Applied Linear Algebra: Problem set-1

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[For $\mathbb{Z}_n$, assume the modulo operations by default, unless otherwise specified.]

1. What are all possible row reduced echelon form matrices in $\mathbb{R}^{2\times 2}$?

2. Prove that there exist exactly three row reduced matrices in $\mathbb{C}^{2\times 2}$ given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that $a+b+c+d=0$.

3. Show that $x^2 = 0$ has a unique solution in $\mathbb{Z}_n$, where $n = pq$ and p, q are distinct prime numbers. Consider the relevant operations to be modulo n. [**Hint**: If a prime number divides a product of two integers, then it must divide at least one of them.]

4. Prove that every subfield of the complex numbers must contain every rational number.

5. For $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \in \mathbb{Z}_7^{2\times 2}$, show through explicit computation of P^{-1}, where $P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \in \mathbb{Z}_7^{2\times 2}$, that $P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \in \mathbb{Z}_7^{2\times 2}$.

6. What is the smallest prime number, $p > 2$, such that $x^2 = 2$ has a solution in $\mathbb{Z}_p$?

7. For the matrix $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, obtain $x \in \mathbb{R}^3$ such that $Ax = \lambda x$ for some scalar λ, if possible.

8. Construct a 2×2 matrix which has an inverse if its entries belong to $\mathbb{R}$, but fails to have an inverse when its entries belong to $\mathbb{Z}_2$. Show the existence of the inverse through explicit computation and prove when not invertible.

9. Consider $A, B \in \mathbb{F}^{n\times n}$. Show that if A is invertible, then $AB = 0 \implies B = 0$. Further, show that if A does not have an inverse, then $\exists B \neq 0$ such that $AB = 0$.

10. Express $\begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \end{bmatrix}$ as a linear combination of $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$. Pose this problem in the form $Ax = b$ and prove that there exists a solution to this problem. Give an example of a vector that cannot be a linear combination of u and v.

11. (*Optimization*) For a particular crop, each square foot of ground requires 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose there are three brands of fertilizer $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, which are all sold in packets weighing 1 kg each. One kg of the three brands contain minerals in the following quantities.
$\mathcal{X}$: 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen.
$\mathcal{Y}$: 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen.
$\mathcal{Z}$: 1 unit of phosphorous and 1 unit of nitrogen.
(a) Suppose farmers can only buy an integral number of packets of each brand. Does this problem have a *meaningful* solution? If it exists, is the solution unique? Justify your answer. Determine all possible solutions to the problem, if any.
(b) Suppose fertilizer of brand $\mathcal{X}$ costs INR 100 per kg, brand $\mathcal{Y}$ costs INR 600 per kg, and brand $\mathcal{Z}$ costs INR 300 per kg. Determine the least expensive solution (even if no exact solution exists) that will satisfy the recommendations (as best as possible).

12. The *characteristic* of a field is defined as the smallest number of times one must add the multiplicative identity in the field to get the additive identity (i.e., for $\underbrace{1+1+\ldots+1}_{n} = 0$, the characteristic is n provided no number smaller than n results in the equality). Prove that the characteristic of a field is either 0 (which is when no finite n exists) or a prime number. [Hint: You may use the field axioms.]

13. Obtain all solutions of $x^2 - 10x + 16 = 0$ over $\mathbb{Z}_2$, $\mathbb{Z}_5$, and $\mathbb{Z}_8$.

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14. Find all solutions of the equation $x^2 - 9x + 18 = 0$ over $\mathbb{Z}_2$, $\mathbb{Z}_5$, and $\mathbb{Z}_8$.

15. Consider the set of integers, $\mathbb{Z}$, with 'addition', $\oplus$ and 'multiplication', $\otimes$ defined as: $x \oplus y = x + y - 1$, and $x \otimes y = x + y - xy$. Determine the multiplicative and additive identities, if they exist. Is $(\mathbb{Z}, \oplus, \otimes)$ an integral domain?

16. Suppose K is a set of numbers and $K(\sqrt{d})$ is a set of all numbers of the form $\alpha + \beta\sqrt{d}$, with $\alpha, \beta \in K$, while $d > 0$ is a 'square-free' number in K ('square-free' numbers are those whose prime decompositions contain no repeated factors). Is $K(\sqrt{d})$ a field for $K = \mathbb{Z}$ or $K = \mathbb{N}$? Explain your answer. Does your answer change if $K = \mathbb{Z}_p$, when p is prime? Explain in any case.

