Lecture 4
Image Transforms

ECE 1390/2390

<u>Learning Objectives:</u>

DFT for images

DCT

Haramord-Walsh transform

Haar transform

Discrete Fourier Transform (DFT)

$$F(w) = \int_{-\infty}^{\infty} dt \cdot f(t) \cdot e^{-iwt}$$

$$F(k) = \sum_{x=0}^{N-1} f(x) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot x / N}$$

$$\mathrm{f}(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) \cdot e^{j \cdot 2 \cdot \pi \cdot k \cdot x / N}$$

DFT

$$F(k) = \sum_{x=0}^{N-1} f(x) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot x/N}$$

$$f(x) = [1\ 2\ 3\ 2\ 1\ 0]$$

$$N = 6$$

$$F(k) = \sum_{x=0}^{5} f(x) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot x/N}$$

$$\begin{aligned} F(k) = {} & 1 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{0}{6}} \\ & + 2 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{1}{6}} \\ & + 3 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{2}{6}} \\ & + 2 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{3}{6}} \\ & + 1 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{4}{6}} \\ & + 0 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{5}{6}} \end{aligned}$$

K=0 = 9
K=1 = -2.11 – 1.53*j
K=2 = 0.12 + 0.36*j
K=3 = 0.12 - 0.36*j
K=4 = -2.11 + 1.53*j
K=4 = 9

DFT-2D

$$F(u,v) = \sum_{y=0}^{M-1} \sum_{x=0}^{N-1} f(x,y) \cdot e^{-j\cdot 2\cdot \pi \cdot (\frac{u}{N} * x + \frac{v}{M} * y)}$$

$$f(x,y) = \frac{1}{M * N} \sum_{v=0}^{M-1} \sum_{u=0}^{N-1} F(u,v) \cdot e^{j\cdot 2\cdot \pi \cdot (\frac{u}{N} * x + \frac{v}{M} * y)}$$

DFT Matrix form

$$H_{2x2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{4x4} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F = H * \mathrm{x}$$

$$H_{8x8} = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-j}{\sqrt{2}} & -j & \frac{-1-j}{\sqrt{2}} & -1 & \frac{-1+j}{\sqrt{2}} & j & \frac{1+j}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & \frac{-1-j}{\sqrt{2}} & j & \frac{1-j}{\sqrt{2}} & -1 & \frac{1+j}{\sqrt{2}} & -j & \frac{-1+j}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+j}{\sqrt{2}} & -j & \frac{1+j}{\sqrt{2}} & -1 & \frac{1-j}{\sqrt{2}} & j & \frac{-1-j}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & -j & -1 & -j \\ 1 & \frac{1+j}{\sqrt{2}} & j & \frac{-1+j}{\sqrt{2}} & -1 & \frac{-1-j}{\sqrt{2}} & -j & \frac{1-j}{\sqrt{2}} \end{bmatrix}$$

DFT Matrix form

$$F = H * \text{x}$$

$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix}$$

$$w = e^{-2\pi j/8}$$

DFT-2D Matrix form

$$F = H * x * H^{T}$$

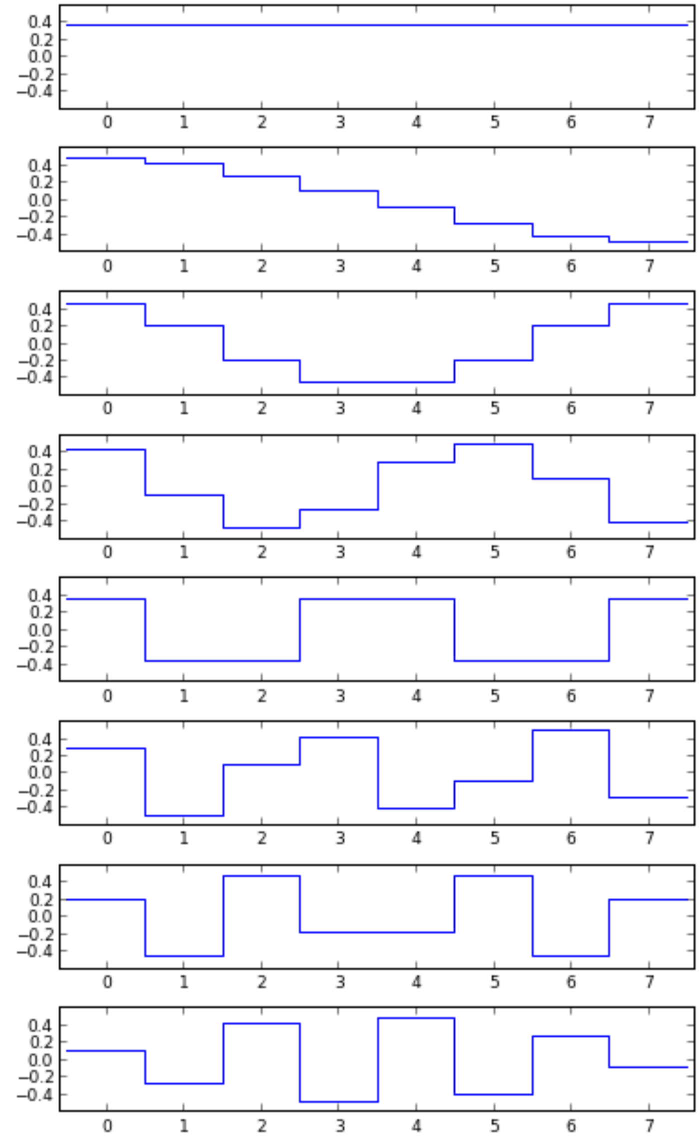
Discrete Cosine Transform (DCT)

