(5) Constraint Based Optimization - Single Constraint:

Revenue function

$R(x_1, x_2, x_3) = 50\sqrt{x_1} + 40\sqrt{x_2} + 30\sqrt{x_3}$

x_1 : budget spent on google Ads.
x_2 : budget spent on Facebook Ads.
x_3 : budget spent on LinkedIn Ads.

we want to maximize $R(x_1, x_2, x_3)$

total budget constraint:

$$x_1 + x_2 + x_3 = 100,000$$

Langrange function:

$$L(x_1, x_2, x_3, \lambda) = 50\sqrt{x_1} + 40\sqrt{x_2} + 30\sqrt{x_3} + \lambda(100000 - x_1 - x_2 - x_3)$$

Compute Gradients

- $\frac{\partial L}{\partial x_1} = \frac{50}{2\sqrt{x_1}} - \lambda = 0$

$$\Rightarrow \lambda = \frac{25}{\sqrt{x_1}}$$

- $\frac{\partial L}{\partial x_2} = \frac{40}{2\sqrt{x_2}} - \lambda \Rightarrow \lambda = \frac{20}{\sqrt{x_2}}$

- $\frac{\partial L}{\partial x_3} = \frac{30}{2\sqrt{x_3}} - \lambda \Rightarrow \lambda = \frac{15}{\sqrt{x_3}}$

- $\frac{\partial L}{\partial \lambda} = 100000 - x_1 - x_2 - x_3 = 0$

$$\Rightarrow x_1 + x_2 + x_3 = 100000$$

Parameter Initialization

$x_1 = x_2 = x_3 = 33\,333.33$

$\alpha = 5000$

$\alpha_\lambda = 0.0001$

$\lambda = 1$

1. for x_1:

$$\frac{\partial L}{\partial x_1} = \frac{25}{\sqrt{x_1^{(0)}}} = \frac{25}{\sqrt{33333.33}} = \frac{25}{182.57} \approx 0.1369$$

$$g_{x_1} = 0.1369 - 1 = -0.8631$$

2. for x_2:

$$\frac{\partial L}{\partial x_2} = 0.1095 - 1 = 0.8905$$

3. for x_3:

$$\frac{\partial L}{\partial x_3} = 0.0822 - 1 = -0.9178$$

Update the variables

- for x_1:

$$x_1^{(1)} = 33333.33 + 3000(-0.8631) = 29017.83$$

- for x_2:

$$x_2^{(1)} = 33333.33 + 3000(-0.8905) = 28880.83$$

- for x_3:

$$x_3^{(1)} = 33333.33 + 3000(-0.9178) \approx = 28744.33$$

after updating values

$$x_1^{(1)} + x_2^{(1)} + x_3^{(1)} \approx 29017.83 + 28880.83 + 28744.33$$
$$= 86643.99$$

constrait violation:

$$g_\lambda = 100000 - 86643.99 = 13356.01$$

$$\lambda^{(1)} = 1 + 0.0001 \times 13356.01$$

$$\approx 1 + 133.56 = 134.56$$

Interpretation

- gradients for x_1, x_2 and x_3 are negative
  $\Rightarrow$ decreasing their allocations
- the total allocation dropped to about 86,644 far below 100,000,

thus λ jumped from 1 to about 2335

Iteration 2

$x_1 = 29018.33$
$x_2 = 28880.83$
$x_3 = 28744.33$
$\lambda^{(1)} = 2.33565$

calculate gradients

- $g_{x_1} = \frac{25}{\sqrt{29018.33}} - \lambda^{(1)}$

$g_{x_1} = 0.1467 - 2.33565 = -2.18895$

- $g_{x_2} = \frac{20}{\sqrt{28880.82}} - \lambda = 0.11765 - 2.33565$

$= -2.2180.$

- $g_{x_3} = \frac{15}{\sqrt{28744.33}} - \lambda^{(1)} = -2.24725$

$g_\lambda = 100000 - (x_1^{(1)} + x_2^{(1)} + x_3^{(1)}) = 100000 - 86643.46$

$= 13356.51$

(B) Update the x values:

Using $\alpha_x = 5000$:

$x_1^{(2)} = 29018.33 + 5000 \times (-2.18895)$

$= 18073.58$

$x_2^{(2)} = 17790.83$

$x_3^{(3)} = 17508.08$

sum = 53372.49

Update λ

$\lambda^{(2)} = \lambda^{(1)} + 0.0001 \times 46627.51 = 6.9984$

thus,

- Sum dropped more due to -ve value of gradients and λ jumped to 6.9984, to push iteration values upward.

we see that for further iteration the model doesnot converge.

to mitigate this problem we can

- reduce learning rate
- use constraint based optimization.

It will not converge due to high learning rate.