Lecture 14 – Laplace Transform 1

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The Laplace transform

- We transform the voltage function into its frequency domain form using the Laplace transform

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

- For circuit analysis, we typically use the one-sided Laplace (assuming the circuit was "switched on" at $t = 0$)
- Thus,

$$F(s) = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

- Also, Laplace transform follows linearity, ie, the transform of the sum of two time-domain functions is the sum of their transforms

$$\begin{aligned} &\mathcal{L}\big(af_1(t) + bf_2(t)\big) \\ &= a\int_{0^-}^{\infty} e^{-st} f_1(t) dt + b\int_{0^-}^{\infty} e^{-st} f_2(t) dt \\ &= aF_1(s) + bF_2(s) \end{aligned}$$

- Also applies to the inverse Laplace:

$$\begin{aligned} \mathcal{L}^{-1}\big(aF_1(s) + bF_2(s)\big) &= \mathcal{L}^{-1}\big(aF_1(s)\big) + \mathcal{L}^{-1}\big(bF_2(s)\big) \\ &= af_1(t) + bf_2(t) \end{aligned}$$

The Laplace Transform

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})u(t)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$u(t)$	$\frac{1}{s}$	$\sin \omega t \, u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos \omega t \, u(t)$	$\frac{s}{s^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} u(t), n = 1, 2, \ldots$	$\frac{1}{s^n}$	$\sin(\omega t + \theta) \, u(t)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{s + \alpha}$	$\cos(\omega t + \theta) \, u(t)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$	$e^{-\alpha t} \sin \omega t \, u(t)$	$\frac{\omega}{(s+\alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t}u(t), n = 1, 2, \ldots$	$\frac{1}{(s+\alpha)^n}$	$e^{-\alpha t} \cos \omega t \, u(t)$	$\frac{s + \alpha}{(s+\alpha)^2 + \omega^2}$

Time Differentiation Theorem

- A key aspect for circuit analysis is the use of time differentiation/integration of Laplace functions
- Say we have: $\mathcal{L}\big(v(t)\big) = V(s)$
- Then, we can find the Laplace of dv/dt:

$$\mathcal{L}\left\{\frac{dv}{dt}\right\} = \int_{0^-}^{\infty} e^{-st}\,\frac{dv}{dt}\,dt$$

- We define: $A = e^{-st}$ and $dB = \frac{dv}{dt}dt$

$$\mathcal{L}\left\{\frac{dv}{dt}\right\} = v(t)e^{-st}\,\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} v(t)e^{-st}dt$$

- The fact that $V(s)$ exists means that $v(t)e^{-st}$ is zero at infinity
- Thus,

$$\boldsymbol{\mathcal{L}\left\{\frac{dv}{dt}\right\} = sV(s) - v(0^-)}$$

$$\mathcal{L}\left\{\frac{d^2v}{dt^2}\right\} = s^2V(s) - sv(0^-) - v'(0^-)$$

Time Integration Theorem

- A key aspect for circuit analysis is the use of time differentiation/integration of Laplace functions
- Say we have: $\mathcal{L}\big(v(t)\big) = V(s)$
- Then, we can find the Laplace of $\int_{0^-}^{t} v(x)dx$:

$$\mathcal{L}\left\{\int_{0^-}^{t} v(x)dx\right\} = \int_{0^-}^{\infty} e^{-st}\left[\int_{0^-}^{t} v(x)dx\right]dt$$

- We define: $A = \int_{0^-}^{t} v(x)dx$ and $B = -\frac{1}{s}e^{-st}$

$$\mathcal{L}\left\{\int_{0^-}^{t} v(x)dx\right\} = \left\{\left[\int_{0^-}^{t} v(x)dx\right]\left[-\frac{1}{s}e^{-st}\right]\right\}\Big|_{t=0^-}^{t=\infty} + \frac{1}{s}\int_{0^-}^{\infty} v(t)e^{-st}dt$$

- The fact that $V(s)$ exists means that $v(t)e^{-st}$ is zero at infinity, and at $t = 0$, the integral becomes zero
- Thus,

$$\boldsymbol{\mathcal{L}\left\{\int_{0^-}^{t} v(x)dx\right\} = \frac{1}{s}V(s)}$$

Time Shift Theorem

- Say we have a function $f(t)u(t)$ with a Laplace transform $F(s)$
- We want to know the transform of $f(t-a)u(t-a)$ for $a \geq 0$

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} e^{-st} f(t-a)u(t-a)dt$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{a}^{\infty} e^{-st} f(t-a)dt$$

Choosing a new variable of integration, $\tau = t - a$, we get:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} e^{-s(\tau+a)} f(\tau)d\tau$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-sa} \int_{0^-}^{\infty} e^{-s\tau} f(\tau)d\tau = \boldsymbol{e^{-sa}F(s)}$$

Initial Value Theorem

- Say we have a function $f(t)$ with a Laplace transform $F(s)$
- We can know the value of $f(0^+)$ using $F(s)$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt$$

