Lecture 16 – Sinusoidal sources

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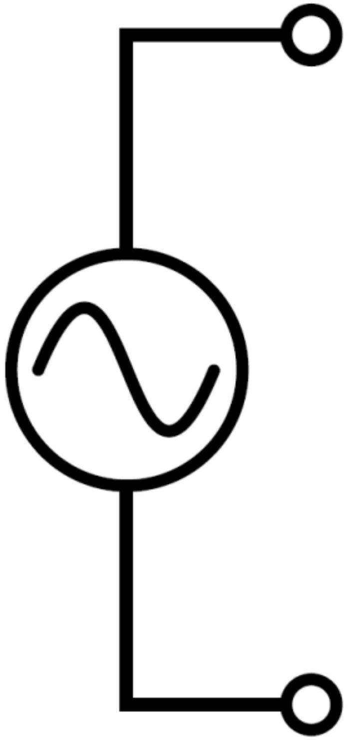
Sinusoidal sources

- It is very common to apply sinusoidal sources to circuits – because power is produced in sine, rotating things are easier to represent in sine, sound is easier to represent as sine etc.
- Thus, the response of a circuit to sine excitation should be studied and understood
- The sine source is represented as shown

$$V = V_m \sin \omega t$$

- At any given time, the magnitude of the voltage can be calculated
- Because it is a time varying function, application of it to inductors and capacitors can lead to interesting responses

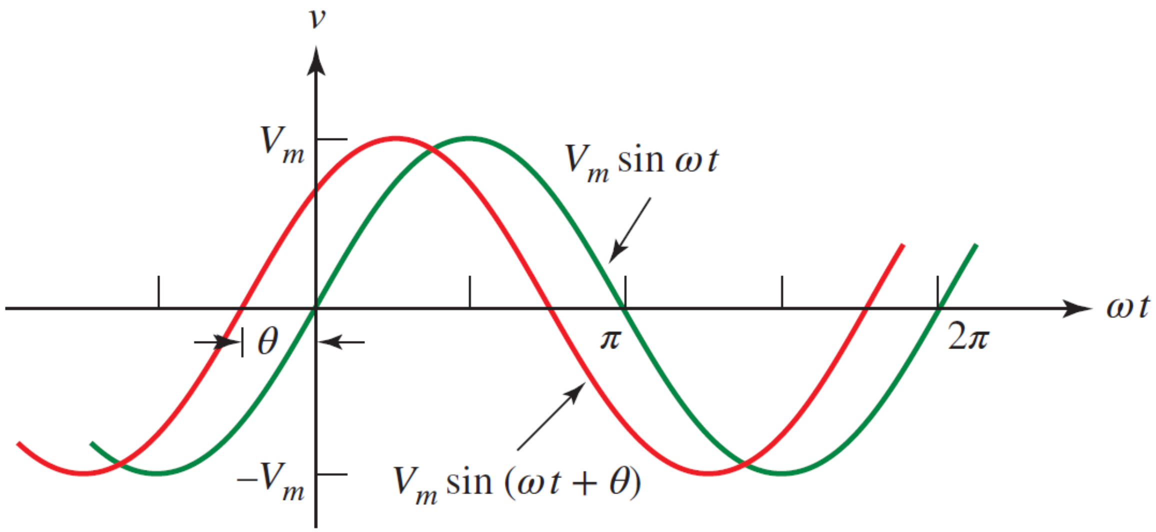


Leading and Lagging

- A sine wave can be shifted in time

- The shift is defined by the phase angle, if a sine wave is not zero at $t = 0$, then it is represented by
$$v(t) = V_m \sin(\omega t + \theta)$$

- If θ is positive, the wave is said to be lagging $V_m \sin \omega t$, if θ is negative, the wave is said to be leading



RL circuit

- Let us start with a simple RL circuit
- We focus on the "steady-state", so the assumption is that the circuit has been in this state for a long time

