Implementation Details on Solution Algorithms

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1. C&CG Algorithm for Robust Optimization Model

Recall that model RO can be rewritten as the following noncompact MILP, referred to as model ROMILP:

$$
\begin{aligned}
\text{[ROMILP]}\ \min\quad & \sum_{k\in\mathcal{K}}\sum_{(i,j)\in\mathcal{A}}(c_{ij}^k q^k)\cdot x_{ij}^k+\sum_{(i,j)\in\mathcal{A}}\sum_{r=1}^{|\mathcal{K}|}f_{ij}\cdot y_{ijr}+\phi && (1.1)\\
s.t.\quad & \phi\geq\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{N}}(h^k q^k)\cdot w_i^{k(\boldsymbol{\delta})}+\sum_{k\in\mathcal{K}}g^k\cdot s^{k(\boldsymbol{\delta})},\quad\forall\ \boldsymbol{\delta}\in\mathbb{U}(\Gamma), && (1.2)\\
& (\boldsymbol{v}^{(\boldsymbol{\delta})},\boldsymbol{b}^{(\boldsymbol{\delta})},\boldsymbol{w}^{(\boldsymbol{\delta})},\boldsymbol{s}^{(\boldsymbol{\delta})})\in\mathcal{Q}(\boldsymbol{\delta}),\quad\forall\ \boldsymbol{\delta}\in\mathbb{U}(\Gamma), && (1.3)\\
& (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\overline{\boldsymbol{v}},\overline{\boldsymbol{b}})\in\mathcal{X}. && (1.4)
\end{aligned}
$$

Here, ϕ is a newly introduced decision variable, and $(\boldsymbol{v}^{(\boldsymbol{\delta})},\boldsymbol{b}^{(\boldsymbol{\delta})},\boldsymbol{w}^{(\boldsymbol{\delta})},\boldsymbol{s}^{(\boldsymbol{\delta})})$ represents a vector of second-stage decision variables associated with each possible scenario $\boldsymbol{\delta}$ in $\mathbb{U}(\Gamma)$. Constraints (1.2) and (1.3) ensure that ϕ equals the worst-case second-stage cost. As a result, solving the min-max-min model RO is reduced to solving the above noncompact MILP model ROMILP. Model ROMILP can also be relaxed by replacing $\mathbb{U}(\Gamma)$ in constraints (1.2) and (1.3) with any of its subsets $\Lambda\subseteq\mathbb{U}(\Gamma)$. The resulting relaxation is referred to as model ROMILP(Λ). The relaxation can be strengthened by appending to Λ more scenarios $\boldsymbol{\delta}$ in $\mathbb{U}(\Gamma)$.

Similar to RS-C&CG algorithm for model RS, our C&CG Algorithm for model RO (or RO-C&CG algorithm in short) also follows the C&CG framework. In each iteration n, where $n=1,2,\cdots$, it first solves model ROMILP(Λ), which is referred to as the master problem, for a particular subset Λ of $\mathbb{U}(\Gamma)$. Let $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}},\phi)$ indicate the optimal solution obtained for the master problem. Accordingly,

$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ forms a nominal timely-implementable first-stage solution to model RO. For this first-stage solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$, our RO-C&CG algorithm then solves the following maximization MILP model, which is referred to as the subproblem, to compute the worst-case second stage cost $F_{RP}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ and to identify the corresponding worst-case scenario $\boldsymbol{\delta}^{(n)}$. Note that M_3 is a sufficiently large constant.

