3. Image Compression using SVD

Define image matrix

$$A = \begin{bmatrix} 5 & 10 & 15 \\ 20 & 25 & 30 \\ 35 & 40 & 45 \end{bmatrix}$$ (matrix represents pixel insities $[0, 255]$)

A^T (transpose of A)

- Row 1 of A^T is column 1 of A
- Row 2 of A^T is column 2 of A
- Row 3 of A^T is column 3 of A

$$A^T = \begin{bmatrix} 5 & 20 & 35 \\ 10 & 25 & 40 \\ 15 & 30 & 45 \end{bmatrix}$$

- Multiply $A^T A$

$A^T A$ is a 3×3 matrix:

$$(A^T A)_{ij} = \sum_{l=1}^{3} (A^T)_{ik} A_{kj}$$

$$A^T = \begin{bmatrix} 5 & 20 & 35 \\ 20 & 25 & 40 \\ 15 & 30 & 45 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & 20 & 15 \\ 20 & 25 & 30 \\ 35 & 40 & 45 \end{bmatrix}$$

$C = A^T A$

$C_{11} = 5.5 + 20.20 + 35.35 = 25 + 400 + 1225 = 1650$

$C_{12} = 5.20 + 20.25 + 35.40 = 100 + 500 + 1400 = 2000$

$C_{13} = 5.15 + 20.30 + 35.45 = 75 + 600 + 1575 = 2250$

$C_{21} = 20.5 + 25.20 + 40.35 = 100 + 500 + 1400 = 2000$

$C_{22} = 20.20 + 25.25 + 40.40 = 2625$

$C_{23} = 2850$

$C_{31} = 2250$

$C_{32} = 2850$

$C_{33} = 3150$

- Final A^TA matrix

$$C = \begin{bmatrix} 1650 & 2000 & 2250 \\ 2000 & 2625 & 2850 \\ 2250 & 2850 & 3150 \end{bmatrix}$$

Date / /
Page

- Multiply AA^T

- $(AA^T)_{11} = 650$
- $(AA^T)_{12} = 1050$
- $(AA^T)_{13} = 1650$
- $(AA^T)_{21} = 1050$
- $(AA^T)_{22} = 1925$
- $(AA^T)_{2,3} = 3050$
- $(AA^T)_{3,1} = 1650$
- $(AA^T)_{3,2} = 3050$
- $(AA^T)_{3,3} = 4850$

$$AA^T = \begin{bmatrix} 650 & 1050 & 1650 \\ 1050 & 1950 & 3050 \\ 1650 & 3050 & 4850 \end{bmatrix}$$

- Setup characteristic polynomial (find eigen values & vector)

$$\det(A^TA - \lambda I) = 0$$

$$A^TA - \lambda I = \begin{bmatrix} 1650-\lambda & 2000 & 2250 \\ 2000 & 2625-\lambda & 2850 \\ 2250 & 2850 & 3150-\lambda \end{bmatrix}$$

take 25 common to make calc. easier

$$M = \begin{bmatrix} 66 & 80 & 90 \\ 80 & 105 & 114 \\ 90 & 114 & 126 \end{bmatrix}$$

$$\therefore A^TA = 25 \begin{bmatrix} 66 & 80 & 90 \\ 80 & 105 & 114 \\ 90 & 114 & 126 \end{bmatrix}$$

characteristic polynomial of M

$$p(\mu) = \det(M - \mu I)$$

$$M - \mu I = \begin{bmatrix} 66-\mu & 80 & 90 \\ 80 & 105-\mu & 114 \\ 90 & 114 & 126-\mu \end{bmatrix}$$

$$\Rightarrow (66-\mu)\left[(105-\mu)(126-\mu) - 114\times 114\right] - 80\left[80(126-\mu) - 114-90\right]$$
$$+ 90\left[80\times 114 - (105-\mu)90\right]$$

