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The linear group pursuit problem with fractional derivatives, simple matrices and different opportunities of the players

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The linear group pursuit problem with fractional derivatives, simple matrices and different opportunities of the players

Alena I. Machtakova [1] ·
Nikolai N. Petrov[2]

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Abstract In finite-dimensional Euclidean space, an analysis is made of the problem of pursuit of a single evader by a group of pursuers, which is described by a system of the form

$$D^{(\alpha)}x_i = a_i x_i + u_i,\ u_i \in U_i, \quad D^{(\alpha)}y = ay + v, v \in V,$$

where $D^{(\alpha)}f$ is a Caputo derivative of order α of the function f. The sets of admissible controls U_i, V are convex compacts, and a_i, a are real numbers. The terminal sets are convex compacts. Sufficient conditions for solvability of pursuit-evasion problems are obtained. The method of resolving functions is used as the basic approach.

Keywords capture · multiple capture capture · pursuit · evasion · differential games · conflict-controlled processes · fractional derivatives

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1 Introduction

The work of Rufus Isaacs [18] has laid the foundation for the theory of differential two-player pursuit-evasion games, a fundamental and insightful theory in which various approaches for analysis of conflict situations [29, 4, 19, 14, 20, 17,

[1] Alena I. Machtakova
Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia
Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, S. Kovalevskaya Str., 16, Ekaterinburg,620108, Russia
E-mail: bichurina.alyona@yandex.ru

[2] Nikolai N. Petrov
Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia
Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, S. Kovalevskaya Str., 16, Ekaterinburg, 620108, Russia
E-mail: kma3@list.ru

21, 22, 32] are proposed. The problems of conflict interaction of a group of pursuers and one or several evaders [6, 16, 3, 31] are a generalization of the theory of two-player pursuit-evasion games. It should be noted that there have been no constructive approaches to date that would allow the theory of differential two-player games to be used for analysis of group pursuit game problems. One of the reasons for this is that the union of sets of attainability of all pursuers and the union of all goal sets are sets that are not convex and, moreover, not connected. On the other hand, there are some applications of these games to problems concerning the motion of vehicles, avoidance of collisions of ships, etc. One of the directions in the development of group pursuit theory at the present time is the search for new problems to which the previously developed methods are applicable. In particular, the authors of [13, 7, 8, 9] address the pursuit problems involving two players, which are described by equations with fractional derivatives, where sufficient conditions for capture are obtained. In [15], a proof is given of the existence of the price of the game in a differential game described by an equation with fractional derivatives. Group pursuit-evasion problems with fractional derivatives under the condition that all participants have equal opportunities are treated in [23, 24, 25, 2, 26].

This paper deals with the problem of capture of an evader by a group of pursuers in a differential game with fractional derivatives without the assumption that all opportunities of the participants in the conflict are equal. Sufficient conditions for capture are obtained.

2 Formulation of the problem

Definition 1 [5] Let $\alpha \in (0,1)$, $f\colon [0,\infty) \to \mathbb{R}^n$ be a function such that f' is absolutely continuous on $[0,\infty)$. A Caputo derivative of order α of the function f is a function $D^{(\alpha)}f$ of the form

$$\big(D^{(\alpha)}f\big)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)}{(t-s)^{\alpha}}\,ds, \quad \text{where } \Gamma(\beta) = \int_0^{\infty} e^{-s}s^{\beta-1}\,ds.$$

In the space $\mathbb{R}^k (k \geqslant 2)$ we consider a $G(n+1)$-player differential game which involves n pursuers $P_1, \ldots, P_n$ and evader E. The law of motion of each of pursuers P_i has the form

$$D^{(\alpha)}x_i = a_i x_i + u_i, \quad x_i(0) = x_i^0, \quad u_i \in U_i. \tag{1}$$

The law of motion of evader E has the form

$$D^{(\alpha)}y = ay + v, \quad y(0) = y^0, \quad v \in V. \tag{2}$$

Here $i \in I = \{1, \ldots, n\}$, $x_i, y, u_i, v \in \mathbb{R}^k$, U_i, V are convex compacts $\mathbb{R}^k$, $\alpha \in (0,1)$, $D^{(\alpha)}f$ is the Caputo derivative of the function f of order α, and a_i and a are real numbers. We assume that $x_i^0 - y^0 \notin M_i$ for all $i \in I$, where $M_i, i \in I$ are given convex compacts.

3 Sufficient conditions for capture

Let $v : [0,\infty) \to V$ be a measurable function. Let us call the restriction of the function v on $[0,t]$ the prehistory $v_t(\cdot)$ of the function v at time t. The measurable function $v : [0,\infty) \to V$ is called admissible.

Definition 2 We will say that a quasi-strategy $\mathcal{U}_i$ of pursuer P_i is given if a map U_i^0 is defined which associates the measurable function $u_i(t)$ with values in U_i to the initial positions $z^0 = (x_1^0,\ldots,x_n^0,y^0)$, time t and an arbitrary prehistory of control $v_t(\cdot)$ of evader E.

