Understanding and using SAT and SMT solvers

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The traveller Eve's problem

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Eve is eager to make scientific visits.

- She has 100 travel wishes $A_1, \ldots, A_{100}$.
- She is allowed to make only 5 travels.
- She wants to be physically at A_1.
- To coordinate a project, she needs to visit either A_2 or A_3.
- Travel A_i costs C_i EUR.
- Eve can spend up to C EUR.
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$$\left(\sum_{i=1}^{100} a_i \leq 5 \right) \wedge (a_1 = 1) \wedge (a_2 = 1 \vee a_3 = 1) \wedge \left(\sum_{i=1}^{100} c_i \leq C \right) \wedge \left(\sum_{i=1}^{100} t_i \geq T \right)$$

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Logic: Linear real arithmetic.

Some technologies for satisfiability checking

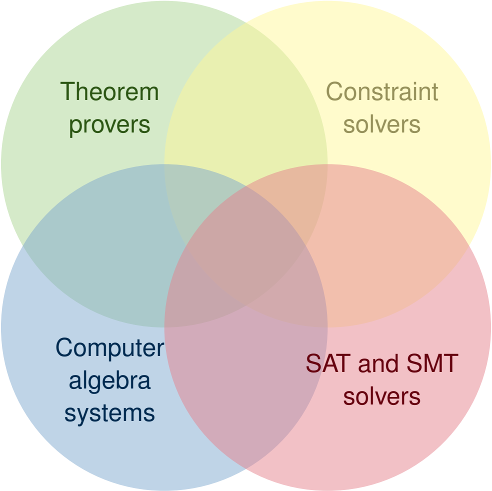
Theorem provers

Constraint solvers

Computer algebra systems

SAT and SMT solvers

Some technologies for satisfiability checking



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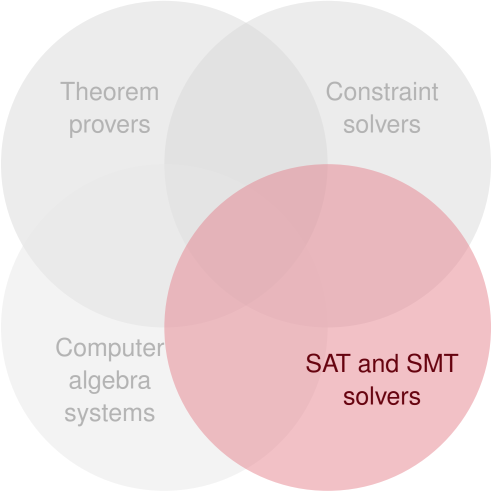
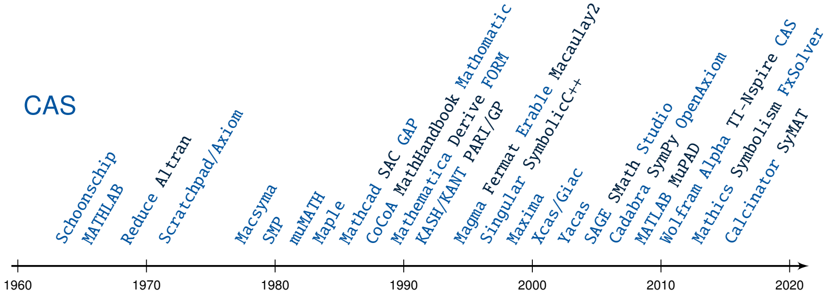


Tool development



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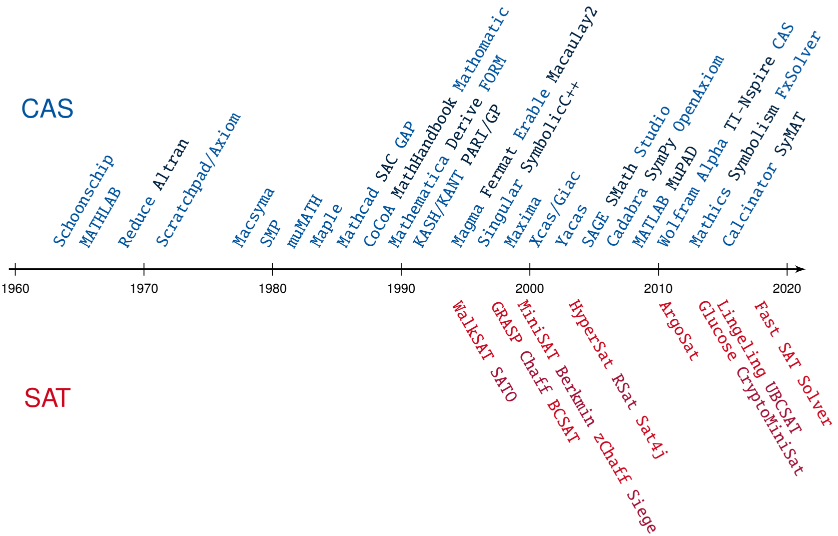

Satisfiability checking for propositional logic

Success story: SAT-solving

- Practical problems with millions of variables are solvable.
- A wide range of applications, e.g., verification, synthesis, combinatorial optimisation, etc.

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Community support:

- Standard input language.
- Large benchmark library.
- Competitions since 2002.
- SAT Live! forum as community platform, dedicated conferences, journals, etc.

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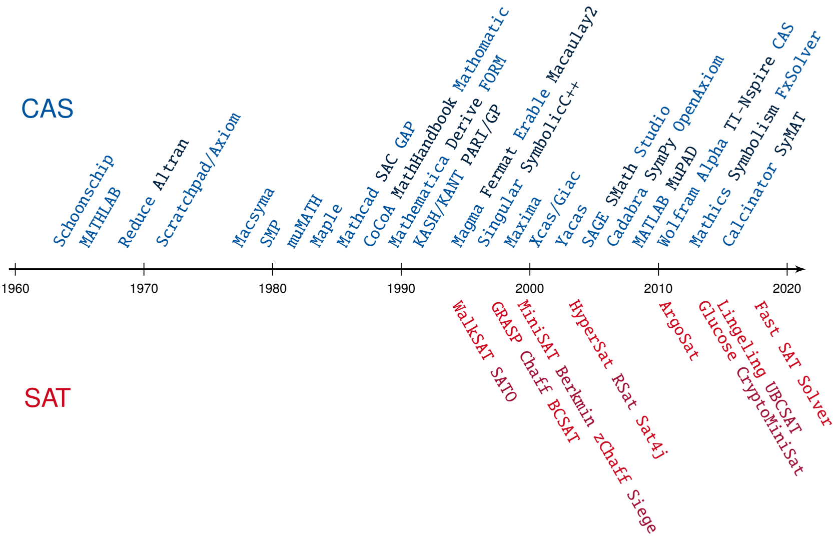

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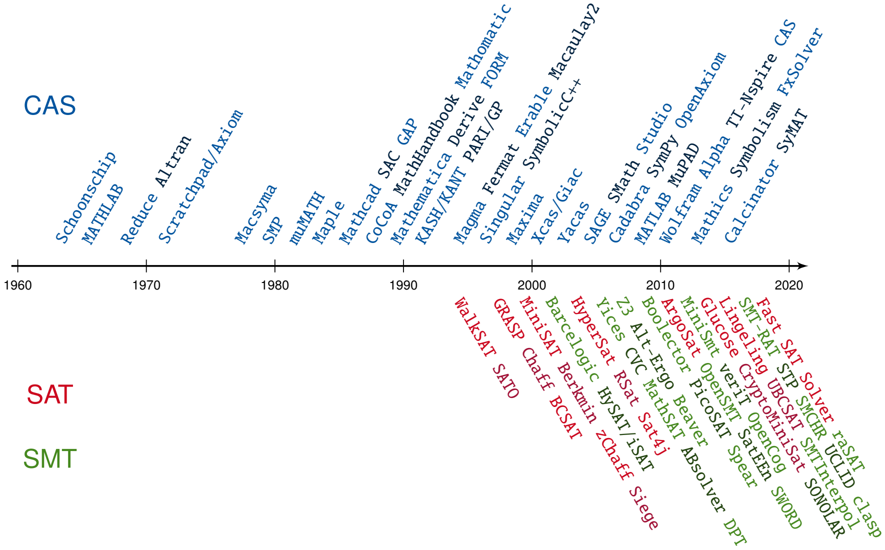

Satisfiability modulo theories (SMT) solving

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- Propositional logic is sometimes too weak for modelling.
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Community support:

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Syntax of propositional logic

Abstract syntax of well-formed propositional formulae:

$$\varphi \;:=\; a \;\mid\; (\neg\varphi) \;\mid\; (\varphi \wedge \varphi)$$

where *AP* is a set of (atomic) propositions (Boolean variables) and $a \in AP$.

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Syntactic sugar:

$$
\begin{array}{rcl}
\bot & := & (a \land \neg a) \\
\top & := & (a \lor \neg a) \\
(\varphi_1 \lor \varphi_2) & := & \neg((\neg\varphi_1) \land (\neg\varphi_2)) \\
(\varphi_1 \rightarrow \varphi_2) & := & ((\neg\varphi_1) \lor \varphi_2) \\
(\varphi_1 \leftrightarrow \varphi_2) & := & ((\varphi_1 \rightarrow \varphi_2) \land (\varphi_2 \rightarrow \varphi_1)) \\
(\varphi_1 \oplus \varphi_2) & := & (\varphi_1 \leftrightarrow (\neg\varphi_2))
\end{array}
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Semantics of propositional logic

- Truth tables define the semantics (=meaning) of the operators.
  They can be used to define the semantics of formulae inductively over their structure.

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p	q	$\neg p$	$p \land q$	$p \lor q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
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Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

Conjunctive normal form

- A literal is either a variable or the negation of a variable.
- A clause is a disjunction of literals.
- A formula in Conjunctive Normal Form (CNF) is a conjunction of clauses.

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- A clause is a disjunction of literals.
- A formula in Conjunctive Normal Form (CNF) is a conjunction of clauses.
- Every propositional logic formula can be converted to an equi-satisfiable CNF in linear time and space on the cost of (linearly many) new variables.

Tseitin's CNF encoding

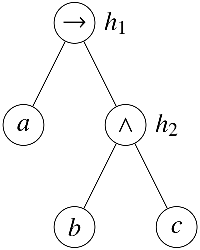
Consider the formula $\varphi = (a \rightarrow (b \wedge c))$.

Tseitin's encoding:

$(h_1 \leftrightarrow (a \rightarrow h_2))\wedge$

$(h_2 \leftrightarrow (b \wedge c))\wedge$

(h_1)



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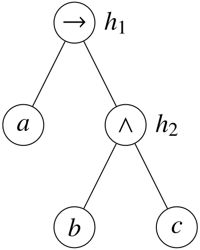


- Each node's encoding has a CNF representation with 3 or 4 clauses.

  $h_1 \leftrightarrow (a \rightarrow h_2)$ in CNF: $(h_1 \vee a) \wedge (h_1 \vee \neg h_2) \wedge (\neg h_1 \vee \neg a \vee h_2)$
  $h_2 \leftrightarrow (b \wedge c)$ in CNF: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Satisfiability problem

Given:

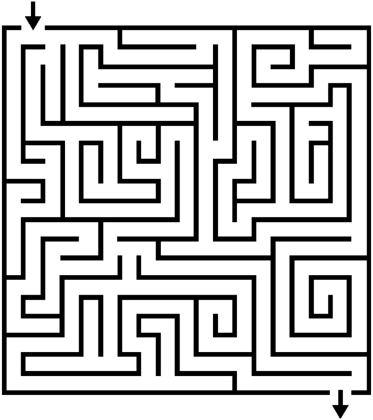
- Propositional logic formula φ in CNF.

Question:

- Is φ satisfiable?

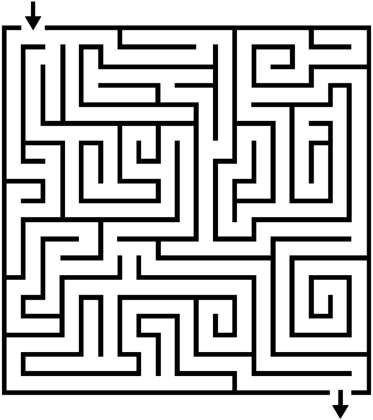
  (Is there a model for φ?)

SAT solving: The DPLL+CDCL idea



SAT solving: The DPLL+CDCL idea

Proof system



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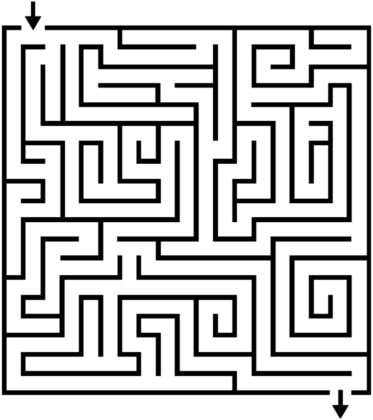

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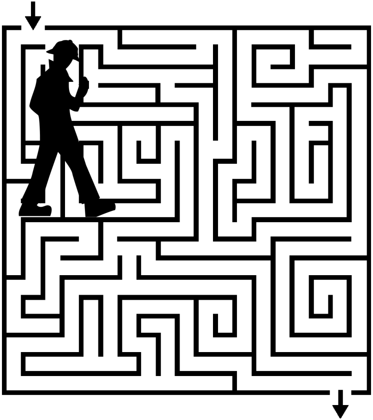

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Look-ahead

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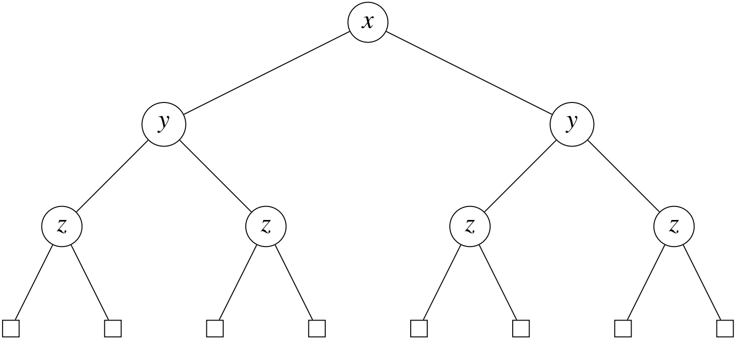
Static variable order $x < y < z$ (smallest first), sign: try positive first



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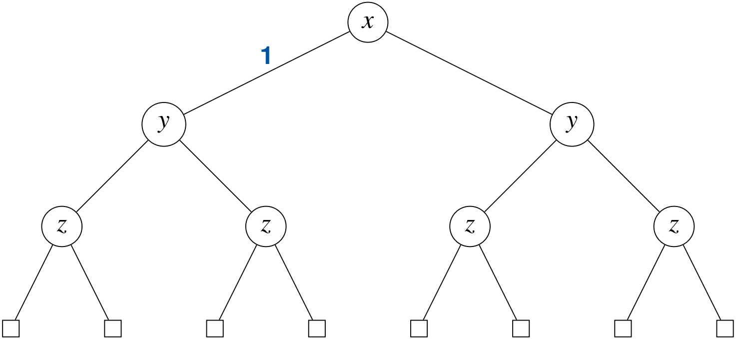

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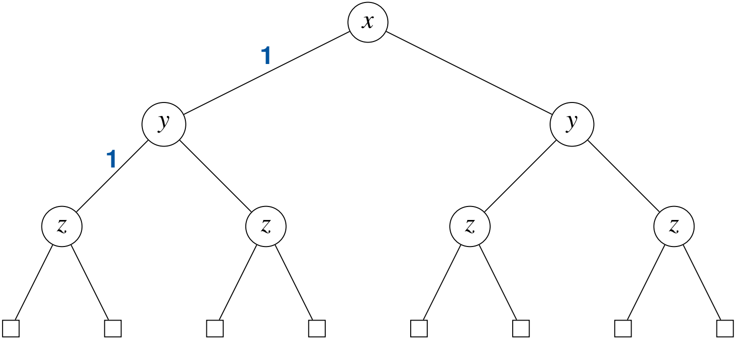