17. Based on the ideas in Problem 16, can you suggest how to constructively cook up a family of fields, $\{\mathbb{F}_1, \mathbb{F}_2, \dots\}$ satisfying the relation $\mathbb{Q} \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{R}$? Outline your steps through proper arguments. Note that $\mathbb{F}_1 \subset \mathbb{F}_2$ implies that $\mathbb{F}_1$, the sub-field of $\mathbb{F}_2$ is a field under the same same addition and multiplication as defined in $\mathbb{F}_2$ and the set $\mathbb{F}_1$ is a subset of $\mathbb{F}_2$. Construct a field, $\mathbb{F} \not\subset \mathbb{R}$, such that $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{C}$.

18. Argue whether a field is a vector space over all its subfields.

19. On $\mathbb{R}^n$, let $x \oplus y := x - y$ and $\alpha \otimes x := -\alpha x$ for $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Verify all the properties of a vector space to check if $(\mathbb{R}^n, \oplus, \otimes)$ is a vector space over $\mathbb{R}$.

20. Give an example of a vector space having exactly 81 elements.

21. Let $\mathbb{V}$ be a set containing pairs of numbers (u, v), where u and v belong to a field $\mathbb{F}$. Suppose $(x_1, y_1) \oplus (x_2, y_2) := (x_1 + x_2, 0)$ and $\alpha \otimes (x, y) = (\alpha x, 0)$ for $\alpha \in \mathbb{F}$. Verify whether $(\mathbb{V}, \oplus, \otimes)$ is a vector space over $\mathbb{F}$.

22. Consider $\mathbb{W}_1 := \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{F} \right\}$, and $\mathbb{W}_2 := \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{F} \right\}$, where $\mathbb{F}$ is a field. Show that $\mathbb{W}_1 \cap \mathbb{W}_2 = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$ and $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{F}^{2\times 2}$.

23. Determine whether each of the following is a subspace of $\mathbb{F}^3$, where $\mathbb{F}$ is a field, and justify your answer in each case:
(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 6\}$
(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$
(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 6x_3\}$

24. Give an example of matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$ and vectors $b_1, b_2 \in \mathbb{R}^2$ such that the solutions sets of $A_1 x = b_1$, say $\mathcal{S}_1$, and $A_2 x = b_2$, say $\mathcal{S}_2$, satisfy the conditions $\mathcal{S}_1 \subseteq \mathcal{S}_2$, but $\mathcal{S}_2 \neq \mathcal{S}_2$.

25. (*Traffic flow*) Consider the traffic flow depicted in the map shown in Fig. 1 over a one-hour open window while

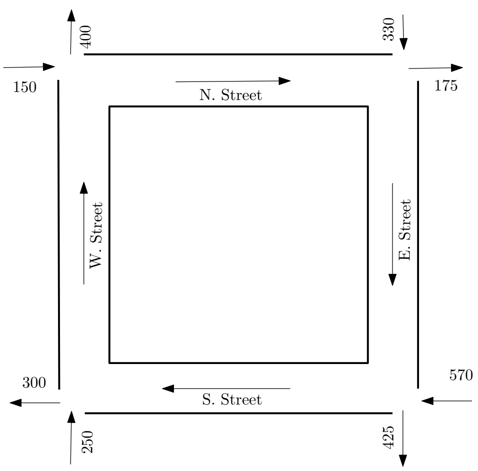

Figure 1: Traffic flow over a 1-hour period

there is no flow of traffic during the remainder of the day. It is known that the streets are one-way, there are no residences in the map, and the entire area is a 'No Parking' zone.

(a) Does there exist a solution to the problem of determining the number of vehicles passing through each of the four main roads? Justify your answer mathematically. If it is solvable, is the solution unique? In any case outline what the solution could be, if it exists.

(b) Suppose the number of vehicles entering the north-east junction were increased by 50 and the number of vehicles leaving through the south-west junction increased by 30. Can you solve for the problem in (a)? Justify your answer with mathematical arguments.

(c) For problem in (a), does there exist a solution if it is given that the total number of vehicles through W. Street was 200 during the one-hour period? Justify your answer.