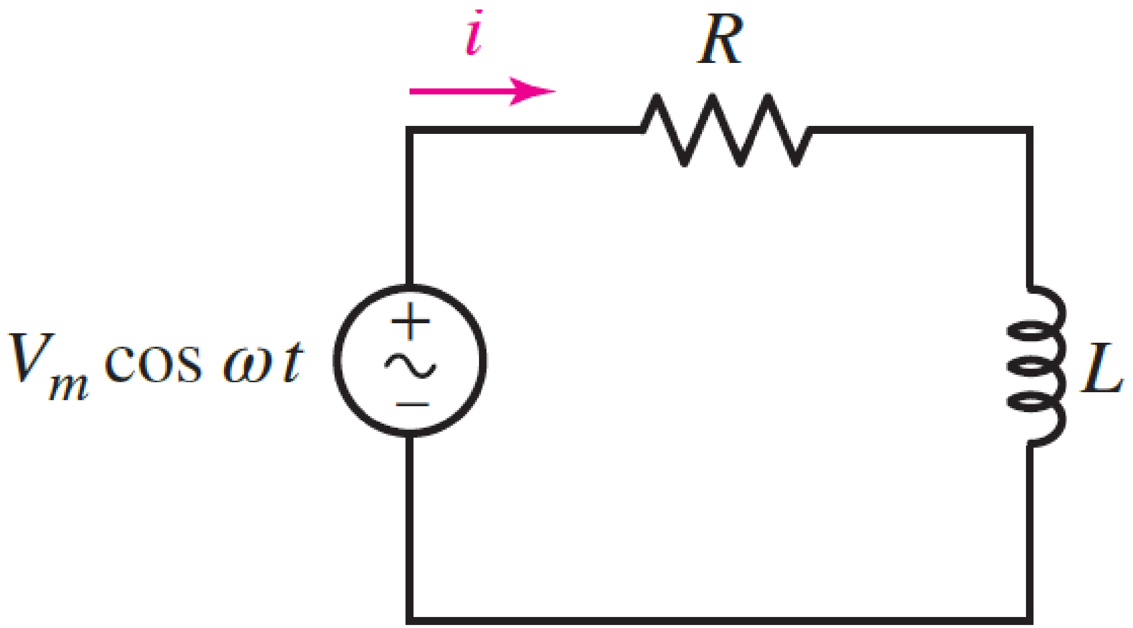
$$L\frac{di}{dt} + Ri = V_m \cos \omega t$$

- The general solution to this is of the form:

$$i(t) = A_1 \cos \omega t + A_2 \sin \omega t$$

- Substituting in the equation:

$$L\omega(-A_1 \sin \omega t + A_2 \cos \omega t)$$
$$+ R(A_1 \cos \omega t + A_2 \sin \omega t)$$
$$= V_m \cos \omega t$$



RL circuit

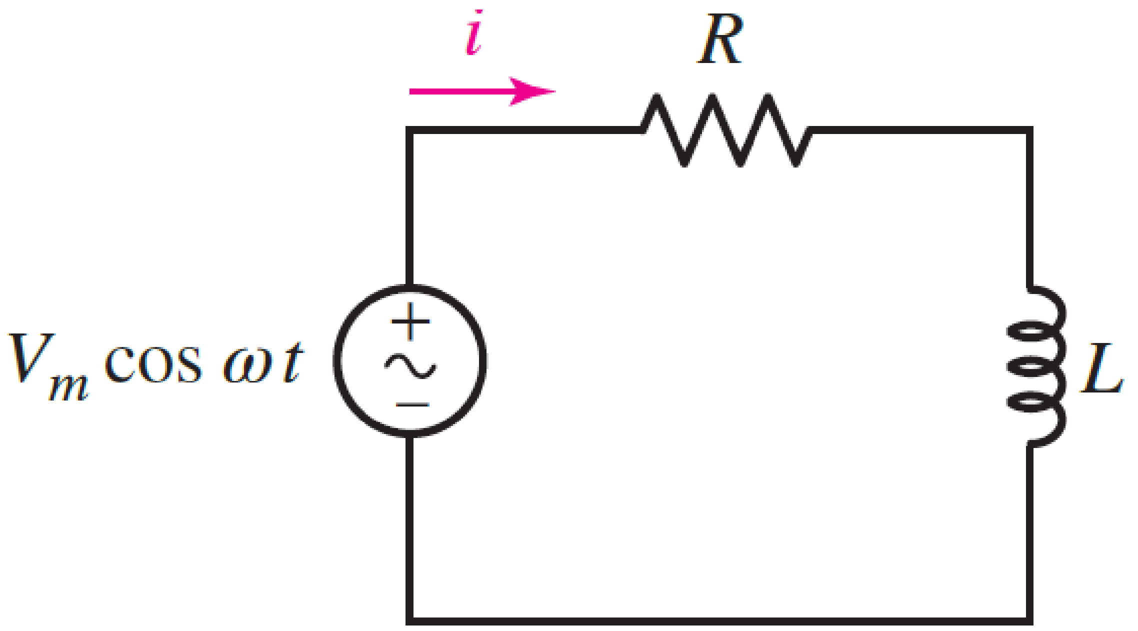
$$L\omega(-A_1 \sin \omega t + A_2 \cos \omega t) \\ + R(A_1 \cos \omega t + A_2 \sin \omega t) \\ = V_m \cos \omega t$$

$$(-LA_1\omega + RA_2) \sin \omega t \\ + (LA_2\omega + RA_1 - V_m) \cos \omega t = 0$$