Coefficients will only be real-valued (c.f. DFT)

$$\mathrm{F} = \mathrm{C} * \mathrm{x}$$

$$C = \begin{bmatrix} C(0,0) & C(1,0) & C(2,0) \\ C(0,1) & C(1,1) & \\ C(0,2) & & \ddots \end{bmatrix}$$

$$C(u,v) = \left[\begin{array}{c} \sqrt{\dfrac{1}{N}} \\ \sqrt{\dfrac{2}{N}} \cos\left(\dfrac{(2v+1)\cdot\pi\cdot u}{2N}\right) \end{array}\right.$$



DCT

DCT^{-1}

$$C \text{ basis is orthogonal}$$

$$I = C^T * C$$

$$C^{-1} = C^T$$

1D transform

$$F = C * x$$

$$x = C^T * F$$

2D transform

$$F = C * x * C^T$$

$$x = C^T * F * C$$

Hadamard Transform

$$H_{2x2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathrm{F} = \frac{1}{N^2}\mathrm{H}_{\mathrm{nxn}} * \mathrm{x} * \mathrm{H}_{\mathrm{nxn}}{}^{\mathrm{T}}$$

The 2x2 is identical to the DFT 2x2

$$H_{2x2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathrm{x} = \mathrm{H} * \mathrm{F} * \mathrm{H}^{\mathrm{T}}$$

$$H_{2x2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Integrator: Low-pass filter

Differentiator: High-pass filter

Hadamard Transform

$$F = \frac{1}{N^2} \mathrm{H}_{\mathrm{nxn}} * \mathrm{x} * \mathrm{H}_{\mathrm{nxn}}{}^{\mathrm{T}}$$

Size of image
must be powers
of 2

$$H_{2x2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{4x4} = (H_{2x2} \otimes H_{2x2}) \qquad H_{4x4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_{8x8} = (H_{2x2} \otimes H_{4x4}) \qquad H_{8x8} = \begin{bmatrix} H_{4x4} & H_{4x4} \\ H_{4x4} & -H_{4x4} \end{bmatrix}$$

$$F = \frac{1}{N * M} \mathrm{H}_{\mathrm{nxn}} * \mathrm{x} * \mathrm{H}_{\mathrm{mxm}}{}^{\mathrm{T}}$$

Welch Transform

Hadamard Matrix

$$H_{4x4} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{array}{l} \text{Number of sign changes} \\ \longrightarrow 0 \\ \longrightarrow 3 \\ \longrightarrow 1 \\ \longrightarrow 2 \end{array}$$

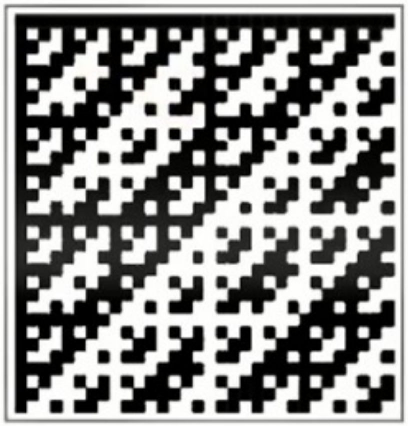
Welch Matrix

$$W_{4x4} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\mathrm{F} = \mathrm{W} * \mathrm{x} * \mathrm{W}^{\mathrm{T}}$$

$$\mathrm{x} = \mathrm{W} * \mathrm{F} * \mathrm{W}^{\mathrm{T}}$$

Hadamard **Natural Ordering**



32x32

$$H_{4x4} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Welch **Sequency Ordering**

