EE-635

(Applied Linear Algebra)

Assignment-1

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Soln 3:

Assignment #1

Sol. 3] Given, $n = p.q$, $p \neq q$ and both prime.

$\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$. Clearly, $0 \in \mathbb{Z}_n$ and

$x^2 \big|_{x=0} = 0 \cdot_{(\text{mod } n)} 0 = 0$. $\therefore$ 0 is a solution to $x^2 = 0$.

Now, assuming contrary to the stated:
Suppose $\alpha \in \mathbb{Z}_n$, $\alpha \neq 0$ and $\alpha^2 = \alpha \cdot_{(\text{mod } n)} \alpha = 0$.

Observing that for any two numbers $x, y \in \mathbb{Z}_n$,
for $x \cdot_{(\text{mod } n)} y = 0$, it means that $n (= pq)$ divides the product, which further implies that x or y are divisible by either p or q respectively (as stated in hint).

So for $\alpha \cdot_{(\text{mod } n)} \alpha = 0$ to hold, 'α' must be divisible by n $(= pq)$ which just means that α must be divisible by p as well as q.

$\alpha^2 = \alpha \cdot \alpha = 0 \Rightarrow \alpha \text{ mod } n = 0 \Rightarrow \alpha \text{ mod } p = 0$ AND $\alpha \text{ mod } q = 0$.
$\because p, q \rightarrow$ PRIMES, $\nexists\ \alpha < n \ni \alpha \text{ mod } p = 0$ AND $\alpha \text{ mod } q = 0$.
$\text{LCM}(p, q) = n$.
$\because \alpha \in \mathbb{Z}_n \Rightarrow \alpha < n$. Thus, there is no number in $\mathbb{Z}_n$ satisfying $x^2 = 0$ other than $x = 0 \Rightarrow x^2 = 0$ has a UNIQUE solution.
$\therefore$ Proved.

Soln 5:

Sol. 5] Given $P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \in \mathbb{Z}_7^{2\times 2}$ and $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \in \mathbb{Z}_7^{2\times 2}$.

For P_{11} (= 3 currently) to be = 1, we have to scale it to such a value that it satisfies $P_{11}' \bmod 7 = 1$.
In this manner, we use 'elementary row operations' to convert P to $I^{2\times 2}$ so as to get P^{-1}.

So,

$\because 3\times 5 = \underbrace{15}_{\bmod 7 = 1}$, $\therefore M_1(5) \cdot P = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$. For $(M_1(5)\cdot P)_{21}$ to be 0, adding 6 times 1 to it so that $7 \bmod 7 = 0$.

$\therefore E_{21}(6) = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ \cancel{7} \; 0 & \cancel{32} \; 4 \end{bmatrix}.$$

Now to make 4 to 1, $8 \bmod 7 = 1$ $\therefore M_2(2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ Now, $5+2=7$, $7 \bmod 7 = 0$ $\therefore E_{12}(2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$\because E_{12}(2)\, M_2(2)\, E_{21}(6)\, M_1(5)\, P = I$,

$\therefore P^{-1} = E_{12}(2) M_2(2) E_{21}(6) M_1(5)$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = \boxed{\begin{bmatrix} 6 & 4 \\ 4 & 2 \end{bmatrix}}$$

Soln 7:

Sol. 7] Given $Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$

$\therefore |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 1 & 0 \\ 0 & 5-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)^3 = 0 \Rightarrow \lambda = 5, 5, 5.$

So, $(A - 5I)x = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\because (A-5I)$ here clearly is in ~~a row-rank '2' matrix~~, RREF form with all zeroes in 1^{st} column, $\therefore x_1$ is a free variable.

$\therefore x_1 = c, \; x_2 = 0, \; x_3 = 0.$

$\therefore \boxed{x = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}} \in \mathbb{R}^3.$

Soln 11:

Sol. 11] Requirement: 10 Phosphorus, 9 Potassium, 19 Nitrogen.

$\begin{bmatrix} \text{Pho} \\ \text{Pot} \\ \text{N} \end{bmatrix} \therefore X = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \; Y = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \; Z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

a) $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \rightarrow$ No. of bags of X, Y and Z respectively.