$$\Rightarrow p(\mu) = \mu^3 - 297\mu^2 + 980\mu - 144 = 0$$

- Numerical approximation of roots

$\mu_1 \approx 0.154$
$\mu_2 \approx 3.18$
$\mu_3 \approx 293.66$

- $\mu_1 + \mu_2 + \mu_3 = \text{trace}(M) = 66 + 105 + 126$

- $\mu_1 \mu_2 \mu_3 = \det(M) = 144$

- $\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = 980$

Date
Page

Eigen Values of A^TA

$\because A^TA = 25M$, each eigen value is multiplied by 25

$\lambda_1 \approx 25 \times 0.154 = 3.85$
$\lambda_2 \approx 25 \times 3.18 = 79.5$
$\lambda_3 \approx 25 \times 293.66 = 7341.5$

Extract the Singular Values:

$\sigma_1 = \sqrt{\lambda_3} \approx \sqrt{7341.5} \approx 85.68$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{79.5} \approx 8.93$

$\sigma_3 = \sqrt{\lambda_1} = \sqrt{3.85} \approx 1.96$

$\sigma_1 > \sigma_2 > \sigma_3$, singular values

Σ matrix:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 85.68 & 0 & 0 \\ 0 & 8.93 & 0 \\ 0 & 0 & 1.96 \end{bmatrix}$$

- Largest Eigen Value:

$\lambda_3 = 7341.5$

eigen-equation for M is

$$(M - \mu_3 I)v = 0$$

$$\begin{cases} (66-293.66)x + 80y + 90z = 0 \\ 80x + (105-293.66)y + 114z = 0 \\ 90x + 114y + (126-293.66)z = 0 \end{cases}$$

$$\begin{cases} -227.66x + 80y + 90z = 0 \\ 80x - 188.66y + 114z = 0 \\ 90x + 114y - 167.66z = 0 \end{cases}$$

we get,

$x = 1, \quad y \approx 1.27, \quad z \approx 1.40$

$\|v_1\| = \sqrt{1^2 + (1.27)^2 + (1.40)^2} \approx \sqrt{1 + 1.6129 + 1.96} \approx \sqrt{4.5729}$

$= 2.14$ (norm)

$$v_1 = \left(\frac{1}{2.14}, \frac{1.27}{2.14}, \frac{1.40}{2.14}\right) = (0.467, 0.593, 0.654)$$

- for $\mu_2 \approx 79.5$, $(\lambda_2 = 79.5)$

$$(M - 3.18 I) v = 0$$

$$\begin{bmatrix} 66-3.18 & 80 & 90 \\ 80 & 105-3.18 & 114 \\ 90 & 114 & 126-3.18 \end{bmatrix} v = 0$$

$$v_2 \approx (1, -0.741, -0.04)$$

$$\|v_2\| \approx \sqrt{1^2 + (-0.741)^2 + (-0.04)^2} \approx 1.245$$

$$v_2 \approx (0.803, -0.595, -0.032)$$

- for $\mu_1 \approx 0.154$ $(\lambda_1 \approx 3.85)$

$$(M - 0.154 I) v = 0$$

$$\begin{bmatrix} 65.846 & 80 & 90 \\ 80 & 104.846 & 114 \\ 90 & 114 & 125.846 \end{bmatrix} v = 0$$

$$v_3 \approx (1, 0.935, -1.561)$$

$$\|v_3\| \approx 2.077$$

$$v_3 \approx (0.481, 0.450, -0.752)$$

$$V \approx \begin{bmatrix} 0.461 & 0.803 & 0.481 \\ 0.593 & -0.595 & 0.450 \\ 0.654 & -0.032 & -0.752 \end{bmatrix}$$

Finding U (Left Singular Vector)

$$u_i = \frac{1}{\sigma_i} A v_i$$

$\sigma_1 \approx 85.68$ $\quad \sigma_2 \approx 8.93$ $\quad \sigma_3 \approx 1.96$

(a) Compute u_1 from v_1.

