DTW (Dynamic Time warping)

$T_1 : (0, 0, 0.5, 0.8, 0.5, 1)$

$T_2 : (0, 0.5, 0.8, 0.5, 1)$

$T_1 = (0, T_2) \Rightarrow T_2 \subset T_1$

DTW works by finding the optimal alignment i.e. either aligning T_2 to T_1 or T_1 to T_2.

In order to align two time series, there is some cost associated, we use dynamic programming to minimize that cost.

$$c(i,j) = |T_{1_i} - T_{2_j}| + \min\big(c(i-1,j-1),\ c(i-1,j),\ c(i,j-1)\big)$$

for $i=1,\ j=1$

We have $c(i-1,j-1) = c(i-1,j) = c(i,j-1) = 0$

$c(i,j) = c(1,1) = |0-0| = 0$

We consider the cost matrix below:

$$T_2 \left\{ \begin{array}{ll} 1 & 5 \\ .5 & 4 \\ .8 & 3 \\ .5 & 2 \\ \boxed{0} & 1 \end{array} \right.$$

$$\begin{array}{llllll} 0 & & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \boxed{0} & 0 & .5 & .8 & .5 & 1 \end{array} \Big\} T_1$$

for $i=1,\ j=2$

$c(2,1) = |0-0| + \min\{0\} = 0$

1 5

·5 4

·8 3

·5 2

$\boxed{0}$ 1 0 0

1	2	3	4	5	6
0	$\boxed{0}$	·5	·8	·5	1

$c(3,1) = |T_{13} - T_{21}|_{\cancel{2}} + \min\{0\}$
$= |·5 - 0| + 0 = ·5$

$c(4,1) = |T_{14} - T_{21}|_2 + \min\{·5\}$
$= |·8| + ·5 = 1·3$

$c(5,1) = |·5| + 1·3 = 1·8$

$c(6,1) = 1 + 1·8 = 2·8$

1 5

·5 4

·8 3

·5 2   ~~0~~   ·5   1·3   1·8   2·8

0 1   0   0

1	2	3	4	5	6
0	0	·5	·8	·5	1

Now, for $j=1, i=2$

$c(1,2) = |0 - 0·5| + \min\{0\} = 0·5$

similarly, $c(1,3) = |0 - 0·8| + \min\{0·5\} = 0·8 + 0·5$
$= 1·3$

$c(1,4) = 1·8$ & $c(1,5) = 2·8$

1	5	2.8					
.5	4	1.8					
.8	3	1.3					
.5	2	.5					
0	1	0	0	.5	1.3	1.8	2.8
		1	2	3	4	5	6
		0	0	.5	.8	.5	1

$$C(2,2) = |T_{12} - T_{22}| + \min\{C(1,1),\ C(1,2),\ C(2,1)\}$$
$$= |0 - 0.5| + \min\{0, 0.5, 0\}$$
$$= 0.5$$

$$C(3,2) = |T_{13} - T_{22}| + \min\{C(2,1),\ C(2,2),\ C(3,1)\}$$
$$= 0 + \min\{0, .5, 0.5\}$$
$$= 0$$

1	5	2.8					
0.5	4	1.8					
0.8	3	1.3					
$\boxed{.5}$	2	.5	.5	0			
0	1	0	0	.5	1.3	1.8	2.8
		1	2	3	4	5	6
		0	0	$\boxed{.5}$	.8	.5	1

If we want to calculate $C(4,2)$, it is just $|T_{14} - T_{22}| + \min\{C(3,1),\ C(3,2),\ C(4,1)\}$

$$= .3 + \min\{.5, 0, 1.3\}$$
$$= .3 + 0 = .3$$

1	5	2·8					
·5	4	1·8					
·8	3	1·3					
$\boxed{·5}$	2	0·5	·5	0	0·3		
0	1	0	0	·5	1·3	1·8	2·8
		1	2	3	4	5	6
		0	0	·5	$\boxed{·8}$	·5	1

Similarly, we find entries / cost for all the combinations. Hence the final matrix is:

1	5	2·8	2·8	·8	·5	·5	0
·5	4	1·8	1·8	·3	·3	0	·5
·8	3	1·3	1·3	·3	0	·3	·5
·5	2	·5	·5	0	·3	·3	·8
0	1	0	0	·5	1·3	1·8	2·8
		1	2	3	4	5	6
		0	0	·5	·8	·5	1

Now, our goal is to find the optimal alignment such that the total cost is minimized.