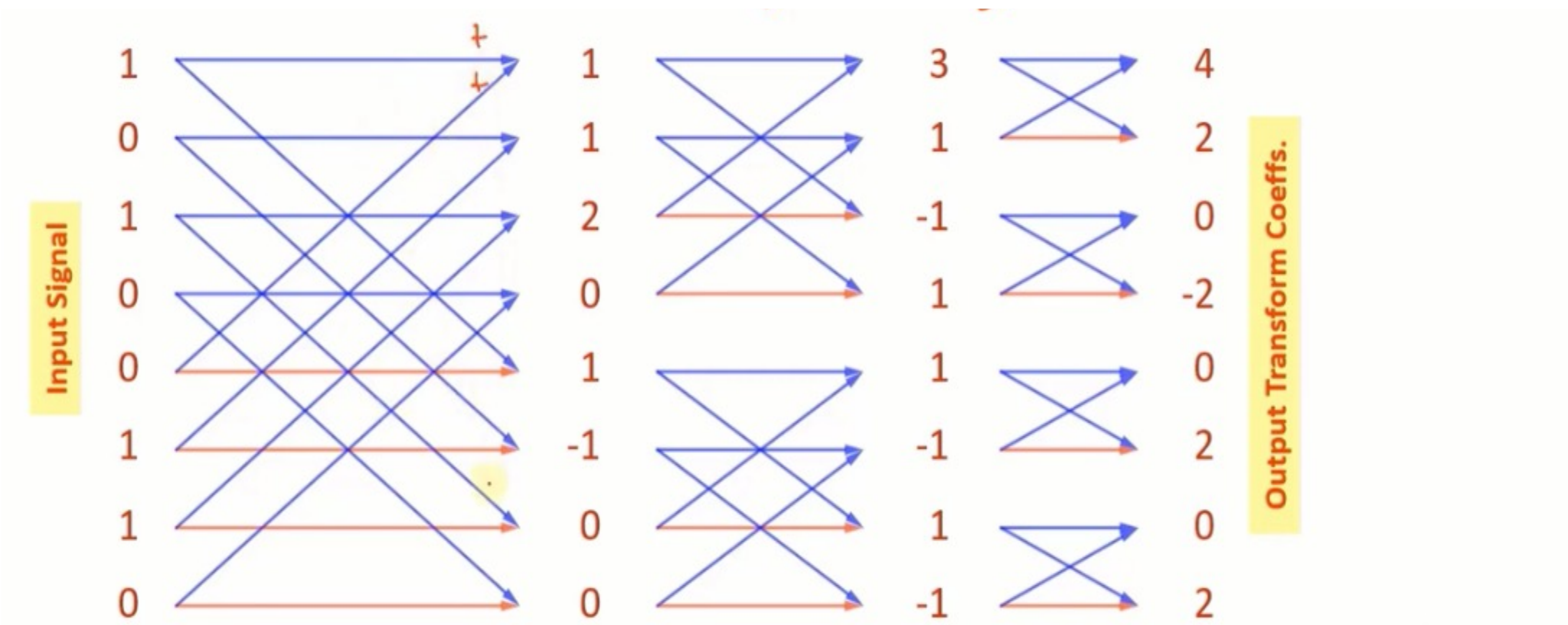
32x32

$$W_{4x4} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Fast Welch Transform

Akin to FFT, FWT is done using only shift and addition/subtraction operations allowing it to be done quickly

