Network Autocorrelation Models

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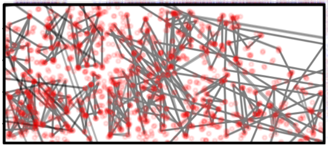
EPIC - SNA



The Network Autocorrelation Model

Inference with the Network Autocorrelation Model

Based on Carter Butts. Social Network Methods. University of California, Irvine.

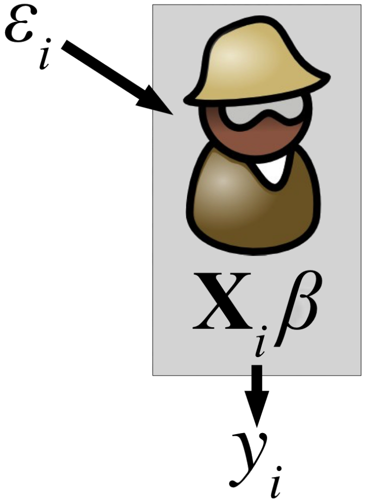
From Indices to Autocorrelation

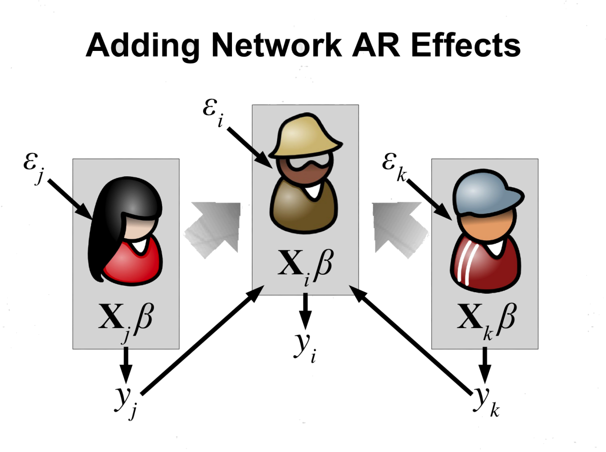
- Last time, we discussed how properties of positions could be related to covariates....
  - Does being a gatekeeper help your career?
  - Do high-degree organizations make all the decisions?
- Today: how covariates relate to each other through ties
  - Network autocorrelation: (usually linear) dependence of individual attributes across positions
- Applicable to many phenomena
  - E.g., Social influence, organizational isomorphism, work- family spillover, neighborhood quality, economic productivity, etc.

Network Autocorrelation Model

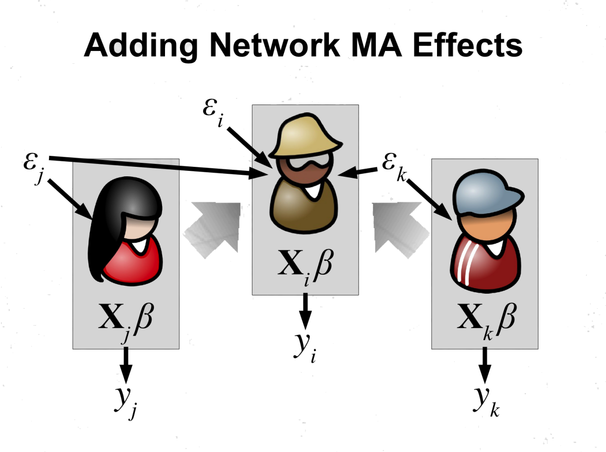
- Simplest tool for modeling network autocorrelation: the (linear) network autocorrelation model
  - Builds on standard (OLS) regression
  - Closely analogous to ARMA models in the time series literature
  - Identical to the spatial ARMA models of Cliff and Ord (1973), Anselin (1988), etc. (But we use a different name!)
- Basic components
  - Response variable (to be modeled)
  - "Weight" matrices (identical to or derived from original networks) Predictive covariates
  - Parameters measuring nature/extent of dependence, covariate effects