$$
\begin{aligned}
F_{RP}(\boldsymbol{x},\boldsymbol{z}) = \max \quad & \sum_{(j,i)\in\mathcal{A}} \sum_{r=1}^{|\mathcal{K}|} \varphi_{jir} - \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} (M_1 x_{ij}^k)\cdot \eta_{ij}^k \\
& + \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} \sum_{r=1}^{|\mathcal{K}|} [M_1(z_{ijr}^k - 1)]\cdot(\theta_{ijr}^k + \xi_{ijr}^k) \\
& + \sum_{k\in\mathcal{K}} e^k \cdot (\gamma^k - \lambda_{o^k}^k) + \sum_{k\in\mathcal{K}} l^k \cdot (\lambda_{d^k}^k - \psi^k) && (1.5)\\
\text{s.t.} \quad & (\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\xi}, \boldsymbol{\lambda}) \in \Omega, && (1.6)\\
& \zeta_{ijr,-1} + \zeta_{ijr,0} + \zeta_{ijr,1} = 1, \quad \forall\ (i,j)\in\mathcal{A}, r\in\{1,2,...,|\mathcal{K}|\}, && (1.7)\\
& \Big(\sum_{k\in\mathcal{K}_i} z_{jir}^k(\beta_i^k - \lambda_i^k) + \sum_{k\in\mathcal{K}_i^d} z_{jir}^k(\psi^k - \lambda_i^k)\Big)\tilde{\tau}_{jir,\ell} - M_3(1-\zeta_{jir,\ell}) \le \varphi_{jir} \\
& \quad \le \Big(\sum_{k\in\mathcal{K}_i} z_{jir}^k(\beta_i^k - \lambda_i^k) + \sum_{k\in\mathcal{K}_i^d} z_{jir}^k(\psi^k - \lambda_i^k)\Big)\tilde{\tau}_{jir,\ell} + M_3(1-\zeta_{jir,\ell}), \\
& \qquad \forall\ (j,i)\in\mathcal{A}, r\in\{1,2,...,|\mathcal{K}|\}, \ell\in\{-1,0,1\}, && (1.8)\\
& \sum_{(i,j)\in\mathcal{A}} \sum_{r=1}^{|\mathcal{K}|} (\zeta_{ijr,-1} + \zeta_{ijr,1}) \le \Gamma. && (1.9)\\
& \zeta_{ijr,\ell} \in \{0,1\}, \quad \forall\ (i,j)\in\mathcal{A}, r\in\{1,2,...,|\mathcal{K}|\}, \ell\in\{-1,0,1\}. && (1.10)
\end{aligned}
$$

Here, recall that $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\xi}, \boldsymbol{\lambda})$ consists of dual variables of the linear programming formulation of $F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}))$. The feasible domain of $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\xi}, \boldsymbol{\lambda})$, indicated by Ω, is a convex polyhedron, which can be defined by the following linear constraints:

$$
\begin{aligned}
& \beta_i^k - \beta_j^k - \eta_{ij}^k - \sum_{r=1}^{|\mathcal{K}|} \theta_{ijr}^k + \sum_{r=1}^{|\mathcal{K}|} \xi_{ijr}^k - \lambda_i^k + \lambda_j^k \le 0,\ \forall\ k\in\mathcal{K}, (i,j)\in\mathcal{A}, i\ne o^k, j\ne d^k, && (1.11)\\
& -\beta_j^k + \gamma^k - \eta_{o^k j}^k - \sum_{r=1}^{|\mathcal{K}|} \theta_{o^k jr}^k + \sum_{r=1}^{|\mathcal{K}|} \xi_{o^k jr}^k - \lambda_{o^k}^k + \lambda_j^k \le 0,\ \forall\ k\in\mathcal{K}, (o^k,j)\in\mathcal{A}, j\ne d^k, && (1.12)\\
& \beta_i^k - \psi^k - \eta_{id^k}^k - \sum_{r=1}^{|\mathcal{K}|} \theta_{id^k r}^k + \sum_{r=1}^{|\mathcal{K}|} \xi_{id^k r}^k - \lambda_i^k + \lambda_{d^k}^k \le 0,\ \forall\ k\in\mathcal{K}, (i,d^k)\in\mathcal{A}, i\ne o^k, && (1.13)\\
& \gamma^k - \psi^k - \eta_{o^k d^k}^k - \sum_{r=1}^{|\mathcal{K}|} \theta_{o^k d^k r}^k + \sum_{r=1}^{|\mathcal{K}|} \xi_{o^k d^k r}^k - \lambda_{o^k}^k + \lambda_{d^k}^k \le 0,\ \forall\ k\in\mathcal{K}, (o^k,d^k)\in\mathcal{A}, && (1.14)\\
& \sum_{k\in\mathcal{K}} \theta_{ijr}^k - \sum_{k\in\mathcal{K}} \xi_{ijr}^k \le 0,\ \forall\ (i,j)\in\mathcal{A}, r\in\{1,2,...,|\mathcal{K}|\}, && (1.15)\\
& \lambda_i^k \le h^k q^k,\ \forall\ i\in\mathcal{N}, k\in\mathcal{K}, && (1.16)
\end{aligned}
$$

