# Lecture 7



## Cruise Performance - Range and Endurance

Rutan Voyager -- Image Courtesy of NASA

# Propeller Aircraft Cruise Performance



- **Power and Fuel Flow**
  - $\dot{W}_f$ **= fuel flow (lbs/hr)**
  - $c_p$ **= specific fuel consumption (lbs/shp/hr)**
  - $\eta_{pr}$ **= propeller efficiency**
  - $P_{reqd}$ **= Power required for level flight**

$$\dot{W}_f = SHP_{reqd} c_p = \frac{P_{reqd}}{\eta_{pr}} c_p$$

- **Aircraft endurance (time aloft), depends on the fuel flow required to maintain flight**

$$\frac{dt}{dW} = \frac{-\eta_{pr}}{P_{reqd} c_p} \qquad \textbf{(hrs/lbs)}$$

- **Aircraft range (distance traveled), depends on distance traveled per unit fuel consumed (relationship above multiplied by velocity)**

$$\frac{ds}{dW} = \frac{-V\eta_{pr}}{P_{reqd} c_p} \qquad \textbf{(ft/lbs)}$$

# Range Derivation for Propeller Aircraft



- **Integrating the weight vs. distance relationship from initial to final weight**

$$ds = \frac{-V_\infty \eta_{pr}}{c_p P_{reqd}} dW \qquad R = \int_{W_i}^{W_f} ds \qquad R = \int_{W_f}^{W_i} \frac{\eta_{pr} V_\infty}{c_p V_\infty D} dW$$

- **Multiply top and bottom by W, and assuming L=W:**

$$R = \int_{W_f}^{W_i} \frac{\eta_{pr}}{c_p} \left(\frac{L}{D}\right) \frac{dW}{W}$$

Breguet Range Equation

- **Integrate:**

$$\boxed{R = \frac{\eta_{pr}}{c_p} \frac{C_L}{C_D} \ln\left(\frac{W_i}{W_f}\right)}$$

# Maximizing Range for Propeller Aircraft



- **To maximize range:**
  - **High propeller efficiency**
  - **Low specific fuel consumption**
  - **High Lift-to-drag ratio**
  - **Large weight fraction of fuel**

$$R = \frac{\eta_{pr}}{c_p} \frac{C_L}{C_D} \ln\left(\frac{W_i}{W_f}\right)$$

- **Note that density does not appear in this equation. However, engine performance varies with altitude, so range will depend on altitude**

Aerosonde UAV
Range: 1100 nmi

# Propeller Aircraft - Endurance



- **To find equation for endurance, integrate time vs. weight relationship**

$$\boxed{dt = \frac{-\eta_{pr}}{P_{reqd} c_p} dW} \qquad E = \int_{W_i}^{W_f} dt \longrightarrow E = \int_{W_f}^{W_i} \frac{\eta_{pr} dW}{c_p P_{reqd}} = \int_{W_f}^{W_i} \frac{\eta_{pr} W dW}{c_p V_\infty D W} = \int_{W_f}^{W_i} \frac{\eta_{pr} L}{c_p V_\infty D} \frac{dW}{W}$$

- **Rewrite velocity in terms of lift coefficient, density, and wing loading:**

$$V_\infty = \sqrt{\frac{2}{\rho_\infty C_L}\left(\frac{W}{S}\right)}$$

$$E = \int_{W_f}^{W_i} \frac{\eta_{pr} C_L}{c_p C_D} \sqrt{\frac{\rho_\infty S C_L}{2W}} \frac{dW}{W} \longrightarrow E = \int_{W_f}^{W_i} \frac{\eta_{pr} C_L^{3/2}}{c_p C_D} \sqrt{\frac{\rho_\infty S}{2}} \frac{dW}{W^{3/2}}$$

- **If we assume propeller efficiency, $c_p$, and $C_L^{3/2}/C_D$ is constant, integrating from initial to final weight yields:**

$$E = \frac{\eta_{pr} C_L^{3/2}}{c_p C_D} \sqrt{2\rho_\infty S} \left( \frac{1}{W_f^{1/2}} - \frac{1}{W_i^{1/2}} \right)$$

