These problems are for practice and understanding purpose.

1 Groups

Problem 1. *Let $(G, \circ)$ be a group. Define $*$ on G by $a * b = b \circ a$. Is $(G, *)$ a group?*

Problem 2. *Let $R = \mathbb{R} \setminus \{0\}$, where $\mathbb{R}$ is the set of real numbers. Define $*$ on R by*

$$x * y = \begin{cases} xy, & \text{if } x > 0, \\ \frac{x}{y}, & \text{if } x < 0. \end{cases}$$

Show that $(R, *)$ is a non-abelian group.

Problem 3. *Let G be the set*

$$G = \{\pm e, \pm a, \pm b, \pm c\}$$

where

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Show that G forms a group under matrix multiplication.

Problem 4. *Give an example of a group with 105 elements. Give two examples of groups with 44 elements.*

Problem 5. *Show that a group G is abelian if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$.*

Problem 6. *For any group elements a and b, prove that $|ab| = |ba|$.*

Problem 7. *Let $G = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$, and for $x, y \in G$, let $x * y$ be the fractional part of $x + y$ (i.e., $x * y = x + y - \lfloor x + y \rfloor$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to a).*

Prove that $*$ is a well-defined binary operation on G and that G is an abelian group under $*$ (called the real numbers mod 1).

Problem 8. *Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$.*

(a) Prove that G is a group under multiplication (called the group of roots of unity in $\mathbb{C}$).

(b) Prove that G is not a group under addition.

Problem 9. *Let $G = \{a + b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.*

(a) Prove that G is a group under addition.

(b) Prove that the nonzero elements of G are a group under multiplication. ("Rationalize the denominators" to find multiplicative inverses.)

Problem 10. *If a and b are commuting elements of G, prove that $(ab)^n = a^nb^n$ for all $n \in \mathbb{Z}$. (Do this by induction for positive n first.)*

Problem 11. *Prove that if $x^2 = 1$ for all $x \in G$, then G is abelian.*

Problem 12. *If $(A, *)$ and $(B, \diamond)$ are groups, we can form a new group $A \times B$, called their direct product, whose elements are those in the Cartesian product*

$$A \times B = \{(a, b) \mid a \in A,\, b \in B\}$$

and whose operation is defined componentwise:

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 * a_2, b_1 \diamond b_2).$$

Prove that $A \times B$ is a group.

Problem 13. *Prove that $A \times B$ is an abelian group if and only if both A and B are abelian.*

Problem 14. *If x is an element of infinite order in G, prove that the elements x^n, for $n \in \mathbb{Z}$, are all distinct.*

Problem 15. *If x is an element of finite order n in G, use the Division Algorithm to show that any integral power of x equals one of the elements in the set $\{1, x, x^2, \ldots, x^{n-1}\}$ (so these are all the distinct elements of the cyclic subgroup generated by x in G).*

Problem 16. *The quaternion group, Q_8, is defined by*

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with the product $\cdot$ computed as follows:

$$\begin{aligned}
&1 \cdot a = a \cdot 1 = a, \quad \text{for all } a \in Q_8,\\
&(-1) \cdot (-1) = 1, \quad (-1) \cdot a = a \cdot (-1) = -a, \quad \text{for all } a \in Q_8,\\
&i \cdot i = j \cdot j = k \cdot k = -1,\\
&i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j,\\
&j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j.
\end{aligned}$$

Verify all the group axioms. And then compute the order of all the elements in Q_8. Is this group abelian or not? Also check that it is cyclic or not.

Problem 17. *Let G be an Abelian group, and let $H = \{x \in G \mid |x| \text{ is odd}\}$. Prove that H is a subgroup of G.*

Problem 18. *Let G be an Abelian group, and let $H = \{x \in G \mid |x| \text{ is 1 or even}\}$. Give an example to show that H need not be a subgroup of G.*

Problem 19. *Consider the elements*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

from $SL(2, \mathbb{R})$, , the group of 2×2 real matrices with determinant 1. Find $|A|$, $|B|$, and $|AB|$. Does your answer surprise you?

Problem 20. *There is always a 1–1 onto mapping between any two right cosets of H in G.*

Problem 21. *Give an example of an infinite group in which every element is of finite order. (Think about quotient group Q/Z where $(Q, +)$ is group of rational numbers with addition operation and $(Z, +)$ is group of integers with addition operation.)*

2 Homomorphism & Isomorphism

Problem 22. *If $f : G \to G'$ is a homomorphism, then prove the following:*

1. $f(e) = e'$ *(where e and e' are the identity elements of G and G' respectively),*
2. $f(x^{-1}) = (f(x))^{-1}$ *for all $x \in G$, and*
3. $f(x^n) = (f(x))^n$ *for all $x \in G$ and n an integer.*

Problem 23. *If $\varphi : G \to H$ is an isomorphism, prove that $|\varphi(x)| = |x|$ for all $x \in G$. Deduce that any two isomorphic groups have the same number of elements of order n for each $n \in \mathbb{Z}^+$. Is the result true if φ is only assumed to be a homomorphism?*

Problem 24. *Prove that the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are not isomorphic.*

Problem 25. *Prove that the additive groups $\mathbb{R}$ and $\mathbb{Q}$ are not isomorphic.*

Problem 26. *Prove that the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.*

Problem 27. *Let A and B be groups. Prove that $A \times B$ is isomorphic to $B \times A$.*

Problem 28. *Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^2$ is a homomorphism if and only if G is abelian.*

Problem 29. *Suppose that f is an isomorphism from a group G onto a group G'.*

1. *f carries the identity of G to the identity of G'.*
2. *For every integer n and for every group element a in G, $f(a^n) = [f(a)]^n$.*
3. *For any elements a and b in G, a and b commute if and only if $f(a)$ and $f(b)$ commute.*
4. *$G \cong \langle a \rangle$ if and only if $G' \cong \langle f(a) \rangle$.*
5. *$|a| = |f(a)|$ for all a in G (isomorphisms preserve orders).*
6. *For a fixed integer k and a fixed group element b in G, the equation $x^k = b$ has the same number of solutions in G as does the equation $x^k = f(b)$ in G'.*
7. *If G is finite, then G and G' have exactly the same number of elements of every order.*

Problem 30. *Suppose that f is an isomorphism from a group G onto a group G'.*

1. *f^{-1} is an isomorphism from G' onto G.*
2. *G is Abelian if and only if G' is Abelian.*
3. *G is cyclic if and only if G' is cyclic.*
4. *If K is a subgroup of G, then $f(K) = \{f(k) \mid k \in K\}$ is a subgroup of G'.*
5. *If K is a subgroup of G, then $f^{-1}(K) = \{g \in G \mid f(g) \in K\}$ is a subgroup of G.*
6. *$f(Z(G)) = Z(G)$.*