

International Institute of Information Technology, Hyderabad  
(Deemed to be University)  
MA4.101-Real Analysis (Monsoon-2025)  
Mid-Semester Exam

Time: 90 Minutes

Total Marks: 40

General Guidelines
<ul style="list-style-type: none"><li>• Attempt any four questions only.</li><li>• One A4 cheat sheet (both sides) is allowed.</li><li>• Unless stated otherwise, standard theorems from class may be used.</li><li>• Show all steps clearly; unsupported answers may not receive full credit.</li></ul>



Question (1) [10 Marks] Answer the following.

- (a) [3 Marks] Prove that for all natural numbers  $n \geq 0$ ,

$$(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$$

is an even number.

- (b) [3 Marks] Consider  $c \in \mathbb{R}$ . Let  $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$  be two neighbourhoods of  $c$ . Then prove that  $S_1 \cup S_2$  is a neighbourhood of  $c$ .
- (c) [4 Marks] Let  $S \subset \mathbb{R}$ . Then  $S$  is an open set iff  $S = \text{int}(S)$ , where  $\text{int}(S)$  is the set of all interior points of  $S$ .

Question (2) [10 Marks] Answer the following.

- (a) [3 Marks] Let  $N \geq 1$  be an integer. Show that there exists an integer  $m$  with

$$\frac{m}{N} \leq \sqrt{2} < \frac{m+1}{N}.$$

Deduce that  $0 \leq \sqrt{2} - \frac{m}{N} < \frac{1}{N}$ .

- (b) [3 Marks] Conclude that  $\sqrt{2}$  can be approximated arbitrarily well by rationals of the form  $m/N$  with  $m \in \mathbb{Z}$ , and hence that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .
- (c) [4 Marks] For  $N = 50$ , produce an explicit rational with denominator 50 lying below  $\sqrt{2}$  and verify the error bound  $< 1/50$  using inequalities.

✓ Question (3) [10 Marks] Answer the following.

- (a) [3 + 3 Marks] Consider two sequences  $(s_n)$  and  $(t_n)$  such that  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . Then prove that

(i)  $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$ .

(ii)  $\lim_{n \rightarrow \infty} (s_n t_n) = st$ .

- (b) [4 Marks] Let  $(u_n), (v_n), (w_n)$  be three sequences of real numbers and  $\forall n \in \mathbb{N}$ , suppose

$$u_n < v_n < w_n.$$

If  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} w_n = l$ , then prove that  $\lim_{n \rightarrow \infty} v_n = l$ .

Question (4) [10 Marks] Let  $(x_n)_{n \geq 1}$  be a sequence and define

$$y_n = \frac{x_n}{x_{n+1}} \quad (n \geq 1). \quad (1)$$

Answer the following.

- (a) [2 Marks] Suppose the sequence  $(x_n)$  is convergent and  $\lim_{n \rightarrow \infty} x_n = L$  with  $L \in \mathbb{R}$ . Can one conclude that  $\lim_{n \rightarrow \infty} y_n = 1$ ? Prove it if yes, or give the correct necessary condition(s) if not.
- (b) [3 Marks] Give an example of  $(x_n)_{n \geq 1}$  such that  $\lim_{n \rightarrow \infty} x_n = 0$  but  $\lim_{n \rightarrow \infty} y_n = -1$ .

- (c) [3 Marks] Give another example of  $(x_n)_{n \geq 1}$  such that  $\lim_{n \rightarrow \infty} x_n = 0$  but  $(y_n)_{n \geq 1}$  does not converge to any finite value.
- (d) [2 Marks] Suppose instead that  $(x_n)$  is monotonically increasing and  $x_n \rightarrow L \neq 0$ . Do your conclusions change?

✓ Question (5) [10 Marks] Consider the sequence  $(a_n)_{n \geq 1}$  with  $a_n = \frac{(-1)^n}{2}$ . Answer the following questions.

- (a) [1 Mark] Does the sequence  $(a_n)_{n \geq 1}$  converge?
- (b) [5 Mark] Let  $L \geq 1/2$  be a positive rational number. Define

$$d_n = \inf_{k \geq n} |a_k - L|.$$

Does the sequence  $(d_n)_{n \geq 1}$  converge? If no, why? If yes, what is  $\lim_{n \rightarrow \infty} d_n$ ?  
What are the limit points of the sequence  $(d_n)_{n \geq 1}$ ?

- (c) [4 Mark] Define

$$e_n = \min_{1 \leq k \leq n} |a_k - L|.$$

Does the sequence  $(e_n)_{n \geq 1}$  converge? If no, why? If yes, what is  $\lim_{n \rightarrow \infty} e_n$ ?

✓ Question (6) [10 Marks] Define the sequence  $(a_n)_{n \geq 1}$  by  $a_1 = 4$  and

$$a_n = \begin{cases} -1 + \frac{1}{k}, & \text{if } n = k! \text{ for some integer } k \geq 2, \\ 1 - \frac{1}{n}, & \text{otherwise.} \end{cases}$$

Here  $k! = k \times (k-1) \times \cdots \times 2 \times 1$  for  $k \geq 2$ . Answer the following questions.

- (a) [1 Mark] Write out the first six terms  $a_1, \dots, a_6$ .
- (b) [2 Marks] Show that the sequence  $(a_n)_{n \geq 1}$  is both bounded from below and bounded from above.
- (c) [3 Marks] Compute  $\sup(a_n)_{n \geq 1}$  and  $\inf(a_n)_{n \geq 1}$ .
- (d) [4 Marks] Compute  $\limsup_{n \rightarrow \infty} a_n$  and  $\liminf_{n \rightarrow \infty} a_n$ .

**Question (7) [10 Marks]** A sequence  $(x_n)_{n \geq 1}$  of real numbers is said to be *monotone increasing* if  $x_{n+1} \geq x_n$  for all  $n \geq 1$ . We say  $(x_n)_{n \geq 1}$  is *quasi-monotone increasing* if  $x_{n+1} \geq x_n - \frac{1}{6}$  for all  $n \geq 1$ . Answer the following.

(a) [2 Marks] Prove that every monotone increasing sequence is quasi-monotone increasing as well.

(b) [4 Marks] Let  $x_n = 1 + \frac{(-1)^n}{18n}$  and

$$y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{18} \left(1 + \frac{1}{n}\right) & \text{if } n \text{ is odd} \end{cases}$$

Prove that  $(x_n)_{n \geq 1}$  and  $(y_n)_{n \geq 1}$  are quasi-monotone increasing sequences that are not monotone increasing.

(c) [2 Marks] Prove or disprove: every bounded quasi-monotone increasing sequence converges.

(d) [2 Marks] Compare this with the Monotone Convergence Theorem. What essential feature of true monotonicity is lost in the quasi-monotone case, and why does this allow bounded quasi-monotone sequences to fail to converge?

✓ **Question (8) [10 Marks]** Prove that in  $\mathbb{R}$

(a) [5 Marks] Every Cauchy sequence is convergent. ✓

(b) [5 Marks] Every convergent sequence is Cauchy. ✓