Breguet Endurance Equation

# Maximizing Endurance for Propeller Aircraft



$$E = \frac{\eta_{pr} C_L^{3/2}}{c_p C_D} \sqrt{2\rho_\infty S} \left( \frac{1}{W_1^{1/2}} - \frac{1}{W_0^{1/2}} \right)$$

- **To maximize endurance:**
  - **Maximize propeller efficiency**
  - **Minimize $c_p$**
  - **Maximize $C_L^{3/2}/C_D$**
  - **Fly at lower altitudes**
  - **Maximize weight fraction of fuel**


E-2D Hawkeye
Endurance: 6-8 hours

# Range Examples

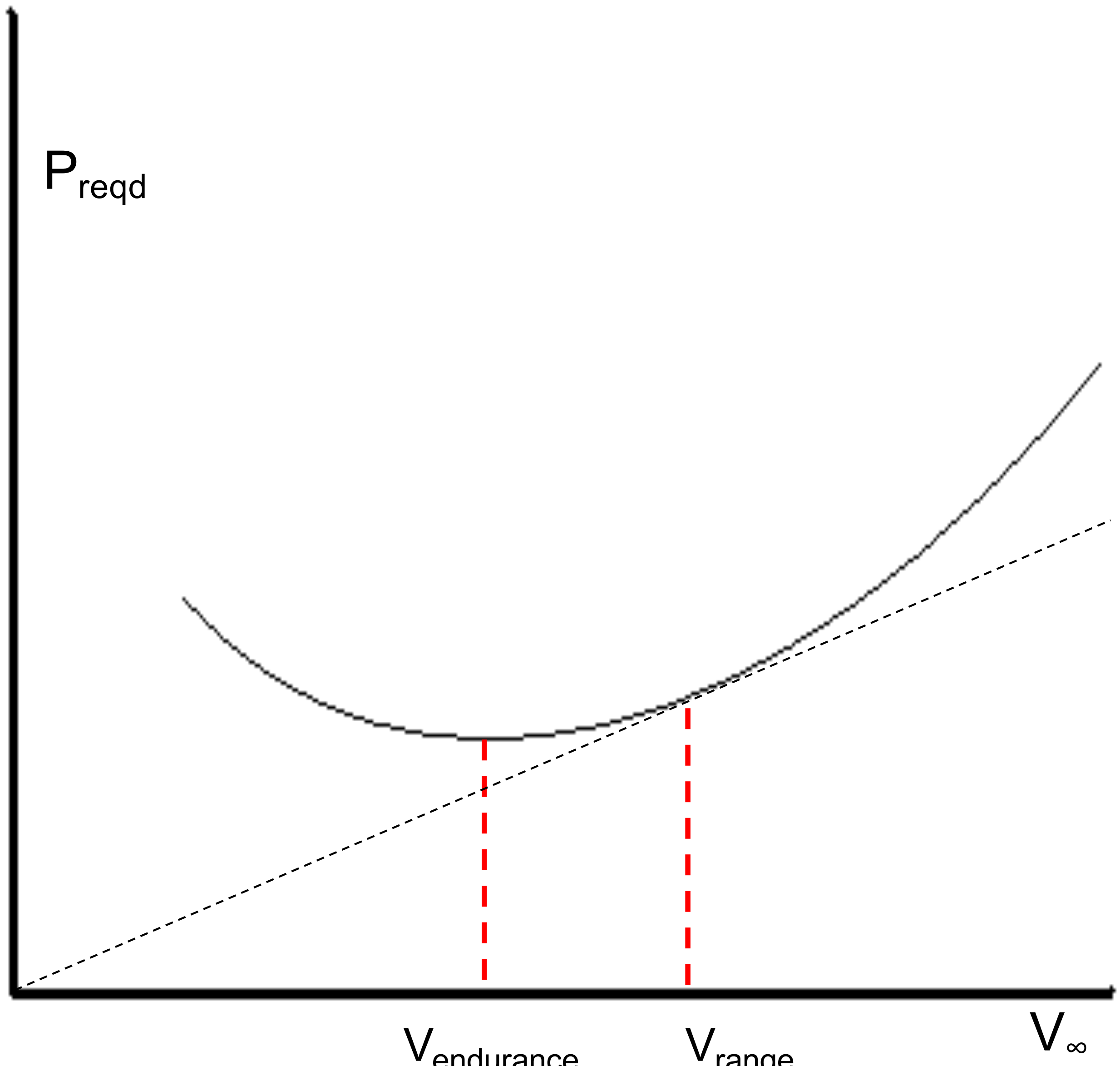