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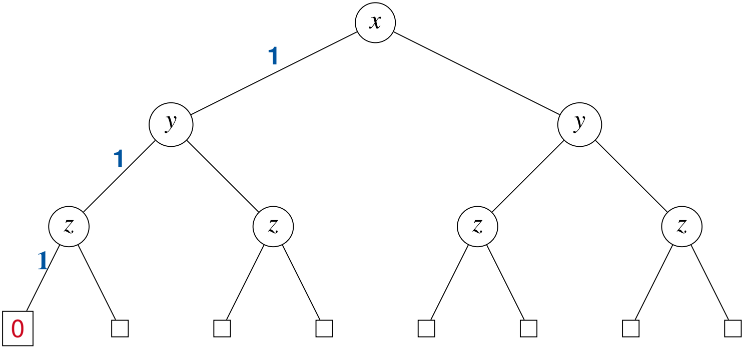

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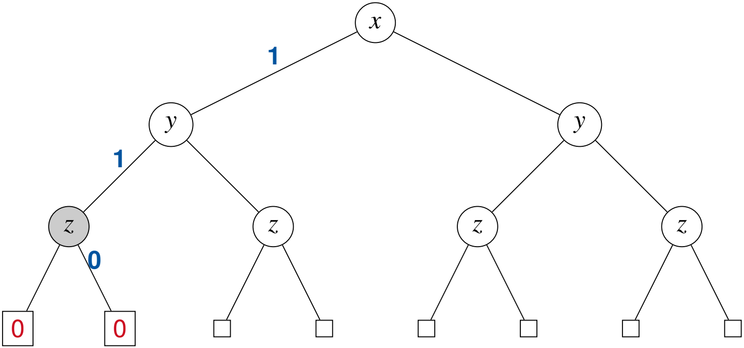

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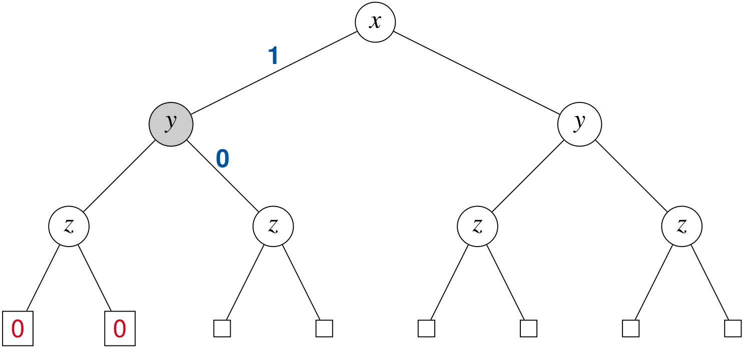

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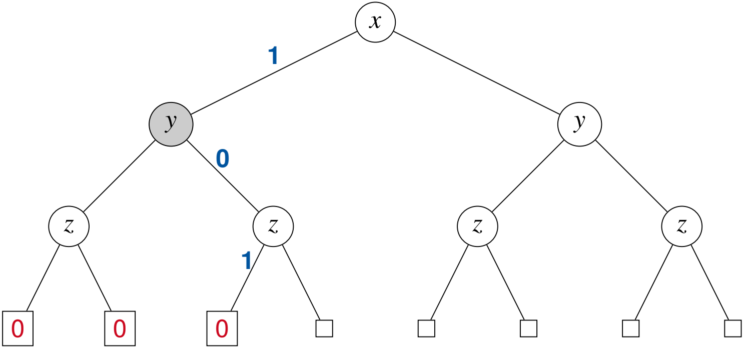

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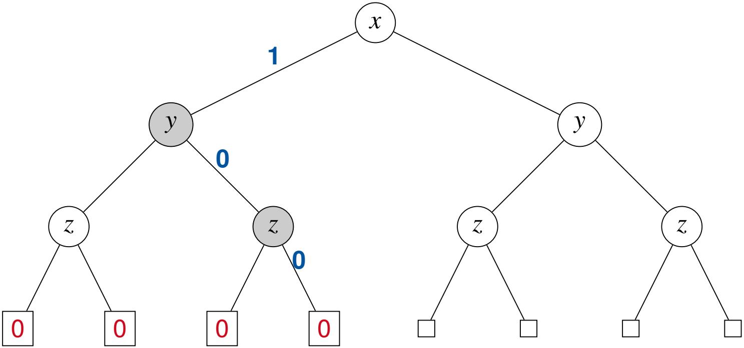

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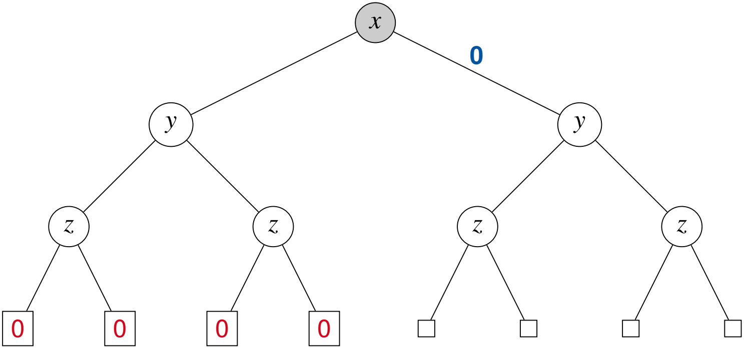

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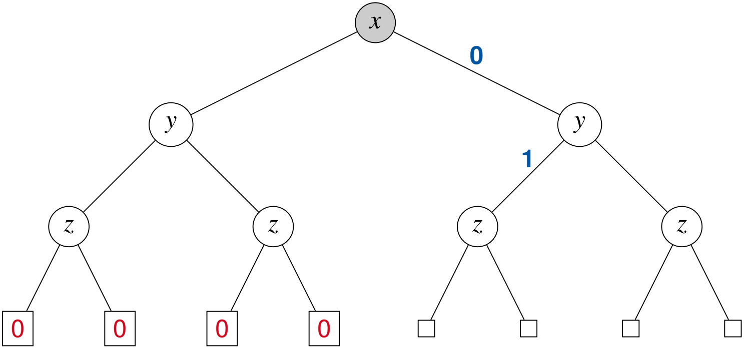

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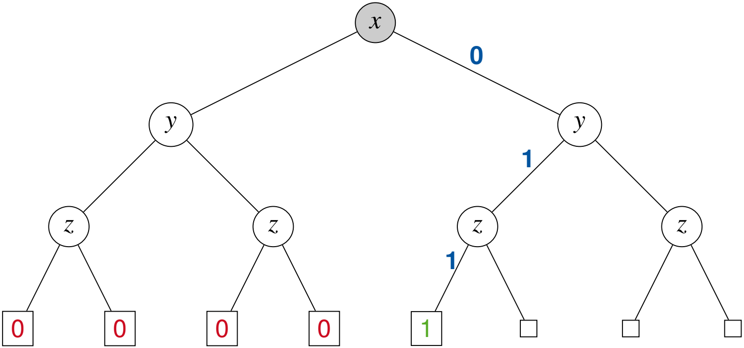

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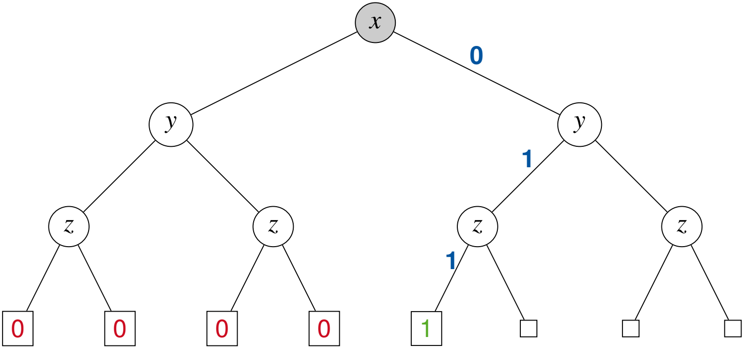

For unsatisfiable problems, all assignments need to be checked.
For satisfiable problems, variable and sign ordering might strongly influence the running time.

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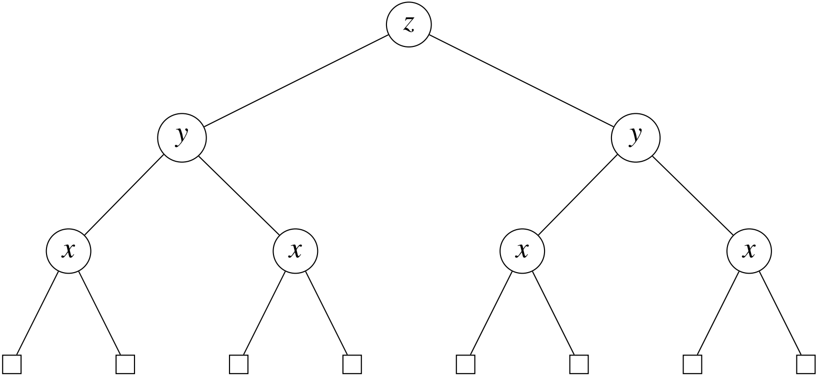
Static variable order $z < y < x$, sign: try negative first



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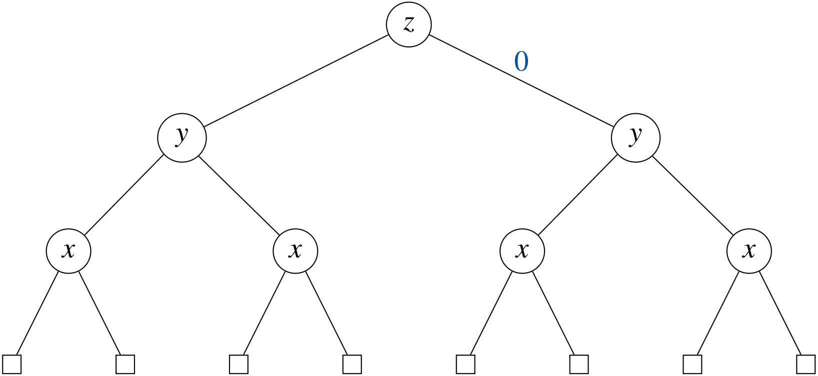

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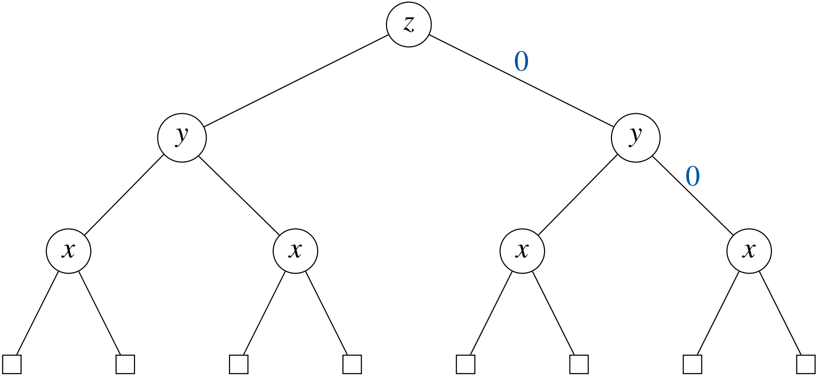

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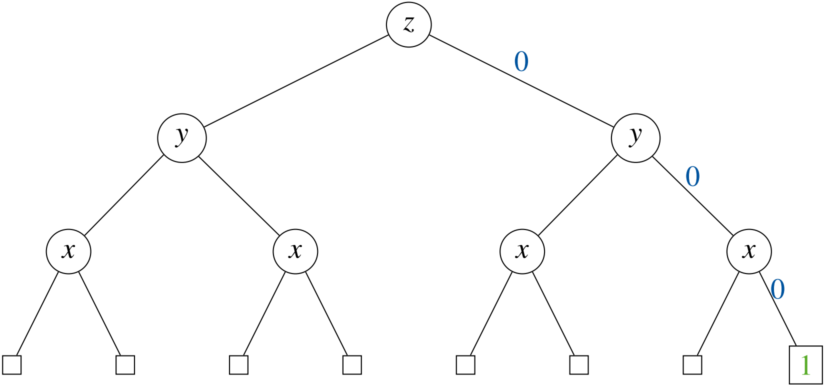

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Status of a clause

- Assume in the following: all literals in a clause have different variables

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satisfied:	at least one literal is satisfied
unsatisfied:	all literals are assigned but none are statisfied
unit:	all but one literals are assigned but none are satisfied
unresolved:	all other cases

Example :

x_1	x_2	x_3	$c = (x_1 \vee x_2 \vee x_3)$
1	0		satisfied
0	0	0	unsatisfied
0	0		unit
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BCP: Unit clauses are used to imply consequences of decisions.

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Some notations:

Decision Level (DL) is a counter for decisions

Antecedent(ℓ): unit clause implying the value of literal ℓ (nil if decision)

Boolean constraint propagation: Example

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

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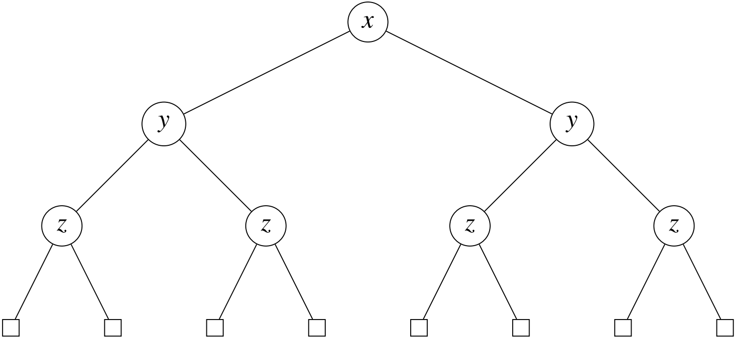

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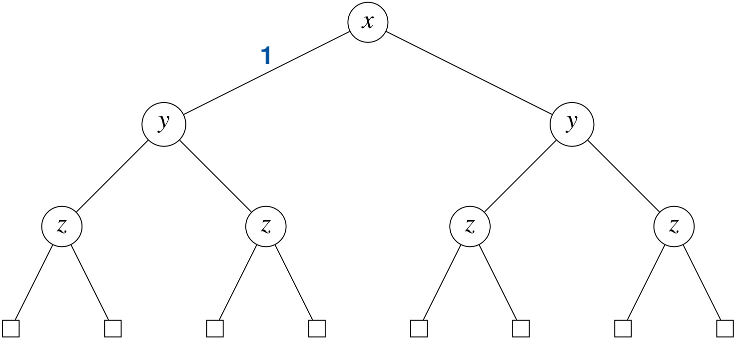

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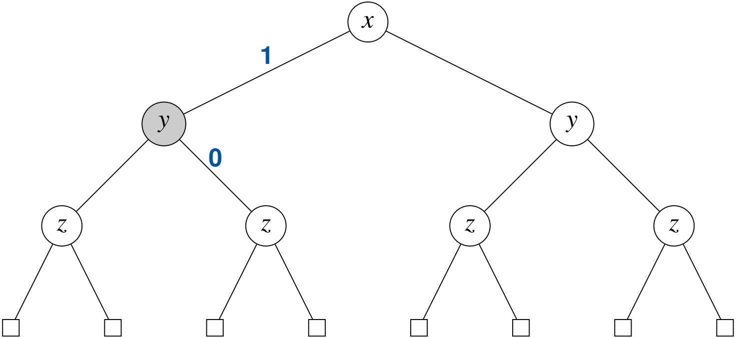

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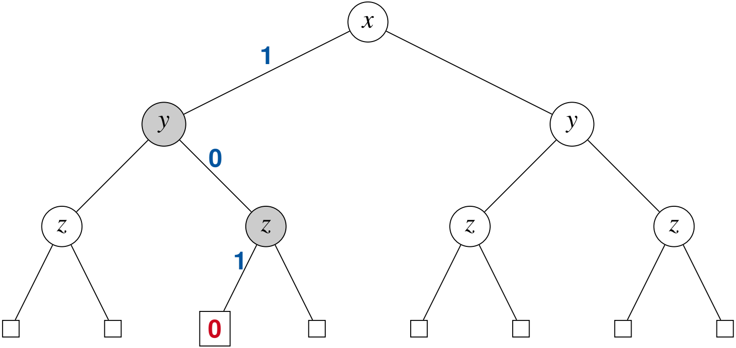

