Lecture 11

Stability Derivatives

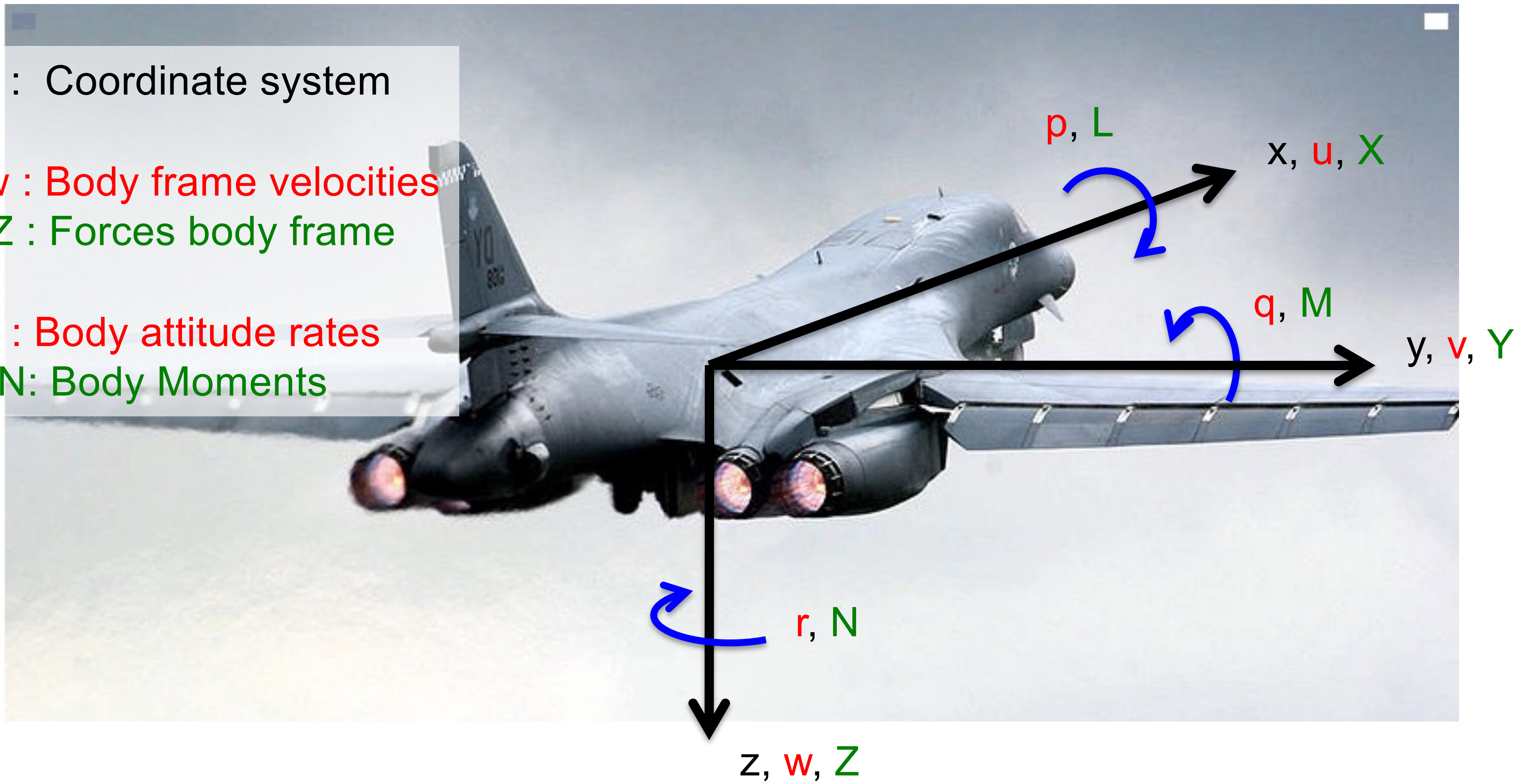
Image Courtesy of NASA

Coordinate System & Definitions

x, y, z : Coordinate system

u, v, w : Body frame velocities
X, Y, Z : Forces body frame

p, q, r : Body attitude rates
L, M, N: Body Moments



Aircraft Equations of Motion

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + m(\boldsymbol{\omega} \times \mathbf{v}) \qquad \mathbf{M} = \frac{d\mathbf{H}}{dt} + (\boldsymbol{\omega} \times \mathbf{H})$$

