Lecture 14

Lateral-Directional Motion

Image Courtesy of NASA

Lateral-Directional Equations of Motion

$$\begin{bmatrix} \Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\Phi} \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & -\left(u_0 - Y_r\right) & g\cos\theta_0 \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\Phi \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\Phi \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

Lateral-Directional Equations of Motion

- **The equations of motion can be rewritten in terms of sideslip angle instead of y velocity component, v**

$$\Delta\beta \approx \frac{\Delta v}{u_0}$$

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\Phi} \end{bmatrix} = \begin{bmatrix} \frac{Y_\beta}{u_0} & \frac{Y_p}{u_0} & -\left(1-\frac{Y_r}{u_0}\right) & \frac{g\cos\theta_0}{u_0} \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\Phi \end{bmatrix} + \begin{bmatrix} 0 & \frac{Y_{\delta_r}}{u_0} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

State-space Form

- **The homogeneous solution will be of the form:**

$$\mathbf{x} = \mathbf{x}_r e^{\lambda_r t}$$

- **Substituting the solution in the state-space equation**

$$\left[\lambda_r \mathbf{I} - \mathbf{A}\right]\mathbf{x}_r = 0$$

- **A nontrivial solution exists when the determinant is zero**

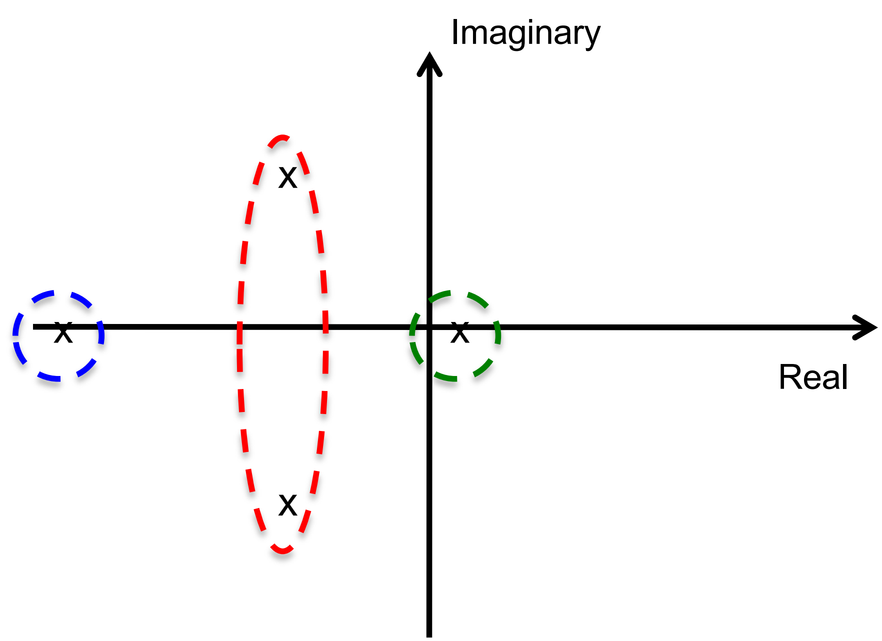
$$\left|\lambda_r \mathbf{I} - \mathbf{A}\right| = 0$$

- **Expanding the determinant leads to a 4th order characteristic equation. Therefore, there will be 4 roots**
  - **Two real roots representing the spiral mode and the roll mode**
  - **A pair of complex roots representing the Dutch roll mode**

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

Lateral-Directional Modes

- **Solving for the roots leads to two real roots and a complex pair**
  - **Roll mode**
  - **Dutch Roll**
  - **Spiral Mode**



Pure Rolling Motion

- **The roll mode can be approximated by looking at a pure rolling motion simplification - the aircraft is free to rotate about its x-axis but all other motion is constrained**

$$\sum RollingMoments = I_x \ddot{\Phi}$$

$$\frac{\partial L}{\partial \delta_a}\Delta\delta_a + \frac{\partial L}{\partial p}\Delta p = I_x \Delta\ddot{\Phi}$$

- **There is one real root to the characteristic equation (no oscillations)**

$$-\frac{\Delta\dot{p}}{L_p} + \Delta p = -\frac{L_{\delta_a}\Delta\delta_a}{L_p}$$

$$\boxed{\begin{aligned} L_{\delta_a} &= \frac{\partial L/\partial \delta_a}{I_X} \\ L_p &= \frac{\partial L/\partial p}{I_X} \end{aligned}}$$

Aileron Step Input

- **Applying a step aileron input, the solution for the roll rate will be**

$$\Delta p(t) = -\frac{L_{\delta_a}}{L_p}\left(1 - e^{-t/\tau}\right)\Delta\delta_a \qquad \tau = -\frac{1}{L_p}$$

- **τ is the time constant, indicating how fast the system approaches a steady state condition after being disturbed**