$$\psi^k - \lambda^k_{d^k} \le g^k,\ \forall\ k \in \mathcal{K}, \tag{1.17}$$

$$\boldsymbol{\beta} \ge \mathbf{0}, \boldsymbol{\gamma} \ge \mathbf{0}, \boldsymbol{\psi} \ge \mathbf{0}, \boldsymbol{\eta} \ge \mathbf{0}, \boldsymbol{\theta} \ge \mathbf{0}, \boldsymbol{\xi} \ge \mathbf{0}, \boldsymbol{\lambda} \ge \mathbf{0}, \tag{1.18}$$

where $\mathcal{K}_i = \{k \in \mathcal{K} : i \ne o^k \text{ and } i \ne d^k\}$ and $\mathcal{K}^d_i = \{k \in \mathcal{K} : i = d^k\}$.

Since ROMILP(Λ) is a relaxation of model RO, its optimal objective value obtained is a lower bound on the optimal objective value of model RO. Since $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ forms a nominal timely-implementable first-stage solution to model RO, the sum of its first-stage total cost $(\sum_{k\in\mathcal{K}}\sum_{(i,j)\in\mathcal{A}}(c^k_{ij}q^k)\cdot\hat{x}^k_{ij} + \sum_{(i,j)\in\mathcal{A}}\sum_{r=1}^{|\mathcal{K}|} f_{ij}\cdot\hat{y}_{ijr})$ and its second stage total cost $F_{RP}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ provides an upper bound on the optimal objective value of model RO.

If the lower bound equals the upper bound, then model RO is solved to optimum, and our RO-C&CG algorithm terminates with an optimal solution given by $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$. Otherwise, it appends the identified worst-case scenario $\boldsymbol{\delta}^{(n)}$ to the subset Λ. As a result, model ROMILP(Λ) of the master problem is extended and strengthened with new decision variables $(\boldsymbol{v}^{(\boldsymbol{\delta}^{(n)})}, \boldsymbol{b}^{(\boldsymbol{\delta}^{(n)})}, \boldsymbol{w}^{(\boldsymbol{\delta}^{(n)})}, \boldsymbol{s}^{(\boldsymbol{\delta}^{(n)})})$ and their new constraints in (1.2) and (1.3). Our RO-C&CG algorithm then proceeds to the next iteration. Our RO-C&CG algorithm is summarized in Algorithm 1, along with its correctness and convergence in Theorem 1.1.

Algorithm 1 RO-C&CG Algorithm for Solving Model RO

1. Initially, set n to 1, and set the subset Λ of $\mathbb{U}(\Gamma)$ to $\{\mathbf{0}\}$.
2. Solve the master problem, i.e., model ROMILP(Λ), to obtain its optimal objective value denoted by LB and its optimal solution denoted by $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, \phi)$.
3. Solve the subproblem, i.e., the maximization MILP model defined by (1.5)–(1.10) for $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$, to obtain its optimal objective value that equals $F_{RP}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$, and to compute a worst-case scenario $\boldsymbol{\delta}^{(n)}$ of $\boldsymbol{\delta}$ according to (1.19) below:

$$\delta_{ijr} = -\zeta_{ijr,-1} + \zeta_{ijr,1}, \quad \forall\ (i,j) \in \mathcal{A}, r \in \{1, ..., |\mathcal{K}|\}. \tag{1.19}$$

   Let UB denote the sum of $(\sum_{k\in\mathcal{K}}\sum_{(i,j)\in\mathcal{A}}(c^k_{ij}q^k)\cdot\hat{x}^k_{ij} + \sum_{(i,j)\in\mathcal{A}}\sum_{r=1}^{|\mathcal{K}|} f_{ij}\cdot\hat{y}_{ijr})$ and $F_{RP}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$.
4. If $LB = UB$, then the algorithm terminates and returns an optimal solution given by $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$. Otherwise, update $\Lambda = \Lambda \bigcup \{\boldsymbol{\delta}^{(n)}\}$, update $n = n+1$, and go to Step 2 for the next iteration.

