B])$, and (b) $\bar{\mathcal{O}} = \ker([C^T\ A^TC^T\ \dots\ (A^{n-1})^TC^T]^T)$.

7. Let $\mathbb{W} \subseteq \mathbb{R}^n$ be an A-invariant subspace such that $\langle w_1, w_2, \dots, w_k\rangle = \mathbb{W}$. Show that $\exists\ S \in \mathbb{R}^{k\times k}$ such that $AW = WS$. where $W = [w_1\ w_2\ \dots\ w_k]$. Further, prove that A and S share at least r eigenvalues, where $r = \text{rank}(W)$.

8. For $\beta \in \mathbb{C}$, which is not an eigenvalue of $A \in \mathbb{C}^{n\times n}$, show that $v \in \mathbb{C}^n$ is an eigenvector of A if and only if it is an eigenvector of $(A - \beta I)^{-1}$.

9. Show that for a matrix $A \in \mathbb{F}^{n\times n}$, with n distinct eigenvalues, there are 2^n A-invariant subspaces.

10. The *spectral* norm of a matrix, $A \in \mathbb{R}^{m\times n}$, is defined as $||A||_2 := \max_{x\neq 0} \dfrac{||Ax||_2}{||x||_2}$, where $x \in \mathbb{R}^n$ and the vector norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ result from the conventional inner products on those spaces. If the non-zero singular values of A are given by $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ ($r \leq \min(m,n)$), then prove that $||A||_2 = \sigma_1$.

11. The *Frobenius* norm of a matrix $A \in \mathbb{R}^{m\times n}$ is given by $||A||_F := \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$. If the non-zero singular values of A are given by $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ ($r \leq \min(m,n)$), then prove that $||A||_F = (\sigma_1^2 + \sigma_2^2 + \dots \sigma_r^2)^{\frac{1}{2}}$.

12. For a matrix $A \in \mathbb{R}^{m\times n}$ whose SVD is given by $A = U \begin{bmatrix} \Sigma_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} V^T$, the Moore-Penrose pseudoinverse is given by $A^\dagger = V \begin{bmatrix} \Sigma_{r\times r}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$. Prove the following:
    (a) $A^{-1} = A^\dagger$ when A is invertible
    (b) $(A^\dagger)^\dagger = A$
    (c) $(A^\dagger)^T = (A^T)^\dagger$
    (d) $A^\dagger = (A^TA)^{-1}A^T$ when A has full column rank and $A^\dagger = A^T(AA^T)^{-1}$ when A has full row rank.
    (e) $(A^TA)^\dagger = A^\dagger(A^T)^\dagger$ and $(AA^T)^\dagger = (A^T)^\dagger A^\dagger$

13. (a) Argue why complex matrices A and A^H (for matrices over $\mathbb{C}$, $(\cdot)^H$ denotes entry-wise conjugation of the transpose of a matrix) must have eigenvalues that are conjugates of one another.

    (b) If u and v are eigenvectors of complex matrices A and A^H, respectively, for two eigenvalues that are not complex conjugates of one another, show that u and v must be orthogonal to each other.

    (c) For a matrix $A \in \mathbb{R}^{n\times n}$, prove that if λ is an eigenvalue with algebraic multiplicity of one (also called a *simple* eigenvalue), the left and right eigenvectors corresponding to λ cannot be orthogonal, irrespective of whether A is diagonalizable or not (left eigenvector is defined as $v \neq 0$ such that $v^*A = \lambda v^*$ for some eigenvalue λ).

14. Suppose we have an ordered basis, $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$, for $\mathbb{V}$ such that $\langle v_1, v_2, \dots, v_k\rangle$ is A-invariant for all $k = 1, 2, \dots, n$. What can you say about the structure of the matrix representation of a linear operator $\varphi : \mathbb{V} \to \mathbb{V}$, given by $[\varphi]_{\mathcal{B}}$?

*Asst. Professor, Electrical Engineering, Office: EE 214D, e-mail: dm@ee.iitb.ac.in

15. For a vector space $\mathbb{V}$ over $\mathbb{C}$ and a polynomial $p \in \mathbb{C}[s]$, consider a linear operator $A : \mathbb{V} \to \mathbb{V}$ and $z \in \mathbb{C}$. Show that z is an eigenvalue of $p(A)$ if and only if $z = p(\lambda)$ for some eigenvalue, λ, of A.

16. For a linear operator $A : \mathbb{V} \to \mathbb{V}$ on a finite dimensional vector space, a positive integer m, and a vector $v \in \mathbb{V}$, suppose $A^{m-1}v \neq 0$, but $A^m v = 0$. Show that $\{v, Av, A^2v, \ldots, A^{m-1}v\}$ is linearly independent.

