Lecture 11 – RLC circuit

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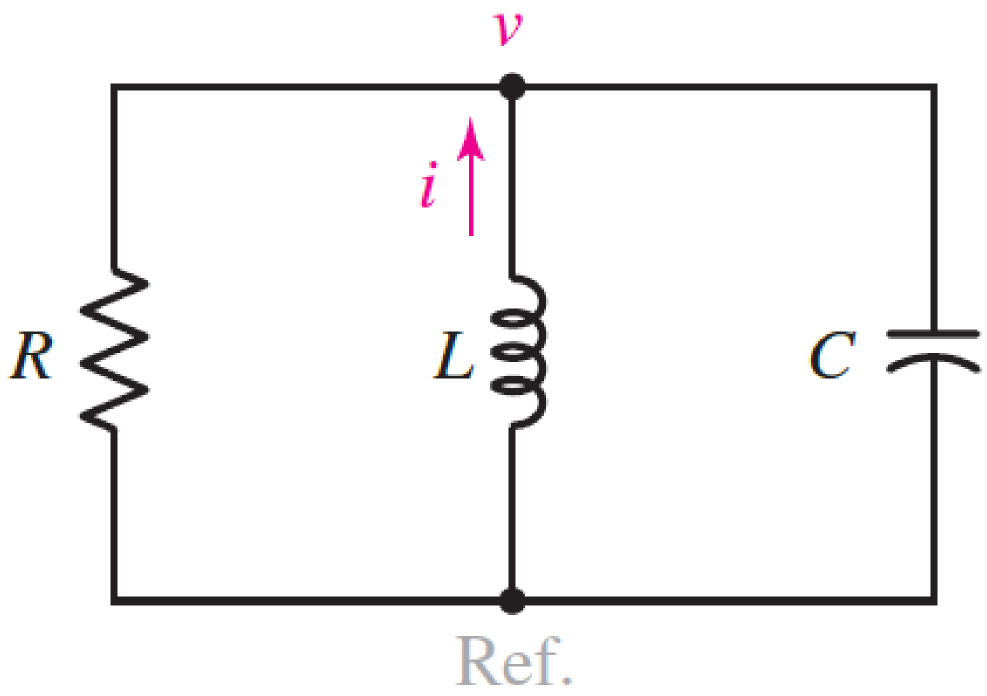
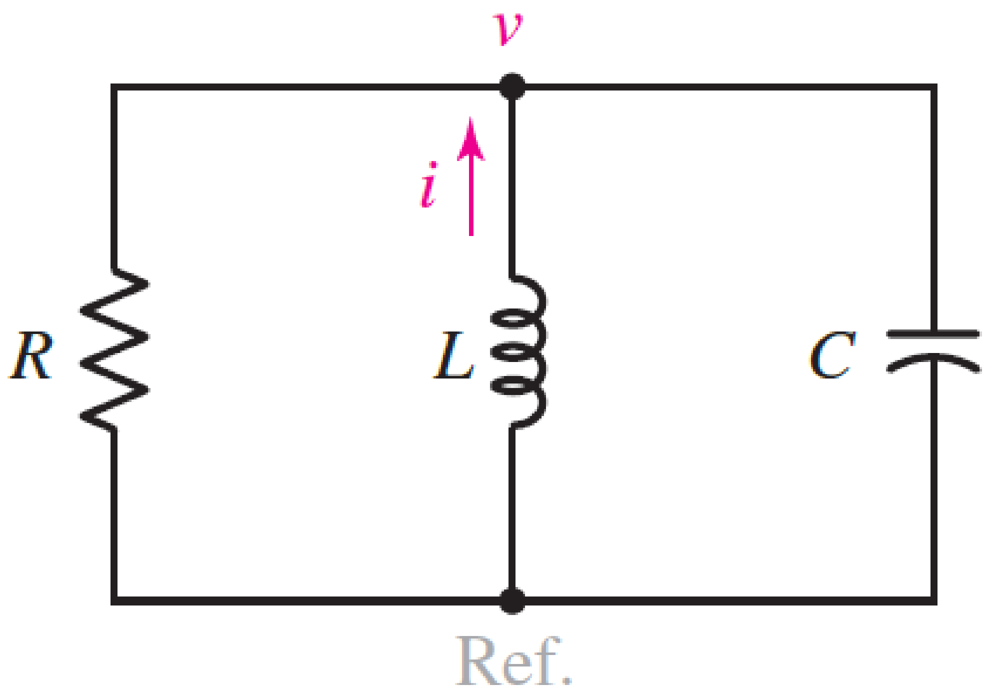
Parallel RLC circuit