This equation needs to be true for all values of t

Thus,

$$-LA_1\omega + RA_2 = 0$$
$$LA_2\omega + RA_1 - V_m = 0$$



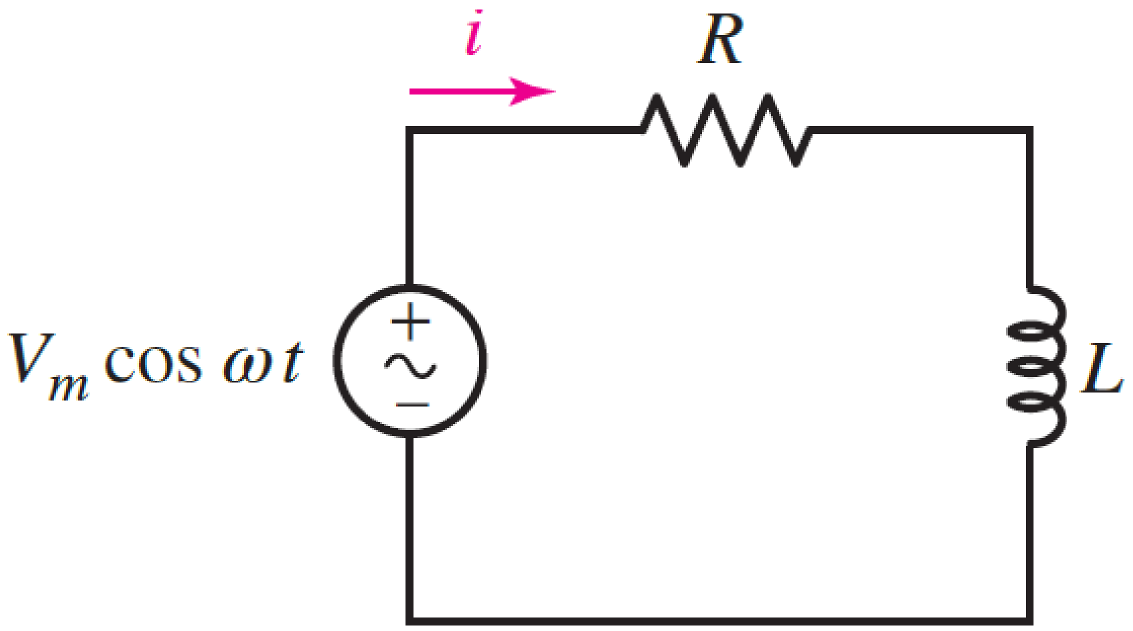
RL circuit

$$-LA_1\omega + RA_2 = 0$$
$$LA_2\omega + RA_1 - V_m = 0$$

$$A_1 = \frac{RV_m}{R^2 + \omega^2 L^2}$$

$$A_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

$$i(t) = \frac{RV_m}{R^2 + \omega^2 L^2}\cos\omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin\omega t$$



RL circuit

Turns out, you can express the current as a cosine function with a phase

$$i(t) = A\cos(\omega t + \theta)$$

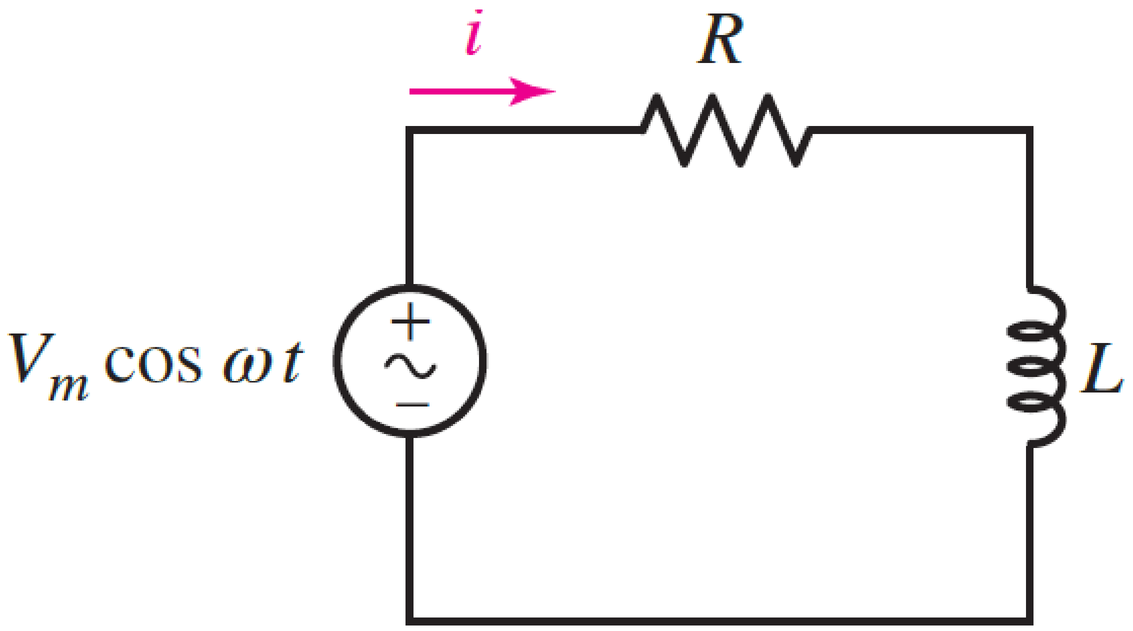
$$A\cos\theta\cos\omega t + A\sin\theta\sin\omega t$$
$$= \frac{RV_m}{R^2 + \omega^2 L^2}\cos\omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin\omega t$$