17. Prove that $A = uv^T \in \mathbb{R}^{n\times n}$, for $u, v \in \mathbb{R}^n$, is diagonalizable if and only if $v^T u \neq 0$.

18. Using the principle of mathematical induction, show that a set of eigenvectors corresponding to distinct eigenvalues is linearly independent.

19. If $A \in \mathbb{C}^{n\times n}$ has a set of n orthonormal eigenvectors, it must be a Hermitian matrix. Prove it or give a counterexample if it is not true.

20. Which of the following are ideals and why (or why not)?
    (a) {all polynomials in $\mathbb{C}[x]$ having the constant term equal to zero}
    (b) {all polynomials in $\mathbb{C}[x]$ containing only even degree terms}
    (c) $\{0, 2, 4\} \subseteq \mathbb{Z}_6$
    (d) {all polynomials in $\mathbb{Z}[x]$ with even coefficients}
    (e) $\mathbb{R} \subseteq \mathbb{R}[x]$

21. Explain why for the Jordan form representation of a nilpotent matrix, $N \in \mathbb{C}^{n\times n}$, the number of Jordan blocks of size $i \times i$ or greater is equal to $\operatorname{rank}(N^{i-1}) - \operatorname{rank}(N^i)$.

22. For a non-zero vector $v \in \mathbb{C}^n$, and a matrix $A \in \mathbb{C}^{n\times n}$, the sequence of vectors $\{v, Av, A^2v, \ldots, A^jv\}$ is called a Krylov sequence (named after the Russian mathematician Alexei Nikolaevich Krylov), and the subspace $\mathcal{K}_j$ spanned by vectors in the sequence is called a Krylov subspace. Suppose a Krylov subspace for vector v, say $\mathcal{K}_{n-1}$, spans $\mathbb{C}^n$, while the characteristic polynomial for A is given by $\chi_A(x) = \sum_{i=0}^{n} \alpha_i x^i$. Represent the matrix A in terms of the ordered basis given by the corresponding Krylov sequence. What is the minimal polynomial, $\mu_A(x)$, and why?

23. For $A \in \mathbb{R}^{n\times n}$, suppose $\{w_1, w_2, \ldots, w_n\}$ is a set of n linearly independent eigenvectors and for $w = \sum_{k=1}^{k=n} k w_k$, we have $Aw = \sum_{k=1}^{k=n} (k^2 + k) w_k$. Then the minimal polynomial, $\mu_A(x)$, and characteristic polynomial, $\chi_A(x)$, of A are identical. Prove or disprove this assertion.

24. Is $\mathbb{Z}[x]$ a PID? Justify your answer.

25. Show that for any matrix $A \in \mathbb{C}^{n\times n}$, we have $\lim_{k\to\infty} A^k = 0$ if and only if the spectral radius (largest modulus among all eigenvalues) of A is less than unity.

26. Show that a matrix is nilpotent if and only if all its eigenvalues are zero. Can a non-zero diagonalizable matrix be nilpotent? Justify your answer. Suppose for $N \in \mathbb{R}^{n\times n}$, we have $\operatorname{trace}(N^k) = 0 \ \forall k \in \mathbb{Z}_+$. Show that N must be nilpotent.

27. Consider two diagonalizable matrices, $A, B \in \mathbb{R}^{n\times n}$. Show that the following statements are equivalent: 1. $AB = BA$, and 2. There exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.

28. Evaluate $\cos(A)$, when $A = \begin{bmatrix} -\pi/2 & \pi/2 \\ \pi/2 & -\pi/2 \end{bmatrix}$.

29. Show that e^A is an orthogonal matrix whenever $A = -A^T$.

30. (*Population migration*) Consider the current populations of two cities, **A** and **B**, to be given by a_0 and b_0, respectively. We assume (quite unrealistically, of course!) that the total population of the two cities remains constant year after year. Suppose every year a fraction $0 < p_A < 1$ of the population in city **A** migrates to city **B**, while a fraction $0 < p_B < 1$ of the population in city **B** migrates to city **A**.
    (a) Write down a discrete time representation for the population in the two cities after $k+1$ years, in terms of the population after k years and other relevant parameters, in the form $\begin{bmatrix} a(k+1) \\ b(k+1) \end{bmatrix} = \Phi\left(\begin{bmatrix} a(k) \\ b(k) \end{bmatrix}\right)$, where $\begin{bmatrix} a(k) \\ b(k) \end{bmatrix} \in \mathbb{R}^2$ denotes the population of the two cities after k years. Argue why $\Phi(\cdot)$ is a linear map.
    (b) Show that $\begin{bmatrix} a(k) \\ b(k) \end{bmatrix} = \Phi^k\left(\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}\right)$.
    (c) Obtain the eigenvalues and eigenvectors of $\Phi(\cdot)$ (or of $[\Phi(\cdot)]_{\mathcal{B}}$, i.e., the representation of $\Phi(\cdot)$ under some basis).
    (d) What happens to the populations of the two cities in the long run (as $k \to \infty$)? If the total initial population remained the same, i.e., $a_0 + b_0 = c$, but we had instead started with $\hat{a}_0$ people in city **A**, and $\hat{b}_0$ people in city **B** (with $\hat{a}_0 + \hat{b}_0 = c$), how would the final population be redistributed?

