International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Quiz 1

Time: 45 Minutes

Total Marks: 20

Question (1) [4 Marks] Define $a_0 = 1$, $a_{n+1} = 2a_n + 1$.

(a). **[1 Mark]** Compute the first 5 terms.

(b). **[1 Marks]** Guess a closed form for a_n.

(c). **[2 Marks]** Prove it by induction.

Question (2) [4 Marks] Place-value notation works because expansions are unique. This fact underpins decimal/binary representation and later decimal expansions of real numbers. Fix $b \geq 2$. Prove that every $n \in \mathbb{N}$ has a unique expansion

$$n = a_0 + a_1 b + \cdots + a_k b^k,$$

with $0 \leq a_i < b$ and $a_k \neq 0$. *[Hint: Use division algorithm.]*

Question (3) [3 Marks] Let us consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each $x \in \mathbb{Z}$, $f(x) = x^k \in \mathbb{Z}$ for some $k \geq 2$.

(a). **[1 Mark]** For which values of k is f injective?

(b). **[1 Mark]** For which values of k is f bijective?

(c). **[1 Mark]** For $k = 2$, what is the inverse image of $\mathbb{N}$ under f, that is, what is $f^{-1}(\mathbb{N})$?

Question (4) [4 Marks] How close can a rational number get to the nearest integer? **Distance measures define closeness. For each rational x, define its distance with the** set of integers as

$$d(x) = \min_{m \in \mathbb{Z}} |x - m|.$$

(a). **[2 Marks]** Show $0 \leq d(x) \leq \frac{1}{2}$.

(b). **[1 Mark]** Give two examples of $x \in \mathbb{Q}$ with $d(x) = \frac{1}{2}$.

(c). **[1 Mark]** Show that $d(x) = 0$ iff $x \in \mathbb{Z}$.

Question (5) [5 Marks] **For a finite set with** n **elements, the power set has** 2^n **elements, i.e., its size grows exponentially. But what if the set is infinite, like** $\mathbb{N}$? Cantor's theorem shows that $\mathcal{P}(\mathbb{N})$ **is strictly larger, revealing that even infinities come in different sizes. In particular, can you show that there is no surjection from** $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$? Proving this will imply that the cardinality of $\mathbb{N}$, **which is infinite, is** smaller than cardinality of $\mathcal{P}(\mathbb{N})$.

[Hint: Assume there is a surjection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Then construct a subset S of $\mathbb{N}$ as $S = \{n \in \mathbb{N} : n \notin f(n)\}$. Use this to show a contradiction.]