KHARAGPUR **DATA**
ANALYTICS GROUP

CDC·101

THINK TANK

" A Curated Collection of Probability Problems & Puzzles "

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BASIC PUZZLES

1. The Sock Drawer

A drawer contains red socks and black socks. When two socks are drawn at random, the probability that both are red is 1/2.

(a) How small can the number of socks in the drawer be?

(b) How small if the number of black socks is even?

Answer : 4, 21

Solution :

Just to set the pattern, let us do a numerical example first. Suppose there were 5 red and 2 black socks; then the probability of the first sock's being red would be $5/(5+2)$. If the first were red, the probability of the second's being red would be $4/(4+2)$, because one red sock has already been removed. The product of these two numbers is the probability that both socks are red:

$$\frac{5}{5+2} \times \frac{4}{4+2} = \frac{5(4)}{7(6)} = \frac{10}{21}.$$

This result is close to $\frac{1}{2}$, but we need exactly $\frac{1}{2}$. Now let us go at the problem algebraically.

Let there be r red and b black socks. The probability of the first sock's being red is $r/(r+b)$; and if the first sock is red, the probability of the second's being red now that a red has been removed is $(r-1)/(r+b-1)$. Then we require the probability that both are red to be $\frac{1}{2}$, or

$$\frac{r}{r+b} \times \frac{r-1}{r+b-1} = \frac{1}{2}.$$

One could just start with $b = 1$ and try successive values of r, then go to $b = 2$ and try again, and so on. That would get the answers quickly. Or we could play along with a little more mathematics. Notice that

$$\frac{r}{r+b} \times \frac{r-1}{r+b-1} \geq \frac{1}{2}, \quad \text{for} \quad b > 0.$$

Therefore we can create the inequalities

$$\left(\frac{r}{r+b}\right)^2 \geq \frac{1}{2} \quad \text{or} \quad \left(\frac{r-1}{r+b-1}\right)^2.$$

Solution :

Taking square roots, we have, for $r > 1$,

$$\frac{r}{r+b} > \frac{1}{\sqrt{2}} > \frac{r-1}{r+b-1}$$

From the first inequality we get

$$r > \frac{1}{\sqrt{2}}(r+b)$$

$$r > \frac{1}{\sqrt{2}-1}b = (\sqrt{2}+1)b.$$

$$(\sqrt{2}+1)b > r-1$$

or

$$(\sqrt{2}+1)b + 1 > r.$$

From the second we get

$$(\sqrt{2}+1)b > r-1$$

or all told

$$(\sqrt{2}+1)b + 1 > r > (\sqrt{2}+1)b.$$

For $b = 1$, r must be greater than 2.414 and less than 3.414, and so the candidate is $r = 3$. For $r = 3, b = 1$, we get

$$P(2 \text{ red socks}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

And so the smallest number of socks is 4.
Beyond this we investigate even values of b.

Solution :

b	r is between	eligible r	P(2 red socks)
2	5.8, 4.8	5	$\frac{5(4)}{7(6)} \neq \frac{1}{2}$
4	10.7, 9.7	10	$\frac{10(9)}{14(13)} \neq \frac{1}{2}$
6	15.5, 14.5	15	$\frac{15(14)}{21(20)} = \frac{1}{2}$

And so 21 socks is the smallest number when b is even. If we were to go on and ask for further values of r and b so that the probability of two red socks is 4, we would be wise to appreciate that this is a problem in the theory of numbers. It happens to lead to a famous result in Diophantine Analysis obtained from Pell's equation.

2. Coin in square

In a common carnival game a player tosses a penny from a distance of about 5 feet onto the surface of a table ruled in 1-inch squares. If the penny (3/4 inch in diameter) falls entirely inside a square, the player receives 5 cents but does not get his penny back; otherwise he loses his penny. If the penny lands on the table, what is his chance to win?

Adobe

Answer : 1/16

Solution :

When we toss the coin onto the table, some positions for the center of the coin are more likely than others, but over a very small square we can regard the probability distribution as uniform. This means that the probability that the center falls into any region of a square is proportional to the area of the region, indeed, is the area of the region divided by the area of the square. Since the coin is $\frac{3}{8}$ inch in radius, its center must not land within $\frac{3}{8}$ inch of any edge if the player is to win. This restriction generates a square of side $\frac{1}{4}$ inch within which the center of the coin must lie for the coin to be in the square. Since the probabilities are proportional to areas, the probability of winning is

$$\frac{1}{4}^{2} = \frac{1}{16}$$

Of course, since there is a chance that the coin falls off the table altogether, the total probability of winning is still smaller. Also the squares can be made smaller by merely thickening the lines. If the lines are $\frac{1}{16}$ inch wide, the winning central area reduces the probability to

$$\frac{3}{16}^{2} = \frac{9}{256}$$

or less than $\frac{1}{28}$.

3. Perfect Bridge Hand

We often read of someone who has been dealt 13 spades at a bridge. With a well shuffled pack of cards, what is the chance that you are dealt a perfect hand(13 of one suit)? (Bridge is played with an ordinary pack of 52 cards, 13 in each of 4 suits, and each of 4 players is dealt 13.)

Answer : $4 \times \frac{13!39!}{52!}$

Solution :

The chances are mighty slim. Since the cards are well shuffled, we might as well deal your 13 off the top. To get 13 of one suit you can start with any card, and thereafter you are restricted to the same suit. So the number of ways to be dealt 13 of one suit is

$$52 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 52 \times 12!$$

The total number of ways to get a bridge hand is

$$52 \times 51 \times 50 \times 49 \times 48 \times 47 \times 46 \times 45 \times 44 \times 43 \times 42 \times 41 \times 40 = \frac{52!}{39!}$$

The desired probability is

$$\frac{52 \times 12!}{\frac{52!}{39!}} = \frac{12!39!}{51!}$$

OR In our problem the number of ways to choose 13 cards out of 52 is

$$\binom{52}{13} = \frac{52!}{13!39!}$$

The number of ways to get 13 spades is

$$\binom{13}{13} = \frac{13!}{13!0!} = 1$$

We multiply by 4 because of the 4 suits, and the final probability is

$$4 \times \frac{13!39!}{52!}$$

, as we already found.

4. Rolling The Bullet

Two bullets are loaded into a gun's round barrel consecutively. The barrel has a capacity of 6. The gun is fired once, but no bullet is shot. Does rolling the barrel (shuffling) before next shot increase the probability of firing a bullet?

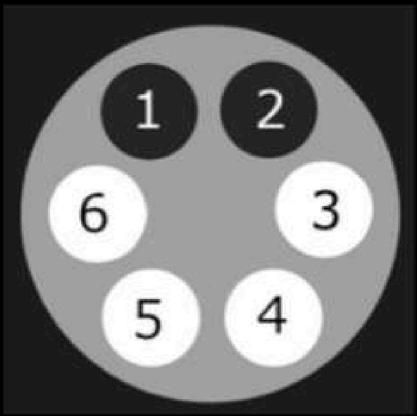
accenture

Answer : YES

Solution :

Initial Misstep: If the two bullets are randomly put instead of consecutively, then, after firing one empty shot, there are 2 bullets and 5 total slots. The probability would be $2/5 = 40\%$, but that's not the case here. **Correct step:**

The probability of firing a bullet without a shuffle is $1/4 = 25\%$. To understand this, imagine that the firing pin was on one of the empty slots $(3, 4, 5, 6)$, and the first shot was taken, but no bullet was fired. Now assuming that the barrel rotates clockwise, the pin will move to one of these slots: $(2, 3, 4, 5)$. Out of these four slots, only the slot (1) has a bullet. Hence, the probability of firing a bullet is $1/4 = 25\%$. Note that the same is true in the anti-clockwise direction.



After the shuffle, the state is reset. There are 6 total slots with 2 bullets, and the probability of firing a bullet after a shuffle is $2/6 = 1/3 \approx 33\%$.

Thus, shuffling does increase the probability of firing a bullet (from 25% to 33%).

5. All Girls World

In a world where everyone wants a girl child, each family continues having babies till they have a girl. What do you think will the boy-to-girl ratio be eventually?

Assuming probability of having a boy or a girl is the same and there is no other gender at the time of birth.

Morgan Stanley

Answer : 1:1

Solution :

Suppose there are N couples. First time, $N/2$ girls and $N/2$ boys are born. $N/2$ couples retire, and the rest half try for another child.

Next time, $N/4$ couples give birth to $N/4$ girls and $N/4$ boys. Thus, even in the second iteration, the ratio is $1:1$. It can now be seen that this ratio will always remain the same, no matter how many times people try to give birth to a favored gender.

6. Half Time

The probability of having accidents on a road in one hour is ¾. What is the probability of accidents in half an hour?

JPMorgan Chase & Co.

Morgan Stanley

Answer : 1/2

Solution :

The probability of no accident in 1 hour can be written as:

$$P(\text{no accident in 1 hour}) = P(A) \times P(B).$$

where,
A = no accident in the first half hour,
B = no accident in the next half hour.

This implies:

$$1 - \frac{3}{4} = p^2 \implies p = \frac{1}{2}.$$

Hence, the probability of an accident in half an hour is:

$$1 - \frac{1}{2} = \frac{1}{2}.$$

7. Craps

The game of craps, played with two dice, is one of America's fastest and most popular gambling games. Calculating the odds associated with it is an instructive exercise.

The rules are as follows: Only the totals for the two dice count. The player throws the dice and:

Wins immediately if the total for the first throw is 7 or 11.

Loses immediately if the total is 2, 3, or 12.

Any other total becomes the player's "point." If the first throw results in a point, the player continues to throw the dice repeatedly until one of the following occurs:

The player wins by throwing the point again.

The player loses by throwing a 7.

What is the player's chance to win?

Answer : 3/9

Solution :

The game is surprisingly close to even, as we shall see, but slightly to the player's disadvantage.

Let us first get the probabilities for the totals on the two dice. Regard the dice as distinguishable, say red and green. Then there are $6 \times 6 = 36$ possible equally likely throws whose totals are shown in the table below. By counting the cells in the table, we get the probability distribution of the totals:

Total	$P(\text{Total})$
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Solution :

Throw of red die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

For example, The probability of a win on the first throw is:

$$\Pr(7) + \Pr(11) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}.$$

The probability of a loss on the first throw is:

$$\Pr(2) + \Pr(3) + \Pr(12) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}.$$

For later throws, we need the probability of making the point. Since no throws except either the point or 7 matter, we can compute for each of these the conditional probability of making the point given that it has been thrown initially. Sometimes such an approach is called the method of reduced sample spaces because,

Solution :

although the actual tosses produce the totals 2 through 12, we ignore all but the point and 7.

For example, for 4 as the point, there are 3 ways to make the point and 6 ways to make a seven, so the probability of making the point is:

$$\frac{3}{3+6} = \frac{3}{9}.$$

Similarly, we get the conditional probabilities for the other points and summarize:

Point	Probability
4	$\frac{1}{3}$
5	$\frac{2}{5}$
6	$\frac{5}{11}$
8	$\frac{5}{11}$
9	$\frac{2}{5}$
10	$\frac{1}{3}$

Each probability of winning must be weighted by the probability of throwing the point on the initial throw to give the unconditional probability of winning for that

Solution :

point. Then we sum to get the total probability of winning by throwing a point:

Pr(win) $= \frac{2}{9} + (\frac{2}{5} \times \frac{4}{36}) + (\frac{1}{3} \times \frac{3}{36}) + (\frac{5}{11} \times \frac{5}{36}) + (\frac{2}{5} \times \frac{4}{36}) + (\frac{1}{3} \times \frac{3}{36}) \approx 0.27071.$

To this, we add the probability of winning on the first throw, $\frac{2}{9} \approx 0.22222$, to get approximately 0.49293 as the player's probability of winning. His expected loss per unit stake is:

$$0.50707 - 0.49293 = 0.01414,$$

or 1.41%.

I believe that this is the most nearly even of house gambling games that have no strategy. And 1.41% doesn't sound like much, but as I write, the stock of General Motors is selling at 71, and their dividend for the year (before extras) is quoted as \$2, or about 2.8%. So per two plays at craps, your loss is at a rate equal to the yearly dividend payout by America's largest corporation.

Some readers may not be satisfied with the conditional probability approach used for points and may

Solution :

wish to see the series summed.

Let the probability of throwing the point be P, and let the probability of a toss that does not count be R, where $R = 1 - P - \frac{1}{6}$. The $\frac{1}{6}$ is the probability of throwing a 7. The player can win by throwing a number of tosses that do not count and then throwing his point. The probability that he makes his point in the $(r+1)$th throw (after the initial throw) is:

$$R^r \times P, \text{ where } r = 0, 1, 2, \ldots$$

To get the total probability, we sum over the values of r:

$$P + RP + R^2P + \cdots = P(1 + R + R^2 + \ldots)$$

Summing this infinite geometric series gives:

$$\text{Probability of making point} = \frac{P}{1-R}.$$

For example, if the point is 4:

$$P = \frac{3}{36}, \quad R = 1 - \frac{3}{36} - \frac{6}{36} = \frac{27}{36}, \quad 1 - R = \frac{9}{36}.$$

$$\Pr(\text{making the point 4}) = \frac{3/36}{9/36} = \frac{3}{9}.$$

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BAYES THEOREM

8. Father of Lies

A father claims about snowfall last night. First daughter tells that the probability of snowfall on a particular night is 1/8. Second daughter tells that 5 out of 6 times the father is lying! What is the probability that there actually was a snowfall?

Answer : 1/36

Solution :

Let S be the event that snowfall occurred, and C be the event that the father is claiming snowfall occurred. The probability of snowfall given the claim, $P(S|C)$, can be calculated using Bayes' Theorem:

$$P(S|C) = \frac{P(C|S) \cdot P(S)}{P(C)}$$

We are given:

$$P(S) = \frac{1}{8}$$

$$P(C|S) = 1 - \frac{5}{6} = \frac{1}{6} \quad \text{(since the father lies } \frac{5}{6} \text{ times)}$$

Next, calculate $P(C)$:

$$P(C) = P(\text{True Claim}) + P(\text{False Claim})$$

$$P(C) = P(C \cap S) + P(C \cap S')$$

where S' represents that it did not snow.

$$P(C) = P(C|S) \cdot P(S) + P(C|S') \cdot P(S')$$

Now, substitute the values:

$$P(C) = \left(\frac{1}{6} \cdot \frac{1}{8}\right) + \left(\frac{5}{6} \cdot \frac{7}{8}\right)$$

Finally, calculate $P(S|C)$:

$$P(S|C) = \frac{P(C|S) \cdot P(S)}{P(C)} = \frac{\frac{1}{6} \cdot \frac{1}{8}}{\left(\frac{1}{6} \cdot \frac{1}{8}\right) + \left(\frac{5}{6} \cdot \frac{7}{8}\right)}$$

$$P(S|C) = \frac{1}{36}$$

9. 50 marbles

Given two boxes, B1 and B2, one containing 50 red marbles and the other containing 50 blue marbles, a ball is selected randomly from either box. The task is to maximize the probability of selecting a red ball by reshuffling the marbles between both boxes.

AMERICAN EXPRESS

Goldman Sachs

Answer : 0.75

Solution :

Let $P(R)$ be the probability of picking a red marble.

$$P(R) = P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2)$$

Where $P(B_1)$ and $P(B_2)$ refer to the probabilities of selecting boxes B_1 and B_2, respectively. The probability of selecting each box is $\frac{1}{2}$.
$R|B_1$ and $R|B_2$ refer to the probability of picking a red ball from boxes B_1 and B_2, respectively.
Without reshuffling any balls, we have:

$$\begin{aligned} P(R) &= \left(\frac{1}{2} \cdot 1\right) + \left(\frac{1}{2} \cdot 0\right) \\ &= 0.5 \end{aligned}$$

If we decrease the number of red balls in box B_1 and increase the number of red balls in box B_2, the probability of getting a red ball will be maximized.
Let us move 49 red marbles from B_1 to B_2. This will result in 1 red ball remaining in B_1, and in B_2, there will be 49 red balls and 50 blue balls, for a total of 99 balls in B_2.
Now, the probability becomes:

$$\begin{aligned} P(R) &= \left(\frac{1}{2} \cdot \frac{1}{1}\right) + \left(\frac{1}{2} \cdot \frac{49}{99}\right) \\ &= 0.747474 \end{aligned}$$

Thus, the maximum probability of choosing a red ball is 0.747474.

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EXPECTATION

10. Expected number of tosses until getting consecutive heads or tails.

Suppose that we have a fair coin and we toss it until we have two consecutive heads (H) or tails (T). What is the expected number of tosses until the game stops?

Adobe

Answer : 4

Solution :

Let X denote the required random variable, and let AB denote events A followed by B, where $A, B \in \{H, T\}$. Then, by the standard technique of conditional expectation, we have:

$$E[X] = E[X \mid TT]P(TT) + E[X \mid HH]P(HH) + E[X \mid TH]P(TH) + E[X \mid HT]P(HT)$$

Substituting the values:

$$E[X] = 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + (2 + E[X])\left(\frac{1}{4}\right) + (2 + E[X])\left(\frac{1}{4}\right)$$

Simplifying further:

$$E[X] = 2 + \frac{1}{2}E[X]$$

which gives:

$$E[X] = 4$$

11: Trials until first success.

On the average, how many times must a die be thrown until one gets a 6?

JPMorgan Chase & Co.