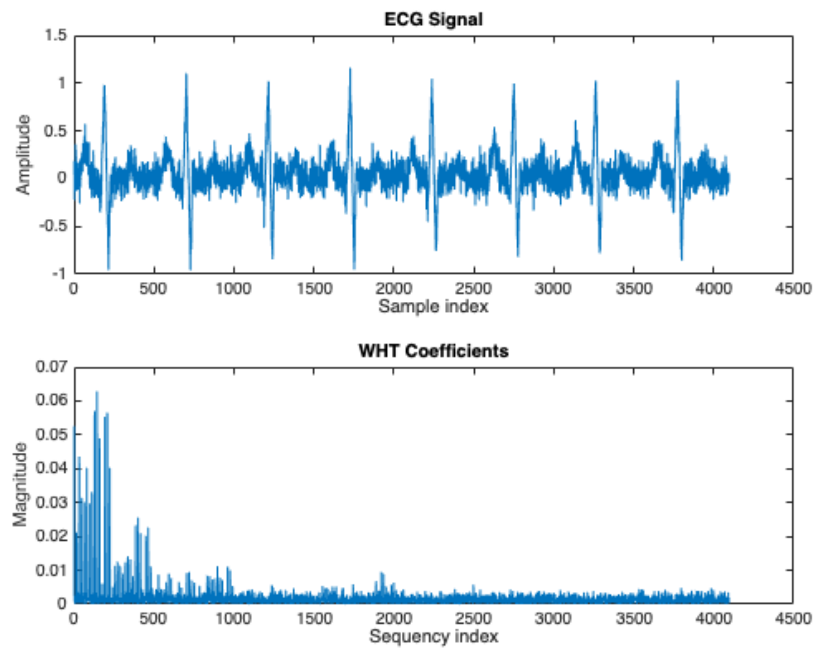
BLUE IS PLUS
RED IS MINUS



- MATLAB/Examples/R2023a/signal/WalshHadamardExample

```
x1 = ecg(512); % Single ecg wave
x = repmat(x1,1,8);
x = x + 0.1.*randn(1,length(x)); % Noisy ecg signal
y = fwht(x); % Fast Walsh–Hadamard transform
```

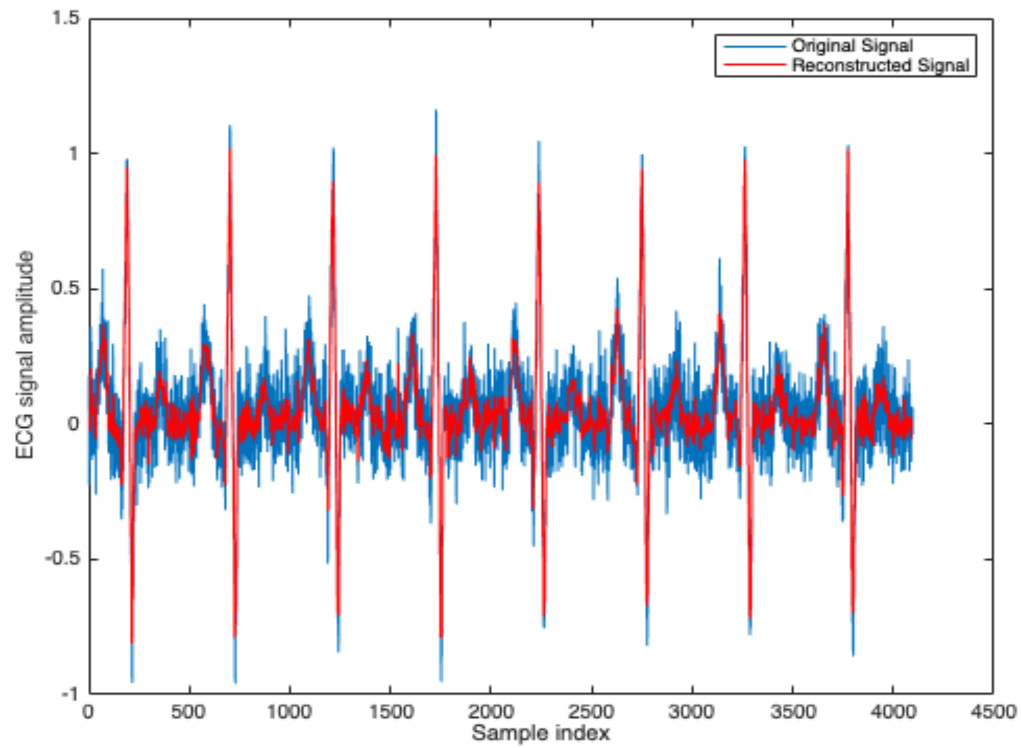
```
subplot(2,1,1)
plot(x)
xlabel('Sample index')
ylabel('Amplitude')
title('ECG Signal')
subplot(2,1,2)
plot(abs(y))
xlabel('Sequency index')
ylabel('Magnitude')
title('WHT Coefficients')
```



• MATLAB/Examples/R2023a/signal/WalshHadamardExample

```
y(1025:length(x)) = 0; % Zeroing out the higher coefficients
xHat = ifwht(y); % Signal reconstruction using inverse WHT
```