Network Autocorrelation Model

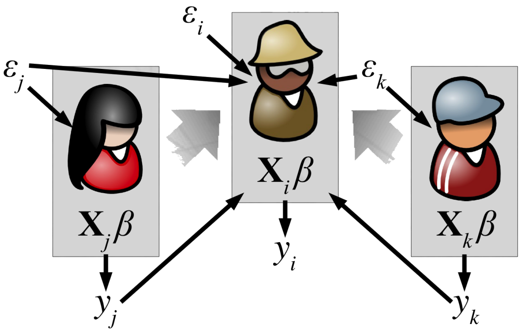



Adding Network AR Effects
ε_i
ε_j
ε_k
$\mathbf{X}_i\beta$
$\mathbf{X}_j\beta$
$\mathbf{X}_k\beta$
y_i
y_j
y_k

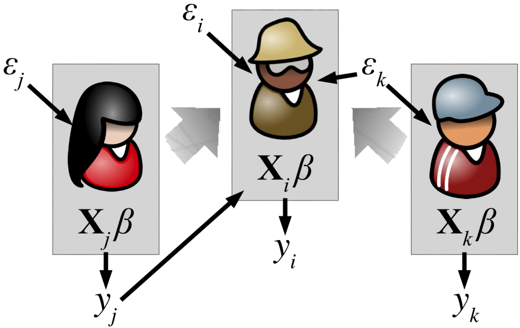
Network Autocorrelation Model



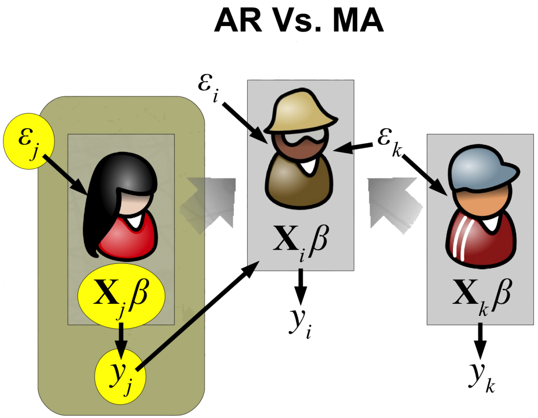
Network ARMA Model

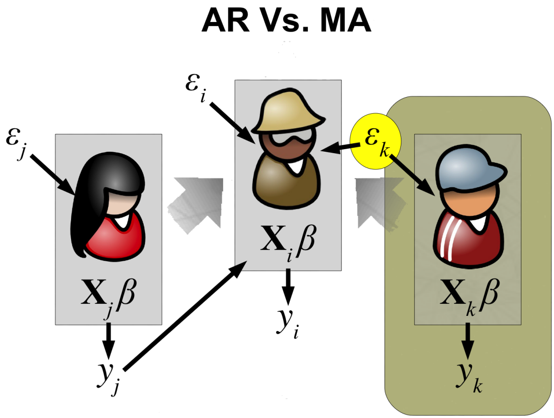


AR Vs. MA

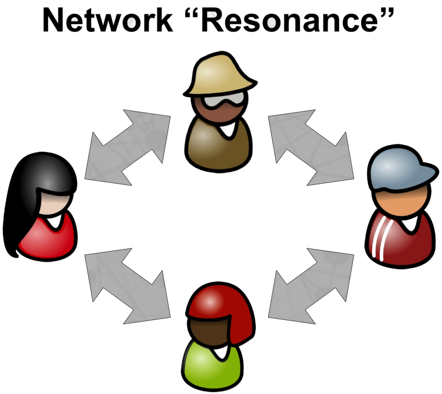


ε_i
ε_j
ε_k
$\mathbf{X}_i\beta$
$\mathbf{X}_j\beta$
$\mathbf{X}_k\beta$
y_i
y_j
y_k