Answer : 6

Solution :

Let p be the probability of a 6 on a given trial.
Then the probabilities of success for the first time on each trial are (let $q = 1 - p$):

$$p + pq + pq^2 + pq^3 + \cdots$$

The sum of the probabilities is:

$$p(1 + q + q^2 + q^3 + \cdots) = \frac{p}{1-q} = 1$$

The mean number of trials, m, is by definition:

$$m = p + 2pq + 3pq^2 + 4pq^3 + \cdots$$

Now, note that the usual trick for summing a geometric series works:

$$qm = pq + 2pq^2 + 3pq^3 + 4pq^4 + \cdots$$

Subtracting the second expression from the first gives:

$$m - qm = p + pq + pq^2 + pq^3 + \cdots$$

or

$$m(1-q) = 1$$

Consequently:

$$mp = 1$$

Thus,

$$m = \frac{1}{p}$$

In our example, $p = \frac{1}{6}$, and so:

$$m = 6$$

12. Chuck a Luck

In this gambling game, a player can buy a ticket for Rs 1 on any number from 1 to 6. Three identical and unfair dice are rolled. If the booked number appears on 0, 1, 2, or 3 dice, the player wins Rs 0, 1, 2, or 3 respectively, without returning the original Rs 1. What is the expected money you can win after buying a ticket for Rs 1?

Answer : 0.5

Solution :

We bet Rs 1 on each number from 1 to 6. In any case, we get Rs 3 back, which gives us:

$$\frac{1}{2} \text{ per ticket}$$

Hence, the expected amount of money we can win is:

$$\frac{1}{2}$$

A Brute-Force way to arrive at this answer is to calculate the probability of getting 1, 2, or 3 faces common to our booking.

This involves calculating the likelihood of matching different numbers of faces from our selected booking and deriving the final expected value.

13. Second Chance

Roll a die, and you get paid what the dice shows. But if you want, you can request a second chance & roll the die again; get paid what the second roll shows instead of the first. What is the expected value?

Morgan Stanley

Answer : 4.25

Solution :

The expected value of a single roll is:

$$\frac{1}{6}(1+2+3+4+5+6) = 3.5$$

Now, for the strategy where we can roll a second time, the optimal strategy is to roll again when the first roll is below the expected value (i.e., 1, 2, or 3) because, on average, the second roll should be higher.

So, if we roll a 1, 2, or 3, we should roll again, and the expected value of that roll is again 3.5. If we roll a 4, 5, or 6, we should keep it.

Now, the expected value of the game under this optimal strategy would be:

$$\frac{1}{6}(3.5+3.5+3.5+4+5+6) = 4.25$$

14. All Girls World

In a world where everyone wants a girl child, each family continues having babies till they have a girl. What do you think the girl-to-boy ratio will be eventually?

Answer : 1:1

Solution :

Suppose there are N couples. The first time, $\frac{N}{2}$ girls and $\frac{N}{2}$ boys are born. Afterward, $\frac{N}{2}$ couples retire, and the remaining $\frac{N}{2}$ couples try for another child.

In the second attempt, $\frac{N}{4}$ couples give birth to $\frac{N}{4}$ girls and $\frac{N}{4}$ boys. Thus, even in the second iteration, the ratio of girls to boys remains $1 : 1$.

It can now be seen that the ratio will remain the same, no matter how many times people attempt to give birth to a favored gender.

For any subsequent iteration k, the number of girls and boys will always maintain the ratio $1 : 1$, mathematically expressed as:

$$\frac{\text{Girls}}{\text{Boys}} = 1$$

15. You have a train to catch

Spiderman has two close friends, Mary Jane & Gwen Stacy. After every mission, he rushes to the central subway. One line heads towards Mary's place, and another towards Stacy. Trains from each line leave every 10 minutes. Spiderman, being impartial, always boards the first train that leaves.
However, he observes that he ends up visiting Mary Jane nine times more often than Gwen Stacy. Can you decipher why?

Goldman Sachs

Answer :

The gap from MM to GG is 1 minute, and the gap from GG to MM is 9 minutes

Solution :

Since both trains leave at the same frequency of 10 minutes, the solution likely lies in the gap between two consecutive trains of each line. Suppose the train M (towards Mary) leaves at the minutes:

$$0, 10, 20, \ldots$$

And the train GGG (towards Gwen) leaves at the minutes:

$$x, (x+10), (x+20), \ldots$$

where $0 < x < 10$.

Suppose Spiderman randomly reaches the station at some time $t+10 \cdot n$ minutes, where n is an arbitrary integer and t is the fraction that determines his fate, with $0 < t \leq 10$.

Solution(Conti) :

Note that if $0 < t \leq x$, it means that he just missed M and now has to wait for G. While if $x < t \leq 10$, it means the opposite, and he just missed G.

Thus, the probability that Spiderman takes the train G is:

$$P(G) = P(0 < t \leq x)$$

Given:

$$0.1 = \frac{x - 0}{10} \quad \Rightarrow \quad x = 1$$

This means that the train heading to Mary Jane's location leaves precisely 1 minute before the train heading to Gwen Stacy's location. Therefore, 9 out of 10 times, Spiderman reaches the station when the next train is leaving towards Mary.

16. n+1 and n coins

Two persons A and B have respectively $n + 1$ and n coins, which they toss simultaneously. Then the probability that A will have more heads than B is:

A) 0.5

B) > 0.5

C) < 0.5

D) None of these

Morgan Stanley

JPMorgan Chase & Co.

Answer : 0.5

Solution :

Let a and a' be the number of heads and tails by A, so:

$$a + a' = n + 1$$

Let b and b' be the number of heads and tails by B, so:

$$b + b' = n$$

The required probability is that A has more heads than B. Let:

$$P(a > b) = P(a' > b') = p$$

Now, express $P(a > b)$ as:

$$P(a > b) = 1 - P(a \leq b)$$

For $a \leq b$, we have:

$$n + 1 - a' \leq n - b'$$

which simplifies to:

$$1 + b' \leq a'$$

Solution(Conti) :

Thus, $b' < a'$, so we can conclude:

$$P(a \leq b) = P(b' < a')$$

Therefore:

$$P(a > b) = 1 - P(b' < a')$$

Given that $2p = 1$, we find:

$$p = \frac{1}{2}$$

Thus, the correct answer is option (A).

17. Innocent Monkey

A very innocent monkey throws a fair die. The monkey will eat as many bananas as are shown on the die, from 1 to 5. But if the die shows '6', the monkey will eat 5 bananas and throw the die again. This process may continue indefinitely. What is the expected number of bananas the monkey will eat?

Answer : 4

Solution :

Let E be the expected number of bananas the monkey will eat.

The possible outcomes when the die is rolled are as follows:

- If the die shows 1, the monkey eats 1 banana.
- If the die shows 2, the monkey eats 2 bananas.
- If the die shows 3, the monkey eats 3 bananas.
- If the die shows 4, the monkey eats 4 bananas.
- If the die shows 5, the monkey eats 5 bananas.
- If the die shows 6, the monkey eats 5 bananas and throws the die again adding to the total number of bananas eaten.

Thus, we can write the equation for the expected value as:

$$E = \frac{1}{6}(1 + 2 + 3 + 4 + 5) + \frac{1}{6}(5 + E)$$

Solution :

Simplifying this:

$$E = \frac{1}{6}(15) + \frac{1}{6}(5 + E)$$

$$E = \frac{15}{6} + \frac{5}{6} + \frac{E}{6}$$

$$E = \frac{20}{6} + \frac{E}{6}$$

$$6E = 20 + E$$

$$5E = 20$$

$$E = 4$$

Thus, the expected number of bananas the monkey will eat is 4.

18. No of Double Heads

A coin is tossed 10 times, and the result is recorded as a string of heads (H) and tails (T). What is the expected number of occurrences of "HH" (two consecutive heads) in the string?
Note: In the string "HHH", the number of occurrences of "HH" is 2. (For example, the expected number of "HH" in 2 tosses is 0.25, and in 3 tosses, it is 0.5.)

Answer : 2.25

Solution :

Let X be the total number of occurrences of "HH" in the sequence of 10 tosses. Define indicator random variables X_i for $i = 1, 2, \ldots, 9$ (since we can only have a pair of consecutive heads starting from the 1st position up to the 9th position):

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th and } (i+1)\text{-th tosses are "HH"} \\ 0 & \text{otherwise} \end{cases}$$

The total number of "HH" is then:

$$X = \sum_{i=1}^{9} X_i$$

Next, we compute the expected value of each X_i. Since the tosses are independent, the probability that two consecutive tosses are both heads is:

$$P(X_i = 1) = P(\text{H on toss } i) \times P(\text{H on toss } i+1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Thus, the expected value of X_i is:

$$E[X_i] = 1 \times \frac{1}{4} + 0 \times \left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

The expected total number of "HH" is the sum of the expectations of the X_i's:

Solution :

$$E[X] = \sum_{i=1}^{9} E[X_i] = 9 \times \frac{1}{4} = \frac{9}{4} = 2.25$$

Thus, the expected number of occurrences of "HH" in 10 tosses is 2.25.

19. Collecting Coupons

Coupons in cereal boxes are numbered from 1 to 5, and a complete set of one of each coupon is required to claim a prize. With one coupon per box, how many boxes on average are required to collect a complete set?

Answer : 11.42

Solution :

We get one of the coupons in the first box. Now, the chance of getting a new coupon from the next box is $\frac{4}{5}$. Using the result from the previous problem, the expected number of boxes required to get the second new coupon is:

$$\frac{1}{\frac{4}{5}} = \frac{5}{4}$$

Similarly, the third new coupon requires:

$$\frac{1}{\frac{3}{5}} = \frac{5}{3}$$

The fourth new coupon requires:

$$\frac{1}{\frac{2}{5}} = \frac{5}{2}$$

And finally, the fifth new coupon requires:

$$\frac{1}{\frac{1}{5}} = 5$$

Thus, the average total number of boxes required to collect all 5 coupons is:

$$E[X] = 1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + 5 = 11.42$$

Solution :

Euler's Approximation for Harmonic Sums

While it is easy to add up the reciprocals in this case, for larger sets of coupons, Euler's approximation for the partial sum of the harmonic series is useful:

$$H_n \approx \ln n + \gamma + \frac{1}{2n}$$

where H_n is the n-th harmonic number, $\gamma \approx 0.57721$ is Euler's constant, and n is the total number of coupons.

For $n = 5$ coupons, the average number of boxes is approximately:

$$E[X] \approx 5 \times (\ln 5 + \gamma) = 5 \times (1.6094 + 0.57721) \approx 11.43$$

This result is very close to the exact value of 11.42 obtained earlier.

Note that often the term $\frac{1}{2n}$ is omitted in Euler's approximation for simplicity when n is large.

20. The Theatre Row

Eight eligible bachelors and seven beautiful models happen to have purchased single seats in the same 15-seat row of a theater. On average, how many pairs of adjacent seats are ticketed for marriageable couples?

Answer : 7.47

Solution :

Total Number of Adjacent Pairs

There are 15 seats in the row, so there are 14 adjacent pairs of seats. Each adjacent pair can either consist of two bachelors, two models, or one bachelor and one model (which we will call a "marriageable couple").

Probability of a Marriageable Couple in Each Pair

Each adjacent pair of seats is filled randomly, so we need to calculate the probability that a given adjacent pair consists of a bachelor and a model. Since there are 8 bachelors and 7 models, the probability of one seat being occupied by a bachelor is $\frac{8}{15}$, and the probability of the other seat being occupied by a model is $\frac{7}{14}$ (since one seat is already occupied by a bachelor, there are only 14 remaining seats).

Thus, the probability that any adjacent pair is occupied by one bachelor and one model is:

$$P(\text{marriageable couple}) = \frac{8}{15} \times \frac{7}{14} + \frac{7}{15} \times \frac{8}{14}$$

Simplifying:

$$P(\text{marriageable couple}) = 2 \times \frac{8}{15} \times \frac{7}{14} = 2 \times \frac{56}{210} = \frac{112}{210} = \frac{8}{15}$$

Expected Number of Marriageable Couples

Since there are 14 adjacent pairs of seats and the probability of a marriageable couple occupying any pair is $\frac{8}{15}$, the expected number of adjacent pairs occupied by one bachelor and one model is:

$$E[\text{number of marriageable couples}] = 14 \times \frac{8}{15}$$

Simplifying:

$$E[\text{number of marriageable couples}] = \frac{112}{15} \approx 7.47$$

KHARAGPUR **DATA ANALYTICS** GROUP

STRATEGY

21. The Three-Cornered Duel

A, B, and C are to fight a three-cornered pistol duel. All know that A’s chance of hitting his target is 0.3, C’s is 0.5, and B never misses. They are to fire at their choice of target in succession in the order A, B, C, cyclically (but a hit man loses further turns and is no longer shot at) until only one man is left un-hit. What should A's strategy be?

Solution :

A naturally is not feeling cheery about this enterprise. Having the first shot, he sees that if he hits C, B will then surely hit him, and so he is not going to shoot at C. If he shoots at B and misses him, then B clearly shoots the more dangerous C first, and A gets one shot at B with probability 0.3 of succeeding. If he misses this time, the less said the better.

On the other hand, suppose A hits B. Then C and A shoot alternately until one hits. A's chance of winning is given by the infinite series:

$$0.5 \cdot 0.3 + (0.5)^2 \cdot 0.7 \cdot 0.3 + (0.5)^3 \cdot (0.7)^2 \cdot 0.3 + \dots$$

Each term corresponds to a sequence of misses by both C and A, ending with a final hit by A. Summing the geometric series, we get:

$$0.5 \cdot 0.3 \left[1 + (0.5 \cdot 0.7) + (0.5 \cdot 0.7)^2 + \dots\right]$$

Solution :

The series sum simplifies to:

$$0.5 \cdot 0.3 \cdot \frac{1}{1-(0.5 \cdot 0.7)} = \frac{0.15}{0.65} = \frac{3}{13} < \frac{3}{10}$$

Thus, hitting B and then finishing off with C has a lower probability of winning for A than just missing the first shot intentionally.

So, A fires his first shot into the ground and then tries to hit B with his next shot. C is out of luck.

In discussing this with Thomas Lehrer, a question was raised about whether this was an honorable solution under the *code duello*. Lehrer replied that the honor involved in three-cornered duels has never been established, and so we are on safe ground to allow A a deliberate miss.

22. Should you sample with or without replacement?

Two urns contain red and black balls, all alike except for color. Urn 4 has 2 reds and 1 black, and Urn B has 101 reds and 100 blacks. An urn is chosen at random, and you win a prize if you correctly name the urn on the basis of the evidence of two balls drawn from it. After the first ball is drawn and its color reported, you can decide whether or not the ball shall be replaced before the second drawing. How do you order the second drawing, and how do you decide on the urn?

Solution :

If the first ball drawn is red, then no matter which urn is being drawn from, it now has half red and half black balls, and the second ball provides no discrimination. Therefore, if a red ball is drawn first, replace it before drawing again. If a black ball is drawn, do not replace it. When this strategy is followed, the probabilities associated with the outcomes are given below:

	Outcome	Probability Urn A	Probability Urn B	Decision
Urn A	2 reds	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3}$	$\frac{1}{2}\cdot\frac{101}{201}\cdot\frac{101}{201}$	$UrnA$
Urn B	Red then black	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{3}$	$\frac{1}{2}\cdot\frac{100}{201}\cdot\frac{100}{201}$	$UrnB$
Urn A	Black then red	$\frac{1}{2}\cdot\frac{1}{3}\cdot 1$	$\frac{1}{2}\cdot\frac{101}{200}\cdot\frac{101}{200}$	$UrnA$
Urn B	2 black	$\frac{1}{2}\cdot\frac{1}{3}\cdot 0$	$\frac{1}{2}\cdot\frac{99}{200}\cdot\frac{99}{200}$	$UrnB$

Table 1: Probabilities associated with each outcome for Urns A and B.

The total probability of deciding correctly is approximately (replacing $\frac{100}{201}$ by $\frac{1}{2}$, etc.):

$$\frac{1}{2}\left[\frac{4}{9}+\frac{1}{4}+\frac{1}{3}+\frac{1}{4}\right]=\frac{23}{36}\approx 0.64$$

Drawing both balls without replacement gives about $\frac{5}{8}$, while drawing both with replacement gives about $\frac{21.5}{36}$.

23. Guess The Toss

A and B are in a team called AB, playing against C. If AB team wins they win Rs 3, nothing otherwise.

The Game: A and B are placed in 2 separate rooms far away. A will toss a coin and B will also toss a coin; A will have to guess the outcome of B's toss and B will guess A's. If both guesses are right, team AB wins Rs 3, nothing otherwise. Should they play the game, by paying Rs 1 at the start?

Solution :

Let's list down all the possibilities:

Person A	Person B
H	H
H	T
T	H
T	T

Table 1: All possible outcomes for Person A and Person B's coin flips.

Observe that in 2 out of the 4 possibilities, their coins have the same face, i.e., either both are heads or both are tails.

If each person speaks their own coin's face as their guess, they win the game with a probability of $\frac{1}{2}$.

The expected pay-off will be positive:

$$\frac{1}{2} \times 3 - 1 = 0.5$$

Since the expected pay-off is positive, they should play this game.

24. Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. Now, do you want to pick door No. 2? What is the probability to win the car if you switch?

Solution :

The probability that your initial choice did not have a car is indeed $\frac{2}{3}$.

Initial Misstep

After one door is opened, there are exactly two doors left, and one of them has a car. So the probability that the car is behind either door is $\frac{1}{2}$. This is incorrect because the host knows which door has a car and which door has a goat. The host always opens a door with a goat.