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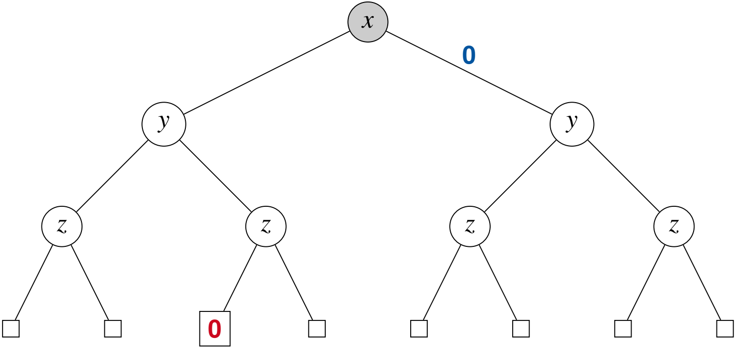
Static variable order $x < y < z$, sign: try positive first



Boolean constraint propagation: Example

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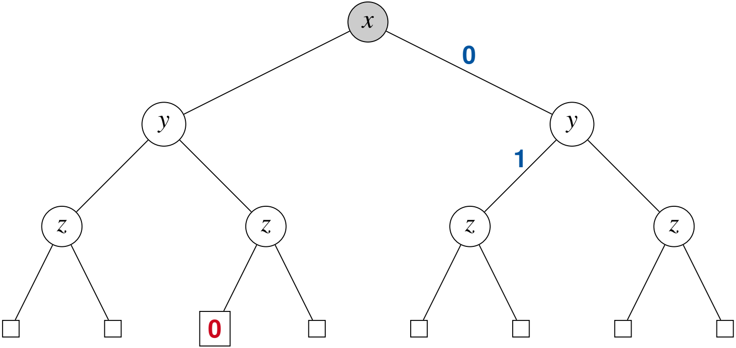

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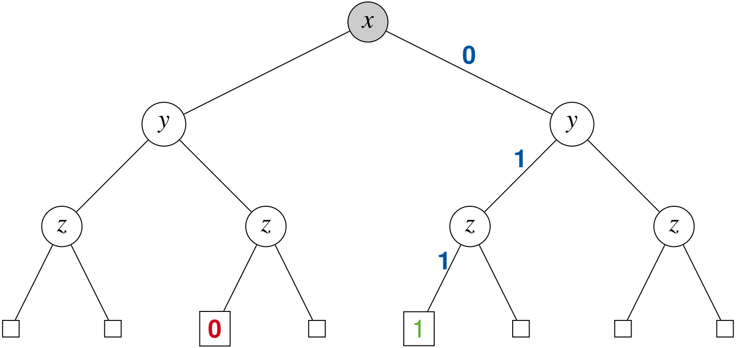

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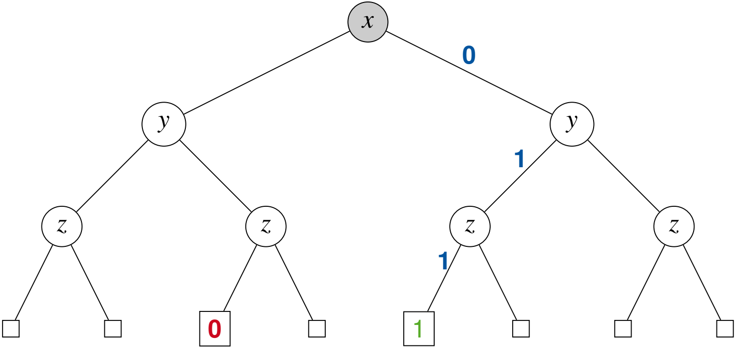

Efficient propagation with the watched literal scheme.

Contents

Implication graph: Example

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$x = 1@1$

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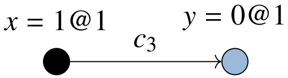

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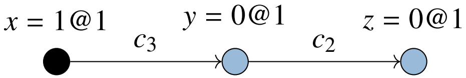

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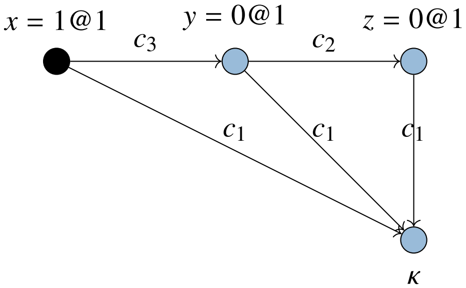

Implication graph: Example

Decisions: $\{ \qquad \}$

$$
\begin{aligned}
c_1 &= (\neg x_1 \lor x_2) \\
c_2 &= (\neg x_1 \lor x_3 \lor x_7) \\
c_3 &= (\neg x_2 \lor \neg x_3 \lor x_4) \\
c_4 &= (\neg x_4 \lor x_5 \lor x_8) \\
c_5 &= (\neg x_4 \lor x_6 \lor x_9) \\
c_6 &= (\neg x_5 \lor \neg x_6)
\end{aligned}
$$

Implication graph: Example

Decisions: $\{x_7 = 0@1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \}$

$$
\begin{aligned}
c_1 &= (\neg x_1 \vee x_2) \\
c_2 &= (\neg x_1 \vee x_3 \vee x_7) \\
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c_5 &= (\neg x_4 \vee x_6 \vee x_9) \\
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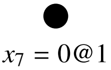
\end{aligned}
$$



Implication graph: Example

Decisions: $\{x_7 = 0@1,\ x_8 = 0@2 \qquad\qquad\qquad\qquad \}$

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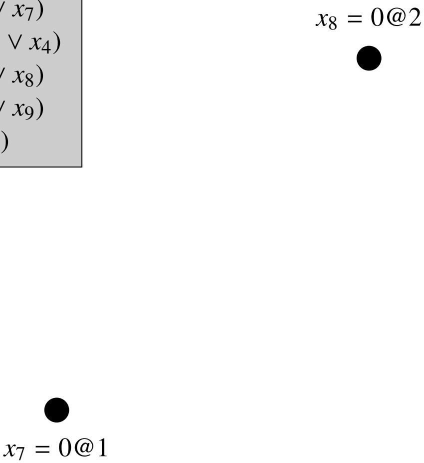

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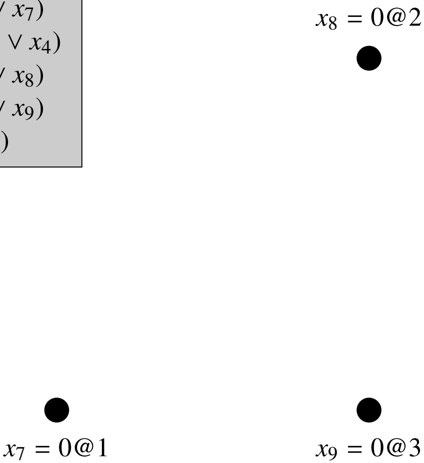

Implication graph: Example

Decisions: $\{x_7 = 0@1,\ x_8 = 0@2,\ x_9 = 0@3,\ x_1 = 1@4\}$

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\begin{aligned}
c_1 &= (\neg x_1 \lor x_2)\\
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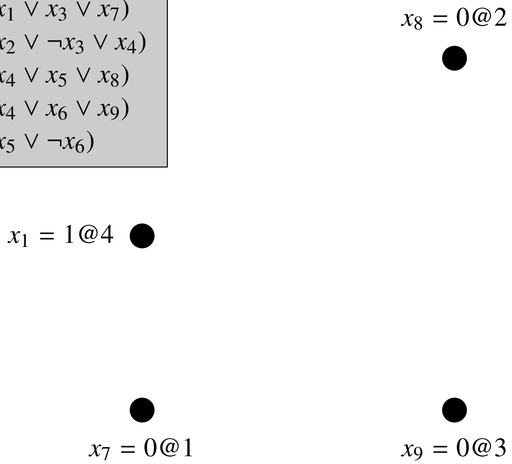

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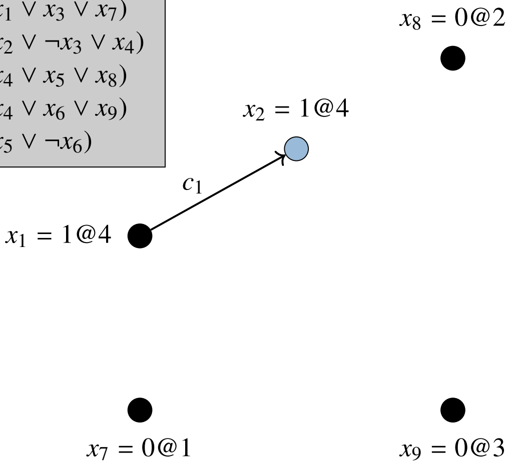

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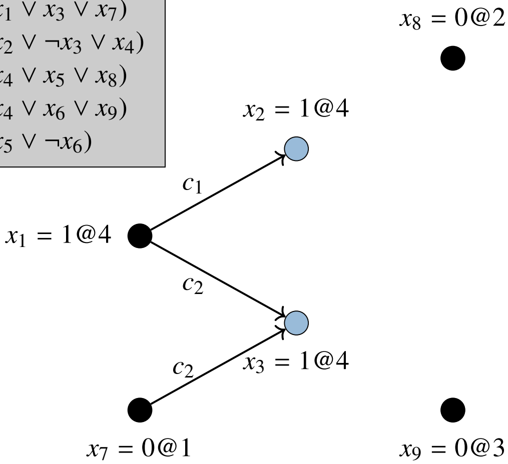

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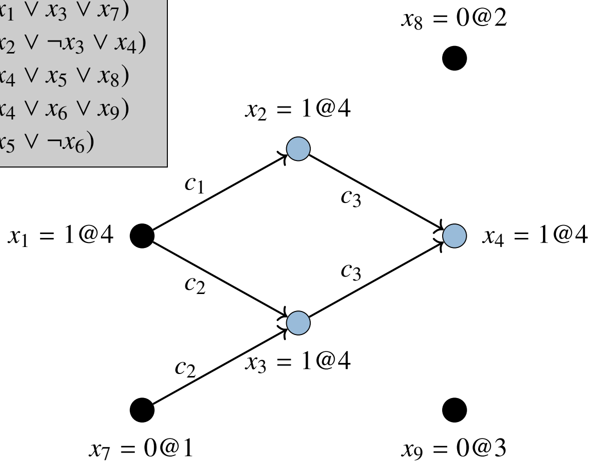

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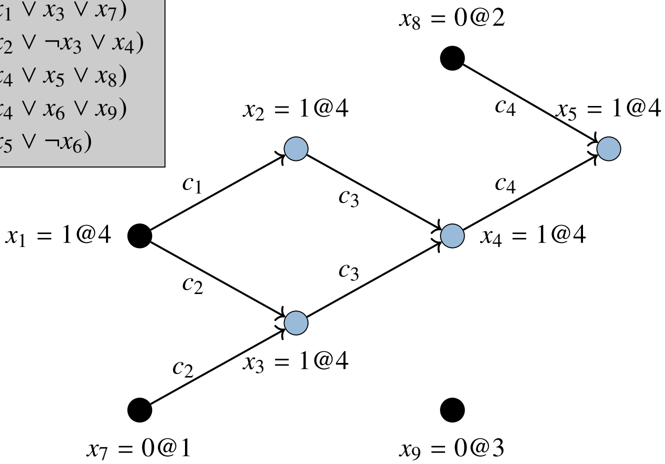

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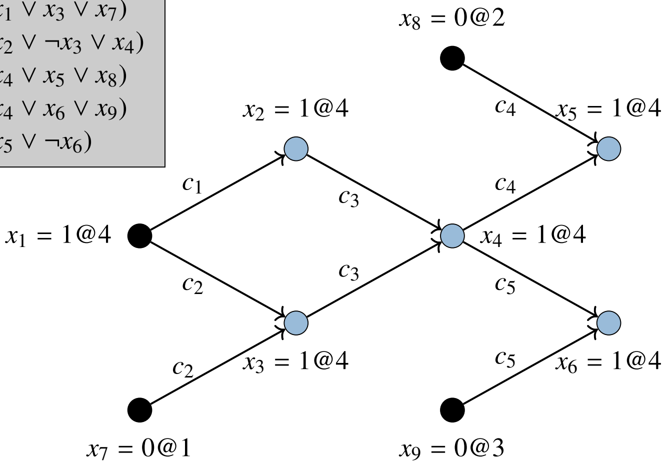

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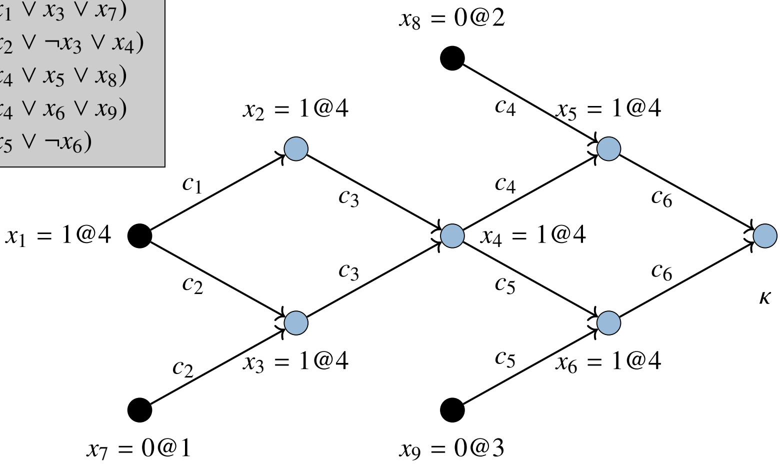

Implication graph cuts

Assume an implication graph $G = (V, E, L)$ with $\kappa \in V$.

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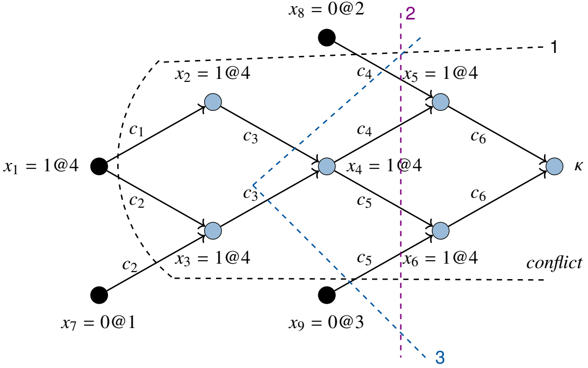
A node set $C \subseteq V$ is a cut in G iff there exists $V' \subseteq V$ such that

- each predecessor of κ is reachable from V' in G,
- $\kappa \notin V'$,
- C consists of those nodes in V' that are sources of an edge with target not in V':

$$C = \{s \in V' \mid \exists t \in V \setminus V'.\ (s, t) \in E\}\ .$$

- Assume an implication graph $G = (V, E, L)$ with $\kappa \in V$.

- Let C be a cut in G.
- $(\vee_{n \in C} \neg literal(n))$ is called a conflict clause:
  it is false under the current assignment but its satisfaction is necessary for the satisfaction of the formula.



1.$(x_8 \vee \neg x_1 \vee x_7 \vee x_9)$

2.$(x_8 \vee \neg x_4 \vee x_9)$

3.$(x_8 \vee \neg x_2 \vee \neg x_3 \vee x_9)$

$\vdots$

$\vdots$

Conflict resolution

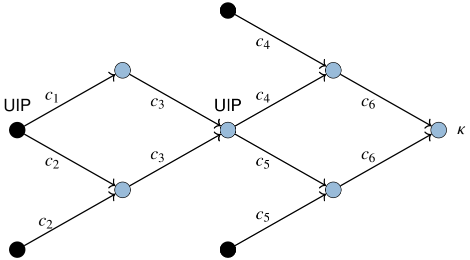
- Which conflict clauses should we consider?

Conflict resolution

- Which conflict clauses should we consider?
- An asserting clause is a conflict clause with a single literal from the current decision level.
  Backtracking (to the right level) makes it a unit clause.
- Modern solvers consider only asserting clauses.