Now, let s approach infinity

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt\right] = \lim_{s\to\infty}\left[\int_{0^-}^{0^+} e^{-st}\frac{df}{dt}dt + \int_{0^+}^{\infty} e^{-st}\frac{df}{dt}dt\right]$$

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}\left[\int_{0^-}^{0^+} \frac{df}{dt}dt + \int_{0^+}^{\infty} e^{-st}\frac{df}{dt}dt\right]$$

As $s \to \infty$, the second integration vanishes for all values of t

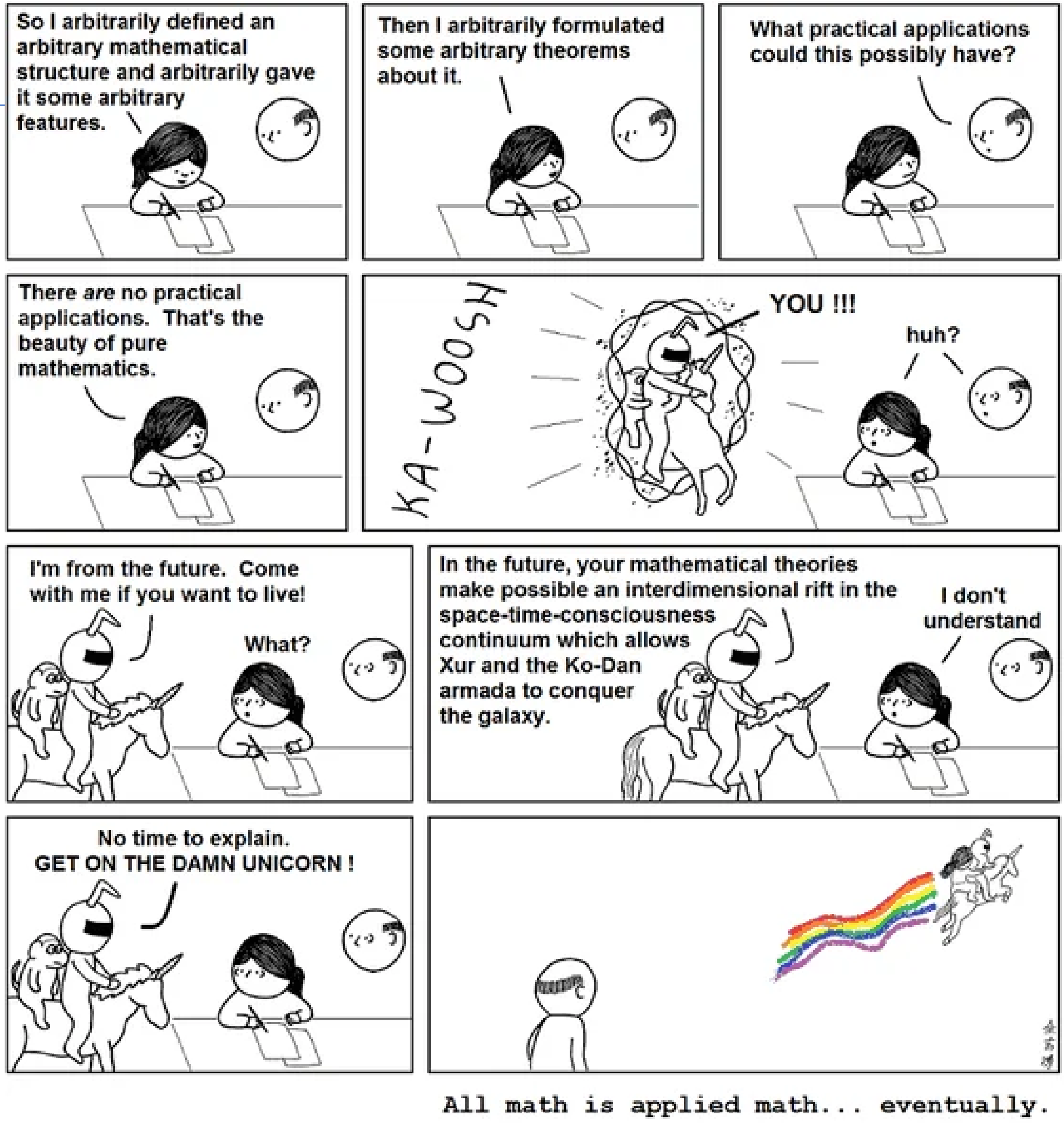
Thus,

$$\lim_{s\to\infty}[sF(s)] - f(0^-) = \lim_{s\to\infty}[f(0^+) - f(0^-)] = f(0^+) - f(0^-)$$

$$\boldsymbol{f(0^+) = \lim_{s\to\infty}[sF(s)]}$$

Back to Circuit Analysis

- Surely, Laplace did not know how useful his transform will be in circuit analysis!
- Because we can now use the s-domain as a method to get away from complex integrodifferential equations, and solve them using algebraic expressions
- Not only that, we can also get insights into a circuit simply by writing down its “transfer function” in the s-domain



