Q1

$q(t) = 10t^2 - 22t \quad mC$

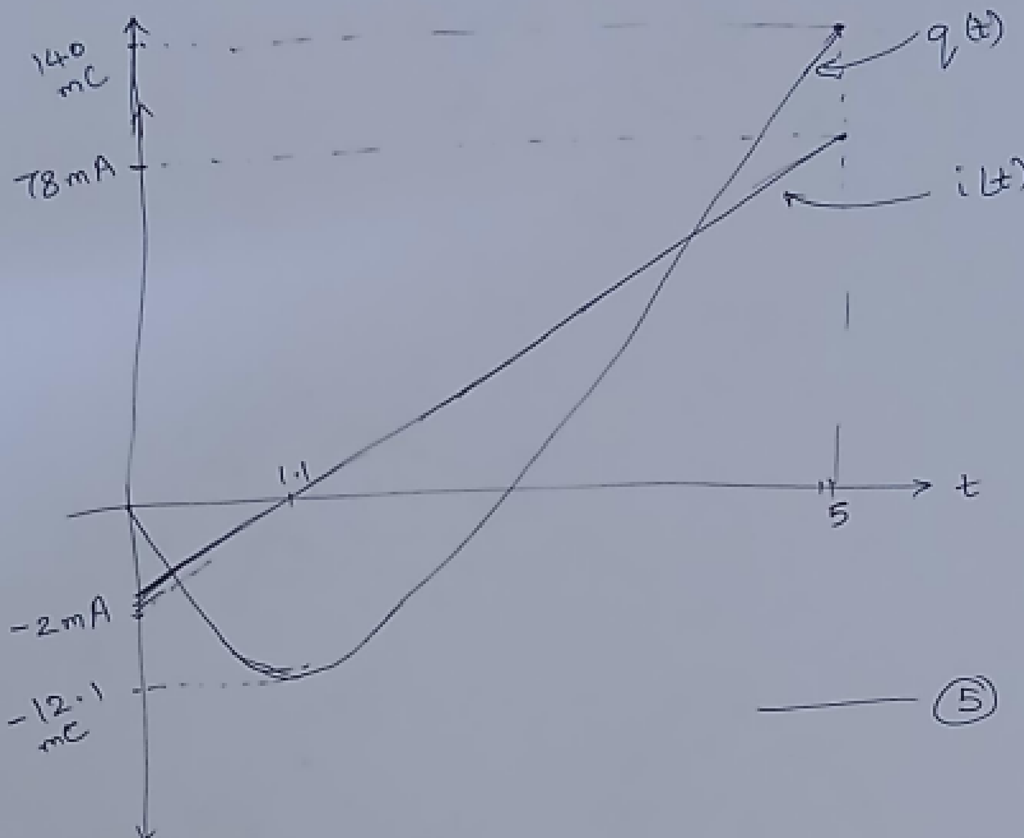
$\therefore i(t) = \dfrac{dq}{dt}$

$= 20t - 22 \quad mA$

$i(t) = 0$

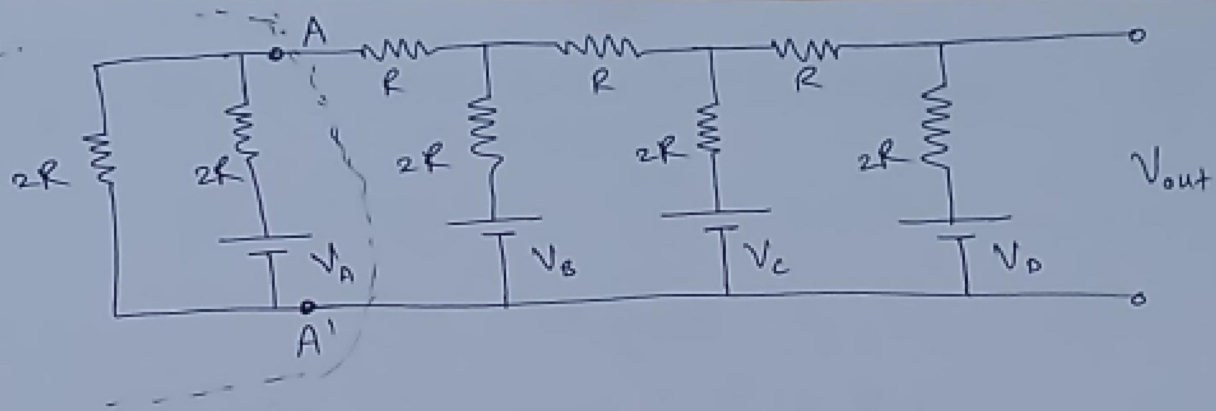
$\therefore 20t - 22 = 0$

$\therefore t = 1.1$ sec. — 5



<Make sure y-axis labels & units are correct>

Q2.



Consider the network in the dotted line (AA')

Thevenin resistance $R_{th_A} = 2R \parallel 2R = R$.

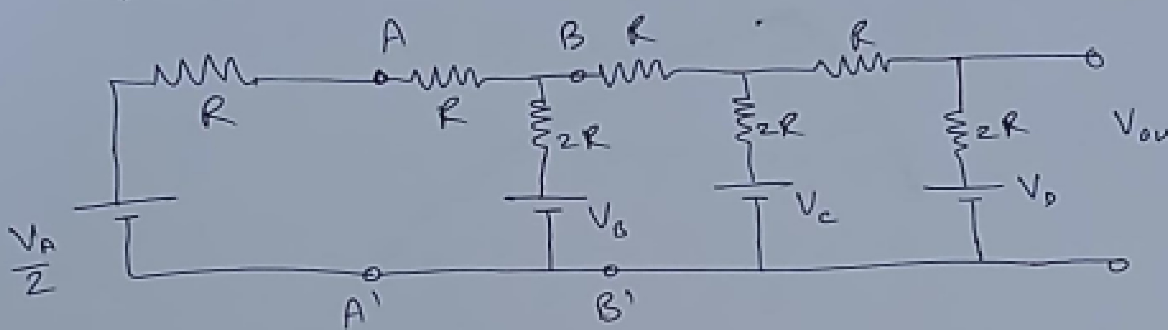
Thevenin voltage $V_{th_A} = V_A - i_A \times 2R$