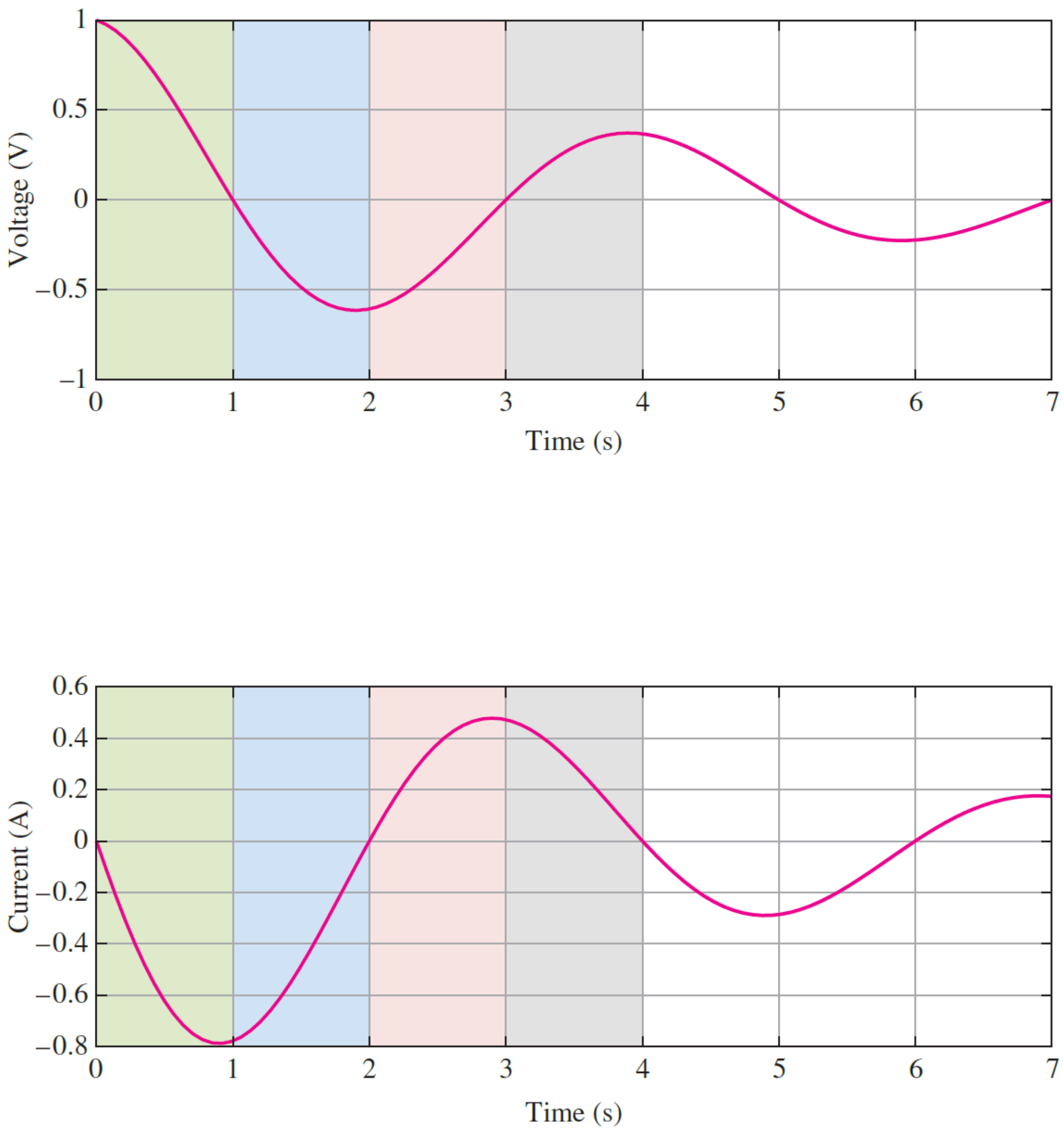
- We start by connect the three basic passive linear elements in a simple parallel circuit
- We assume that there is no applied voltage
- This is a fundamental circuit used everywhere from wireless communication to signal processing
- The R in the circuit can also be assumed to be the non-ideality associated with the L and C elements in an LC circuit



