Lecture 13

Longitudinal Motion
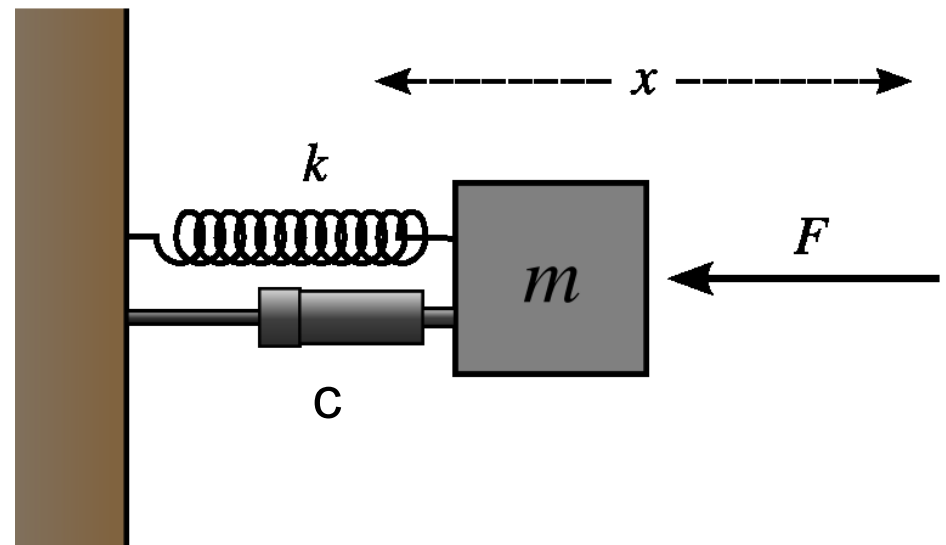
2nd Order Differential Equations

- Many systems can be modeled by second order differential equations

- When driven by a forcing function, F(t), the response is referred to as the forced response

- The free response occurs when the forcing function is zero. The homogenous solution is the solution to the differential equation when the right side of the equation is zero

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = F(t)$$

Spring Mass Damper System – Image by I.Karonen

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2nd Order Differential Equations

- **The solution to the differential equation can be found by making the following substitution:**

$$x = Ae^{\lambda t}$$

- **The equation can then be written as**

$$\lambda^2 Ae^{\lambda t} + \frac{c}{m}\lambda Ae^{\lambda t} + \frac{k}{m}Ae^{\lambda t} = 0$$

- **Getting rid of the exponent terms produces the characteristic equation:**

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \qquad \lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- **The solution to the differential equation is**

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

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2nd Order Differential Equations

- **The type of motion depends on the value of λ, which depends on the physical constants of the problem**

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- **Three possible cases for λ are:**

  - $\left(c/2m\right) > \sqrt{k/m}$ **Real roots, over-damped condition**
  - $\left(c/2m\right) = \sqrt{k/m}$ **Repeated roots – critically damped condition**
  - $\left(c/2m\right) < \sqrt{k/m}$ **Complex roots – under-damped condition**

Over-damped Condition

- **When** $(c/2m) > \sqrt{k/m}$ **the roots are real, and assuming** $(c/2m)$ **>0, the roots will be negative**

- **This is the over-damped condition, where the motion will die out exponentially with time**

- **The solution will take the form:**

$$x(t) = C_1 \exp\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\, \right]t + C_2 \exp\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\, \right]t$$

Under-damped Condition

- **When** $\left(c/2m\right) < \sqrt{k/m}$, **the roots will be complex**

- **The equation of motion is**

$$x(t) = C_1 \exp\left[-\frac{c}{2m} - i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,\right]t + C_2 \exp\left[-\frac{c}{2m} + i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,\right]t$$

- **And can be re-written as**

$$x(t) = \exp\left(-\frac{c}{2m}t\right)\left[A\cos\left[\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t\right] + B\sin\left[\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t\right]\right]$$

- **The sinusoid motion has a frequency**

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

Critically Damped Case

- $(c/2m) = \sqrt{k/m}$ **represents the boundary between the over-damped and the under-damped condition**

- **The roots are identical:**

$$\lambda_{1,2} = -\frac{c}{2m}$$

- **The general solution for repeated roots becomes**

$$x(t) = (C_1 + C_2 t)e^{\lambda t}$$

Critically Damped Case

- **At the critically damped case**

$$\frac{k}{m} = \left(\frac{c}{2m}\right)^2 \quad \longrightarrow \quad c = 2\sqrt{km} = c_{cr}$$

 c_{cr} is the critical damping constant

- **The damping ratio, ζ, is the ratio between the oscillating damping constant and the critical damping constant**