Theorem 1.1 *Algorithm 1 terminates in a finite number of iterations and returns an optimal solution for model RO.*

Proof. At each iteration of Algorithm 1, UB and LB are updated by solving the corresponding master problem and subproblem, while a new worst-case scenario $\boldsymbol{\delta}$ in $\mathbb{U}(\Gamma)$ is obtained and added to the scenario subset Λ. Algorithm 1 stops when $UB = LB$. As model ROMILP(Λ) is a relaxation of the reformulation ROMILP of model RO, the value of LB, which equals the optimal objective value of model ROMILP(Λ), is a valid lower bound on the optimal objective value of model RO. As UB is the worst-case total cost of the first-stage solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ obtained from the master problem, it provides a valid upper bound on the optimal objective value of model RO. Thus, when $UB = LB$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ must be an optimal solution to model RO. This implies that when Algorithm 1 terminates with $UB = LB$, it returns an optimal solution to model RO.

We next show as follows that Algorithm 1 terminates with $UB = LB$ in a finite number of iterations. To show this, we note that, at each iteration n, if the worst-case scenario $\boldsymbol{\delta}^{(n)}$ identified in Step 3 of Algorithm 1 is not in the current scenario subset Λ, it will be added to Λ. According to Proposition 4.2 in the main text of the paper, $\boldsymbol{\delta}^{(n)}$ satisfies that $\boldsymbol{\delta}^n_{ijr} \in \{-1, 0, 1\}$ for all $(i,j) \in \mathcal{A}$ and $r \in \{1, 2, \cdots, |\mathcal{K}|\}$, and has a finite number of possible values. Therefore, in a finite number of iterations, $\boldsymbol{\delta}^{(n)}$ identified in Step 3 of Algorithm 1 must be included in the current scenario subset Λ. In such a situation, both LB and UB must be equal to the optimal objective value of the current master problem, implying that $LB = UB$ and $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ forms an optimal solution to model RO. This completes the proof of Theorem 1.1. $\square$

2. Acceleration Strategies for C&CG Algorithms

In this section, we illustrate several acceleration strategies employed in our implementation of the newly proposed C&CG algorithms.

2.1. Breaking Symmetry for Master Problems

First, for each commodity $k \in \mathcal{K}$, and for each pair of nodes i' and j' of the network $\mathcal{D} = (\mathcal{N}, \mathcal{A})$, let $\underline{\tau}^k(i', j')$ denote the length of the shortest-time path from node i' to node j' under the nominal travel times in the flat network, such that the origin o^k and destination d^k of commodity k are not included in between the start and end nodes of the path. It can be seen that for each arc $(i,j) \in \mathcal{A}$, if $\underline{\tau}^k(o^k, i) + \overline{\tau}_{ij} + \underline{\tau}^k(j, d^k) > l^k - e^k$, then under the nominal travel times, commodity k cannot pass arc (i,j) without violating its earliest time for departure from origin o^k or its due time for arrival at destination d^k. Therefore, in every nominal timely-implementable first-stage solution of the robust CTSNDP, commodity k can pass arc $(i,j) \in \mathcal{A}$ only if the following condition is satisfied:

$$\underline{\tau}^k(o^k, i) + \overline{\tau}_{ij} + \underline{\tau}^k(j, d^k) \le l^k - e^k. \tag{2.1}$$

Define $\mathcal{K}_{ij}$ as the set of such commodities k that satisfy (2.1). Accordingly, variables and constraints associated with commodity $k \in \mathcal{K} \setminus \mathcal{K}_{ij}$ can be safely eliminated from the master problems for each $(i,j) \in \mathcal{A}$.

