Tutorial: Group Theory & Generating Functions

Properties of Group

1. **[Easy]** An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out, so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out?

2. **[Hard]** Let G be a group with the property that for any x, y, z in the group, $xy = zx$ implies $y = z$. Prove that G is Abelian.

3. **[Medium]** Prove the assertion that the set $\{1, 2, ..., n-1\}$ is a group under multiplication modulo n if and only if n is prime. (You may use the following identity for the proof- $\exists x, y \in \mathbb{Z}$ s.t. $\gcd(a, b) = ax + by$)

4. **[Easy]** Determine if the following sets G with the operation indicated form a group. If not, point out which of the group axioms fail.

   (a) $G =$ set of all integers, $a * b = a - b$.

   (b) $G =$ set of all integers, $a * b = a + b + ab$.

   (c) $G =$ set of all rational numbers $\neq -1$, $a * b = a + b + ab$.

5. **[Easy]** If G is any group, show that:

   (a) e is unique (i.e., if $f \in G$ also acts as a unit element for G, then $f = e$).

   (b) Given $a \in G$, then $a^{-1} \in G$ is unique.

Subgroups

1. **[Easy]** $(\mathbb{Z}, +)$ is a group - set of integers with '+' operation. Prove that $(H, +)$ is a subgroup where $H = \{2k | k \in \mathbb{Z}\}$ i.e. H is set of even integers.

2. **[Medium]** If A, B are subgroups of an abelian group G, let $AB = \{ab | a \in A, b \in B\}$. Prove that AB is a subgroup of G.

3. **[Medium]** Let H and K be subgroups of G and $x, y \in G$ with $Hx = Ky$. Then show that $H = K$.

4. [**Medium**] If A, B are subgroups of G such that $b^{-1}Ab \subset A$ for all $b \in B$, show that AB is a subgroup of G.

5. [**Hard**] Let M, N be subgroups of G such that $x^{-1}Mx \subset M$ and $x^{-1}Nx \subset N$ for all $x \in G$. Prove that MN is a subgroup of G and that $x^{-1}(MN)x \subset MN$ for all $x \in G$.

Cyclic Groups

1. [**Easy**] Prove that a cyclic group is abelian.

2. [**Easy**] If G is cyclic, show that every subgroup of G is cyclic.

3. [**Medium**] Prove that a group has exactly 3 subgroups if and only if it is cyclic of order p^2 for some prime p.

4. [**Easy**] Prove that a group G of prime order is cyclic.

5. [**Easy**] Let a and b be group elements such that $|a| = 10$ and $|b| = 21$. Prove that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Cosets & Lagrange's Theorem

1. [**Easy**] Let n be a positive integer. Let $H = \{0, \pm n, \pm 2n, \pm 3n, ...\}$. Find all left cosets of H in $\mathbb{Z}$. How many are there?

2. [**Easy - Medium**] If p is a prime number of the form $4n + 3$ show that we cannot solve $x^2 \equiv -1 \pmod{p}$.

3. [**Easy - Medium**] Given that p is a prime, solve $x^2 \equiv 1 \pmod{p}$.

4. [**Hard**] If $aH = bH$ forces $Ha = Hb$ in G, show that $aHa^{-1} = H$ for every $a \in G$.

Generating Functions

1. [**Easy**] Find the generating functions of the following sequences in closed form.

   (a) $1, 0, 1, 0, 1, 0, ...$

   (b) $2, -4, 6, -8, 10, -12, ...$

   (c) $\binom{2}{0}, \binom{2}{1}, \binom{2}{2}, 0, 0, 0, ...$

2. [**Easy**] Write a generating function that expresses the number of solutions of $y_1 + y_2 + y_3 = 17$, where $2 \le y_1 \le 5$, $3 \le y_2 \le 6$, and $4 \le y_3 \le 7$.

3. **[Medium]** Find generating functions that express the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars when:

   (a) The order in which the tokens are inserted does not matter.

   (b) The order in which the tokens are inserted matters.

   You may use the extended binomial theorem for the following two questions.

4. **[Medium]** How many integer solutions to the equation $a + b + c = 6$ satisfy $-1 \le a \le 2$ and $1 \le b, c \le 4$?

5. **[Easy]** There are 30 identical prizes, to be distributed among the 50 different students, and each student may get zero or more prizes. How many ways are there to distribute the 30 prizes among the 50 students?

6. **[Hard]** Let A_n denote the number of binary strings (strings containing only 0 and 1) of length n such that:

   i. There are no 3 consecutive 0s, i.e. no 000 in the string.

   ii. There are no 2 consecutive 1s, i.e. no 11 in the string.

   iii. The string ends with 0.

   (a) Find a generating function for A_n. (**Hint:** Try to find a recurrence relation)

   (b) Now suppose we remove the last condition, i.e. now the string can end in 0 or 1. Write down the generating function in this case.

7. **[Easy]** Consider the following recurrence relation: $A_{n+2} = 5A_{n+1} - 6A_n$ with $A_0 = 1$, $A_1 = 5$.

   (A) Find a generating function for this recurrence.

   (B) Find a closed form solution for A_n.

8. **[Medium - Hard]**Given positive integers n, k; define $f(n, k)$ as follows: for each way of writing n as an ordered sum of exactly k nonnegative integers, let S be the product of those k integers. Then $f(n, k)$ is the sum of all of the S's that are obtained in this way. Find the ordinary power series generating function of f and an explicit, simple formula for it.

9. **[Extra]** The Catalan numbers are defined as: $C_0 = 1$ and for $n \ge 0$, $C_{n+1} = \sum_{i=0}^{n}(C_i \cdot C_{n-i})$.

   (a) Find the generating function for the Catalan numbers.

   (b) Use the generating function to show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

   You may use the following identities:

- Expansion of $\sqrt{1-x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-x)^n$
- $\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$ is valid for fractional n.