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 0 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 19 \end{bmatrix}$$

$\therefore$ Using EROs, $R_3 \rightarrow R_3 - R_2 - R_1$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} \xrightarrow{R_2 \to R_2/3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix}$$

$$\underline{R_1 \to R_1 - R_2} \therefore \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}, \cancel{R_2 \to R_2 - R_1}$$

$$R_2 \to R_2 - R_1 \therefore \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 0 \end{bmatrix}$$

$\therefore$ We have it equivalent to the system of eqns:

$\alpha + \gamma = 7$ and $\beta - \gamma = -4 \implies \alpha + \beta = 3$

$\because \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$, ~~$\therefore$ for $\alpha + \gamma = 7$,~~

~~$(\alpha, \gamma) = ($~~

The possible triplets (α, β, γ) are:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$\therefore$ The solution is not unique, there are 4 solutions as represented in the column matrices above.

b) Considering (α, β, γ) to be weights of (X, Y, Z) in kgs, the NET expense would be:

Case 1: $0 \times 100 + 3 \times 600 + 7 \times 300 = ₹\,3900$

Case 2: $1 \times 100 + 2 \times 600 + 6 \times 300 = ₹\,3{,}100$

Case 3: $2 \times 100 + 1 \times 600 + 5 \times 300 = ₹\,2{,}300$

Case 4: $3 \times 100 + 0 \times 600 + 4 \times 300 = \boxed{₹\,1{,}500}$

$\therefore$ Least expensive solution is 3 kg X along with 4 kg of Z.

Soln 12:

ChaaF

12] Given, the characteristic of field (let's say we represent it by ~~n~~) of the case n being infinite is 0.

So, considering finite n: $n \to$ prime, $\to$ composite

Case 1: n is composite.

Let $n = a \cdot b$ for ~~some~~ some $a > 1, b > 1$,

By definition of characteristic, $\underbrace{1+1+\dots+1}_{(n)} = n \cdot 1 = 0$.

$\because n = a \cdot b \Rightarrow n \cdot 1 = (n_1 \cdot 1)(n_2 \cdot 1) = 0$

But we are also given that $\mathbb{Z}_n$ is a field,

$\therefore$ using field axioms

either $(n_1 \cdot 1)$ should be $= 0$ or $(n_2 \cdot 1) = 0$ as every field is an integral domain.

but $(n_1 \cdot 1)$ or $(n_2 \cdot 1) \neq 0$, as it will be CONTRADICTION to the minimality of n.

Case 2: n is prime.

Consider any non-zero element 'α' $\in \mathbb{Z}_n$; (n is prime)

then $GCD(\alpha, n) = 1$

$\therefore$ Using Bezout's identity, $\exists\, x, y \in \mathbb{Z} : \alpha x + ny = 1$

Now,

taking mod n on both sides,

$x \cdot (\alpha \bmod n) + y \cdot 0 = 1$

$\Rightarrow \alpha \pmod{n} = x^{-1} \Rightarrow \alpha \cdot 1 = x^{-1} \quad (\alpha \neq 0, \alpha < n)$

This implies that $\underbrace{1+1+\dots+1}_{\alpha} = x^{-1} \neq 0 \quad (\because x \in \mathbb{Z})$

$\therefore \nexists\, \alpha \in \mathbb{Z}_n$ where n is prime $\ni \alpha \cdot 1 = 0$ or char $F = \alpha$

$\therefore$ CharF is nothing but n, in this case. (which is a prime)

$\therefore$ charF $= 0$ or charF $= n \ni n$ is prime. QED

Soln 15:

Sol. 15] Let I_a and I_m be additive and multiplicative identity (assume).

then, $x \oplus I_a = x$ AND $x \otimes I_m = x$

$\Rightarrow \cancel{x} + I_a - 1 = \cancel{x}$ AND $\cancel{x} + I_m - x I_m = \cancel{x}$

$\Rightarrow I_a = 1$ AND $I_m (1-x) = 0$

x CANNOT be fixed, $\therefore I_m = 0$

VERIFYING:

$x \oplus 1 = x + 1 - 1 = x$ ✓ $\qquad x \otimes 0 = x + 0 - x(0) = x$ ✓

$1 \oplus x = 1 + x - 1 = x$ ✓ $\qquad 0 \otimes x = 0 + x - 0(x) = x$ ✓

$\therefore \boxed{I_a = 1}$ AND $\boxed{I_m = 0}$

Now, ~~we~~ checking whether it is an integral domain:

$(x \otimes y = 0 \Leftrightarrow x = 0 \text{ or } y = 0)$

$\therefore x + y - xy = 0 \Rightarrow x = y(x-1) \Rightarrow y = \frac{x}{x-1}$ → Satisfied by (0,0), (2,2) !