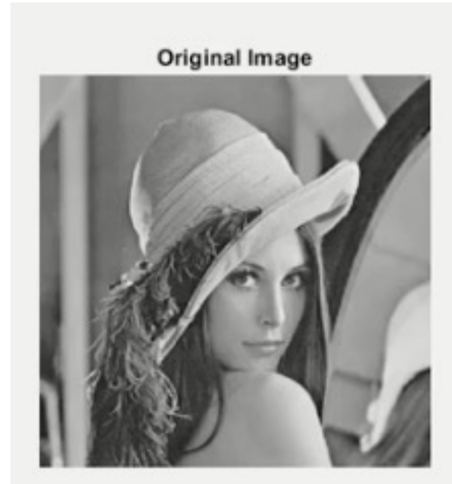
```
figure
plot(x)
hold on
plot(xHat,'r')
xlabel('Sample index')
ylabel('ECG signal amplitude')
legend('Original Signal’,…
'Reconstructed Signal')
```

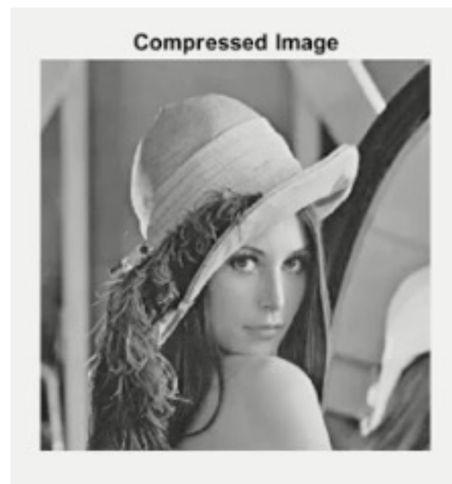


```
img = imread('Lena.jpg')
whcoef = fwht( fwht(img)')';

whcoef(end/2+1:end, end/2+1:end) = 0

img_recovered = ifwht( ifwht(whcoef)')';
```





Peak signal-to-noise ratio (PSNR)

$$MSE = \frac{1}{n * m * (3)} \sum_{color} \sum_{i=0}^{n} \sum_{j=0}^{m} (\underbrace{I_1}_{\text{Original image}}(i,j) - \underbrace{I_2}_{\text{New image}}(i,j))^2)$$

