Lecture 15 – Laplace Transform 2

Dr. Aftab M. Hussain,

Associate Professor, PATRIoT Lab, CVEST

Circuit Analysis

- We can go from time-domain to s-domain and back in order to simplify our circuit analysis
- Simply convert every element in the circuit to the equivalent circuit in the s-domain, including voltage and current sources for the initial conditions
- Then, use linear algebra to obtain the value of the desired quantity as a function of s
- The time-domain answer can be obtained by taking the inverse Laplace transform

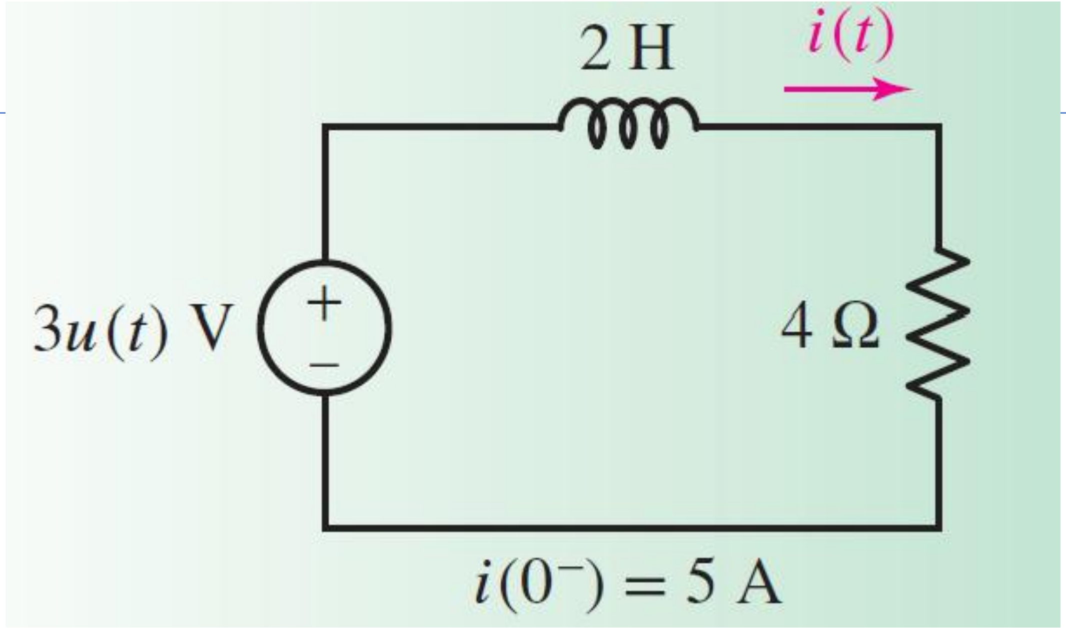
$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})u(t)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$u(t)$	$\frac{1}{s}$	$\sin \omega t \, u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos \omega t \, u(t)$	$\frac{s}{s^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} u(t), n = 1, 2, \ldots$	$\frac{1}{s^n}$	$\sin(\omega t + \theta) \, u(t)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
$e^{-\alpha t} u(t)$	$\frac{1}{s + \alpha}$	$\cos(\omega t + \theta) \, u(t)$	$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$
$te^{-\alpha t} u(t)$	$\frac{1}{(s+\alpha)^2}$	$e^{-\alpha t} \sin \omega t \, u(t)$	$\frac{\omega}{(s+\alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t), n = 1, 2, \ldots$	$\frac{1}{(s+\alpha)^n}$	$e^{-\alpha t} \cos \omega t \, u(t)$	$\frac{s + \alpha}{(s+\alpha)^2 + \omega^2}$

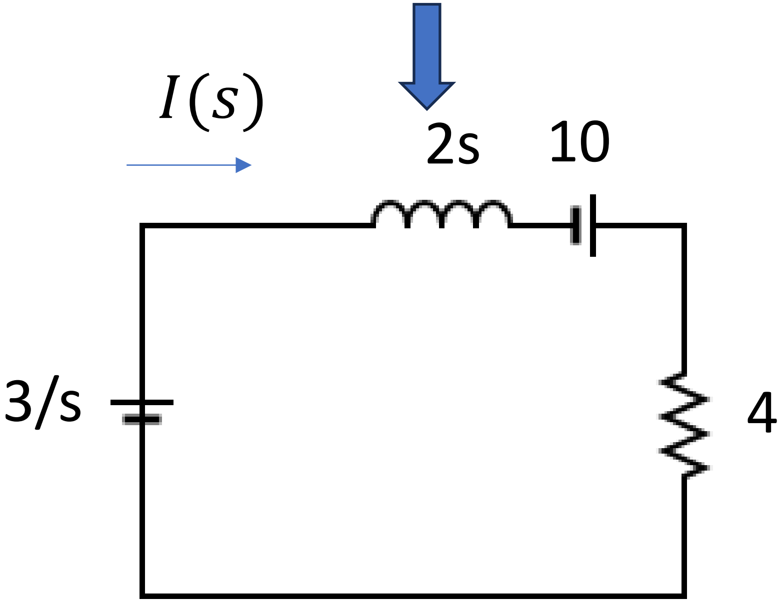
Operation	$f(t)$	$\mathbf{F}(\mathbf{s})$
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(\mathbf{s}) \pm \mathbf{F}_2(\mathbf{s})$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(\mathbf{s})$
Time differentiation	$\dfrac{df}{dt}$	$\mathbf{s}\mathbf{F}(\mathbf{s}) - f(0^-)$
	$\dfrac{d^2f}{dt^2}$	$\mathbf{s}^2\mathbf{F}(\mathbf{s}) - \mathbf{s}f(0^-) - f'(0^-)$
	$\dfrac{d^3f}{dt^3}$	$\mathbf{s}^3\mathbf{F}(\mathbf{s}) - \mathbf{s}^2f(0^-) - \mathbf{s}f'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^{t} f(t)dt$	$\frac{1}{\mathbf{s}}\mathbf{F}(\mathbf{s})$
	$\int_{-\infty}^{t} f(t)dt$	$\frac{1}{\mathbf{s}}F(\mathbf{s}) + \frac{1}{\mathbf{s}}\int_{-\infty}^{0^-} f(t)dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(\mathbf{s})\mathbf{F}_2(\mathbf{s})$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-a\mathbf{s}}\mathbf{F}(\mathbf{s})$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(\mathbf{s}+a)$
Frequency differentiation	$tf(t)$	$-\dfrac{d\mathbf{F}(\mathbf{s})}{d\mathbf{s}}$
Frequency integration	$\dfrac{f(t)}{t}$	$\int_{\mathbf{s}}^{\infty} \mathbf{F}(\mathbf{s})d\mathbf{s}$
Scaling	$f(at),\ a \geq 0$	$\frac{1}{a}\mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$
Initial value	$f(0^+)$	$\lim_{\mathbf{s}\to\infty} \mathbf{s}\mathbf{F}(\mathbf{s})$