Then, for each $(i,j) \in \mathcal{A}$, we define the set $\mathcal{F}_{i,j} = \{(k,k') : e^k + \underline{\tau}^k(o^k,i) + \tau_{i,j} + \underline{\tau}^{k'}(j,d^{k'}) > l^{k'}$ or $e^{k'} + \underline{\tau}^{k'}(o^{k'},i) + \tau_{i,j} + \underline{\tau}^k(j,d^k) > l^k, k,k' \in \mathcal{K}_{i,j}, k < k'\}$. A pair of commodities $(k,k') \in \mathcal{F}_{i,j}$ signifies that commodities k and k' can never be consolidated on arc (i,j) due to time window constraints. We thus can tighten the master problem with the following valid inequalities:

$$z^k_{ijr} + z^{k'}_{ijr} \leq 1, \qquad \forall (k,k') \in \mathcal{F}_{i,j}. \tag{2.2}$$

Moreover, we apply the asymmetric representatives strategy, which is widely used in the graph coloring literature (Campêlo et al. 2008, Malaguti et al. 2009), to effectively break solution symmetry and enhance tractability for master problems. The core of the asymmetric representatives strategy is to sort the commodities in $\mathcal{K}_{ij}$ for each arc (i,j) based on the maximum conflict subset of $\mathcal{K}_{ij}$, and associate a representative commodity with each consolidation, enforcing that the r'-th commodity can be included in the r-th consolidation only if $r' \leq r$ and the representative commodity of the r-th consolidation is also assigned to it.

Specifically, a subset of $\mathcal{K}_{i,j}$ is designated as a conflict subset if every pair of commodities within the subset forms an element in $\mathcal{F}_{i,j}$. To facilitate the formulation, we re-index the commodities in $\mathcal{K}_{i,j}$ based on the conflict structure. We sort $\mathcal{K}_{i,j}$ as $(\tilde{k}_1,...,\tilde{k}_n,\bar{k}_1,...,\bar{k}_m)$, where the set $\{\tilde{k}_1,...,\tilde{k}_n\}$ constitutes a maximal conflict subset of $\mathcal{K}_{i,j}$ (i.e., $\forall k,k' \in \{\tilde{k}_1,...,\tilde{k}_n\}$ with $k<k'$, we have $(k,k') \in \mathcal{F}_{i,j}$), and $\{\bar{k}_1,...,\bar{k}_m\}$ are the remaining commodities. For notational convenience, we re-index all elements in $\mathcal{K}_{i,j}$ as $\mathcal{K}_{i,j} = \{k_{i,j,1}, k_{i,j,2}, ..., k_{i,j,n+m}\}$. We then define $k_{i,j,r} \in \mathcal{K}_{i,j}$ as the representative of the r-th consolidation, enforcing that any commodity $k_{i,j,r'} \in \mathcal{K}_{i,j}$ can be included in the r-th consolidation only if $r' \leq r$ and the representative $k_{i,j,r}$ is also assigned to it. Moreover, without loss of optimality, we can restrict that if $(k,k') \in \mathcal{F}_{i,j}$, commodity k will never be included in the consolidation with representative k'. We then implement this asymmetric representatives strategy for each $(i,j) \in \mathcal{A}$ as follows:

- for each $k_{i,j,h}$ with $h \in \{1,...,|\mathcal{K}_{i,j}|\}$, remove all variables $z^{k_{i,j,h}}_{ijr}$ with $r > h$;
- for each $r \in \{1,...,|\mathcal{K}_{i,j}|\}$, remove all variables $z^{k_{i,j,h}}_{ijr}$ with $(k_{i,j,r}, k_{i,j,h}) \in \mathcal{F}_{i,j}$ or $(k_{i,j,h}, k_{i,j,r}) \in \mathcal{F}_{i,j}$;
- all redundant constraints relevant to these removed redundant variables can be accordingly removed;
- add the following inequalities with the reduced $\boldsymbol{z}$ variables:

$$z^{k_{i,j,h'}}_{ijh} \leq z^{k_{i,j,h}}_{ijh}, \quad \forall h \in \{1,...,|\mathcal{F}_{i,j}|\}, h' \in \{h+1,...,|\mathcal{F}_{i,j}|\}, \tag{2.3}$$

which further break the symmetric structure of the solutions of the remaining model by restricting that a commodity $k_{i,j,h'}$ can be included in the h-th consolidation on arc (i,j) only if the representative commodity $k_{i,j,h}$ is included in the h-th consolidation.

