International Pi Thunder Tournament Solutions

Pi Tournament lasts 40 minutes, and is divided into 7 sets of questions. You will start with Set 1, and will receive Set $k+1$ after submitting Set k. You can take as much time as you wish for each set, but you cannot go back to a previous set after submitting it.

Sets 1 through 6 consist of short answer questions. Either your answer is correct or incorrect, and points are given accordingly. Tiebreaker question will only be taken into consideration when breaking ties.

Set 7 (the last set) consists of estimation questions. Your score is based on how close your answer is to the correct answer. Formula for score calculating will be provided in Set 7.

Each question within the set is equally weighted, but each set is weighted differently.

Set 1	10 marks each question;
Set 2	12 marks each question;
Set 3	15 marks each question;
Set 4	18 marks each question/1 addition mark for tiebreaker;
Set 5	23 marks each question/ 2 addition mark for tiebreaker;
Set 6	30 marks each question/ 3 addition mark for tiebreaker;
Set 7	20 marks(Maximum) each question

Acknowledgment to:
Jack Sun
Sny from Numberbasher
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Set 1

1. Find the sum of the roots of equation $\pi x^2 - 2\pi^2 x + 3\pi^3 = e$.

Answer: $\boxed{2\pi}$

Solution: Since this is a quadratic equation, we can solve with $x = (-b \pm \sqrt{b^2 - 4ac})/2a$. Notice that the sum of the two solutions would therefore be $-b/a$, a commonly known result the Vieta Theorem, which evaluates to $\boxed{2\pi}$.

More information can be found here.

2. Given $\odot P$ and $\odot Q$, with radius 3 and 2 respectively, are internally tangent at A. If MP $\perp$ PQ, find the length of MP.

Answer: $\boxed{\sqrt{3}}$

Solution: Since $MP \perp PQ$, we can use the Pythagorean Theorem. Notice that QM and QA are both the radius of $\odot Q$ and PA is the radius of $\odot P$. You can take it from here trivially: $PA = PQ - QA = 1$, $QM = 2$, so $PM = \sqrt{PQ^2 + QM^2} = \boxed{\sqrt{3}}$.

3. A group of 7 people of different heights can be arranged in any order, but they cannot be shorter than both the people on either side of them. How many ways are there to arrange them?

Answer: $\boxed{64}$

Solution: Since no person can be shorter than both people adjacent to him, we are very restricted in the permutations. Notice that there cannot be more than one peak, and that all one-peak formulations work. You can evaluate the number of one-peak formations easily since the tallest person must be the peak and the other people may be on either side; this evaluates to $2^6 = \boxed{64}$.

Set 2

4. Find the smallest positive integer k such that there is exactly one prime number of the form $kx + 60$ for the integers $0 \leq x \leq 10$.

Answer: $\boxed{17}$

Solution: Since the prime number must have either $x = 1$ or $x = 7$

and $k = 1, 7, 11, 13, 17, \cdots$. Notice how $kx + 60$ cannot have small prime factors and must have factors of 7, 11, or higher. You can simply find the smallest k and x such that $kx + 60 = 77$, which gives $x = 1$ and $k = \boxed{17}$.

5. How many ways are there to put mints(at least 1) in a 2x4 grid, but no two mints are adjacent to each other?

Answer: $\boxed{40}$

Solution: Since the number of ways to place in a $2 \times n$ grid can be reduced to an $2 \times (n-1)$ grid with several conditions, we can utilize a dynamic programming approach by tabulating the amount of ways to place mints into a $2 \times n$ grid with placing either 0 or 1 mint in the final column, that is, to define a recursive function $f(n, x)$ = the number of ways to put mints into a $2 \times n$ grid, leaving empty the last column if x is zero and otherwise filling the top-right block. Notice that $f(1, 0) = f(1, 1) = 1$, $f(n, 0) = f(n-1, 0) + 2f(n-1, 1)$ and $f(n, 1) = f(n-1, 0) + f(n-1, 1)$. You can easily get $f(4, 0) + 2f(4, 1) = \boxed{40}$.

6. In cyclic quadrilateral ABCD, BC=CD, $\angle ADC = 2\angle BAD = 100°$. CE is the angle bisector of $\angle BCD$. Find $\angle BED$.