$\because$ non-zero numbers (2,2) satisfy, i.e. $2 \otimes 2 = 0$

$\Rightarrow (\mathbb{Z}, \oplus, \otimes)$ is an ~~integral~~ integral domain.

Soln 16:

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16] (Being solved here, so as to assist solution to Qn 17])

Given, $K = \{\text{set of Nos}\}$, $K(\sqrt{d}) := \{(\alpha+\beta\sqrt{d}) : \alpha, \beta \in K, d>0 \text{ is sq-free}\}$

Now, for $K(\sqrt{d})$ to be a field, it is sufficient to prove that it is an integral domain and also has a multiplicative inverse for every non-zero element in it.

Consider $\alpha+\beta\sqrt{d}$, $\alpha, \beta, d \neq 0$.

~~Now~~, Then, its inverse may be taken as $\gamma+\delta\sqrt{d}$.

$\therefore (\alpha+\beta\sqrt{d})(\gamma+\delta\sqrt{d}) = 1 \Rightarrow (\gamma+\delta\sqrt{d}) = \dfrac{1}{(\alpha+\beta\sqrt{d})}$

$(\gamma+\delta\sqrt{d}) = \dfrac{\alpha-\beta\sqrt{d}}{\alpha^2-\beta^2 d} \Rightarrow \gamma = \dfrac{\alpha}{\alpha^2-\beta^2 d}$, $\delta = \dfrac{-\beta}{\alpha^2-\beta^2 d}$

Okay,

now check for ① $K=\mathbb{Z}$: $\mathbb{Z}(\sqrt{d})$: $d, \alpha, \beta \in \mathbb{Z}$.

Then $\gamma+\delta\sqrt{d}$ NEED NOT necessarily belong to $\mathbb{Z}$,

for example: $\alpha = 13$, $\beta = 2$, $d = 15$

(Clearly, $\alpha, \beta, d \in \mathbb{Z}$ and $d = 3\times 5 \rightarrow$ sq-free)

then

$$\gamma+\delta\sqrt{d} = \frac{13}{169-(+2)^2\times 15} + \frac{-(+2)\sqrt{15}}{13^2-(+2)^2\times 15} = \frac{13}{109} - \frac{2\sqrt{15}}{109}$$

Clearly, γ, δ here $\notin \mathbb{Z}$.

$\therefore \mathbb{Z}(\sqrt{d})$ is NOT a field.

All of the same lines above also explain how ② $\mathbb{N}(\sqrt{d})$ is NOT a field.

Soln 17:

17] Constructing $\mathbb{F}_1$, the smallest field in the family that contains rational numbers $\mathbb{Q}$:

$\therefore \mathbb{F}_1 = \alpha + \beta\sqrt{d}$, (noting that $\mathbb{Q}$ is already a field)

Now, $\alpha, \beta, d \in \mathbb{Q}$ is a ~~field~~ FIELD ✓

assuming $\mathbb{F}_n$ to be field, we deduce $\mathbb{F}_{n+1}$ to be a field
(Proof by induction)

$\therefore \mathbb{F}_{n+1} = \{\alpha + \beta\sqrt{d} : \alpha, \beta \in \mathbb{F}_n \text{ and } d = d_n\}$ (d corresponding to sq-free component of $\mathbb{F}_n$)

$\forall x \in \mathbb{F}_{n+1}$, $x = x + 0 \cdot \sqrt{d_n} \in \mathbb{F}_{n+1}$

$\Rightarrow x \in \mathbb{F}_{n+1}$

Thus, $\mathbb{F}_n \in \mathbb{F}_{n+1}$.

