V^i, R^i, I and $D_1 \neq D_2^i$ for all i. Hence $D_1 \notin R(G)$ and $D_2 \notin R(G)$. Finally $HR^2 = V = R^2H$. But $H \neq V^i$, R^i, D_1^i, D_2^i, I and $R^2 \neq H^i$ for all i. Hence $H \notin R(G)$. Thus $AC(G)$: I, R, R^2, and R^3.

The converses of Theorem 2 and Theorem 4 are false as Example 2 indicates. Finally the $AC(G)$ is, in general nontrivial, as is demonstrated by Example 3.

In a later paper, the author hopes to obtain results concerning $G/AC(G)$, some relations between the center and the anticenter, and to explore more fully the effect of isomorphism and homomorphism on the anticenter.

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CLASSROOM NOTES

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A DERIVATION OF n-DIMENSIONAL SPHERICAL COORDINATES

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An instructive example in linear algebra is the derivation of n-dimensional spherical coordinates without appealing to geometric intuition. The method of derivation is based on concepts from linear algebra; namely, bases of a vector space, scalar product, angle between vectors and projection of a vector onto a subspace. Spherical coordinates in n-dimensions are a generalization of the usual three-dimensional spherical coordinates and are particularly useful in evaluating certain integrals taken over the surface of an n-dimensional sphere. Later we shall give an example of such an integration.

Let E_n denote real n-dimensional euclidean space. Vectors in E_n will be denoted by bold-faced letters. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in E_n with components ξ_j and η_j, $j=1, \cdots, n$, respectively, we define the scalar product of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x}\cdot\mathbf{y} = \sum_{j=1}^{n} \xi_j\eta_j.$$

The nonnegative number $\|\mathbf{x}\| = (\mathbf{x}\cdot\mathbf{x})^{1/2}$ is called the norm of $\mathbf{x}$. The angle between $\mathbf{x}$ and $\mathbf{y}$ is defined by $\cos\phi = \mathbf{x}\cdot\mathbf{y}/\|\mathbf{x}\|\|\mathbf{y}\|$, where ϕ is restricted to the range $0 \leqq \phi \leqq \pi$. A set of vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ is an orthonormal set in E_n if $\mathbf{x}_i\cdot\mathbf{x}_j = 0$ or 1 accordingly as $i \neq j$ or $i = j$. Any set of n orthonormal vectors forms a basis for E_n.

Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be any orthonormal basis in E_n. Let $\mathbf{x}$ be any vector on the n-dimensional sphere of radius r about the origin, that is, $\|\mathbf{x}\|=r$. If $\mathbf{x}=\sum_{i=1}^n \xi_i\mathbf{e}_i$ then $\|\mathbf{x}\|^2=\sum_{i=1}^n \xi_i^2$. If θ_i is the angle between $\mathbf{x}$ and $\mathbf{e}_i$ then $\xi_i=\mathbf{x}\cdot\mathbf{e}_i=r\cos\theta_i$. Hence $\mathbf{x}=\sum_{i=1}^n r\cos\theta_i\mathbf{e}_i$ and $\mathbf{x}$ can be specified by giving its length r and the n angles θ_i. But since $r^2=\mathbf{x}\cdot\mathbf{x}=r^2\sum_{i=1}^n\cos^2\theta_i$ we see that the θ_i are not independent of each other. Spherical coordinates in n-dimensions show us how to pick out $n-1$ angles $\phi_1, \cdots, \phi_{n-2}, \theta$ which are independent of each other and which, when combined with the norm r, completely describe the vector $\mathbf{x}$ with respect to the given orthonormal basis.

Derivation of the coordinates. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ and $\mathbf{x}$ be as above. Let ϕ_1 be the angle between $\mathbf{x}$ and $\mathbf{e}_1$, $0\leqq\phi_1\leqq\pi$. Then $\xi_1=\mathbf{x}\cdot\mathbf{e}_1=r\cos\phi_1$ and

$$\mathbf{x} = r\cos\phi_1\mathbf{e}_1 + \sum_{j=2}^n \xi_j\mathbf{e}_j.$$

Now

$$r^2 = \|\mathbf{x}\|^2 = r^2\cos^2\phi_1 + \sum_{j=2}^n \xi_j^2 \quad\text{or}\quad \sum_{j=2}^n \xi_j^2 = r^2\sin^2\phi_1.$$

Setting $\xi_j=\alpha_j r\sin\phi_1$, $j=2, \cdots, n$, we have

$$\mathbf{x} = r\cos\phi_1\mathbf{e}_1 + r\sin\phi_1\sum_{j=2}^n \alpha_j\mathbf{e}_j,$$

where $\sum_{j=2}^n\alpha_j^2=1$. (If ϕ_1 is 0 or π, then $\mathbf{x}=\pm r\mathbf{e}_1$.)

