EE 325
Probability and Random Processes

Quiz 4
November 6, 2023
11:35 pm – 12:30 pm

No clarifications. If you think a question is wrong/incomplete, make suitable assumptions. Clearly state and justify these assumptions.

Name:

Roll Number:

	Points Awarded
Question 1	
Question 2	
Question 3	
Question 4	
Total	

1. Let Y be a zero mean random variable with variance σ^2. Show that (10 points)

$$\mathbb{P}(Y \geq a) \leq \frac{\sigma^2}{a^2 + \sigma^2}.$$

Solution: Chevyshev's inequality:

$$\mathbb{P}(|Y - \mathbb{E}[Y]| \geq a) \leq \frac{\sigma^2}{a^2} \qquad \text{for any } a > 0$$
$$\implies \mathbb{P}(Y \geq a) \leq \frac{\sigma^2}{a^2} \rightarrow \text{not tight enough}$$

$$\mathbb{P}(Y \geq a) = \mathbb{P}((Y + c) \geq (a + c)) \leq \mathbb{P}((Y + c)^2 \geq (a + c)^2) \tag{1}$$

Now, applying Markov's inequality to equation 1

$$\mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}[(Y + c)^2]}{(a + c)^2}$$
$$\implies \mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}[Y^2 + c^2 + 2cY]}{(a + c)^2}$$

$$\implies \mathbb{P}(Y \geq a) \leq \frac{\sigma^2 + c^2}{(a + c)^2} \tag{2}$$

Now, to get the tightest possible bound we need to minimize the RHS of equation 2.

$$f(c) = \frac{\sigma^2 + c^2}{(a + c)^2}$$
$$\implies f'(c) = \frac{2c(a + c)^2 + 2(\sigma^2 + c^2)(a + c)}{(a + c)^4}$$
$$\implies f'(c) = \frac{2(a + c)(ac - \sigma^2)}{(a + c)^4}$$

Now,

$$f'(c) = 0 \implies c = \frac{\sigma^2}{a}$$

One can readily ascertain that this constitutes the minimum by confirming the positivity of the second derivative.

$$\therefore \mathbb{P}(Y \geq a) \leq \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{(a + \frac{\sigma^2}{a})^2}$$
$$\implies \mathbb{P}(Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

2. Find the mean and variance of the random variable X, whose moment generating function is given by $M_X(s) = 1 + p(e^s - 1)$. (10 Points)

Solution:

$$\mathbb{E}[X] = \frac{d}{ds} M_X(s)|_{s=0} = pe^s|_{s=0} = p$$
$$\mathbb{E}[X^2] = \frac{d^2}{ds^2} M_X(s)|_{s=0} = pe^s|_{s=0} = p$$
$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2$$

3. Let Y_1, Y_2, $\cdots$ be independent random variables, where $Y_n \sim \text{Bernoulli}\left(\frac{n}{n+1}\right)$. Let A_i be the event $\{Y_i = 0\}$.

   (a) Show that A_i's occur infinitely often. (3 Points)

   Solution: Given A_i is the event $\{Y_i = 0\}$. Hence,

$$P(A_i) = \frac{1}{i} \qquad \text{for } i = 1, 2, ...$$

$$\implies \sum_{i=1}^{\infty} P(A_i) = \infty$$

   Hence, A_i's occur infinitely often.

   (b) Show that $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = 1$. (3 Points)

   Solution: As A_i occurs infinitely often with probability 1 by Borel-Cantelli lemma 2.

$$\implies P(\bigcup_{i=1}^{\infty} A_i) = 1$$

   (c) We define the sequence $\{X_n, n = 2, 3, 4, \cdots\}$ as (4 Points)

$$X_{n+1} = \prod_{i=1}^{n} Y_i, \text{for } n = 1, 2, 3, \cdots .$$

   Does X_n converge in the a.s. sense? If yes, what is the limit?
   (Hint: use part(b))

   Solution: From (b)

$$\implies X_{n+1} \xrightarrow{a.s} 0$$

4. Let $X_1, X_2, \cdots, X_n$ be independent random variables, each following the Uniform$[0,1]$ distribution. For each $n \geq 1$, define $Y_n = \left(\prod_{i=1}^n X_i\right)^{1/n}$. Does the sequence $\{Y_n\}$ converge in probability? If yes, what does it converge to? (Hint: use the WLLN) (10 points)

Solution:

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} e^{\ln Y_n} = \lim_{n\to\infty} e^{\ln\left(\Pi_{i=1}^n X_i\right)^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\Sigma_{i=1}^n \ln(X_i)} = e^{\lim_{n\to\infty}\frac{1}{n}\Sigma_{i=1}^n \ln(X_i)} = e^{\mathbb{E}[\ln(X_i)]} = e^{-1}$$

$$\left(\because \text{ From WLLN, } \lim_{n\to\infty}\frac{1}{n}\Sigma_{i=1}^n \ln(X_i) = \mathbb{E}[\ln(X_i)]\right)$$