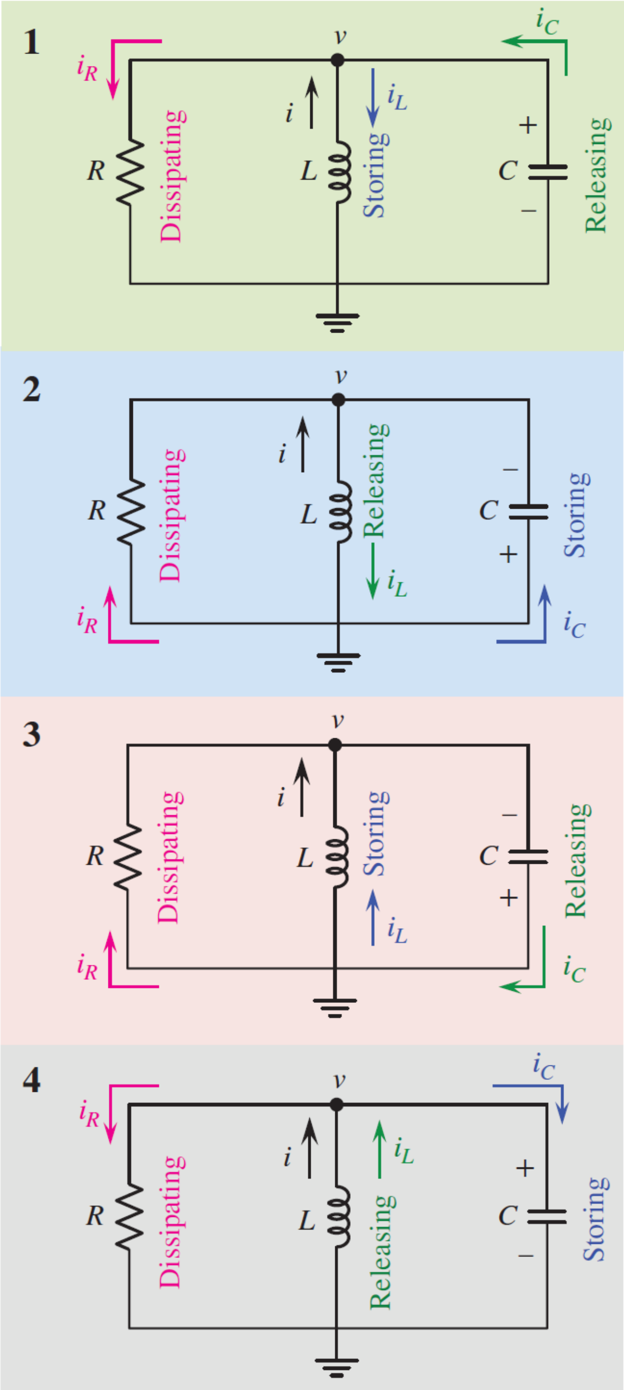
Parallel RLC circuit

- We can assume that there is prior voltage in the capacitor and current in the inductor, at $t = 0$
- If that is so, the capacitor will initially discharge through the resistor and try to build up a current through the inductor
- In time, there will be no charge in the capacitor, however, some current buildup in the inductor that will try to "discharge" through the capacitor, thus providing some charge back to the capacitor
- During this entire time, the resistance is dissipating energy because of the current through it





Voltage (V)
1
0.5
0
−0.5
−1
0 1 2 3 4 5 6 7
Time (s)
Current (A)
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0 1 2 3 4 5 6 7
Time (s)

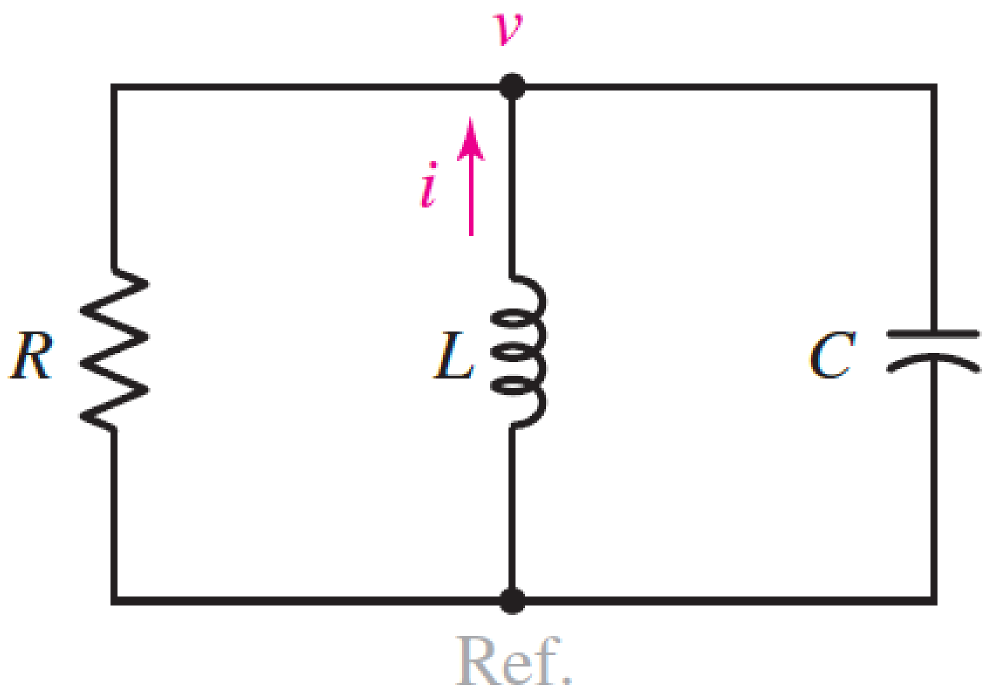


1
v i_R i i_L i_C
R Dissipating
L Storing
C Releasing
2
v i i_R i_L i_C
R Dissipating
L Releasing
C Storing
3
v i i_R i_L i_C
R Dissipating
L Storing
C Releasing
4
v i_R i i_L i_C
R Dissipating
L Releasing
C Storing

Analysis

- For the single node:
$$\frac{v}{R} + \frac{1}{L}\int v\,dt - i(t_0) + C\frac{dv}{dt} = 0$$
- Initial conditions being $i(0) = I_0$ and $v(0) = V_0$
- We can differentiate the equation to get a second-order homogenous differential:
$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$
- We assume the solution to be of the form: $v = Ae^{st}$