Let $\mathbf{u}_2=\sum_{j=2}^n\alpha_j\mathbf{e}_j$. The vector $\mathbf{u}_2$ is a unit vector (that is $\|\mathbf{u}_2\|=1$) in the direction of the projection of $\mathbf{x}$ onto the $(n-1)$-dimensional subspace spanned by $\mathbf{e}_2, \cdots, \mathbf{e}_n$. If ϕ_2 is the angle between $\mathbf{u}_2$ and $\mathbf{e}_2$ then $\cos\phi_2=\mathbf{u}_2\cdot\mathbf{e}_2=\alpha_2$, $0\leqq\phi_2\leqq\pi$, and

$$\mathbf{u}_2 = \cos\phi_2\mathbf{e}_2 + \sum_{j=3}^n \alpha_j\mathbf{e}_j.$$

Hence,

$$1 = \|\mathbf{u}_2\|^2 = \cos^2\phi_2 + \sum_{j=3}^n \alpha_j^2 \quad\text{or}\quad \sum_{j=3}^n \alpha_j^2 = \sin^2\phi_2.$$

If we set $\alpha_j=\beta_j\sin\phi_2$, $j=3, \cdots, n$, then

$$\mathbf{u}_2 = \cos\phi_2\mathbf{e}_2 + \sin\phi_2\sum_{j=3}^n \beta_j\mathbf{e}_j,$$

where $\sum_{j=3}^n\beta_j^2=1$. Thus

$$\mathbf{x} = r\cos\phi_1\mathbf{e}_1 + r\sin\phi_1\cos\phi_2\mathbf{e}_2 + r\sin\phi_1\sin\phi_2\sum_{j=3}^n \beta_j\mathbf{e}_j.$$

In general, let $\mathbf{u}_j$ be the unit vector in the direction of the projection of $\mathbf{x}$ onto the space spanned by $\mathbf{e}_j, \mathbf{e}_{j+1}, \cdots, \mathbf{e}_n$, $j=2, \cdots, n-1$ and let ϕ_{j-1} be the angle between $\mathbf{u}_j$ and $\mathbf{e}_j$, $0 \leqq \phi_j \leqq \pi$, $j=2, \cdots, n-1$. Then

$$\mathbf{x} = \sum_{j=1}^{n-2} r\left(\prod_{k=1}^{j-1} \sin \phi_k\right) \cos \phi_j \mathbf{e}_j + r\left(\prod_{k=1}^{n-2} \sin \phi_k\right) \mathbf{u}_{n-1}.$$

Now $\mathbf{u}_{n-1} = \delta_{n-1}\mathbf{e}_{n-1} + \delta_n \mathbf{e}_n$, where $1 = \|\mathbf{u}_{n-1}\|^2 = \delta_{n-1}^2 + \delta_n^2$. If now we define an angle θ by $\cos \theta = \delta_n$, $\sin \theta = \delta_{n-1}$, we see that $0 \leqq \theta \leqq \pi$ will not suffice since δ_{n-1} can be negative and $\sin \alpha \geqq 0$ for $0 \leqq \theta \leqq \pi$. In order to include all possible combinations of (δ_{n-1}, δ_n) we must have $0 \leqq \theta < 2\pi$.

Thus if $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a given orthonormal basis in E_n and $\mathbf{x}$ is a vector of norm r with components ξ_j with respect to this basis, then

$$(*) \quad \begin{aligned} \xi_1 &= r \cos \phi_1, \\ \xi_j &= r \cos \phi_j \prod_{k=1}^{j-1} \sin \phi_k \qquad (j = 2, \cdots, n-2), \\ \xi_{n-1} &= r \sin \theta \prod_{k=1}^{n-2} \sin \phi_k, \\ \xi_n &= r \cos \theta \prod_{k=1}^{n-2} \sin \phi_k, \end{aligned}$$

where $0 \leqq \phi_j \leqq \pi$, $j=1, \cdots, n-2$; $0 \leqq \theta < 2\pi$; $0 \leqq r < \infty$.

