MTech CSE – 1st Semester
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Question

Prove that if a matrix A is non-singular, then its Schur complement is also non-singular.

Answer

Preliminaries and Definitions

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where:

- $A_{11} \in \mathbb{R}^{k\times k}$,
- $A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$,

and the block dimensions are compatible.

We assume that A_{11} is invertible.

This assumption ensures that the Schur complement of A_{11} in A is well-defined.

Definition: Schur Complement

The Schur complement of A_{11} in A is defined as

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

Goal

We are given that:

- the full matrix A is non-singular,

and we must prove that:

- the Schur complement S is also non-singular.

Key Idea of the Proof

The proof relies on:

- block Gaussian elimination,
- factorization of the matrix A,
- the fact that a product of matrices is invertible if and only if each factor is invertible.

Step 1: Block Factorization of A

Using block Gaussian elimination, the matrix A can be factored as:

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix},$$

where

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

Step 2: Analyze Invertibility of Each Factor

First Factor

Let

$$L = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}.$$

This is a block lower triangular matrix with identity matrices on the diagonal. Hence, L is invertible.

Second Factor

Let

$$U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}.$$

This is a block upper triangular matrix. Its determinant is given by

$$\det(U) = \det(A_{11}) \cdot \det(S).$$

Step 3: Use Non-Singularity of A

Since

$$A = LU,$$

and A is non-singular while L is invertible, it follows that U must also be non-singular. Therefore,

$$\det(A_{11}) \cdot \det(S) \neq 0.$$

Because A_{11} is assumed invertible, we have

$$\det(A_{11}) \neq 0,$$

which implies

$$\det(S) \neq 0.$$

Thus, the Schur complement S is non-singular.

Final Conclusion

If A is non-singular and A_{11} is invertible, then its Schur complement

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

is also non-singular.

Worked Numerical Example

Consider the matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Partition A as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where:

$$A_{11} = [2], \quad A_{12} = [1 \;\; 0], \quad A_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Since $\det(A) = 8 \neq 0$, the matrix A is non-singular, and A_{11} is invertible with $A_{11}^{-1} = \frac{1}{2}$.

The Schur complement of A_{11} is:

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{5}{2} & 1 \\ 1 & 2 \end{bmatrix}$$

The determinant of S is:

$$\det(S) = \frac{5}{2} \cdot 2 - 1 = 4 \neq 0$$

Thus, the Schur complement is non-singular, illustrating the result.

Converse Result

We now prove the converse statement.

Theorem

If A_{11} and its Schur complement

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

are both non-singular, then the block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is non-singular.

Proof

Using block matrix factorization, we write:

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}$$

The first factor is invertible since it is block lower triangular with identity matrices on the diagonal. The second factor is invertible because both A_{11} and S are invertible.

Since A is a product of two invertible matrices, it is itself invertible. Hence, A is non-singular. $\square$

Relation to LU and LUP Decomposition

The Schur complement arises naturally during LU and LUP decomposition of block matrices.

In block LU decomposition, a matrix A is factorized as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}$$

where S is the Schur complement of A_{11}.

Thus, the Schur complement represents the remaining block after eliminating the variables corresponding to A_{11} using Gaussian elimination. Non-singularity of S ensures that the elimination process can proceed without breakdown.

In the presence of pivoting, the same structure appears in LUP decomposition, where row permutations are applied before computing the Schur complement. The non-singularity of Schur complements is therefore essential for the stability and correctness of LU/LUP-based algorithms. This block factorization underlies the $O(n^3)$ complexity of LU decomposition and the correctness of subsequent $O(n^2)$ solve phases.

Remarks

This result is fundamental in:

- block LU decomposition,
- numerical linear algebra,
- matrix inversion formulas,
- optimization and control theory.

A symmetric result holds if A_{22} is invertible, in which case the Schur complement of A_{22} is also non-singular.

Interpretation (Intuition)

Schur complements arise naturally when eliminating variables in block linear systems. Non-singularity of the full system implies that no information is lost during elimination. Hence, the reduced system represented by the Schur complement must also be non-singular.