$$PSNR = 20 * log_{10}\left(\frac{Max(I_1)}{\sqrt{MSE}}\right)$$

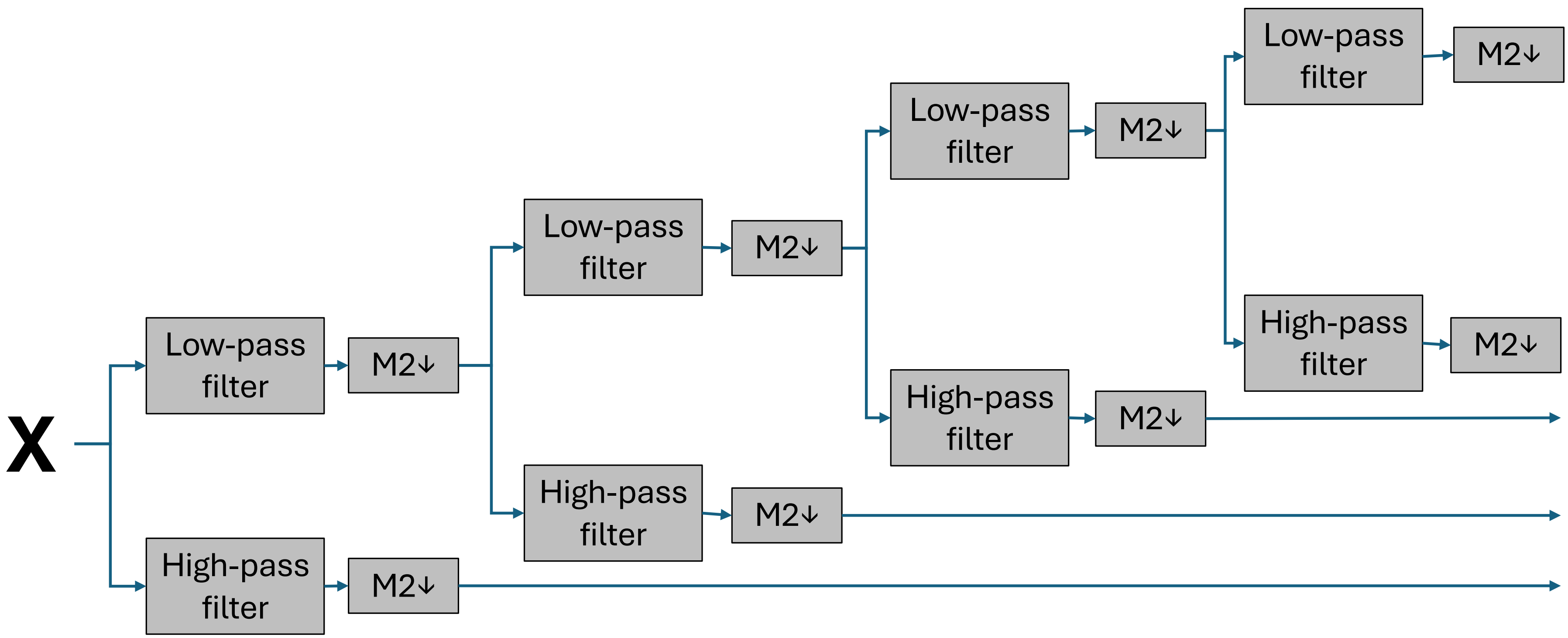
Wavelet Transform

$$F(w) = \int_{-\infty}^{\infty} dt \cdot f(t) \cdot e^{-iwt}$$

$$CWT(\tau, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} dt \cdot x(t) \cdot \psi\left(\frac{t-\tau}{s}\right)$$

Tau- time/shift
Scale – frequency
Large S = stretched wavelet (slow changes)
Small S = compressed (fine details)