Conflict resolution

- Assume an implication graph G with a conflict node κ.
  A unique implication point (UIP) for κ in G is a node $n \neq \kappa$ in G such that all paths from the last decision to κ go through n.
- The first UIP is the UIP closest to the conflict node.
- If a cut C contains a UIP then the clause $(\vee_{n \in C} \neg \mathit{literal}(n))$ is asserting.



Conflict-driven backtracking

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For example, from $(x \vee \neg y) \wedge (x \vee y)$ and decision $x = 0$, we get (x).

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For example, from $(x \vee \neg y) \wedge (x \vee y)$ and decision $x = 0$, we get (x).
A: Backtrack to DL0.

Q: What happens if the conflict appears at decision level 0?
A: The formula is unsatisfiable.

The DPLL+CDCL algorithm

```
if (!BCP()) return UNSAT;
while (true)
{
        if (!decide()) return SAT;
        while (!BCP())
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The DPLL+CDCL algorithm

Choose the next variable and value.
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Boolean constraint propagation.
Return false if reached a conflict.

Conflict resolution and backtracking. Return false if impossible.

Conflict clauses and (binary) resolution

- The (binary) resolution is a sound (and complete) inference rule:

$$\frac{(\beta \vee a_1 \vee \ldots \vee a_n) \qquad (\neg\beta \vee b_1 \vee \ldots \vee b_m)}{(a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m)} \text{(Binary Resolution)}$$

- Example:

$$\frac{(x_1 \vee x_2) \qquad (\neg x_1 \vee x_3 \vee x_4)}{(x_2 \vee x_3 \vee x_4)}$$

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- Example:

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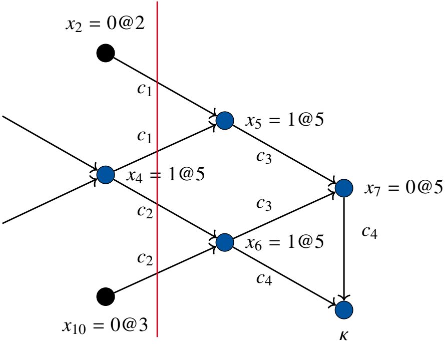
What is the relation of binary resolution and conflict clauses?

Conflict clauses and (binary) resolution

- Consider the following example:

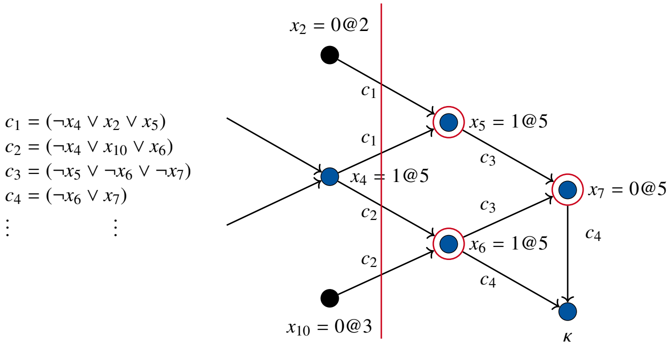
$c_1 = (\neg x_4 \vee x_2 \vee x_5)$
$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$
$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$
$c_4 = (\neg x_6 \vee x_7)$
$\vdots \qquad \vdots$



- Asserting conflict clause: $c_5 : (x_2 \vee \neg x_4 \vee x_{10})$

Conflict clauses and (binary) resolution

- Assigment order: x_4, x_5, x_6, x_7 Conflict clause: $c_5 : (x_2 \vee \neg x_4 \vee x_{10})$



- Starting with the conflicting clause, apply resolution with the antecedent of the last assigned literal, until we get an asserting clause:
  - T1 = Res$(c_4, c_3, x_7) = (\neg x_5 \vee \neg x_6)$
  - T2 = Res(T1,$c_2, x_6) = (\neg x_4 \vee \neg x_5 \vee x_{10})$
  - T3 = Res(T2,$c_1, x_5) = (x_2 \vee \neg x_4 \vee x_{10})$

Finding the asserting conflict clause