Circuit Analysis

- Let us start by analyzing a simple RL series circuit
- We have the initial current in the inductor at 5 A
- And a voltage source is switched on at $t = 0$
- This circuit can be converted into an s-domain equivalent as shown
- Thus,

$$\frac{3}{s} = I(2s + 4) - 10$$





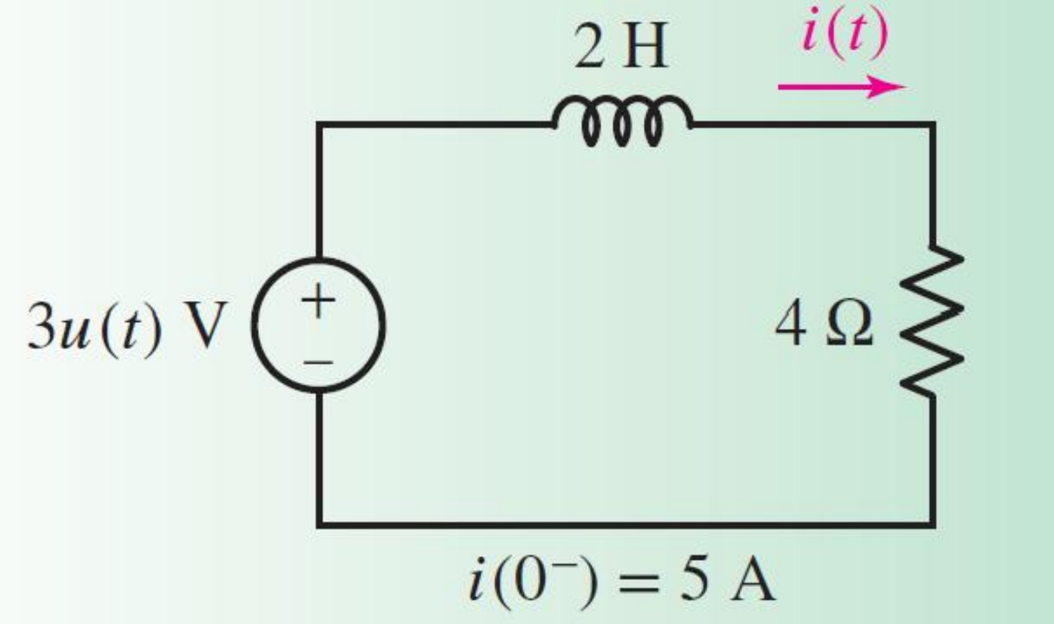
Circuit Analysis

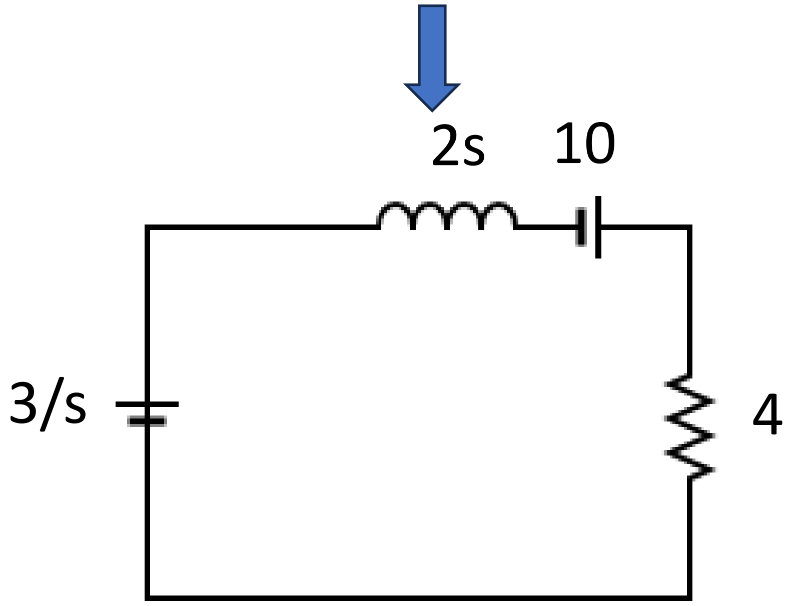
$$\frac{3}{s} = I(2s + 4) - 10$$

$$I = \frac{3 + 10s}{s(2s + 4)}$$

$$I = \frac{1.5 + 5s}{s(s + 2)}$$

$$I = \frac{1.5}{s(s + 2)} + \frac{5}{(s + 2)}$$





Method of residues

- To obtain the inverse Laplace transform, it helps to factorize the denominator as much as possible
- This factorization can be done using method of residues
- Say, we have the following expression:

$$F(s) = \frac{1}{(s+a)(s+b)}$$

- We want to obtain an expression of the form:

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

- What are the values of A and B?

$$A(s+b) + B(s+a) = 1$$

Method of residues

$$A = (s+a)F(s) - \frac{s+a}{s+b}\ B$$

$$\lim_{s\to -a} A = \lim_{s\to -a}\left[(s+a)F(s) - \frac{s+a}{s+b}\ B\right]$$

$$A = (s+a)F(s)\Big|_{s=-a}$$

$$B = (s+b)F(s)\Big|_{s=-b}$$

Method of residues

- In case of repeated poles:

$$V(s) = \frac{N(s)}{(s-p)^n}$$

- We want to expand this in the form:

$$V(s) = \frac{a_n}{(s-p)^n} + \frac{a_{n-1}}{(s-p)^{n-1}} + \cdots + \frac{a_1}{(s-p)}$$

- Thus,

$$(s-p)^n\, V(s) = a_n + a_{n-1}(s-p) + \cdots$$

- Thus,

$$a_n = (s-p)^n V(s)\Big|_{s=p}$$

- To obtain a_{n-1}, we need to eliminate the $(s-p)$ term. We can differentiate the expression wrt s

$$a_{n-1} = \frac{d}{ds}\left[(s-p)^n V(s)\right]\Big|_{s=p}$$

$$a_{n-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left[(s-p)^n V(s)\right]\Big|_{s=p}$$