*A and s can be complex numbers



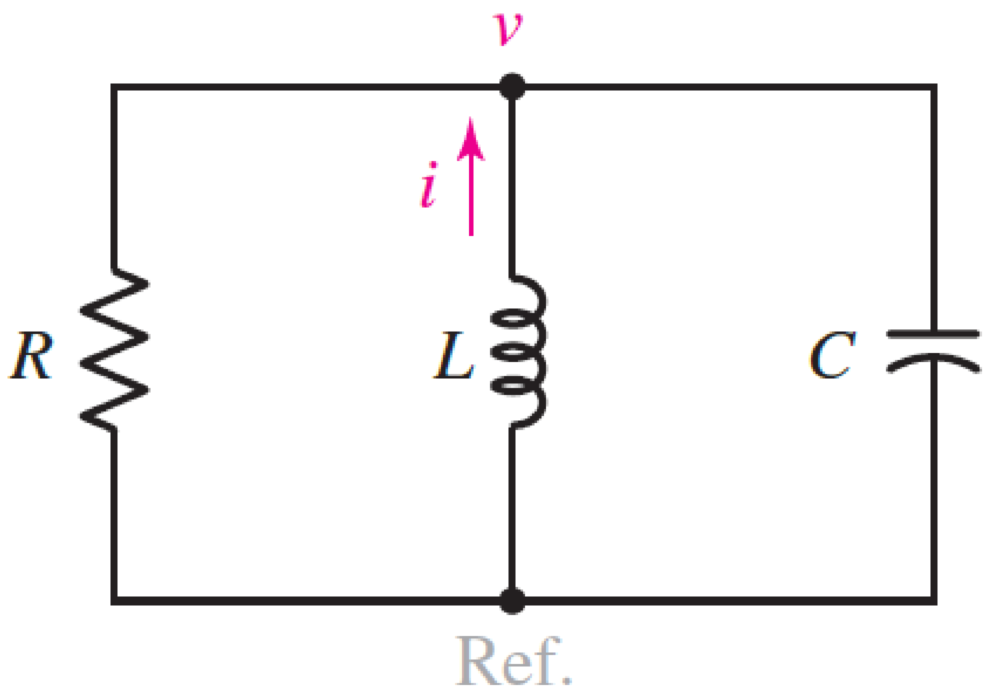
Analysis

- Applying our general solution to the equation, we get:

$$C\frac{d^2(Ae^{st})}{dt^2}+\frac{1}{R}\frac{d(Ae^{st})}{dt}+\frac{1}{L}(Ae^{st})=0$$

$$\left(Ae^{st}\right)\left[Cs^2+\frac{1}{R}s+\frac{1}{L}\right]=0$$

- Either:
  - $A=0$
  - $s=-\infty$
  - The bracket term is zero
- The first two cannot be true because that will mean $v=0$ for all values of t!



Analysis

- Thus, for a non-trivial solution to exist,

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

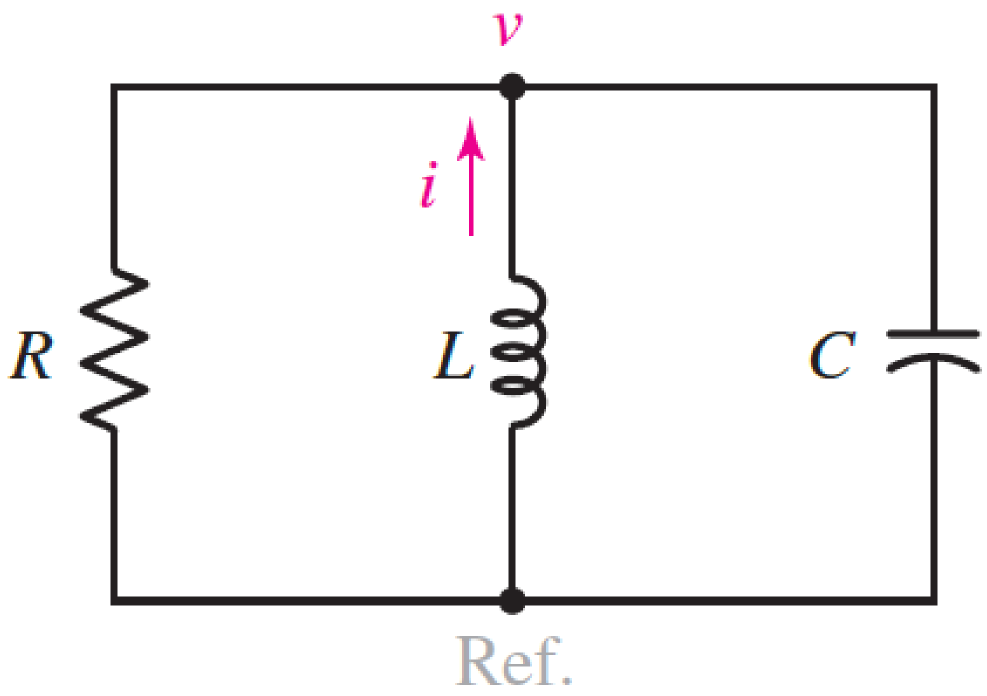
- We get two solutions:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- Because $v = Ae^{st}$ satisfies the original differential with both these values, the general solution is:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