$$A\cos\theta = \frac{RV_m}{R^2 + \omega^2 L^2}$$

$$A\sin\theta = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

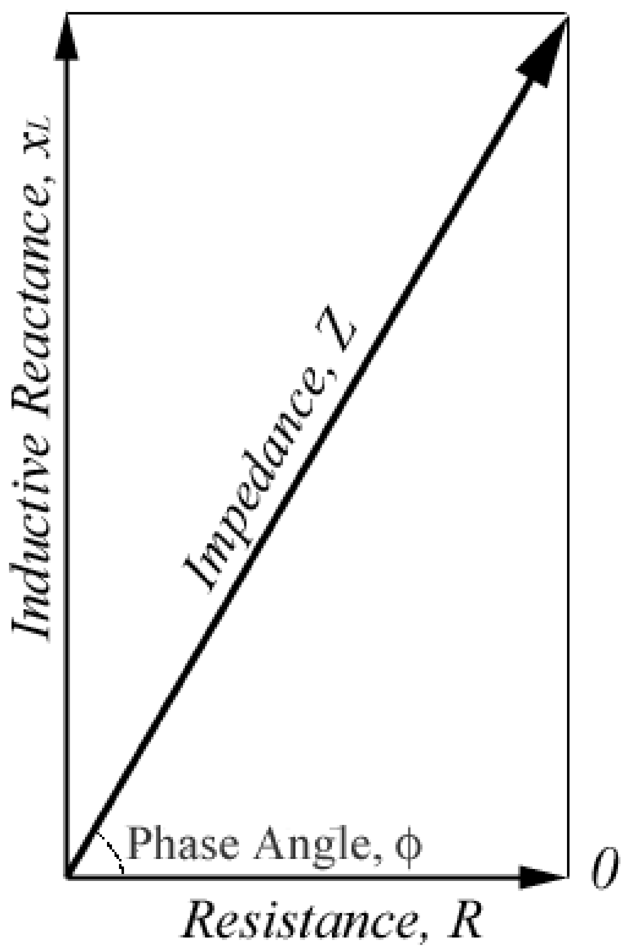
$$A = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \text{ and } \tan\theta = \frac{\omega L}{R}$$

If $Z \equiv R + j\omega L$, $|Z| = \sqrt{R^2 + \omega^2 L^2}$



Complex impedance

- The impedance of inductor in the s-domain is sL, and the complex frequency $s = \sigma + j\omega$
- In AC response, the damping coefficient $\sigma = 0$, thus, inductor impedance is ωL
- Similarly, capacitor inductance is $1/j\omega C$
- The series/parallel sum of these impedances provides the complex impedance of the circuit
- Typically represented in the complex plane



RLC impedance

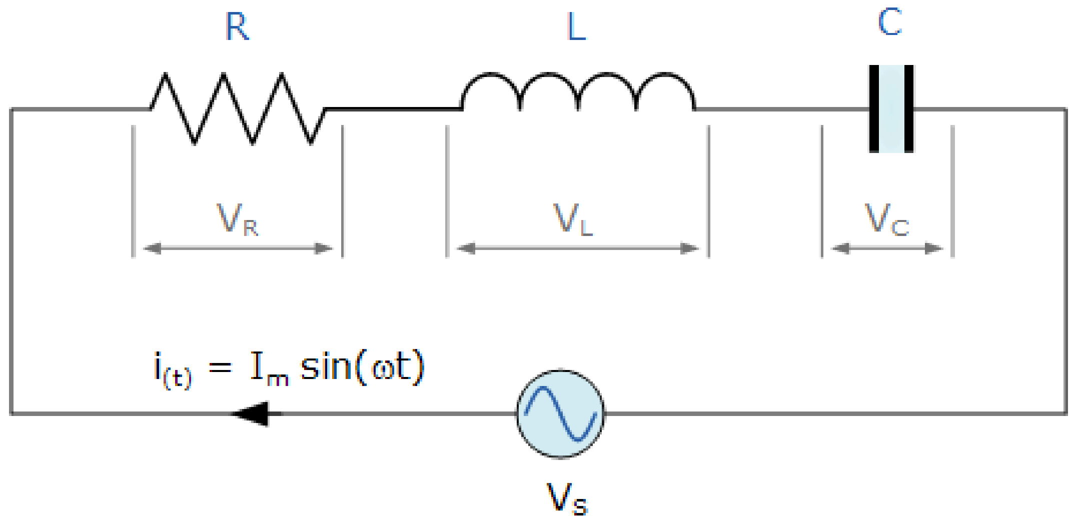
- Say we have a series RLC circuit, what is its complex impedance for a pure AC response?
- For a pure AC response, we can assume $s = j\omega$

