(5) Constraint Based Optimization - Single Constraint:

Revenue function

$$R(x_1, x_2, x_3) = 50\sqrt{x_1} + 40\sqrt{x_2} + 30\sqrt{x_3}$$

x_1 : budget spent on google Ads.
x_2 : budget spent on Facebook Ads.
x_3 : budget spent on LinkedIn Ads.

we want to maximize $R(x_1, x_2, x_3)$

total budget constraint:

$$x_1 + x_2 + x_3 = 100,000$$

Langrange function:

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = 50\sqrt{x_1} + 40\sqrt{x_2} + 30\sqrt{x_3} + \lambda(100000 - x_1 - x_2 - x_3)$$

Partial Derivatives:

- $\frac{\partial L}{\partial x_1} = \frac{50}{2\sqrt{x_1}} - \lambda = 0 \qquad \left(\lambda_1 = \frac{25}{\sqrt{x_1}}\right)$

$\Rightarrow \sqrt{x_1} = \frac{50}{2\lambda} \Rightarrow x_1 = \left(\frac{50}{2\lambda}\right)^2 = \left(\frac{25}{\lambda}\right)^2$

- $\frac{\partial L}{\partial x_2} = \frac{40}{2\sqrt{x_2}} - \lambda = 0 \qquad \left(\lambda_2 = \frac{20}{\sqrt{x_2}}\right)$

$\Rightarrow \sqrt{x_2} = \frac{40}{2\lambda} \Rightarrow x_2 = \left(\frac{20}{\lambda}\right)^2$

- $\frac{\partial L}{\partial x_3} = \frac{30}{2\sqrt{x_3}} - \lambda = 0 \qquad \left(\lambda_3 = \frac{15}{\sqrt{x_3}}\right)$

$\Rightarrow \frac{30}{2\lambda} = \sqrt{x_3} \Rightarrow x_3 = \left(\frac{15}{\lambda}\right)^2$

Solve for λ

$$x_1 + x_2 + x_3 = 100000$$

$$\Rightarrow \left(\frac{25}{\lambda}\right)^2 + \left(\frac{20}{\lambda}\right)^2 + \left(\frac{15}{\lambda}\right)^2 = 100{,}000$$

$$\Rightarrow \frac{625}{\lambda^2} + \frac{400}{\lambda^2} + \frac{225}{\lambda^2} = 100\,000$$

$$\Rightarrow \frac{1250}{\lambda^2} = 100000$$

$$\Rightarrow \lambda^2 = \frac{1250}{100000}$$

$$\Rightarrow \lambda^2 = \frac{1}{80}$$

$$\Rightarrow \lambda = \frac{1}{\sqrt{80}} \approx 0.1118$$

- Compute Optimal Values:

$$x_1 = \left(\frac{25}{\lambda}\right)^2 = \left(\frac{25}{0.1118}\right)^2 \approx (223.7)^2 = 50000$$

$$x_2 = \left(\frac{20}{\lambda}\right)^2 = \left(\frac{20}{0.1118}\right)^2 \approx (178.9)^2 = 32000$$

$$x_3 = \left(\frac{15}{\lambda}\right)^2 = \left(\frac{15}{0.1118}\right)^2 \approx (134.2)^2 = 18000$$

- Verify the solution.

$$x_1 + x_2 + x_3 = 50000 + 32000 + 18000 = 100000$$

Optimal solution most likely is

$x_1 = 50000$

$x_2 = 32000$

$x_3 = 18000$

The company should allocate the budget as follows

Google Ads - 50,000
Facebook Ads - 32,000
LinkedIn Ads - 18,000