$$i_A = \frac{V_A}{4R}$$

$$\therefore V_{th_A} = \frac{V_A}{2}.$$

— (4)

Equivalent circuit:



Now, for the circuit to the left of BB':

Thevenin resistance $R_{th_B} = (R + R) \parallel 2R$

$$= R$$

" Voltage $V_{th_B} = V_B - i_B \times 2R$

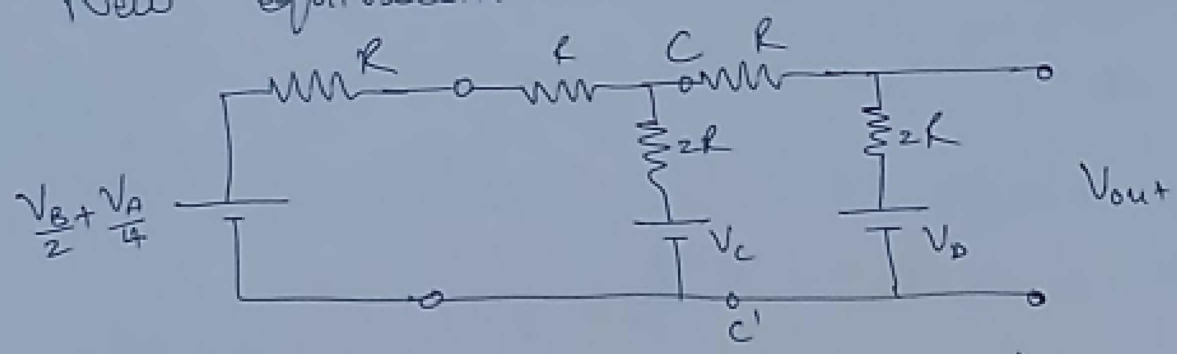
$$i_B = \frac{1}{4R}\left(V_B - \frac{V_A}{2}\right)$$

$$\therefore V_{th_B} = V_B - \frac{V_B}{2} + \frac{V_A}{4}$$

$$= \frac{V_B}{2} + \frac{V_A}{4}$$

— (4)