Correct Solution

The following table shows which door the host might open. Assume that the car is behind Door #1, and we randomly choose one door.

Initial Choice	Reality	Host Opens	Remaining Door	Good to Switch?
Door #1	Car	Door #2 or Door #3	Door #3 or Door #2	No
Door #2	Goat	Door #3	Door #1	Yes
Door #3	Goat	Door #2	Door #1	Yes

Table 1: Outcomes based on initial door choice and host's action.

Solution :

We see that at the end, the remaining unopened door is Door #1 if we start with Door #2 or Door #3. This means that in 2 out of 3 cases, we started with the incorrect door (either Door #2 or Door #3) and we get the option to switch to the correct door at the end (Door #1).

Thus, you should switch to the other door, and you will win the car with a probability of $\frac{2}{3}$.

Generalization

The probability of being initially wrong is the same as the probability of being correct after switching. We can generalize this to n doors. The probability of winning the game by switching after the host has opened $n-2$ doors is:

$$\text{Probability of Winning} = 1 - \frac{1}{n}$$

This probability increases as the number of doors increases, making switching a better strategy as n becomes larger.

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GEOMETRY AND INTEGRAL PUZZLES

25. The Hurried Duelers

Duels in the town of Discretion are rarely fatal. There, each contestant comes at a random moment between 5 AM. and 6 A.M. on the appointed day and leaves exactly 5 minutes later, honor served, unless his opponent arrives within the time interval and then they fight. What fraction of duels lead to violence?

Adobe

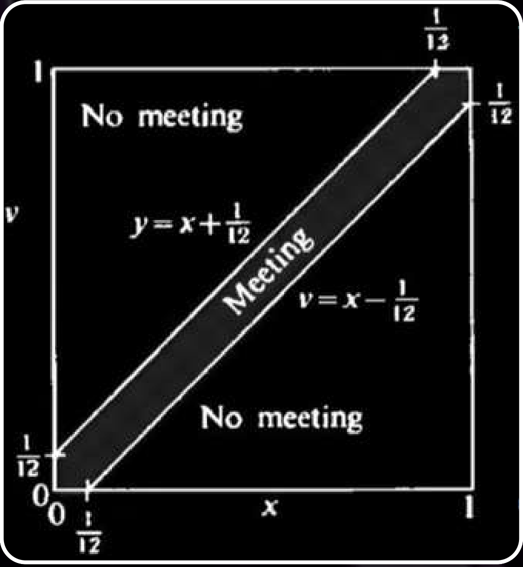
Solution :

Let x and y be the times of arrivals measured in parts of an hour from 5 A.M. The shaded region of the figure shows the arrival times for which the duelists meet. The probability that they do not meet is

$$\frac{11}{12}^2,$$

and so the fraction of duels in which they meet is

$$\frac{23}{144} \approx \frac{1}{6}.$$



26. The Little End of the Stick

(a) If a stick is broken in two at random, what is the average length of the smaller piece?

(b) (For calculus students) What is the average ratio of the smaller length to the larger?

Solution :

(a) Breaking "at random" means that all points of the stick are equally likely as a breaking point (uniform distribution). The breaking point is just as likely to be in the left half as in the right half. If it is in the left half, the smaller piece is on the left, and its average size is half of that half, or one-fourth the length of the stick. The same sort of argument applies when the break is in the right half of the stick, and so the answer is one-fourth of the length.

(b) We might suppose that the point fell in the right-hand half. Then

$$\frac{1-x}{x}$$

is the fraction if the stick is of unit length. Since x is evenly distributed from $\frac{1}{2}$ to 1, the average value, instead of the intuitive $\frac{1}{3}$, is given by:

$$2 \cdot \left(\int_{\frac{1}{2}}^{1} \frac{1-x}{x} \, dx \right) = 2 \cdot \left(\int_{\frac{1}{2}}^{1} \left(\frac{1}{x} - 1 \right) \, dx \right)$$

Calculating this gives:

$$= 2\log(2) - 1 \approx 0.386.$$

27. The Broken Bar

A bar is broken at random in two places. Find the average size of the smallest, of the middle-sized, and of the largest pieces.

Solution :

We might as well work with a bar of unit length. Let x and y be the positions of the two breaking points, x the leftmost one (Fig. 1). We know from the principle of symmetry that each of the three segments (left, middle, and right) averages $\frac{1}{3}$ of the length in repeated drops of two points. But we are asked about the smallest one, for example. If we drop two points at random, let X stand for the position of the first point dropped and Y for the position of the second. Then the random pair (X, Y) is uniformly distributed over a unit square as in Fig. 2, and probabilities can be measured by areas. For example, the probability that $X < 0.2$ and $Y < 0.3$ is given by the area below and to the left of $(0.2, 0.3)$, and it is $0.2 \times 0.3 = 0.06$.

Figure 1: Interval with break points x and y.

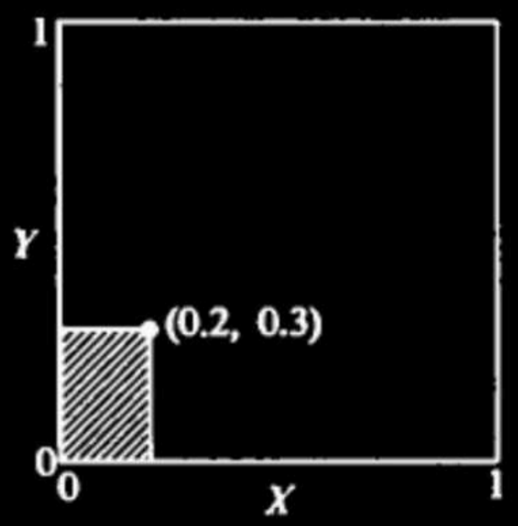


Figure 2: Unit square representing probability distribution for a pair of points (X, Y) dropped on a unit interval.

For convenience, let us suppose that X is to the left of Y, or that $X < Y$. Then the distribution is over the unshaded half-square in Fig. 3. Then probabilities are still proportional to areas, but the area must be multiplied by 2 to get the probability. If we want to get the average length for the

Solution :

segment of the smallest length, then note that either X, $Y - X$, or $1 - Y$ is smallest. Let us suppose X is smallest, so that

$$X < Y - X \quad \text{or, equivalently,} \quad 2X < Y,$$

and

$$X < 1 - Y \quad \text{or, equivalently,} \quad X + Y < 1.$$

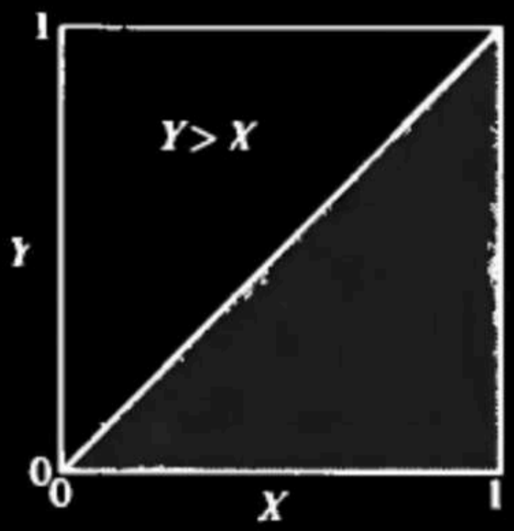


Figure 3: Unshaded area shows where $Y > X$.

In Fig. 4, the triangular region meeting all these conditions is shown heavily outlined. Although X runs from 0 to $\frac{1}{3}$, it must be averaged over the triangular region. The key fact from plane geometry is that the centroid of a triangle is $\frac{1}{3}$ of the way from a base toward the opposite vertex. The base of interest in the heavily outlined triangle is the one on the Y-axis. The altitude parallel to the X-axis is $\frac{1}{3}$. Consequently, the mean of X is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. Therefore the average value of the smallest segment is $\frac{1}{9}$.

Let's see what happens if X is the largest. We want

$$X > Y - X \quad \text{or, equivalently,} \quad 2X > Y,$$

and

$$X > 1 - Y \quad \text{or, equivalently,} \quad X + Y > 1.$$

Solution :

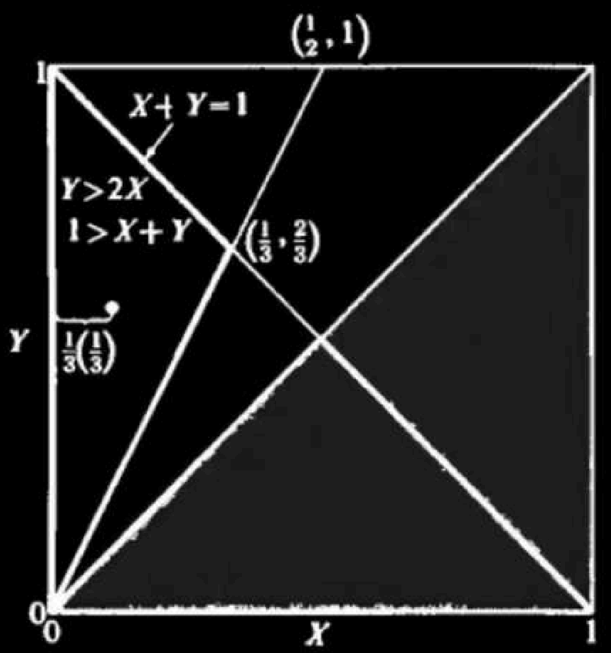


Figure 4: Triangular region where left-most segment is smallest is heavily outlined.

Then we compute the mean for X for each triangle separately and weight the two means by the areas of the triangles to get the final answer.

The mean of X for the right-hand triangle whose base is the dotted line is

$$\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right).$$

That for the left-hand triangle whose base is the dotted line is

$$\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{6}\right).$$

The weights are proportional to the altitudes $\frac{1}{2}$ and $\frac{1}{6}$, respectively, because the triangles have a common base. Finally, the mean of X is

$$\frac{\frac{1}{2}(\frac{1}{3}) + \frac{1}{3}(\frac{1}{3} - \frac{1}{6})}{\frac{1}{2} + \frac{1}{6}} = \frac{11}{18}.$$

Since the mean of the smallest is $\frac{1}{9}$ or $\frac{1}{18}$ and that for the largest $\frac{1}{3}$, the mean for the middle segment is $1 - \frac{1}{18} - \frac{1}{9} = \frac{5}{9}$. You may want to check this by applying the method just used when, for example, $X > Y - X$.

Solution :

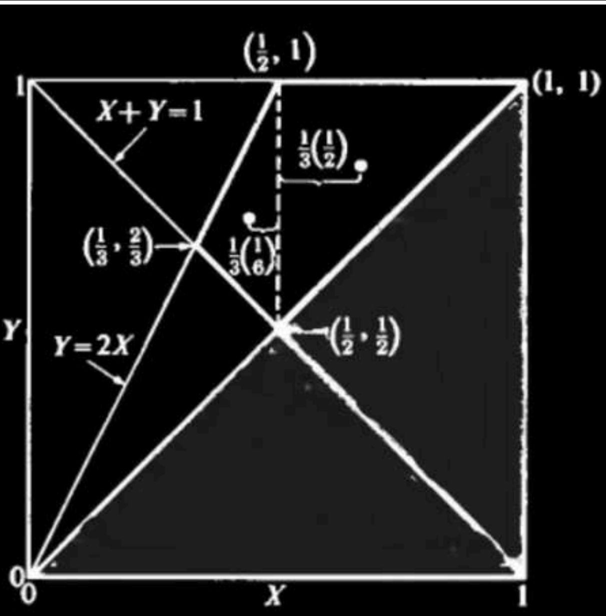


Figure 5: Region where X is greatest is heavily outlined.

So finally, the means of the smallest, middle-sized, and longest segments of the broken bar are proportional to 2, 5, and 11, respectively.

When we break a bar in 2 pieces, the average lengths of the smaller and larger pieces are proportional to

$$\frac{1}{2}, \frac{1}{2},$$

which can be written

$$\frac{1}{2}(2), \frac{1}{2}(2+1).$$

For 3 pieces we have, in order, the proportions

$$\frac{1}{3}, \frac{5}{9}, \frac{11}{18},$$

or

$$\frac{1}{3}(3), \frac{1}{3}(3+\frac{1}{2}), \frac{1}{3}(3+\frac{1}{2}+1).$$

In general, if there are n pieces, the average lengths in order of size are proportional to

$$\text{smallest:} \quad \frac{1}{n}(1),$$

Solution :

$$\text{next largest:} \quad \frac{1}{n}\left(\frac{1}{n}+\frac{1}{n-1}\right),$$

$$\text{3rd:} \quad \frac{1}{n}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right),$$

$$\vdots$$

$$\text{largest:} \quad \frac{1}{n}\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right).$$

But I have no easy proof of this.

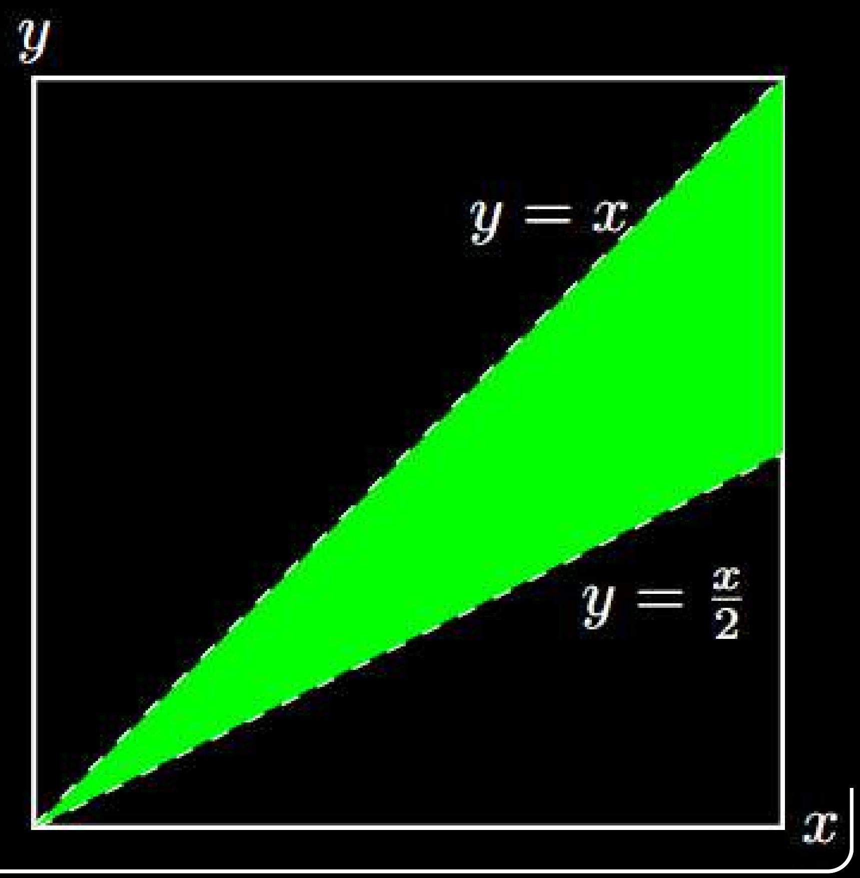
28.Random Ratio

p and q are two points chosen at random between 0 and 1. What is the probability that the ratio p/q lies between 1 and 2?

Answer : 1/4

Solution :

Assume that the points are x and y respectively. We want to know if x/y is between 1 and 2. If we plot this, the limits are x/y=1 and x/y=2. Thus, the desired region is the area between lines x=y and x=y/2



This region is 1/4th of the rest.

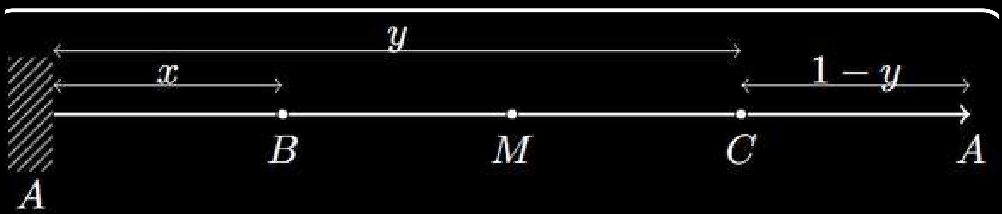
29. Probability the three points on a circle will be on the same semi-circle

Three points are chosen at random on a circle. What is the probability that they are on the same semi circle?

Answer : 3/4

Solution :

Without loss of generality, we can assume the circumference of the circle to be equal to 1. Cut the circle at the first point A and spread it out as a line. Let the other two points B and C be located at distances of x and y from A. The midpoint of this line is M and $0 \leq x, y \leq 1$.



Points B and C can be both located on the same side of M or on either side of M. If both points lie on the same side of M, then all 3 points lie on the same semi-circle.

$$P(\text{same side}) = (2)(\frac{1}{4}) = \frac{1}{2}$$

We are multiplying by 2 as the points can lie on both sides of M.