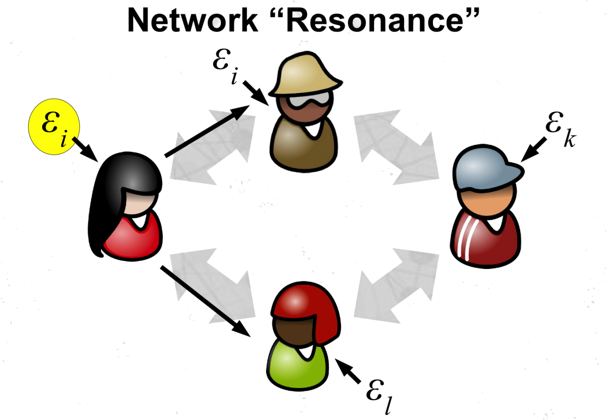

AR Vs. MA
ε_i
ε_j
ε_k
$\mathbf{X}_j\beta$
$\mathbf{X}_i\beta$
$\mathbf{X}_k\beta$
y_j
y_i
y_k

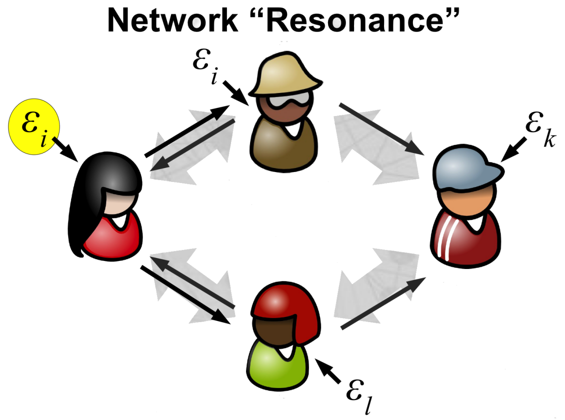

AR Vs. MA
ε_i
ε_j
ε_k
$\mathbf{X}_j\beta$
$\mathbf{X}_i\beta$
$\mathbf{X}_k\beta$
y_j
y_i
y_k

Network Autocorrelation Model

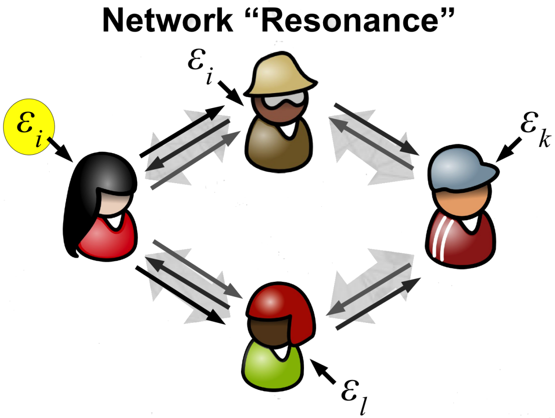


Network Autocorrelation Model

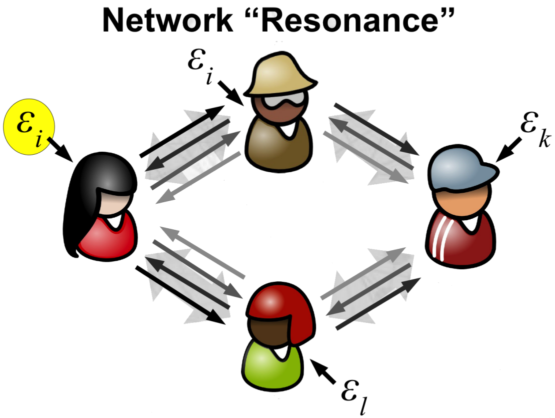



Network "Resonance"
ε_i
ε_i
ε_k
ε_l

Network Autocorrelation Model



Network Autocorrelation Model



The Formal Development

Individual Case Form:

$$y_i = \sum_{j=1}^{p} X_{ij}\beta_j + \epsilon_i$$

Group (Matrix/Vector) Form:

$$y = X\beta + \epsilon$$

Network Autocorrelation Model

The Formal Development

Individual Case Form:

Response

$$y_i = \sum_{j=1}^{p} X_{ij} \beta_j + \epsilon_i$$

Perturbation

Covariates

Group (Matrix/Vector) Form:

$$y = X\beta + \epsilon$$

The Formal Development

Individual Case Form:

$$y_i = \sum_{j=1}^{N} \theta \, W_{ij} \, y_j + \sum_{j=1}^{p} X_{ij} \beta_j + \epsilon_i$$