31. Show that $\det\left(e^A\right) = e^{\text{trace}(A)}$.

32. Show that whenever $AB = BA$, we have $e^{A+B} = e^A e^B$ for $A, B \in \mathbb{R}^{n\times n}$. Provide a counterexample to this assertion if $AB \neq BA$.

33. (*Competition vs cooperation*)
(a) Consider two species, say $\mathcal{P}$ and $\mathcal{Q}$, in an environment competing for the same resources. Thus, each species' population increases in proportion to its current population and decreases in proportion to that of its competitor. Suppose the constants of proportionality for both $\mathcal{P}$ and $\mathcal{Q}$ are 0.02 (for increase) and -0.01 (for decrease), respectively, while their initial populations are 10,000 and 20,000, respectively. Obtain the expressions for the population evolution of the two species. Does any of the two species go extinct within some finite time? If so, which one?
(b) Suppose two symbiotic species, say $\mathcal{R}$ and $\mathcal{S}$, inhabit the same environment so that each species' population increases in proportion to the current population of its symbiotic neighbour, while it decreases in proportion to its own current population. Further, suppose the two constants of proportionality (for both increase and decrease, respectively) are 0.01 and -0.01 for both the species. If the initial populations of $\mathcal{R}$ and $\mathcal{S}$ are 10,000 and 20,000, respectively, obtain the expressions for the population evolution of the two species. What can you say about the long run trend of either population ?

34. For a symmetric matrix, $A \in \mathbb{R}^{n\times n}$ with zeros on its diagonal, suppose $a_{ij} = 0$ or 1 for $i \neq j$, show that the largest eigenvalue, λ_M is bounded by $\min_i \text{Rowsum}_i(A) \leq \lambda_M \leq \max_i \text{Rowsum}_i(A)$, where Rowsum_i denotes the sum of the entries of the i^{th} row.

35. Obtain the minimal polynomial for the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 3 & 6 & 8 \end{bmatrix}$. Hence, evaluate its eigenvalues.

36. (*Graph theory*) The *distance* between two nodes $u, v \in \mathcal{V}$ in an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, denoted $d_{\mathcal{G}}(u, v)$, is the length of the shortest path between them (i.e., the number of edges in that path). The *diameter* of an undirected graph, $\mathcal{G}$, denoted as $\text{dia}(\mathcal{G})$ is the maximum distance between any pair of vertices in $\mathcal{G}$. Show that for a connected, undirected graph $\mathcal{G}$, with n vertices, the adjacency matrix, $A(\mathcal{G})$, satisfies the following properties:
(a) If u, v are vertices of $\mathcal{G}$, with $d_{\mathcal{G}}(u, v) = m$, then $I_n, A(\mathcal{G}), \ldots, A(\mathcal{G})^m$ are linearly independent.
(b) If $A(\mathcal{G})$ has k distinct eigenvalues and $\text{dia}(\mathcal{G}) = q$, then $k > q$.

37. Suppose the minimal polynomial of a matrix $A \in \mathbb{C}^{n\times n}$, say $\mu_A(x)$, admits a coprime factorization given by $\mu_A(x) = (x - \lambda)^k p(x)$, while the algebraic multiplicity of the eigenvalue, λ, equals m. Show that $\dim(\text{Ker}(A - \lambda I)^k) = m$.

38. If $A \in \mathbb{C}^{n\times n}$ is a diagonalizable matrix with characteristic polynomial $\chi_A(x) = (x-1)^{k_1}(x+1)^{k_2}x^{k_3}$, then the rank of A can be increased by adding or subtracting the identity matrix to it. Prove or disprove the assertion.

39. Suppose $A \in \mathbb{R}^{n\times n}$ has a characteristic polynomial given by $\chi_A(x) = (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2}$ and $\text{rank}(A - \lambda_1 I) = n - k_1$. Then A must be diagonalizable. Prove it or give a counterexample if it is not true.

40. Prove that for an operator $\varphi : \mathbb{V} \to \mathbb{V}$, with $\dim(\mathbb{V}) = n < \infty$, we have $\mathbb{V} = \text{Ker}(\varphi^n) \oplus \text{im}(\varphi^n)$.

[I was advised] to read [Camille] Jordan's 'Cours d'analyse'; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant.

—'A Mathematician's Apology (1940)' G. H. Hardy