Solution :

They can also be located on either side of M and in this case, the conditions for all 3 points to lie on the same semicircle are:

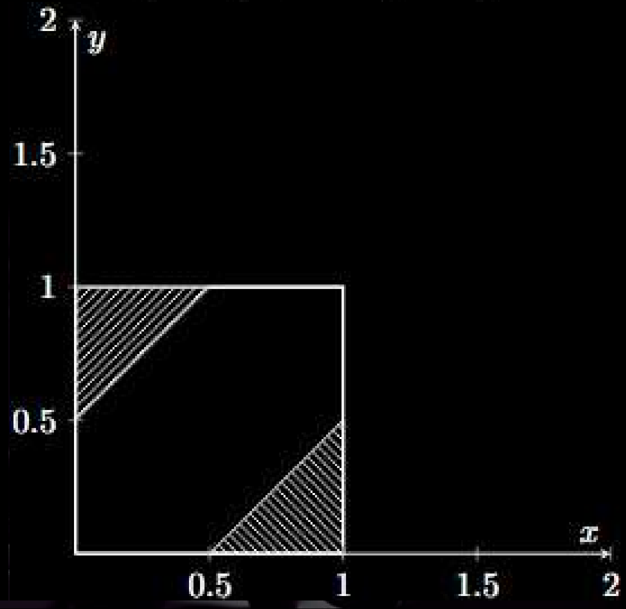
$$x + 1 - y < 0.5$$

$$y + 1 - x < 0.5$$

$$\Rightarrow y > x + 0.5 \quad \text{if } y > x$$

$$\Rightarrow x > y + 0.5 \quad \text{if } x > y$$

Representing these two regions graphically, we get the following area:



The area of the shaded region is $\frac{1}{4}$. Combining all results, we get that the probability of A, B, and C all lying on the same semicircle is $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

30. Stick to Triangle

A stick is broken into 3 parts by choosing 2 points randomly along its length.
With what probability can it form a triangle?

AlphaGrep Adobe

Answer : 1/4

Solution :

For three segments to form a triangle, the sum of the lengths of any two segments must be greater than the length of the third. If we denote the positions of the breakpoints as X and Y, where $X \leq Y$, the lengths of the segments are X, $Y - X$, and $1 - Y$.

To satisfy the triangle inequality, each segment must be less than half the stick's length. Therefore, we require:

$$\min(X, Y - X, 1 - Y) \leq \frac{1}{2}$$

This condition can be simplified to:

1. $\min(X, Y) \leq 0.5$
2. $\max(X, Y) \geq 0.5$
3. $|X - Y| \leq 0.5$

These conditions define a region in the unit square $[0,1]\times[0,1]$. If you divide this square into 8 congruent triangles by drawing lines parallel to the axes and the diagonal Y=X, you'll notice that only 2 of these 8 triangles satisfy all the conditions.

Since 2 out of the 8 triangles meet the criteria, the probability that the three segments can form a triangle is:

$2/8 = 1/4$.

So, the probability is 0.25 or $1/4$.

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31. Witches at the coffee shop

Two witches make a nightly visit to an all-night coffee shop. Each arrives at a random time between 0:00 and 1:00. Each one of them stays for exactly 30 minutes. On any one given night, what is the probability that the witches will meet at the coffee shop?

Answer : $3/4$

Solution :

Let's define X and Y as the arrival times of the two witches, measured in minutes after midnight. Since each witch arrives randomly between 0:00 and 1:00 AM, X and Y are uniformly distributed between 0 and 60 minutes.

We can visualize the possible arrival times as points in a 60×60 square on the XY-plane, where X is the horizontal axis, and Y is the vertical axis. The total area of this square, representing all possible combinations of arrival times, is:

$$60 \times 60 = 3600 \text{ square units.}$$

The witches will meet if their arrival times are close enough so that their 30-minute stays overlap. Mathematically, this condition is:

$$|X - Y| \leq 30.$$

To find the favorable region where the witches meet, we consider two lines:

$$Y = X + 30$$
$$Y = X - 30$$

Solution :

The region where the witches meet is bounded by these two lines, specifically where they fall within the 60×60 square. The favorable region is a parallelogram with vertices at:

$$(0, 30), (30, 0), (60, 30), (30, 60)$$

To calculate the area of this region, we note that it occupies three-fourths of the total area of the square. Therefore, the area of the favorable region is:

$$\frac{3}{4} \times 3600 = 2700 \,\text{square units.}$$

Finally, the probability that the witches will meet is the ratio of the favorable area to the total area:

$$\text{Favorable Area} = 2700, \quad \text{Total Area} = 3600$$

$$\text{Probability} = \frac{2700}{3600} = 0.75$$

Thus, the probability that the witches will meet at the coffee shop is 0.75 or $\frac{3}{4}$.

32. Random Chords

The probability that the length of a randomly chosen chord of a circle lies between ⅔ and ⅚ of its diameter is ?

JPMorgan Chase & Co.

Answer : 1/4

Solution :

If l is the length of the chord, r is the distance of the mid-point of the chord from the centre of the circle and a is the radius of the given circle, then

$$r = a\cos\theta, \quad l = 2a\sin\theta$$

Given:

$$\frac{2}{3}(2a) < 2a\sin\theta < \frac{5}{6}(2a)$$
$$\Rightarrow \frac{\sqrt{11}}{6}a < a\cos\theta < \frac{\sqrt{5}}{3}a$$
$$\Rightarrow \frac{\sqrt{11}}{6}a < r < \frac{\sqrt{5}}{3}a$$

The given condition is satisfied if the midpoint of the chord lies within the region between the concentric circles of radius $\frac{\sqrt{11}}{6}$ and $\frac{\sqrt{5}}{3}$.

$$\text{Hence the probability} = \frac{\text{The area of the circular annulus}}{\text{area of the given circle}}$$
$$= \frac{5}{9} - \frac{11}{36} = \frac{1}{4}$$

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DISTRIBUTIONS

33. Waiting for a Truck

On a given highway, trucks arrive at the station according to a Poisson process with $\lambda = 0.1/\text{minute}$. This means that after a truck has just passed, the time for the next truck to arrive is an exponential random number with an average arrival time of 10 minutes. Your car just broke on this highway, and you are waiting for the next truck for hitchhiking, what is your expected waiting time? On average how many minutes ago the last truck left?

Goldman Sachs

Answer : 10

Solution :

The exponential distribution is memoryless because the past has no bearing on its future behaviour. Every instant is like the beginning of a new random period, which has the same distribution regardless of how much time has already elapsed. Suppose we're observing a stream of events with exponentially distributed inter-arrival times. Because of the memoryless property, the expected time until the next event is always $1/\lambda$, no matter how long we've been waiting for a new arrival to occur.

This behaviour is a bit counter intuitive. We might expect that arrivals get more likely the longer we wait. For example, if the bus is supposed to come every ten minutes, and we have been waiting for nine minutes without seeing a bus, we expect that the next bus should be along very soon. If the time between bus arrivals is exponentially distributed, however, the memoryless property tells us that our waiting time – no matter how long it's been – is of no use in predicting when the next bus will arrive.

Of course, the time between real bus arrivals is never exponentially distributed. People want to know exactly when their buses will come, so bus schedules are nearly deterministic, which justifies our intuition about longer waiting times increasing the probability of an arrival.

34. Sklar's problem

What is $P(X>0|X+Y>0)$ given that X,Y are i.i.d standard normal?

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Answer : 3/4

Solution :

The conditional probability can be written as:

$$P(X > 0 \mid X + Y > 0) = \frac{P(X > 0 \text{ and } X + Y > 0)}{P(X + Y > 0)}$$

Geometric Interpretation

Since X and Y are i.i.d. standard normal random variables, their joint distribution is radially symmetric. This symmetry allows us to divide the plane into eight equally probable regions based on the signs of X, Y, and $X + Y$.

Symmetry and Equal Probabilities

The following eight regions are all equally likely because of the radial symmetry of the joint distribution:

$$0 < Y < X$$
$$0 < X < Y$$
$$X < Y < 0$$
$$Y < X < 0$$
$$0 < Y < -X$$
$$0 < -X < Y$$
$$-X < Y < 0$$
$$Y < -X < 0$$

Relevant Regions for $X + Y > 0$

In exactly four of these regions, we have $X + Y > 0$ (i.e., the upper half-plane where the sum is positive):

$$0 < Y < X$$
$$0 < X < Y$$
$$0 < Y < -X$$
$$-X < Y < 0$$

Solution :

Regions Where $X > 0$ and $X + Y > 0$

Out of these four regions, in exactly three, $X > 0$ holds:

$$0 < Y < X$$
$$0 < X < Y$$
$$0 < Y < -X$$

Final Calculation

Since each of the eight regions is equally probable, the probability of $X + Y > 0$ is $\frac{1}{2}$ (four out of eight regions). The probability of $X > 0$ and $X + Y > 0$ is $\frac{3}{8}$ (three out of eight regions). Thus, the conditional probability is:

$$P(X > 0 \mid X + Y > 0) = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$$

35. Catching the Greedy Counterfeiter

The king's minter boxes his coins n to a box. Each box contains m false coins. The king suspects the minter and randomly draws 1 coin from each of n boxes and has these tested. What is the chance that the sample of n coins contains exactly r false ones?

Solution :

Each of the coins in the king's sample is drawn from a new box and has probability $\frac{m}{n}$ of being counterfeit. The drawings are independent, and so we get the binomial probability for r false (and $n-r$ true) coins to be:

$$P(\text{r false coins}) = \binom{n}{r}\left(\frac{m}{n}\right)^r\left(1-\frac{m}{n}\right)^{n-r}$$

For large n, with r and m fixed, we approximate the binomial distribution by a Poisson distribution. We rewrite the probability expression as:

$$P(\text{r false coins}) = \frac{1}{r!}\cdot n(n-1)\cdots(n-r+1)\cdot\left(\frac{m}{n}\right)^r\cdot\left(1-\frac{m}{n}\right)^{n-r}$$

As n grows large, the terms simplify as follows:

$$\frac{n(n-1)\cdots(n-r+1)}{n^r}\to 1$$

$$\left(1-\frac{m}{n}\right)^n \to e^{-m}$$

$$\left(1-\frac{m}{n}\right)^{-r}\to 1$$

Thus, for large n:

$$P(\text{r false coins}) \approx \frac{e^{-m}m^r}{r!}$$

Poisson Distribution

The probabilities follow a Poisson distribution with parameter m, and the final expression for the probability of exactly r false coins is:

$$P(r) = \frac{e^{-m}m^r}{r!}, \quad r = 0, 1, 2, \ldots$$

36.Correlated Variables

Assume that the random variables X and Y are normally distributed: $X \sim N(\mu_X, \sigma_X^2)$, and $Y \sim N(\mu_Y, \sigma_Y^2)$. The correlation between X and Y is ρ. How can you choose constants a and b such that you minimize the variance of the random variable sum $S = aX + bY$ under the constraints that $a + b = 1$, $0 \le a \le 1$, and $0 \le b \le 1$?

Solution :

The obvious application is to proportions of a portfolio invested in risky assets. Make the substitution $b = 1 - a$. Then the variance of the sum is

$$V(S) = a^2\sigma_X^2 + 2a(1-a)\rho\sigma_X\sigma_Y + (1-a)^2\sigma_Y^2.$$

The first-order condition is

$$\frac{\partial V(S)}{\partial a} = 0.$$

The partial derivative is:

$$\frac{\partial V(S)}{\partial a} = 2a\sigma_X^2 + 2\rho\sigma_X\sigma_Y - 4a\rho\sigma_X\sigma_Y + 2(1-a)(-1)\sigma_Y^2$$

$$= 2\left[a(\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2) + \rho\sigma_X\sigma_Y - \sigma_Y^2\right].$$

Thus, the particular a that satisfies the first-order condition is

$$a^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}.$$

We should check the second-order condition

$$\left.\frac{\partial^2 V(S)}{\partial a^2}\right|_{a=a^*} > 0,$$

to make sure this is a minimum, not a maximum. This is straightforward:

$$\frac{1}{2}\frac{\partial^2 V(S)}{\partial a^2} = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2$$

$$\geq \sigma_X^2 - 2(+1)\sigma_X\sigma_Y + \sigma_Y^2$$

$$= (\sigma_X - \sigma_Y)^2 \geq 0.$$

Solution :

and the first inequality is strict unless $\rho = +1$.

In fact, I have solved the unconstrained problem—ignoring the constraint $0 \le a \le 1$. If a^* breaches the constraints, the constrained solution for a is either 1 or 0, depending upon whether σ_X or σ_Y is the smaller respectively.

In the special case where $\rho = -1$ (perfect negative correlation), the solution for a^* is given by

$$\begin{aligned} a^* &= \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2} \\ &= \frac{\sigma_Y^2 + \sigma_X\sigma_Y}{\sigma_X^2 + 2\sigma_X\sigma_Y + \sigma_Y^2} \\ &= \frac{\sigma_Y(\sigma_X + \sigma_Y)}{(\sigma_X + \sigma_Y)^2} \\ &= \frac{\sigma_Y}{\sigma_X + \sigma_Y}, \end{aligned}$$

and this particular a^* gives variance of $aX + bY$ equal to zero.

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PIGEONHOLE PRINCIPLE

37. hemisphere lemma

Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them.

Morgan Stanley

Solution :

Introduction

Consider the problem of selecting five distinct points on a sphere. The goal is to demonstrate that at least four of these points must lie in the same hemisphere. We approach this by using the pigeonhole principle and basic properties of great circles on the sphere.

Procedure

Step 1: Select Two Distinct Points

Pick any two distinct points out of your five points. If all five points are identical, they clearly lie in a single hemisphere, and the problem is trivial. However, for distinct points, we can proceed by selecting any two. These two points will define at least one great circle on the sphere.

Step 2: Define a Great Circle

A great circle is the largest possible circle that can be drawn on a sphere. The great circle defined by the two distinct points divides the sphere into two hemispheres. If the two points are antipodal (i.e., directly opposite on the sphere), they define infinitely many great circles, but we can arbitrarily pick any one of them. Otherwise, they define a unique great circle.

Step 3: Apply the Pigeonhole Principle

The great circle cuts the sphere into two hemispheres. Now, place the other three points relative to these two hemispheres. According to the pigeonhole principle, since there are only two hemispheres and three additional points, at least two of the three remaining points must lie in the same hemisphere as one another. This ensures that at least four points, including one of the original two distinct points, must be in the same hemisphere.

38. Tiling Problem

Given a “2 x n” board and tiles of size “2 x 1”, count the number of ways to tile the given board using the 2 x 1 tiles. A tile can either be placed horizontally i.e., as a 1 x 2 tile or vertically i.e., as 2 x 1 tile.

Solution :

Let "count(n)" be the count of ways to place tiles on a "2 x n" grid, we have following two ways to place first tile.

1) If we place first tile vertically, the problem reduces to "count(n-1)"

2) If we place first tile horizontally, we have to place second tile also horizontally. So the problem reduces to "count(n-2)"

Therefore, count(n) can be written as below.

count(n) = n if n = 1 or n = 2

count(n) = count(n-1) + count(n-2)

The above recurrence is nothing but Fibonacci Number expression.

39. Birthday Pairings

What is the least number of persons required if the probability exceeds 1/2 that two or more of them have the same birthday? (Year of birth need not match)

Solution :

The usual simplifications are that February 29 is ignored as a possible birthday and that the other 365 days are regarded as equally likely birth dates. Let us solve a somewhat more general problem. Let N be the number of equally likely days, r the number of individuals, and let us compute the probability of no like birthdays. Then we can get the probability of at least one pair of like birthdays by taking the complement.

There are N days for the first person to have a birthday, $N-1$ for the second so that they do not match the first, $N-2$ for the third so that they match neither of the first two, and so on down to $N-r+1$ for the r-th person. Then, applying the multiplication principle, the number of ways for no matching birthdays is:

$$N(N-1)(N-2)\cdots(N-r+1)$$

which has r factors.

To get the probability of no matching birthdays, we also need the number of ways r people can have birthdays without restriction. There are N ways for each person to have a birthday. Thus, by the multiplication principle, the total number of different ways the birthdays can be assigned to r people is:

$$N^r$$

The number in expression (1) divided by that in expression (2) gives the probability of no matching birthdays, because we assume that all birthdays, and therefore all ways of assigning birthdays, are equally likely. Thus, the probability is:

$$\frac{N(N-1)(N-2)\cdots(N-r+1)}{N^r}$$

Solution :

likely. The complement of this ratio is the probability of at least one pair of like birthdays.

Thus

$$P_R = P(\text{at least 1 matching pair}) = 1 - \frac{N(N-1)\cdots(N-r+1)}{N^r}. \tag{2}$$

To evaluate expression (3) for large values of N such as 365 requires some courage or, better, some good tables of logarithms. T. C. Fry in *Probability and its engineering uses*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1928, gives tables of logarithms of factorials, and so it is convenient to evaluate the probability of no like birthdays in the form

$$\frac{N!}{(N-r)!N^r}.$$

The following data help:

$$\begin{aligned}
&\log 365! && = 778.39975\\
&\log 365 && = 2.56229286\\
r = 20,\quad &\log 345! && = 727.38410\\
r = 21,\quad &\log 344! && = 724.84628\\
r = 22,\quad &\log 343! && = 722.30972\\
r = 23,\quad &\log 342! && = 719.77442\\
r = 24,\quad &\log 341! && = 717.24040\\
r = 25,\quad &\log 340! && = 714.70764
\end{aligned}$$

A short bout with tables of logarithms shows that for $r = 23$, the probability of at least one success is 0.5073, but for $r = 22$, the probability is 0.4757. Thus $r = 23$ is the least number that gives a 50-50 chance of getting some like birthdays. Most persons are

Solution :

surprised that the number required is so small for they expected about 365/2.

We discuss that notion in our next problem, but let us do a bit more with the current one.