```
bool resolve_conflict() {
    if (current_decision_level = 0) then { return false; }
    cl := current_conflicting_clause;
    while (cl is not asserting) do {
        lit := last_assigned_literal(cl);
        var := variable_of_literal(lit);
        ante := antecedent(var);
        cl := resolve(cl, ante, var);
    }
    add_clause_to_database(cl);
    return true;
}
```

Applied to our example:

name	*cl*	*lit*	*var*	*ante*
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2
	$(\neg x_4 \vee x_{10} \vee \neg x_5)$	$\neg x_5$	x_5	c_1
c_5	$(\neg x_4 \vee x_2 \vee x_{10})$			

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An unsatisfiable core of an unsatisfiable CNF formula is an unsatisfiable subset of the original set of clauses.

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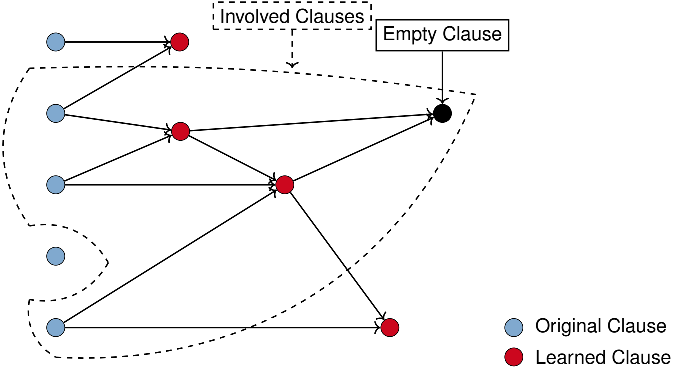
- The set of all original clauses is an unsatisfiable core.
- The set of those original clauses that were used for resolution in conflict analysis during SAT-solving (inclusively the last conflict at decision level 0) gives us an unsatisfiable core which is in general much smaller.
- However, this unsatifiable core is still not always minimal (i.e., we can remove clauses from it still having an unsatisfiable core).

The resolution graph

A resolution graph gives us more information to get a minimal unsatisfiable core.



Termination

Theorem

It is never the case that the solver enters decision level dl again with the same partial assignment.

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Proof.

Define a partial order on partial assignments: $\alpha < \beta$ iff either α is an extension of β or α has more assignments at the smallest decision level at that α and β do not agree.

BCP decreases the order, conflict-driven backtracking also. Since the order always decreases during the search, the theorem holds. $\square$

Contents

Decision heuristics: VSIDS

- VSIDS (variable state independent decaying sum)
- Gives priority to variables involved in recent conflicts.
- "Involved" can have different definitions. We take those variables that occur in clauses used for conflict resolution.

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- Gives priority to variables involved in recent conflicts.
- "Involved" can have different definitions. We take those variables that occur in clauses used for conflict resolution.

1. Each variable has a counter initialized to 0.
2. We define an increment value (e.g., 1).
3. When a conflict occurs, we increase the counter of each variable, that occurs in at least one clause used for conflict resolution, by the increment value.
   Afterwards we increase the increment value (e.g., by 1).
4. For decisions, the unassigned variable with the highest counter is chosen.
5. Periodically, all the counters and the increment value are divided by a constant.

Decision heuristics: VSIDS

VSIDS is a ’quasi-static’ strategy:

- static because it doesn’t depend on current assignment
- dynamic because it gradually changes. Variables that appear in recent conflicts have higher priority.

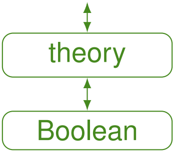
  This strategy is a conflict-driven decision strategy.

”...employing this strategy dramatically (i.e., an order of magnitude) improved performance...”

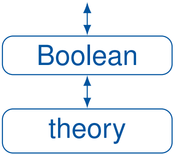
Contents

Three SMT solving approaches

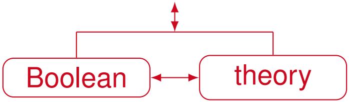
Eager SMT solving



Lazy SMT solving



Model-constructing satisfiability calculus (MCSAT)



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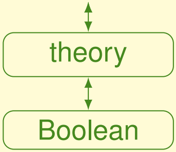

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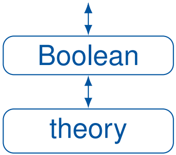

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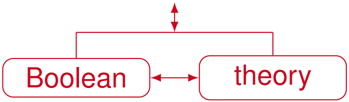

Eager example [Bryant and Velev, 2000]

$$\varphi^E \;=\; x_1 = x_2 \;\land\; x_2 = x_3 \;\land\; x_1 \neq x_3$$

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$$\varphi^{prop} \;:=$$

φ^E *is satisfiable* *iff* φ^{prop} *is satisfiable*

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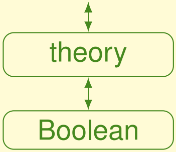
φ^E *is satisfiable* *iff* φ^{prop} *is satisfiable*

Similar approaches are available for uninterpreted functions, bit-vector arithmetic (“bit-blasting”), floating-point arithmetic and others.

Three SMT solving approaches

Eager SMT solving



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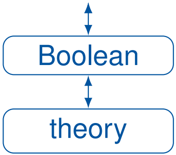

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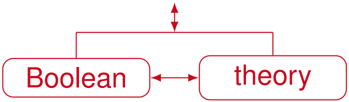

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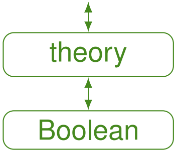

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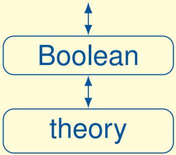

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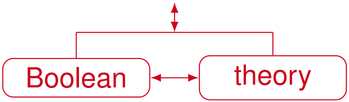

The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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$$\begin{aligned}(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\ (p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\ 3p_1 + 2p_2 + p_3 \leq 300\end{aligned}$$

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Logic:

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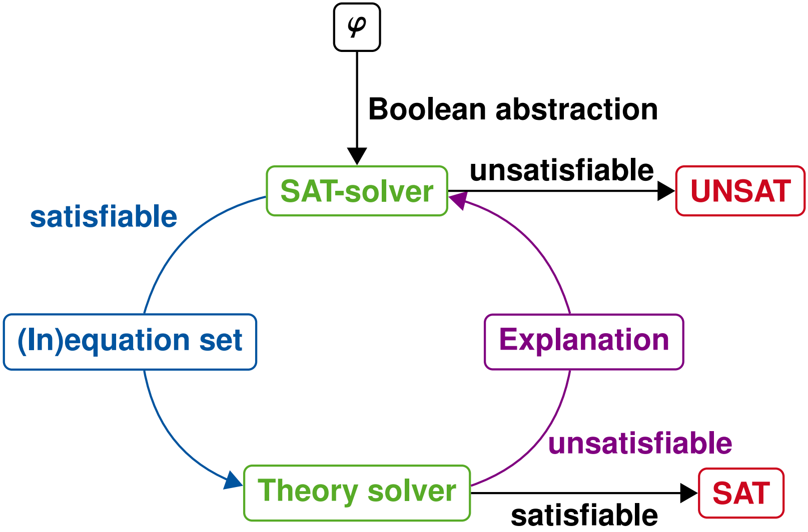
- He has 300 EUR for presents in total.

$$\begin{aligned}
(p_1 = 0 \lor p_2 = 0 \lor p_3 = 0) \land p_1 + p_2 + p_3 \geq 100 \land \\
(p_1 \geq 5 \lor p_2 \geq 5) \land p_3 \geq 10 \land p_1 + 2p_2 + 5p_3 \leq 180 \land \\
3p_1 + 2p_2 + p_3 \leq 300
\end{aligned}$$

Logic: First-order logic over the integers with addition.

Full lazy SMT solving



Boolean abstraction

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Boolean abstraction

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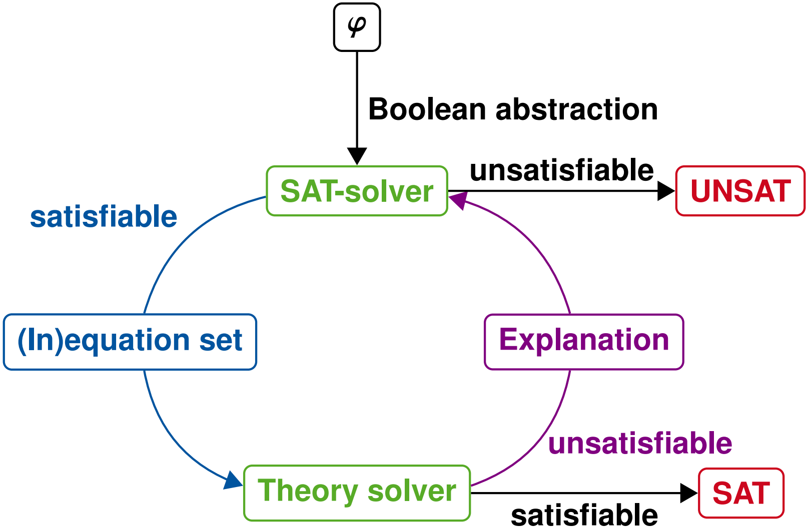
$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$


φ
Boolean abstraction
SAT-solver
unsatisfiable
UNSAT
satisfiable
(In)equation set
Explanation
unsatisfiable
Theory solver
SAT
satisfiable

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
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$DL0$:

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
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$DL0 : a_4 : 1, a_7 : 1, a_8 : 1$

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Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 :$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 :$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
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$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$

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Assume a fixed variable order:	$a_1, \ldots, a_9$
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$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 :$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
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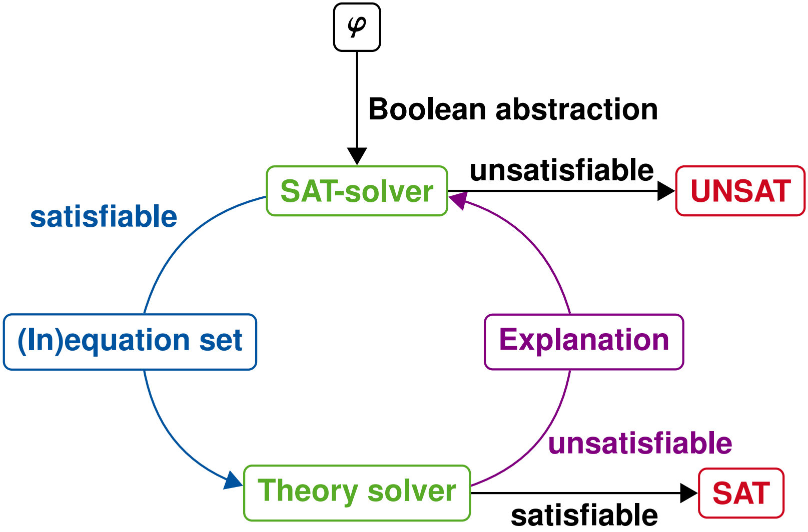
$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $\quad DL1 : a_1 : 0$

$DL2 : a_2 : 0, a_3 : 1$ $\quad DL3 : a_5 : 0, a_6 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $\quad DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$ $\quad DL3 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1 \quad DL1 : a_1 : 0$
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True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1 \quad DL1 : a_1 : 0$
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True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
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Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_3 : p_3 = 0 \quad a_6 : p_2 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_3 : p_3 = 0$

$a_6 : p_2 \geq 5$

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

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$a_3 : p_3 = 0$

$a_6 : p_2 \geq 5$

No.

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_3 : p_3 = 0$

$a_6 : p_2 \geq 5$

No.

Reason:

Theory solving

Is the conjunction of the following constraints satisfiable?

a_4 :$p_1 + p_2 + p_3 \geq 100$

a_7 :$p_3 \geq 10$

a_8 :$p_1 + 2p_2 + 5p_3 \leq 180$

a_9 :$3p_1 + 2p_2 + p_3 \leq 300$

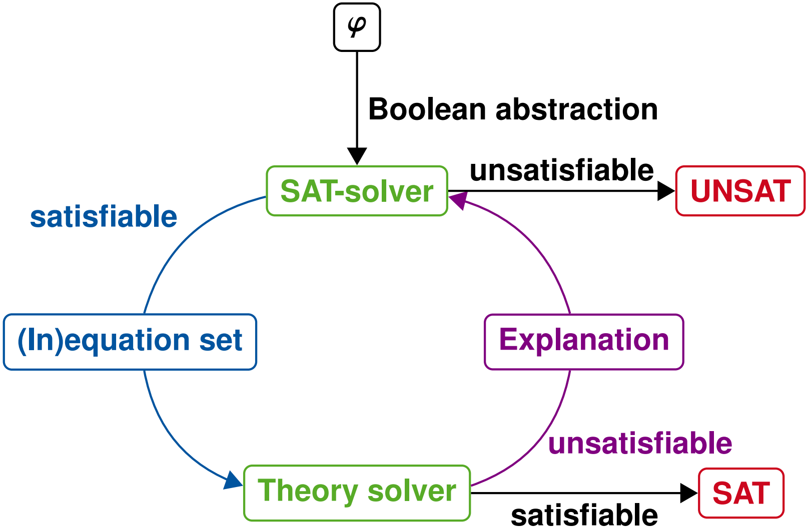
a_3 :$p_3 = 0$

a_6 :$p_2 \geq 5$

No.

Reason: $\underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_3 \geq 10}_{a_7}$ are conflicting.

Full lazy SMT solving



SAT solving

Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 :$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 :$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

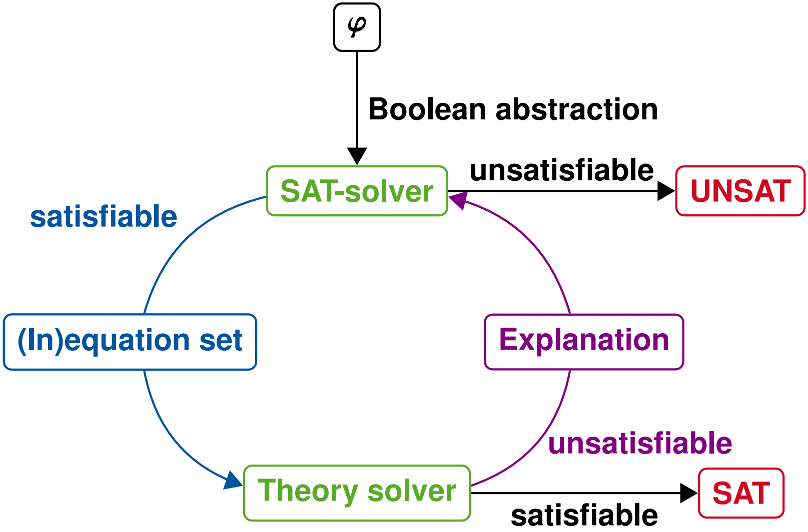
SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7)$$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$
\begin{aligned}
&(\underbrace{p_1 = 0}_{a_1} \lor \underbrace{p_2 = 0}_{a_2} \lor \underbrace{p_3 = 0}_{a_3}) \land \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \land \\
&(\underbrace{p_1 \geq 5}_{a_5} \lor \underbrace{p_2 \geq 5}_{a_6}) \land \underbrace{p_3 \geq 10}_{a_7} \land \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \land \\
&\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \land (\neg a_3 \lor \neg a_7)
\end{aligned}
$$

Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_2 : p_2 = 0 \quad a_6 : p_2 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

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Theory solving

Is the conjunction of the following constraints satisfiable?

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$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_6 : p_2 \geq 5$

No.

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_6 : p_2 \geq 5$

No.

Reason:

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

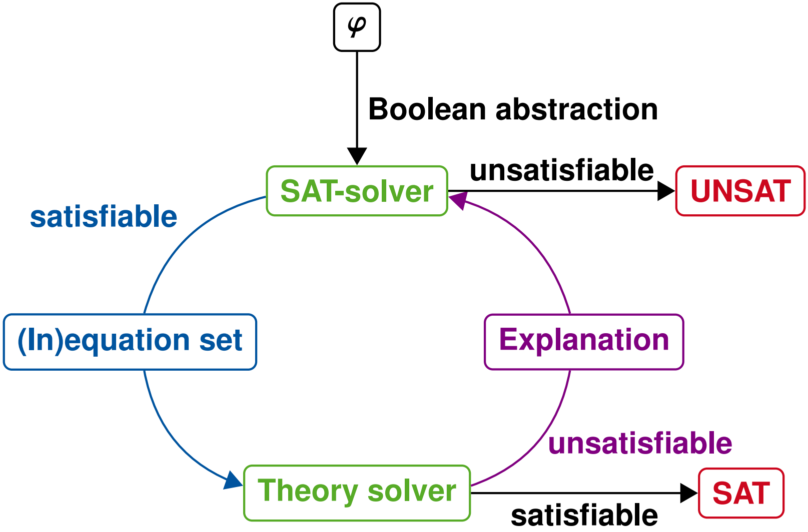
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_6 : p_2 \geq 5$

No.

Reason: $\underbrace{p_2 = 0}_{a_2} \wedge \underbrace{p_2 \geq 5}_{a_6}$ are conflicting.


φ
Boolean abstraction
SAT-solver
unsatisfiable
UNSAT
satisfiable
(In)equation set
Explanation
unsatisfiable
Theory solver
SAT
satisfiable

Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1, a_6 : 0$

SAT solving

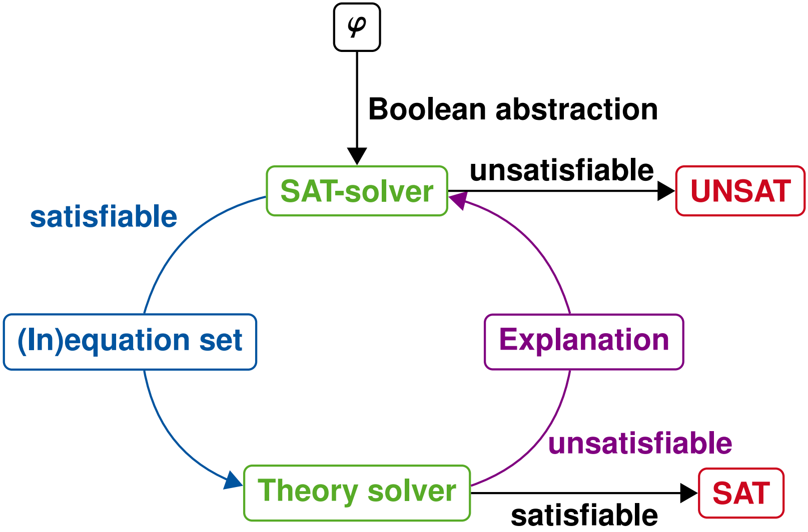