Definition 3 A capture occurs in the game $G(n+1)$ if there exist time $T>0$ and quasi-strategies $\mathcal{U}_1,\ldots,\mathcal{U}_n$ of pursuers $P_1,\ldots,P_n$ such that for any measurable function $v(\cdot)$, $v(t)\in V$, $t\in[0,T]$ there exist time $\tau\in[0,T]$ and a number $l\in I$ for which $x_l(\tau)-y(\tau)\in M_l$.

We introduce the following notation:

$$E_\rho(B,\mu)=\sum_{k=0}^{\infty}\frac{B^k}{\Gamma(k\rho^{-1}+\mu)} \text{ is a generalized Mittag-Leffler function}$$

$$(\rho>0,\mu\in\mathbb{R}^1, B \text{ is a square matrix}),$$

$$f_i(t)=E_{1/\alpha}(a_it^\alpha,1),\quad f(t)=E_{1/\alpha}(at^\alpha,1),$$

$$g_i(t,\tau)=(t-\tau)^{\alpha-1}E_{1/\alpha}\big(a_i(t-\tau)^\alpha,\alpha\big),\quad F_i(t)=\int_0^t g_i(t,\tau)\,d\tau,$$

$$g(t,\tau)=(t-\tau)^{\alpha-1}E_{1/\alpha}\big(a(t-\tau)^\alpha,\alpha\big),\quad F(t)=\int_0^t g(t,\tau)\,d\tau,$$

$\mathrm{Int}A$ and $\mathrm{co}A$ are, respectively, the interior and the convex hull of the set A. Note that it follows from Theorem 4.1.1.[30] that for all $i\in I, t\geqslant 0$, $\tau\in[0,t]$ the inequalities $g_i(t,\tau)\geqslant 0$ and $g(t,\tau)\geqslant 0$ hold.

Let $\gamma_i(t,\tau), i\in I\ t\geqslant 0$, $\tau\in[0,t]$ be some bounded functions measurable in (t,τ) and locally summable in τ (at each t). Following [11], we will call them *translation functions*. Let us fix some set of translation functions $\gamma(t,\tau)=\{\gamma_i(t,\tau), i\in I\}$ and denote

$$\xi_i(t)=f_i(t)x_i^0-f(t)y^0+\int_0^t\gamma_i(t,\tau)\,d\tau.$$

Consider the multivalued maps

$$W_i(t,\tau,v,\gamma_i)=\big\{\lambda\geqslant 0\big|\ g_i(t,\tau)U_i-g(t,\tau)v-\gamma_i(t,\tau)\cap\lambda(M_i-\xi_i(t))\neq\emptyset\big\},$$

$$W_i(t,\tau,\gamma_i)=\bigcap_{v\in V}W_i(t,\tau,v,\gamma_i).$$

Assumption 1. There exists translation functions $\gamma(t,\tau) = \{\gamma_i(t,\tau), i \in I\}$ such that for all $i \in I$, $t \geqslant 0$, $0 \leqslant \tau \leqslant t$ the following condition holds:

$$0 \in W_i(t,\tau,\gamma_i).$$

Theorem 1 *Suppose that Assumption 1 holds and there exist $T > 0$ and $l \in I$ such that $\xi_l(T) \in M_l$. Then a capture occurs in the game $G(n+1)$.*

Proof Consider the multivalued map

$$U_l(T,\tau,v) = \{u_l \in U_l \mid g_l(T,\tau)u_l - g(T,\tau)v - \gamma_l(T,\tau) = 0\}.$$

By virtue of Assumption 1, $U_l(T,\tau,v) \neq \varnothing$ for all $\tau \in [0,T]$, $v \in V$. It follows from the measurable choice theorem [1] that there exists a measurable selector $u_l^2(\tau,v) \in U_l(T,\tau,v)$. Assume that the control of pursuer P_l is

$$u_l(\tau) = u_l^2(T,\tau,v(\tau)),\ \tau \in [0,T].$$

We define the controls of the other pursuers in an arbitrary way. The solution to the Cauchy problem for the systems (1) and (2) can be represented in the form [10]

$$x_l(t) - y(t) = \xi_l(t) + \int\limits_0^t \Big(g_l(t,\tau)u_l(\tau) - g(t,\tau)v(\tau) - \gamma_l(t,\tau)\Big)\,d\tau.$$

Therefore, $x_l(T) - y(T) = \xi_l(T) \in M_l$. This proves the theorem.

Corollary 1 *Suppose there exists a number $l \in I$ such that*
1. $0 \in g_l(t,\tau)U_l - g(t,\tau)v$ *for all* $v \in V$, $t \geqslant 0$, $\tau \in [0,t]$;
2. $a_l < 0$, $a < 0$;
3. $0 \in \mathrm{Int} M_l$.
Then a capture occurs in the game $G(n+1)$.