Now, consider $y = \alpha + \beta\sqrt{d_n}$ where $\alpha = 0$, $\beta = c$, $c \in \mathbb{N}$.

then $c\sqrt{d_n} \in \mathbb{F}_{n+1}$ but $c\sqrt{d_n} \notin \mathbb{F}_n$

$\therefore \mathbb{F}_n \subset \mathbb{F}_{n+1}$.

Now, for $\mathbb{F}_{n+1}$ to be a field, consider any two $f_1, f_2 \in \mathbb{F}_{n+1}$. ($\alpha_1 + \beta_1\sqrt{d_n}$, $\alpha_2 + \beta_2\sqrt{d_n}$)

★ Checking closure under addn:

$$f_1 + f_2 = \alpha_1 + \alpha_2 + (\beta_1 + \beta_2)\sqrt{d_n} = \alpha_3 + \beta_3\sqrt{d_n}$$

$\because \alpha_1, \alpha_2 \in \mathbb{F}_n \Rightarrow (\alpha_1 + \alpha_2) = \alpha_3 \in \mathbb{F}_n$. ∼ly, $(\beta_1 + \beta_2) = \beta_3 \in \mathbb{F}_n$.

$\therefore f_1 + f_2 \in \mathbb{F}_{n+1}$. ✓

★ Checking closure under multiplication:

$$f_1 \cdot f_2 = (\alpha_1 + \beta_1\sqrt{d_n})(\alpha_2 + \beta_2\sqrt{d_n}) = \underbrace{\alpha_1\alpha_2}_{\in \mathbb{F}_n} + \underbrace{\beta_1\beta_2 d_n}_{\in \mathbb{F}_n} + \underbrace{(\alpha_1\beta_2 + \beta_1\alpha_2)}_{\in \mathbb{F}_n}\sqrt{d_n}$$

$\Rightarrow f_1 \cdot f_2 \in \mathbb{F}_{n+1}$.

★ Additive ~~inf~~ inverse: let $\underbrace{\alpha_2 = -\alpha_1}_{\in \mathbb{F}_n}$, $\underbrace{\beta_2 = -\beta_1}_{\in \mathbb{F}_n}$.

$\therefore f_1 + f_2 = 0 \therefore$ exists. ✓

★ Multiplicative inverse:

now consider $f_1 \in \mathbb{F}_{n+1} \setminus \{0\} := \alpha + \beta\sqrt{d_n}$, $\alpha, \beta \in \mathbb{F}_n$

Let $f^{-1}f = 1 \Rightarrow f^{-1}(\alpha + \beta\sqrt{d_n}) = 1$

$$\Rightarrow f^{-1} = \frac{1}{(\alpha + \beta\sqrt{d_n})} = \frac{\alpha}{\alpha^2 - \beta^2 d_n} - \frac{\beta\sqrt{d_n}}{\alpha^2 - \beta^2 d_n} = a - b\sqrt{d_n}$$

Now, $\because \alpha, \beta, \sqrt{d_n} \in \mathbb{F}_n \Rightarrow \alpha^2, \beta^2, d_n \in \mathbb{F}_n \therefore a, b \in \mathbb{F}_n$.

$\Rightarrow f^{-1} \in \mathbb{F}_{n+1}$. Thus, proved that $\mathbb{F}_{n+1}$ is a FIELD whensoever $\mathbb{F}_n$ is a field.

Thus, we cooked up $\mathbb{F}_i \ \forall i \in \mathbb{Z}^+ \ni \mathbb{Q} \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \mathbb{F}_i \subset \mathbb{F}_n \subset \mathbb{R}$.

To show that $\mathbb{F}_n \subset \mathbb{R}$ and NOT $\mathbb{F} \subseteq \mathbb{R}$:

consider any transcendental number, say, 'e' $(= 2.718\ldots)$

Clearly, $e \in \mathbb{R}$ but $e \notin \mathbb{F}_n$.