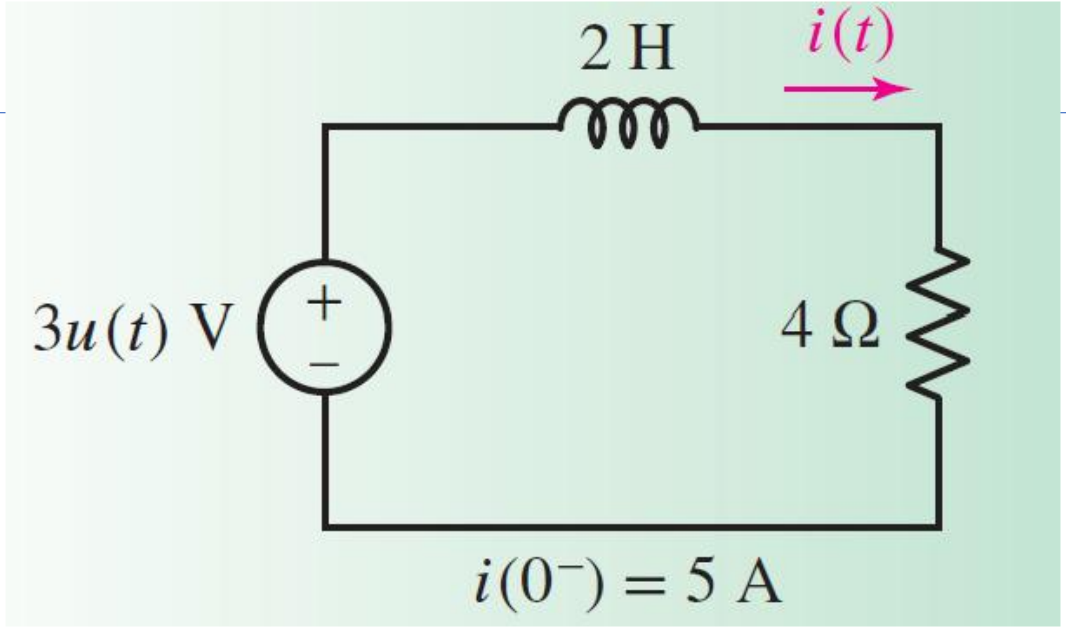
Circuit Analysis

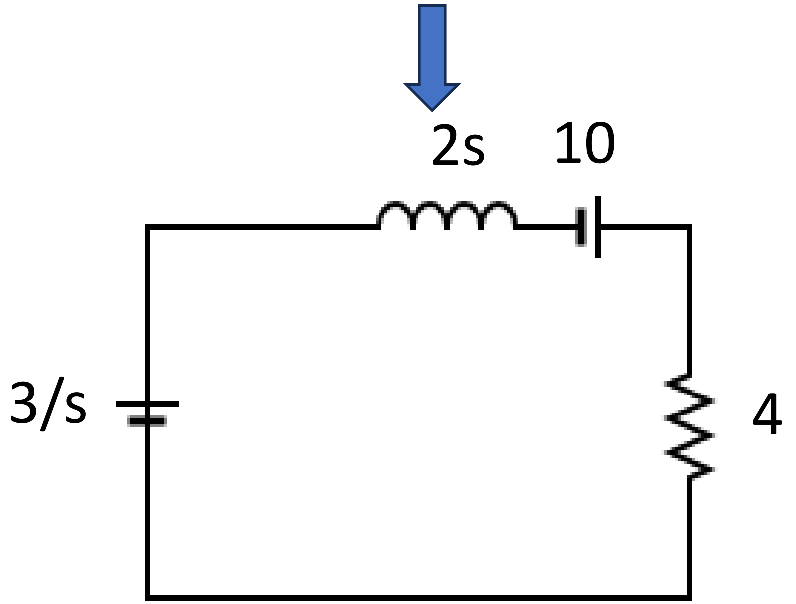
$$I = \frac{1.5}{s(s+2)} + \frac{5}{(s+2)}$$

$$I = \frac{0.75}{s} - \frac{0.75}{(s+2)} + \frac{5}{(s+2)}$$

$$I = \frac{0.75}{s} + \frac{4.25}{(s+2)}$$

- Taking the inverse transform:
  $i(t) = \left(0.75 + 4.25\, e^{-2t}\right)u(t)$





Circuit Analysis

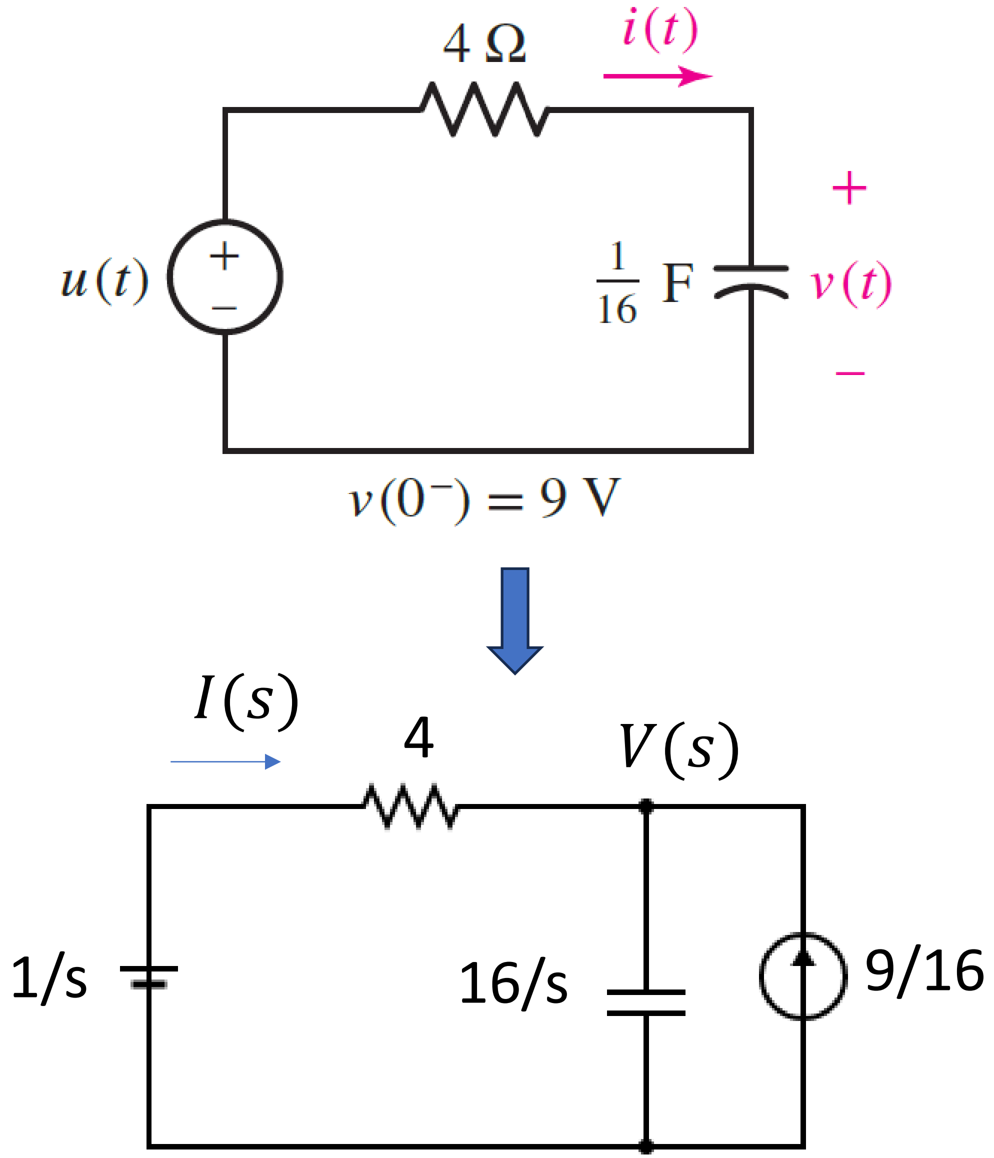
- A series RC circuit with initial voltage on the capacitor
- The s-domain circuit contains a parallel current source