Group (Matrix/Vector) Form:

$$y = \theta \, W \, y + X \beta + \epsilon$$

The Formal Development

Individual Case Form:

AR Term

Covariates

$$y_i = \sum_{j=1}^{N} \theta\, W_{ij}\, y_j + \sum_{j=1}^{p} X_{ij}\, \beta_j + \epsilon_i$$

Response

Perturbation

Group (Matrix/Vector) Form:

$$y = \theta\, W\, y + X\beta + \epsilon$$

The Formal Development

Individual Case Form:

$$y_i = \sum_{j=1}^{N} \theta \, W_{ij} y_j + \sum_{j=1}^{p} X_{ij} \beta_j + \epsilon_i$$

$$\epsilon_i = \sum_{j=1}^{N} \psi \, Z_{ij} \epsilon_j + \nu_i$$

Group (Matrix/Vector) Form:

$$y = \theta \, W y + X \beta + \epsilon$$

$$\epsilon = \psi \, Z \epsilon + \nu$$

Network Autocorrelation Model

The Formal Development

Individual Case Form:

AR Term

Covariates

$$y_i = \sum_{j=1}^{N} \theta W_{ij} y_j + \sum_{j=1}^{p} X_{ij} \beta_j + \epsilon_i$$

Response

MA Term

$$\epsilon_i = \sum_{j=1}^{N} \psi Z_{ij} \epsilon_j + \nu_i$$

Total Perturbation

Individual Perturbation

Group (Matrix/Vector) Form:

$$y = \theta W y + X \beta + \epsilon$$

$$\epsilon = \psi Z \epsilon + \nu$$

Resolving the Resonance: Total Perturbations

$$\epsilon = \psi Z \epsilon + \nu$$

Resolving the Resonance: Total Perturbations

$$\epsilon = \psi Z \epsilon + \nu$$
$$\epsilon - \psi Z \epsilon = \nu$$

Resolving the Resonance: Total Perturbations

$$\begin{aligned} \epsilon &= \psi Z \epsilon + \nu \\ \epsilon - \psi Z \epsilon &= \nu \\ (\mathrm{I} - \psi Z)\epsilon &= \nu \end{aligned}$$

Resolving the Resonance: Total Perturbations

$$
\begin{aligned}
\epsilon &= \psi Z \epsilon + \nu \\
\epsilon - \psi Z \epsilon &= \nu \\
\left(I - \psi Z\right) \epsilon &= \nu \\
\epsilon &= \left(I - \psi Z\right)^{-1} \nu
\end{aligned}
$$

Resolving the Resonance: Total Perturbations

$$
\begin{aligned}
\epsilon &= \psi Z \epsilon + \nu \\
\epsilon - \psi Z \epsilon &= \nu \\
\left(\mathrm{I} - \psi Z\right) \epsilon &= \nu \\
\epsilon &= \left(\mathrm{I} - \psi Z\right)^{-1} \nu
\end{aligned}
$$

Resolving the Resonance, II: Responses

$$y = \theta \, W \, y + X \beta + \epsilon$$

Resolving the Resonance, II: Responses

$$y = \theta W y + X\beta + \epsilon$$
$$y - \theta W y = X\beta + \epsilon$$

Resolving the Resonance, II: Responses

$$
\begin{aligned}
y &= \theta W y + X\beta + \epsilon \\
y - \theta W y &= X\beta + \epsilon \\
(I - \theta W) y &= X\beta + \epsilon
\end{aligned}
$$

Resolving the Resonance, II: Responses

$$
\begin{aligned}
y &= \theta W y + X\beta + \epsilon \\
y - \theta W y &= X\beta + \epsilon \\
(I - \theta W) y &= X\beta + \epsilon \\
y &= (I - \theta W)^{-1} (X\beta + \epsilon)
\end{aligned}
$$