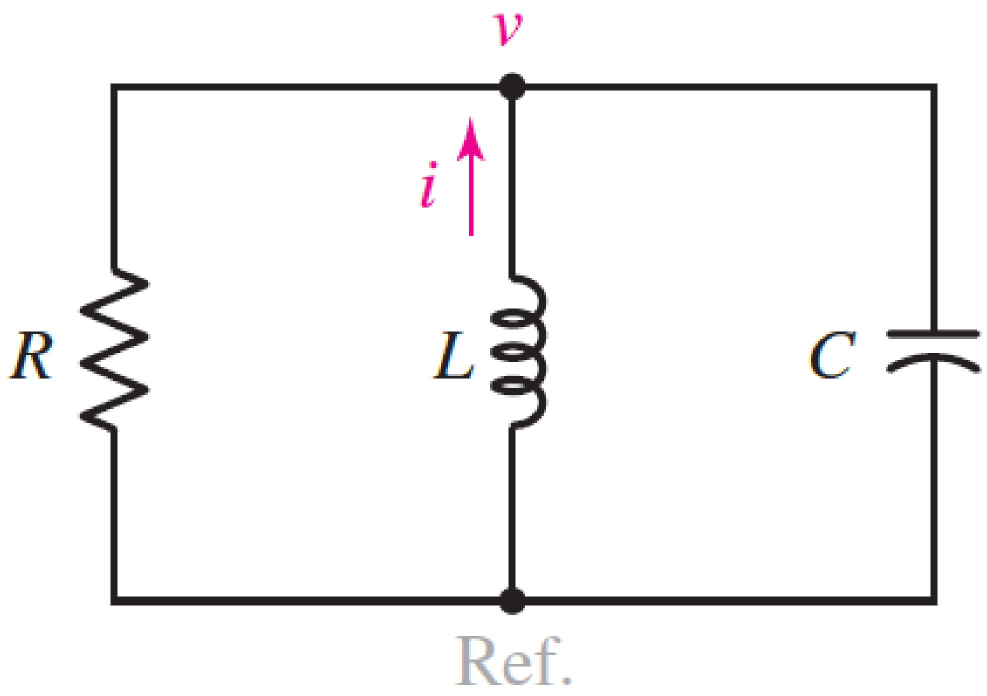
Analysis

- We have the solution

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

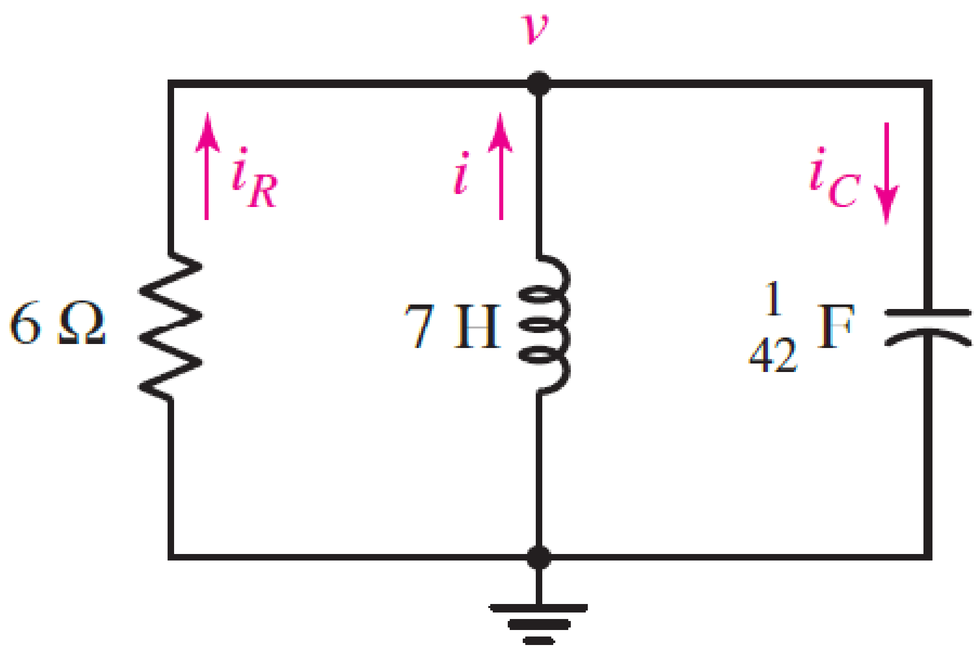
$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- However, this is not easy to visualize or to draw conclusions from
- We start to make cases – say $s_{1,2}$ are real, i.e., $\frac{1}{4R^2C^2} > \frac{1}{LC}$
- We realize that $s_{1,2}$ are *negative* real numbers
- This means that as $t \to \infty$, the value of $v \to 0$ for all values of R, L, C that satisfy $\frac{1}{4R^2C^2} > \frac{1}{LC}$