$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Solution found for the Boolean abstraction.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Boolean abstraction
SAT-solver
unsatisfiable
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Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

Theory solving

$DL0: a_4: 1, a_7: 1, a_8: 1, a_9: 1, a_3: 0 \quad DL1: a_1: 0, a_2: 1, a_6: 0, a_5: 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
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Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_2 : p_2 = 0 \quad a_5 : p_1 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

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Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_5 : p_1 \geq 5$

Yes.

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_5 : p_1 \geq 5$

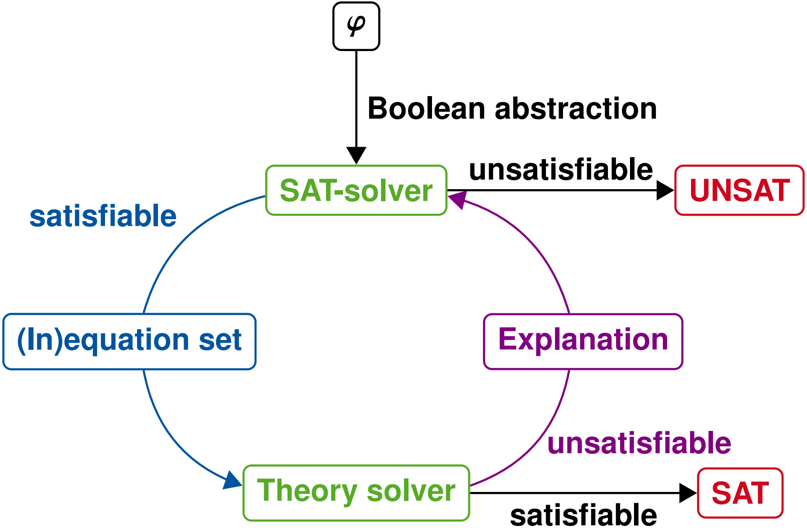
Yes. E.g.,

Is the conjunction of the following constraints satisfiable?

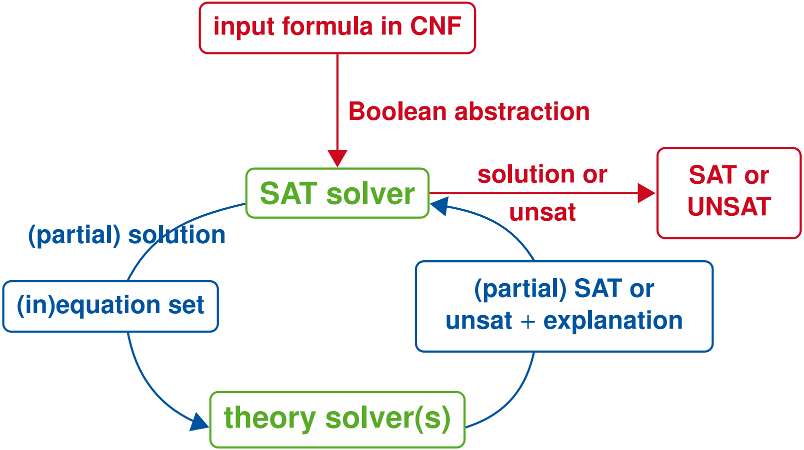
a_4 :$p_1 + p_2 + p_3 \geq 100$
a_7 :$p_3 \geq 10$
a_8 :$p_1 + 2p_2 + 5p_3 \leq 180$
a_9 :$3p_1 + 2p_2 + p_3 \leq 300$
a_2 :$p_2 = 0$
a_5 :$p_1 \geq 5$

Yes. E.g., $p_1 = 90$, $p_2 = 0$, $p_3 = 10$ is a solution.

Full lazy SMT solving



Less lazy SMT solving

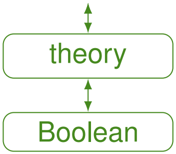


Requirements on the theory solver

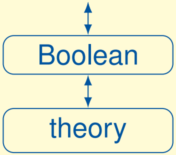
1. Incrementality: In less lazy solving we extend the set of constraints. The solver should make use of the previous satisfiability check for the check of the extended set.
2. (Preferably minimal) infeasible subsets: Compute a reason for unsatisfaction
3. Backtracking: The theory solver should be able to remove constraints in inverse chronological order.

Three SMT solving approaches

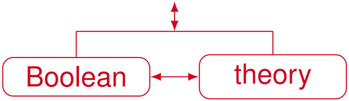
Eager SMT solving



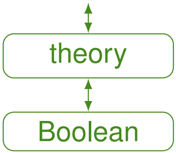
Lazy SMT solving



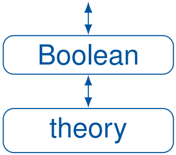
Model-constructing satisfiability calculus (MCSAT)



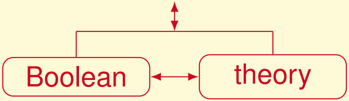
Eager SMT solving



Lazy SMT solving



Model-constructing satisfiability calculus (MCSAT)



The DPLL+CDCL idea [Davis et al., ’60/61] [Marques-Silva et al., ’96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

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$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate -

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Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate -

$\mathbb{B}$-decision $a = \mathit{false}$

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Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a =$ *false*
$\mathbb{B}$-propagate	-

The DPLL+CDCL idea [Davis et al., ’60/61] [Marques-Silva et al., ’96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \lor b \lor c) \land (a \lor b \lor \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a = false$
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$b = false$

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a =$ *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$b =$ *false*
$\mathbb{B}$-propagate	$c =$ *true* $\lightning$

Exploration:	$\mathbb{B}$-decision
Look-ahead:	$\mathbb{B}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* ↯
$\mathbb{B}$-conflict resolution	$(a \vee b)$

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* $\lightning$
$\mathbb{B}$-conflict resolution	$(a \vee b)$

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c) \qquad \ldots\ x \cdot y^2 < 0\ \ldots$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* ↯
$\mathbb{B}$-conflict resolution	$(a \vee b)$

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad\quad$ $\ldots\; x \cdot y^2 < 0 \;\ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a =$ *false*		
$\mathbb{B}$-propagate	-		
$\mathbb{B}$-decision	$b =$ *false*		
$\mathbb{B}$-propagate	$c =$ *true* $\lightning$		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad \ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a =$ *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-		
$\mathbb{B}$-decision	$b =$ *false*		
$\mathbb{B}$-propagate	$c =$ *true* ⚡		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c) \qquad \ldots\ x \cdot y^2 < 0\ \ldots$$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	b = *false*		
$\mathbb{B}$-propagate	c = *true* ↯		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad\quad$ $\ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a = \mathit{false}$	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	$b = \mathit{false}$	$\mathbb{T}$-decision	$x = 1$
$\mathbb{B}$-propagate	$c = \mathit{true}$ $\lightning$		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\qquad \ldots\; x \cdot y^2 < 0 \;\ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	b = *false*	$\mathbb{T}$-decision	$x = 1$
$\mathbb{B}$-propagate	c = *true* $\lightning$	$\mathbb{T}$-propagate	$y \in \emptyset$ $\lightning$
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad\quad$ $\ldots\; x \cdot y^2 < 0 \;\ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	b = *false*	$\mathbb{T}$-decision	$x = 1$
$\mathbb{B}$-propagate	c = *true* $\lightning$	$\mathbb{T}$-propagate	$y \in \emptyset$ $\lightning$
$\mathbb{B}$-conflict resolution	$(a \vee b)$	$\mathbb{T}$-conflict resolution	$(x \cdot y^2 < 0 \rightarrow x < 0)$

Contents

Our SMT-RAT library [SAT'12, SAT'15]

SMT solver

Strategic composition of SMT-RAT modules

SMT real-algebraic toolbox

collection of solver modules

CArL

real-arithmetic computations

gmp, Eigen3, boost

- MIT licensed source code: `github.com/smtrat/smtrat`
- Documentation: `smtrat.github.io`

Solver modules in SMT-RAT [SAT'12, SAT'15]

CArL library: basic arithmetic datatypes and computations [Sapientia'18, NFM'11, CAI'11]

Basic modules

SAT solver | CNF converter | Preprocessing/simplifying modules

Non-algebraic decision procedures

Equalities and uninterpreted functions | Bit-vectors | Bit-blasting

Interval constraint propagation | Pseudo-Boolean formulas

Algebraic decision procedures

Gauß+Fourier-Motzkin, FMplex [GandALF'23]

Gröbner bases [CAI'13] | MCSAT (FM,VS,CAD) [2xSC^2'19] | Simplex [ISSAC'21]

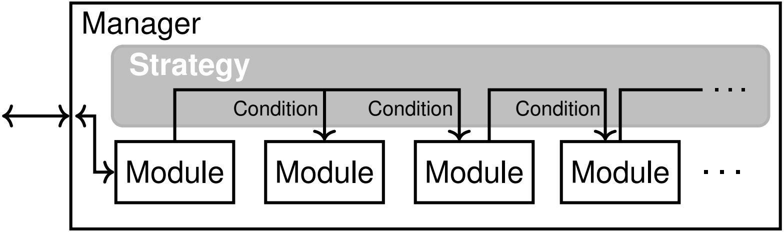
Cylindrical algebraic decomposition [SC^2'21, CADE-24, JSC'19, SC^2'17, 3 PhDs]

Cylindrical algebraic covering [SMT'23, JLAMP'21, SYNASC'21, PhD Kremer]

Virtual substitution [FCT'11, SC^2'17, 1 PhD] | Subtropical satisfiability [NFM'23]

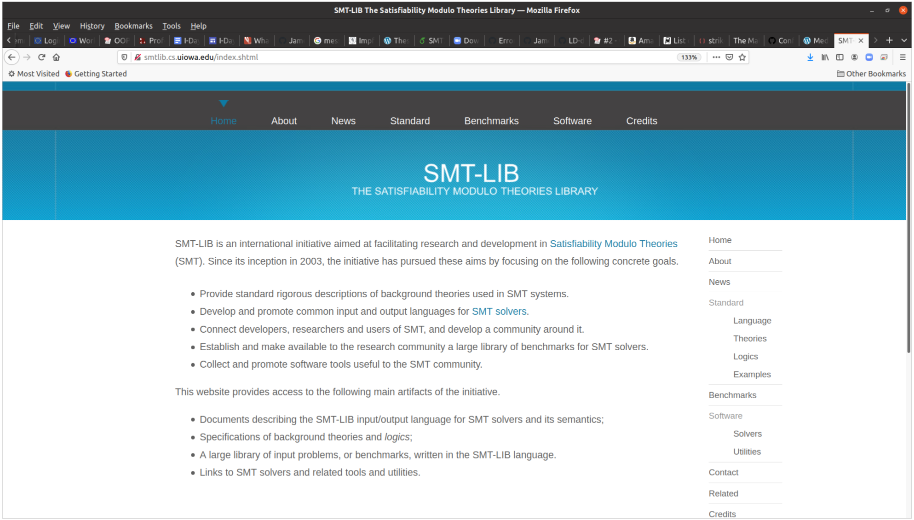
Generalized branch-and-bound [CASC'16] | Cube tests | Linearization

Strategic composition of solver modules in SMT-RAT

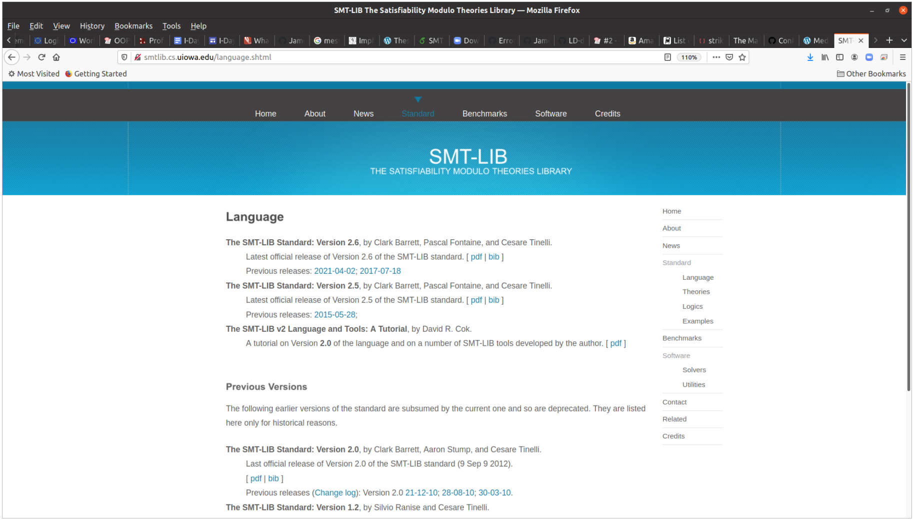


Contents

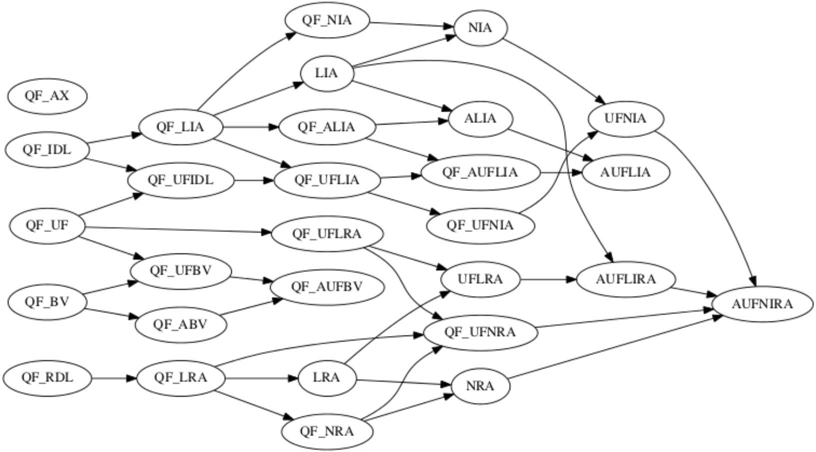
The Satisfiability Modulo Theories Library



The Satisfiability Modulo Theories Library



SMT-LIB theories



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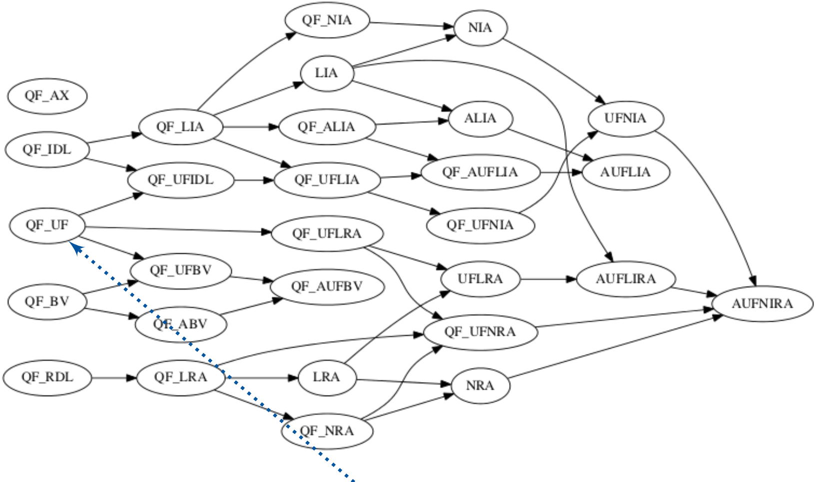
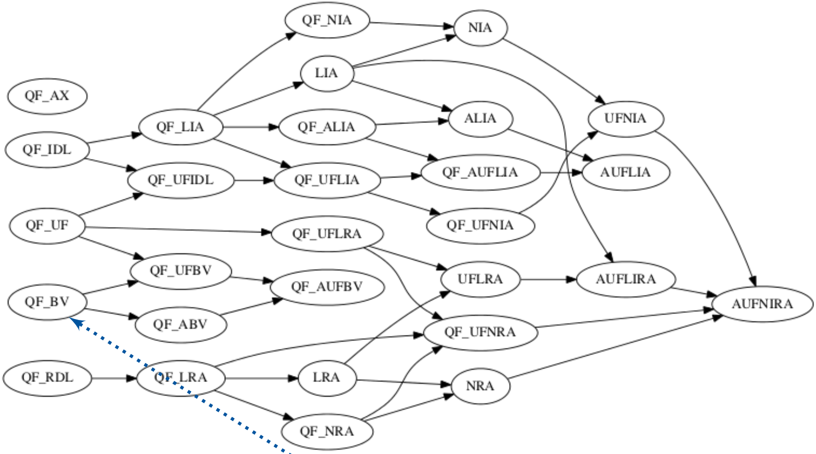


Quantifier-free equality logic with uninterpreted functions

$$(a = c \wedge b = d) \rightarrow f(a, b) = f(c, d)$$

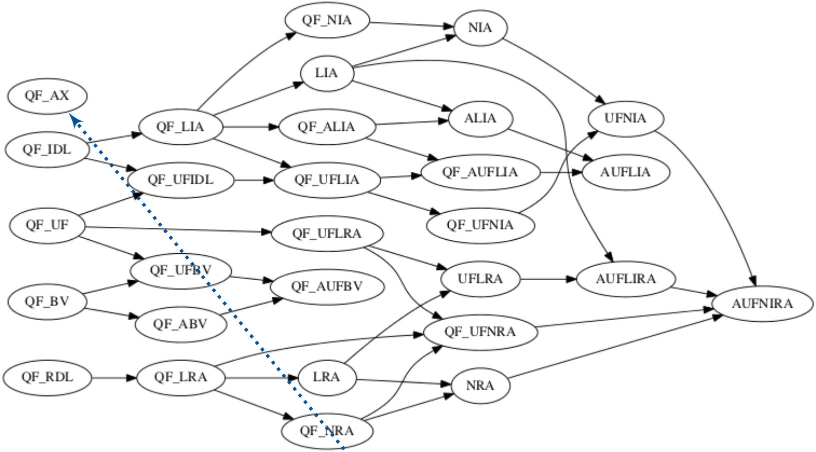
Source: `http://smtlib.cs.uiowa.edu/logics.shtml`



Quantifier-free bit-vector arithmetic

$$(a|b) \leq (a\&b)$$

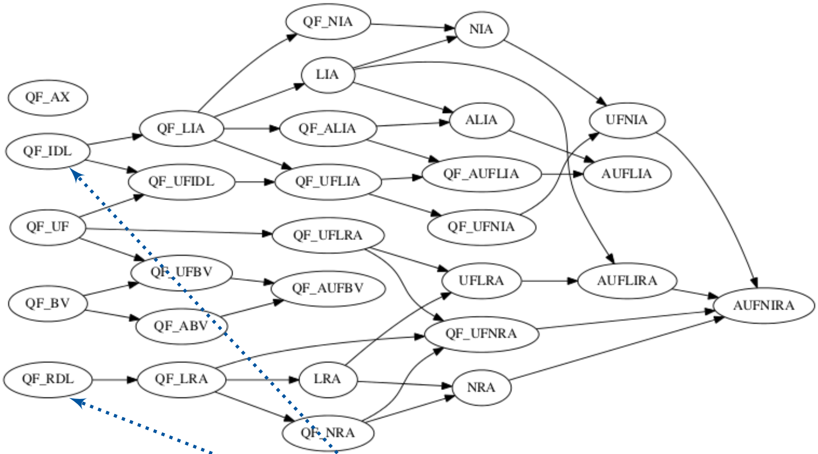
Source: `http://smtlib.cs.uiowa.edu/logics.shtml`



Quantifier-free array theory

$i = j \rightarrow \mathit{read}(\mathit{write}(a, i, v), j) = v$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

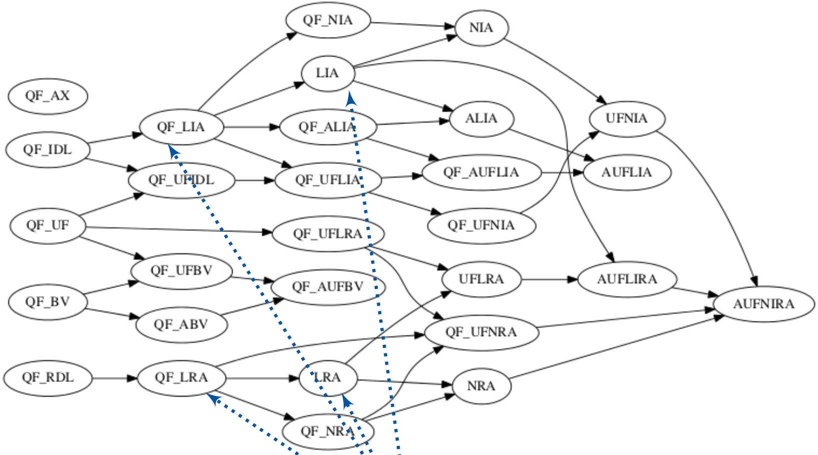


Quantifier-free integer/rational difference logic

$x - y \sim 0, \ \sim \in \{<, \leq, =, \geq, >\}$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

SMT-LIB theories

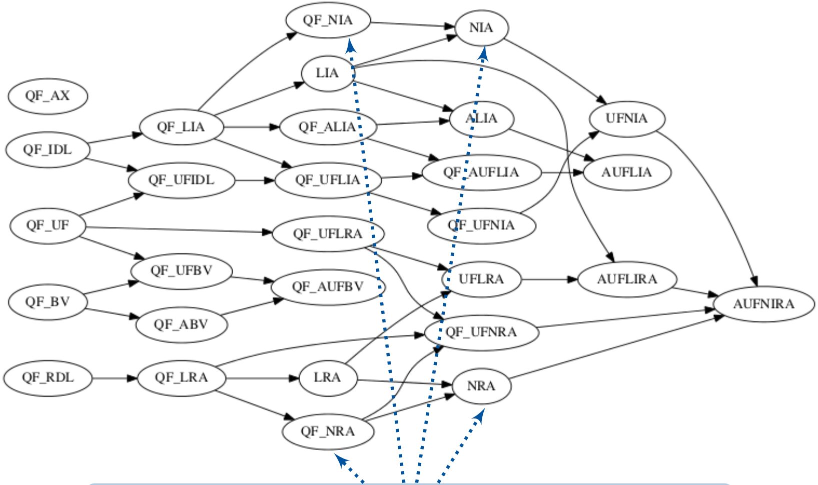


(Quantifier-free) real/integer linear arithmetic

$$3x + 7y = 8$$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

SMT-LIB theories

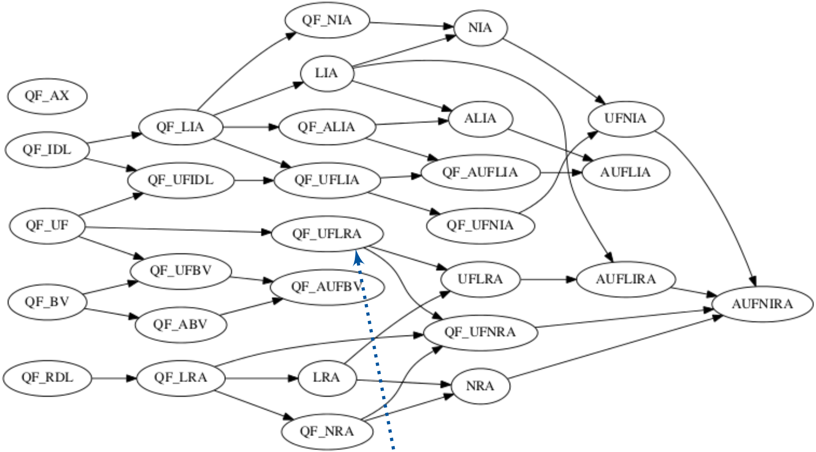


(Quantifier-free) real/integer non-linear arithmetic

$$x^2 + 2xy + y^2 \geq 0$$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

SMT-LIB theories



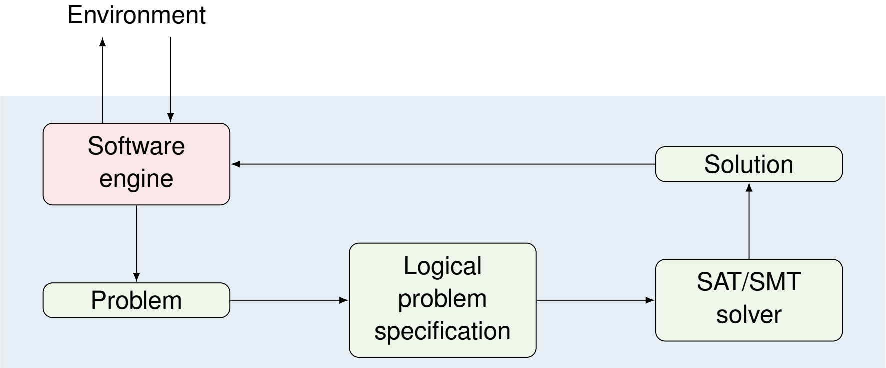
Combined theories

$2f(x) + 5y > 0$

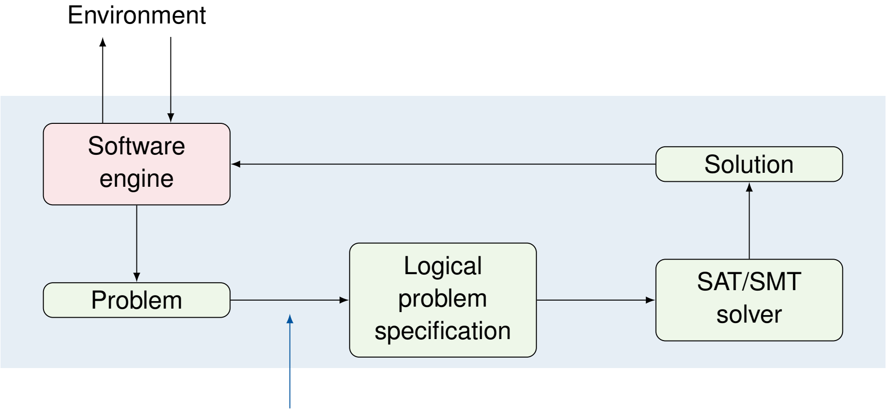
Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

Contents

Embedding SAT/SMT solvers



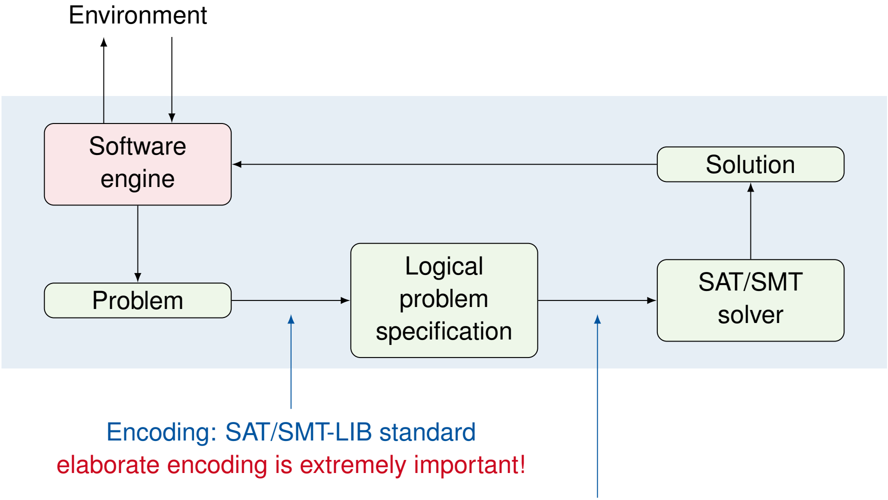
Embedding SAT/SMT solvers



Encoding: SAT/SMT-LIB standard

elaborate encoding is extremely important!

Embedding SAT/SMT solvers

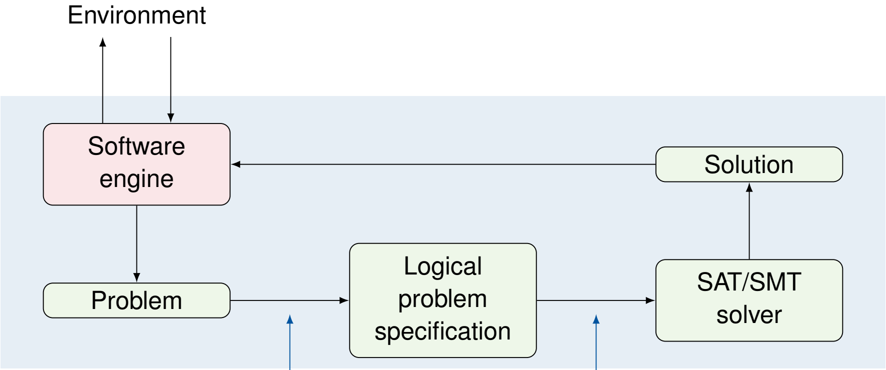


Encoding: SAT/SMT-LIB standard

elaborate encoding is extremely important!

standard input syntax → free solver choice

Embedding SAT/SMT solvers

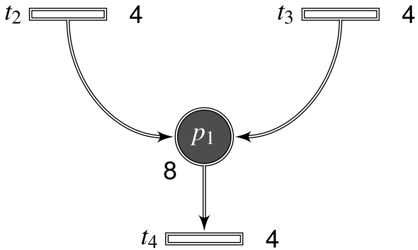


Encoding: SAT/SMT-LIB standard
elaborate encoding is extremely important!

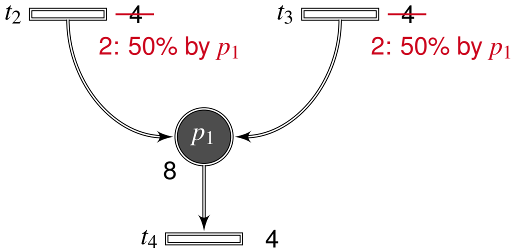
standard input syntax → free solver choice

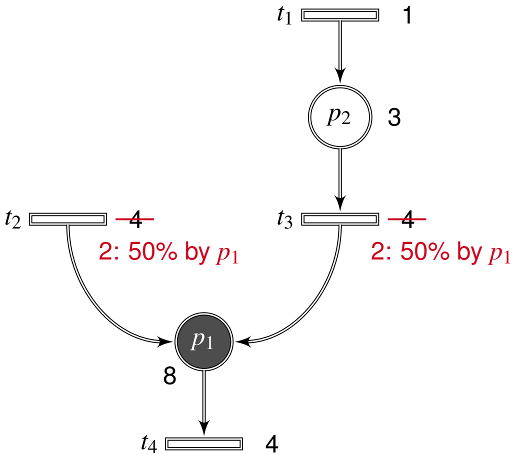
Next: some applications of SMT solvers

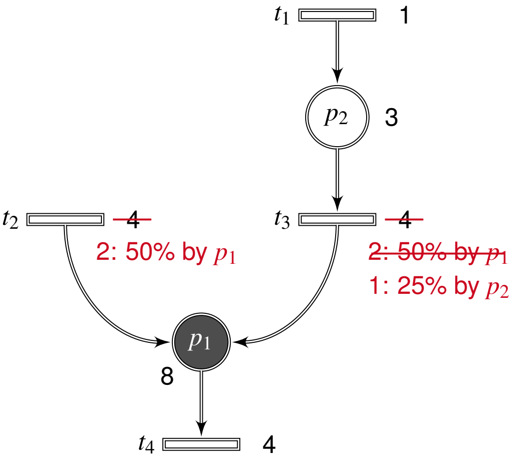
1. Rate adaption in hybrid Petri nets

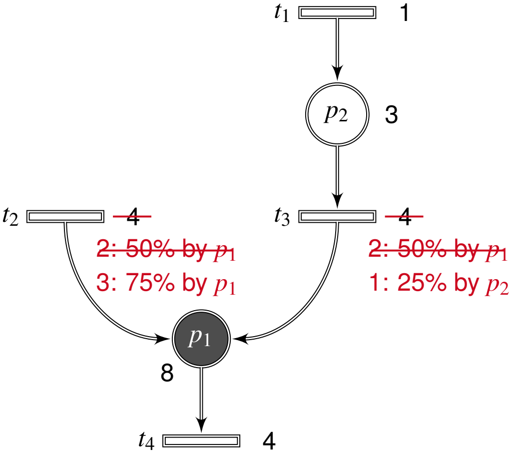


1. Rate adaption in hybrid Petri nets




t_1
1
p_2
3
t_2
~~4~~
2: 50% by p_1
t_3
~~4~~
2: 50% by p_1
p_1
8
t_4
4


t_1
1
p_2
3
t_2
~~4~~
2: 50% by p_1
t_3
~~4~~
~~2: 50% by p_1~~
1: 25% by p_2
p_1
8
t_4
4


t_1
1
p_2
3
t_2
~~4~~
~~2: 50% by p_1~~
3: 75% by p_1
t_3
~~4~~
~~2: 50% by p_1~~
1: 25% by p_2
p_1
8
t_4
4

1. SMT encoding of rate adaption fixedpoint

$$(1)\quad \Big[\bigwedge_{p\in P} 0 \le \mathtt{factor}_p \le 1\Big] \wedge \Big[\bigwedge_{t\in T_a} 0 \le \mathtt{factor}_t \le 1\Big] \wedge$$

$$(2)\quad \Big[\bigwedge_{t\in T_a} ((\mathtt{owner}_t = \mathit{source}(t) \wedge \mathtt{owner}_t \in P_{\mathit{empty}}) \vee (\mathtt{owner}_t = \mathit{target}(t) \wedge \mathtt{owner}_t \in P_{\mathit{full}}))\Big] \wedge$$

$$(3)\quad \Big[\bigwedge_{p\in P} \mathtt{in}_p = (\textstyle\sum_{t\in \mathit{In}(p)\cap T_a} \mathtt{factor}_t \cdot \mathit{nominal\_rate}(t)) + (\textstyle\sum_{t\in \mathit{In}(p)\cap T_{na}} \mathit{nominal\_rate}(t)) \wedge$$

$$\mathtt{out}_p = (\textstyle\sum_{t\in \mathit{Out}(p)\cap T_a} \mathtt{factor}_t \cdot \mathit{nominal\_rate}(t)) + (\textstyle\sum_{t\in \mathit{Out}(p)\cap T_{na}} \mathit{nominal\_rate}(t))\Big] \wedge$$