Resolving the Resonance, II: Responses

$$y = \theta W y + X\beta + \epsilon$$

$$y - \theta W y = X\beta + \epsilon$$

$$(I - \theta W) y = X\beta + \epsilon$$

$$y = (I - \theta W)^{-1} (X\beta + \epsilon)$$

Responses

Matrix of "Equilibrium AR Weights"

Covariates

Total Perturbations

Resolving the Resonance, II: Responses

$$y = \theta W y + X\beta + \epsilon$$

$$y - \theta W y = X\beta + \epsilon$$

$$(I - \theta W) y = X\beta + \epsilon$$

$$y = (I - \theta W)^{-1} (X\beta + \epsilon)$$

$$\boxed{y = (I - \theta W)^{-1} \left(X\beta + (I - \psi Z)^{-1} \nu \right)}$$

Resolving the Resonance, II: Responses

Matrix of "Equilibrium MA Weights"

Matrix of "Equilibrium AR Weights"

Covariates

Individual Perturbations

Responses

$$\mathrm{y} = (\mathrm{I} - \theta\,\mathrm{W})^{-1}\left(\mathrm{X}\beta + (\mathrm{I} - \psi\,\mathrm{Z})^{-1}\nu\right)$$

Using Multiple Weight Matrices

$$\theta \, \mathrm{W} \rightarrow \sum_{i=1}^{w} \theta_{\,i} \mathrm{W}_{\mathrm{i}} \qquad \psi \, \mathrm{Z} \rightarrow \sum_{i=1}^{z} \psi_{\,i} \mathrm{Z}_{\mathrm{i}}$$

Using Multiple Weight Matrices

$$\theta\, \mathrm{W} \rightarrow \sum_{i=1}^{w} \theta_i \mathrm{W}_\mathrm{i} \qquad \psi\, \mathrm{Z} \rightarrow \sum_{i=1}^{z} \psi_i \mathrm{Z}_\mathrm{i}$$

$$\mathrm{y} = \left(\mathrm{I} - \sum_{i=1}^{w} \theta_i \mathrm{W}_\mathrm{i}\right)^{-1} \left(\mathrm{X}\beta + \left(\mathrm{I} - \sum_{i=1}^{z} \psi_i \mathrm{Z}_\mathrm{i}\right)^{-1} \nu\right)$$

Inference with the Network Autocorrelation Model

- In practice, rarely given parameters
  - Usually observe y, X, and W and/or Z, want to infer β, θ, ψ
  - Need each $I - W$, $I - Z$ invertible for solution to exist
- Typical approach: maximum likelihood
  - Generally take v as iid, $v_i \sim N(0, \sigma^2)$
  - Standard errors based on the inverse information matrix at the MLE
  - Compare models in the usual way (e.g., AIC, BIC)
  - (Could also do this in a Bayesian way, could substitute other forms for v, etc.)

Choosing the Weight Matrix

- An important modeling issue: choosing the form for W and/or Z
  - Must satisfy invertibility condition
  - i, j cell must express the dependence of i on j
- Many suggestions given by Leenders (2002)

- Ex: Friedkin-Johnson criteria
  - Based on small-group social influence experiments
  - Induce quasi-convexity in equilibrium attitudes
  - $W_{ij} \geq 0$, $W_{ij} \leq 1$
  - $\sum_j W_{ij} \leq 1$
  - Diagonal can be > 0

Important Errors to Avoid

- Using NAMs when you need NLIs (or vice versa)
  - Am I affected by my position, or by the properties of my neighbors?
  - Nearly all influences theories imply some sort of autocorrelation; ditto for some theories of competition
- Confusing correlation with influence
  - If influence occurs through direct ties, structurally equivalent positions will correlate; NAM will shown an SE effect
- Confusing AT and MA terms
  - AR terms imply diffusion of response values, MA terms imply diffusion of perurbations
  - Do my neighbors' covariates affect me? If so, this is an AR process! (If not, it's an MA process.)
- Forgetting selection effects
  - NAM assumes that weight matrix is fixed
  - If selection is present, NAM results will be misleading