First, the table gives probabilities of at least one pair of like birthdays for various values of r:

r	P_R
5	0.027
10	0.117
20	0.411
23	0.507
30	0.706
40	0.891
60	0.994

Second, let us learn a tricky way to approximate the probability of failure. Recall that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

If x were very small, then the terms beyond $1 - x$ would not amount to much. Consequently, for small values of x, we might approximate e^{-x} by $1 - x$, or, as in what follows, $1 - x$ by e^{-x}. Note that $N(N-1)\cdots(N-r+1)/N^r$ is a product of factors $(N-k)/N$, where k is much smaller than N. These factors can be written as $1 - k/N$, where $0 \leq k \leq r$.

Therefore,

$$N(N-1)\cdots(N-r+1)/N^r \approx e^{-(0+1+\cdots+(r-1))/N} = e^{-(r-1)r/2N}.$$

40. Finding Your Birthmate

You want to find someone whose birthday matches yours. What is the least number of strangers whose birthdays you need to ask about to have a 50-50 chance?

Answer : 253

Solution :

I think this personal birthmate problem is what most persons think of when they are asked about Birthday Pairings, Problem 40. From their notions about the personal birthmate problem stems their surprise at $r = 23$ for the previous problem. In the current birthmate problem, it is of no use to you if two other persons have the same birthday unless it matches yours. For this problem, most people reason that the number should be about half of 365, or say, 183. Since they have confused the pairings problem with this one, they regard 23 as very small.

While good marks should be given for 183 for the birthmate problem to persons working it in their heads, even here that number is not close to the correct value because the sampling of births is done with replacement. If your first candidate is born on the Fourth of July, that does not use up the date, and later candidates may also be born on that date. Indeed, each candidate's chance to miss matching your birthday is $(N-1)/N$, where $N = 365$, the number of days in a year. When you examine n people, the probability that none of them have your birthday is $[(N-1)/N]^n$, and so the probability that at least one matches is:

$$P_s = 1 - \left(\frac{N-1}{N}\right)^n. \tag{1}$$

Solution :

We need to find the smallest n so that P_s is at least $\frac{1}{2}$. The logarithm of 364 is 2.56110, of $\frac{1}{2}$ is -0.30103.

If we solve the resulting problem in logarithms, we find that n should be 253, quite a bit more than 183.

Alternatively, we could use again the approximation

$$\frac{N-1}{N} = 1 - \frac{1}{N} \approx e^{-1/N}.$$

Then we require approximately

$$P_s \approx 1 - e^{-n/N} = \frac{1}{2}.$$

Consequently,

$$e^{-n/N} = \frac{1}{2}.$$

Taking natural logarithms gives us

$$\frac{n}{N} \approx 0.693,$$
$$n \approx 0.693N.$$

And for $N = 365$, $n \approx 253$.

This birthmate problem is easier to solve than the pairings problem, and so it would be nice to have a relation between the two answers.

Solution :

To see the approximation in action, try it on $r = 23$ and get about 0.500 instead of 0.507. Or set $r(r-1)/2(365)$ equal to $-\log 0.5 \approx 0.693$ and solve for r.

Third, suppose the original problem were extended so that you wanted the least number to achieve at least one pair of either identical birthdays or adjacent birthdays (December 31 is adjacent to January 1). Try this problem on your own.

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RANDOM WALK

41. The Cliff-Hanger

A drunkard takes a forward step with a probability of 0.4 and a backward step with a probability of 0.6. What is the probability they will be one step away from the origin after 11 steps?

Morgan Stanley

Answer : 1

Solution :

For $n = 2$, the 1st person survives by shooting the 2nd. For $n = 4$, the 1st person survives again.

Let us define "round" as one complete circle of shootings. Logically, it makes sense that the first person always survives as the number of people keeps halving evenly in each round. Let us prove this by induction.

Let $S(n)$ be the safe position for a given $n = 2^k$.

Base Case: When $k = 0$, we have one pirate ($2^0 = 1$). The only pirate, which is the first, is the last to survive, so $S(1) = 1$.

Inductive Step: Assume our hypothesis is true for $k = m$. That is, $S(2^m) = 1$.

Solution :

We need to prove this holds for $k = m + 1$, i.e., for $n = 2^{m+1}$ pirates.

In the first round of shooting, all pirates in even-numbered positions are eliminated. Therefore, at the end of the first round, there are 2^m pirates left, all in odd-numbered positions.

Now, the pirates' positions are reset, and the first pirate (who was initially in position 1) is still in position 1.

Therefore, we now have a circle of 2^m pirates, starting from the first pirate. The inductive hypothesis states that the first pirate will survive till the end.

Thus, $S(2^k) = 1$ is true for any integer $k \geq 0$.

For $n = 1024 = 2^{10}$, the position of the surviving pirate would still be the 1st position. Therefore, someone should stand in the first position to survive.

42. Drunkard Problem

A drunkard takes a forward step with a probability of 0.4 and a backward step with a probability of 0.6. What is the probability they will be one step away from the origin after 11 steps?

AMERICAN EXPRESS

43. Gambler's Ruin

Player M has \$1, and Player N has \$2. Each play gives one of the players \$1 from the other. Player M is enough better than Player N that he wins 2/3 of the plays. They play until one is bankrupt. What is the chance that Player M wins?

Solution :

Let us restate the problem generally.

Player A has a units; Player B has b units. On each play of a game, one player wins and the other loses 1 unit. On each play, the probability that Player A wins is p, and the probability that Player B wins is $q = 1 - p$. Play continues until one player is bankrupt. The figure represents the amount of money Player A has at any time. He starts at a. When $a = 0$, he is bankrupt; when $a = a + b$, Player B is bankrupt.

We know that, had Player A played against a bank with unlimited resources, he would have become bankrupt with probability q/p. In the course of a trip to bankruptcy, either he attains an amount of money $a + b$ (where $a + b$ is now finite), or he is never that well off. Let the probability that he loses to Player B be P_{loss} (this is equivalent to the infinite bank winning without Player A ever reaching $a + b$). Then

$$P_{loss} = \frac{q}{p} + \left(1 - \frac{q}{p}\right) \cdot P_{loss}1$$

because P_{loss} is the fraction of the sequences that are absorbed before reaching $a + b$, and of the fraction $\left(1 - \frac{q}{p}\right)$ that do reach $a + b$, the portion P_{loss} is also absorbed at 0 if the game is allowed to proceed indefinitely.

Solution :

Then $P_{win} = 1 - P_{loss}$ is the probability that Player A wins. Making substitutions into eq. (1) and solving for P_{win} gives:

$$P_{win} = 1 - \frac{q}{p} \cdot \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^{a+b}} 2$$

For our players $a = 1$, $b = 2$, $p = \frac{2}{3}$, and $q = \frac{1}{3}$. So in this instance, it is better to be twice as good a player rather than twice as wealthy.

If $p = q = \frac{1}{2}$, then P_{win} in eq. (2) takes the indeterminate form $\frac{0}{0}$. When L'Hospital's rule is applied, we find:

$$P_{win} = \frac{a}{a+b}$$

Thus, had the players been evenly matched, Player A's chance would be $\frac{1}{3}$ and his expectation would be $\frac{1}{3}$. Thus, the game is fair, meaning it has 0 expectation of gain for each player.

44. Bold Play vs. Cautious Play

At Las Vegas, a man with \$20 needs \$40, but he is too embarrassed to wire his wife for more money. He decides to invest in roulette (which he doesn't enjoy playing) and is considering two strategies: bet the \$20 on "evens" all at once and quit if he wins or loses, or bet on "evens" one dollar at a time until he has won or lost \$20. Compare the merits of the strategies.

Solution :

In this problem, we consider two different strategies for a gambler aiming to double his initial money (\$20) in a game of Red-and-Black.

Bold Play: With bold play, the gambler bets all 20 dollars at once. This gives a probability of:

$$P_{\text{bold}} = \frac{4}{7} \approx 0.474$$

for achieving his goal.

Cautious Play: With cautious play, the gambler bets one dollar at a time. This scenario can be modeled using the **gambler's ruin problem** with parameters:

$$m = 20, \quad n = 20, \quad p = \frac{3}{7}, \quad q = \frac{4}{7}.$$

Here: - m is the initial stake (20). - n is the target gain (doubling his stake, so 20). - p is the probability of winning a single dollar. - q is the probability of losing a single dollar.

Calculating the Probability of Success: Using the formula for the probability of reaching a goal n starting from an initial stake m, we have:

$$P = \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^{m+n}}.$$

Substitute the values:

Solution :

$$P = \frac{1 - \left(\frac{4/7}{3/7}\right)^{20}}{1 - \left(\frac{4/7}{3/7}\right)^{40}}.$$

Simplifying:

$$P \approx 0.11.$$

Thus, the probability of success with cautious play is significantly lower than that of bold play:

$$P_{\text{cautious}} \approx 0.11.$$

Intuition Behind the Results: Cautious play has reduced the chances of reaching the goal to less than one-fourth of that for bold play. The intuitive explanation is that **bold play is also fast play**, and fast play reduces the exposure of the money to the house's percentage edge.

Comments by Dubins and Savage: Dubins and Savage argue that no general proof of the superiority of bold play is based solely on this intuitive argument. However, Dubins notes that for the specific case of doubling one's money at Red-and-Black, the explanation provided by Savage is valid, albeit with some mathematical nuances regarding the attainability of bounds.

45. Amoeba Problem

A population of amoebas starts with 1. After 1 period that amoeba can divide into 1, 2, 3, or 0 (it can die) with equal probability. What is the probability that the entire population dies out eventually?

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MISCELLANEOUS

46. Twin Knights

Suppose King Arthur holds a jousting tournament where the jousts are in pairs as in a tennis tournament (see diagram below for the tournament ladder).
The 8 knights in the tournament are evenly matched, and they include the twin knights Balin and Balan.

(a) What is the chance that the twins meet in a match during the tournament?
(b) Replace 8 by 2^n in the above problem. Now what is the chance that they meet?

Solution :

(a) Designate the twins as A and B. Put A in the top bracket (first line of the ladder). Then B is in the same bracket (pair of lines), or in the next bracket, or in the bottom half. The chance that B is adjacent to A is $\frac{1}{4}$, and then the chance they meet is 1. The chance that B is in the next pair from A is $\frac{1}{2}$, and then the chance they meet is $\frac{1}{4}$, because, to meet, each must win his first match. Finally, the chance that B is in the bottom half is $\frac{1}{4}$, and then their chance to meet is $\frac{1}{2^4} = \frac{1}{16}$ because both must win 2 matches. Thus the total probability of their meeting is:

$$\frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{16} = \frac{1}{4}.$$

(b) Note that for a tournament of size 2 they are sure to meet. For $2^2 = 4$ entries, their chance of meeting is $1/2$; for $2^3 = 8$ entries, we have computed their chance to be $1/4 = 1/2^2$. Thus a reasonable conjecture is that for a tournament of size 2^n, their chance of meeting is $1/2^{n-1}$.

Let us prove this conjecture by induction. We consider first the case where the knights are in opposite halves of the ladder, then the case where they are in the same half. The chance that both A and B are in opposite halves of the ladder is $\frac{2^{n-1}}{2^n - 1}$, as we know from the tennis problem immediately above. If they are in opposite halves, A and B can meet only in the finals. A knight has chance $1/2^{n-1}$ of getting to the finals because he must win $n-1$ jousts. The chance that both A and B make the finals is $(1/2^{n-1})^2 = 1/2^{2n-2}$. Therefore the chance of their being in opposite halves and meeting is

$$\frac{2^{n-1}}{2^n - 1}\left(\frac{1}{2^{2n-2}}\right).$$

To this probability must be added the chance of their being in the same half and meeting. Their chance of being in the same half is $\frac{2^{n-1}-1}{2^n-1}$, and according to the induction hypothesis, their chance of meeting in a tournament of $n-1$ rounds is $1/2^{n-2}$. If the induction hypothesis is true, their

47. The Ballot Box

In an election, two candidates ,Albert and Benjamin , have i an ballot box a and b votes respectively , a>b, for example ,3 and 2 .If ballots are randomly drawn and tallied ,what is the chance that at least once after the first tally the candidates have the same number of tallies?

Solution :

Solution for The Ballot Box

For $a = 3$ and $b = 2$, the equally likely sequences of drawings are

$$\begin{array}{ccc} AAABB & *AABBA & *ABBAA \\ AABAB & *ABABA & *BABAA \\ *ABAAB & *BAABA & *BBAAA \end{array}$$

where the starred sequences lead to ties, and thus the probability of a tie in this example is $\frac{7}{10}$.

More generally, we want the proportion of the possible tallying sequences that produce at least one tie. Consider those sequences in which the *first tie* appears when exactly $2n$ ballots have been counted $n \leq b$. For every sequence in which A (for Albert) is always ahead until the tie, there is a corresponding sequence in which B (for Benjamin) is always ahead until the tie. For example, if $n = 4$, corresponding to the sequence

$$AABABABB$$

in which A leads until the tie, there is the complementary sequence

$$BBABABAA$$

in which B always leads. This second sequence is obtained from the first by replacing each A by a B and each B by an A.

Given a tie sometime, there is a first one. The number of sequences with A ahead until the first tie is the same as the number with B ahead until the first tie. The trick is to compute the probability of getting a first tie with B ahead until then.

Since A has more votes than B, A must ultimately be ahead. If the first ballot is a B, then there must be a tie sooner or later; and the only way to get a first tie with B leading at first is for B to receive the first tally.

Solution :

The probability that the first ballot is a B is just

$$\frac{b}{a+b}.$$

But there are just as many tie sequences resulting from the first ballot's being an A. Thus the probability of a tie is exactly

$$P(\text{tie}) = \frac{2b}{a+b} = \frac{2}{r+1},$$

where $r = \frac{a}{b}$. We note that when a is much larger than b, that is, when r gets large, the probability of a tie tends to zero (a result that is intuitively reasonable). And the formula holds when $b = a$, because we must have a tie and the formula gives unity as the probability.

48. Drunk Passenger

A line of 100 airline passengers is waiting to board a plane. They each hold a ticket to one of the 100 seats on that flight. For convenience, let's say that the nthnth passenger in line has a ticket for seat number 'n'. Being drunk, the first person in line picks a random seat (equally likely for each seat). All of the other passengers are sober, and will go to their assigned seats unless it is already occupied; If it is occupied, they will then find a free seat to sit in, at random. What is the probability that the last (100th) person to board the plane will sit in their own seat (#100)?

Solution :

Note that the last passenger can only take seat #1 or #100.

- If any passenger takes seat #1, the cycle stops, and all the subsequent passengers take their own seats (including the last).
- Otherwise, if the #100 seat is taken before #1, the cycle is paused, i.e., the subsequent passengers do take their own seats, but the last passenger would take seat #1.

Now, any passenger from 1st to 99th who is picking a random vacant seat will choose between #1, #100 or any other seat equally likely. Thus, by symmetry, #1 or #100, anyone will be taken first - with equal probability.

Hence the last person ends up in their own seat with a probability of 0.5.

Follow-up Question

What's the probability that the second-last person sits on their seat?

Follow-up Answer

$\frac{2}{3}$. The answer follows from the same logic of symmetry between the choice of seat #1, seat #100 and seat #99.

The probability that the k^{th} person from the last will find their seat vacant is $\frac{k}{k+1}$.

KHARAGPUR **DATA ANALYTICS** GROUP

II . PUZZLES

KHARAGPUR **DATA ANALYTICS** GROUP

GENERAL

1. Burning Cords

You are given two cords that both burn in exactly one hour, not necessarily at a uniform rate. How should you light the cords to determine a time interval of exactly 15 minutes?

FOLLOW-UP QUESTION:

Instead of having two cords, what if you only had one cord, how would you measure 15 minutes?

AMERICAN EXPRESS

Solution :

First, let's try to measure thirty minutes. To do that, we can burn both ends of a cord.

Even if it burns non-uniformly, it will still finish in exactly 30 minutes.

To measure 15 minutes, burn the first cord at one end and the second cord at both ends. Half an hour later, when the second cord finishes burning, the first cord has exactly 30 minutes of length left.

Now we can burn the other end of the first cord, and it shall finish in exactly 15 minutes.

FOLLOW-UP ANSWER:

Break the cord into approximately half, and burn these two cords from both of their ends. If both these cords finish burning together, that means exactly 15 minutes have passed. If any one of these cords finishes first, break the other cords from approximately the middle, and further burn all ends of these little cords. Continuing this way theoretically leads to exactly 15 minutes!

2. Dark Room Deck

In a dark room, there is a deck of 52 cards, with exactly 10 cards facing up, rest facing down. You need to split this into two decks with an equal number of cards facing up!

Morgan Stanley

Solution :

Create a deck of k cards randomly from the original 52-card deck, (k=10 here) and then turn over the k-card deck. Goal is achieved!

3. Eight Balls

You are given eight balls. They appear identical, but one is heavier than the rest. You have a pair of balancing scales. How do you find the heavy ball?

Morgan Stanley

Solution:

Step I:

1. Pick 6 balls.
2. Weigh 3 balls on each side of the scale.

Step II:

1. If the scale balances:
   (a) Weigh the remaining 2 balls.
   (b) The heavier ball will tilt the scale.
2. If the scale does not balance:
   (a) Take the 3 balls on the heavier side.
   (b) Discard one ball.
   (c) Weigh the remaining 2 balls.
   (d) The heavier ball will tilt the scale.