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

$$Z = R + j\omega L - \frac{j}{\omega C}$$

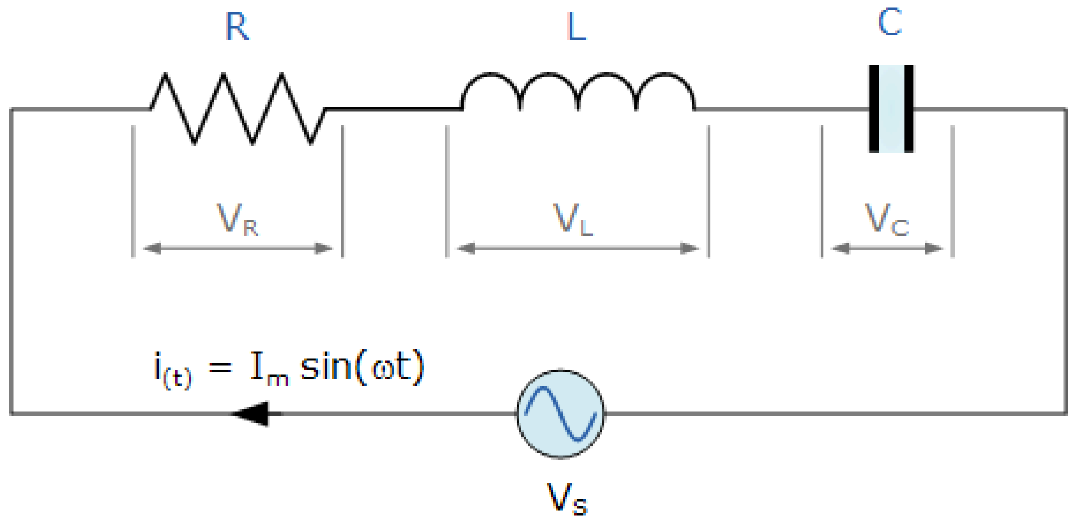
$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$



RLC impedance

- What is the minimum value of Z wrt ω?
- When $\omega = \omega_0 = 1/\sqrt{LC}$, we get $|Z| = R$. This is called resonance and ω_0 is called resonant frequency
- Thus, just like the s-domain analysis, impedances can be approximated as $j\omega L$ and $1/j\omega C$ for ***pure sinusoidal responses***
- In case of other stimuli, a combination of pure AC and pure DC response can also be studied (superposition theorem)