Force Equations

$$X - mg\sin\theta = m(\dot{u} + qw - rv)$$
$$Y + mg\cos\theta\sin\phi = m(\dot{v} + ru - pw)$$
$$Z + mg\cos\theta\cos\phi = m(\dot{w} + pv - qu)$$

Moment Equations

$$L = I_X\dot{p} - I_{XZ}\dot{r} + qr(I_Z - I_Y) - I_{XZ}pq$$
$$M = I_Y\dot{q} + rp(I_X - I_Z) + I_{XZ}(p^2 - r^2)$$
$$N = -I_{XZ}\dot{p} + I_Z\dot{r} + pq(I_Y - I_X) + I_{XZ}qr$$

Body Rates

$$p = \dot{\phi} - \dot{\psi}\sin\theta$$
$$q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$
$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

Euler Rates

$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta$$

Stability Derivatives

$$\Delta X = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta \delta_e + \frac{\partial X}{\partial \delta_t}\Delta \delta_t$$

$$\Delta Y = \frac{\partial Y}{\partial v}\Delta v + \frac{\partial Y}{\partial p}\Delta p + \frac{\partial Y}{\partial r}\Delta r + \frac{\partial Y}{\partial \delta_r}\Delta \delta_r$$

$$\Delta Z = \frac{\partial Z}{\partial u}\Delta u + \frac{\partial Z}{\partial w}\Delta w + \frac{\partial Z}{\partial \dot{w}}\Delta \dot{w} + \frac{\partial Z}{\partial q}\Delta q + \frac{\partial Z}{\partial \delta_e}\Delta \delta_e + \frac{\partial Z}{\partial \delta_t}\Delta \delta_t$$

Aerodynamic forces and moments can be represented with stability coefficients

$$\Delta L = \frac{\partial L}{\partial v}\Delta v + \frac{\partial L}{\partial p}\Delta p + \frac{\partial L}{\partial r}\Delta r + \frac{\partial L}{\partial \delta_r}\Delta \delta_r + \frac{\partial L}{\partial \delta_a}\Delta \delta_a$$

$$\Delta M = \frac{\partial M}{\partial u}\Delta u + \frac{\partial M}{\partial w}\Delta w + \frac{\partial M}{\partial \dot{w}}\Delta \dot{w} + \frac{\partial M}{\partial q}\Delta q + \frac{\partial M}{\partial \delta_e}\Delta \delta_e + \frac{\partial M}{\partial \delta_t}\Delta \delta_t$$

$$\Delta N = \frac{\partial N}{\partial v}\Delta v + \frac{\partial N}{\partial p}\Delta p + \frac{\partial N}{\partial r}\Delta r + \frac{\partial N}{\partial \delta_r}\Delta \delta_r + \frac{\partial N}{\partial \delta a}\Delta \delta_a$$

Derivatives due to Change in Forward Speed

- **The aircraft drag, lift, and pitching moments will vary as the forward speed changes**

- **The change in the X force is due to changes in drag and thrust**

$$\Delta X = \frac{\partial X}{\partial u}\Delta u \qquad\qquad \frac{\partial X}{\partial u} = -\frac{\partial D}{\partial u} + \frac{\partial T}{\partial u}$$

- **Rewriting the X force in terms of drag and thrust, and taking the partial derivative with respect to speed:**

$$\frac{\partial X}{\partial u} = -\frac{1}{2}\rho S\left(u_0^2\frac{\partial C_D}{\partial u} + 2u_0 C_{D_0}\right) + \cancel{\frac{\partial T}{\partial u}}$$

Neglecting thrust variations with speed

- $\dfrac{\partial X}{\partial u}$ **is called the speed damping derivative**

- **The stability derivative as can be written as:**

$$X_u = \frac{1}{m}\frac{\partial X}{\partial u} = \frac{-\bar{q}S}{mu_0}\left(\cancel{C_{D_u}} + 2C_{D_0}\right) \qquad\qquad C_{D_u} = u_0\frac{\partial C_D}{\partial u}$$

Usually C_D variations with speed are small

Derivatives due to Change in Forward Speed (cont.)