Application to integration. Let $f(\xi_1, \cdots, \xi_n)$ be a continuous real-valued function defined in E_n which may be written in the form

$$f(\xi_1, \cdots, \xi_n) = g(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, \xi_1^2 + \cdots + \xi_n^2),$$

where the α_i are constants independent of the ξ's. We wish to compute the integral of f over the surface of the n-dimensional sphere of radius r with the origin as center. If $\mathbf{x}$ is the vector with coordinates ξ_j and $\mathbf{a}$ the vector with coordinates α_j (these coordinates being with respect to some given orthonormal basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$) then

$$\int_{\xi_1^2+\cdots+\xi_n^2=r^2} f(\xi_1, \cdots, \xi_n) dS = \int_{\|\mathbf{x}\|=r} g(\mathbf{a}\cdot\mathbf{x}, \|\mathbf{x}\|^2) dS,$$

where dS is the surface differential.

Let $\mathbf{a}_1 = \mathbf{a}/\|\mathbf{a}\|$* and choose vectors $\mathbf{a}_2, \cdots, \mathbf{a}_n$ to complete an orthonormal basis in E_n. Let the coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{a}_1, \cdots, \mathbf{a}_n$ be $\zeta_1, \cdots, \zeta_n$. Then $\mathbf{a}_1 \cdot \mathbf{x} = \zeta_1$. Make the spherical coordinate transformation

* If $\mathbf{a}=0$, then $\mathbf{a}_1$ may be any unit vector.

given by (*) with ξ_j replaced by ζ_j, $j=1, \cdots, n$.

The Jacobian of the transformation is

$$J = r^{n-1}\prod_{k=1}^{n-2}\sin^k \phi_{n-1-k}.$$

Also, $\mathbf{a}_1\cdot\mathbf{x}=\zeta_1=r\cos\phi_1$. Thus the integral becomes

$$2\pi r^{n-1}\left[\prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi_{n-1-k}d\phi_{n-1-k}\right]\int_0^\pi g(\|\mathbf{a}\|r\cos\phi_1,\ r^2)\sin^{n-2}\phi_1 d\phi_1$$
$$=\frac{2r^{n-1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\int_0^\pi g(\|\mathbf{a}\|r\cos\phi_1,\ r^2)\sin^{n-2}\phi_1 d\phi_1.$$

Thus we have reduced the integral over the surface of an n-dimensional sphere to a single integral on the real line. In particular, if $f\equiv 1$, we obtain $[2\pi^{n/2}/\Gamma(n/2)]r^{n-1}$ for the surface area of an n-dimensional sphere of radius r and, integrating from 0 to r, we obtain $[2\pi^{n/2}/(n\Gamma(n/2))]r^n$ for the volume of the sphere.

MATRIX INTEGRATION OF $x^k \exp(-\beta^2x^2)$

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Let V be the vector space of finite linear combinations of $x^k\exp(-\beta^2x^2)$, fixed β, $k=0, 1, \cdots$, with basis $\{x^k\exp(-\beta^2x^2)\}$. Let D be a linear transformation on V which differentiates a vector belonging to V.

Since $(x^k\exp(-\beta^2x^2))D=kx^{k-1}\exp(-\beta^2x^2)-2\beta^2x^{k+1}\exp(-\beta^2x^2)$, the matrix of D is

$$\begin{bmatrix} 0 & -2\beta^2 & \cdot & \cdot & \cdots & 0 & \cdots \\ 1 & 0 & -2\beta^2 & \cdot & \cdots & \cdot & \cdots \\ \cdot & 2 & 0 & -2\beta^2 & \cdots & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ \cdot & \cdot & 0 & k & 0 & -2\beta^2 & \cdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

V is closed under D and the kernel of D consists of the zero vector alone. The calculation of D^{-1} may be carried out algebraically, giving an interesting equation for $\int x^k\exp(-\beta^2x^2)dx$.

Because of the nature of D, D^{-1} may be calculated in four independent steps depending on whether k and j are even or odd, where $\|D^{-1}\|=\|a_{kj}\|$. Using $DD^{-1}=I$, we obtain the following expressions for a_{kj}.

(1) j odd, k odd: $\qquad a_{kj}=0;$