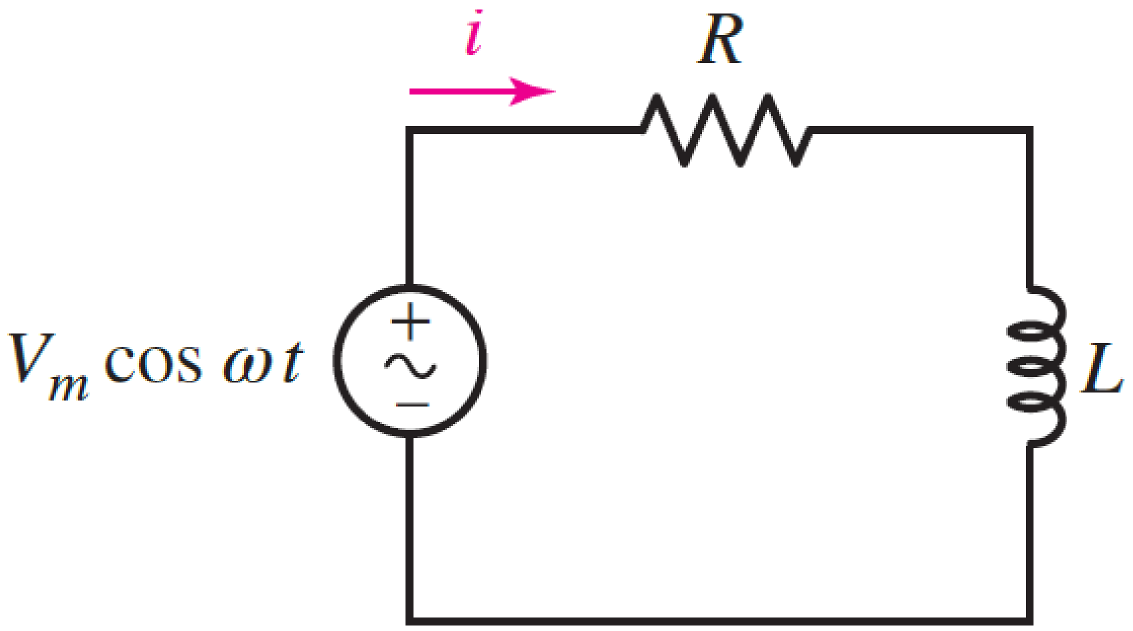
RL circuit

- For AC input, Let us see if we can perform that analysis in the s-domain

$$I(s) = \frac{\left(\frac{V_m s}{s^2 + \omega^2}\right)}{R + sL}$$

$$I(s) = V_m \left[\frac{s}{(s^2+\omega^2)(R + sL)}\right]$$

$$I(s) = V_m \left[\frac{s}{(s + j\omega)(s - j\omega)(R + sL)}\right]$$

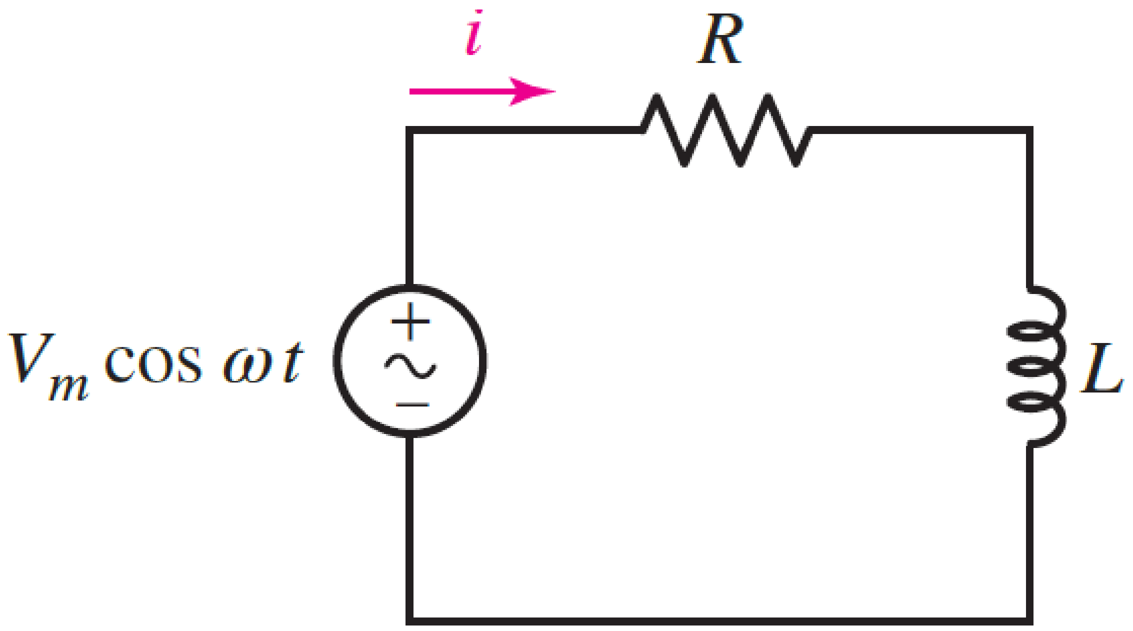


RL circuit

$$I(s) = V_m\left[\frac{s}{(s+j\omega)(s-j\omega)(R+sL)}\right]$$

$$I(s) = V_m\left[\frac{1}{2(R-j\omega L)(s+j\omega)} + \frac{1}{2(R+j\omega L)(s-j\omega)} + \frac{\left(-\frac{R}{L}\right)}{(R^2/L^2+\omega^2)(R+sL)}\right]$$

$$I(s) = V_m\left[\frac{1}{2(R-j\omega L)(s+j\omega)} + \frac{1}{2(R+j\omega L)(s-j\omega)} + \frac{\left(-\frac{R}{L}\right)}{(R^2/L^2+\omega^2)(R+sL)}\right]$$