# Power Curve (propeller aircraft)



- **Max endurance at velocity that minimizes $P_{reqd}$**
- **Max range minimizes $T_{reqd}$**
  - **Since T=P/V, $T_{reqd}$ represents slope of power curve**
  - **Minimum $T_{reqd}$ when line through origin is tangent with power curve (when slope is most shallow)**



# Carson's Speed



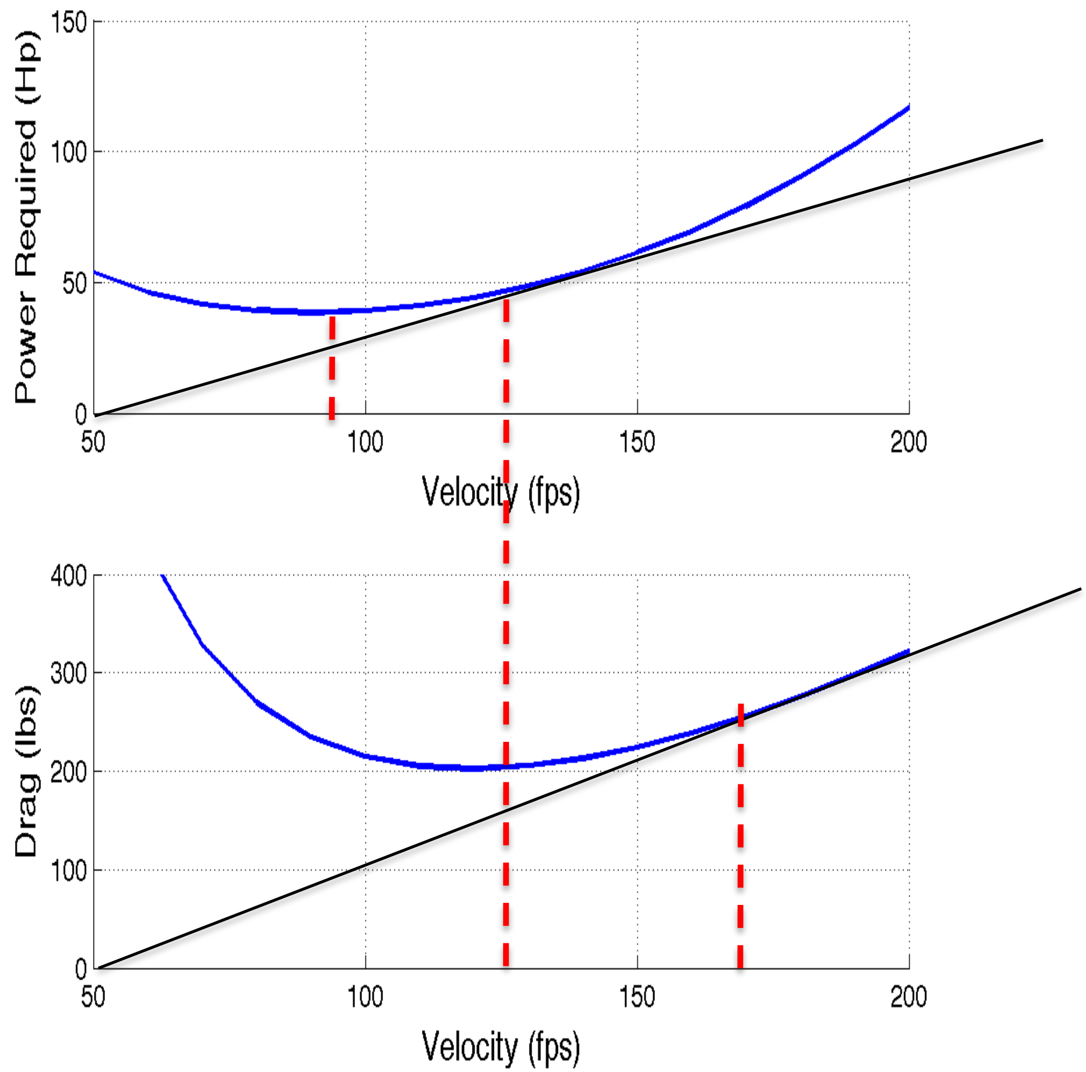
- **To maximize range, propeller aircraft would maximize L/D, but in most situations, propeller aircraft fly faster than velocity for max L/D**