Proof Taking $\gamma_l(t,\tau) = 0$ for all $t \geqslant 0$, $\tau \in [0,t]$, we find that $0 \in W_l(t,\tau,\gamma_l)$ for all $t \geqslant 0$, $\tau \in [0,t]$. Then $\xi_l(t) = f_l(t)x_l^0 - f(t)y^0$. It follows from [30, p. 12] that the following asymptotic estimates hold for $t \to +\infty$:

$$f(t) = -\frac{1}{at^\alpha \Gamma(1-\alpha)} + O\left(\frac{1}{t^{2\alpha}}\right), \quad f_l(t) = -\frac{1}{a_l t^\alpha \Gamma(1-\alpha)} + O\left(\frac{1}{t^{2\alpha}}\right).$$

Therefore, $\lim\limits_{t\to+\infty} \xi_l(t) = 0$. Consequently, there exists $T > 0$ for which $\xi_l(T) \in M_l$. It remains to apply Theorem 1. This proves the corollary.

In what follows, we will assume that $\xi_i(t) \notin M_i$ for all $i \in I$, $t \geqslant 0$. To each pursuer $P_i, i \in I$ we associate the resolving function

$$\lambda_i(t,\tau,v) = \sup\{\lambda \geqslant 0,\ \big|\lambda \in W_i(t,\tau,v,\gamma_i)\}.$$

Theorem 2 *Suppose Assumption 1 holds and there exists $T > 0$ such that*

$$\inf_{v(\cdot)} \max_{i \in I} \int_0^T \lambda_i(T, \tau, v(\tau))\, d\tau \geqslant 1.$$

Then a capture occurs in the game $G(n+1)$.

Proof Let $v(\cdot)$ be an arbitrary admissible function. Define the functions

$$h_i(t, v(\cdot)) = 1 - \int_0^t \lambda_i(T, \tau, v(\tau))\, d\tau,$$

and the sets and time instants

$$T_i(v(\cdot)) = \{\tau \in [0, T] \mid h_i(\tau) = 0\},$$
$$t_i^*(v(\cdot)) = \begin{cases} \inf\{\tau \mid \tau \in T_i(v(\cdot))\} & \text{if } T_i(v(\cdot)) \neq \emptyset, \\ +\infty, & \text{if } T_i(v(\cdot)) = \emptyset. \end{cases}$$

It follows from the conditions of the theorem that there exists a number $l \in I$ for which $t_l^*(v(\cdot)) \leqslant T$. Consider the multivalued maps

$$U_i^1(\tau, v) = \{u_i \in U_i \mid g_i(T, \tau)u_i - g(T, \tau)v - \gamma_i(T, \tau) \in \lambda_i(T, \tau, v)(M_i - \xi_i(T))\},$$
$$U_i^2(\tau, v) = \{u_i \in U_i \mid g_i(T, \tau)u_i - g(T, \tau)v - \gamma_i(T, \tau) = 0\}.$$

It follows from the measurable choice theorem [1] that there exist measurable selectors $u_i^1(\tau, v) \in U_i^1(\tau, v)$, $u_i^2(\tau, v) \in U_i^2(\tau, v)$. Specify the controls of pursuers $P_i, i \in I$ as follows. If $t_i^*(v(\cdot)) < T$, we assume

$$u_i(t) = \begin{cases} u_i^1(t, v(t)), & t \in [0, t_i^*(v(\cdot))], \\ u_i^2(t, v(t)), & t \in (t_i^*(v(\cdot)), T]. \end{cases}$$

If $t_i^*(v(\cdot)) \geqslant T$, we assume

$$u_i(t) = u_i^1(t, v(t)), \ t \in [0, T].$$

The solution to the Cauchy problem for the systems (1) and (2) can be represented as [10]

$$x_i(t) - y(t) = \xi_i(t) + \int_0^t \Big(g_i(t, \tau)u_l(\tau) - g(t, \tau)v(\tau) - \gamma_i(t, \tau) \Big)\, d\tau.$$

Therefore,

$$x_l(T) - y(T) \in \xi_l(T) + \int_0^{t_l^*(v(\cdot))} \lambda_l(T, \tau, v(\tau))(M_l - \xi_l(T))\, d\tau =$$

$$\xi_l(T)\Big(1 - \int_0^{t_l^*(v(\cdot))} \lambda_l(T, \tau, v(\tau))\, d\tau\Big) + \int_0^{t_l^*(v(\cdot))} \lambda_l(T, \tau, v(\tau)) M_l\, d\tau \subset M_l.$$

Consequently, a capture occurs in the game $G(n+1)$. This proves the theorem.

Theorem 3 *Suppose that $M_i = \{0\}$ for all $i \in I$, Assumption 1 holds and there exists $T > 0$ such that*

$$\inf_{v(\cdot)} \max_i \int_0^T \frac{\lambda_i^0(T, \tau, v(\tau))}{\|\xi_i(T)\|}\, d\tau \geqslant 1,$$

where $\xi_i^0(t) = \xi_i(t)/\|\xi_i(t)\|$,

$$\lambda_i^0(t, \tau, v) = \sup\{\lambda \geqslant 0 \big| -\lambda \xi_i^0(t) \in g_i(t, \tau)U_i - g(t, \tau)v - \gamma_i(t, \tau)\}.$$

Then a capture occurs in the game $G(n+1)$.