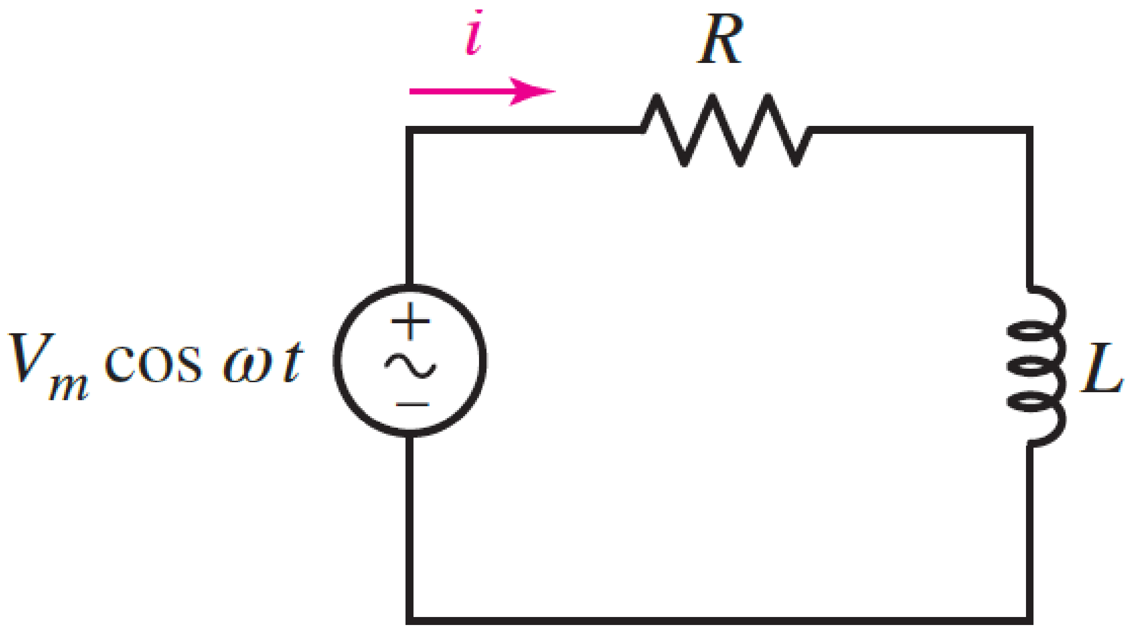
RL circuit

$$I(s) = V_m \left[\frac{sR + \omega^2 L^2}{(R^2+\omega^2 L^2)(s^2 + \omega^2)} + \frac{\left(-\frac{R}{L}\right)}{(R^2/L^2 + \omega^2)(R + sL)} \right]$$

- Thus, the response of the circuit is:

$$i(t) = \left(A \sin \omega t + B \cos \omega t + C\, e^{-\frac{R}{L}t}\right) u(t)$$

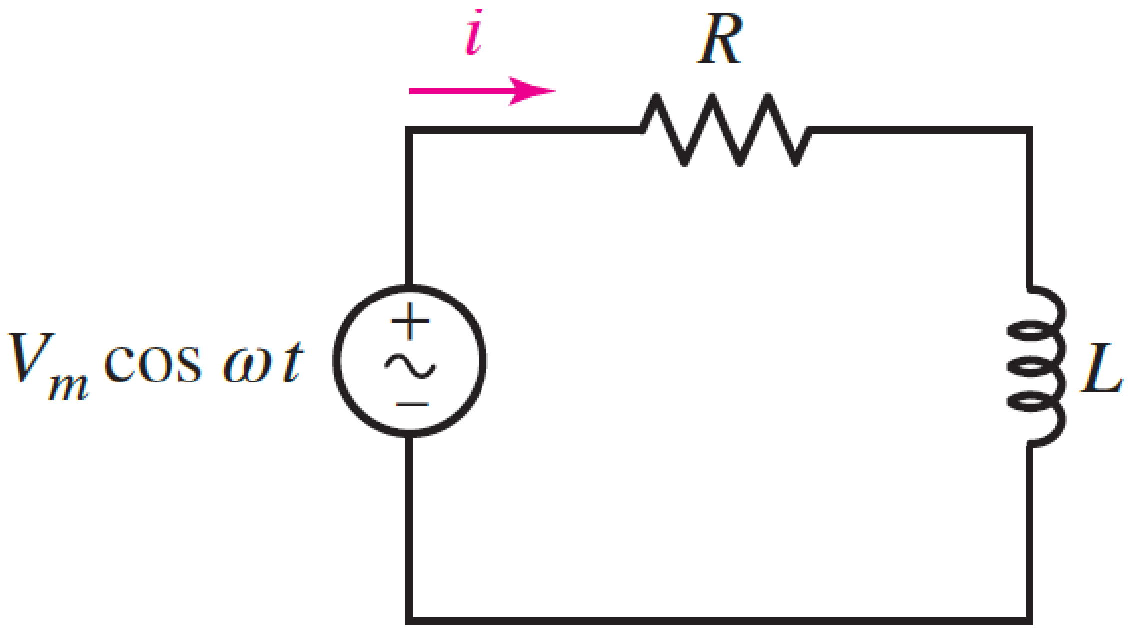
- What is the initial current?