- **Carson's Speed: Highest speed to thrust ratio, also called optimum cruise**
  - **Whereas max range minimized the fuel burned for each mile traveled, Carson's speed minimizes how much fuel we burn for each knot of velocity**
  - **We've seen that before from the jet aircraft range derivation**
  - **Carson's speed for propeller aircraft is same as velocity for max range for jet aircraft**

# Power and Thrust Curves

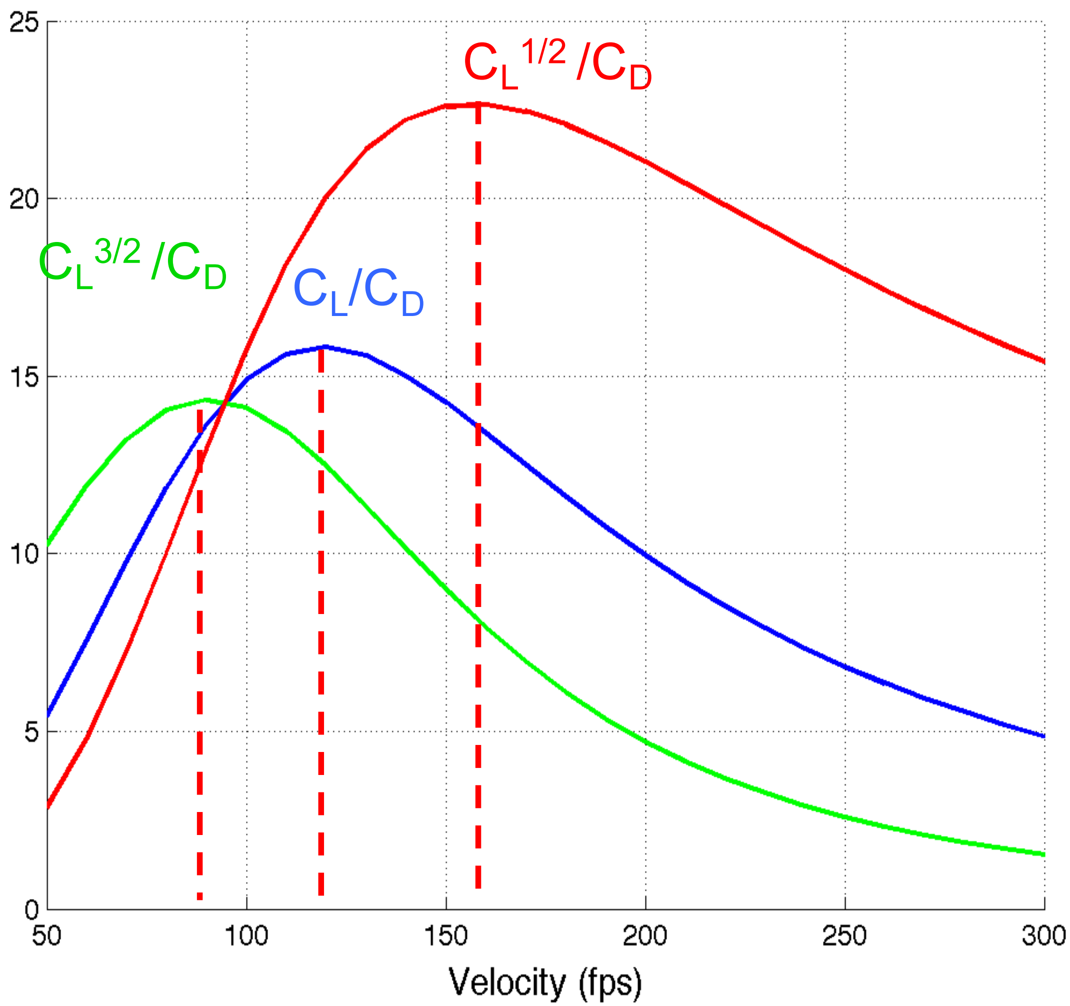


- **Minimum of Power Curve**
  - **Best endurance for propeller aircraft**
- **Minimum of thrust curve (and tangent line of power curve)**
  - **Best range for propeller aircraft.**
  - **Best endurance for jets**
- **Tangent line of thrust curve**
  - **Best range for jets**
  - **Optimal Cruise for propeller aircraft (Carson's Speed)**



# $C_L$ Relationships






Endurance speed, range speed and Carson's speed are all related by $3^{1/4}$

$$\frac{V_{C_L/C_D}}{V_{C_L^{3/2}/C_D}} = 3^{1/4} = 1.32$$

$$\frac{V_{C_L^{1/2}/C_D}}{V_{C_L/C_D}} = 3^{1/4} = 1.32$$

# Jet Aircraft Breguet Equations



- **Jet engines produce thrust directly, and thrust is roughly constant with velocity**
  - **Thrust is directly related to throttle setting ( for propellers, power is tied to throttle setting)**

- **As a result, fuel efficiency for jets is based on thrust, not power**

- **Thrust and fuel flow:**
  - **$c_t$ = thrust specific fuel consumption (lbs/hr/lbs)**
  - **$T_{reqd}$ = thrust required for level flight (lbs)**
  - $\dot{W}_f$ **= Fuel flow (lbs/hr)**

$$\dot{W}_f = T_{reqd} c_t$$