Analysis

- Now, to obtain the final response, we need to know the values of $A_{1,2}$
- We take the values of R, L and C to obtain the values of s
- $s_1 = -1$ and $s_2 = -6$
- We also assume that there was no initial charge in the capacitor and a 10 A current in the inductor
- The voltage response is given by:
$$v = A_1 e^{-t} + A_2 e^{-6t}$$
- Because there was no initial voltage, $A_1 + A_2 = 0$



Analysis

- We use the initial current condition
- Say we take the derivative of v at $t = 0$

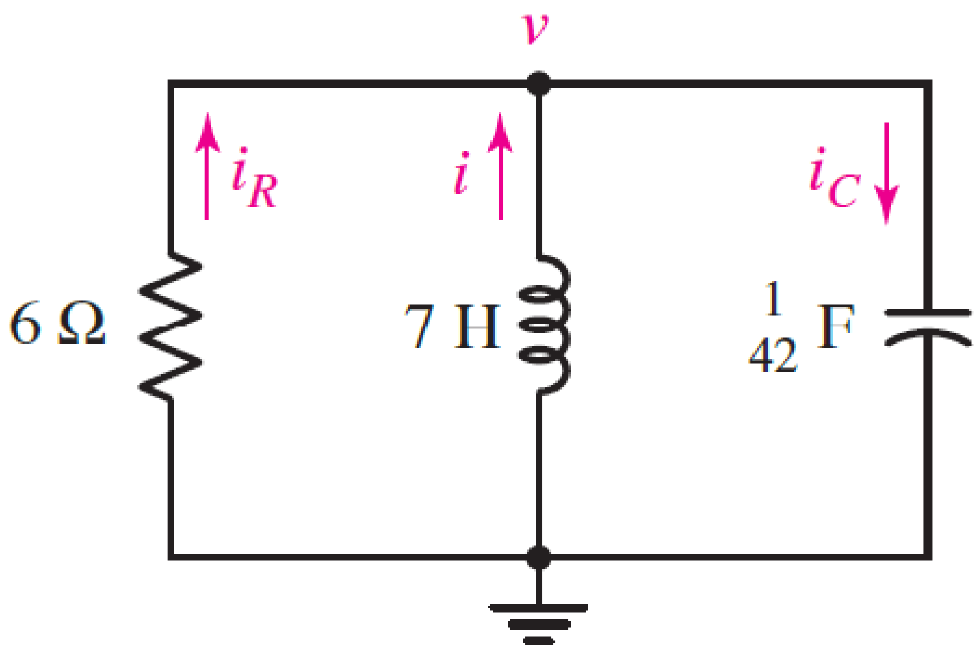
$$\frac{dv}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$$

$$\left.\frac{dv}{dt}\right|_{t=0} = -A_1 - 6A_2$$

- But we don't know the value of this derivative at $t = 0$
- What we know is:

$$i_c = C\frac{dv}{dt}$$

$$i_c(0) = C\left.\frac{dv}{dt}\right|_{t=0}$$



Analysis

- We know the initial current in the inductor was 10 A
- Thus,

$$i(0) = i_c(0) + i_R(0)$$

- The current through the resistance at $t = 0$ is zero

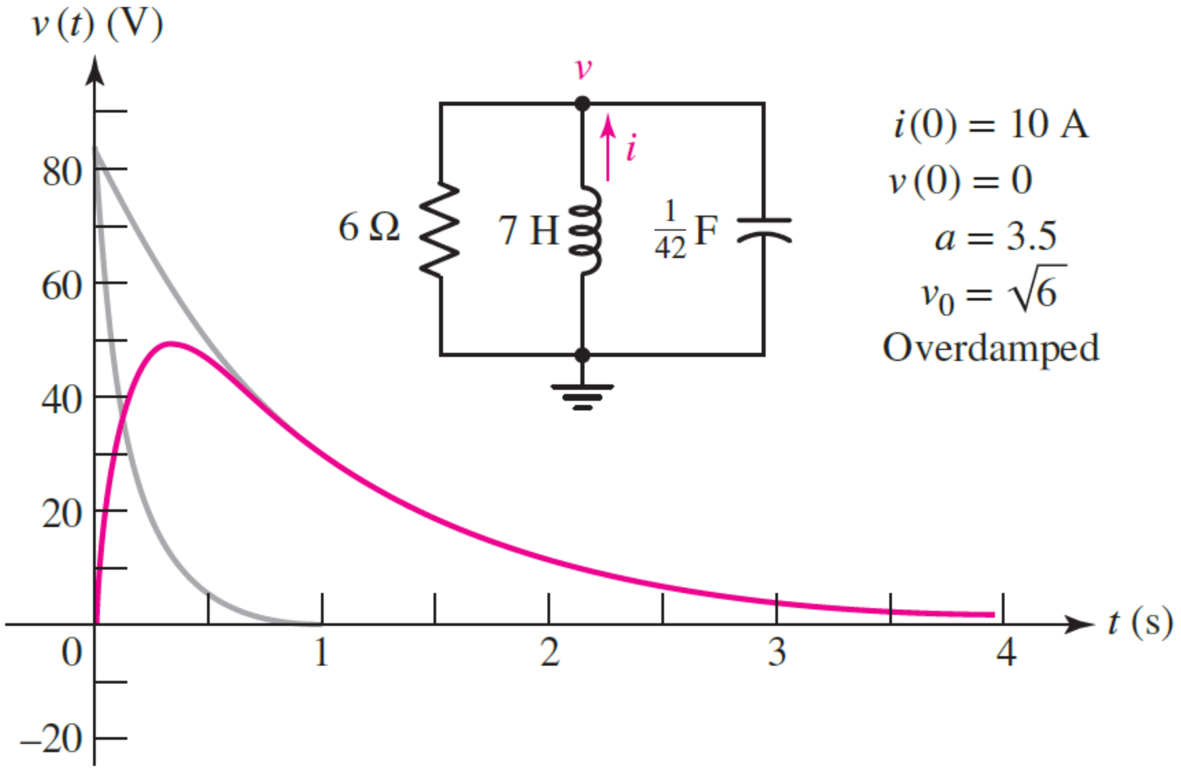
$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{1}{C} i_0 = 420 = -A_1 - 6A_2$$