$$c = \zeta c_{cr} \qquad \zeta = \frac{c}{c_{cr}}$$

- **If the damping term is zero, the frequency becomes**

$$\omega = \sqrt{\frac{k}{m}} = \omega_n$$

 where ω_n is the un-damped natural frequency

Damping Ratio & Natural Frequency

- **The characteristic equation can be written in terms of the damping ratio and the natural frequency**
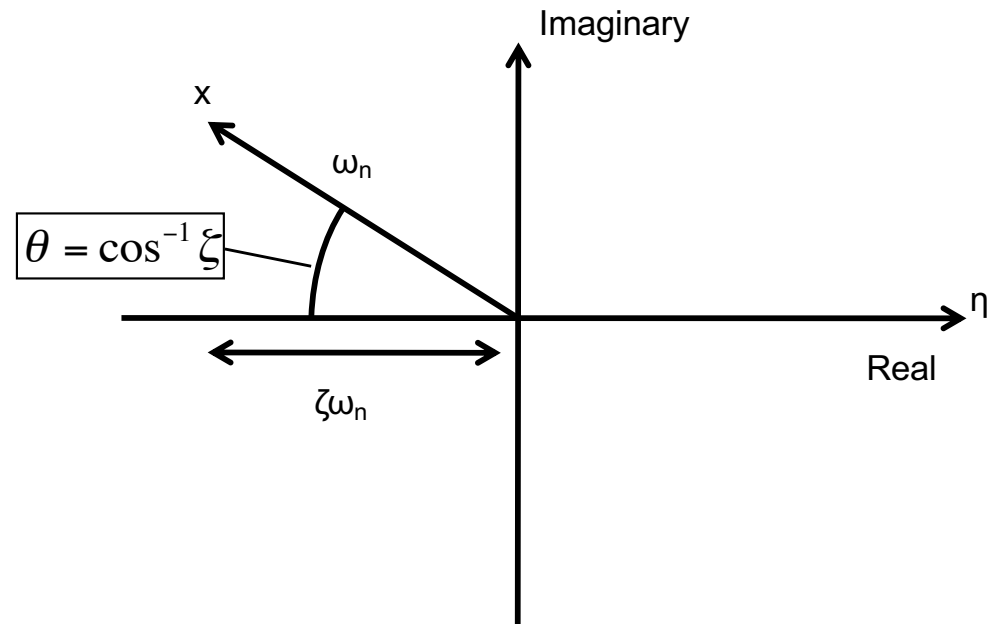
- $$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

  $$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

- **The real part of λ determines the damping of the response**

  $$\eta = -\zeta\omega_n$$

$$\boxed{\theta = \cos^{-1}\zeta}$$

Imaginary

x

ω_n

$\zeta\omega_n$

η

Real

- **The imaginary part is ω, is the damped natural frequency**

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

Longitudinal Motion – State Space

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- **Rewriting the longitudinal equations of motion in state-space form:**

$$\begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{w} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ Z_{\delta_e} & Z_{\delta_T} \\ M_{\delta_e} + M_{\dot{w}}Z_{\delta_e} & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_T \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$ **is the state vector**

$$X_u = \frac{-1}{m}\frac{\overline{q}S}{u_0}\left(C_{D_u} + 2C_{D_0}\right)$$

$$\mathbf{\eta} = \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_T \end{bmatrix}$$ **is the control vector**

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Longitudinal Motion – State Space

- **The homogeneous solution will be of the form:**

$$\mathbf{X} = \mathbf{X}_r e^{\lambda_r t}$$

- **Substituting the solution in the state-space equation:**

$$\left[\lambda_r \mathbf{I} - \mathbf{A}\right]\mathbf{x}_r = 0$$

- **A nontrivial solution exists when the determinant is zero:**
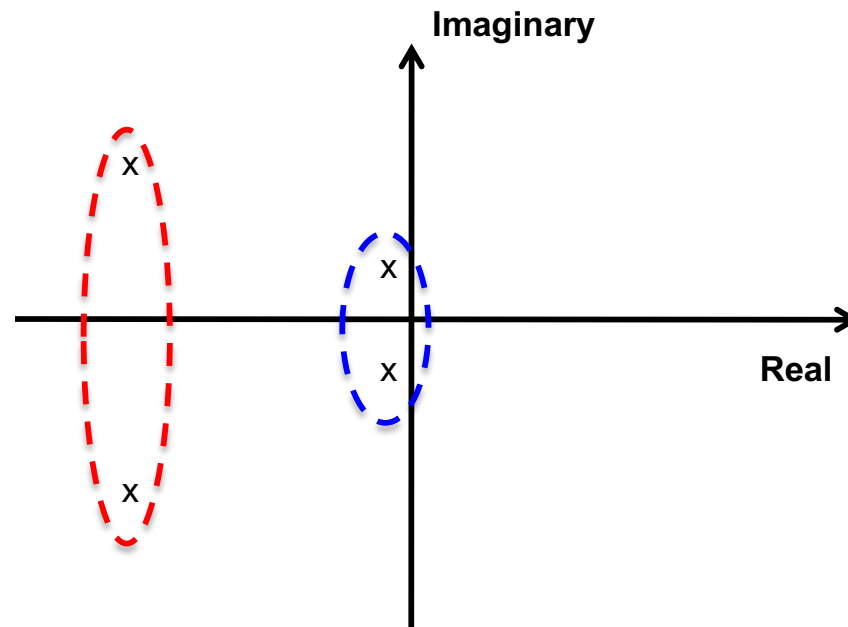
$$\left|\lambda_r \mathbf{I} - \mathbf{A}\right| = 0$$