26. Denote $\mathcal{F}[\mathbb{N};\mathbb{F}]$, the vector space of all infinite sequences of elements in $\mathbb{F}$, as $\mathbb{F}^\infty$. Which of the following subsets of $\mathbb{R}^\infty$ are subspaces of $\mathbb{R}^\infty$? Justify your answer in each case.
(a) $\mathbb{W} := \{f \in \mathbb{R}^\infty : f(n+1) \leq f(n)\ \forall n \in \mathbb{N}\}$
(b) $\mathbb{W} := \{f \in \mathbb{R}^\infty : \lim_{n\to\infty} f(n) = 0\}$
(c) $\mathbb{W} := \{f \in \mathbb{R}^\infty : \exists a_f, d_f \in \mathbb{R} \text{ so that } f(n) = a_f + (n-1)d_f\}$
(d) $\mathbb{W} := \{f \in \mathbb{R}^\infty : \exists a_f, r_f \in \mathbb{R} \text{ so that } f(n) = a_f r_f^{n-1}\}$
(e) $\mathbb{W} := \{f \in \mathbb{R}^\infty : f(n) \neq 0 \text{ only for finitely many } n \in \mathbb{N}\}$
(f) $\mathbb{W} := \{f \in \mathbb{R}^\infty : f(n) = 0 \ \text{ for infinitely many } n \in \mathbb{N}\}$

27. Suppose X is a non-empty set and $\mathcal{F}[X,\mathbb{F}]$ is the vector space of all functions from X to $\mathbb{F}$. For any $f \in \mathcal{F}[X,\mathbb{F}]$, define:

$$S_f := \{x \in X : f(x) \neq 0\}$$

as the *support* of the function $f(\cdot)$. Let $\mathcal{F}_F \subseteq \mathcal{F}[X,\mathbb{F}]$ such that

$$\mathcal{F}_F := \{f \in \mathcal{F}[X,\mathbb{F}] : S_f \text{ is a finite subset of } X\}.$$

In other words, $\mathcal{F}_F$ is a collection of all functions in $\mathcal{F}[X,\mathbb{F}]$ having a finite support. Argue why $\mathcal{F}_F$ must be a subspace of $\mathcal{F}[X,\mathbb{F}]$.

28. Suppose $\mathbb{V}$ is the vector space (over $\mathbb{R}$) of all functions from $\mathbb{R}$ to $\mathbb{R}$ while $\mathbb{V}_e$ and $\mathbb{V}_o$ are subsets of $\mathbb{V}$ containing functions that satisfy the property $f(x) = f(-x)$ and $f(x) = -f(-x)$, respectively.

(a) Prove that $\mathbb{V}_e$ and $\mathbb{V}_o$ are subspaces of $\mathbb{V}$.

(b) Show that $\mathbb{V} = \mathbb{V}_e \oplus \mathbb{V}_o$.

29. Let $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ and $C = AB$. Show that $\mathrm{Ker}(B)$ is a subspace of $\mathrm{Ker}(C)$. Also show that $\mathrm{Im}(C)$ is a subspace of $\mathrm{Im}(A)$.

30. For two subspaces $\mathbb{U}$ and $\mathbb{W}$ of the vector space $\mathbb{V}$, if every vector belonging to $\mathbb{V}$ either belong to $\mathbb{U}$ or to $\mathbb{W}$ (or both), show that either $\mathbb{V} = \mathbb{U}$, or $\mathbb{V} = \mathbb{W}$ (or both).

31. Provide an example of each of the following:
(a) A non-empty subset of $\mathbb{R}^2$, say $\mathbb{V} \subseteq \mathbb{R}^2$, which is closed under vector addition and under taking additive inverse, but is not a subspace of $\mathbb{R}^2$.
(b) A non-empty subset of $\mathbb{R}^2$, say $\mathbb{U} \subseteq \mathbb{R}^2$, which is closed under scalar multiplication, but is not a subspace of $\mathbb{R}^2$.

32. Let $\mathbb{V}$ be a vector space such that $p, q, r, s \in \mathbb{V}$. Suppose $\mathbb{S} := \langle\{r, s\}\rangle$, $\mathbb{P} := \langle\{p, r, s\}\rangle$, $\mathcal{Q} := \langle\{q, r, s\}\rangle$ are subspaces of $\mathbb{V}$. If $q \in \mathbb{P}$ but $q \notin \mathbb{S}$, show that $p \in \mathcal{Q}$.

What is best in mathematics deserves not merely to be learnt as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the ideal and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs.

—**'The Study of Mathematics (1907)', Bertrand Russell**