- Thus, we get $A_1 = -A_2 = 84$

$$v = 84(e^{-t} - e^{-6t})$$

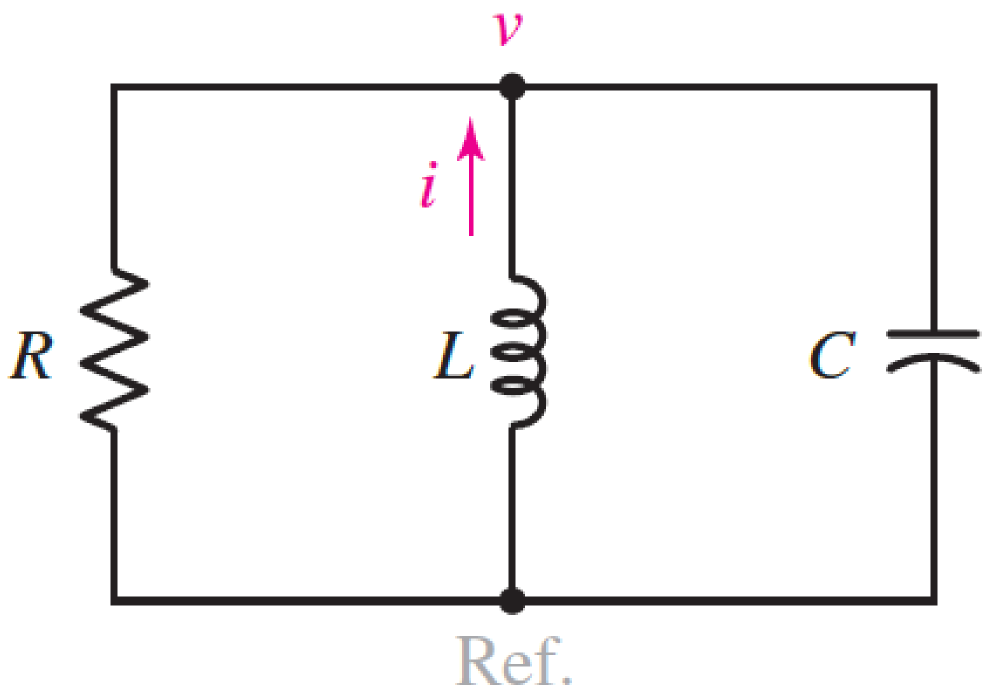
- $t_m = 0.358\, s$ and $v_m = 48.9\, V$
- Such a system is called "**overdamped**" system