New equivalent:

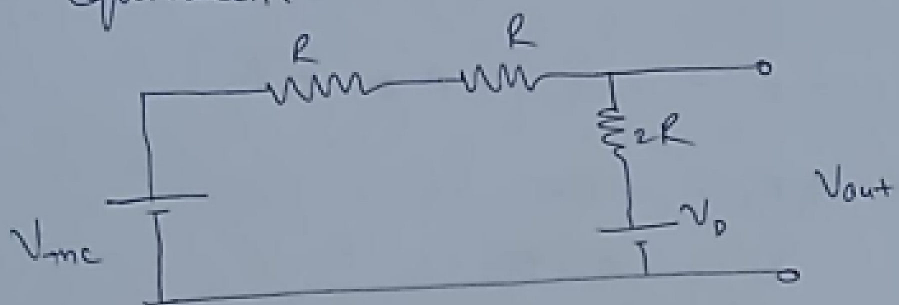


Similar analysis is done for CC' circuit.

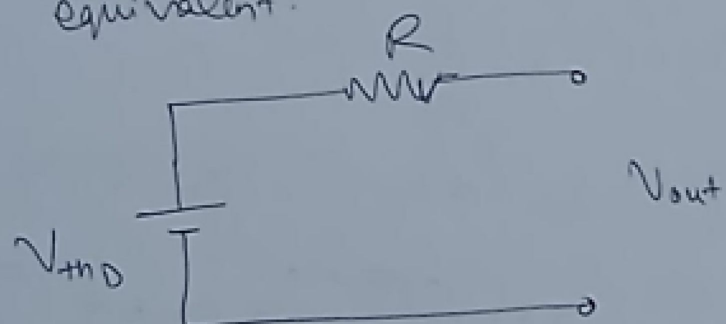
we get $R_{thC} = R$ —— (2)

$$V_{thC} = \frac{V_C}{2} + \frac{V_B}{4} + \frac{V_A}{8}$$

New equivalent:



Final equivalent:



Thevenin equivalent resistance $= R$

" " voltage $= \frac{V_D}{2} + \frac{V_C}{4} + \frac{V_B}{8} + \frac{V_A}{16}$

—— (2)

Q3. For $t < 0$:

The DC source is in series with the RLC circuit. The capacitor acts as open circuit.

∴ Voltage across the capacitor $V_c(0^-) = 7.2\,V$

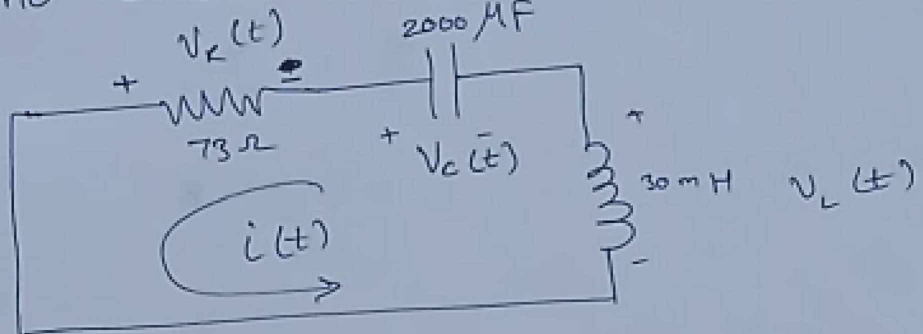
current through the inductor $i_L(0^-) = 0$

3

For $t > 0$:

The DC source is shorted out.