- **As t increases, $e^{-t/\tau}$ goes to zero. Therefore the steady state value for p is found to be:**

$$p_{ss} = -\frac{L_{\delta_a}}{L_p}\Delta\delta_a \qquad p_{ss} = \frac{-C_{l_{\delta_a}} QSb/I_X}{C_{l_p}(b/2u_0)QSb/I_X}\Delta\delta_a$$

$$\boxed{L_p = \frac{QSb^2 C_{l_p}}{2I_X u_0} \qquad L_{\delta_a} = \frac{QSbC_{l_{\delta_a}}}{I_X}}$$

$$\frac{p_{ss}b}{2u_0} = \frac{-C_{l_{\delta_a}}}{C_{l_p}}\Delta\delta_a$$

- **The term** $\frac{p_{ss}b}{2u_0}$ **can be used to size the aileron**

Roll Mode Approximation

- **Setting the control input to zero:**

$$\tau \Delta \dot{p} + \Delta p = 0$$

- **which leads to the following characteristic equation:**

$$\lambda + \frac{1}{\tau} = 0 \longrightarrow \lambda_{roll} = -\frac{1}{\tau} = L_p$$

- **The size of L_p depends on the size of the wing and tail surfaces.**

$$L_p = \frac{QSb^2 C_{l_p}}{2 I_X u_0}$$

What does $C_{l,p}$ depend on?

Spiral Mode

- **The spiral mode is like the phugoid mode in that it is usually very slow. It can be unstable but often not that important since the pilot has sufficient time to compensate**

- **Spiral Mode is influenced by changes in the bank angle and heading angle**

- **If directional stability is too large, or if lateral stability (dihedral effect) too small, the plane can be unstable in spiral mode**

Spiral Mode Approximation

- **To obtain an approximation for the spiral mode, the side force equation and ΔΦ are neglected**

- **Sideslip remains small, but its effect on roll and yaw is important.**

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} L_\beta & L_r \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix}$$

$$L_\beta \Delta\beta + L_r \Delta_r = 0$$

$$\Delta\dot{r} = N_\beta \Delta\beta + N_r \Delta r$$

$$\Delta\dot{r} + \frac{L_r N_\beta - L_\beta N_r}{L_\beta} \Delta r = 0$$

Spiral Mode Approximation

- **The characteristic equation:**

$$\lambda + \frac{L_r N_\beta - L_\beta N_r}{L_\beta} = 0$$

- **Solving for the root:**

$$\lambda_{spiral} = \frac{L_\beta N_r - L_r N_\beta}{L_\beta}$$

- **For spiral stability ($\lambda_{spiral} < 0$):**

$$L_\beta N_r > L_r N_\beta$$

L_β	<0	Dihedral Effect
N_r	<0	Yaw damping
L_r	>0	Roll moment due to yaw rate
N_β	>0	Directional Stability

Dutch Roll Approximation

- **The Dutch roll mode is coupled roll and yaw motion**

- **If Dutch roll mode is not sufficiently damped, flight can be very uncomfortable**

- **A rough approximation of the Dutch roll mode can be made by assuming that it is primarily driven by side-slipping and yawing motions. If the rolling moment equation is neglected, the equations of motion simplify to:**

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta / u_0 & -\left(1 - Y_r / u_0\right) \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix}$$

Dutch Roll Mode Approximation

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta / u_0 & -(1 - Y_r / u_0) \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix}$$

- **Solving for the characteristic equation:**

$$|\lambda_r \mathbf{I} - \mathbf{A}| = 0 \qquad \lambda^2 - \left(\frac{Y_\beta + u_0 N_r}{u_0} \right) \lambda + \frac{Y_\beta N_r - N_\beta Y_r + u_0 N_\beta}{u_0} = 0$$

- **From the characteristic equation, the un-damped natural frequency and the damping ratio can be obtained:**

$$\omega_{n_{DR}} = \sqrt{\frac{Y_\beta N_r - N_\beta Y_r + u_0 N_\beta}{u_0}} \qquad \zeta_{DR} = -\frac{1}{2\omega_{n_{DR}}} \left(\frac{Y_\beta + u_0 N_r}{u_0} \right)$$

Stability tradeoffs

- **High directional stability tends to help stabilize Dutch roll mode, but it destabilizes spiral mode**

- **Large dihedral effect will stabilize spiral mode, but will make Dutch roll worse**
  - **Large transport aircraft sometimes have problems with this, and that is one reason why you see anhedral on their wings**
  - **They also often have yaw damper systems that automatically actuate the rudder to dampen yaw motion**

Image Courtesy of USAF

Lateral-Directional Motion Summary

- **Aircraft lateral-directional motion is a 4^{th} order system. Solving the 4^{th} order characteristic equation produces two real roots and a complex pair**

- **Roll mode: dominated by the roll damping derivative, it is used to approximate the aircraft's roll response to an aileron deflection**

- **Spiral mode: a slow, sometimes unstable mode where the aircraft bank angle slowly increases if left uncorrected, which causes more banking and turning (spiraling)**

- **Dutch Roll: Coupled roll and yaw oscillating motion**

References

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