Solution:

Generalization:
Given n balls, we can use the following strategy:

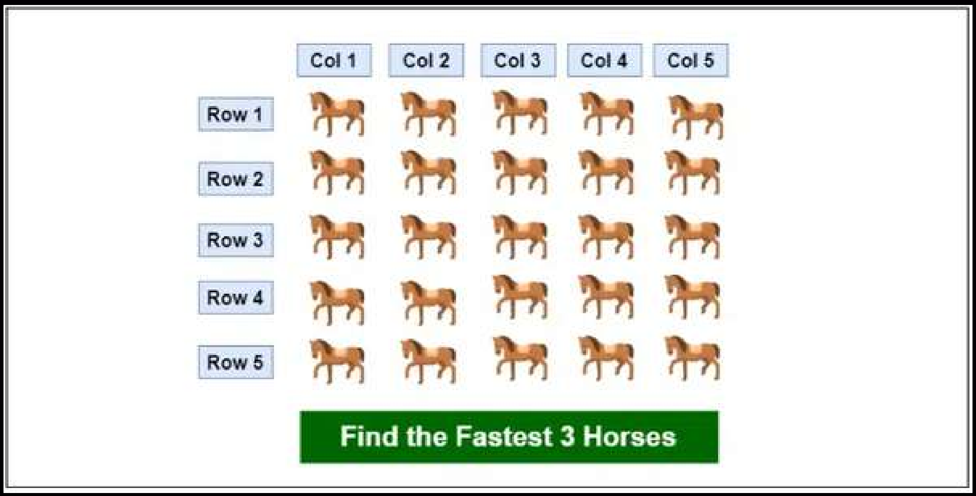
1. If $n = 3k$:
   - Divide the balls into three groups of k balls each.
   - Weigh two groups against each other.
2. If $n = 3k + 1$:
   - Divide the balls into two groups of k balls and one group of $k + 1$ balls.
   - Weigh the two groups of k balls against each other.
3. If $n = 3k + 2$:
   - Divide the balls into two groups of $k + 1$ balls and one group of k balls.
   - Weigh the two groups of $k + 1$ balls against each other.

4. Find the fastest 3 horses

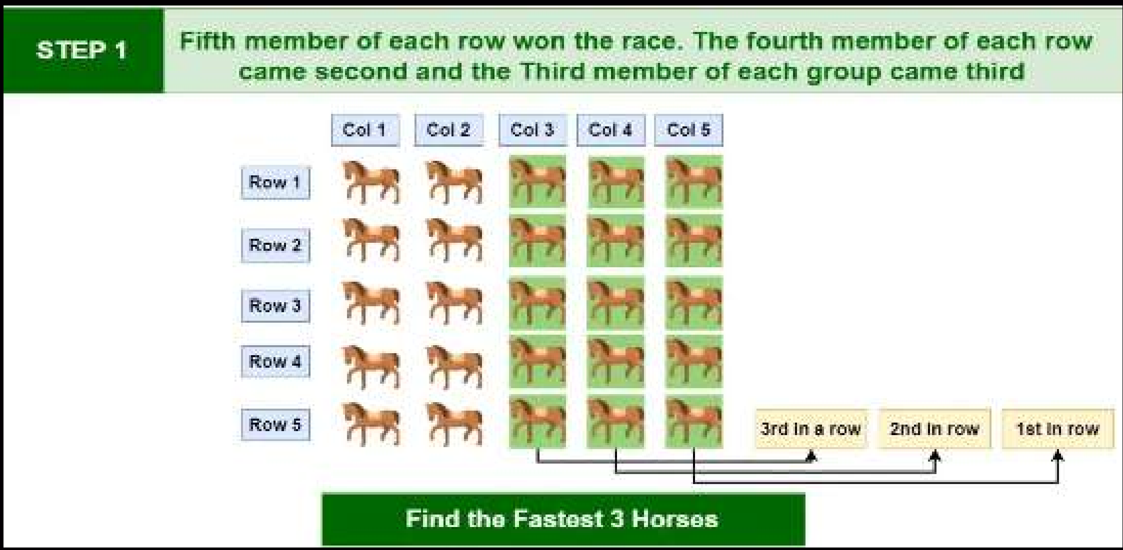
There are 25 horses among which you need to find out the fastest 3 horses. You can conduct a race among at most 5 to find out their relative speed. At no point, you can find out the actual speed of the horse in a race. Find out the minimum no. of races which are required to get the top 3 horses.

Solution:

First, we group the horses into groups of 5 and race each group on the race course. This gives us 5 races (see image below).

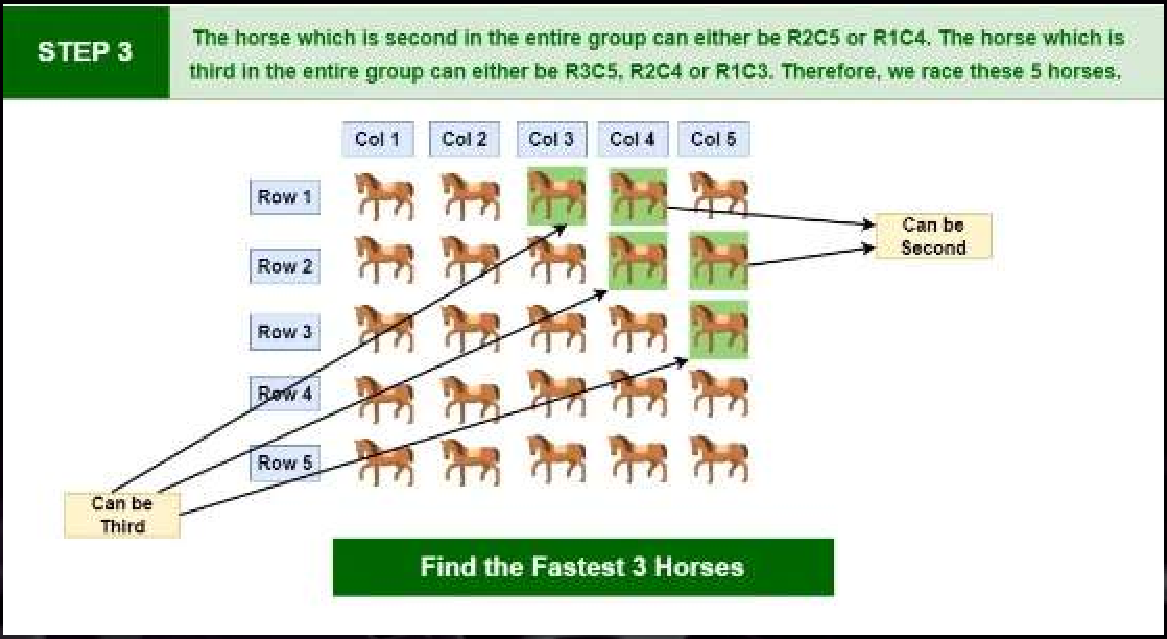


In the image, each row represents one race of 5 horses. For convenience, let us name the horses using the row and column index. Therefore, the first race(row 1) was contested between the horses R1C1, R1C2, R1C3, R1C4 and R1C5. The second race (row 2) was contested between the horses R2C1, R2C2 and so on. Let us assume that the fifth member of each row won the race (R1C5 won the first race, R2C5 won the second race and so on), the fourth member of each row came second (R1C4 came second in the first race, R2C4 came second in the second race and so on) and the third member of each group came third (R1C3 came third in the first race, R2C3 came third in the second race and so on).



Next, we race the 5 level 1 winners (R1C5, R2C5, R3C5, R4C5 and R5C5). Let's say R1C5 wins this race, R2C5 comes second and R3C5 comes third.

The winner of this race (R1C5) is the fastest horse of the entire group. Now, the horse which is second in the entire group can either be R2C5 or R1C4. The horse which is third in the entire group can either be R3C5, R2C4 or R1C3. Therefore, we race these 5 horses.



Therefore, the horse R1C5 is the fastest horse. The horses which come first and second in the last race are the horses which are second and third in the entire group respectively. in this way, the minimum number of races required to determine the first, second and third horses in the entire group is 7.

5. Mislabelled Jars

There are 3 jars, namely, A, B, C. All of them are mislabeled. Following are the labels of each of the jars:

- A: Candies
- B: Sweets
- C: Candies and Sweets (mixed in a random proportion)

You can put your hand in a jar and pick only one eatable at a time. Tell the minimum number of eatable(s) that has/have to be picked in order to label the jars correctly.

Goldman Sachs

Answer :

1 pick of an eatable is required to correctly label the Jars.

Solution :

1. You have to pick only one eatable from **Jar C**. Suppose the eatable is a **candy**, then **Jar C** contains **candies only** (because all the jars were mislabeled).
2. Now, since **Jar C** has candies only, **Jar B** can contain **sweets** or **mixture**. However, **Jar B** can contain only the **mixture** because its label reads **"Sweets"**, which is incorrect.
3. Therefore, **Jar A** must contain **sweets**. Thus, the correct labels are:
   - **Jar A**: Sweets.
   - **Jar B**: Candies and Sweets (Mixture).
   - **Jar C**: Candies.

Solution :


You
Jar A (Candies)
Jar B (Sweets)
Jar C (Candies)
Jar C is picked, if it is a candy, then rightly label it as Candies. Now Jar B might contain Sweets or Mixture.


You
Jar A (Sweets)
Jar B (Candies & Sweets)
Jar C (Candies)
Since Jar B is also mis-labeled, it must contain the Mixtures and Jar A will rightly be labeled as Sweets.

6. 12 Marbles

You are given a set of scales and 12 marbles. The scales are of the old balance variety. That is, a small dish hangs from each end of a rod that is balanced in the middle. The device enables you to conclude either that the contents of the dishes weigh the same or that the dish that falls lower has heavier contents than the other. The 12 marbles appear to be identical. In fact, all of them are identical, and one is of a different weight. Your task is to identify the unusual marble and discard it. You are allowed to use the scales three times if you wish, but no more. Note that the unusual marble may be heavier than the others, or it may be lighter; you do not know which. You are asked to both identify it and determine whether it is heavy or light.

Solution :

The first step is to split the 12 marbles into three groups of four. Each group of four has two subgroups: a singleton and a triplet: $\{1\}_A, \{3\}_A$, $\{1\}_B, \{3\}_B$, and $\{1\}_C, \{3\}_C$.

1. Compare $\{1\}_A, \{3\}_A$ with $\{1\}_B, \{3\}_B$. If they balance, then the odd ball is in group C.
2. In this case, compare $\{3\}_C$ to $\{3\}_B$. If $\{3\}_C$ is heavier (or lighter), then comparing any two marbles from within $\{3\}_C$ immediately locates the odd one. If $\{3\}_C$ balances $\{3\}_B$, then compare $\{1\}_C$ to $\{1\}_B$ to see whether $\{1\}_C$ is heavier or lighter.
3. If the initial comparison is unbalanced, say $\{1\}_A, \{3\}_A$ is heavier than $\{1\}_B, \{3\}_B$, then rotate groups $\{3\}_A, \{3\}_B, \{3\}_C$ and compare grouping $\{1\}_A, \{3\}_B$ with $\{1\}_B, \{3\}_C$ (while holding out $\{1\}_C, \{3\}_A$).
4. If they balance, then a heavy marble is in $\{3\}_A$, and comparing any two marbles from within $\{3\}_A$ immediately locates the odd one.
5. Suppose they do not balance. If $\{1\}_A, \{3\}_B$ is heavy, then either $\{1\}_A$ is heavy, or $\{1\}_B$ is light. Compare $\{1\}_A$ with $\{1\}_C$ to finish. If $\{1\}_A, \{3\}_B$ is light, then $\{3\}_B$ is light, and comparing any two marbles within $\{3\}_B$ immediately locates the light one. In each case, only three weighings are needed.

7. Which Switch

There is a light bulb inside a room and three switches outside. All switches are currently in off state and only one switch controls the light bulb. You may turn any number of switches on or off any number of times you want. How many times do you need to go into the room to figure out which switch controls the light bulb?

Solution :

The Bulb gets hot slowly when turned on. Turn on the switch #1 for 10 minutes, turn it off and turn on the switch #2 and get into the room. There are 3 possible cases:

- If the bulb is on, then switch #2 controls the bulb.
- If the bulb is off but hot, then switch #1 controls the bulb.
- If the bulb is off and cool, then switch #3 controls the bulb.

Thus, we can infer the switch that controls the bulb in one entry.

8. Shooting in Circles

1024 pirates stand in a circle. They start shooting alternately in a cycle such that the 1st pirate shoots the 2nd, the 3rd shoots the 4th, and so on. The pirates who got shot are eliminated from the game. They continue in circles, shooting the next standing pirate, until only one pirate is left. Which position should someone stand to survive?

Solution :

Let us try a few numbers:

For $n = 2$, the 1st person survives by shooting the 2nd.

For $n = 4$, the 1st person survives again.

Let us define a "round" as one complete circle of shootings. Logically, it makes sense that the first person always survives as the number of people keeps halving evenly in each round. Let us prove this by induction.

Let $S(n)$ be the safe position for a given $n = 2^k$.

Base Case: When $k = 0$, we have one pirate ($2^0 = 1$). The only pirate, which is the first, is the last to survive, so $S(1) = 1$.

Inductive Step: Assume our hypothesis is true for $k = m$. That is, $S(2^m) = 1$.

We need to prove this holds for $k = m + 1$, i.e., for $n = 2^{m+1}$ pirates.

In the first round of shooting, all pirates in even-numbered positions are eliminated. Therefore, at the end of the first round, there are 2^m pirates left, all in odd-numbered positions.

Now, the pirates' positions are reset, and the first pirate (who was initially in position 1) is still in position 1.

Therefore, we now have a circle of 2^m pirates, starting from the first pirate. The inductive hypothesis states that the first pirate will survive till the end.

Thus, $S(2^k) = 1$ is true for any integer $k \geq 0$.

For $n = 1024 = 2^{10}$, the position of the surviving pirate would still be the 1st position. Therefore, someone should stand in the first position to survive.

9. Chocolate Bar

There is a 6x8 rectangular chocolate bar made up of small 1x1 bits. We want to break it into 48 bits. We can break one piece of chocolate horizontally or vertically, but cannot break two pieces together! What is the minimum number of breaks required?

Solution :

For a chocolate of size mxn, we need mn - 1 steps. By breaking an existing piece horizontally or vertically, we merely increase the total number of pieces by one. Starting from 1 piece, we need mn - 1 steps to get to mn pieces. Another way to reach the same conclusion is to focus on "bottom left corners of squares": Keep the chocolate rectangle in front of you and start drawing lines corresponding to cuts. Each cut "exposes" one new bottom left corner of some square. Initially, only one square's bottom left corner is exposed. In the end, all mn squares have their bottom left corners exposed.

10 .Clock's Perfect Alignment

What is the first time after 3pm when an hour and minute hands of a clock are exactly on top of each other?

Solution :

Rotation covered by minute hand (MH) per minute $= \frac{360}{60} = 6$
Rotation covered by hour hand (HH) per minute $= \frac{30}{60} = 0.5$

For every minute the change (decrease) in angle $= 6 - 0.5 = 5.5$

At the time of overlap, the angle between them is zero.
At 3:00 AM, HH will be on 3 and MH on 12.
Angle between HH and MH $= 6 \times 15 = 90$

Let the angle be zero after t minutes past 3:00 AM:

$$0 = 90 - t(5.5)$$

$$t = \frac{90}{5.5} = 16.36 \text{ min}$$

Ans: At 3:16 min 21.8 sec, the hands overlap.

11. Tower of Hanoi

In front of you there are three poles. One pole is stacked with 64 rings ranging in weight from one ounce (at the top) to 64 ounces (at the bottom). Your task is to move all the rings to one of the other two poles so that they end up in same order. The rules are that you can move only ring at a time, you can move a ring only from one pole to another, and you cannot even temporarily place a ring on top of a lighter ring.

What is the minimum number of moves you need to make to achieve the task?

Solution :

Let $V(n)$ denote the minimum number of moves needed for n rings. Assert that $V(n) = 2^n - 1$, for all positive integers n. The proof uses mathematical induction.

Case n = 1: With one ring, it certainly takes exactly one move. Assertion is thus true for the case $n = 1$.

Case n = N: Suppose that the assertion is true for $n = N$, and consider the case $n = N+1$. By assumption, it takes $V(N) = 2^N - 1$ moves to get the first N rings to pole #2. Use one additional move to get ring #(N+1) to pole #3. Now use $V(N) = 2^N - 1$ moves to move the N rings on pole #2 to pole #3. The total number of moves used is $2V(N)+1 = 2(2^N - 1)+1 = 2^{(N+1)} - 1$. However, this is just $V(N+1)$. That is, if the assertion is true for $n = N$, it is also true for $n = (N+1)$.

The result follows, because the assertion is true for $n = 1$, and showed that if the assertion is true for $n = N$, then it is also true for $n = N + 1$. In particular, because when showed the assertion to be true for $n = 1$, it follows that it must be true for $n = 2$. It then follows that because the assertion is true for $n = 2$, it is also true for $n = 3, \ldots$ and so on, up to ∞.

Solution :

With $n = 64$ rings, it takes $V(64) = 2^{64} - 1 = 18,446,744,073,709,551,615$ moves. At one move per second, it would take you 584.5 billion (i.e., 584.5 thousand million) years to finish this task. The Earth will fall into the Sun in less than one-hundredth of this time period.

12. Cheating Husbands

A certain town comprises of 100 married couples. Some husbands secretly cheat on their wives. All wives know about the nature of every husband except their own. When a wife concludes that her husband cheated, she kicks her husband into the street at midnight. All husbands remain silent about their secret. One day, the mayor of the town announces to the whole town that there is at least 1 cheating husband in the town. After announcement, no one talks, waiting for someone to get kicked. Till 9th night from announcement, no husband was kicked, but on the 10th night, some husbands got kicked out simultaneously. How many are they?