- **Similar to the X force, the derivative of the Z force with respect to forward speed is:**

$$\frac{\partial Z}{\partial u} = -\frac{1}{2}\rho S u_0\left(C_{L_u} + 2C_{L_0}\right) \qquad Z_u = -\frac{\bar{q}S}{mu_0}\left(\cancel{C_{L_u}} + 2C_{L_0}\right)$$

  - **In general, lift curve slope will vary with Mach No., but this effect can be neglected for small perturbations in speed**

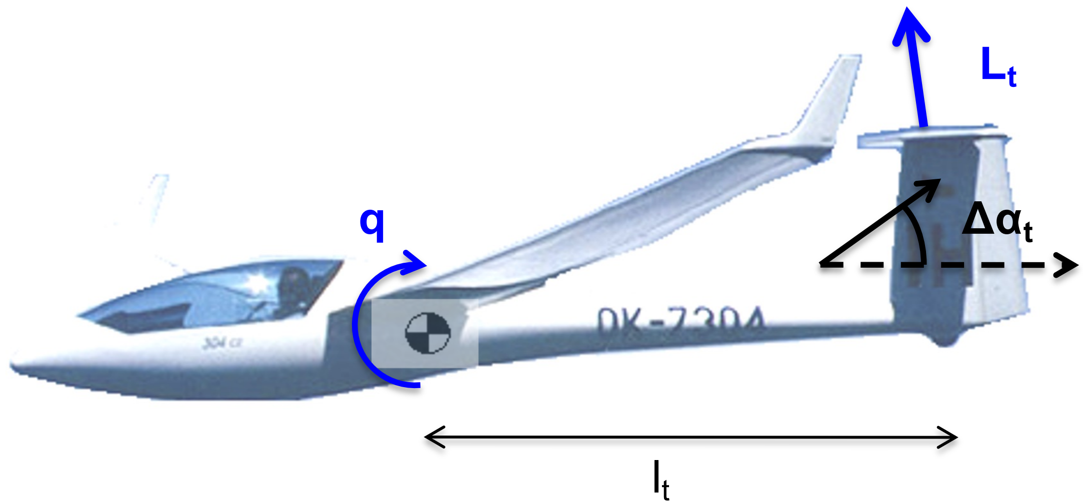
$$C_L = \frac{C_L\big|_{M=0}}{\sqrt{1-\mathrm{M}^2}}$$

- **The pitching moment can also vary with changing speed:**

$$M_u = \frac{\bar{q}S\bar{c}}{J_Y u_0}\left(2C_{M_0} + \cancel{C_{M_u}}\right) \qquad C_{M_u} = u_0\frac{\partial C_M}{\partial u}$$

Derivatives due to Pitch Rate

- **What forces and moments are influenced by pitch rate?**



- **Change in Z-force due to incremental lift on tail surface**

$$\Delta Z = -\Delta L_t = -C_{L_{\alpha_t}} \Delta\alpha_t \frac{1}{2}\rho u_0^2 S_t \qquad \Delta\alpha_t = \frac{q l_t}{u_0}$$

η: captures changes in dynamic pressure from wing to tail. Often we ignore it (set = 1).

$$C_{z_q} = -2C_{L_{\alpha_t}} \eta V_H$$

Derivatives due to Pitch Rate

- **The additional tail lift produces a pitching moment about the CG**

$$\Delta M_{cg} = -l_t \Delta L_t$$

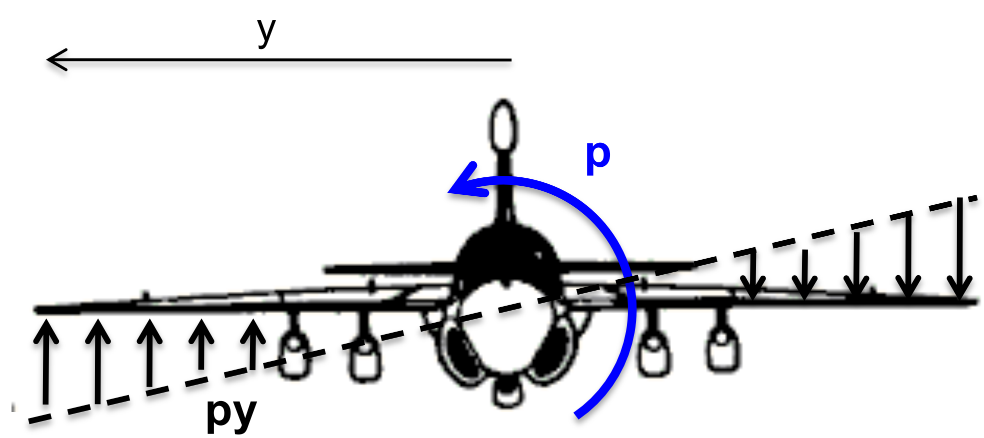
$$C_{m_q} = -2C_{L_{\alpha_t}} \eta V_H \frac{l_t}{\bar{c}}$$