$$\frac{1}{s} = 4I + \frac{16}{s}\left(I + \frac{9}{16}\right)$$

$$-\frac{8}{s} = 4I + \frac{16}{s}I$$

$$I = -\frac{2}{s+4}$$

$$i(t) = -2e^{-4t}u(t)$$



Circuit Analysis

- The voltage is given by:

$$V = \frac{16}{s}\left(I + \frac{9}{16}\right)$$

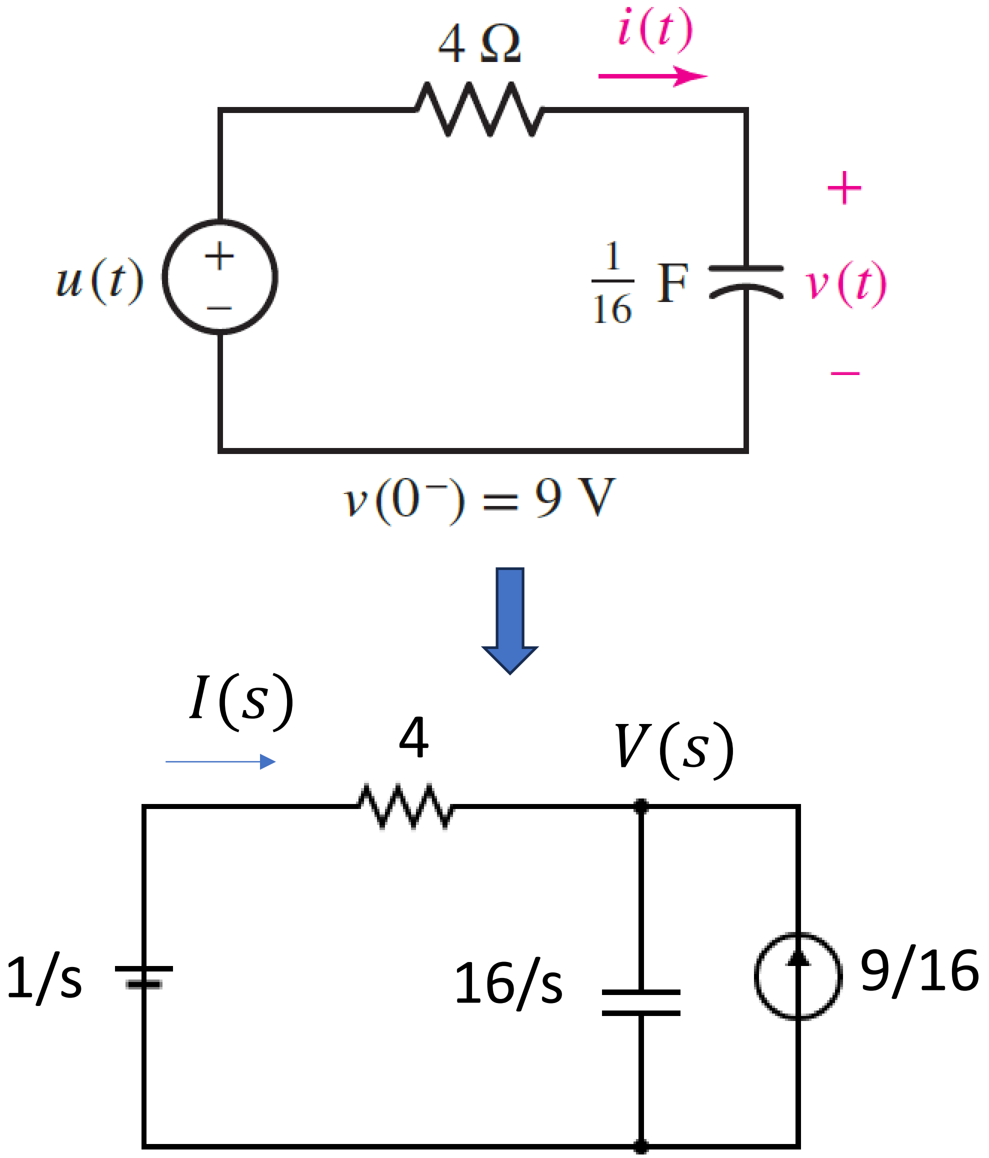
$$V = \frac{16}{s}\left(-\frac{2}{s+4} + \frac{9}{16}\right)$$