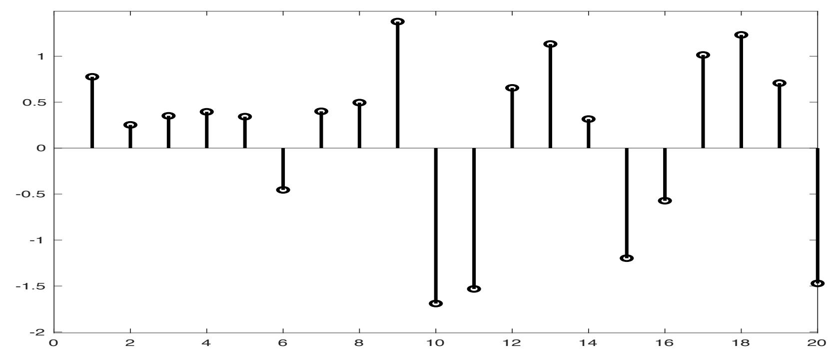
Wavelet Transform



Decimator

Take every M^{th} sample

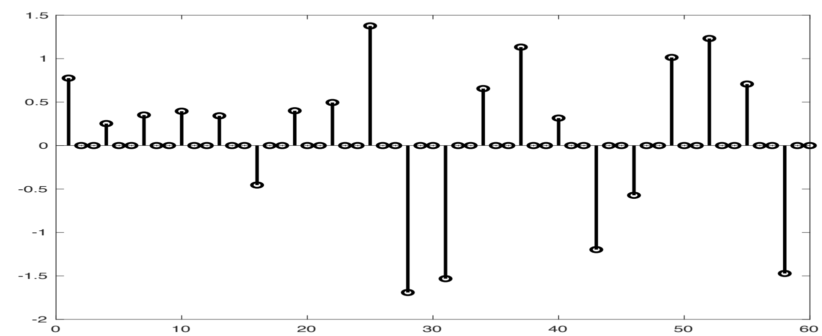
M↓



Interpolater

Expand by M samples

M↑

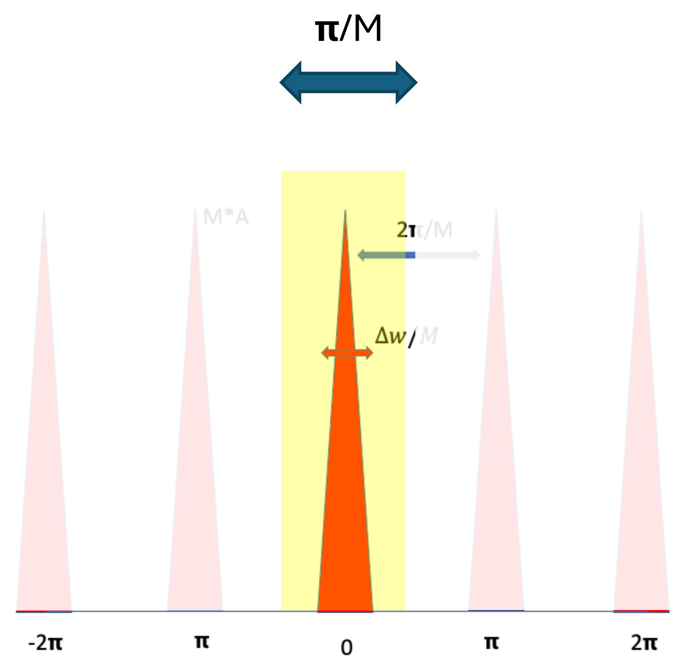


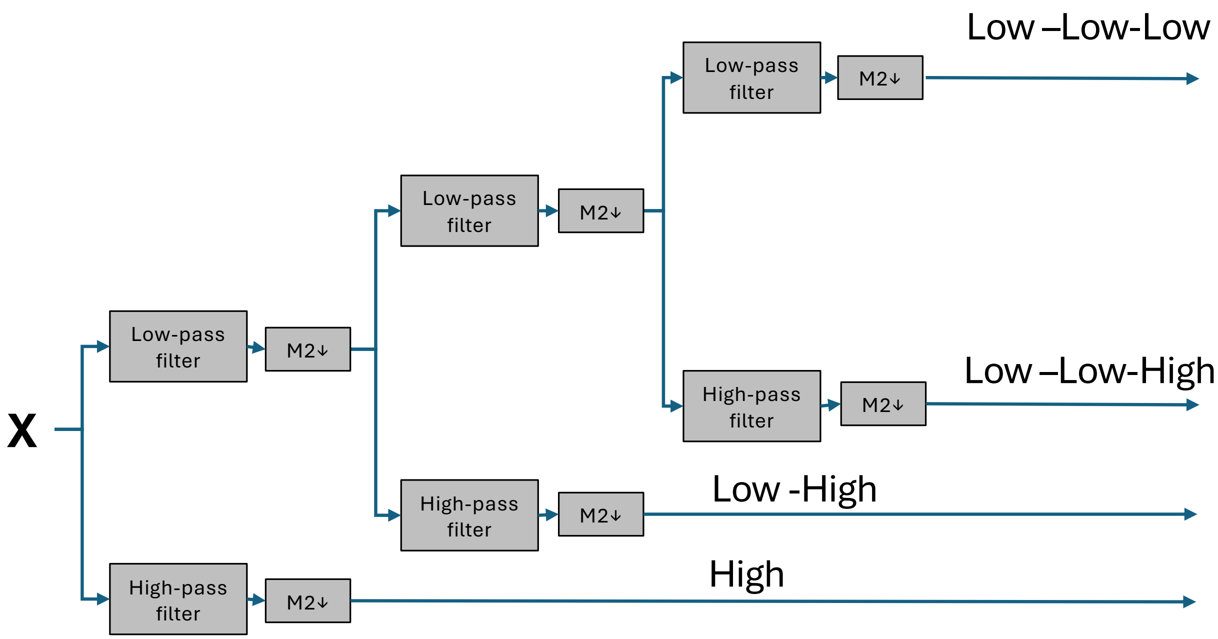
- LPF with cutoff at **π**/M

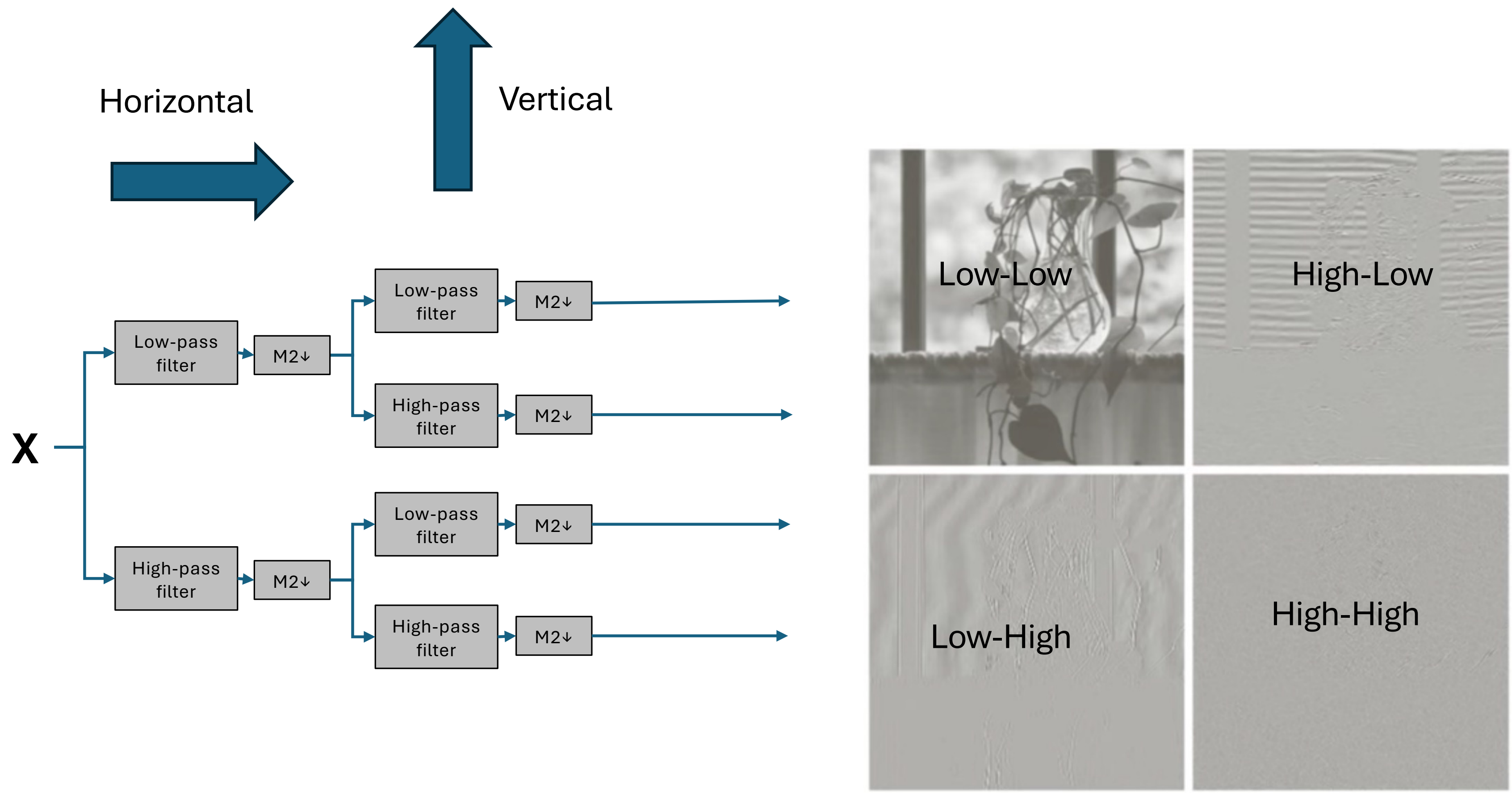
$$\text{LPF} \rightarrow M\downarrow$$

- LPF with cutoff at **π**/M

$$M\uparrow \rightarrow \text{LPF}$$




X
Low-pass filter
M2↓
High-pass filter
M2↓
Low-pass filter
M2↓
High-pass filter
M2↓
Low-pass filter
M2↓
High-pass filter
M2↓
Low –Low-Low
Low –Low-High
Low -High
High

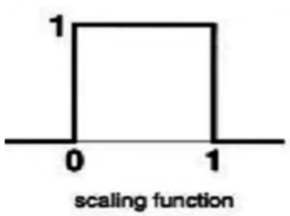

Horizontal
Vertical
X
Low-pass filter
M2↓
High-pass filter
M2↓
Low-pass filter
M2↓
High-pass filter
M2↓
Low-pass filter
M2↓
High-pass filter
M2↓
Low-Low
High-Low
Low-High
High-High

Haar wavelet

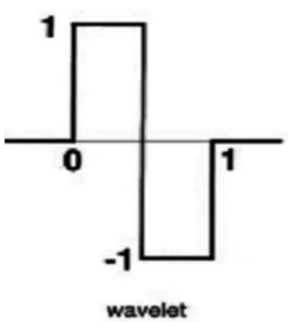
$$H_{2x2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Integrator: Low-pass filter

Differentiator: High-pass filter



$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & else \end{cases}$$



$$\psi(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1 \\ 0 & else \end{cases}$$

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} H_2 \otimes [1 \quad 1] \\ I_2 \otimes [1 \quad -1] \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$H_8 = \frac{1}{\sqrt{2}}\begin{bmatrix} H_4 \otimes [1 \quad 1] \\ I_4 \otimes [1 \quad -1] \end{bmatrix}$$

Haar transform

- $H^T * H = I$ Real and Orthogonal

- Y = $H * \mathrm{X} * H^T$ Forward 2D Haar transform
- X = $H^T * \mathrm{Y} * H$ Inverse 2D Haar transform