- **Since the moment acts in a direction opposite to the pitch rate, the moment is a damping term**
  - C_{m_q} **is called the pitch damping derivative**

Derivatives due to Roll Rate

- **When an aircraft rolls, the local angle of attack for the left and right wing are different, and the differential lift produces a rolling moment**

- **Like C_{m_q} ,the rolling moment due to roll rate is a damping derivative**



$$C_{l_p} = -\frac{4C_{L_{\alpha_w}}}{Sb^2}\int_0^{b/2} cy^2 dy$$

$$\Delta\alpha(y) = \frac{py}{u_0}$$

- **What other forces and moments are produced?**

Derivatives due to Yaw Rate

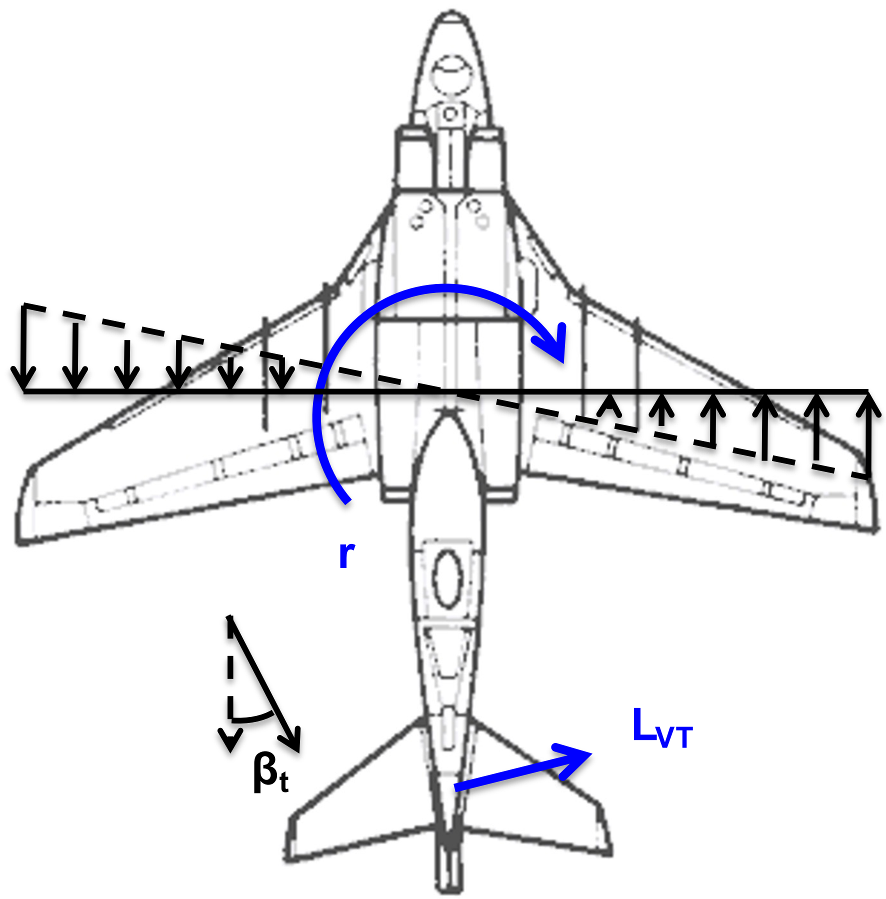
- **What forces and moments are generated by a yaw rate?**
  - **Yaw damping derivative:**

    $$C_{n_r} \approx 2C_{y_{\beta tail}} \left(\frac{l_v}{b}\right)^2$$

  - **Yaw rate leads to side force on vertical tail**

    $$C_{y_r} = -2C_{y_{\beta_t}} \frac{l_t}{b}$$

  - **Is a rolling moment produced?**



Derivatives due to Time Rate of Change of Angle of Attack

- **Downwash due to the main wing influences the local angle of attack on the tail, and that the greater the angle of attack on the main wing, the larger the downwash effect will be**

- **If the angle of attack is changing, the downwash will change, but there is a lag due to the time it takes the air to travel from the wing to the tail**

