International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Practice Problems

Question (1) Induction isn't just for sums. It lets us compare growth rates. The fact that exponentials beat polynomials is central in analysis. Prove that for every $n \in \mathbb{N}$, we have

$$n < 2^n.$$

Question (2) Recursive definitions often hide beautiful closed formulas. This problem shows how induction transforms a recursive process into a neat algebraic expression. Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(0) = 0, \quad f(n+1) = f(n) + (n+1).$$

Find a closed formula for $f(n)$ and prove its correctness.

Question (3) Recursive sequences often grow quickly, but how quickly? Induction lets us compare Fibonacci with exponentials, foreshadowing analysis of growth rates. Let $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$. Prove that

$$F_n < 2^n \quad \text{for all } n \geq 1.$$

Question (4) The rationals look "sparse," but they're so densely packed that between any two rationals lie infinitely many others. This prepares us for how rationals approximate the reals. Answer the following questions. (a) Between any two integers $a < b$, show there are infinitely many rationals. (b) Between any two rationals $p < q$, show there are infinitely many rationals.

Question (5) A set T is called countable if there is a bijective mapping from $\mathbb{N}$ to

T. Further, the union of countable sets is countable. The set T is said to be dense in another set S iff for each $a, b \in S$ with $a < b$, there exists at least one element $x \in T$ such that $a < x < b$. The rationals of the form $p/2^k$ (dyadic rationals) are countable, yet dense. They foreshadow binary expansions and approximations to real numbers. Let

$$T = \left\{ \tfrac{p}{2^k} : p \in \mathbb{Z}, k \in \mathbb{N} \right\}.$$

(a) Show that T is countable. (b) Show that T is dense in $\mathbb{Q}$.

Question (6) Prove that the set $\mathbb{Z}$ of integers is not dense in set $\mathbb{Q}$ of rationals.

Question (7) Infinity behaves differently: removing infinitely many elements may leave a set "the same size." This problem illustrates that infinite sets do not follow finite intuition. Let

$$S = \{\, n \in \mathbb{N} : n \text{ is not a multiple of } 3 \,\}.$$

Show that S has the same cardinality as $\mathbb{N}$.

Question (8) With infinitely many pigeons and finitely many holes, at least one hole contains infinitely many pigeons. This principle underlies compactness arguments later. Let $f : \mathbb{N} \to \{1, 2, 3, 4, 5\}$. Prove that some value in $\{1, 2, 3, 4, 5\}$ is taken infinitely often by f.
[*Hint: Partition* $\mathbb{N}$ *into subsets* $S_i = \{n : f(n) = i\}$ *for* $i = 1, \ldots, 5$.]

Question (9) Telescoping and binomial bounds prepare us for convergence. This is the first glimpse of analysis of infinite processes. Let $0 < r < 1$ be rational. Prove that for each $n \geq 2$, show that

$$0 < 1 - (1 - r)^n < nr.$$

Question (10) Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ with $p, r \in \mathbb{Z}$ and $q, s > 0$. Prove that if $\frac{p}{q} \neq \frac{r}{s}$, then

$$\left| \frac{p}{q} - \frac{r}{s} \right| \geq \frac{1}{qs}.$$

Write the condition for equality in above inequality.