Solution :

It must be 10 husbands kicked out. If there was only 1 cheating husband in the town, there will be 99 women who know exactly who the cheater is. The 1 remaining woman, who is being cheated on, would have assumed there are no cheaters. But now that the mayor has confirmed that there is at least one cheater, she realizes that her own husband must be cheating on her. So her husband gets kicked on the day of the announcement.

Now let's assume there are 2 cheaters in the town. There will be 98 women in the town who know who the 2 cheaters are. The 2 wives, who are being cheated on, would think that there is only 1 cheater in the town. Since neither of these 2 women know that their husbands are cheaters, they both do not report their husbands on the day of the announcement. The next day, when the 2 women see that no husband was kicked, they realize that there could only be one explanation – both their husbands are cheaters. Thus, on the second day, 2 husbands are kicked.

Through induction, it can be proved that when this logic is applied to n cheating husbands, they are all kicked on the n-th day after the mayor's announcement. Hence it must be 10 husbands kicked in our case.

13. Gold Bar Problem

you hire a man to work in your yard for seven days. you wish to pay him in gold. you have one gold bar with seven parts - like a chocolate bar. you wish to pay him one gold part per day. but you may snap the bar in only two places. Where do you snap the bar so that you may pay him at the end of each day, and on successive days he may use what you paid him previously to make change?

Answer : Size of 1, 2 and 4 units

Solution :

Using these pieces, you can pay the man as follows:

- **Day 1:** Pay 1 (Piece of size 1)
- **Day 2:** Take back Piece 1 and pay 2 (Piece of size 2)
- **Day 3:** Pay 1 again (Piece of size 1)
- **Day 4:** Take back Pieces 1 and 2, and pay 4 (Piece of size 4)
- **Day 5:** Pay 1 (Piece of size 1)
- **Day 6:** Take back Piece 1 and pay 2 (Piece of size 2)
- **Day 7:** Pay 1 (Piece of size 1)

This way, you can pay him each day while allowing him to use the pieces received on previous days to make change.

14. Newpapers Puzzle

A newspaper made of 16 large sheets of paper folded in half. The newspaper has 64 pages altogether. The first sheet contains pages 1, 2, 63, 64. If we pick up a sheet containing page number 45. What are the other pages that this sheet contains?

Solution :

On the back of 45, it is 46. The pages are such that for each page p, $65 - p$ will be also on the same page. Then,

$$65 - 45 = 20$$

$$65 - 46 = 19$$

So the four pages in this sheet are 19, 20, 45, 46.

KHARAGPUR **DATA ANALYTICS** GROUP

MATHS PUZZLES

15. Water and Wine

We have two jars, each containing an equal volume V of liquid. One jar contains water and the other contains wine. First, we transfer an amount x of water from the water jar to the wine jar. After mixing, we transfer the same amount x of the mixture (now consisting of both water and wine) back into the water jar. The question is: Is there now more water in the wine jar or more wine in the water jar?

Adobe

Answer : Equal

Solution :

1. **First Transfer (Water to Wine Jar):** We transfer x volume of water from the water jar to the wine jar.
   - The water jar now has $V - x$ volume of water.
   - The wine jar now contains V volume of wine and x volume of water, totaling $V + x$.
2. **Second Transfer (Back to Water Jar):** Now we take x volume of this mixture (which contains both water and wine) and transfer it back to the water jar.
   - Since the concentration of water in the wine jar is $\frac{x}{V+x}$, the amount of water transferred back is:

     $$\text{Water transferred} = x \cdot \frac{x}{V+x} = \frac{x^2}{V+x}$$

   - The concentration of wine in the wine jar is $\frac{V}{V+x}$, so the amount of wine transferred back is:

     $$\text{Wine transferred} = x \cdot \frac{V}{V+x} = \frac{Vx}{V+x}$$

Solution :

3. **Final Volumes in Each Jar:**
   - In the **water jar**:
     - Final water amount = original V, minus x, plus $\frac{x^2}{V+x}$.
     - Final wine amount = $\frac{Vx}{V+x}$.
   - In the **wine jar**:
     - Final water amount = $x - \frac{x^2}{V+x}$.
     - Final wine amount = original V, minus $\frac{Vx}{V+x}$.
4. **Equating Water and Wine**: We compare the amount of water in the wine jar with the amount of wine in the water jar:

$$x - \frac{x^2}{V+x} = \frac{Vx}{V+x}$$

Simplifying, we find both amounts are equal, so the **water in the wine jar equals the wine in the water jar**.

16. Rabbit on the Staircase

A rabbit sits at the bottom of a staircase with n stairs. The rabbit can hop up only one or two stairs at a time. What is the number of different ways possible for the rabbit to ascend to the top of the stairs of length $n = 1, 2, 3, \ldots$?

JPMorgan Chase & Co.

Answer : Fibonacci Number

Solution :

The number of ways to reach the top of a staircase with n stairs is given by the Fibonacci sequence:

$$f(n) = f(n-1) + f(n-2)$$

with the initial conditions:

$$f(1) = 1, \quad f(2) = 2.$$

Thus, the number of ways the rabbit can ascend to the top of n stairs is the n-th Fibonacci number.

17. Comparison of e^{π} and π^{e}

Morgan Stanley

JPMorgan Chase & Co.

Solution :

Step 1: Define the Function

We define the function:

$$f(x) = \frac{\ln x}{x}$$

Step 2: Find the Derivative

To analyze this function, we compute its derivative:

$$f'(x) = \frac{1 - \ln x}{x^2}$$

Solution :

Step 3: Analyze the Critical Points

The critical points occur when $f'(x) = 0$:

$$1 - \ln x = 0 \implies \ln x = 1 \implies x = e$$

We will analyze the behavior of $f'(x)$ around this critical point:
- For $x < e$, $\ln x < 1$ so $f'(x) > 0$. Thus, $f(x)$ is increasing. - For $x > e$, $\ln x > 1$ so $f'(x) < 0$. Thus, $f(x)$ is decreasing.

This indicates that $f(x)$ has a maximum at $x = e$.

Step 4: Compare $f(e)$ and $f(\pi)$

Now, we compare the values of f at e and π: - Since $e < \pi$ and $f(x)$ is decreasing for $x > e$, we have:

$$f(e) > f(\pi)$$

18. 2 Equations & 3 Unknowns

I guessed 3 natural numbers x, y, z. You can ask me 2 sums of these numbers with any integer coefficients (a, b, c). That is, you give me a, b, and c, and I tell you the result of the expression:

$$a \cdot x + b \cdot y + c \cdot z.$$

Seeing the answer, you then give me the 2nd triplet of (a, b, c) and I will tell you $a \cdot x + b \cdot y + c \cdot z$.

The goal is to find x, y, and z.

Goldman Sachs

Solution :

1. Use the first calculation with $(a, b, c) = (1, 1, 1)$:

$$S_1 = x + y + z.$$

Let d be the number of digits in this result.

2. Set the second triplet of coefficients:

$$(a, b, c) = (1, 10^d, 10^{2d}).$$

Let the sum be S_2:

$$S_2 = x + 10^d y + 10^{2d} z.$$

3. From S_2, extract the values: - x is the first d digits of S_2, - y is the digits from $[d+1]$ to $[2d]$, - z is the digits from $[2d+1]$ to $[3d]$.

Thus, we note that it is possible to solve for n natural numbers $x_1, x_2, \ldots, x_n$ with just 2 questions.

19. Dead Men Walking

Assume 100 zombies are walking on a straight line, all moving with the same speed. Some are moving towards left, and some towards right. If a collision occurs between two zombies, they both reverse their direction. Initially all zombies are standing at 1 unit intervals. For every zombie, you can see whether it moves left or right, can you predict the number of collisions?

Solution :

Analysis

When two zombies collide, they reverse their direction. However, this can be treated as if they passed through each other without changing direction.

Key Observations

1. Each collision occurs when a zombie moving right meets a zombie moving left. 2. The number of collisions can be determined by counting how many zombies are moving right when a zombie moving left is encountered, and vice versa.

Solution :

Collision Count

To predict the total number of collisions: - Let R be the number of zombies moving to the right. - Let L be the number of zombies moving to the left. - The total number of collisions C can be given by the formula:

$$C = R \times L$$

This is because each right-moving zombie will collide with each left-moving zombie.

20.Primes

Why it is that if p is a prime number greater than 3, then p^2 - 1 is always divisible by 24 with no remainder.

Solution :

A prime number has no factors other than itself and 1. Thus, 4 is not prime because it has factors: (1,4) and (2,2). Drawing a number line might be a good way to explain this to an interviewer. I will just use words.

1. A prime p bigger than 2 cannot be an integer multiple of 2, else it would not be prime. Thus, a prime bigger than 2 must be odd. Thus, $p-1$ is even. Thus, $p-1=2n$ for some positive integer n. Thus, $p=2n+1$.

2. A prime p bigger than 3 cannot be an integer multiple of 3, else it would not be prime. However, draw a number line and it must be that either $p-1$ or $p+1$ (but not both) is a multiple of 3. That is, p is 1 away from a multiple of 3, but we do not know in which direction. Thus, $p \pm 1 = 3m$ for some positive integer m, where $\pm$ means exactly one of $+$ or $-$, but not both. Thus, $p = 3m \pm 1$.

3. The question asks about p^2-1. From #1 we see that $p^2-1 = 4n^2+4n = 4n(n+1)$. One of n or $n+1$ must be even, and with that 4 there, we see that p^2-1 contains a factor of 8 (i.e., $2 \times 2 \times 2$).

4. From #2, we see that $p^2-1 = 9m^2 \pm 6m = 3m(m \pm 2)$. Thus, p^2-1 contains 3 as a factor.

5. If we picture p^2-1 factored out into all possible numbers of smallest possible size, then the results from #3 and #4 cannot overlap. That is, p^2-1 contains factors of $2 \times 2 \times 2$ and 3; thus, p^2-1 is an integer multiple of 24.

21. Sharing Wood

A. B & C live together and share everything equally. One day A brings home 5 logs of wood, B brings 3 logs and C brings none. Then they use the wood to cook together and share the food. Since C did not bring any wood, he gives $8 instead. How much to A and how much to B?

Answer :

Out of the 8 dollars, A gets 7 and B gets 1

Solution :

Since each person consumed $\frac{8}{3}$ woods, A gave $5 - \frac{8}{3} = \frac{7}{3}$ woods to C, and B gave $3 - \frac{8}{3} = \frac{1}{3}$ woods to C.

22. Multilingual

A group has 70 members. For any two members X and Y .There is a language that X speaks but Y does not, and there is a language that Y speaks but X does not. At least how many different languages are spoken by this group?

Answer : 8

Solution :

Let's go with hit and trial.

1. Everyone knows the same language, that is a contradiction. Therefore it must be more than 1 language.
2. So, P_1 knows one language and P_2 knows another language. But P_3 will know the same language as either P_1 or P_2. We cannot extend this to 3 people.
3. Might be true, but it is hard to confirm.

Let's try a different approach. Let's calculate the largest group possible, given the number of languages.

3. If there are three languages (L_1, L_2, L_3), we can construct the following table.

	L_1	L_2	L_3
P_1	knows		
P_2		knows	
P_3			knows

4. If there are four languages (L_1, L_2, L_3, L_4), we can construct the following table.

	L_1	L_2	L_3	L_4
P_1	knows			
P_2		knows		
P_3			knows	

Solution :

We need to prove this holds for $k = m + 1$, i.e., for $n = 2^{m+1}$ pirates.

In the first round of shooting, all pirates in even-numbered positions are eliminated. Therefore, at the end of the first round, there are 2^m pirates left, all in odd-numbered positions.

Now, the pirates' positions are reset, and the first pirate (who was initially in position 1) is still in position 1.

Therefore, we now have a circle of 2^m pirates, starting from the first pirate. The inductive hypothesis states that the first pirate will survive till the end.

Thus, $S(2^k) = 1$ is true for any integer $k \geq 0$.

For $n = 1024 = 2^{10}$, the position of the surviving pirate would still be the 1st position. Therefore, someone should stand in the first position to survive.

23. Relative Distance

Two trains are on same track and they are coming toward each other. The speed of the first train is 50 km/h and the speed of the second train is 70 km/h. A bee starts flying between the trains when the distance between two trains is 100 km. The bee first flies from first train to second train. Once it reaches the second train, it immediately flies back to the first train … and so on until trains collide. Calculate the total distance travelled by the bee. Speed of bee is 80 km/h.

Answer : 66.7 KM

Solution :

Let the first train A move at u km/h.
Let the second train B move at v km/h.
Let the distance between two trains be d km.
Let the speed of bee be b km/h.

Therefore, the time taken by trains to collide $= d/(u+v)$ Now putting all the known values into the above equation, we get,

$$u = 50km/hr$$

$$v = 70km/hr$$

$$d = 100km$$

$$b = 80km/hr$$

Therefore, the total distance travelled by bee

$$= b * d/(u+v)$$

$$= 80 * 100/(50+70)$$

$$= 66.67km(approx)$$

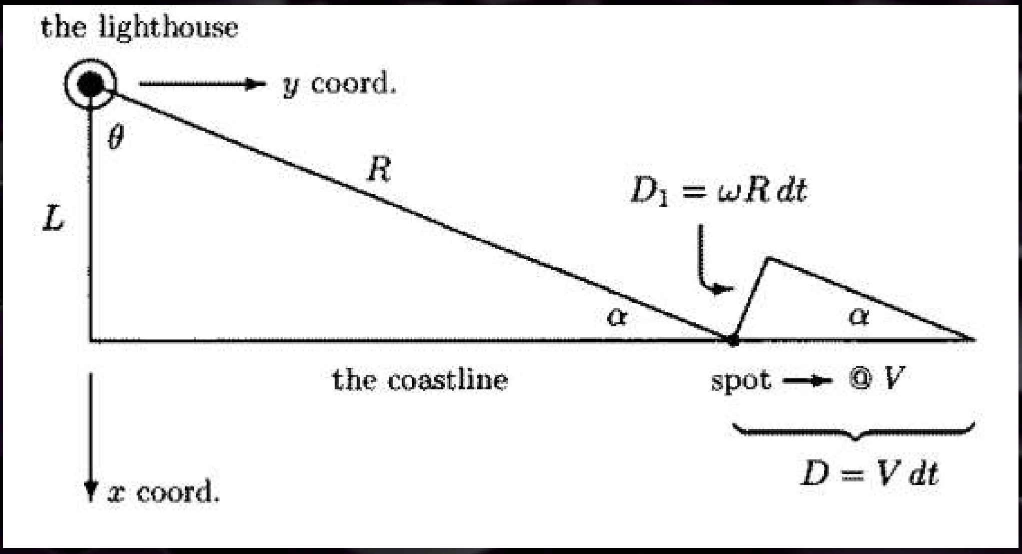
24. Lighthouse Problem

Suppose there is a straight coastline and a lighthouse that is L=3 miles away from the coast. The light revolves at one revolution per minute. How fast is the beam of light traveling along the coastline? When the beam is 3L away from the coastal point closest to the light, how fast is the light traveling along the coast?

Answer : $V = \dfrac{\omega R^2}{L}$

Solution :

We are given a lighthouse problem, where a lighthouse is located a distance L from the coast. The beam of light creates a "spot" that moves a distance R across the sea from the lighthouse (see Figure below).



Solution:

We follow the beam's course for a small time interval dt. In Figure, we observe that the beam's spot covers a distance:

$$D = V dt$$

along the coast, while the beam's "perpendicular motion" covers a distance:

$$D_1 = \omega R dt$$

where:

- V is the speed of the spot along the coast.
- $\omega = \frac{2\pi}{60}$ radians per second is the angular velocity of the beam.

For small dt, the distance triangle forms a right-angled triangle. From this, we have:

$$\sin(\alpha) = \frac{D_1}{D} = \frac{\omega R dt}{V dt} = \frac{\omega R}{V}$$

Solution:

Thus, equating both expressions for $\sin(\alpha)$, we find:

$$V = \frac{\omega R^2}{L}$$

This gives the speed of the spot along the coast in terms of the angular velocity ω, the distance from the lighthouse to the coast L, and the distance from the lighthouse to the spot R.

25. Derangement

Supposing 4 letters are placed in 4 different envelopes. In how many ways can they be taken out from their original envelopes and distributed among the 4 different envelopes so that no letter remains in its original envelope?

Goldman Sachs

Answer : 9

Solution :

We are tasked with finding the number of ways to rearrange 4 letters such that no letter remains in its original envelope. This is known as the *derangement* problem.

The formula to calculate the number of derangements D_n for n objects is:

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

For $n = 4$, we have:

$$D_4 = 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

First, calculate 4!:

$$4! = 24$$

Solution :

Now, calculate the terms inside the parentheses:

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$$

$$= 0 + 0.5 - 0.1667 + 0.04167 = 0.375$$

Finally, multiply by 24:

$$D_4 = 24 \times 0.375 = 9$$

Thus, the number of ways to distribute the letters such that no letter remains in its original envelope

26. Domino Covering

An 8x8 chessboard can be entirely covered by 32 dominoes of size 2x1. Suppose we cut off two opposite corners of chess (i.e. two white blocks or two black blocks). Prove that now it is impossible to cover the remaining chessboard with 31 dominoes.