Impedance

- In general, impedance is the ratio of applied voltage to current for a given circuit element
- We can define impedance in the s-domain as:

$$Z(s) = \frac{V(s)}{I(s)}$$

- For instance, for a resistor, the current-voltage relationship is given by:

$$v(t) = Ri(t)$$

- Thus,

$$V(s) = RI(s)$$

- Thus, for a resistor,

$$Z(s) = \frac{V(s)}{I(s)} = R$$

- Note, the units of $V(s)$ and $I(s)$ are volt-second and ampere-second respectively. $Z(s)$ is in Ohms

Impedance

- Now, for an inductor, the current-voltage relationship is given by:

$$v(t) = L\frac{di}{dt}$$

- Taking the Laplace transform of both sides:

$$V(s) = L\,\mathcal{L}\left\{\frac{di}{dt}\right\}$$

$$V(s) = L\left(sI(s) - i(0^-)\right)$$

- For zero initial current in the inductor,

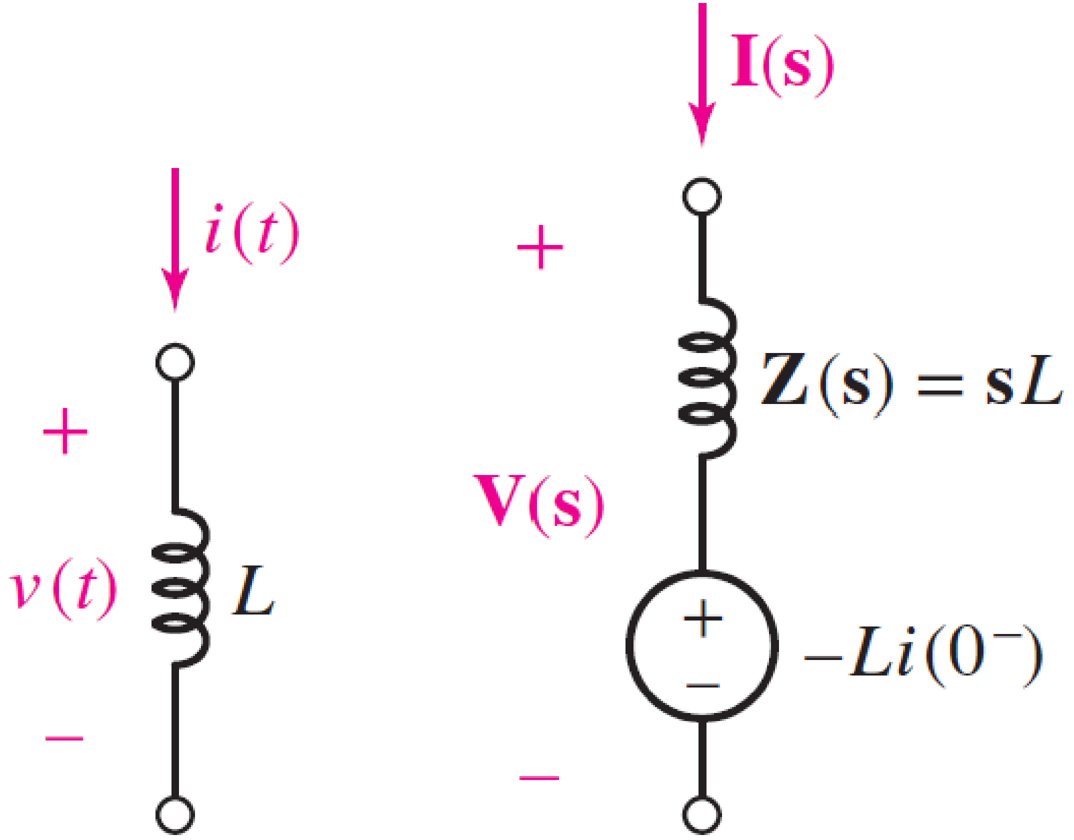
$$V(s) = sLI(s)$$

$$Z(s) = \frac{V(s)}{I(s)} = sL$$

Impedance

$$V(s) = L\left(sI(s) - i(0^-)\right)$$

- Because this expression has two terms, with the second term independent of s
- Thus, we can model this as an impedance in series with a constant voltage source
$$V(s) = sLI(s) - Li(0^-)$$
- In this case, the inductor is assumed to have zero initial current and simply an impedance of sL



Impedance

- Now, for a capacitor, the current-voltage relationship is given by:

$$i(t) = C\,\frac{dv}{dt}$$

- Taking the Laplace transform of both sides:

$$I(s) = C\,\mathcal{L}\left\{\frac{dv}{dt}\right\}$$

$$I(s) = C\left(sV(s) - v(0^-)\right)$$

- For zero initial voltage on the capacitor,

$$I(s) = sCV(s)$$

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{sC}$$

Impedance

$$I(s) = C\left(sV(s) - v(0^-)\right)$$

- For complete modelling, the constant term can be put as an additional current source in parallel to the capacitor
- It can also be modelled as a series voltage source with voltage $v(0^-)/s$
- ***<u>With these transformations for resistor, inductor and capacitor, any RLC circuit becomes a combination of impedances and ideal sources – only linear equations and no integrodifferential equations!</u>***