Discrete Structures

Problem Set 1

Release Date: August 25th, 2025
Deadline: September 3rd, 2025

Instructions:

- The marks for each question (or sub-question) has been specified alongside it.
- Take care to ensure that if you use variables that have not been defined in the question, you state them clearly, and the domain they belong to clearly as well(if applicable)
- Answer questions precisely, maintaining mathematical rigor as much as possible
- **Both** digital (**Written , Not Typed**) and handwritten submissions will be accepted.
- The deadline for this set is **September 3rd , 11:59 p.m.**
- Submissions to be done on moodle with the format:
  <rollnumber>_DS_A1 (for example 2023113019_DS_A1)
- Plagiarism and AI use shall be penalized heavily.

Problem 1: **(3+1=4)**

Defining the ternary conditional operator, $C(p, q, r)$, as *"If p is true, the result is q; otherwise, the result is r".*

(a) One of the following propositions is logically equivalent to $C(p, q, r)$:

(i) $(p \to q) \land (\neg p \to r)$

(ii) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

Identify which proposition is equivalent to $C(p, q, r)$ by constructing and comparing the necessary truth tables.

(b) Using only the proposition you identified in the previous part as being equivalent to $C(p, q, r)$, show how you can define the **negation** operator $(\neg p)$ by choosing specific constant values for q and r from the set $\{T, F\}$ (True, False).
For example, demonstrate that an expression like $C(p, F, T)$ or $C(p, T, T)$ (these are just examples) is equivalent to $\neg p$. Use truth table from (i) for this.

Problem 2: **(1+1+1+1=4)**

A is a countably infinite square matrix, where all entries are non-negative integers from the set $\mathbb{W} = \{0, 1, 2, \ldots\}$.

Given the following ternary predicate:
$E(i, j, k)$: *"The entry in row i and column j of matrix A is the integer k."*
Translate the following English statements about the matrix A into formal logical expressions. Your expressions should use the predicate, quantifiers $(\forall, \exists)$, logical connectives (eg., $\land, \neg$), and standard arithmetic operators and relations like $<$ and $>$.

(a) There exists at least one entry in the matrix with the value 5.

(b) There exists a row that consists entirely of zeros.

(c) There is no single greatest value in the entire matrix.

(d) There is a unique column that contains the value 3. (No other column contains 3, and the entire column has every entry as 3)

Note that these 4 are independent statements, and are to be treated as such.

Problem 3: **(1+1.5+1.5=4)**

"There exists some number n such that for every number p, at least one of the following is true: p is less than or equal to n, or p is not a prime number, or $p + 2$ is not a prime number."

Let the domain be the set of all integers, $\mathbb{Z}$. predicate be $P(x)$: "x is a prime number."

(a) Translate the italicized statement into a formal logical expression using quantifiers $(\forall, \exists)$ and the predicate $P(x)$.

(b) Write the negation of this formal expression from task 1. Simplify and use De Morgan's laws.

(c) Translate your final, simplified negated expression from task 2 back into English.

Problem 4: **(4)**

Let α be any real number such that $\alpha + \frac{1}{\alpha} \in \mathbb{Z}$. Prove that $\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}$ for any $n \in \mathbb{N}$.

Problem 5: **(1+1+1+1=4)**

Prove or disprove the following:

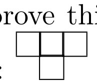
(a) $P(n) = n^2 + n + 1$ where $n \in \mathbb{N} = \{1, 2, 3, ...\}$ is always prime for any natural number $n \in \mathbb{N}$.

(b) Product of 2 rational numbers is rational

(c) If an integer $x > 10$, then $x^2 + 1 > 0$.

(d) If an integer x is both less than 5 and greater than 10, then $x^2 = 3$.

Problem 6: **(5)**

Prove or disprove this statement: A 10×10 chessboard cannot be covered by 25 T-tetrominoes. (T-tetromino: )

Problem 7: **(5)**

Prove that for every integer n, the number $n^5 - n$ is divisible by 30.

Problem 8: **(5)**

Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is less than 1/2.