Discrete Structures

Problem Set 3

Release Date: October 16th, 2025
Deadline: October 31st, 2025

1 Instructions:

- The marks for each question (or sub-question) has been specified alongside it.
- Take care to ensure that if you use variables that have not been defined in the question, you state them clearly, and the domain they belong to clearly as well (if applicable)
- Answer questions precisely, maintaining mathematical rigor as much as possible
- Both digital (Written, Not Typed) and handwritten submissions will be accepted.
- The deadline for this set is October 31st, 11:59 p.m.
- Submissions to be done on moodle with the format:
  `{rollnumber}_DS_A3` (for example `2023113019_DS_A3`)
- Plagiarism and AI use shall be penalized heavily.

2 Problems

1. Let G be a set with an operation $*$ such that:

   (1) G is closed under $*$.

   (2) $*$ is associative.

   (3) There exists an element $e \in G$ such that $e * x = x$ for all $x \in G$.

   (4) Given $x \in G$, there exists a $y \in G$ such that $y * x = e$.

   Prove that G is a group. (Thus you must show that right inverse and right identity also hold for this set under $*$.) **[6 marks]**

2. Prove that G is an abelian group if every element of the group G is its own inverse. **[5 marks]**

3. If G has no proper subgroups, prove that G is cyclic. **[5 marks]**

4. Find all of the left cosets of $\{1, 11\}$ in $U(30)$. **[5 marks]**

5. Let $H \subseteq G$ and $g \in G$. If $|g| = n$ and $g^m \in H$, where n and m are co-prime integers, then show that $g \in H$. **[6 marks]**

6. Show that $U(12)$ is isomorphic to $U(8)$. **[5 marks]**

7. Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication, and let r be a positive integer. Show that the mapping that takes x to x^r is a homomorphism from $\mathbb{R}^*$ to $\mathbb{R}^*$ and determine the kernel. Which values of r yield an isomorphism? **[7 marks]**

8. Given a ring R, show that: **[5 Marks]**

   a. For all $a \in R, a0 = 0a = 0$

   b. For all $a, b \in R, a(-b) = (-a)b = -(ab)$

   c. For all $a, b \in R, (-a)(-b) = ab$

   d. If R has a unit element, then for all $a \in R, (-1)a = -a$

   e. If R has a unit element, then $(-1)(-1) = 1$

9. Show that every non-zero element of the ring Z_n is a unit or zero-divisor. **[5 marks]**

10. Suppose that there is an integer $n > 1$ such that $x^n = x$ for all elements x of some ring. If m is a positive integer and $a^m = 0$ for some a, show that $a = 0$. **[5 marks]**

11. An integral domain D is called a *principal ideal domain* if every ideal of D has the form $\langle a \rangle = \{ad \mid d \in D\}$ for some a in D. Show that Z is a principal ideal domain. **[5 marks]**

12. Show that a field is an integral domain. **[5 marks]**

13. Let F be a field of characteristic $p \neq 0$. Show that

    a. for all $a, b \in F$, we have $(a + b)^p = a^p + b^p$.

    b. for any positive integer n, if $m = p^n$, then $(a + b)^m = a^m + b^m$ for all $a, b \in F$.

[6 marks]