MTech CSE – 1st Semester
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Question

Prove that every connected component of the symmetric difference of two matchings in a graph G is either a path or an even-length cycle.

Answer

Preliminaries and Definitions

Let $G = (V, E)$ be an undirected graph.

Definition 1: Matching

A matching $M \subseteq E$ is a set of edges such that no two edges in M share a common endpoint.

Let M_1 and M_2 be two matchings in G.

Definition 2: Symmetric Difference

The symmetric difference of M_1 and M_2 is defined as:

$$M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1).$$

That is, $M_1 \oplus M_2$ consists of edges that belong to exactly one of the two matchings.

Goal

We must show that every connected component of the graph

$$H = (V,\, M_1 \oplus M_2)$$

is either:

- a path, or
- a cycle of even length.

Key Observations

Observation 1: Degree Bound

Fix any vertex $v \in V$.

Since M_1 is a matching, at most one edge of M_1 is incident to v. Since M_2 is a matching, at most one edge of M_2 is incident to v.

Therefore, in the symmetric difference $M_1 \oplus M_2$,

$$\deg_H(v) \leq 2.$$

Observation 2: Structure of Graphs with Maximum Degree 2

Any graph in which every vertex has degree at most 2 consists of connected components that are:

- paths, or
- cycles.

Thus, each connected component of H must be either a path or a cycle. It remains to prove that every cycle has even length.

Alternating Structure of the Symmetric Difference

Observation 3: Alternation of Edges

Consider any vertex v with $\deg_H(v) = 2$.

One incident edge must belong to M_1, and the other must belong to M_2. This is because no matching can contribute more than one incident edge at v.

Hence, along any connected component of H, edges alternate between:

$$M_1, M_2, M_1, M_2, \ldots$$

Analysis of Connected Components

Case 1: The Component Is a Path

If a connected component contains a vertex of degree 1, then it is a path. Such a path may start and end at vertices that are unmatched in one or both matchings.

No restriction is imposed on the length of such a path. Hence, paths are valid connected components.

Case 2: The Component Is a Cycle

Suppose a connected component is a cycle. Then every vertex on the cycle has degree exactly 2.

Edges on the cycle alternate between M_1 and M_2. Let the cycle have length k.

Since the edges alternate, exactly half of the edges belong to M_1 and half to M_2. This is possible only if k is even.

Therefore, every cycle in $M_1 \oplus M_2$ has even length.

Final Conclusion

We have shown that:

- every vertex in $M_1 \oplus M_2$ has degree at most 2,
- hence, every connected component is either a path or a cycle,
- all cycles must be of even length due to edge alternation.

Thus, every connected component of $M_1 \oplus M_2$ is either a path or an even-length cycle.

Worked Example (Textual Diagram)

Consider the graph with vertices:

$$V = \{a, b, c, d, e\}.$$

Let the two matchings be:

$$M_1 = \{(a, b), (c, d)\}, \quad M_2 = \{(b, c), (d, e)\}.$$

Textual Diagram Representation

We describe the graph structure textually:

$$a — b — c — d — e$$

Edges in the matchings:

- Edges in M_1: (a, b) and (c, d),
- Edges in M_2: (b, c) and (d, e).

The symmetric difference is:

$$M_1 \oplus M_2 = \{(a, b), (b, c), (c, d), (d, e)\}.$$

Resulting Component

The symmetric difference forms a single connected component:

$$a \to b \to c \to d \to e,$$

which is a *path*.

The edges alternate between M_1 and M_2:

$$M_1,\ M_2,\ M_1,\ M_2.$$

This example illustrates how a connected component of the symmetric difference can form a path with alternating edges.

Worked Example (Even-Length Cycle)

Consider the graph with vertices:

$$V = \{v_1, v_2, v_3, v_4\}.$$

Let the two matchings be:

$$M_1 = \{(v_1, v_2), (v_3, v_4)\}, \quad M_2 = \{(v_2, v_3), (v_4, v_1)\}.$$

Textual Diagram Representation

The graph can be described as a cycle:

$$v_1 — v_2 — v_3 — v_4 — v_1$$

Edges alternate between the two matchings around the cycle.

Symmetric Difference

Since no edge is common to both matchings, the symmetric difference is:

$$M_1 \oplus M_2 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}.$$

This forms a single connected component that is a cycle of length 4.

Observation

The edges alternate between M_1 and M_2, and the cycle length is even. This illustrates the second possible structure of a connected component in the symmetric difference.

Relation to Augmenting Paths and Maximum Matching

This structural result has a direct and fundamental connection to augmenting paths in matching theory.

Let M be a matching and M^* be a maximum matching in a graph G. Consider the symmetric difference:

$$M \oplus M^*.$$

By the result proved above, every connected component of $M \oplus M^*$ is either a path or an even-length cycle with alternating edges.

Augmenting Paths

An *augmenting path* is a path that:

- starts and ends at vertices unmatched by M,
- alternates between edges not in M and edges in M,
- begins and ends with edges not in M.

In $M \oplus M^*$, such augmenting paths correspond exactly to path components whose endpoints are unmatched in M.

Flipping the matching along such a path increases the size of the matching by one.

Even-Length Cycles

Even-length cycle components in $M \oplus M^*$:

- alternate between M and M^*,
- do not change the size of the matching when flipped,
- represent different but equally sized matchings.

Thus:

- path components explain how a matching can be improved,
- cycle components explain structural differences between maximum matchings.

This decomposition underlies the correctness of classical matching algorithms, including augmenting-path-based algorithms for maximum matching.

Intuition

Matchings pair vertices without overlap. Taking the symmetric difference highlights exactly where the two matchings disagree. At these disagreement points, edges alternate cleanly between the two matchings, forcing the structure to be either paths or even cycles—and nothing else.

Remark (Bipartite Graphs)

In bipartite graphs, all alternating cycles are even, which aligns naturally with the structure of the symmetric difference.