$$v_1 \approx (0.461, 0.593, 0.654)$$

$$Av_1 = \begin{bmatrix} 5 \times 0.461 + 20 \times 0.593 + 15 \times 0.654 \\ 20 \times 0.467 + 25 \times 0.593 + 30 \times 0.654 \\ 35 \times 0.467 + 40 \times 0.593 + 45 \times 0.654 \end{bmatrix} \approx \begin{bmatrix} 24.005 \\ 43.785 \\ 69.495 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1} \approx \frac{(24.005, 43.785, 69.495)}{85.68}$$

$$\approx (0.280, 0.511, 0.811)$$

(b) Compute u_2 from v_2

$$v_2 \approx (0.803, -0.595, -0.032)$$

$$Av_2 \approx \begin{bmatrix} 5\times0.803 + 20\times(-0.595) + 15.(-0.032) \\ 20\times0.803 + 25\times(-0.595) + 30\times(-0.032) \\ 35\times0.803 + 40\times(-0.595) + 45\times(-0.032) \end{bmatrix}$$

$$\approx \begin{bmatrix} -8.365 \\ 0.225 \\ 2.865 \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} \approx \frac{(-8.365, 0.225, 2.865)}{8.93}$$

$$\approx (-0.937, 0.025, 0.321)$$

(c) Compute u_3 from v_3

$$v_3 \approx (0.481, 0.450, -0.752)$$

$$Av_3 \approx \begin{bmatrix} 5\times0.481 + 20\times0.450 + 15\times(-0.752) \\ 20\times0.481 + 25\times0.450 + 30\times(-0.752) \\ 35\times0.481 + 40\times0.450 + 45\times(-0.752) \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.125 \\ -1.69 \\ 0.995 \end{bmatrix}$$

$$u_3 = \frac{Av_3}{\sigma_3} \approx \frac{(0.125, -1.69, 0.995)}{1.96} \approx (0.064, -0.862, 0.508)$$

3. Final SVD components

- Right singular vectors (V):

$$V = \begin{bmatrix} 0.461 & 0.803 & 0.481 \\ 0.593 & -0.595 & 0.450 \\ 0.659 & -0.032 & -0.752 \end{bmatrix}$$

- Singular values (diagonal of Σ):

$$\Sigma = \begin{bmatrix} 85.68 & 0 & 0 \\ 0 & 8.93 & 0 \\ 0 & 0 & 1.96 \end{bmatrix}$$

- Left singular vectors (U):

$$U = \begin{bmatrix} 0.280 & -0.937 & 0.064 \\ 0.511 & 0.025 & -0.862 \\ 0.811 & 0.321 & 0.508 \end{bmatrix}$$

$$A = U \Sigma V^T$$

Interpretation:

- σ_1 is significantly larger than σ_2 and σ_3.

if we choose to keep only the largest singular value ($k=1$)

$$A_1 = \sigma_1 u_1 v_1^T$$

we keep dominant features while ignoring finer details.

- σ_1 represents most important feature.
- σ_2 and σ_3 add finer details.

Rank 1 - approximation

$$A_1 = \sigma_1 u_1 v_1^T$$

- $\sigma_1 \doteq 85.68$
- $u_1 = \begin{bmatrix} 0.280 \\ 0.511 \\ 0.811 \end{bmatrix}$
- $v_1 = \begin{bmatrix} 0.467 \\ 0.593 \\ 0.654 \end{bmatrix}$

- $(u_1 v_1^T)_{ij} = (u_1)_i \times (v_1)_j$

$$u_1 v_1^T = \begin{bmatrix} 0.13076 & 0.16604 & 0.1831 \\ 0.23854 & 0.30292 & 0.3341 \\ 0.37834 & 0.48052 & 0.5300 \end{bmatrix}$$

multiply by $\sigma_1 = 85.68$

$$A_1 = \begin{bmatrix} 11.20 & 14.23 & 15.69 \\ 20.42 & 25.96 & 28.65 \\ 32.41 & 41.19 & 45.40 \end{bmatrix}$$

A_1 represents compressed structure it contains dominant structure of A.

- we achieve a significant compression. with using rank 1 compression.