The circuit simply behaves like a series RLC with no external source



$$\alpha = \frac{R}{2L} = \frac{73}{2 \times 0.03} = 1216.67 \quad s^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.03 \times 0.002}} = 129.09 \quad s^{-1}$$

2

∴ $\alpha > \omega_0 \Rightarrow$ The circuit is overdamped.

∴ General solution is $A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

$$s_{1/2} = -1216.67 \pm \sqrt{(1216.67)^2 - (129.09)^2}$$

$$= -1216.67 \pm 1209.8$$

$$\therefore s_1 = -2426.47 \quad s^{-1}$$

$$s_2 = -6.87 \quad s^{-1}$$

2

$\therefore \quad i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

At $t = 0, \quad i_L(0^-) = i(0) = 0$

$\therefore \quad A_1 + A_2 = 0$ ——— (1)

Now, $\quad V_L(0) = L \left.\dfrac{di}{dt}\right|_{t=0}$

& $\quad V_R(t) + V_C(t) + V_L(t) = 0$

Because $\quad V_R(0) = 0$

$$V_L(0) = -V_C(0)$$
$$= -7.2 \text{ V}$$

$\therefore \quad -7.2 = 0.03 \left[s_1 A_1 + s_2 A_2 \right]$

$s_1 A + s_2 A_2 = -240$

$\therefore \quad A_1 = 0.0986 \quad A$

$A_2 = -0.0986 \quad A$

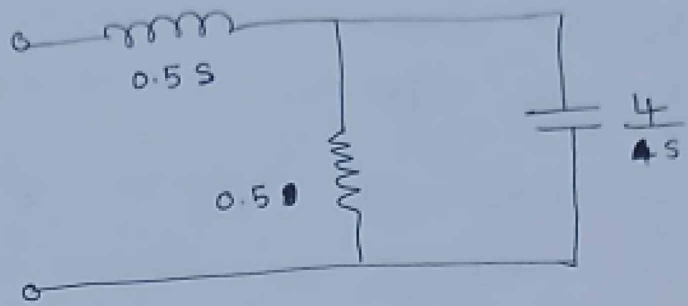
$\therefore \quad V_L(t) = L \dfrac{di}{dt}$

$$= L\left(s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}\right)$$

$$= -7.17\, e^{-2426.47t} + 0.02\, e^{-6.87t} \quad V$$

——— (3)

Q5. In the s-domain



$\therefore$ Impedance $= \frac{1}{2}s + \left(\frac{1}{2} \parallel \frac{4}{s}\right)$

$$Z_{in}(s) = \frac{s}{2} + \frac{4}{8+s}$$

$$= \frac{s}{2} + \frac{4}{8+s}$$

$$= \frac{s(8+s)+8}{2(8+s)} \qquad \text{—— } (3)$$

$$\therefore \; Y_{in}(s) = \frac{2(s+8)}{s^2+8s+8}$$

$$= \frac{2(s+8)}{(s+4)^2 - 8}$$

$$= 2\left[\frac{(s+4)}{(s+4)^2-8}\right] + \frac{8}{(s+4)^2-8}$$

$$= 2e^{-4t}\cosh(2\sqrt{2}t) + 2\sqrt{2}\,e^{-4t}\sinh(2\sqrt{2}t)$$

—— (5)

Starts at 2 Ohm^{-1} at $t=0$
Decreases because e^{-4t} overpowers other exp terms

—— (5)

Q6.

- Say we have a function $f(t)$ with a Laplace transform $F(s)$
- We can know the value of $f(0^+)$ using $F(s)$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt$$

Now, let s approach infinity

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt\right] = \lim_{s\to\infty}\left[\int_{0^-}^{0^+} e^{-st}\frac{df}{dt}dt + \int_{0^+}^{\infty} e^{-st}\frac{df}{dt}dt\right]$$

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}\left[\int_{0^-}^{0^+} \frac{df}{dt}dt + \int_{0^+}^{\infty} e^{-st}\frac{df}{dt}dt\right]$$

As $s \to \infty$, the second integration vanishes for all values of t

Thus,

$$\lim_{s\to\infty}[sF(s)] - f(0^-) = \lim_{s\to\infty}[f(0^+) - f(0^-)] = f(0^+) - f(0^-)$$

$$\boldsymbol{f(0^+) = \lim_{s\to\infty}[sF(s)]}$$

End Sem : NeSS.