$$(4)\quad \Big[\bigwedge_{p\in P_{\mathit{empty}}} \Big((\mathtt{factor}_p = 1 \vee \bigvee_{t\in \mathit{Out}(p)} \mathtt{owner}_t = p) \wedge$$

$$\Big(\bigwedge_{t\in \mathit{Out}(p)} (\mathtt{owner}_t = p \rightarrow \mathtt{factor}_t = \mathtt{factor}_p) \wedge$$

$$(\mathtt{owner}_t \ne p \rightarrow \mathtt{factor}_t < \mathtt{factor}_p)\quad\Big) \wedge$$

$$\mathtt{in}_p \ge \mathtt{out}_p \wedge (\mathtt{factor}_p < 1 \rightarrow \mathtt{in}_p = \mathtt{out}_p)\quad\Big)\Big] \wedge$$

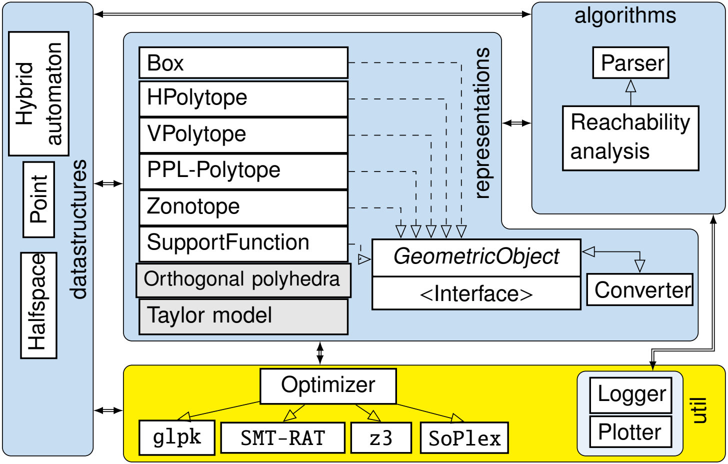
$$(5)\quad \Big[\bigwedge_{p\in P_{\mathit{full}}} \Big((\mathtt{factor}_p = 1 \vee \bigvee_{t\in \mathit{In}(p)} \mathtt{owner}_t = p) \wedge$$

$$\Big(\bigwedge_{t\in \mathit{In}(p)} (\mathtt{owner}_t = p \rightarrow \mathtt{factor}_t = \mathtt{factor}_p) \wedge$$

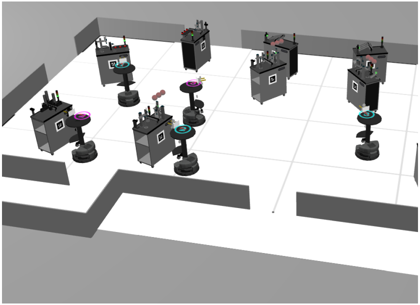
$$(\mathtt{owner}_t \ne p \rightarrow \mathtt{factor}_t \le \mathtt{factor}_p)\quad\Big) \wedge$$

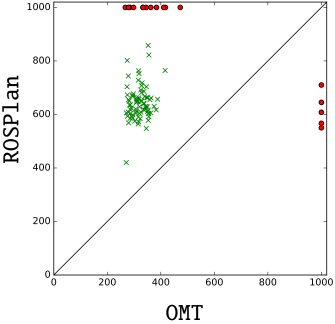
$$\mathtt{in}_p \le \mathtt{out}_p \wedge (\mathtt{factor}_p < 1 \rightarrow \mathtt{in}_p = \mathtt{out}_p)\quad\Big)\Big]$$

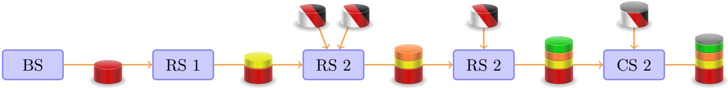
2. Reachability analysis for hybrid systems with HyPro



Source: E. Ábrahám, X. Chen, S. Sankaranarayanan, S. Schupp. PhD Chen, PhD Schupp, Information and Computation'22, IRI'18, SEFM'18, TACAS'18, NFM'17, QAPL'17, ARCH'15, CyPhy'15, NFM'15, FMCAD'14, CAV'13, FTSCS'13, NOLCOS'13, RTSS'12, EUROCAST'11, RP'11.

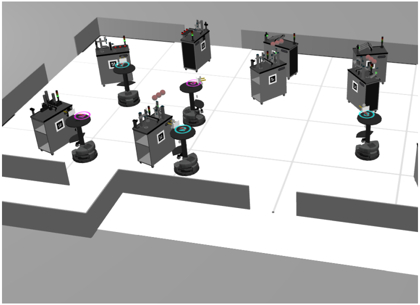


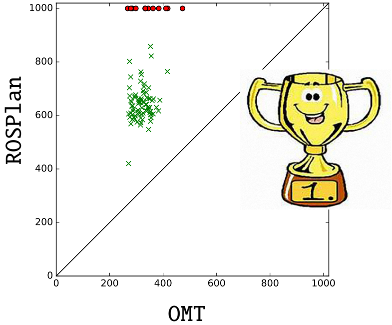


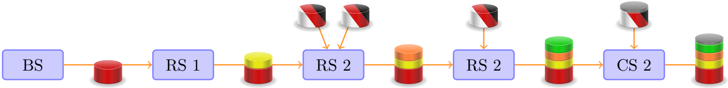


Source: E. Ábrahám, G. Lakemeyer, F. Leofante, T. D. Niemüller, A. Tacchella.

PhD Leofante, IJCAI'20, Information Systems Frontiers 2019, ECMS'19, AAAI'18, iFM'18, ICAPS'17, PlanRob'17, IRI'17.







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4. Relevant domains for testing (Siemens)

System model

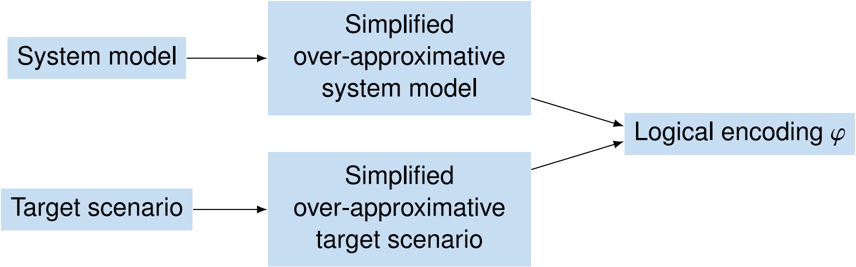
Target scenario

4. Relevant domains for testing (Siemens)

System model → Simplified over-approximative system model

Target scenario → Simplified over-approximative target scenario

4. Relevant domains for testing (Siemens)



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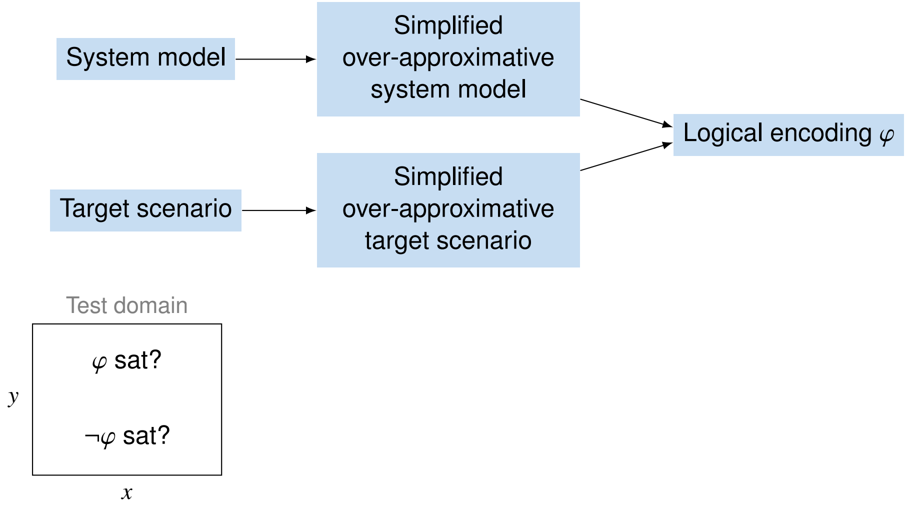

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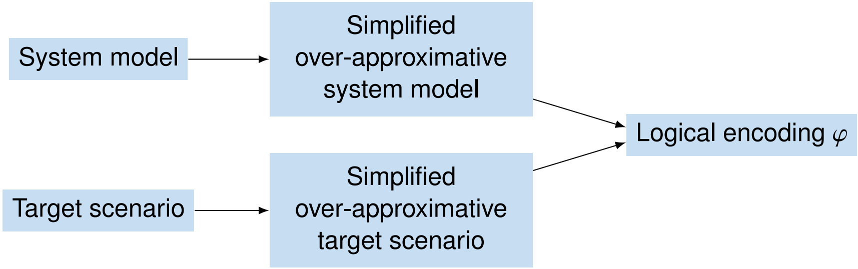
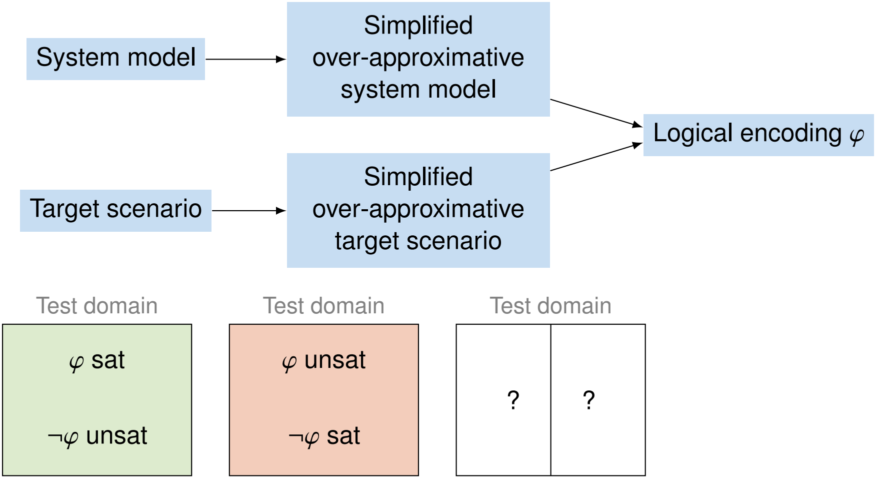


Test domain	Test domain	Test domain
φ sat	φ unsat	φ sat
$\neg\varphi$ unsat	$\neg\varphi$ sat	$\neg\varphi$ sat

4. Relevant domains for testing (Siemens)



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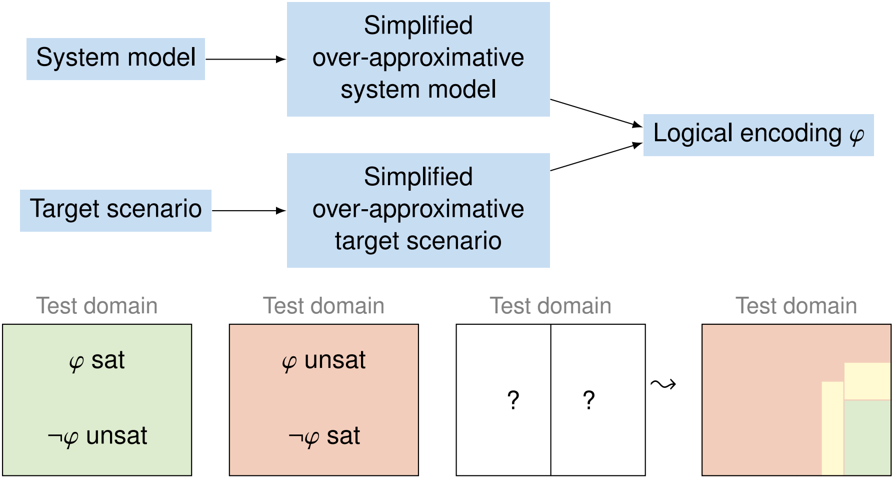
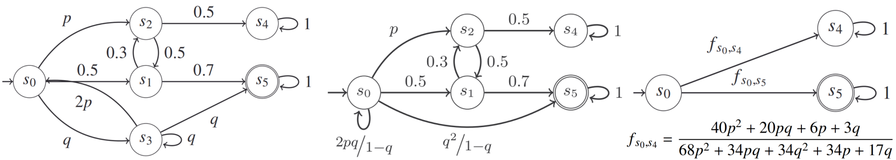
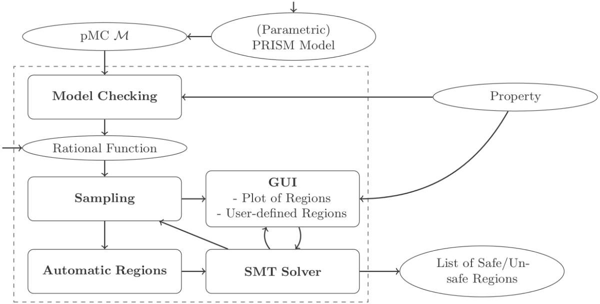
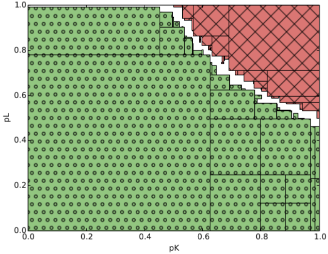








Source: C. Dehnert, S. Junges, N. Jansen, F. Corzilius, M. Volk, H. Bruintjes, J.-P. Katoen, E. Ábrahám.

PROPhESY: A probabilistic parameter synthesis tool.

In Proc. of CAV'15.

Contents

Usage of SMT solvers

- Standard input language, benchmarks
- Online usage, command-line, programming interfaces
- Black-box usage possible, but specific knowledge is advantageous
  - for efficient usage and
  - selection of the best fitting tool (e.g. fast vs complete).

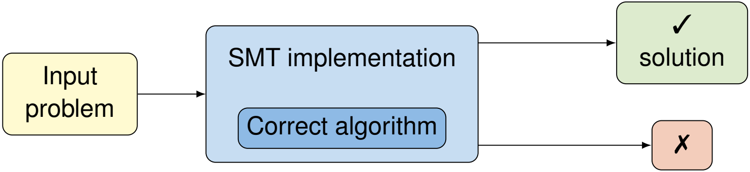
Proof generation

- Theoretical basics: algorithms with correctness proofs.

Correct algorithm

Proof generation

- Theoretical basics: algorithms with correctness proofs.
- Reliable tools: in QF_NRA for SMT-COMP'21, no bugs discovered on large benchmark sets.



Proof generation

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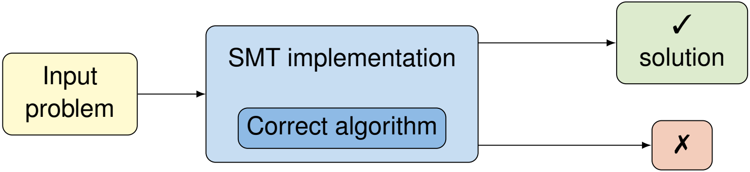
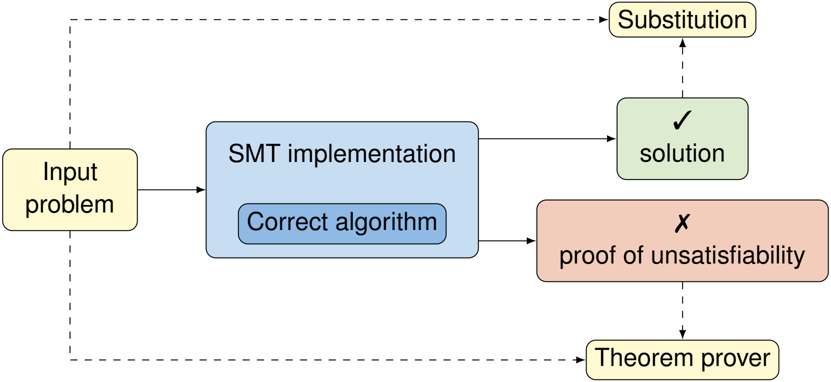
- But still: bugs can remain undetected for a long time.



Proof generation

- Theoretical basics: algorithms with correctness proofs.
- Reliable tools: in QF_NRA for SMT-COMP'21, no bugs discovered on large benchmark sets.
- But still: bugs can remain undetected for a long time.
- Solution: automatically checkable proof certificates.



Further functionalities

- Model generation
- Explanations of unsatisfiability (unsat cores, interpolants)
- Optimization

- Satisfiability for quantified formulas
- Quantifier elimination (get all solutions symbolically)

- Scalability
  - Preprocessing
  - Heuristics, especially variable ordering
  - Machine learning
  - Closer integration of decision procedures
  - Parallelization

Contents

You need to have installed...

- Python
- Z3

  https://github.com/exercism/z3/blob/main/docs/INSTALLATION.md

Contents

SAT encodings

- Suppose we can solve the satisfiability problem... how can this help us?

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- There are numerous problems in the industry that are solved via the satisfiability problem of propositional logic
  - Logistics
  - Planning
  - Electronic Design Automation industry
  - Cryptography
  - ...

Example 1: Placement of wedding guests

- Three chairs in a row: $1, 2, 3$
- We need to place Aunt, Sister and Father.
- Constraints:
  - Aunt doesn't want to sit near Father
  - Aunt doesn't want to sit in the left chair
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- Q: Can we satisfy these constraints?

Example 1 (continued)

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- Notation:

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  Left chair = 1, Middle chair = 2, Right chair = 3

  Introduce a propositional variable for each pair (person, chair):

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$$((x_{1,1} \vee x_{1,3}) \rightarrow \neg x_{3,2}) \wedge (x_{1,2} \rightarrow (\neg x_{3,1} \wedge \neg x_{3,3}))$$

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$$\bigwedge_{p=1}^{3} \bigvee_{c=1}^{3} x_{p,c}$$

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At most one person per chair:

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At most one person per chair:

$$\bigwedge_{p1=1}^{3} \bigwedge_{p2=p1+1}^{3} \bigwedge_{c=1}^{3} (\neg x_{p1,c} \vee \neg x_{p2,c})$$

Example 2: Assignment of frequencies

- n radio stations
- For each station assign one of k transmission frequencies, $k < n$.
- E – set of pairs of stations, that are too close to have the same frequency.

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- Q: Can we assign to each station a frequency, such that no station pairs from E have the same frequency?

Example 2 (continued)

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  Every station is assigned at least one frequency:

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Every station is assigned at most one frequency:

$$\bigwedge_{s=1}^{n} \bigwedge_{f1=1}^{k-1} \bigwedge_{f2=f1+1}^{k} \left(\neg x_{s,f1} \vee \neg x_{s,f2} \right)$$

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  Close stations are not assigned the same frequency:

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  Close stations are not assigned the same frequency:

  For each $(s1, s2) \in E$,

$$\bigwedge_{f=1}^{k} \left(\neg x_{s1,f} \vee \neg x_{s2,f} \right)$$

Example 3: Seminar topic assignment

- n participants
- n topics
- Set of preferences $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$

  $(p, t) \in E$ means: participant p would take topic t

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- n topics
- Set of preferences $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$

  $(p, t) \in E$ means: participant p would take topic t

- Q: Can we assign to each participant a topic which he/she is willing to take?

Example 3 (continued)

- Notation:

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- Notation: $x_{p,t}$ = "participant p is assigned topic t"

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$$\bigwedge_{p=1}^{n}\left(\bigvee_{t=1}^{n} x_{p,t}\right)$$

Example 3 (continued)

- Notation: $x_{p,t}$ = "participant p is assigned topic t"
- Constraints:

  Each participant is assigned at least one topic:

$$\bigwedge_{p=1}^{n} \left(\bigvee_{t=1}^{n} x_{p,t} \right)$$

  Each participant is assigned at most one topic:

Example 3 (continued)

- Notation: $x_{p,t}$ = “participant p is assigned topic t”
- Constraints:

  Each participant is assigned at least one topic:

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  Each participant is assigned at most one topic:

$$\bigwedge_{p=1}^{n}\ \bigwedge_{t1=1}^{n-1}\ \bigwedge_{t2=t1+1}^{n}\left(\neg x_{p,t1} \vee \neg x_{p,t2}\right)$$

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  Each participant is willing to take his/her assigned topic:

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- Notation: $x_{p,t}$ = "participant p is assigned topic t"
- Constraints:

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$$\bigwedge_{p=1}^{n} \bigwedge_{t1=1}^{n-1} \bigwedge_{t2=t1+1}^{n} \left(\neg x_{p,t1} \vee \neg x_{p,t2} \right)$$

  Each participant is willing to take his/her assigned topic:

$$\bigwedge_{p=1}^{n} \bigwedge_{(p,t)\notin E} \neg x_{p,t}$$

Example 3 (continued)

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Each topic is assigned to at most one participant:

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Each topic is assigned to at most one participant:

$$\bigwedge_{t=1}^{n} \bigwedge_{p1=1}^{n} \bigwedge_{p2=p1+1}^{n} \left(\neg x_{p1,t} \vee \neg x_{p2,t}\right)$$

DIMACS input syntax for SAT solvers

The DIMACS format for SAT solvers has three types of lines:

- header: "p cnf n m" in which
  - n denotes the highest variable index and
  - m the number of clauses.
- clauses: a sequence of integers ending with "0"
- comments: any line starting with "c "

Example:

	c example
	p cnf 2 4
$(a \vee b) \quad \wedge$	$1 \quad 2 \quad 0$
$(\neg a \vee b) \quad \wedge$	$-1 \quad 2 \quad 0$
$(a \vee \neg b) \quad \wedge$	$1 \quad -2 \quad 0$
$(\neg a \vee \neg b) \quad \wedge$	$-1 \quad -2 \quad 0$

Solving propositional logic with SMT solvers

- SMT-LIB format:
  https://microsoft.github.io/z3guide/docs/logic/propositional-logic
- Python interface:
  https://ericpony.github.io/z3py-tutorial/guide-examples.htm
- Both:
  https://cvc5.github.io/tutorials/beginners/

Contents

SMT-LIB theories

Syntax of core theory

