# DS Tutorial-2 : Set Theory Problems

## Problem 1 (Finite Set)

Let $A = \{f : \{1, 2, \ldots, n\} \to \{0, 1\} \mid f \text{ is injective}\}$.

(a) Express $|A|$ in terms of $n$.

(b) Prove that $A$ is finite for any fixed $n \in \mathbb{N}$.

(c) What happens to $|A|$ as $n$ increases beyond 2?

## Problem 2 (Finite Set)

Consider the set $B = \{S \subseteq \mathbb{N} \mid S \text{ is finite and } \sum_{s \in S} s \leq 100\}$.

(a) Prove that $B$ is finite.

(b) Determine whether the set $C = \{S \subseteq \mathbb{N} \mid S \text{ is finite and } |S| \leq 5\}$ is finite or infinite.

(c) Compare the cardinalities of $B$ and $C$.

## Problem 3 (Infinite Set)

Let $D = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \mid a^2 + b^2 = 2\}$.

(a) Prove that $D$ is infinite.

(b) Is $D$ countably infinite or uncountably infinite? Justify your answer.

(c) Construct an explicit bijection between $D$ and a proper subset of itself.

## Problem 4 (Infinite Set)

Let $E = \{n \in \mathbb{N} \mid n \equiv 1 \pmod 3\} \cup \{n \in \mathbb{N} \mid n \equiv 2 \pmod 5\}$.

(a) Write out the first 10 elements of $E$.

(b) Prove that $E$ is infinite.

(c) Is $E$ countably infinite? Justify your answer.

## Problem 5 (Cantor's Diagonalization)

Consider the set $T = \{f : \mathbb{N} \to \{0, 1\}\}$ of all functions from natural numbers to $\{0, 1\}$.

(a) Assume $T$ is countably infinite and list the functions as $f_1, f_2, f_3, \ldots$

(b) Define a new function $g : \mathbb{N} \to \{0, 1\}$ by $g(n) = 1 - f_n(n)$.

(c) Show that $g$ cannot be in the list $\{f_1, f_2, f_3, \ldots\}$.

(d) What does this contradiction tell us about the cardinality of $T$?