Proof Let $v(\cdot)$ be an arbitrary admissible function. Define the functions

$$h_i(t) = 1 - \int_0^t \frac{\lambda_i^0(T, \tau, v(\tau))}{\|\xi_i(T)\|}\, d\tau$$

and the sets and time instants

$$T_i(v(\cdot)) = \{\tau \in [0, T] \bigm| h_i(\tau) = 0\},$$
$$t_i^*(v(\cdot)) = \begin{cases} \inf\{\tau \bigm| \tau \in T_i(v(\cdot))\} & \text{if } T_i(v(\cdot)) \neq \emptyset, \\ +\infty, & \text{if } T_i(v(\cdot)) = \emptyset. \end{cases}$$

It follows from the condition of the theorem that there exists a number $l \in I$ for which $t_l^*(v(\cdot)) \leqslant T$. Consider the multivalued maps

$$U_i^1(\tau, v) = \{u_i \in U_i \bigm| g_i(T, \tau)u_i - g(T, \tau)v - \gamma_i(T, \tau) = -\lambda_i^0(T, \tau, v)\xi_i^0(T))\},$$
$$U_i^2(\tau, v) = \{u_i \in U_i \bigm| g_i(T, \tau)u_i - g(T, \tau)v - \gamma_i(T, \tau) = 0\}.$$

It follows from the measurable choice theorem [1] that there exist measurable selectors $u_i^1(\tau, v) \in U_i^1(\tau, v)$, $u_i^2(\tau, v) \in U_i^2(\tau, v)$. Specify the controls of pursuers $P_i, i \in I$ as follows. If $t_i^*(v(\cdot)) < T$, we assume

$$u_i(t) = \begin{cases} u_i^1(t, v(t)), & t \in [0, t_i^*(v(\cdot))], \\ u_i^2(t, v(t)), & t \in (t_i^*(v(\cdot)), T]. \end{cases}$$

If $t_i^*(v(\cdot)) \geqslant T$, we assume

$$u_i(t) = u_i^1(t, v(t)), \ t \in [0, T].$$

The solution to the Cauchy problem for the systems (1) and (2) can be represented in the form [10]

$$x_i(t) - y(t) = \xi_i(t) + \int_0^t \Big(g_i(t, \tau)u_l(\tau) - g(t, \tau)v(\tau) - \gamma_i(t, \tau)\Big)\, d\tau$$

Therefore,

$$x_l(T) - y(T) = \xi_l(T) - \int\limits_0^{t_l^*(v(\cdot))} \lambda_l^0(T, \tau, v(\tau))\xi_l^0(T)\, d\tau =$$

$$= \xi_l^0(T)\|\xi_l(T)\|\Big(1 - \int\limits_0^{t_l^*(v(\cdot))} \frac{\lambda_l^0(T, \tau, v(\tau))}{\|\xi_l(T)\|}\, d\tau\Big) = 0.$$

This proves the theorem.

Next, we consider in more detail the situation in which for all $i \in I$

$$M_i = \{0\}, \quad U_i = \{u_i \mid \|u_i - b_i\| \leqslant R_i\}, \quad V = \{v \mid \|v - b\| \leqslant R\}, \qquad (3)$$

where $b, b_i \in R^k, i \in I, R_i, i \in I, R$ are positive real numbers and the Euclidean norm is used.

Assumption 2. $R_i g_i(t, \tau) \geqslant R g(t, \tau)$ for all $i \in I$, $t \geqslant 0$, $\tau \in [0, t]$.

It follows from this assumption that Assumption 1 is satisfied if we take $\gamma_i(t, \tau) = g_i(t, \tau) b_i - g(t, \tau) b$ as $\gamma_i(t, \tau)$. Then

$$\xi_i(t) = f_i(t) x_i^0 - f(t) y^0 + F_i(t) b_i - F(t) b.$$

Further, let $\delta(t) = \min\limits_{\|p\|=1} \max\limits_i (p, \xi_i^0(t))$, where (a, b) is the scalar product of the vectors a and b.

Assumption 3. There exists time $T > 0$ such that

a) $0 \in \operatorname{Intco}\{\xi_i^0(T), i \in I\}$,

b) the following inequality holds:

$$\sum_{i \in I} \|\xi_i(T)\| \leqslant 2\delta(T) R F(T). \qquad (4)$$

Lemma 1 *Suppose that M_i, U_i and V are defined by relation* (3) *and Assumptions 2 and 3 are satisfied. Then for any admissible function $v(\cdot)$ the following inequality is satisfied:*

$$\sum_{i \in I} \Big(\|\xi_i(T)\| - \int\limits_0^T \lambda_i^0(T, \tau, v(\tau))\, d\tau \Big) \leqslant 0.$$

Proof It follows from the definition of the functions λ_i^0 that

$$\lambda_i^0(T, \tau, v) = g(T, \tau)(v - b, \xi_i^0(T)) +$$
$$\sqrt{g^2(T, \tau)(v - b, \xi_i^0(T))^2 + R_i^2 g_i^2(T, \tau) - \|v - b\|^2 g^2(T, \tau)}.$$