Answer: $\boxed{30}$

Solution: Since $EB = ED$ and $\angle EBC = \angle EDC$ we have that $\triangle EDC$ and $\triangle EBC$ are congruent. Notice that $\angle EBA = \angle EBC - \angle ABC = \angle EDC - (180° - \angle ADC)$ and therefore $\angle BED = \angle BAD - \angle EBA = 180° + \angle BAD - 2\angle ADC$. You can easily find $\angle EBA = \boxed{30°}$ with the given that $\angle ADC = 2\angle BAD = 100°$.

Set 3

7. Let $s(n)$ be the sum of digits of n. Find,

$$\sum_{k=1}^{314} s(n).$$

Answer: $\boxed{3105}$

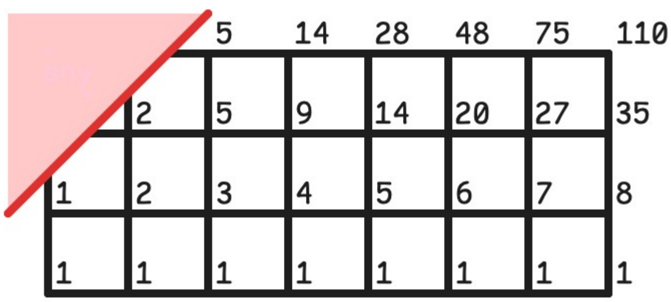
Solution: Since we want to find the sum of the digits, we consider the hundred, ten, and unit places separately for $\sum_{k=0}^{299} s(n)$ and then add

$\sum_{k=300}^{314} s(n)$. Notice that for $\sum_{k=0}^{299} s(n)$, the hundred place has 0, 1, and 2 each 100 times, and the other two places has 0 through 9 each 30 times, which totals to $300 + 1350 \times 2 = 3000$. You can get the final answer by adding $s(300301302303 \cdots 314) = 105$ to get $\boxed{3105}$.

8. The π Tournament committee starts with 2 apples on day 0. Each day, the committee will either eat 1 apple or buy 1 apple. The committee experiences 'panic' if their apple count ever hits 0. After 10 days, the committee has exactly 6 apples and did not experience 'panic' at any point during those 10 days. How many different ways could the committee have chosen to eat or buy apples over these 10 days?

Answer: $\boxed{110}$

Solution: Since we hit 6 apples from 2 apples after 10 days, we buy 7 apples and eat 3 apples in total. Notice that this problem has relatively small numbers, so we use a tabular approach with a 3×7 table, eliminating the cells with 2 eats and no buys or with 3 eats and at most 1 buys. You can finish the table easily and read off the answer $\boxed{110}$ at $(7, 3)$.



9. The 314th multiple of 7 that only have digits 0,1 and 2 is n. Find $\lfloor \log_{10} n \rfloor$.

Answer: $\boxed{7}$

Solution: Since we only have the digits 0, 1, and 2, let us think about base 3. Notice that a number in base 3, when interpreted as base 10, has the same value mod 7. You can get the answer with $\lfloor \log_3(314 \times 7) \rfloor = \boxed{7}$.

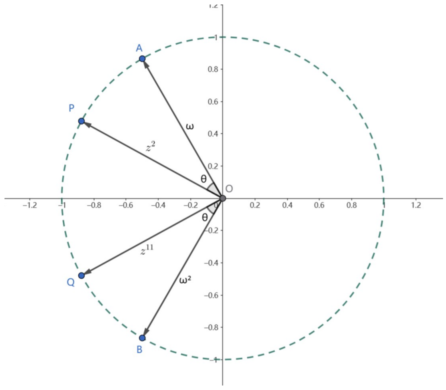
Set 4

10. Let $\omega = \frac{-1+\sqrt{3}i}{2}$. Find the sum of all real numbers t, $0 \le t < 2$, such that $z = \cos(\pi t) + i\sin(\pi t)$ and $|z^2 + \omega| = |z^{11} + \omega^2|$.

Answer: $\boxed{\dfrac{62}{3}}$

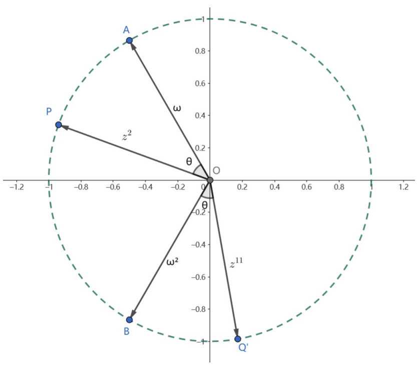
Solution: Note that $|\omega| = |z| = 1$, consider ω, ω^2, z^2, z^{11} as points on the unit circle in the complex plane. $\left|z^2 + \omega\right| = \left|z^{11} + \omega^2\right| \Rightarrow$ the angle between z^2 and ω equals to the angle between z^{11} and ω^2.