$\therefore \mathbb{F}_i = \{\alpha + \beta\sqrt{d_i} : \alpha, \beta \in \mathbb{F}_{i-1} \ \forall i \geq 1\}$

② Constructing $\mathbb{F} \not\subset \mathbb{R}, \ni \mathbb{Q} \subset \mathbb{F} \subset \mathbb{C}$:

$\mathbb{F} = \{\alpha + i\beta : \alpha, \beta \in \mathbb{Q}, \ i = \sqrt{-1}\}$

$\forall q \in \mathbb{Q}, \quad q = q + 0 \cdot i \in \mathbb{F} \Rightarrow \mathbb{Q} \subseteq \mathbb{F}$

Now, consider $p = \alpha + i\beta$ where $\alpha = 0, \ \beta = c \in \mathbb{N}$.

Then ~~$c\sqrt{i}$~~ $ci \in \mathbb{F}$ but $ci \notin \mathbb{Q}$ and $ci \notin \mathbb{R}$.

Thus, $\mathbb{F} \not\subset \mathbb{R}$, and $\mathbb{Q} \subset \mathbb{F}$.

Soln 22:

22] Let $w \in \overset{ⓐ}{W_1} \cap \overset{ⓑ}{W_2} \Rightarrow w \in W_1$ and $w \in W_2$

$\therefore$ $w_{22} = 0$ (from ⓐ) and $w_{12} = w_{21} = 0$ (from ⓑ)

$$\therefore \quad w = \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix}, \ z \in F. \Rightarrow w = z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \forall z \in F$$

Thus, any element $w \in W_1 \cap W_2$ can be spanned by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore \ W_1 \cap W_2 = span\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Now,

consider $v \in W_1 + W_2$.

$$\therefore \quad v = \begin{bmatrix} \alpha + x & \beta + 0 \\ \gamma + 0 & 0 + y \end{bmatrix}, \quad \alpha, \beta, \gamma, x, y \in F$$

$$= \begin{bmatrix} a & \beta \\ \gamma & y \end{bmatrix}, \quad a, \beta, \gamma, y \in F$$

$$\in F^{2\times 2}.$$

Thus, $W_1 + W_2 = F^{2\times 2}$.

Soln 26:

26] a) $X_n \leq X_{n+1}$, $\{x\} \in W \subset \mathbb{R}^\infty$

considering sequence $a\{x\} = y$,

$y_n = a x_n$, $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^\infty$

(note: if $\alpha < 0$ then reverse sequence, $y_n \geq y_{n+1}$)

$\therefore y_n \notin W$, $\therefore \alpha x_n$ NEED NOT $\in W$ $\forall \alpha \in \mathbb{R}$.

$\therefore$ W is NOT a subspace.

b) $W : \lim_{n \to \infty} f(n) = 0$, let $\{x\}, \{y\} \in W$.

$\lim_{n \to \infty} x_n = 0$, $\lim_{n \to \infty} y_n = 0$. $\alpha, \beta \in \mathbb{R}$.

Let $z_n = \alpha x_n + \beta y_n$

$\lim_{n \to \infty} z_n = \alpha \lim_{n \to \infty} x_n + \beta \lim_{n \to \infty} y_n = 0$

$\{z\} \in W$

$\therefore$ W is subspace.

c) $W := \{ f \in \mathbb{R}^\infty : \exists\, a_f, d_f \in \mathbb{R}$ so that $f_n = a_f + (n-1) d_f \}$

$\left.\begin{array}{l} \alpha a_f = a'_f \in \mathbb{R} \\ \alpha d_f = d'_f \in \mathbb{R} \end{array}\right\} \Rightarrow \alpha f \in W$. $\therefore$ W is a subspace.

d) $W := \{ f \in \mathbb{R}^\infty : \exists\, a_n, r_n \ni f(n) = a_f r_f^{\,n-1} \}$

$f_1(n) = a_{f_1} r_{f_1}^{\,n-1}$ $\quad f_2(n) = a_{f_2} r_{f_2}^{\,n-1}$

$f_1(n) + f_2(n) \neq AR^{n-1}$ $\quad (A, R \in \mathbb{R})$

$\therefore$ NOT a subspace.

e) Here,

$\alpha f + \beta g \in W$ as no. of non-zero elements are FINITE

$\therefore$ $W \rightarrow$ is a subspace.

f) $\alpha f(n) = 0$, $\alpha f + \beta g) = 0 + 0 = 0 \quad \forall n$

$\therefore$ $\alpha f + \beta g \in W$

$\therefore$ W is a subspace

Soln 28:

28] Let $\mathbb{V}$ be the vector space of all $f: \mathbb{R} \to \mathbb{R}$;
Let $\mathbb{V}_e \to$ Subset of all even fns : $f(-x) = f(x)$.
$\mathbb{V}_o \to$ subset of all odd fns : $f(-x) = -f(+x)$.

a)
Consider $f_{e_1}, f_{e_2} \in \mathbb{V}_e$ (even fns)
Then for any scalar c,

$$(cf_{e_1} + f_{e_2})(-x) = cf_{e_1}(-x) + f_{e_2}(-x)$$
$$= cf_{e_1}(x) + f_{e_2}(x)$$
$$= (cf_{e_1} + f_{e_2})(x).$$

$\because cf_{e_1} + f_{e_2} \in \mathbb{V}$.
$\therefore \mathbb{V}_e$ is a subspace of $\mathbb{V}$.

Similarly, consider f_{o_1} and $f_{o_2} \in \mathbb{V}_o$ (odd fns)
then for any scalar c,

$$(cf_{o_1} + f_{o_2})(-x) = cf_{o_1}(-x) + f_{o_2}(-x)$$
$$= -cf_{o_1}(x) - f_{o_2}(x) = -(cf_{o_1} + f_{o_2})(x).$$

$\therefore \mathbb{V}_o$ is also a subspace of $\mathbb{V}$.

b) T.P.T $\mathbb{V} = \mathbb{V}_e \oplus \mathbb{V}_o$
Proof: Let $f \in \mathbb{V}$ be arbitrary.

Let $f_e \in \mathbb{V}$, $f_e = \dfrac{f(x) + f(-x)}{2}$,

Let $f_o \in \mathbb{V}$, $f_o = \dfrac{f(x) - f(-x)}{2}$.

Clearly, $f_e \in \mathbb{V}_e$ and $f_o \in \mathbb{V}_o$.

$\because f(x) = \cancel{f_e + f}\ f_e(x) + f_o(x) \ \forall x \Rightarrow f(x) \in \mathbb{V}_e \oplus \mathbb{V}_o$ or $\mathbb{V} \subseteq \mathbb{V}_e \oplus \mathbb{V}_o$

$\therefore \boxed{\mathbb{V} = \mathbb{V}_e \oplus \mathbb{V}_o}$ ($\because f(x) \in \mathbb{V}$, $\therefore \mathbb{V}_e + \mathbb{V}_o \subseteq \mathbb{V}$)

Soln 32:

32] $p, q, r, s \in V$, $S := \langle\{r, s\}\rangle$, $P := \langle\{p, r, s\}\rangle$,
$Q := \langle\{q, r, s\}\rangle$ → all subspaces of V.

T.P.T if $q \in P$ and $q \notin S \Rightarrow p \in Q$.

Proof : $q = \alpha p + \beta r + \gamma s \quad : \alpha, \beta, \gamma \in F$.

CONCEPT : α CANNOT be 0 !!

if $\alpha = 0$, then $q = \beta r + \gamma s$
which contradicts to given that $q \notin S$.
(q CANNOT be spanned by r and s)

$\therefore \boxed{\alpha \neq 0}$.

So,

$$\rightarrow p = \frac{q}{\alpha} - \frac{\beta r}{\alpha} - \frac{\gamma s}{\alpha}, \quad \alpha, \beta, \gamma \in F,$$

$$p = aq - br - cs, \quad a, b, c \in F.$$

$$\Rightarrow p \in \langle\{q, r, s\}\rangle$$

$\Rightarrow p \in Q$ $\quad \therefore$ Proved.

26] a) $x_n \leq x_{n+1}$, $\{x\} \in W \subset \mathbb{R}^\infty$

considering sequence $a\{x\} = y$,

$$y_n = \alpha x_n, \quad \alpha \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^\infty$$

(note : if $\alpha < 0$ then reverse sequence, $y_n \geq y_{n+1}$)

$\therefore y_n \notin W$, $\therefore \alpha x_n$ NEED NOT $\in W$ $\forall \alpha \in \mathbb{R}$.

$\therefore W$ is NOT a subspace.