Denote $\hat{v} = v - b$. Then $\|\hat{v}\| \leqslant R$. It follows from Assumption 2 that for all $t \in [0, T]$, $i \in I$, $v \in V$ the following inequalities hold:

$$\lambda_i^0(T, t, v) \geqslant g(T, t)\Big((\hat{v}, \xi_i^0(T)) + |(\hat{v}, \xi_i^0(T))|\Big). \qquad (5)$$

The functions $\lambda_i^0(T,t,v)$ are concave in v [16] for each fixed t. Therefore, there exists $\hat{v}_0, \|\hat{v}_0\| = 1$ for which

$$\sum_{i\in I} \lambda_i^0(T,t,\hat{v}) \geqslant \min_{\|\hat{v}\|\leqslant R} \sum_{i\in I} \lambda_i^0(T,t,\hat{v}) = \sum_{i\in I} \lambda_i^0(T,t,R\hat{v}_0).$$

From item a) of Assumption 3 and from Ref. [27] it follows that for each $v, \|v\| = 1$ there is a number $l \in I$ for which $(v, \xi_l^0(T)) \geqslant \delta(T) > 0$. Hence, from (5) we obtain

$$\lambda_l^0(T,t,R\hat{v}_0) \geqslant 2g(T,t)R \cdot (\hat{v}_0, z_l^0(T)) \geqslant 2R\delta(T)g(T,t).$$

Then for each $t \in [0,T]$, $v \in V$ the following inequality holds:

$$\sum_{i\in I} \lambda_i^0(T,t,v) \geqslant \sum_{i\in I} \lambda_i^0(T,t,R\hat{v}_0) \geqslant \lambda_l^0(T,t,R\hat{v}_0) \geqslant 2Rg(T,t)\delta(T).$$

Next, we have

$$\sum_{i\in I} \|\xi_i(T)\| - \int_0^T \sum_{i\in I} \lambda_i^0(T,t,v(t))\,dt \leqslant \sum_{i\in I} \|\xi_i(T)\| - \int_0^T 2\delta(T)Rg(T,t)\,dt =$$
$$= \sum_{i\in I} \|\xi_i(T)\| - 2R\delta(T)F(T) \leqslant 0.$$

This proves the lemma.

Corollary 2 *Let the conditions of Lemma 1 be satisfied. Then for any admissible function $v(\cdot)$ there is a number $l \in I$ for which t*

$$\|\xi_l(T)\| - \int_0^T \lambda_l^0(T,t,v(t))\,dt \leqslant 0.$$

Theorem 4 *Suppose that M_i, U_i and V are defined by relation (3) and Assumptions 2 and 3 are satisfied. Then a capture occurs in the game $G(n+1)$.*

Proof Let $v(\cdot)$ be an admissible function. By virtue of Corollary 1 there exists a number $l \in I$ for which

$$\|\xi_l(T)\| - \int_0^T \lambda_l^0(T,t,v(t))\,dt \leqslant 0 \text{ or } 1 - \int_0^T \frac{\lambda_l^0(T,t,v(t))}{\|\xi_l(T)\|}\,dt \leqslant 0.$$

It remain to apply Theorem 3. This proves the theorem.

Corollary 3 *[23]. Let $a_i = a \leqslant 0$, $b_i = b = 0$, $R_i = R > 0$, $M_i = \{0\}$ for all $i \in I$ and*

$$y^0 \in \operatorname{Intco}\{x_1^0, \ldots, x_n^0\}.$$

Then a capture occurs in the game $G(n+1)$.

Proof In this case, $f_i(t) = f(t)$, $F_i(t) = F(t) = t^\alpha E_{1/\alpha}(at^\alpha, \alpha + 1)$. For all $i \in I$, $t \geqslant 0$. Taking $\gamma_i(t,\tau) = 0$, we find that $\xi_i(t) = f(t)(x_i^0 - y^0)$. Therefore, for all $T > 0$

$$\sum_{i\in I} \|\xi_i(T)\| = |f(T)| \sum_{i\in I} \|\xi_i^0(0)\| = f(T) \sum_{i\in I} \|\xi_i^0(0)\|,$$

$\delta(T) = \delta(0) > 0$ by virtue of [27]. Hence, inequality (4) can be represented as

$$\sum_{i\in I} \|\xi_i(0)\| \leqslant 2\delta(0) \frac{F(T)}{f(T)}. \tag{6}$$

If $a < 0$, then the following asymptotic estimates [30, p. 12] hold for $t \to +\infty$:

$$f(t) = -\frac{1}{at^\alpha \Gamma(1-\alpha)} + O\left(\frac{1}{t^{2\alpha}}\right), \quad F(t) = -\frac{1}{a} + O\left(\frac{1}{t^\alpha}\right).$$

Consequently,

$$\frac{F(T)}{f(T)} = T^\alpha \Gamma(1-\alpha) + O\left(T^\alpha\right),$$

and hence inequality (6) will be satisfied automatically for sufficiently large T.

If $a = 0$, then $f(t) = 1$, $F(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$. Hence, $\frac{F(T)}{f(T)} = \frac{T^\alpha}{\Gamma(1+\alpha)}$, and therefore inequality (6) will be satisfied automatically for sufficiently large T. This proves the corollary.