Case 1: z^2 and z^{11} are on different sides of ω and ω^2



Note that ω and ω^2 are symmetric about the x-axis. And since the angle between z^2 and ω equals to the angle between z^{11} and ω^2, we have $Arg(z^{11}) + Arg(z^2) = 2k\pi$. Thus, $t = \frac{2k}{13}$, $k = 0, 1, 2, \cdots, 12$.

Case 2: z^2 and z^{11} are on the same side of ω and ω^2



Similarly, this time we have $Arg(z^{11}) - Arg(z^2) = \frac{2\pi}{3} + 2k\pi$. Thus, $t = \frac{2}{27} + \frac{2k}{9}$, $k = 0, 1, \cdots, 8$.

Therefore, $S = \frac{2}{13}(0+1+\cdots+12) + \frac{2}{27}\cdot 9 + \frac{2}{9}(0+1+\cdots+8) = \boxed{\frac{62}{3}}$.

Alternative Method:
First note that $|z| = |\omega| = 1$, $\omega = \overline{\omega^2}$, $\omega^2 = \overline{\omega}$.

$$\left(z^2+\omega\right)\left(\overline{z^2}+\overline{\omega}\right) = \left(z^{11}+\omega^2\right)\left(\overline{z^{11}}+\overline{\omega^2}\right)$$

$$|z|^4 + \omega\overline{z^2} + z^2\overline{\omega} + |\omega|^2 = |z|^{22} + \omega^2\overline{z^{11}} + z^{11}\overline{\omega^2} + |\omega|^4$$

$$\omega\overline{z^2} + z^2\overline{\omega} = \omega^2\overline{z^{11}} + z^{11}\overline{\omega^2}$$

$$\omega(\overline{z^2} - z^{11})\cdot\frac{z^{11}}{\omega} = \omega^2(\overline{z^{11}} - z^2)\cdot\frac{z^{11}}{\omega}$$

$$z^9(1-z^{13}) = \omega(1-z^{13})$$

$$(1-z^{13})(\omega - z^9) = 0$$

Thus, $z^{13} = 1$ or $z^9 = \omega$, which gives the same result of $S = \boxed{\frac{62}{3}}$.

11. If a and b are selected uniformly from $\{0, 1, 2\cdots 63\}$ with replacement, find the expected number of 1's in the binary representation of $a+b$.

Answer: $\boxed{\frac{447}{128}}$

Solution: Let $f(n)$ be the answer for a and b selected from $\{0, 1, \ldots, 2^n - 1\}$. Note that $f(1) = \frac{3}{4}$. Now, consider $f(n+1)$. We can think of a and b as being selected from $\{0, 1, \ldots, 2^n - 1\}$ and then with probability $\frac{1}{2}$ adding 2^n to their sum and with probability $\frac{1}{4}$ adding 2^{n+1}. If nothing is added, the number of ones is just the number of ones in $a+b$. If 2^{n+1} is added, it is that number plus another one in the 2^{n+1}'s place. If 2^n is added, that's an extra 1 if and only if $a+b$ has a 0 in its 2^n's place, i.e. $a+b < 2^n$. This happens with probability $\frac{2^n(2^n+1)}{2\cdot 2^n \cdot 2^n} = \frac{2^n+1}{2^{n+1}}$. Thus, we have

$$f(n+1) = f(n) + \frac{1}{4} + \frac{1}{2}\cdot\frac{2^n+1}{2^{n+1}} = f(n) + \frac{1}{2}\left(1 + \frac{1}{2^{n+1}}\right)$$

From here it is easy to find the explicit formula $f(n) = \frac{n+1}{2} - \frac{1}{2^{n+1}}$. So $f(6) = \boxed{\frac{447}{128}}$.

12. Let I be the incenter of triangle ABC. AC and AB are tangent to $\odot I$ at E and F respectively. EF meets BI at G. M is the middle point of AC. If $\angle ABC = 70°$ and $\angle ACB = 35°$, find $\angle AMG$.

