MTech CSE – 1st Semester
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Question

Prove that positive-definite matrices are suitable for LU decomposition and do not require pivoting to avoid division by zero in the recursive strategy.

Answer

Overview and Strategy

To answer this question, we must show two things:

- Existence of LU decomposition without pivoting for positive-definite matrices.
- Absence of zero pivots during Gaussian elimination, so no division by zero occurs.

The key idea is that positive-definite matrices have strictly positive leading principal minors, which guarantees that all pivots encountered in LU decomposition are non-zero and positive.

Preliminaries and Definitions

Definition: Positive-Definite Matrix

A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive-definite if

$$x^T Ax > 0 \quad \text{for all non-zero } x \in \mathbb{R}^n.$$

Definition: LU Decomposition (Without Pivoting)

An LU decomposition of a matrix A is a factorization

$$A = LU,$$

where:

- L is a unit lower triangular matrix,
- U is an upper triangular matrix.

In Gaussian elimination, the diagonal entries of U are the pivot elements. Pivoting is required only if a pivot becomes zero (or numerically unstable).

Key Theoretical Result

Theorem (Sylvester's Criterion)

A real symmetric matrix A is positive-definite if and only if all its leading principal minors are strictly positive, that is,

$$\det(A_k) > 0 \quad \text{for } k = 1, 2, \ldots, n,$$

where A_k denotes the $k \times k$ leading principal submatrix of A.

Step 1: Connection Between LU Decomposition and Leading Principal Minors

For an LU decomposition without pivoting, the pivots satisfy:

$$u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})}, \quad k = 1, 2, \ldots, n,$$

with the convention $\det(A_0) = 1$.

Thus, a zero pivot $u_{kk} = 0$ occurs if and only if $\det(A_k) = 0$. Therefore, non-zero leading principal minors guarantee non-zero pivots.

Step 2: Apply Positive-Definiteness

Since A is positive-definite:

- A is symmetric,
- by Sylvester's criterion, $\det(A_k) > 0$ for all k.

Hence,

$$u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})} > 0 \quad \text{for all } k.$$

This shows that:

- all pivots are strictly positive,
- no division by zero can occur during LU decomposition.

Step 3: Suitability for the Recursive LU Strategy

The recursive LU algorithm computes:

$$u_{kk} = a_{kk}^{(k)}, \quad l_{ik} = \frac{a_{ik}^{(k)}}{u_{kk}}.$$

Since $u_{kk} > 0$ for all k:

- each division is well-defined,
- the recursive elimination proceeds safely without pivoting.

Step 4: Why Pivoting Is Unnecessary

Pivoting is used to:

- avoid division by zero,
- improve numerical stability.

For positive-definite matrices:

- division by zero cannot occur,
- pivots are guaranteed to be positive,
- the matrix is well-conditioned in theory.

Thus, pivoting is unnecessary to ensure correctness.

Relation to Numerical Stability

In numerical linear algebra, pivoting is often introduced not only to avoid division by zero but also to improve numerical stability by controlling the growth of rounding errors.

For positive-definite matrices, the pivots in LU decomposition are guaranteed to be strictly positive and bounded away from zero. As a result:

- division by zero cannot occur,
- large element growth is avoided,
- the amplification of rounding errors is limited.

Consequently, LU decomposition without pivoting is numerically stable for positive-definite matrices in exact arithmetic and performs reliably in floating-point computations.

Worked Numerical Example

Consider the symmetric matrix:

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

We first verify that A is positive-definite. All leading principal minors are:

$$\det(A_1) = 4 > 0, \quad \det(A_2) = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 16 > 0, \quad \det(A_3) = 36 > 0.$$

Hence, A is positive-definite.

LU Decomposition Without Pivoting

Applying Gaussian elimination without pivoting:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

All pivots (4, 4, and 2) are strictly positive, and no division by zero occurs. This confirms that LU decomposition proceeds safely without pivoting.

Additional Insight: Relation to Cholesky Decomposition

Every positive-definite matrix admits a Cholesky decomposition:

$$A = LL^T,$$

which is a special case of LU decomposition with

$$U = L^T.$$

In this case:

- all pivots are square roots of positive numbers,
- no pivoting is required.

This further reinforces that positive-definite matrices are inherently suitable for triangular factorizations.

Comparison: LU vs Cholesky Decomposition

Both LU and Cholesky decompositions can be used for positive-definite matrices, but Cholesky decomposition is more specialized.

- **LU Decomposition** applies to a broader class of matrices and factors A as LU.
- **Cholesky Decomposition** applies only to symmetric positive-definite matrices and factors A as LL^T.

From a computational perspective:

- Cholesky decomposition requires roughly half the number of arithmetic operations compared to LU decomposition.
- Cholesky is inherently stable and does not require pivoting.
- LU decomposition without pivoting is safe for positive-definite matrices but is less efficient than Cholesky.

Therefore, while positive-definite matrices admit LU decomposition without pivoting, Cholesky decomposition is generally preferred in practice.

Final Conclusion

Positive-definite matrices admit LU decomposition without pivoting because:

- all leading principal minors are strictly positive,
- all pivots in Gaussian elimination are non-zero,
- division by zero cannot occur in the recursive strategy.

Interpretation (Intuition)

Positive-definite matrices represent strictly convex quadratic forms. Eliminating variables never collapses dimensionality, and each step of Gaussian elimination preserves positivity. Hence, the algorithm proceeds safely without row exchanges.

Remark

Positive-definite matrices arising from physical or optimization problems are often well-conditioned, which further enhances the numerical reliability of direct factorization methods.