RL circuit

$$i(0^+) = \lim_{s\to\infty}\left\{sV_m\left[\frac{sR+\omega^2L^2}{(R^2+\omega^2L^2)(s^2+\omega^2)} + \frac{\left(-\frac{R}{L}\right)}{\left(\frac{R^2}{L^2}+\omega^2\right)(R+sL)}\right]\right\}$$

$$i(0^+) = \frac{R}{R^2+\omega^2L^2} - \frac{R}{R^2+\omega^2L^2} = 0$$



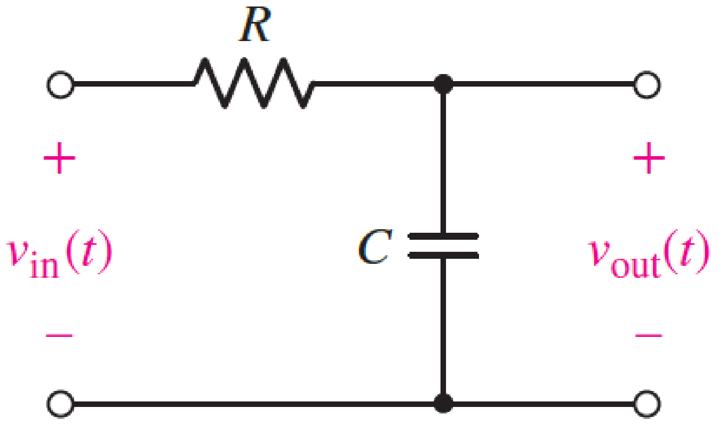
Transfer function

- We can convert any circuit in the s-domain and analyse its response to various input stimuli
- For this, a common technique is to detach the forcing function and obtain a “transfer function” for the circuit for any arbitrary forcing function
- In the given circuit

$$\frac{V_{in} - V_{out}}{R} = \frac{V_{out}}{1/sC}$$

$$V_{out}\left(\frac{1}{R} + sC\right) = \frac{V_{in}}{R}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + sRC} \equiv H(s)$$



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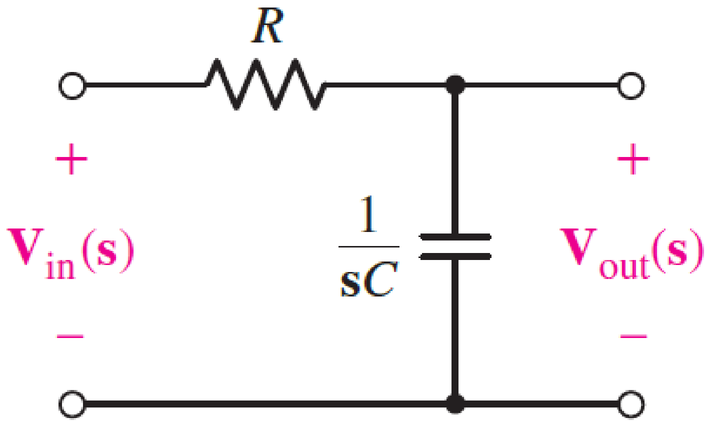
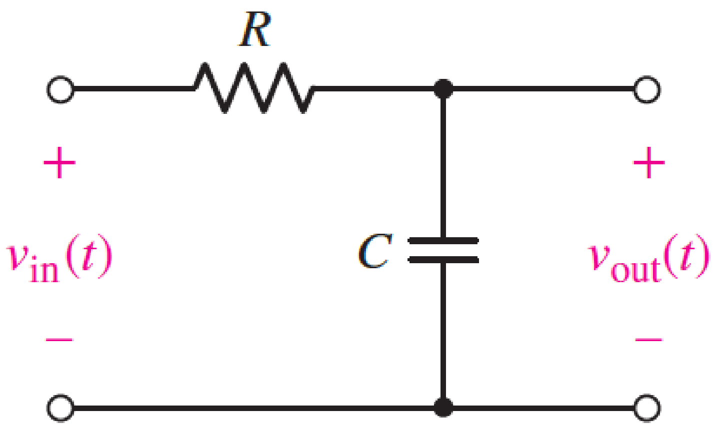


Transfer function

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + sRC} \equiv H(s)$$

- We could just as easily specify a particular current as either the input or output quantity, leading to a different transfer function for the same circuit
- Thus, for any input, the output of the circuit is $V(s)H(s)$
- The inverse Laplace of this provides the response
- However, stability, frequency response etc. can be directly inferred from the poles and zeros of the transfer function



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