Answer: $\boxed{75°}$

Solution: Find the midpoint N of segment BC, as shown.

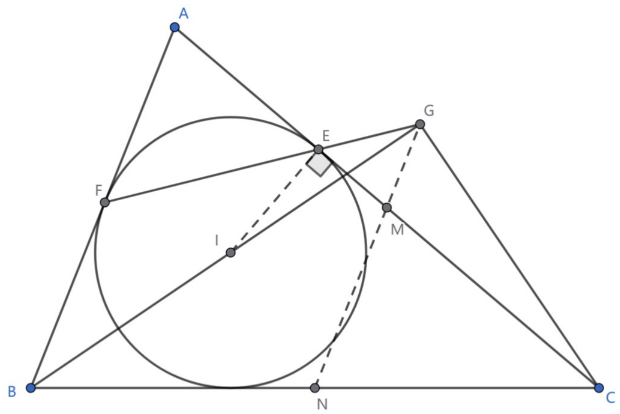
$\angle GEC = \angle AEF = \frac{1}{2}(180° - \angle BAC) = \frac{1}{2}(\angle ABC + \angle ACB) = \angle IBC + \angle ICB = \angle GIC$

$\Rightarrow$ G, E, I, C are cyclic

$\Rightarrow \angle BGC = \angle IGC = \angle IEC = 90°$

Since N is the midpoint of the hypotenuse of right triangle BGC, we have NB=NG=NC. Thus, $\angle BGN = \angle GBN = \angle ABG \Rightarrow GN \parallel AB$.

Additionally, since $MN \parallel AB$, we have G, M, N are collinear. Therefore, $\angle AMG = \angle BAC = 180° - \angle ABC - \angle ACB = \boxed{75°}$.



13(Tiebreaker). $\mathcal{P}$ is a regular 314-gon in the coordinate plane. Let s be the number of distinct x-coordinates that vertices of $\mathcal{P}$ take. Compute the sum of all possible s.

Answer: $\boxed{629}$

Solution: Let $\mathcal{P}$ have vertices $P_1P_2\dots P_{314}$. If no two vertices have the same x-coordinate, then s is 314. Otherwise, two vertices P_i and P_j have the same x-coordinate. Then P_k and P_{i+j-k} also have the same x -coordinate, as $P_iP_j \parallel P_kP_{i+j-k}$.If $i+j$ is odd, the 314 vertices of $\mathcal{P}$ pair up into 157 pairs of the form (P_k, P_{i+j-k}), so s is 157. If $i+j$ is even, then the vertices $P_{\frac{i+j}{2}}$ and $P_{\frac{i+j}{2}+5}$ do not pair up, and the remaining 312 vertices form 156 pairs, so s is 158. Thus, the answer is $314 + 157 + 158 = \boxed{629}$.

Set 5

14. The value of,

$$\prod_{k=1}^{2025}(2^{2^k}-2^{2^{k-1}}+1)$$

Can be expressed as $\frac{aA^2+bA+c}{d}$. Find $a+b+c+d$.

Answer: $\boxed{10}$

Solution: First note that $(a^2+a+1)(a^2-a+1)=(a^2+1)^2-a^2=a^4+a^2+1$.

$$\prod_{k=1}^{2025}(2^{2^k}-2^{2^{k-1}}+1)$$

$$=\frac{1}{7}(2^2+2+1)(2^2-2+1)\prod_{k=2}^{2025}(2^{2^k}-2^{2^{k-1}}+1)$$

$$=\frac{1}{7}(2^{2^2}+2^{2^1}+1)(2^{2^2}-2^{2^1}+1)\prod_{k=3}^{2025}(2^{2^k}-2^{2^{k-1}}+1)$$

$$=\cdots\cdots\cdots$$

$$=\frac{1}{7}(2^{2^{2025}}+2^{2^{2024}}+1)(2^{2^{2025}}-2^{2^{2024}}+1)$$

$$=\frac{(2^{2^{2025}})^2+2^{2^{2025}}+1}{7}$$

Thus, $a+b+c+d=1+1+1+7=\boxed{10}$.

