EE 325
Probability and Random Processes

Quiz 3
October 21, 2023
2:30 pm – 3:15 pm

No clarifications. If you think a question is wrong/incomplete, make suitable assumptions. Clearly state and justify these assumptions.

Name:

Roll Number:

	Points Awarded
Question 1	
Question 2	
Question 3	
Question 4	
Total	

1. Let X, Y, Z be independent and uniformly distributed random variables on $[0, 1]$. Find the probability that $Z^2 \leq XY$. (10 Points)

Solution -

$$\begin{aligned}
\mathbb{P}(Z^2 \leq XY) &= \mathbb{P}(|Z| \leq \sqrt{XY}) \\
&= \mathbb{P}(Z \leq \sqrt{XY}) \\
&= \int_0^1 \int_0^1 \mathbb{P}(Z \leq \sqrt{xy}) f_X(x) f_Y(y) \mathrm{d}x\mathrm{d}y \qquad (\because 0 \leq Z \leq 1) \\
&= \int_0^1 \int_0^1 \sqrt{xy} \ \ \mathrm{d}x\mathrm{d}y \\
&= \frac{4}{9}
\end{aligned}$$

2. (a) Prove that $\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2 \times \mathrm{cov}(X_1, X_2)$. (8 points)

Solution -

$$\begin{aligned}
\mathrm{var}(X_1 + X_2) &= \mathbb{E}[(X_1 + X_2 - \mathbb{E}[X_1 + X_2])^2] \\
&= \mathbb{E}[\{(X_1 - \mathbb{E}[X_1]) + (X_2 - \mathbb{E}[X_2])\}^2] \\
&= \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2 + (X_2 - \mathbb{E}[X_2])^2 + 2 \cdot (X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])] \\
&= \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2 \cdot \mathrm{cov}(X_1, X_2)
\end{aligned}$$

(b) Prove that $\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$ if X is independent of Y. (2 points)

Solution- For independent random variables X and Y $\mathrm{cov}(X, Y) = 0$.

3. Let

$$f_X(x) = \begin{cases} \dfrac{x^2}{243} & \text{for } 0 \leq x \leq 9 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \min\{\sqrt{X}, 3 - \sqrt{X}\}$. Compute $f_Y(y)$. (10 Points)

Solution -

$$\begin{aligned}
\mathbb{P}(Y \geq y) &= \mathbb{P}(\sqrt{X} \geq y, 3 - \sqrt{X} \geq y) \\
&= \mathbb{P}(y \leq \sqrt{X}, \sqrt{X} \leq 3 - y) \\
&= \mathbb{P}(y \leq \sqrt{X} \leq 3 - y) \qquad (0 \leq y \leq \frac{3}{2}) \\
&= \mathbb{P}(y^2 \leq X \leq (3 - y)^2) \qquad (0 \leq y \leq \frac{3}{2}) \\
&= \int_{y^2}^{(3-y)^2} \frac{x^2}{243} \mathrm{d}x \qquad (0 \leq y \leq \frac{3}{2}) \\
&= \frac{(3 - y)^6 - y^6}{729}
\end{aligned}$$

$\Longrightarrow$

$$f_Y(y) = \begin{cases} \frac{2}{243}(3-y)^5 + y^5 & \text{for } 0 \le y \le \frac{3}{2} \\ 0 & \text{otherwise.} \end{cases}$$

4. Let N_1 and N_2 denote the number of calls arriving at a call center from two different localities in a certain interval of time. Suppose that N_1 and N_2 are independent Poisson random variables with parameters λ_1 and λ_2 respectively.

   (a) What is the PMF of the total number of calls received at the call center? (5 Points)

   Solution - For any $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}
\mathbb{P}(N_1 + N_2 = n) &= \sum_{k=0}^{n} \mathbb{P}(N_1 + N_2 = n | N_1 = k)\mathbb{P}(N_1 = k) \\
&= \sum_{k=0}^{n} \mathbb{P}(N_2 = n - k | N_1 = k)\mathbb{P}(N_1 = k) \\
&= \sum_{k=0}^{n} \mathbb{P}(N_2 = n - k)\mathbb{P}(N_1 = k) \\
&= \sum_{k=0}^{n} \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\frac{e^{-\lambda_1}\lambda_1^{k}}{(k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}
\end{aligned}$$

   The PMF of the total number of calls received at the call center is Poisson with parameter $\lambda_1 + \lambda_2$.

   (b) Obtain the conditional PMF of N_1 given that a total of n calls arrived at the call center. (5 Points)

   Solution - For any $0 \le k \le n$

$$\begin{aligned}
\mathbb{P}(N_1 = k | N_1 + N_2 = n) &= \frac{\mathbb{P}(N_1 + N_2 = n, N_1 = k)}{\mathbb{P}(N_1 + N_2 = n)} \\
&= \frac{\mathbb{P}(N_2 = n - k, N_1 = k)}{\mathbb{P}(N_1 + N_2 = n)} \\
&= \frac{\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\frac{e^{-\lambda_1}\lambda_1^{k}}{(k)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}$$

   The conditional PMF of N_1 given that a total of n calls arrived is Binomial$(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$.