EE 325
Probability and Random Processes

Quiz 1
August 17, 2023
9:35 – 10:15 am

No clarifications. If you think a question is wrong/incomplete, make suitable assumptions. Clearly state and justify these assumptions.

Name:

Roll Number:

	Points Awarded
Question 1	
Question 2	
Question 3	
Question 4	
Total	

1. Express each of the following events in terms of events A, B, and C as well as the operations of complementation, union, and intersection. Also, draw the corresponding Venn diagrams for each case.

   (a) At least one of the events A, B, C occurs; (2 points)

   (b) At most one of the events A, B, C occurs; (2 points)

   (c) None of the events A, B, C occurs; (2 points)

   (d) Exactly one of the events A, B, C occurs; (2 points)

   (e) Events A and B occur, but not C. (2 points)

   Answer:

   (a) $A \cup B \cup C$

   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$

   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

   (d) $A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

   (e) $A \cap B \cap C^c$

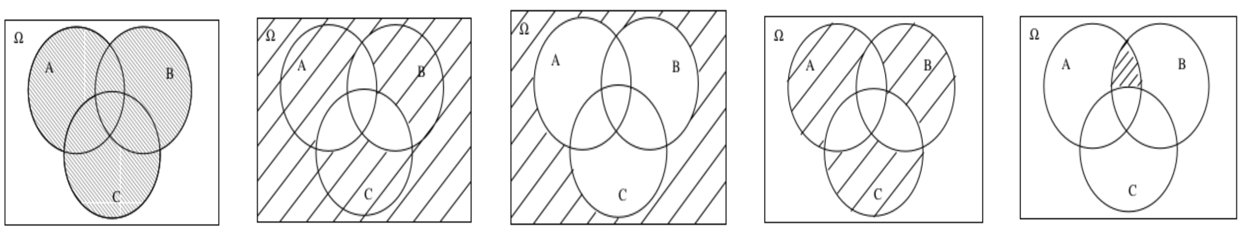

Figure 1: Venn diagram.

2. Find $\mathbb{P}(A \cup (B^c \cup C^c)^c)$ in each of the following cases:

   (a) A, B, C are mutually exclusive events, and $\mathbb{P}(A) = 3/7$. (3 points)

   (b) $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B \cap C) = 1/3$, $\mathbb{P}(A \cap C) = 0$. (3 points)

   (c) $\mathbb{P}(A^c \cap (B^c \cup C^c)) = 0.65$. (4 points)

   Answer:

   (a) We are given $\mathbb{P}(A) = \frac{3}{7}$, $\mathbb{P}(B \cap C) = 0$ and $\mathbb{P}(A \cap B \cap C) = 0$. Using De Morgan's laws, we know $(B^c \cup C^c)^c = B \cap C$. Therefore, $\mathbb{P}(A \cup (B^c \cup C^c)^c) = \mathbb{P}(A \cup (B \cap C)) = \mathbb{P}(A) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap (B \cap C)) = \frac{3}{7}$.

   (b) We are given $\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B \cap C) = \frac{1}{3}$ and $\mathbb{P}(A \cap C) = 0$. Therefore, again applying De Morgan's laws, $\mathbb{P}(A \cup (B^c \cup C^c)^c) = \mathbb{P}(A \cup (B \cap C)) = \mathbb{P}(A) + P(B \cap C) - \mathbb{P}(A \cap (B \cap C)) = \frac{5}{6}$ where we deduce $A \cap B \cap C = \phi$ (and thus $\mathbb{P}(A \cap B \cap C) = 0$) because $A \cap C = \phi$ and $A \cap B \cap C \subseteq A \cap C$.

   (c) We are given $\mathbb{P}(A^c \cap (B^c \cup C^c)) = 0.65$ and De Morgan's laws imply $(A^c \cap (B^c \cup C^c))^c = A \cup (B^c \cup C^c)^c$, which is the event of interest. Therefore $\mathbb{P}(A \cup (B^c \cup C^c)^c) = 1 - \mathbb{P}(A^c \cap (B^c \cup C^c)) = 0.35$.

3. State the axioms of probability and prove the following *only* using the axioms of probability. (10 points)

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C).$$

Answer:

Axioms of Probability:

(a) For any event A, $\mathbb{P}(A) \geq 0$.

(b) Probability of the sample space S is $\mathbb{P}(S) = 1$.

(c) If $A_1, A_2, A_3, \cdots$ are disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup A_3 \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \cdots$

As proved in class: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Hence, we can write $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}((A \cap C) \cup (B \cap C))$ (using distributive law)$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$.

4. Consider the sample space $\Omega = \mathbb{N}$. Find the values of the constant C for which the following are probability measures:

(a) $\mathbb{P}(\{x\}) = C2^{-x}$ for all $x \in \Omega$. (5 points)

(b) $\mathbb{P}(\{x\}) = Cx^{-2}$ for all $x \in \Omega$. (5 points)

Answer:

(a) The probability measure of the entire sample space $\Omega = \mathbb{N}$ should be equal to 1. Summing up for all, $\sum_{x \in \Omega} \mathbb{P}(\{x\}) = 1 \implies \sum_{x \in \Omega} C2^{-x} = 1 \implies C \sum_{x \in \mathbb{N}} 2^{-x} = 1 \implies C = 1$. Hence $C = 1$.

(b) Similarly we can write $\sum_{x \in \Omega} Cx^{-2} = 1 \implies C \cdot \pi^2/6 = 1 \implies C = 6/\pi^2$