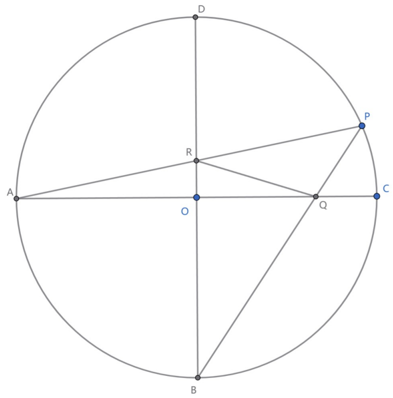
15. Let AC and BD be two perpendicular diameters of $\odot O$. Point P is a moving point on the minor arc CD. Line PA intersects BD at point R, and line PB intersects AC at point Q. Let the areas of $\triangle OQR$ and $\triangle PQR$ be S_1 and S_2, respectively. Given that

$$\frac{1}{2S_1}-\frac{1}{S_2}=\frac{1}{2025}$$

Find the radius of $\odot O$.

Answer: $\boxed{45}$

Solution:



Assume O as the origin $(0,0)$ and $A(-r,0)$, $B(0,-r)$, $C(r,0)$, $D(0,r)$, $P(r\cos\theta, r\sin\theta)$.

$$\Rightarrow Q(\frac{r\cos\theta}{1+\sin\theta}, 0),\ \ R(0, \frac{r\sin\theta}{1+\cos\theta})$$

$$\Rightarrow \overrightarrow{PR} = (-r\cos\theta, \frac{-r\sin\theta\cos\theta}{1+\cos\theta}),\ \ \overrightarrow{PQ} = (\frac{-r\sin\theta\cos\theta}{1+\sin\theta}, -r\sin\theta)$$

$$\Rightarrow S_1 = \frac{r^2\sin\theta\cos\theta}{2(1+\cos\theta)(1+\sin\theta)}$$

$$S_2 = \frac{1}{2}\left(r^2\sin\theta\cos\theta - \frac{r^2\sin^2\theta\cos^2\theta}{(1+\sin\theta)(1+\cos\theta)}\right) = \frac{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}{2(1+\sin\theta)(1+\cos\theta)}$$

$$\Rightarrow \frac{1}{2S_1} - \frac{1}{S_2}$$

$$= \frac{(1+\cos\theta)(1+\sin\theta)}{r^2\sin\theta\cos\theta} - \frac{2(1+\sin\theta)(1+\cos\theta)}{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}$$

$$= \frac{(\sin\theta+\cos\theta-1)(1+\sin\theta)(1+\cos\theta)}{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}$$

$$= \frac{(\sin\theta+\cos\theta-1)(\sin\theta+\cos\theta+1+\sin\theta\cos\theta)}{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}$$

$$= \frac{(\sin\theta+\cos\theta)^2-1+\sin\theta\cos\theta(\sin\theta+\cos\theta-1)}{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}$$

$$= \frac{2\sin\theta\cos\theta+\sin\theta\cos\theta(\sin\theta+\cos\theta-1)}{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}$$

$$= \frac{\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}{r^2\sin\theta\cos\theta(1+\sin\theta+\cos\theta)}$$
$$= \frac{1}{r^2}$$

Thus, $\frac{1}{r^2} = \frac{1}{2025} \Rightarrow r = \boxed{45}$.

Alternative method:

Since P can be any point on $\overset{\frown}{CD}$, let P be the midpoint of $\overset{\frown}{CD}$. Note that $\angle APB = \frac{\pi}{4}$. By symmetry, $\angle PAO = \angle PBO = \frac{\pi}{8}$, $\overset{\frown}{CP}=\overset{\frown}{DP}$.

$\angle DRP = \angle DBP + \angle APB = \overset{\frown}{DP} + \overset{\frown}{AB} = \overset{\frown}{CP} + \overset{\frown}{CB} = \overset{\frown}{PB} = \angle RDP$

Thus, $\triangle PDR$ is an isosceles triangle. By symmetry, $PD = PR = PQ = PC$.