(2+4+1=7 marks) Prove that if $x(t)$ is periodic, so is $x^2(t)$. Find the Fourier series (FS) coefficients of $x^2(t)$ in terms of those of $x(t)$ (note: derive from first principles, do not use FS properties directly). Check your answer for $x(t) = sin(t)$.

(a) As $x(t)$ is periodic
WKT $\exists$ a smallest T such that

$$x(t+T) = x(t), \forall t \in \mathbb{R}. \rightarrow (1)$$

Squaring (1), we see that $x^2(t+T) = x^2(t), \forall t \in \mathbb{R}$

Thus, if $y(t) = x^2(t)$,

$$y(t+T) = y(t), \forall t \in \mathbb{R}.$$

Hence, $y(t) = x^2(t)$ is periodic.

(b) Let the FS of $x(t)$ be

$$x(t) = \sum_{k \in \mathbb{Z}} a_k e^{j2\pi k\omega_0 t},$$

where $\omega_0 = 2\pi / T$ (T being period of $x(t)$)

Then $x^2(t) = \left(\sum_{k \in \mathbb{Z}} a_k e^{j2\pi k\omega_0 t} \right)^2$

$$= \left(\sum_{k_1 \in \mathbb{Z}} a_{k_1} e^{j2\pi k_1 \omega_0 t} \right) \left(\sum_{k_2 \in \mathbb{Z}} a_{k_2} e^{j2\pi k_2 \omega_0 t} \right)$$

$$= \left(\ldots a_{-2} e^{j2\pi(-2)\omega_0 t} + a_{-1} e^{j2\pi(-1)\omega_0 t} + a_0 e^{j2\pi(0)\omega_0 t} + a_1 e^{j2\pi(1)\omega_0 t} + \ldots \right) \left(\ldots \text{same} \ldots \right)$$

$$= \sum_{k \in \mathbb{Z}} \left(\sum_{\ell \in \mathbb{Z}} a_\ell \, a_{k-\ell} \right) e^{j2\pi k \omega_0 t}$$

Thus the FS coefficients of the square signal

are $b_k = \left(\sum_{\substack{k_1, k_2 \in \mathbb{Z}: \\ k_1 + k_2 = k}} a_{k_1} a_{k_2} \right), \forall k \in \mathbb{Z}.$ $\longrightarrow$ (A)

(c) Let $x(t) = \sin(t) = \dfrac{e^{j2\pi\omega_0 t} - e^{-j2\pi\omega_0 t}}{2j}$

where $\omega_0 = \frac{1}{2\pi} \cdot$ (since $T = 2\pi$)

Thus, FS coefficients are

$$a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}$$

$$\& \; a_k = 0 \;\; \forall k \in \mathbb{Z} \setminus \{1, -1\}.$$

Now, the FS coefficients $b_k : k \in \mathbb{Z}$ of $\sin^2(t)$, as per (A)

would be as follows

$$b_{-2} = a_{-1} \cdot a_{-1} = \frac{1}{4j^2} = \frac{-1}{4}$$

$$b_2 = a_1 \cdot a_1 = -1/4$$

$$b_0 = a_{-1} \cdot a_1 + a_1 \cdot a_{-1} = \frac{1}{4} + \frac{1}{4} = 1/2.$$

$$b_k = 0 \quad , \forall k \in \mathbb{Z} \setminus \{-2, 0, 2\}.$$

Thus we get

$$\sum b_k e^{j2\pi k w_0 t} = b_{-2} e^{-j2t} + b_0 + b_2 e^{j2t}$$

$$= \frac{-1}{4} e^{-j2t} + \frac{1}{2} - \frac{1}{4} e^{j2t}$$

$$= \frac{1}{2}\left[1 - \left(\frac{e^{j2t} + e^{-j2t}}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 - \cos 2t\right]$$

$$= \frac{1}{2}\left[2 \sin^2 t\right]$$

$$= \sin^2 t.$$

Hence we can see that (A) is true, for this example.