$$V = -\frac{32}{s(s+4)} + \frac{9}{s}$$

$$V = -\frac{8}{s} + \frac{8}{s+4} + \frac{9}{s}$$

$$V = \frac{1}{s} + \frac{8}{s+4}$$

$$v(t) = \left(1 + 8e^{-4t}\right)u(t)$$

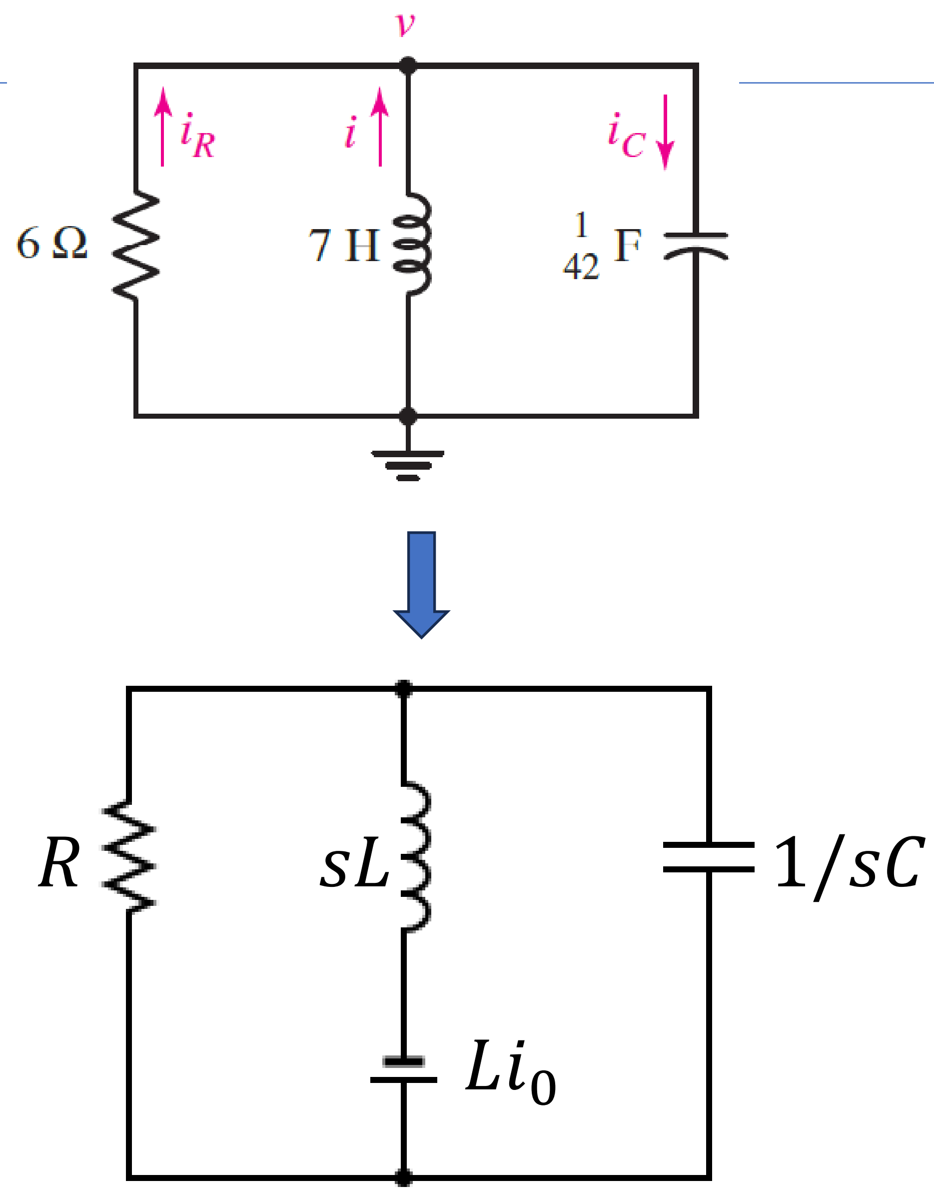


Parallel RLC circuit

- Consider the parallel RLC circuit from before
- We have, $i(0^-) = -10$ (current leaving at node V)
- The equivalent s-domain circuit can be drawn as shown
- Here all the elements are impedances, hence, we can consider RC in parallel as:

$$Z_{RC} = \frac{1}{\frac{1}{R} + sC} = \frac{R}{1 + sRC}$$

- Then the circuit becomes a simple potential divider



Parallel RLC circuit

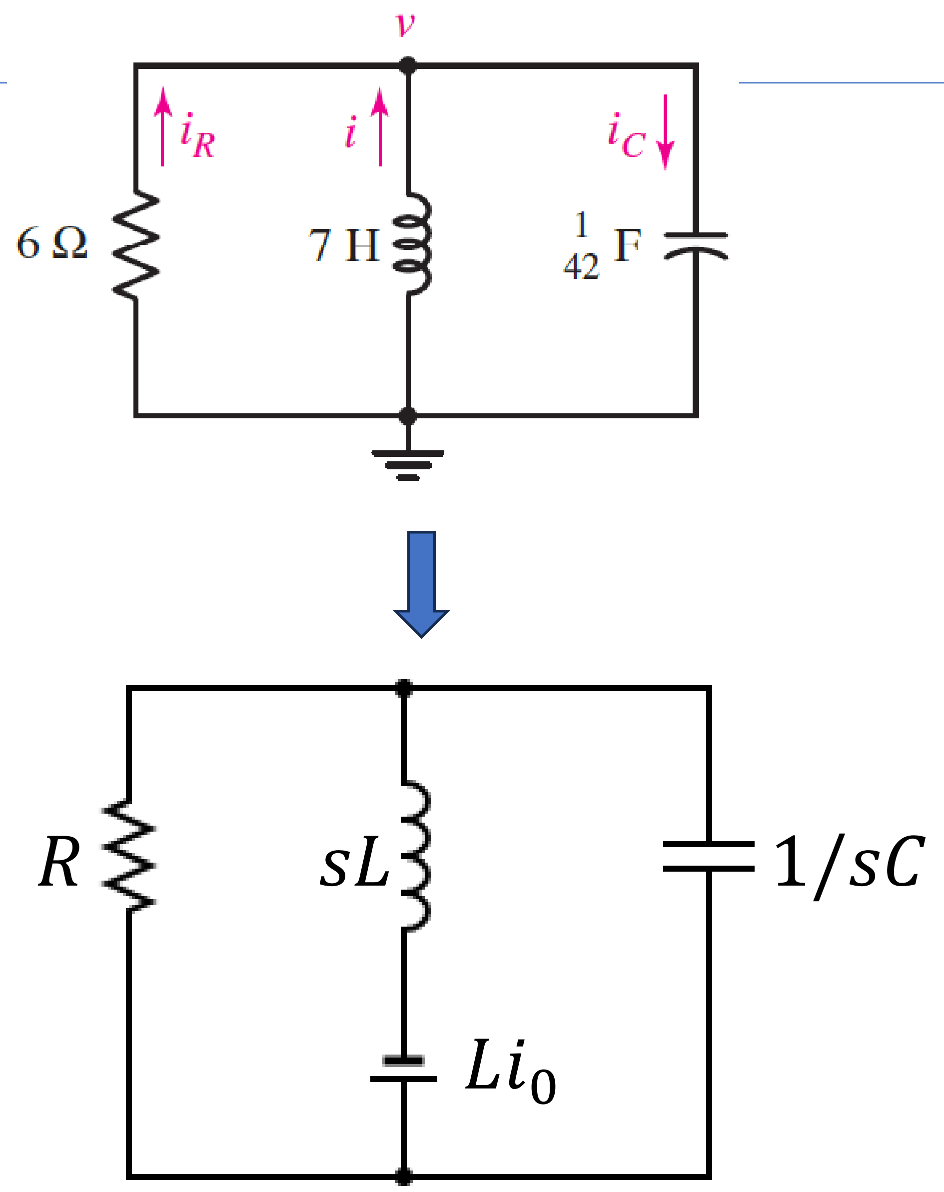
- Thus, the voltage response is given by:

$$V = -Li_0\left(\frac{Z_{RC}}{sL + Z_{RC}}\right)$$

$$V = -Li_0\left(\frac{R}{sL(1 + sRC) + R}\right)$$

$$V = -LRi_0\left(\frac{1}{s^2RLC + sL + R}\right)$$

$$V = -i_0\left(\frac{1}{s^2C + \frac{s}{R} + \frac{1}{L}}\right)$$



Parallel RLC circuit

$$V = -i_0\left(\frac{1}{s^2C + \frac{s}{R} + \frac{1}{L}}\right)$$

- This will have roots:

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

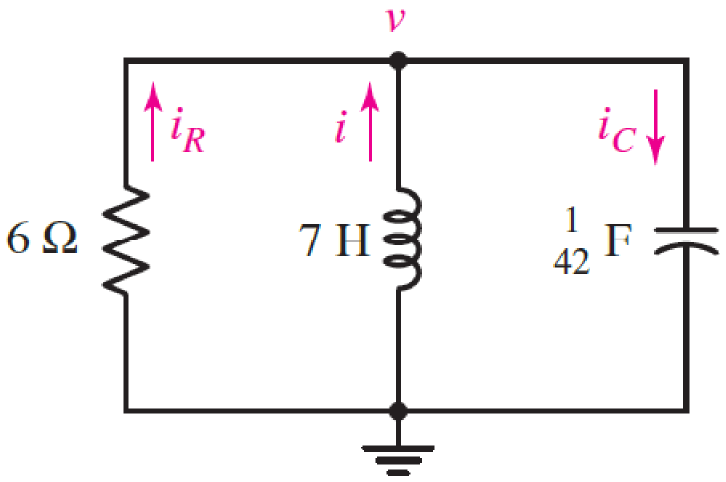
- Real roots if:

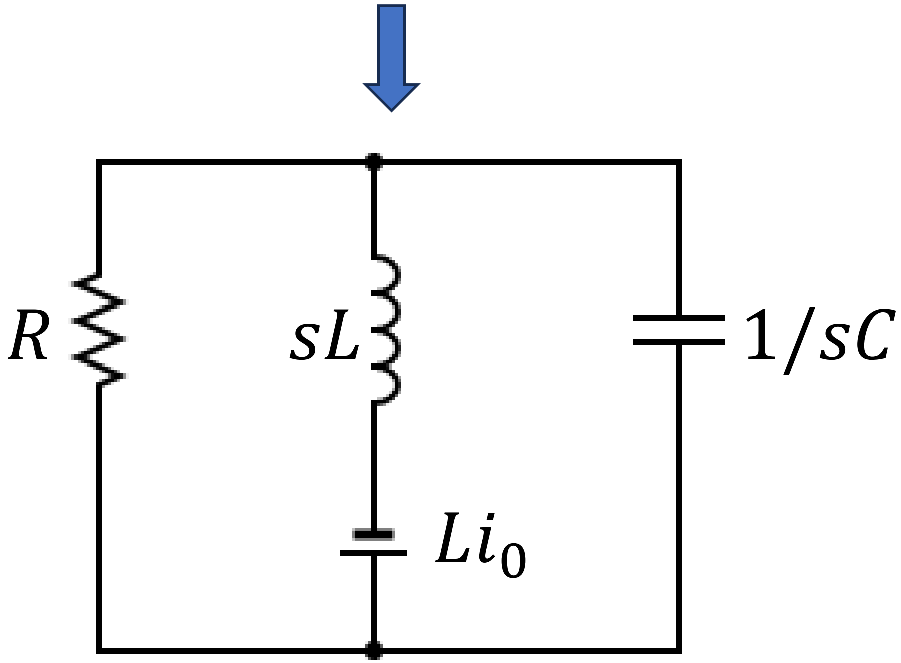
$$\frac{1}{4R^2C^2} > \frac{1}{LC}$$

- For the given values, the becomes:

$$V = \frac{10}{\frac{s^2}{42} + \frac{s}{6} + \frac{1}{7}} = \frac{420}{s^2 + 7s + 6}$$

- Roots are -1 and -6





Parallel RLC circuit

$$V = \frac{420}{(s+1)(s+6)}$$

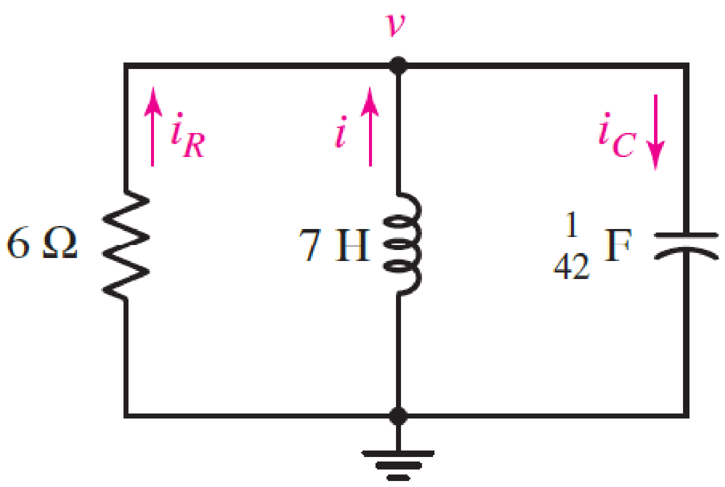
- Using the method of residues:

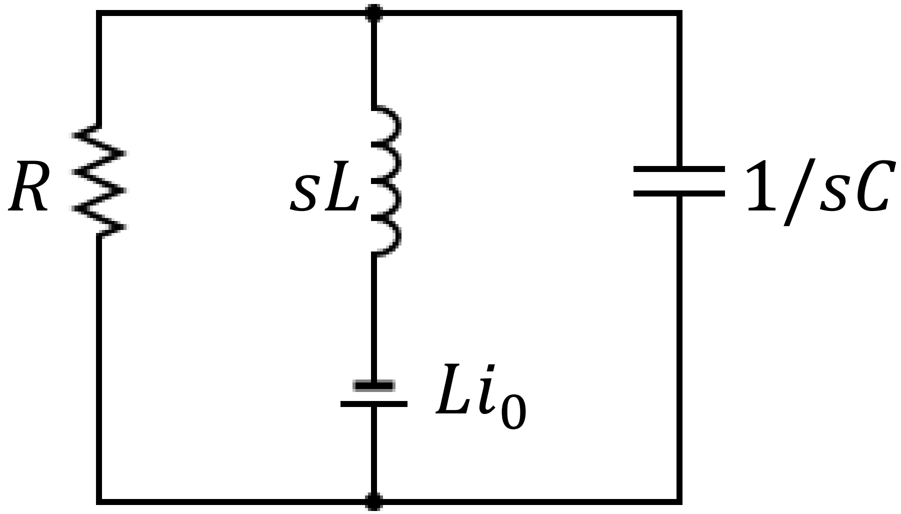
$$V = \frac{84}{s+1} - \frac{84}{s+6}$$

- Thus, the voltage response is:

$$v(t) = 84\left(e^{-t} - e^{-6t}\right)$$

- Initial condition is already taken care of!