$\Rightarrow S_{\triangle PQR} = \frac{1}{2}PR \cdot PQ\sin\angle RPQ = \frac{\sqrt{2}}{4}PD^2 = \frac{\sqrt{2}}{4}(DB \cdot \sin DBP)^2 = \sqrt{2}r^2\sin^2\frac{\pi}{8}$

$S_{\triangle OQR=} = \frac{1}{2}OR \cdot OQ = \frac{1}{2}(AO \cdot \tan\frac{\pi}{8})(BO \cdot \tan\frac{\pi}{8}) = \frac{1}{2}r^2\tan^2\frac{\pi}{8}$

$\Rightarrow \frac{1}{2S_1} - \frac{1}{S_2} = \frac{1}{r^2\tan^2\frac{\pi}{8}} - \frac{1}{\sqrt{2}r^2\sin^2\frac{\pi}{8}} = \frac{2\cos^2\frac{\pi}{8}-\sqrt{2}}{2r^2\sin^2\frac{\pi}{8}} = \frac{\cos\frac{\pi}{4}+1-\sqrt{2}}{r^2(1-\cos\frac{\pi}{4})} = \frac{1-\frac{\sqrt{2}}{2}}{r^2(1-\frac{\sqrt{2}}{2})} = \frac{1}{r^2}$

Thus, $r^2 = 2025 \Rightarrow r = \boxed{45}$.

16. Find the smallest positive integer n, such that for any positive odd number a,
$$2^{2025} \mid a^{2^n} - 1.$$

Answer: $\boxed{2023}$

Solution:

$$a^{2^n} - 1 = (a^{2^{n-1}} - 1)(a^{2^{n-1}} + 1) = \cdots = (a-1)(a+1)(a^2+1)\cdots(a^{2^{n-1}}+1)$$

Since a is an odd integer, $a^{2^k} + 1 \equiv 2 \pmod 4$ when $k \geq 1$. Thus, $v_2(a^{2^n}-1) = v_2(a-1)+v_2(a+1)+n-1 \geq 2025$. As $v_2(a-1)+v_2(a+1) \geq 3$, the minimum value of n is $2025+1-3 = \boxed{2023}$.

17(Tiebreaker). What is the value of:
$$\int_0^1 \frac{e^x - e^{-x}}{2}\mathrm{d}x + \int_0^{\frac{e^2-1}{2e}} \ln(x+\sqrt{x^2+1})\mathrm{d}x.$$

Answer: $\boxed{\dfrac{e^2-1}{2e}}$

Solution: Note that,

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a)$$

And $\frac{e^x-e^{-x}}{2} = \sinh x$, $\ln(x+\sqrt{x^2+1}) = \sinh^{-1} x$, so the answer is $1\cdot\frac{e^2-1}{2e} - 0 = \boxed{\dfrac{e^2-1}{2e}}$.

Set 6

18. Suppose a, b, c are nonzero real numbers such that,

$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$

Find $a+b+c$.

Answer: $\boxed{-\dfrac{\sqrt[3]{3}}{2}}$

Solution: Note that,

$$abc + 1 = \frac{a}{a+b+c},\ abc+2 = \frac{b}{a+b+c},\ abc+7 = \frac{c}{a+b+c}.$$

$$\Rightarrow 3abc + 10 = \frac{a+b+c}{a+b+c} = 1\ \Rightarrow abc = -3$$

$$\Rightarrow \frac{-3}{a} + \frac{1}{a} = \frac{-3}{b} + \frac{2}{b} = \frac{-3}{c} + \frac{7}{c}$$

$$\Rightarrow a = 2b,\ c = -4b$$

$$(-4b)\cdot 2b + \frac{2}{b} = \frac{1}{2b+b-4b} \Rightarrow b = \frac{\sqrt[3]{3}}{2}$$

Thus, $a+b+c = 2b+b-4b = -b = \boxed{-\dfrac{\sqrt[3]{3}}{2}}$

19. An 8×8 chessboard has alternating black and white squares. How many distinct rectangles, with sides on the grid lines of the checkerboard, that contains at least 3 black squares can be drawn on the chessboard?

Answer: $\boxed{863}$

Solution: There are $\left(\frac{8(8+1)}{2}\right)^2 = 1296$ rectangles in total. We delete those that contain less than 3 black squares.

1×1: $8 \times 8 = 64$ rectangles;
1×2: $7 \times 16 = 112$ rectangles;
1×3: $6 \times 16 = 96$ rectangles;
1×4: $5 \times 16 = 80$ rectangles;
1×5: $2 \times 16 = 32$ rectangles;
2×2: $7 \times 7 = 49$ rectangles;

Thus, there are $1296 - 64 - 112 - 96 - 80 - 32 - 49 = \boxed{863}$ rectangles in total.

20. Let $f(x) = 2^x + 3^x$. For how many integers $1 \leq n \leq 2025$ is $f(n)$ relatively prime to all of $f(0), f(1), \cdots, f(n-1)$?