~~(7 marks)~~ Prove or disprove the following claims. (Note : To disprove a claim, you **may** need to give a 'counterexample', i.e., any example for which you should then prove that the claim is not true. For example, if 'all even numbers are positive' is the claim, then you can show the example 0, which is even, but not positive.)

(a) (2.5) ~~(2 marks)~~ The input $x(t)$ and output $y(t)$ of an LTI system satisfies the equation (in the s-domain)

$$Y(s) = X(s)H(s),$$

where $H(s)$ is the system transfer function. (You are free to assume that the time-domain relationship between $x(t)$ and $y(t)$ via the impulse response).

(b) (2.5) ~~(2 marks)~~ The ROC of a causal LTI system does not contain any $s \in \mathbb{C}$ such that $Re(s) < 0$.

(c) (3 marks) Let $x_1(t)$ and $x_2(t)$ be the inputs to an LTI system and the respective outputs be $y_1(t)$ and $y_2(t)$. Then, corresponding to the output of the same system being $y_1(t-t_1) + y_2(t-t_0)$, the input is unique and must be $x_1(t-t_1) + x_2(t-t_0)$

(a) We know that, the I/O relationship in an LTI system in time-domain is given by

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Now, applying LT on both sides,

$$\int_{t=-\infty}^{\infty} y(t)e^{-st}\, dt = Y(s) = \int_{t=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \right) e^{-st} dt$$

$$\underset{\text{(interchange integrals)}}{=} \int_{\tau=-\infty}^{\infty} x(\tau) \left(\int_{t=-\infty}^{\infty} h(t-\tau)e^{-st}\, dt \right) d\tau \rightarrow \text{I}$$

Now $$\int_{t=-\infty}^{\infty} h(t-\tau)e^{-st}\, dt = \int_{t_1=-\infty}^{\infty} h(t_1)e^{-s(t_1+\tau)}\, dt_1$$

use $t_1 = t - \tau$
$\Rightarrow t = t_1 + \tau$ & $dt = dt_1$

$$= e^{-s\tau} \underbrace{\int_{t_1=-\infty}^{\infty} h(t_1) e^{-st_1} dt_1}$$

$$= e^{-s\tau} H(s) \longrightarrow \text{II}$$

Using II in I, we get

$$Y(s) = \int_{\tau=-\infty}^{\infty} x(\tau) e^{-s\tau} H(s) d\tau$$

$$= H(s) \underbrace{\int_{\tau=-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau}$$

$$Y(s) = H(s) \; X(s) .$$

Thus the statement is true always.

(b) Consider the system $h(t) = \delta(t)$.

This is a causal system, as $y(t) = x(t) * h(t)$

$$= x(t) , \forall t.$$

Then, $H(s) = 1$, $\forall s$.

Thus ROC $= \mathbb{C}$. Hence the statement is not true.

(Alternatively, any other example can be considered like $h(t) = e^{-at} u(t) \longleftrightarrow H(s) = \frac{1}{s+a}$ (ROC $Re(s) > -a$)

$\hookrightarrow$ (Fix $a > 0$).)

(c). Consider $h(t) = 0$, $\forall t$.

Then for any $x(t)$, $y(t) = h(t) * x(t) = 0$, $\forall t$

This is a trivial LTI system. The input is NOT unique for any given output (as the only possible output is 0).

Thus, $x_1(t-t_1) + x_2(t-t_0)$ is NOT the only i/p which results in $y_1(t-t_1) + y_2(t-t_0) = 0$

(as $y_1(t-t_1) = 0 = y_2(t-t_0)$, $\forall t$.)

Any i/p signal can result in $y_1(t-t_1) + y_2(t-t_0) = 0$ as the o/p, $\forall t$.