We start with top right corner, i.e.

$$
\begin{array}{c}
5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \\
\end{array}
\begin{array}{c}
\begin{bmatrix}
2.8 & 2.8 & .8 & .5 & .5 & \boxed{0} \\
1.8 & 1.8 & .3 & .3 & 0 & .5 \\
1.3 & 1.3 & .3 & 0 & .3 & .5 \\
.5 & .5 & 0 & .3 & .3 & .8 \\
0 & 0 & .5 & 1.3 & 1.8 & 2.8
\end{bmatrix} \\
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\end{array}
$$

Now, we select the minimum value from entries which are neighbours to 0 i.e.

$$
\begin{array}{c|cccccc}
5 & 2.8 & 2.8 & .8 & .5 & .5 & \boxed{0} \\
4 & 1.8 & 1.8 & .3 & .3 & \boxed{0} & .5 \\
3 & 1.3 & 1.3 & .3 & 0 & .3 & .5 \\
2 & .5 & .5 & 0 & .3 & .3 & .8 \\
1 & 0 & 0 & .5 & 1.3 & 1.8 & 2.8 \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

→ we again find minimum from this.

Now, again we find the minimum value from the entries which are neighbour to this '0' and are lower to it.

$$
\begin{array}{c|cccccc}
5 & 2.8 & 2.8 & .8 & .5 & .5 & \boxed{0} \\
4 & 1.8 & 1.8 & .3 & .3 & \boxed{0} & .5 \\
3 & 1.3 & 1.3 & .3 & \boxed{0} & .3 & .5 \\
2 & .5 & .5 & 0 & .3 & .3 & .8 \\
1 & 0 & 0 & .5 & 1.3 & 1.8 & 2.8 \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

By repeating the process, we finally get:

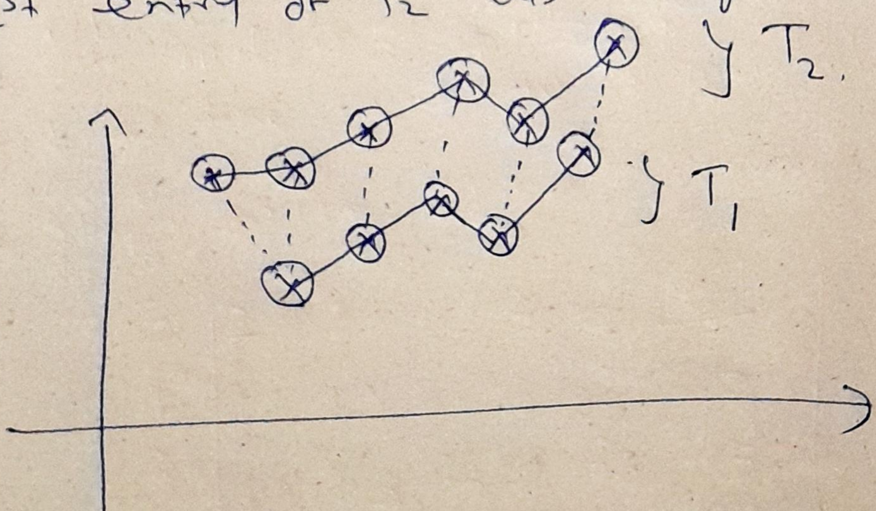
$$
\begin{array}{c|cccccc}
5 & 2.8 & 2.8 & .8 & .5 & .5 & \boxed{0} \\
4 & 1.8 & 1.8 & .3 & .3 & \boxed{0} & 0.5 \\
3 & 1.3 & 1.3 & 0.3 & \boxed{0} & 0.3 & 0.5 \\
2 & .5 & .5 & \boxed{0} & 0.3 & 0.3 & 0.8 \\
1 & \boxed{0} & \boxed{0} & .5 & 1.3 & 1.8 & 2.8 \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

Every highlighted entry indicate alignment of T_1 and T_2 for example:

T_2						
5	2.8	2.8	.8	.5	.5	0
4	1.8	1.8	.3	.3	0	.5
3	1.3	1.3	.3	0	.3	.5
2	.5	.5	0	.3	.3	.8
1	0	0	.5	1.3	1.8	2.8
	1	2	3	4	5	6

T_1

5^{th} entry of T_2 aligns with 6^{th} entry of T_1
4^{th} entry of T_2 align w/t 5^{th} entry of T_1
⋮
1^{st} entry of T_2 aligns with 2^{nd} Entry of T_1
1^{st} entry of T_2 also aligns w/t 1^{st} entry of T_1



$$\text{Total cost} = \frac{\text{Sum of all min costs}}{\text{Count of min costs}} = \frac{0}{6} = 0$$

∴ Hence, cost of alignment is 0.