- **Expanding the determinant leads to a 4th order characteristic equation. Therefore, there will be 4 roots, two pairs of complex roots representing the phugoid mode and the short period mode**

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

Longitudinal Modes

- **Solving for the roots leads to two complex pairs for the two longitudinal modes of motion**
  - **Phugoid (long-period oscillations)**
  - **Short period**

Phugoid Mode

- **The phugoid mode is the long-period mode. It represents the gradual interchange of energy between kinetic and potential energy as the aircraft changes altitude**

- **To approximate the phugoid, the pitching moment equation is neglected, and the change in angle of attack is assumed to be small**

$$\Delta\alpha = \frac{\Delta w}{u_0} = 0$$

- **This leads to a simplified set of equations:**

$$\begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ \dfrac{-Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

Phugoid Mode

- **To solve for the roots:**

$$\left| \lambda_r \mathbf{I} - \mathbf{A} \right| = 0 \quad \longrightarrow \quad \begin{vmatrix} \lambda - X_u & g \\ \dfrac{Z_u}{u_0} & \lambda \end{vmatrix} = 0 \qquad \lambda_p = \left[X_u \pm \sqrt{X_u^2 + 4\dfrac{Z_u g}{u_0}} \right] \Big/ 2$$

- **The natural frequency and damping ratio terms are given as:**

$$\omega_{n_p} = \sqrt{\dfrac{-Z_u g}{u_0}} \qquad\qquad \zeta_p = \dfrac{-X_u}{2\omega_{n_p}}$$

- **It turns out after evaluating the stability derivatives and making a few simplifications, the frequency and damping ratios can be approximated by:**

$$\omega_{n_p} = \sqrt{2}\,\dfrac{g}{u_0} \qquad\qquad \zeta_p = \dfrac{1}{\sqrt{2}}\dfrac{1}{L/D}$$

Short Period

- **To approximate the short period mode, the X-force equation is ignored, and the velocity is assumed to remain constant**

$$\begin{bmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} Z_q \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$\Delta \alpha = \frac{\Delta w}{u_0} \qquad M_{\dot{\alpha}} = u_0 M_{\dot{w}}$$

$$\begin{bmatrix} \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha / u_0 & 1 \\ \left(M_\alpha + M_{\dot{\alpha}} Z_\alpha / u_0 \right) & \left(M_q + M_{\dot{\alpha}} \right) \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix}$$

Short Period

- **To find the roots:**

$$\begin{vmatrix} \lambda - Z_\alpha/u_0 & -1 \\ -M_\alpha - M_{\dot{\alpha}} Z_\alpha/u_0 & \lambda - M_q - M_{\dot{\alpha}} \end{vmatrix} = 0$$

$$\lambda^2 - \left(M_q + M_{\dot{\alpha}} + Z_\alpha/u_0 \right) \lambda + M_q Z_\alpha/u_0 - M_\alpha = 0$$

$$\lambda_{sp} = \left(M_q + M_{\dot{\alpha}} + Z_\alpha/u_0 \right)\Big/ 2 \pm \left[\left(M_q + M_{\dot{\alpha}} Z_\alpha/u_0 \right)^2 - 4\left(M_q Z_\alpha/u_0 - M_\alpha \right) \right]^{1/2} \Big/ 2$$

- **The damping ratio and natural frequency are:**

$$\omega_{n_{sp}} = \sqrt{ \frac{Z_\alpha M_q}{u_0} - M_\alpha }$$

$$\zeta_{sp} = \frac{ -\left(M_q + M_{\dot{\alpha}} + Z_\alpha/u_0 \right) }{ 2\omega_{n_{sp}} }$$

Summary

- **Aircraft longitudinal Motion is a 4th order system. The solution to the 4th order characteristic equation is two complex root pairs**

- **Phugoid mode**
  - **Long period, low damped motion**
  - **AoA is roughly constant as vehicle exchanges kinetic and potential energy (oscillating in altitude, increasing and decreasing velocity)**

- **Short period mode**
  - **Faster dynamics, shorter period, but higher damped motion**

- **The two modes are distinct in frequency. The 4th order system and can be approximated as two separate 2nd order systems**

References

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