Solution :

The two diagonally opposite corners are of the same color. A domino covers adjacent faces & hence a domino always covers 1 black and 1 white square. The 31 dominoes will cover 31 blacks and 31 whites. The chess has 30 & 32 square instead. Hence this can't be done.

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STRATEGY

Pirates and the Treasure

Five pirates need to divide 100 Gold Coins. Pirates have a hierarchy, from Level 1 to level 5. The highest level pirate is the leader. The leader proposes a plan to distribute the gold and all the pirates vote on it (including the leader). If at least 50% of the pirates agree on the plan, the gold is split according to the proposal. If not, level-5 pirate is kicked from the ship, and the level-4 pirate now proposes a new plan. This process continues until a proposal is accepted. All pirates are extremely smart and extremely greedy. How does level-5 Pirate divide the treasure?

Goldman Sachs

Answer : (98, 0, 1, 0, 1)

Solution :

Let the pirates be called P5, P4, P3, P2, P1. P5 is the leader.

Let's study a simpler case with only 2 pirates. P2 can propose to take all the 100 gold coins. Even if P1 votes against this, P2 will vote in favor and the proposal will still be accepted (50 % acceptance) leaving P1 with zero coins. The distribution is (100, 0) for levels (2, 1) respectively.

Now let's look at the case with 3 pirates. P3 knows that if this proposal is not accepted, then P2 will get all the gold and P1 will get nothing. So P3 can bribe P1 with one gold coin. P1 knows that one gold coin is better than zero, and hence votes in favor of this proposal. Therefore, P3 can propose the following distribution Level 3 pirate: 99, Level-2: 0, Level-3: 1. Since pirate 1 and 3 will vote in favor, this proposal will be accepted. The distribution is (99, 0, 1) for levels (3, 2, 1) respectively.

If there are 4 pirates, P4 can extend this pattern by bribing those who would get nothing in 3-pirates case, to ensure their votes. So the distribution will be (99, 0, 1, 0) for levels (4, 3, 2, 1) respectively, this will ensure that P2 will vote in favor and with the help of self voting (P4), we have 50% votes to accept this proposal.

Similarly, P5 can bribe P3 and P1 pirates because they get nothing if the proposal is rejected, as P4 was giving them nothing. Hence, at level-5, the proposal is (98, 0, 1, 0, 1). This proposal will get accepted and provide the maximum amount of gold to the leader.

This puzzle gives a basic idea of game theory and dynamic programming.

Two Eggs

An Egg breaks only if dropped from above a threshold floor, within a 100-story building. Every time you drop an egg, it is counted as an attempt. You are given 2 eggs to deduce the threshold floor, with a minimum number of attempts in the worst case!

JPMorgan Chase & Co.

Answer : 14

Solution :

We can skip some floors and jump ahead when testing with the first egg (egg1). Whenever egg1 breaks, egg2 has to scan linearly from the last safe floor to the floor where egg1 broke.

For example, we can test floors 10, 15, 20 and so on. Suppose egg1 broke at the th 20 th floor, now we need to test from 16 to 19, which can be done one by one using the second egg (egg2). But this is suboptimal, because regardless of when egg1 breaks, egg2 is always taking 4 attempts. We can improve this algorithm by reducing the length of the gap on each attempt of egg1.

For instance, if we test at floor 10, 10+9, 10+9+8, and so on, then in the worst case, the number of attempts will be at-most (1+9)=10 or (2+8)=10 and so on... But, there's a problem, this sequence ends at

$$10 + 9 + 8... = n(n + 1)/2 = 55$$

and does not span the entire range of 100 floors. But the idea is in the right direction.

A solution for minimum steps in the worst case is the smallest integer greater than or equal to the positive solution of

$$x(x + 1)/2 = 100$$

$$x = 13.651$$

Start at floor 14, if the egg breaks start linearly from 1, if it does not break then drop the egg from $14 + 13 = 27th$ floor, and so on.

This ensures at-most 14 attempts.

31: Tigers & The Sheep

Hundred tigers and one sheep are put on a magic island that only has grass. Tigers can live on grass, but they want to eat sheep. If a Tiger bites the Sheep then it will become a sheep itself. If 2 tigers attack a sheep, only the first tiger to bite converts into a sheep. Tigers don't mind being a sheep, but they have a risk of getting eaten by another tiger. All tigers are intelligent and want to survive. Will the sheep survive?

Answer : Survives

Solution :

If there is 1 tiger, then it will eat the sheep because he does not need to worry about being eaten. Sheep will not survive. If there are 2 tigers, both of them knows that if one eats the Sheep, the other tiger will eat him. So, the sheep will survive.

If there are 3 tigers, then they each of them knows that if one tiger eats up the sheep, then Iceland will be left with 1 sheep and 2 tigers and as shown in the previous case, the sheep will survive. Hence each tiger will try to eat up the sheep. The sheep will not survive.

If there are 4 tigers, then the sheep will survive.

And so on....

So, If there are even number of tigers the sheep will survive, else it will die. Hence, if there are 100 tigers the sheep will survive.

32:Duck & Fox

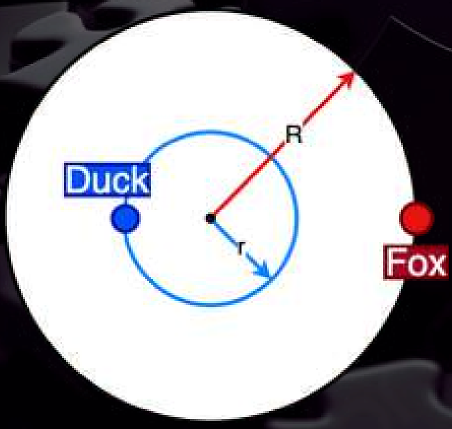
A duck is sitting at the center of a circular lake. A fox is waiting at the shore, not able to swim, wishing to eat the duck. The Fox can move around the whole lake at a speed four times the speed at which the duck can swim. The duck can fly, but only once it reaches the shore of the lake, it can't fly from the water directly. Can the duck always reach the shore without being eaten by the fox?
Note: This is an old duck, and cannot take a flight while swimming. The duck cannot submerge in the water.

Answer : Yes

Solution :

This is a classic problem involving geometry and relative speed, often referred to as the "Duck and Fox Problem". Here's the approach:

1. Swim in Circles: The duck begins by swimming in a circle with a radius of r<R/4r<R/4 where RR is the radius of the lake. The center of this circle is the same as that of the lake. The fox is four times as fast as the duck. By swimming in a small circle, the duck can start creating an angular separation between itself and the fox.
2. Force the Fox to Run: Since the fox is trying to chase the duck, it will continue to run around, trying to move to the point nearest to the duck but on the shore. As the duck goes around slightly faster, the fox lags behind. Eventually, the duck will be at a point where the fox is directly across the lake from it.



Solution contd. :

3. **Wait for the Right Moment:** The duck continues swimming in its smaller circle until the fox is straight across the lake from it. At this point, the duck and fox are separated by half the circumference of the lake, a distance that the fox has to cover to reach the duck if it heads straight for the shore.

4. **Head for the Shore:** While standing directly opposite, the duck can now swim straight to the shore. Duck has to travel around R·3/4 distance, while the fox has to travel πR. Given the ratio of their speeds, it will take 3R and 3.14·R units of time respectively. Hence the duck will have a few extra moments to start the flight.

5. **Escape:** Reaching moments before the fox, the duck can now fly away to safety.

34: Poisonous wine

So there's this king. Someone breaks into his wine cellar where he stores 1000 bottles of wine. This person proceeds to poison one of the 1000 bottles, but gets away too quickly for the king's guard to see which one he poisoned or to catch him.
The king needs the remaining 999 safe bottles for his party in 4 weeks. The king has 10 prisoners who deserve execution. The poison takes about 3 weeks to take effect, and any amount of it will kill whoever drinks it. How can he figure out which bottle was poisoned in time for the party?

Solution :

The King's Poisoned Wine Problem

The king needs to isolate a single poisoned bottle of wine. Here's how the problem is solved:

Setup

There are 10 servants, and after approximately 3 weeks, each servant can either be dead or alive. This means there are $2^{10} = 1024$ possible outcomes. Since $1024 > 1000$, it is possible to devise a scheme that will work for identifying the poisoned wine.

The Scheme

The king assigns each servant a number from 1 to 10. He also assigns each wine bottle a number from 0 to 999. When labeling the bottles, the king writes the number in binary with ten digits. For example:

- 0: 0000000000
- 1: 0000000001
- 2: 0000000010
- 3: 0000000011
- 4: 0000000100
- 5: 0000000101
- ...
- 999: 1111100111

Solution :

The Strategy

The strategy is simple:

1. For each wine bottle, convert its number to binary.
2. Identify which positions in the binary form have a '1'.
3. Make the corresponding servants (based on those positions) drink a small quantity of that wine. For example, if the binary representation of a wine bottle is `0000000001`, only servant 1 drinks it. If the binary representation is `0000000110`, then servants 2 and 3 drink it.

Conclusion

After 3 weeks, observe which servants are dead. Suppose only servants 4 and 7 die. This means the binary representation of the poisoned wine has '1' at positions 4 and 7, and the rest are zeros. In this case, the binary number would be `0000100100`. Converting this binary number to decimal gives the number of the poisoned wine bottle.

35:King's Salary

After the revolution, each of the 66 citizens of a certain city, including the king, has a salary of 1. King cannot vote, but has the power to suggest changes - namely, redistribution of salaries. Each person's salary must be a whole number of dollars, and the salaries must sum to 66. He suggests a new salary plan for every person including himself in front of the city. Citizens are greedy, and vote yes if their salary is raised, no if decreased, and don't vote otherwise. The suggested plan will be implemented if the number of "yes" votes are more than "no" votes. The king is both, selfish and clever. He proposes a series of such plans. What is the maximum salary he can obtain for himself?

Answer : $6

Solution :

The king begins by proposing that 33 citizens have their salaries doubled to $2, at the expense of the remaining 33 (himself included). Next, he increases the salaries of 17 of the 33 salaried voters (to $3 or $4) while reducing the remaining 16 to $0. In successive turns, the number of salaried voters falls to 9, 5,
3, and 2. Finally, the king bribes three paupers with $1 each to help him turn over the two big salaries to himself, thus finishing with a royal salary of $63. It is not difficult to see that the king can do no better at any stage than to reduce the number of salaried voters to just over half the previous number; in particular, he can never achieve a unique salaried voter. Thus, he can do no better than $6

33: Prisoner's Hat

One hundred prisoners are lined up, facing one direction and assigned a random hat, either red or blue. Each prisoner can see the hats in front of them but not behind. Starting with the prisoner at the back of the line and moving forward, they must each, in turn, say only one word which must be "red" or "blue". If the word matches their hat color they are released, if not, they are kept imprisoned. They can hear each others' answers, no matter how far they are on the line, but they do not hear the verdict (whether the answer was correct). A friendly guard warns them one night before, giving them enough time to come up with a strategy. How many prisoners can be freed using the best strategy?

Assume that there is an unknown number of red & blue hats.

Answer : 99

Solution :

We can number prisoners from 100 to 1, with 100 being the last person in the line. Prisoners agree that if the 100th100th person will say "red" if there are an even number of "red" hats visible on the prisoners 1 to 99, and blue otherwise. This way, prisoner number 99 can look ahead and count the red hats. if they add up to an even number and the number 100 said “red”, then 99 must be wearing a blue hat. if they add up to an even number and number 100 said “blue”, signaling an odd number of red hats, number 99 must also be wearing a red hat.
Similarly, number 98 knows that 99 said the correct hat, and so uses that information along with the 97 hats in the front to figure out their own hat's color. This can continue till the first prisoner.
This strategy will free 99 prisoners. But the 100th100th prisoner has to rely on luck and has a 50:50 chance of being right.

36. Apple-Truck problem (1.10-Heard on the street)

There are two cities, A and B, 1,000 miles apart. You have 3,000 apples at City A, and you want to deliver as many as possible of them to City B. The only delivery method available is a truck. There are, however, two problems. The truck can hold at most only 1,000 apples, and if there are any apples at all in the truck, the hungry dishonest driver will steal and eat one apple for every mile he drives. What is the maximum number of apples you can deliver from City A to City B? Note that you are welcome to stop partway, dump off some apples, and then come back and pick them up later.

Answer : 833

Solution :

We are tasked with delivering as many apples as possible from City A to City B, 1,000 miles apart. We have:

- 3,000 apples at City A.
- A truck with a capacity of 1,000 apples.
- The driver steals 1 apple per mile traveled, as long as apples are in the truck.

The solution involves breaking the journey into segments and minimizing apple losses by making multiple trips for each segment.

We start by moving all 3,000 apples a small distance X_1 miles. Each trip moves 1,000 apples, and we need to make 3 trips for every full load.

$$\text{Loss per mile} = 3 \text{ apples per mile}$$

Solution:

The number of apples left after X_1 miles is:

$$\text{Remaining apples} = 3,000 - 3 \times X_1$$

We want at least 2,000 apples left after X_1, so:

$$2,000 = 3,000 - 3 \times X_1$$

$$X_1 = \frac{1,000}{3} = 333.33 \text{ miles}$$

After traveling $X_1 = 333.33$ miles, we have 2,000 apples remaining.

Next, we move 2,000 apples forward. Since we are only moving 2,000 apples, we make 2 trips for each batch.

$$\text{Loss per mile} = 2 \text{ apples per mile}$$

The apples remaining after traveling X_2 miles is:

$$\text{Remaining apples} = 2,000 - 2 \times X_2$$

Solution(Conti) :

We want at least 1,000 apples left after X_2, so:

$$1,000 = 2,000 - 2 \times X_2$$

$$X_2 = \frac{1,000}{2} = 500 \text{ miles}$$

After traveling $X_2 = 500$ miles, we have 1,000 apples remaining.

For the final segment, we move 1,000 apples the remaining distance to City B, which is:

$$1,000 - (X_1 + X_2) = 1,000 - (333.33 + 500) = 166.67 \text{ miles}$$

In this phase, we only lose 1 apple per mile because we make only one trip.

$$\text{Remaining apples} = 1,000 - 167 = 833 \text{ apples}$$

37. (1.16-Heard on the street)

You and I are playing a competitive game. We shall take turns to call out for integers and the first one to call out 50 wins.

Rules :

- The player who starts must call out in the range of 1 to 10 inclusive.
- A new number called out by the next player must be in the range of n+1 to n+10 inclusive, where n is the previous player's called-out number.

What would be the strategy you would use to win the game?

Solution :

We are playing a game where the first person to call out "50" wins. The rules are:

1. The player who starts must call out an integer between 1 and 10, inclusive.
2. A new number called must exceed the most recent number by at least 1 and by no more than 10.

Winning Strategy: The key is to ensure that on your turn, you are always able to call out a number that will force your opponent into a losing position.

We can work backwards to identify the "winning numbers" that will guarantee victory if you call them. These numbers are:

$$6, 17, 28, 39, 50$$

If you can get to one of these numbers, you will be able to reach the next number in the sequence, forcing your opponent into a losing situation.

Solution :

Reasoning:

- If you call out "50," you win.
- To call out "50," your opponent must be forced to call a number between 40 and 49.
- Therefore, calling out "39" forces your opponent into this losing range.
- Similarly, calling out "28" forces your opponent towards 29–38, and so on.
- The full sequence of numbers you want to aim for is:

$$6 \to 17 \to 28 \to 39 \to 50$$

Solution:

Optimal Strategy:

- If you go first, call out "6."
- From here, no matter what your opponent says, you will always be able to reach the next winning number in the sequence.
- If your opponent goes first and does not call out "6," you can respond with the next winning number.

38. (1.26-Heard on the street)
Three men and Hats

Inside of a dark closet are five hats: three blue and two red. Knowing this, three smart men go into the closet, and each selects a hat in the dark and places it unseen upon his head.

Once outside the closet, no man can see his own hat. The first man looks at the other two, thinks, and says, "I cannot tell what color my hat is." The second man hears this, looks at the other two, and says, "I cannot tell what color my hat is either." The third man is blind. The blind man says, "Well, I know what color my hat is." What color is his hat?

Answer : Blue

Solution :

Let the first man who spoke be A, the second man who spoke be B, and the blind person be C.

A does not know what color hat he is wearing. A would only know what color hat he was wearing if he could see that both B and C were wearing red hats, because there are only two red hats in the cupboard. Since this is not the case, either:

- B and C are both wearing blue hats, *or*
- One of B or C is wearing red and the other is wearing blue.

When B speaks, we assume he has thought of all of this. When B looks at C, if C were wearing red, then B would know that he (B) must be wearing blue since they cannot both be wearing red. However, this does not happen, so C must be wearing blue, which causes B to not know if he is wearing red or blue.

Therefore, the third person (C) must be wearing a blue hat.

39.
Kings Conundrum

This is an absolute classic. A king demands a tax of 1,000 gold sovereigns from each of the 10 regions of his nation. The tax collectors for each region bring him the requested bag of gold coins at year's end. An informant tells the king that one tax collector is cheating and giving coins that are consistently 10% lighter than they should be, but he does not know which collector is cheating. The king knows that each coin should weigh exactly one ounce. How can the king identify the cheat by using a weighing device exactly once?

Solution :

The king should take one coin from bag one, two coins from bag two, three coins from bag three, and so on, finishing with ten coins from bag ten. Place this collection on the weighing device, and look for the discrepancy from the expected total weight, which is:

$$1 + 2 + 3 + \cdots + 10 = 55 \text{ ounces.}$$

If the actual weight is, for example, 0.40 ounces short, then bag four is light, and collector four is the cheat.