Answer: $\boxed{11}$

Solution: First note that $f(2^k) \mid f(2^k \cdot m)$ where m is an odd integer,

$$f(2^k \cdot m) = 2^{2^k \cdot m} + 3^{2^k \cdot m} = (2^{2^k})^m + (3^{2^k})^m = (2^{2^k} + 3^{2^k})((2^{2^k})^{m-1} + \cdots + (3^{2^k})^{m-1}).$$

Now we prove that for any $n = 2^k$ $(k = 0, 1, \cdots)$, $f(n)$ is relatively prime to all of $f(0), f(1), \cdots, f(n-1)$. Consider prime number p such that $p \mid f(a)$ and $p \mid f(b)$.

$$3^a + 2^a \equiv 0 \pmod{p} \qquad 3^b + 2^b \equiv 0 \pmod{p}$$

$$3^a \equiv -2^a \pmod{p} \qquad 3^b \equiv -2^b \pmod{p}$$

By Bézout's identity, there exist integers u and v such that $ua + vb = d$, where $d = \gcd(a, b)$.

$$(3^a)^u \equiv (-2^a)^u \pmod{p} \qquad (3^b)^v \equiv (-2^b)^v \pmod{p}$$

$$\Rightarrow 3^{ua} \cdot 3^{vb} \equiv (-1)^u 2^{ua} \cdot (-1)^v 2^{vb} \pmod{p}$$

$$\Rightarrow p \mid 3^{ua+vb} + (-1)^{u+v} 2^{ua+vb}$$

$$\Rightarrow p \mid 3^d + 2^d \quad \textbf{or} \quad p \mid 3^d - 2^d$$

Let $a = 2^k$, $b = t$ where $0 \le t \le 2^k - 1$. As stated above, if $p \mid 3^{2^k} + 2^{2^k}$ and $p \mid 3^t + 2^t$, we have $p \mid 3^{\gcd(2^k,t)} \pm 2^{\gcd(2^k,t)}$. Note that $\gcd(2^k, t) = 2^s$ for some $0 \le s < k$. So we have,

$$p \mid 3^{2^k} + 2^{2^k} \qquad p \mid 3^{2^s} \pm 2^{2^s}$$

When $p \mid 3^{2^s} + 2^{2^s}$, note that

$$3^{2^s} \equiv -2^{2^s} \pmod p \Rightarrow (3^{2^s})^2 \equiv (-2^{2^s})^2 \pmod p \Rightarrow 3^{2^{s+1}} \equiv 2^{2^{s+1}} \pmod p$$

Continue this process we will obtain $3^{2^k} \equiv 2^{2^k} \pmod p \Rightarrow p \mid 3^{2^k} - 2^{2^k}$, which is impossible since $p \mid 3^{2^k} + 2^{2^k}$ and $p \mid 3^{2^k} - 2^{2^k}$ indicates p must simultaneously divide 2^{2^k} and 3^{2^k}.

When $p \mid 3^{2^s} - 2^{2^s}$, note that

$$3^{2^s} - 2^{2^s} = (3+2)(3^2+2^2)\cdots(3^{2^{s-1}} + 2^{2^{s-1}})$$

Thus, $p \mid 3^{2^s} - 2^{2^s}$ indicates p must divide at least one of $3^{2^i} + 2^{2^i}$, where $0 \le i \le s-1$, which goes back to the former case.

Therefore, such p does not exist, and so $f(2^k)$ is relatively prime to all of $f(0), f(1), \cdots, f(2^k - 1)$. When $1 \le n \le 2025$, there are $2^0,\ 2^1,\ \cdots,\ 2^{10} = 1024$, $\boxed{11}$ powers of 2 in total.

21(Tiebreaker). In quadrilateral $HNMS$, let $HS \parallel NM$, and let O be the centroid of $\triangle HNM$. Denote the area of quadrilateral $HOMS$ as L, and let $X = HN \cdot MS$. Given that