2.2. Iterations: Bundling New Scenarios to Add

To further enhance the efficiency of both the RO-C&CG and RS-C&CG algorithms, we also implement a bundle strategy to update the scenario set in each iteration. This approach is similar to the one used by Remli et al. (2019) in their Benders decomposition-based algorithm for a transpiration service procurement problem.

In each iteration of our C&CG algorithms, we solve the master problem to obtain an optimal first-stage solution, as well as a pool of feasible first-stage solutions. We can accomplish this using general optimization solvers such as Gurobi and CPLEX. These first-stage feasible solutions, including the optimal solution, are sorted by their objective values in non-decreasing order. For each of these solutions, we then solve the corresponding subproblem to identify its worst-case scenario, resulting in multiple new scenarios that can be added to the master problem for future iterations. As we add more new scenarios in each iteration, the C&CG algorithm may require fewer iterations to reach the optimum solution. However, as we add new scenarios along with their decision variables and constraints, the size of the master problem increases, which may lead to longer computation times for each iteration of the algorithm.

Accordingly, to strike a better balance between efficiency and accuracy, we add a bundle of at most two new scenarios to the master problem in each iteration. One of these new scenarios is the worst-case scenario of the optimal first-stage solution. To identify the second scenario to add, we evaluate the first-stage solutions in the pool and choose the solution with the least objective value. If this solution's objective value is better than the current best upper bound on the optimal objective value, we update the upper bound accordingly, and we then add the worst-case scenario of this best solution, as the second scenario in the bundle, to the master problem.

2.3. Initial Bounds for Bisection Search Procedure of RS-C&CG Algorithm

In Step 2 of the bisection search procedure utilized in RS-C&CG Algorithm, we need to initialize the values of ρ_l and ρ_h such that $\rho_l \leq \rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq \rho_h$. For this, we establish Lemma 2.1 below.

Lemma 2.1

1. *If* $F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z} > 0$, *then* $\rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = +\infty$;
2. *Otherwise,* $(F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}_l)) - \mathcal{Z})/\|\boldsymbol{\delta}_l\|_1 \leq \rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ *for each* $\boldsymbol{\delta}_l \in \mathbb{U} \setminus \{\mathbf{0}\}$, *and* $\rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq \max\{0, F_1(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{\delta} \in \mathbb{U}}\{F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}))\} - \mathcal{Z}\}$.

Proof. Recall that we slightly abuse the notation to define that $\sigma/\|\mathbf{0}\|_1 = 0$ for $\sigma = 0$, $\sigma/\|\mathbf{0}\|_1 = +\infty$ for $\sigma > 0$, and $\sigma/\|\mathbf{0}\|_1 = -\infty$ for $\sigma < 0$. Consider any $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{X}$. To prove the first statement of Lemma 2.1, we note that if $F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z} > 0$, then

$$\hat{\rho}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \geq \frac{F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z}}{\|\mathbf{0}\|_1} = +\infty,$$

implying that $\hat{\rho}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = +\infty$.

To prove the second statement, consider the case where $F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z} \leq 0$, which implies that $(F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0}) - \mathcal{Z}))/\|\mathbf{0}\|_1 \leq 0$. According to the definition of $\rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, we have that $(F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}_l)) - \mathcal{Z})/\|\boldsymbol{\delta}_l\|_1 \leq \rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{\delta}_l \in \mathbb{U} \setminus \{\mathbf{0}\}$. Moreover,

- If $\max_{\boldsymbol{\delta} \in \mathbb{U}} \{F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta})) - \mathcal{Z}\} \leq 0$, then

$$\hat{\rho}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \max_{\boldsymbol{\delta} \in \mathbb{U}} \frac{F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta})) - \mathcal{Z}}{\|\boldsymbol{\delta}\|_1} \leq 0.$$