```
:sorts ((Bool 0))
:funs (
  (true Bool)
  (false Bool)
  (not Bool Bool)
  (and Bool Bool Bool :left-assoc)
  ...
  (par (A) (= A A Bool :chainable))
  (par (A) (ite Bool A A A))
  ...
```

SMT-LIB theories

Syntax of real theory

```
:sorts ((Real 0))
:funs (
   ...
   (+ Real Real Real :left-assoc)
   (* Real Real Real :left-assoc)
   ...
   (<  Real Real Bool :chainable)
   ...
)
```

SMT-LIB commands

- Lisp-like script language
- Supported by essentially all SMT solvers
- Easy to parse and extend

Boolean example

```
(set-logic QF_UF)
(declare-const p Bool)
(assert (and p (not p)))
(check-sat)
```

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Linear integer example

```
(set-logic QF_LIA)
(declare-const x Int)
(declare-const y Int)
(assert (= (- x y) (+ x (- y) 1)))
(check-sat)
```

SMT-LIB commands

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Unsatisfiable cores

```
(set-logic QF_UF)
(set-option :produce-unsat-cores true)
(declare-const p Bool)
(declare-const q Bool)
(declare-const r Bool)
(assert (! (=> p q) :named a))
(assert (! (=> q r) :named b))
(assert (! (not (=> p r)) :named c))
(assert ...)
(check-sat)
(get-unsat-core)
```

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Optimization

```
(set-logic QF_LIA)
(declare-const x Int)
(declare-const y Int)
(assert (and (< y 5) (< x 2)))
(assert (< (- y x) 1))
(maximize (+ x y))
(check-sat)
(get-objectives)
```

Solving theory formulas with SMT solvers

- https://cvc5.github.io/tutorials/beginners
- SMT-LIB input:
  https://microsoft.github.io/z3guide/docs/logic/intro/
  https://smt-lib.org/examples.shtml
- Z3/cvc5 Python interface:
  https://ericpony.github.io/z3py-tutorial/guide-examples.htm