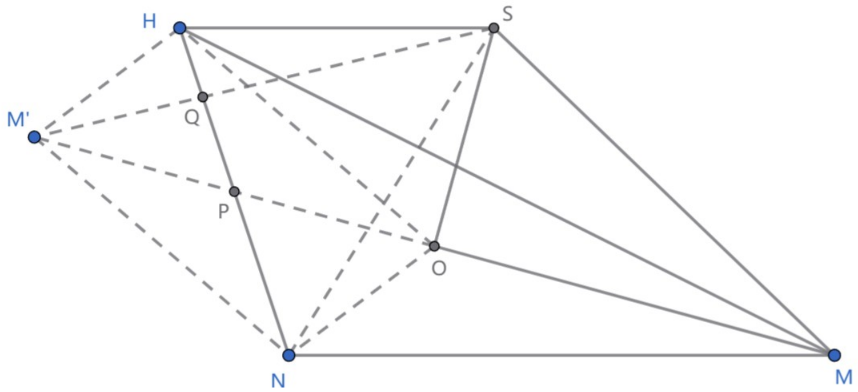
$$\angle MOS = 90^\circ.$$

Find the minimum value of $\frac{X}{L}$.

Answer: $\boxed{2}$

Solution: Let point M' be the reflection of M with respect to line SO as shown. Since MO is perpendicular to SO, M, O and M' are collinear. Let $M'O$ meet HN at P, since O is the centroid of $\triangle HNM$, P is the midpoint of HN and $PO = \frac{1}{2}MO = \frac{1}{2}M'O \Rightarrow PO = PM'$.

$PO = PM'$ and $PN = PH \Rightarrow HONM'$ is a parallelogram.



$L = S_{HOMS} = S_{\triangle HOM} + S_{\triangle HMS} = S_{\triangle HOM'} + S_{\triangle HNS} = S_{\triangle HNM'} + S_{\triangle HNS} = S_{HM'NS}$

Since $M'S = MS$, $S_{HM'NS} = \frac{1}{2}QM' \cdot QH \sin\angle HQM' + \frac{1}{2}QH \cdot QS \sin\angle SQH + \frac{1}{2}QS \cdot QN \sin\angle SQN + \frac{1}{2}QN \cdot QM' \sin\angle NQM' = \frac{1}{2}(QH + QN)(QM' + QS)\sin\angle M'QN = \frac{1}{2}HN \cdot M'S \sin\angle NQM' = \frac{1}{2}HN \cdot MS \sin\angle NQM'$.

Thus, $\frac{X}{L} = \frac{HN \cdot MS}{\frac{1}{2}HN \cdot MS \sin\angle NQM'} = \frac{2}{\sin\angle NQM'} \geq \boxed{2}$. Equality holds if and only if $M'S$ is perpendicular to HN.

Set 7

22. How many numbers less than 3,141,592 are the product of exactly 2 distinct primes? (You will receive $\max(0, \lfloor 20 - 40 \cdot |\frac{N}{A} - 1| \rfloor)$ points, if you submit N and the correct answer is A.)

Answer: $\boxed{627287}$

Solution: Python code is given below:

```
import math

def count_semiprimes():
    n = 3141592
    m = n - 1  # 3141591
    max_q_max = m // 2
    sqrt_n = int(math.isqrt(n))

    sieve = [True] * (max_q_max + 1)
    sieve[0] = sieve[1] = False
```

```
    for i in range(2, int(math.isqrt(max_q_max)) + 1):
        if sieve[i]:
            sieve[i*i : max_q_max+1 : i] = [False] * len(sieve[i*i : max_q_max+1 : i])

    pi = [0] * (max_q_max + 1)
    count = 0
    for i in range(max_q_max + 1):
        if sieve[i]:
            count += 1
        pi[i] = count

    primes_p = [p for p in range(2, sqrt_n + 1) if sieve[p]]

    total = 0
    for p in primes_p:
        max_q = m // p
        if max_q > p:
            total += pi[max_q] - pi[p]

    return total

print(count_semiprimes())
```

23. What is the mean of the answers of all previous 21 questions from Set 1 to Set 6? (Submit a positive integer A. If the correct answer is C and your answer is A, you will receive $\lfloor 20(\min(\frac{A}{C}, \frac{C}{A}))^2 \rfloor$ points.)

Answer: $\boxed{336.363246}$

24. Submit a positive integer $n < 10^5$. Let the product of all valid submission to this question be P_i. (You will receive $\max(0, \left\lfloor 20 - \frac{|(P_i')^\pi - n|}{10\pi} \right\rfloor)$ points, where P_i' is the number of different prime factors of P_i.)

Answer: $\boxed{\textbf{N/A}}$

Solution: P_i' for Pi Tournament 2025 is $\boxed{7}$.