4 Sufficient conditions for evasion

Definition 4 An evasion from an encounter occurs in the game $G(n+1)$ if there exists a programmed control $v(\cdot)$ of evader E such that for any trajectories $x_i(t), i \in I$ of pursuers $P_i, i \in I$ and for any $t > 0$ one has $x_i(t) \neq y(t)$.

Theorem 5 *Suppose that M_i, U_i and V are defined by relation* (3) *and there exists a vector $p \in R^k$, $\|p\| = 1$ such that*

1) for all $i \in I$, $t \geqslant 0$ the following inequalities hold:

$$\|b + Rp\| F(t) \geqslant (\|b_i\| + R_i) F_i(t), \quad f(t) \geqslant f_i(t),$$

2) $(b + Rp, y^0) \geqslant 0, \quad (b + Rp, y^0) \geqslant \max\limits_{i\in I}(b + Rp, x_i^0)$. Then an evasion from an encounter occurs in the game $G(n+1)$.

Proof Specify the control of evader E as follows. Assume $v(t) = b + Rp$ for all $t \in [0, +\infty)$. Let $u_i(t), i \in I$ be arbitrary controls of pursuers $P_i, i \in I$. Define the functions $\hat{u}_i(t) = u_i(t) - b_i$,

$$\widetilde{u}_i(t) = \frac{\int\limits_0^t g_i(t,s)\hat{u}_i(s)\, ds}{R_i F(t)}, \quad \omega_i(t) = f(t)y^0 - f_i(t)x_i^0.$$

Then for all $t \geqslant 0$, $i \in I$

$$\|\hat{u}_i(t)\| \leqslant R_i, \quad \|\widetilde{u}_i(t)\| \leqslant 1,$$
$$y(t) - x_i(t) = \omega_i(t) + (b + Rp)F(t) - (b_i + R_i\widetilde{u}_i(t))F_i(t).$$

We prove that $(b+Rp, \omega_i(t)) \geqslant 0$ for all $i \in I$, $t \geqslant 0$. Indeed, $(b+Rp, y^0) \geqslant (b + Rp, x_i^0)$ for all $i \in I$. Since [28] $f_i(t) \geqslant 0$, $f(t) \geqslant 0$ and $f(t) \geqslant f_i(t)$ for all $i \in I$, $t \geqslant 0$, it follows that

$$f_i(t)(b + Rp, x_i^0) \leqslant f_i(t)(b + Rp, y^0) \leqslant f(t)(b + Rp, y^0) \text{ or } (b + Rp, \omega_i(t)) \geqslant 0.$$

Therefore, for all $i \in I$, $t \geqslant 0$ the following inequalities hold:

$$\|y(t) - x_i(t)\| \geqslant \|\omega_i(t) + (b + pR)F(t)\| - \|(b_i + \widetilde{u}_i(t)F_i(t)R_i\| =$$
$$= \sqrt{\|\omega_i(t)\|^2 + 2(b + Rp, \omega_i(t))F(t) + \|b + Rp\|^2F^2(t)} - F_i(t)(\|b_i\| + R_i) >$$
$$\|b + Rp\|F(t) - (\|b_i + R_i)F_i(t) \geqslant 0.$$

This proves the theorem.

Corollary 4 *Let $M_i = \{0\}$, $b_i = b = 0$ for all $i \in I$. In addition, for all $t \geqslant 0$, $i \in I$ the following inequalities hold:*

$$RF(t) \geqslant R_iF_i(t), \quad f(t) \geqslant f_i(t)$$

and there exists a vector $p \in R^k$, $\|p\| = 1$ such that

$$(p, y^0) \geqslant 0, \quad (p, y^0) \geqslant \max_i (p, x_i^0).$$

Then an evasion from an encounter occurs in the game $G(n + 1)$.

Corollary 5 *Let $a_i = a$, $R_i = R$ $M_i = \{0\}$, $b_i = b = 0$ for all $i \in I$ and*

$$y^0 \notin \operatorname{co}\{x_1^0, \dots, x_n^0\}.$$

Then an evasion from an encounter occurs in the game $G(n + 1)$.

Proof It follows from the conditions of the CΓ́orollary that y^0 and $\operatorname{co}\{x_1^0, \dots, x_n^0\}$ are separable. Therefore, there exists a vector $p \in R^k$, $\|p\| = 1$ such that for all $i \in I$ $(p, y^0) \geqslant (p, x_i^0)$ or $(p, y^0 - x_i^0) \geqslant 0$. Specify the control of evader E, assuming that $v(t) = Rp$ for all $t \geqslant 0$. Then

$$\omega_i(t) = f(t)(y^0 - x_i^0), \; y(t) - x_i(t) = \omega_i(t) + RpF(t) - \widetilde{u}_i(t)RF(t).$$

Therefore,

$$\|y(t) - x_i(t)\| \geqslant \|\omega_i(t) + RpF(t)\| - RF(t) =$$
$$\sqrt{\|\omega_i(t)\|^2 + 2R(p, y^0 - x_i^0)f(t)F(t) + R^2F^2(t)} - RF(t) > 0$$

for all $t > 0$. This proves the corollary.