$$dW = -\dot{W}_f dt = -c_t T_{reqd} dt$$

# Jet Aircraft – Endurance



- **Fuel flow rate is proportional to the thrust required**

$$\frac{dW}{dt} = -T_{reqd} c_t$$

- **Integrate time vs. weight relationship from initial to final weight:**

$$E = \int_{t_0}^{t_1} dt = \int_{W_i}^{W_f} \frac{dW}{-c_t T_{reqd}} \qquad E = \int_{W_f}^{W_i} \frac{dW}{c_t T_{reqd}}$$

- **Multiply top and bottom by W, and since \L=W:**

$$E = \int_{W_f}^{W_i} \frac{dW}{c_t T_{reqd}} = \int_{W_f}^{W_i} \frac{W dW}{c_t D W} = \int_{W_f}^{W_i} \frac{L}{c_t D} \frac{dW}{W}$$

- **Integrate:**

$$\boxed{E = \frac{1}{c_t} \frac{C_L}{C_D} \ln\left(\frac{W_0}{W_1}\right)}$$

# Jet Aircraft – Endurance



$$E = \frac{1}{c_t}\frac{C_L}{C_D}\ln\left(\frac{W_0}{W_1}\right)$$


RQ-4 Global Hawk
Endurance: 34+ hours

- **To maximize endurance for jet aircraft:**
  - **Maximize L/D**
  - **Minimize TSFC**
  - **Make fuel weight fraction as large as possible**

- **According to Breguet equation, altitude is not important**

# Jet Aircraft Range



- **Relationship between range and weight is**

$$\frac{ds}{dW} = \frac{-V}{T_{reqd} c_t}$$

- **To maximize range, maximize $V/T_{reqd}$**
  - **From** $$V_\infty = \sqrt{\frac{2}{\rho_\infty C_L}\left(\frac{W}{S}\right)}$$ $$V \propto C_L^{-1/2}$$
  - **For range, maximize $C_L^{1/2}/C_D$**

# Jet Aircraft - Range



- **To find range, integrate distance vs. weight relationship:**

$$\boxed{\frac{ds}{dW} = \frac{-V}{T_{reqd} c_t}} \qquad R = \int_{W_i}^{W_f} dx = \int_{W_i}^{W_f} \frac{V_\infty dW}{-c_t T_{reqd}}$$