$$4R^2C^2 < LC$$



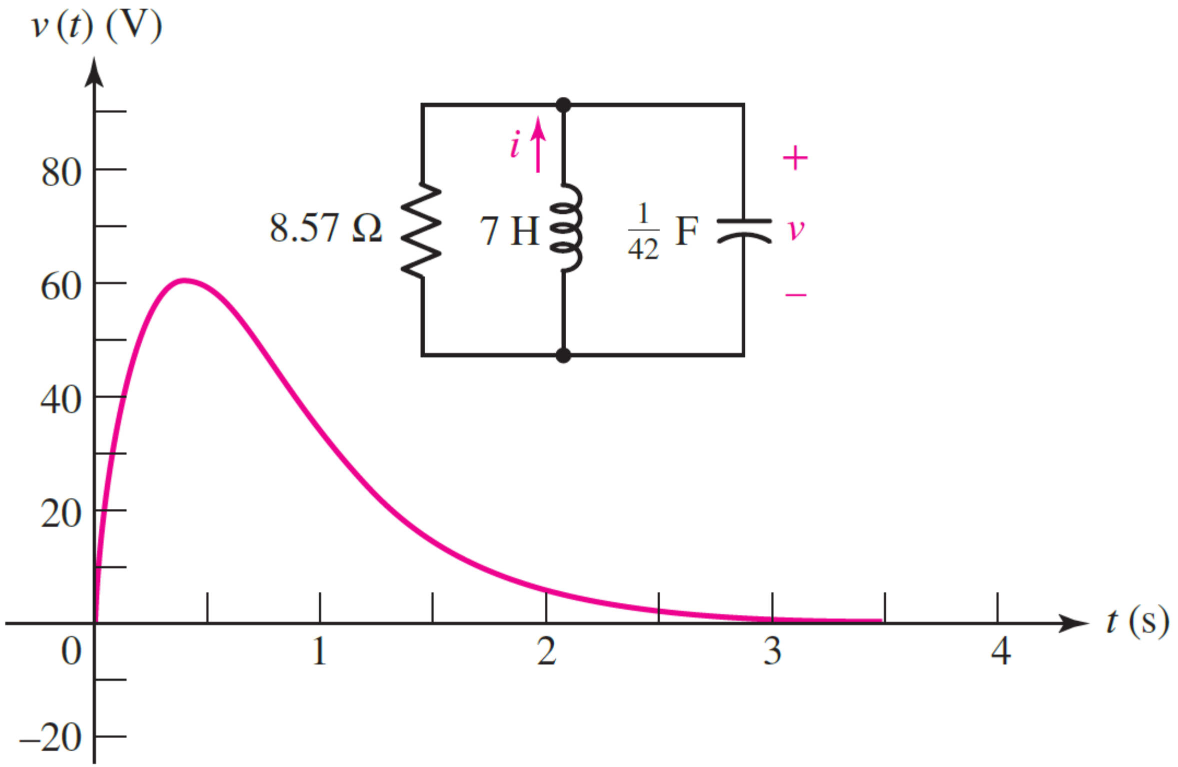
Analysis -2

- The next assumption in our pursuit of the solution is if:
$$4R^2C^2 = LC$$
- In this case, s has only one value
$$s = \frac{1}{2RC}$$
- And the general solution looks like:
$$v = Ae^{st}$$
- For our first initial condition, $v(0) = 0$, we get $A = 0$, thus,
$$v(t) = 0$$
- WAIT, WHAT?



Analysis -2

- For this case, the general form of the solution is given by:
$$v = A_1 t e^{st} + A_2 e^{st}$$
- Say we make our circuit such that it follows the $4R^2C^2 = LC$ condition
- We put $R = \frac{7\sqrt{6}}{2}$, we get $s = -\frac{1}{\sqrt{6}}$
- We can again use the two boundary conditions to find the values of A
- We get:
$$v = 420te^{-\frac{t}{\sqrt{6}}}$$
- This system is called "**critically damped**"



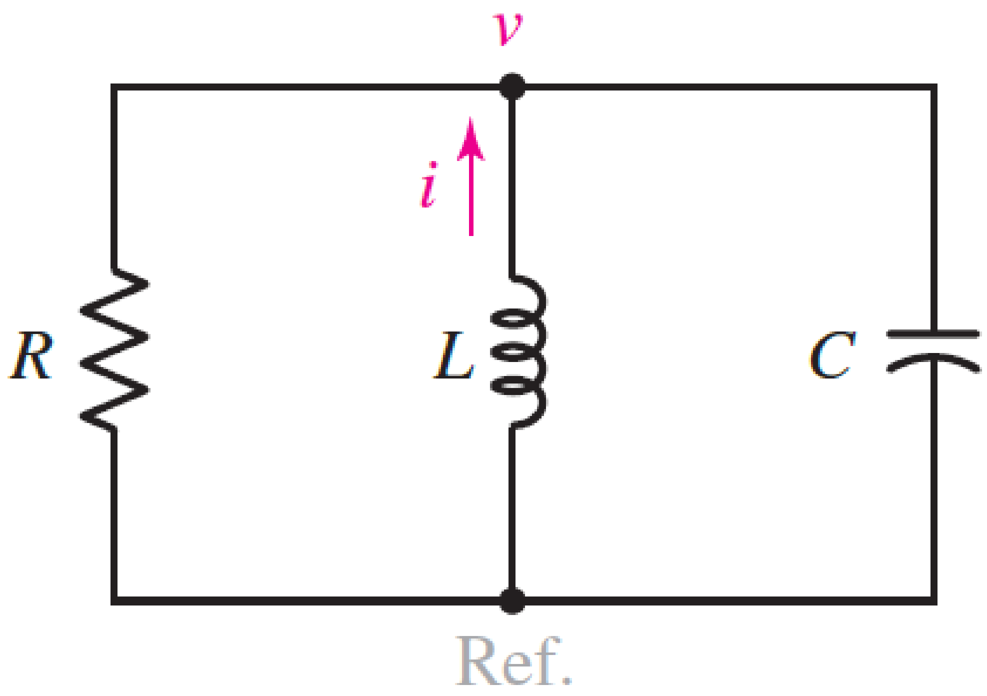
Analysis - 3

- Finally, we look at the case when $4R^2C^2 < LC$
- In this case, both s values are imaginary and complex conjugates
- We can now define some terms for better analysis of this problem
- We define:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\alpha = \frac{1}{2RC}$$

- Thus, $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
- We can define $\omega = \sqrt{\omega_0^2 - \alpha^2}$



Analysis - 3

- Thus, the general solution to the differential equation becomes:
$$v = e^{-\alpha t}\left(A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right)$$