- Otherwise, $\max_{\boldsymbol{\delta} \in \mathbb{U}} \{F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta})) - \mathcal{Z}\} > 0$, implying that there must exist a $\boldsymbol{\delta}^* \in \mathbb{U} \setminus \{\mathbf{0}\}$ with $F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}^*)) - \mathcal{Z} > 0$ and $\hat{\rho}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}^*)) - \mathcal{Z})/\|\boldsymbol{\delta}^*\|_1 > 0$. By Lemma 4.1 and Proposition 4.1 in the main text of the paper, we can assume without loss of generality that $\delta^*_{ijr} \in \{-1, 0, 1\}$ for all $(i,j) \in \mathcal{A}$ and $r \in \{1, 2, ..., |\mathcal{K}|\}$, which, together with $\boldsymbol{\delta}^* \in \mathbb{U} \setminus \{\mathbf{0}\}$, implies that $\|\boldsymbol{\delta}^*\|_1 \geq 1$. Thus, we obtain that

$$0 < \hat{\rho}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq \frac{F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta})^*) - \mathcal{Z}}{1} \leq \max_{\boldsymbol{\delta} \in \mathbb{U}} \{F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta})) - \mathcal{Z}\}.$$

Hence, it can be concluded that if $F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z} \leq 0$, there must be $\hat{\rho}^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq \max\{0, F_1(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{\delta} \in \mathbb{U}} F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta})) - \mathcal{Z}\}$.

Lemma 2.1 is proved. □

Since $F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}))$ is defined by a linear program, it can be obtained directly by an optimization solver. Model $\max_{\boldsymbol{\delta} \in \mathbb{U}} \{F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}))\}$ is equivalent to $F_{RP}(\boldsymbol{x}, \boldsymbol{z})$ defined in (**??**)–(**??**) with $\mathbb{U}(\Gamma)$ being relaxed to $\mathbb{U}$ (i.e., with $\Gamma = +\infty$), which can be transformed to an MILP as shown in (1.5)–(1.10). Thus, it can be solved by an optimization solver.

According to Lemma 2.1, if $(F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z}) > 0$, then the worst-case normalized cost deviation $\rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = +\infty$ and $\mathbf{0}$ is the worst-case scenario for $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. Otherwise, we know that $(F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\mathbf{0})) - \mathcal{Z})/\|\boldsymbol{\delta}\|_1$ for any $\boldsymbol{\delta} \in \mathbb{U} \setminus \{\mathbf{0}\}$ and $\max\{0, F_1(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{\delta} \in \mathbb{U}} \{F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}))\} - \mathcal{Z}\}$ provide a lower bound and an upper bound on the worst-case normalized cost deviation $\rho^*(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, respectively.

Hence, in Step 2 of the bisection search procedure utilized in RS-C&CG Algorithm, we choose any $\boldsymbol{\delta}_l \in \mathbb{U} \setminus \{\mathbf{0}\}$, to set $\rho_l = (F_1(\boldsymbol{x}, \boldsymbol{y}) + F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}_l)) - \mathcal{Z})/\|\boldsymbol{\delta}_l\|_1$ and set $\rho_h = \max\{0, F_1(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{\delta} \in \mathbb{U}} \{F_{LP}(\boldsymbol{x}, \boldsymbol{z}, \tilde{\boldsymbol{\tau}}(\boldsymbol{\delta}))\} - \mathcal{Z}\}$, as their initial values.

References

Campêlo, M., Campos, V.A., Corrêa, R.C., 2008. On the asymmetric representatives formulation for the vertex coloring problem. Discrete Applied Mathematics 156, 1097–1111.

Malaguti, E., Monaci, M., Toth, P., 2009. Models and heuristic algorithms for a weighted vertex coloring problem. Journal of Heuristics 15, 503–526.

Remli, N., Amrouss, A., El Hallaoui, I., Rekik, M., 2019. A robust optimization approach for the winner determination problem with uncertainty on shipment volumes and carriers' capacity. Transportation Research Part B: Methodological 123, 127–148.