- **Insert** $\sqrt{\frac{2}{\rho_\infty C_L}\left(\frac{W}{S}\right)}$ **in place of V:**

$$R = \int_{W_f}^{W_i} \frac{V_\infty dW}{c_t T_{reqd}} = \int_{W_f}^{W_i} \frac{V_\infty W dW}{c_t D W} = \int_{W_f}^{W_i} \frac{V_\infty L}{c_t D} \frac{dW}{W}$$

$$R = \int_{W_f}^{W_i} \frac{C_L}{c_t C_D} \sqrt{\frac{2W}{\rho_\infty S C_L}} \frac{dW}{W} = \int_{W_f}^{W_i} \frac{C_L^{1/2}}{c_t C_D} \sqrt{\frac{2}{\rho_\infty S}} \frac{dW}{W^{1/2}}$$

- **Integrate from $W_i$ to $W_f$:** $R = \frac{2}{c_t} \frac{C_L^{1/2}}{C_D} \sqrt{\frac{2}{\rho_\infty S}} \left(W_0^{1/2} - W_f^{1/2}\right)$

# Maximizing Range for Jet Aircraft



- **In order to increase our range, a jet aircraft should**
  - **Maximize $C_L^{1/2}/C_D$**
  - **Minimize TSFC**
  - **Fly at high altitudes (minimize air density)**
  - **Make fraction of fuel weight as large as possible**

$$R = \frac{2}{c_t}\frac{C_L^{1/2}}{C_D}\sqrt{\frac{2}{\rho_\infty S}}\left(W_0^{1/2} - W_f^{1/2}\right)$$

# Breguet Range Assumptions



- **We've seen that the Breguet Range equation is based on solving the integral:**

$$R = \int_{W_f}^{W_i} \frac{V_\infty L}{c_t D} \frac{dW}{W}$$

- **There are 3 critical parameters that the pilot can control (density, velocity, L/D). To solve the integral, we hold two parameters constant and assume the 3rd will vary as the weight changes (as the weight changes, it's impossible to hold all 3 constant)**
  - **The Breguet Range equations held density and L/D constant (constant altitude, constant angle of attack). This implies that velocity will vary as the weight changes**
  - **However, we could have solved for the range by holding velocity and L/D constant and allowing density to change. How would density change as weight changes? If a pilot is flying at a constant velocity, what must he do to remain at a constant L/D as his weight decreases?**

# Breguet Range Assumptions



Jet Range Equations

**Range equation based on constant L/D and density assumptions**

$$R = \frac{2}{c_t}\frac{C_L^{1/2}}{C_D}\sqrt{\frac{2}{\rho_\infty S}}\left(W_0^{1/2} - W_f^{1/2}\right)$$

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**Range equation based on constant L/D and constant velocity**

$$R = \frac{V}{c_t}\frac{C_L}{C_D}\ln\left(\frac{W_0}{W_f}\right)$$

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**Range equation based on constant velocity and constant altitude (ref 3)**

$$R = \frac{V}{c_t\sqrt{kC_{D_0}}}\left[\tan^{-1}\frac{\sqrt{k}}{\frac{1}{2}\rho V^2 S\sqrt{C_{D_0}}}W_0 - \tan^{-1}\frac{\sqrt{k}}{\frac{1}{2}\rho V^2 S\sqrt{C_{D_0}}}W_1\right]$$

# Electric Aircraft – Endurance



- **For battery-electric aircraft, the aircraft does not lose weight as it flies, simplifying range and endurance derivations**
- **Endurance Derivation**
  - **E = Stored battery energy**
  - $\eta_p$ **= Propeller efficiency**
  - $\eta_m$ **= Motor efficiency**

$$\dot{E} = \frac{dE}{dt} = \frac{P_{reqd}}{\eta_p \eta_m} = \frac{DV_\infty}{\eta_p \eta_m}$$

$$dt = \frac{\eta_p \eta_m}{DV_\infty} dE = \frac{\eta_p \eta_m}{DV_\infty} \frac{L}{W} dE = \frac{\eta_p \eta_m}{DV_\infty} \frac{L}{W} dE$$

$$t = \int \eta_p \eta_m \frac{C_L^{3/2}}{C_D} \sqrt{\frac{\rho_\infty S}{2}} \frac{dE}{W^{3/2}}$$

$$V_\infty = \sqrt{\frac{2}{\rho_\infty C_L}\left(\frac{W}{S}\right)}$$

$$t = \eta_p \eta_m \frac{C_L^{3/2}}{C_D} \sqrt{\frac{\rho_\infty S}{2}} \frac{E}{W^{3/2}}$$

