MTech CSE – 1st Semester
Student ID: **A125023**
Student Name: **SRIJITA VERMA**

Question

Explain the LU decomposition of a matrix using Gaussian Elimination. Clearly describe each step involved in the process.

Answer

Introduction

LU decomposition is a fundamental matrix factorization technique in numerical linear algebra. It expresses a given square matrix A as the product of two triangular matrices:

$$A = LU$$

where:

- L is a lower triangular matrix with unit diagonal entries,
- U is an upper triangular matrix.

LU decomposition is closely related to Gaussian elimination and is widely used for:

- solving systems of linear equations,
- computing matrix inverses,
- efficient numerical computations involving multiple right-hand sides.

Preconditions for LU Decomposition

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

An LU decomposition without pivoting exists if:

- all leading principal minors of A are non-zero, or equivalently,
- Gaussian elimination can be performed without requiring row exchanges.

If row exchanges are required, a permutation matrix P is introduced, leading to:

$$PA = LU.$$

Relationship Between Gaussian Elimination and LU Decomposition

Gaussian elimination transforms a matrix A into an upper triangular matrix U using a sequence of elementary row operations.

LU decomposition captures this process algebraically by:

- storing the elimination multipliers in the matrix L,
- storing the final transformed matrix in U.

Thus, LU decomposition is Gaussian elimination written in matrix form.

Step-by-Step Process of LU Decomposition Using Gaussian Elimination

Step 1: Start with the Original Matrix

Given:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We aim to eliminate entries below the diagonal column by column.

Step 2: Perform Gaussian Elimination (First Column)

To eliminate the entries below a_{11}, we compute multipliers:

$$m_{i1} = \frac{a_{i1}}{a_{11}}, \quad i = 2, 3, \ldots, n.$$

Each row operation is:

$$R_i \leftarrow R_i - m_{i1} R_1.$$

After these operations:

- All entries below a_{11} become zero.
- The matrix begins to take upper triangular form.

Step 3: Store Multipliers in Matrix L

The multipliers used in Gaussian elimination form the sub-diagonal entries of L.

We define:

$$L_{ii} = 1 \quad \text{for all } i,$$

$$L_{i1} = m_{i1} \quad \text{for } i > 1.$$

By convention, the diagonal entries of L are set to 1, making the LU factorization unique when no pivoting is used.

Thus:

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ m_{n1} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Step 4: Repeat for Subsequent Columns

For column k ($k = 2, 3, \ldots, n-1$), compute multipliers:

$$m_{ik} = \frac{u_{ik}}{u_{kk}}, \quad i = k+1, \ldots, n.$$

Apply row operations:

$$R_i \leftarrow R_i - m_{ik} R_k.$$

Store the multipliers in L:

$$L_{ik} = m_{ik}.$$

The updated matrix entries form the upper triangular matrix U.

Step 5: Final Matrices L and U

After completing elimination:

- The resulting upper triangular matrix is U.
- The stored multipliers form the lower triangular matrix L.

Thus:

$$A = LU,$$

where:

- L has ones on the diagonal and multipliers below it,
- U contains the diagonal and upper triangular entries.

Solving Linear Systems Using LU Decomposition

Once $A = LU$ is computed, the system

$$Ax = b$$

is solved in two stages:

- **Forward substitution:** $Ly = b$,
- **Backward substitution:** $Ux = y$.

This approach is computationally efficient, especially when solving for multiple right-hand sides.

Illustrative Example

Let:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

Gaussian elimination gives the multiplier:

$$m_{21} = \frac{4}{2} = 2.$$

Thus:

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}.$$

And indeed:

$$LU = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}.$$

Worked 3×3 Numerical Example

Consider the matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 7 \\ -2 & 4 & 5 \end{bmatrix}$$

We compute the LU decomposition of A using Gaussian elimination without pivoting.

Step 1: Eliminate entries below a_{11}

The multipliers are:

$$m_{21} = \frac{4}{2} = 2, \quad m_{31} = \frac{-2}{2} = -1$$

Applying row operations:

$$R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 + R_1$$

This gives:

$$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 7 & 6 \end{bmatrix}$$

Step 2: Eliminate entries below u_{22}

The multiplier is:

$$m_{32} = \frac{7}{1} = 7$$

Applying:

$$R_3 \leftarrow R_3 - 7R_2$$

We obtain:

$$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & -29 \end{bmatrix}$$

Step 3: Construct the L matrix

The multipliers populate the lower triangular matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 7 & 1 \end{bmatrix}$$

Thus, the LU decomposition is:

$$A = LU$$

A direct multiplication verifies that this decomposition is correct.