5 Examples

Examples

Example 1 In the space $\mathbb{R}^k$ we consider a game $G(3)$ described by a system of the form

$$D^{(\alpha)}x_i = u_i, \quad x_1(0) = (1, 0, \dots, 0), \quad x_2(0) = (-1, 0, \dots, 0),$$
$$D^{(\alpha)}y = v, \quad y(0) = (0, 0, \dots, 0), i \in I = \{1, 2\},$$

where $u_i, v \in V = \{(v_1, \dots, v_k) \in \mathbb{R}^k \mid v_j \in [-1, 1], j = 1, \dots, k\}$, $M_i = \{0\}$.

Then Assumption 1 will be satisfied with zero translation functions. Next, we have $f_i(t) = f(t) = 1$, $g_i(t, s) = g(t, s) = (t - s)^{\alpha-1}$, $\xi_i(t) = x_i^0 - y^0$, $\lambda_i(t, \tau, v) = g(t, \tau)\overline{\lambda}_i(v)$, where $\overline{\lambda}_i(v) = \sup\{\lambda \geqslant 0 \mid -\lambda\xi_i(t) \in V - v\}$. Therefore, $\overline{\lambda}_1(v) = 1 + v_1$, $\overline{\lambda}_2(v) = 1 - v_1$, $\max\{\overline{\lambda}_1(v), \overline{\lambda}_2(v)\} \geqslant 1$. Consequently, for any admissible function $v(\cdot)$ we have

$$\int_0^t \big(\lambda_1(t, \tau, v(\tau)) + \lambda_2(t, \tau, v(\tau))\big)\, d\tau \geqslant$$
$$\int_0^t \max_i \lambda_i(t, \tau, v(\tau))\, d\tau \geqslant \int_0^t g(t, \tau)\, d\tau = \frac{t^\alpha}{\alpha}.$$

Taking T so that the inequality $T^\alpha \geqslant 2\alpha$ is satisfied, we find that for any admissible function $v(\cdot)$ there is a number $l \in \{1, 2\}$ for which

$$\int_0^T \lambda_l(T, \tau, v(\tau))\, d\tau \geqslant 1.$$

Thus, the condition of Theorem 2 is satisfied. Therefore, a capture occurs in the game $G(3)$.

Example 2 In the space $\mathbb{R}^2$ we consider a game $G(4)$ which is described by a system of the form

$$D^{(\alpha)}x_i = ax_i + u_i, \quad \|u_i\| \leqslant 1, \quad D^{(\alpha)}y = ay + v, \quad \|v\| \leqslant 1, \quad i \in I = \{1, 2, 3\},$$

where M_i, U_i and V are defined by relation (3), and $R_i = R > 0$ for all i. Then Assumption 2 is satisfied. Let us fix time $T > 0$ and choose x_i^0, y^0 such that the following equations hold (with $(\mu > 0)$:

$$\xi_1(T) = (0, -\mu), \quad \xi_2(T) = (\frac{1}{2}\mu, -\frac{\sqrt{3}}{2}\mu), \quad \xi_3(T) = (-\frac{1}{2}\mu, -\frac{\sqrt{3}}{2}\mu).$$

This gives

$$0 \in \operatorname{Intco}\{\xi_i^0(T), i \in I\}, \quad \delta(T) = \frac{\sqrt{3}}{2}, \quad \sum_{i \in I} \|\xi_i(T)\| = 3f(T)\mu.$$

Proposition 1 *If $0 < \mu < \frac{\sqrt{3}}{3} RF(T)$, then a capture occurs in the game $G(4)$.*

Example 3 In the space R^2 we consider a game $G(4)$ which is described by a system of the form

$$D^{(\alpha)}x_1 = u_i, \quad \|u_1\| \leqslant 1, \quad D^{(\alpha)}x_2 = u_2, \quad \|u_2\| \leqslant 1,$$

$$D^{(\alpha)}x_3 = u_3, \quad \|u_3\| \leqslant \frac{1}{4}, \quad D^{(\alpha)}y = v, \quad \|v\| \leqslant 1, \quad i \in I = \{1,2,3\},$$

$$x_1^0 = (-\frac{1}{2}\mu, -\frac{\sqrt{3}}{2}\mu), \quad x_2^0 = (-\frac{1}{2}\mu, \frac{\sqrt{3}}{2}\mu), \quad x_3^0 = (\mu, 0), \quad y^0 = (0,0), \quad \mu > 0.$$

Then for all $t \geqslant 0$

$$f_i(t) = f(t) = 1, \quad F(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad \sum_{i \in I} \|\xi_i(t)\| = 3\mu, \quad \delta(t) = \frac{\sqrt{3}}{2}.$$

Define $T > 0$ and choose $\mu > 0$ so that the inequality $\mu < \sqrt{3}F(T)$ is satisfied. In this case, Assumption 3 holds. We prove that an evasion from an encounter occurs in the game $G(4)$. Take a control of evader E in the form $v(t) = v_0 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ for all $t > 0$. We have

$$y(t) = \frac{v_0 t^\alpha}{\Gamma(\alpha+1)}, \quad x_i(t) = x_i^0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u_i(s)\, ds.$$