- **Maximize endurance by minimizing power required**

# Electric Aircraft - Range



- **For battery-electric aircraft, the aircraft does not lose weight as it flies, simplifying range and endurance derivations**
- **Range Derivation**

$$\frac{dE}{ds} = \frac{P_{reqd}}{\eta_p \eta_m V_\infty} = \frac{D V_\infty}{\eta_p \eta_m V_\infty}$$

$$V_\infty = \sqrt{\frac{2}{\rho_\infty C_L}\left(\frac{W}{S}\right)}$$

$$ds = \frac{\eta_p \eta_m}{D} dE = \frac{\eta_p \eta_m}{D} \frac{L}{W} dE = \frac{\eta_p \eta_m}{D} \frac{L}{W} dE$$

$$R = \int \eta_p \eta_m \frac{C_L}{C_D} \frac{dE}{W}$$

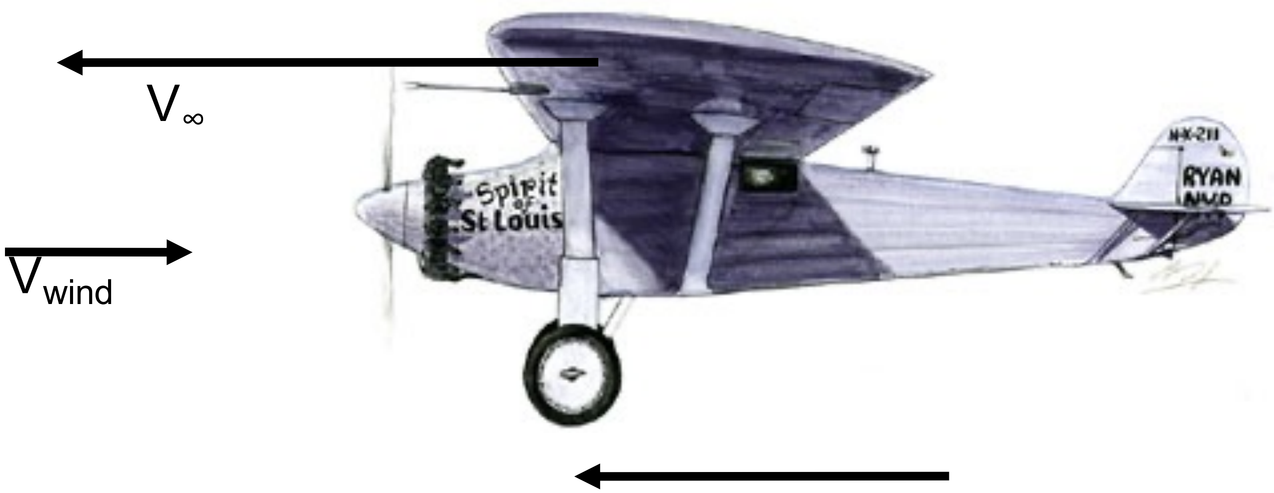
- **Maximize range by minimizing drag (maximize L/D)**

$$\boxed{R = \eta_p \eta_m \frac{C_L}{C_D} \frac{E}{W}}$$

# Wind Effects



- **How does wind affect range?**
- **Does it affect endurance?**




$$V_G = V_\infty + V_{wind}$$

# Wind Effects



- **To maximize range in headwind conditions, an aircraft should fly at an airspeed higher than the one corresponding to max range with no wind**
  - **Rule of thumb is to increase airspeed by approximately half of the wind speed.. (ex. In a 10 kt headwind, increase speed by 5 kts)**

- **In a tailwind, an aircraft should slow down to maximize range**

# References



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