Discrete Structures

Problem Set 2

Release Date: September 13th, 2025
Deadline: September 18th, 2025

Instructions:

- The marks for each question (or sub-question) has been specified alongside it.
- Take care to ensure that if you use variables that have not been defined in the question, you state them clearly, and the domain they belong to clearly as well (if applicable)
- Answer questions precisely, maintaining mathematical rigor as much as possible
- Both digital (Written, Not Typed) and handwritten submissions will be accepted.
- The deadline for this set is September 18th, 11:59 p.m.
- Submissions to be done on moodle with the format:
  ⟨rollnumber⟩_DS_A2 (for example 2023113019_DS_A2)
- Plagiarism and AI use shall be penalized heavily.

Problem 1: (2+2+1=5)

Let $A = \{n \in \mathbb{N} : n \equiv 1 \pmod 3\}$, $B = \{n \in \mathbb{N} : n \equiv 2 \pmod 5\}$, and $C = \{n \in \mathbb{N} : n \text{ is prime}\}$.

(a) Find $A \cap B$ and prove your result.

(b) Determine whether $(A \cup B) \cap C$ is finite or infinite.

(c) Prove that $A \cup B \neq \mathbb{N}$ by finding an explicit element in $\mathbb{N} \setminus (A \cup B)$.

Problem 2: (2+1+2=5)

Consider the set $X = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

(a) Construct $\mathcal{P}(X)$ explicitly.

(b) Prove that $|\mathcal{P}(\mathcal{P}(X))| = 2^{|\mathcal{P}(X)|}$.

(c) Show that there exists no set Y such that $Y \in Y$ using the axiom of regularity.

Problem 3: (2+2+1=5)

Define $f : \mathbb{Q} \to \mathbb{Q}$ by $f(x) = \frac{3x+1}{x+2}$ for $x \neq -2$.

(a) Prove that f is injective.

(b) Determine whether f is surjective by finding its range.

(c) If f is bijective, find $f^{-1}(x)$. If not, modify the codomain to make it bijective.

Problem 4: (1+2+2=5)

Define $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a^2 + b^2 \leq 25\}$.

(a) Prove that R is finite by explicitly bounding its elements.

(b) Calculate $|R|$ by systematic enumeration.

(c) Generalize: prove that for any $k \in \mathbb{N}$, the set $\{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a^2 + b^2 \leq k\}$ is finite.

Problem 5: (2+2+1=5)

Let $E = \{f : \mathbb{N} \to \{0,1\} : f(n) = 0 \text{ for all but finitely many } n\}$.

(a) Prove that E is countably infinite.

(b) Establish a bijection between E and the set of finite subsets of $\mathbb{N}$.

(c) Show that $E \subsetneq \{0,1\}^{\mathbb{N}}$ (E is a proper subset).

Problem 6: (1+2+2=5)

Consider the set $D = \{r \in \mathbb{Q} : 0 < r < 1\}$.

(a) Prove that D is countably infinite.

(b) Construct an explicit bijection between D and $\mathbb{N}$.

(c) Compare $|D|$ with $|\{r \in \mathbb{R} : 0 < r < 1\}|$ and justify your answer.

Problem 7: (5)

Let $M = \{f : \mathbb{N} \to \mathbb{N} : f \text{ is strictly increasing}\}$.

Assume M is countable and list functions as $f_1, f_2, f_3, \ldots$ Define $g(n) = f_n(n) + 1$. Prove that $g \notin \{f_1, f_2, f_3, \ldots\}$ and conclude that M is uncountably infinite. Compare $|M|$ with $|\mathbb{R}|$ using cardinality arguments.