Denote

$$\hat{u}_i(t) = \frac{\int_0^t (t-s)^{\alpha-1} u_i(s)\, ds}{\int_0^t (t-s)^{\alpha-1}\, ds}.$$

Then

$$x_i(t) = x_i^0 + \frac{t^\alpha \hat{u}_i(t)}{\Gamma(\alpha+1)}, \text{ where } \|\hat{u}_1(t)\| \leqslant 1, \quad \|\hat{u}_2(t)\| \leqslant 1 \quad \|\hat{u}_3(t)\| \leqslant \frac{1}{4},$$

$$\|x_1(t) - y(t)\| \geqslant \left\| x_1^0 - \frac{v_0 t^\alpha}{\Gamma(\alpha+1)} \right\| - \frac{t^\alpha}{\Gamma(\alpha+1)} \geqslant$$

$$\sqrt{\mu^2 + \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2} - \frac{t^\alpha}{\Gamma(\alpha+1)} > 0.$$

Similarly, $\|x_2(t) - y(t)\| > 0$ for all $t > 0$.

$$\|x_3(t) - y(t)\| \geqslant \left\| x_3^0 - \frac{v_0 t^\alpha}{\Gamma(\alpha+1)} \right\| - \frac{1}{4}\frac{t^\alpha}{\Gamma(\alpha+1)} =$$

$$= \sqrt{\mu^2 - \sqrt{3}\mu \frac{t^\alpha}{\Gamma(\alpha+1)} + \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2} - \frac{1}{4}\frac{t^\alpha}{\Gamma(\alpha+1)} =$$

$$= \sqrt{\left(\mu - \frac{\sqrt{3}}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2 + \left(\frac{t^\alpha}{2\Gamma(\alpha+1)}\right)^2} - \frac{1}{4}\frac{t^\alpha}{\Gamma(\alpha+1)} \geqslant \frac{1}{4}\frac{t^\alpha}{\Gamma(\alpha+1)} > 0.$$

Note that in this example the assumption for pursuer P_3 is not satisfied.

Example 4 Suppose that M_i, U_i and V are defined by relation (3). In the space $\mathbb{R}^2$ we consider a game $G(4)$ which is described by a system of the form

$$D^{(\alpha)}x_i = u_i, \quad u_i \in U_i, \quad D^{(\alpha)}y = y + v, \quad v \in V, \quad i \in I = \{1,2,3\},$$
$$b_1 = b = (0,0), \quad \|b_2\| = \|b_3\| = \frac{1}{2}R, \quad R_1 = R, \quad R_2 = R_3 = \frac{1}{2}R, R > 0$$
$$x_1^0 = (0,-1.05), \quad x_2^0 = (-1,-3), \quad x_3^0 = (1,-3), \qquad y^0 = (0,-1).$$

We have $f_i(t) = 1$, $f(t) = E_{1/\alpha}(t^\alpha, 1) \geqslant f_i(t)$ $F(t) = t^\alpha E_{1/\alpha}(t^\alpha, \alpha+1) \geqslant F_i(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ for all $i \in I$, $t \geqslant 0$. Take $p = (0,1)$. Then the inequalities $\|b+Rp\|F(t) \geqslant (\|b_i\|+R_i)F_i(t)$, $(y^0, b+Rp) \geqslant \max_i (x_i^0, b+Rp)$ hold. Assume that the control $v(t)$ of evader E is equal to $v(t) = v_0 = b + Rp = (0,1)$ for all $t \geqslant 0$. Prove that pursuer P_1 performs a capture of E using the control $u_1(t) = (0,1)$. We have

$$y(t) = \big(0, -E_{1/\alpha}(t^\alpha, 1) + t^\alpha E_{1/\alpha}(t^\alpha, \alpha+1)\big), \quad x_1(t) = \Big(0, -1,05 + \frac{t^\alpha}{\Gamma(\alpha+1)}\Big).$$

Consider the function

$$H(t) = -E_{1/\alpha}(t^\alpha, 1) + t^\alpha E_{1/\alpha}(t^\alpha, \alpha+1) + 1,05 - \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Then $H(0) > 0$. By virtue of [12, p. 118, formula 4] the following equation holds:

$$E_{1/\alpha}(z, 1) = \frac{1}{\Gamma(1)} + zE_{1/\alpha}(z, \alpha+1).$$

Therefore, $H(1) = 0,05 - \frac{1}{\Gamma(\alpha+1)} < 0$. Consequently, the function H has a root on $(0,1)$ and hence in the game $G(4)$ with such a control v of evader E pursuer P_1 performs a capture of the evader. Note that in this case the condition $(b+Rp, y^0) \geqslant 0$ of Theorem 5 is not satisfied.

6 Conclusions

New sufficient conditions have been obtained for a capture and an evasion from an encounter in the group pursuit problem with fractional derivatives. To solve the problem, the method of resolving functions has been used.

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