Computational Advantages of LU Decomposition

- Reduces repeated computation for multiple right-hand sides,
- Improves numerical efficiency over repeated Gaussian elimination,
- Forms the basis for many advanced numerical algorithms.

Limitations and Pivoting

If a pivot element $u_{kk} = 0$ (or very small), LU decomposition without modification fails.

In such cases:

- Partial pivoting is used,
- The decomposition becomes $PA = LU$, where P is a permutation matrix.

LU Decomposition with Partial Pivoting (PA = LU)

In practice, Gaussian elimination may encounter zero or very small pivot elements, leading to numerical instability or division by zero. To address this, *partial pivoting* is employed.

Partial pivoting involves rearranging the rows of the matrix so that, at each elimination step, the pivot element has the largest absolute value among the entries in the current column below the diagonal.

Permutation Matrix

Row interchanges can be represented using a permutation matrix P. A permutation matrix is obtained by reordering the rows of the identity matrix.

Applying the same row swaps to A yields:

$$PA$$

PA = LU Decomposition

When pivoting is used, the matrix factorization takes the form:

$$PA = LU$$

where:

- P is a permutation matrix,
- L is a unit lower triangular matrix containing the elimination multipliers,
- U is an upper triangular matrix.

Effect on Gaussian Elimination

The Gaussian elimination steps remain the same, except that:

- Rows may be swapped before elimination at each stage,
- The permutation matrix P records these swaps,
- The multipliers are stored in L after row exchanges.

Solving Linear Systems with Pivoting

Given a system:

$$Ax = b$$

Using $PA = LU$, we solve:

$$PAx = Pb$$

This is done in three steps:

1. Forward substitution: $Ly = Pb$
2. Backward substitution: $Ux = y$

Importance of Partial Pivoting

Partial pivoting:

- Improves numerical stability,
- Prevents division by small or zero pivots,
- Is used in most practical implementations of LU decomposition.

Worked 3×3 Numerical Example with Partial Pivoting

Consider the matrix:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -2 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

Direct LU decomposition without pivoting fails since the first pivot $a_{11} = 0$. Hence, partial pivoting is required.

Step 1: Construct the Permutation Matrix

In the first column, the largest entry in absolute value is 2 (row 3). We swap row 1 with row 3.

The permutation matrix is:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Applying the permutation:

$$PA = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

Step 2: Gaussian Elimination on PA

Eliminate entries below the first pivot 2.

Multipliers:

$$m_{21} = \frac{1}{2}, \quad m_{31} = 0$$

Apply row operations:

$$R_2 \leftarrow R_2 - \frac{1}{2}R_1$$

This yields:

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & -\frac{7}{2} \\ 0 & 2 & 1 \end{bmatrix}$$

Step 3: Second Pivot and Elimination

The second pivot is $-\frac{7}{2}$, which is nonzero. The multiplier is:

$$m_{32} = \frac{2}{-\frac{7}{2}} = -\frac{4}{7}$$

Apply:

$$R_3 \leftarrow R_3 + \frac{4}{7} R_2$$

Resulting in:

$$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & -\frac{7}{2} \\ 0 & 0 & -1 \end{bmatrix}$$

Step 4: Construct the L Matrix

The multipliers form the lower triangular matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & -\frac{4}{7} & 1 \end{bmatrix}$$

Thus, the matrix admits the decomposition:

$$PA = LU$$

A direct multiplication verifies the correctness of this factorization.

Remark

The computational cost of LU decomposition using Gaussian elimination is $O(n^3)$. However, once the factorization is computed, each additional system

$$Ax = b$$

can be solved in $O(n^2)$ time using forward and backward substitution.

Conclusion

LU decomposition formalizes Gaussian elimination by separating it into a lower triangular matrix L and an upper triangular matrix U. By recording elimination multipliers in L and the transformed matrix in U, LU decomposition provides an efficient and structured approach to solving linear systems and performing numerical computations.