EE 325
Probability and Random Processes

Quiz 2
September 9, 2023
11:35 am – 12:15 pm

No clarifications. If you think a question is wrong/incomplete, make suitable assumptions. Clearly state and justify these assumptions.

Name:

Roll Number:

	Points Awarded
Question 1	
Question 2	
Question 3	
Question 4	
Total	

1. A company has a new COVID-19 test. If you are infected with the virus, the probability that the test returns a positive result is p. If you are not infected with the virus, the probability that the test returns a positive result is q. Assume that $a \in [0, 1]$ fraction of the population has the virus at a given time. If a person is chosen uniformly at random from the population and is tested and the result comes back positive, what is the probability that the person is infected with the virus? (10 Points)

   Answer: We are given that a fraction of the population is infected, hence $1 - a$ fraction is not infected. It follows that, $\mathbb{P}(\texttt{infected}|\texttt{positive}) = \frac{\mathbb{P}(\texttt{positive, infected})}{\mathbb{P}(\texttt{positive})} = \frac{a \times p}{a \times p + (1-a) \times q}$.

2. Alice and Bob love to challenge each other to coin-tossing contests. On one particular day, Alice brings $2n + 1$ fair coins and lets Bob toss $n + 1$ of those coins, while she tosses the remaining n coins. Compute the probability that after all the coins have been tossed Bob will have gotten more heads than Alice. (10 Points)

   Answer: Let B be the event that Bob tossed more heads, let X be the event that after each has tossed n coins, Bob has more heads than Alice, let Y be the event that under the same conditions, Alice has more heads than Bob, and let Z be the event that they have the same number of heads. Since the coins are fair, we have $P(X) = P(Y)$, and also $P(Z) = 1 - P(X) - P(Y)$. Furthermore, we see that, $P(B|X) = 1$, $P(B|Y) = 0$, $P(B|Z) = 1/2$. Using the law of total probability, $P(B) = P(B|X)P(X) + P(B|Y)P(Y) + P(B|Z)P(Z) = P(X) + \frac{1}{2}P(Z) = \frac{1}{2}(P(X) + P(Y) + P(Z)) = 1/2$.

3. There are four dice in a drawer: one tetrahedron (4 sides), one hexahedron (i.e., cube, 6-sides), and two octahedra (8 sides). Your friend secretly grabs one of the four dice at random. Let S be the number of sides on the chosen die. Now your friend rolls the chosen die without showing it to you. Let R be the result of the roll.

   (a) Plot the probability mass function of S? (4 Points)
   Answer: PMF is given as follows:

S	4	6	8
PMF	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

   The random variable S takes on a value of 0 at all other points.

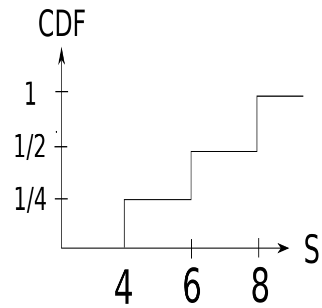
   (b) Plot the cumulative distribution function of S? (4 Points)
   Answer: CDF is given as follows:

S	4	6	8
CDF	$\frac{1}{4}$	$\frac{1}{2}$	1

   This is shown in Fig. 3b

   (c) Compute $\mathbb{P}(S = k|R = 3)$ for $k = 8$. (2 Points)
   Answer: $\mathbb{P}(S = 8|R = 3) = \frac{\mathbb{P}(S=8, R=3)}{\mathbb{P}(R=3)} = \frac{\frac{1}{2} \times \frac{1}{8}}{\frac{1}{2} \times \frac{1}{8} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{6}} = \frac{3}{8}$

4. We call X a Geometric random variable if X takes values $\{1, 2, 3, ...\}$ and $\mathbb{P}(X = m) = pq^{m-1}$, where $0 < p, q < 1$ and also $p + q = 1$. Show that X has memoryless property, i.e., for any two positive integers m, n, $\mathbb{P}(X > m + n|X > n) = \mathbb{P}(X > m)$. (10 Points)



Answer: The CDF of a Geometric random variable X is given as follows:

$$\mathrm{F}_X(k) = \mathbb{P}(X \leq k) = \sum_{k'=1}^{k} \mathbb{P}(X = k') = \sum_{k'=1}^{k} p(1-p)^{k'-1} = 1 - (1-p)^k.$$

Now, $\mathbb{P}(X > m+n | X > n) = \frac{\mathbb{P}(X>m+n)}{\mathbb{P}(X>n)} = \frac{1-\mathrm{F}_X(m+n)}{1-\mathrm{F}_X(n)} = \frac{(1-p)^{m+n}}{(1-p)^n} = (1-p)^m = \mathbb{P}(X > m)$.