Parallel RLC circuit

- Changing the value of R:

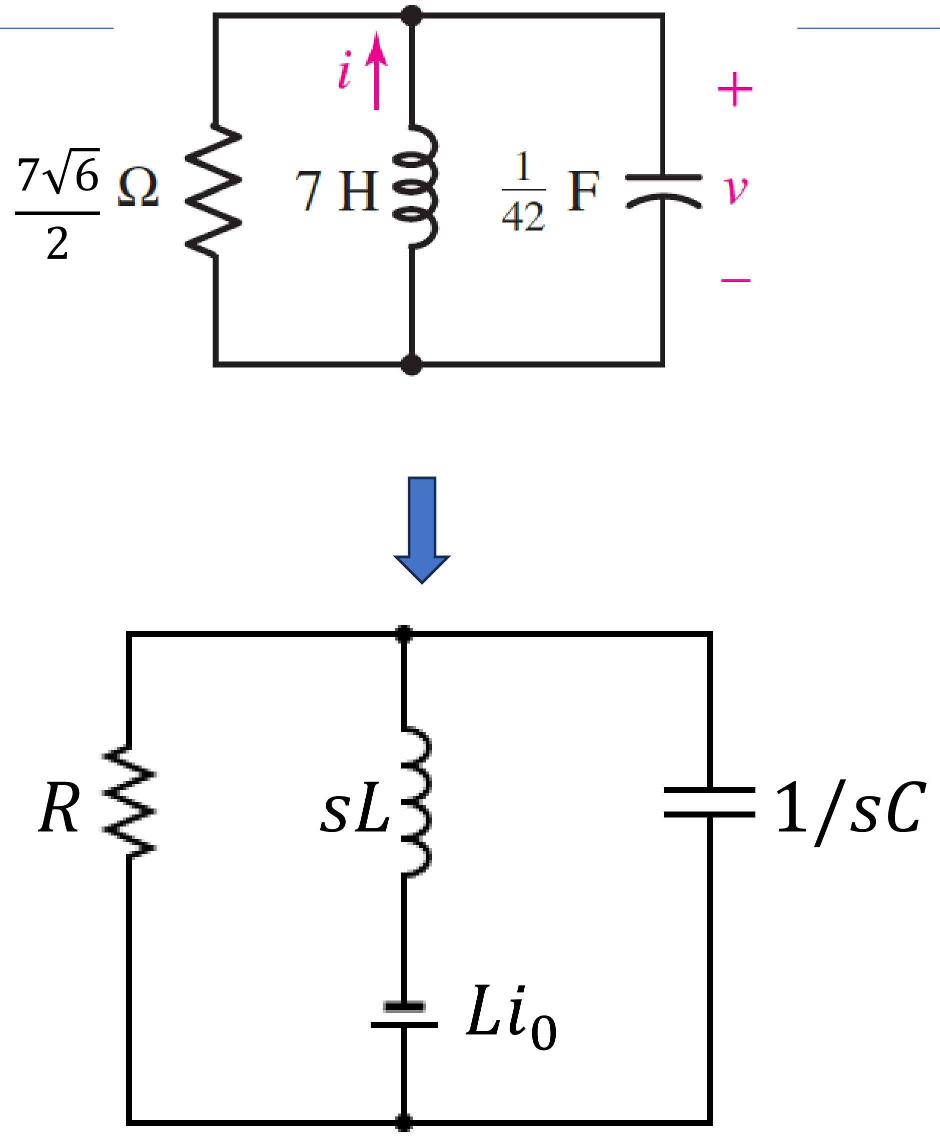
$$V = -i_0\left(\frac{1}{s^2C + \frac{s}{R} + \frac{1}{L}}\right)$$

$$V = \left(\frac{10}{\frac{s^2}{42} + \frac{2s}{7\sqrt{6}} + \frac{1}{7}}\right)$$

$$V = \frac{420}{\left(s + \sqrt{6}\right)^2}$$

Thus, the response voltage is:

$$v(t) = 420te^{-\sqrt{6}t}u(t)$$



$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$

Parallel RLC circuit

- Changing the value of R to 10.5:

$$V = -i_0\left(\frac{1}{s^2C+\frac{s}{R}+\frac{1}{L}}\right)$$

$$V = \left(\frac{10}{\frac{s^2}{42}+\frac{s}{10.5}+\frac{1}{7}}\right)$$

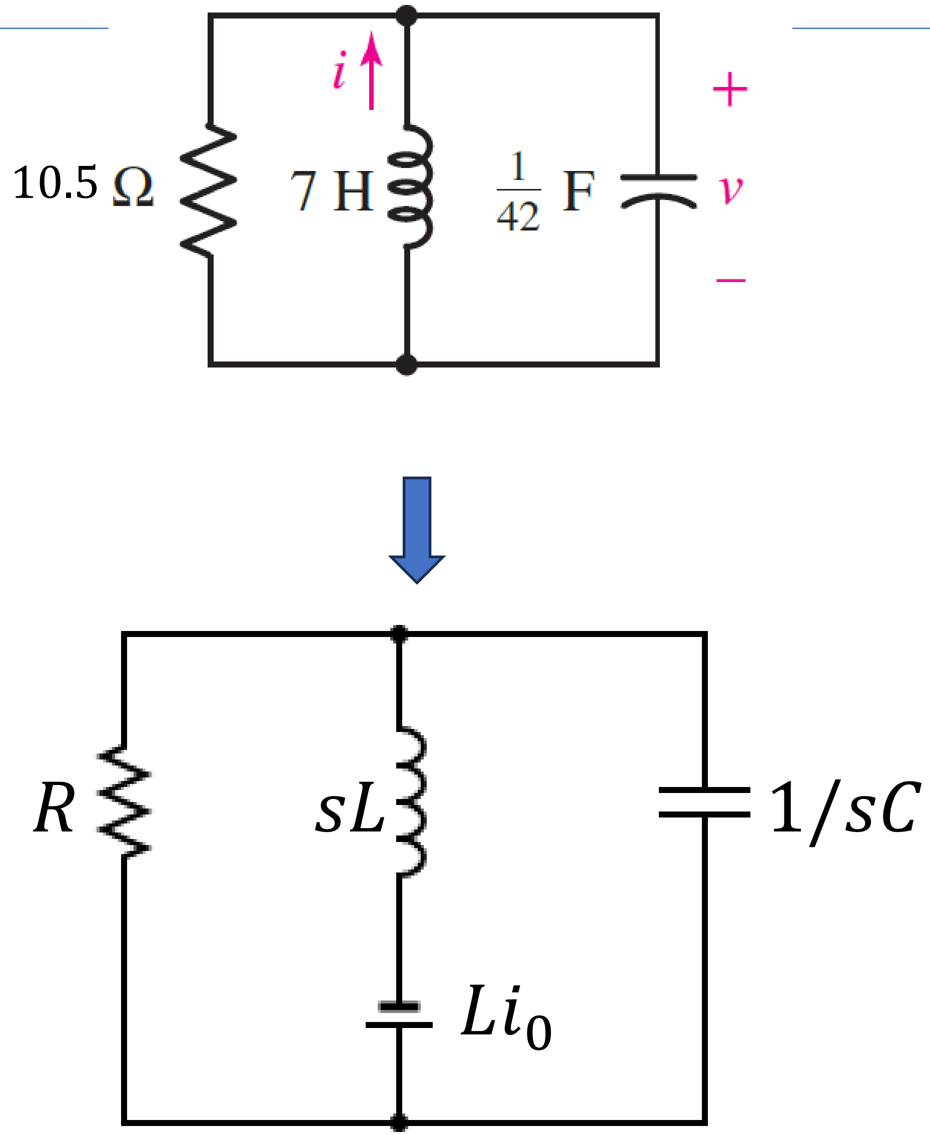
$$V = \frac{420}{s^2+4s+6}$$

No real roots. So, we factorize as:

$$V = \frac{420}{(s+2)^2+2}$$

Thus, the response voltage is:

$$v(t) = 210\sqrt{2}\, e^{-2t} \sin\sqrt{2}t \;\; u(t)$$



$e^{-\alpha t}\sin\omega t\, u(t)$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$

Lossless LC circuit

- Changing the value of R to INF:

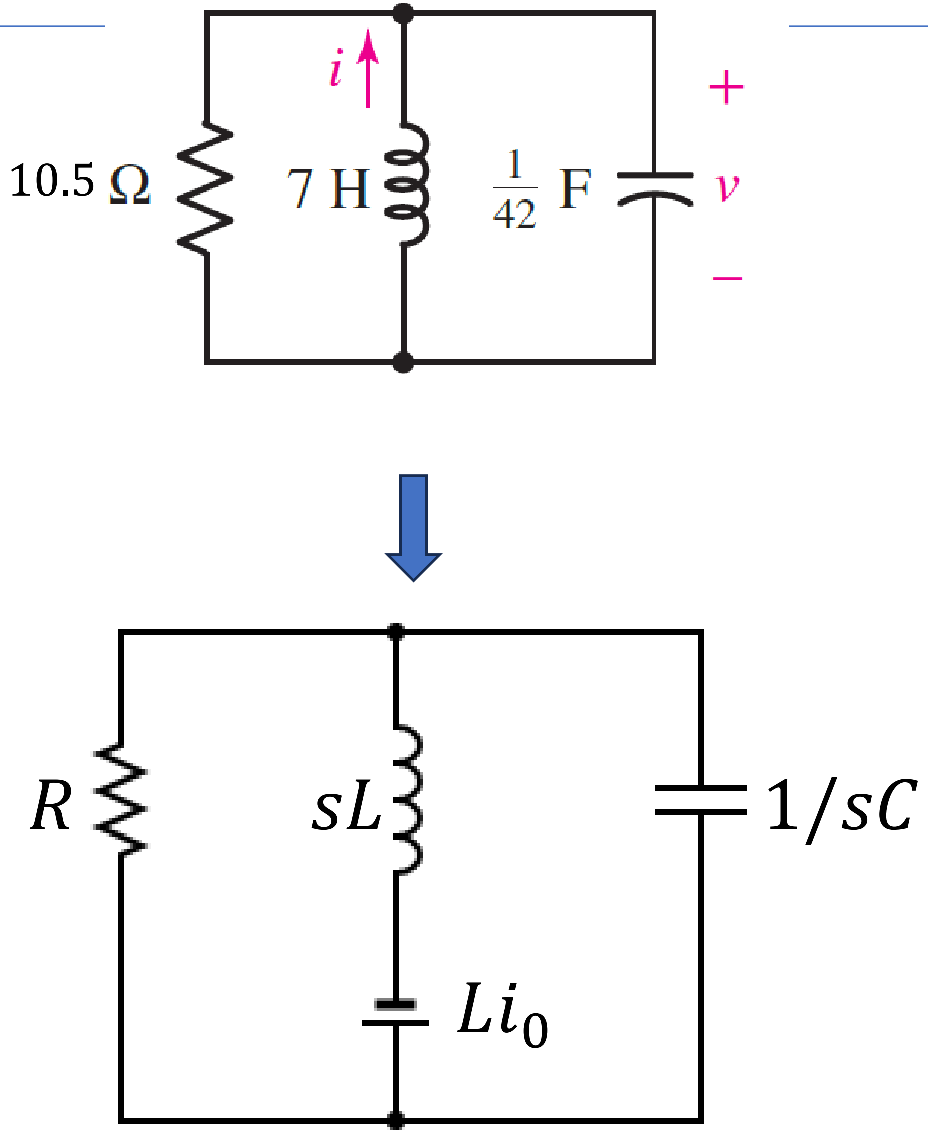
$$V = -i_0\left(\frac{1}{s^2C+\frac{1}{L}}\right)$$

The lossless LC circuit will have the voltage response in the s-domain as:

$$V = -\frac{i_0}{C}\left(\frac{1}{s^2+\frac{1}{LC}}\right)$$

$$V = \frac{420}{s^2+6}$$

$$v(t) = \frac{420}{\sqrt{6}}\sin\sqrt{6}t$$



$e^{-\alpha t}\sin\omega t\,u(t) \qquad \frac{\omega}{(s+\alpha)^2+\omega^2}$