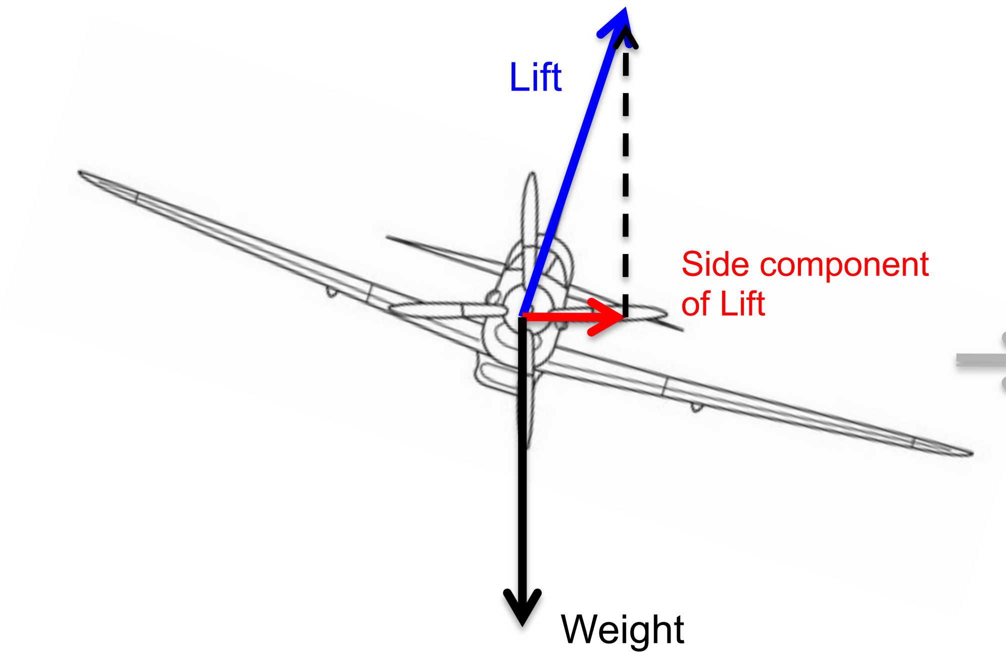
$$\Delta\alpha_t = \frac{d\varepsilon}{dt}\Delta t \qquad \Delta t = \frac{l_t}{u_0}$$

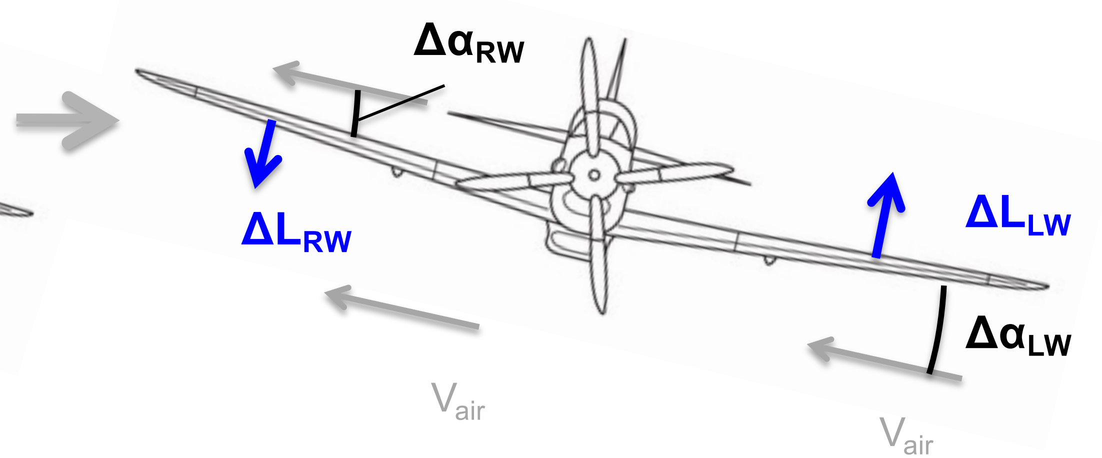
$$C_{z_{\dot{\alpha}}} = -2V_H \eta C_{L_{\alpha_t}} \frac{d\varepsilon}{d\alpha}$$

$$C_{m_{\dot{\alpha}}} = -2C_{L_{\alpha_t}} \eta V_H \frac{l_t}{\bar{c}} \frac{d\varepsilon}{d\alpha}$$

Roll Stability & Wing Dihedral

The lateral velocity changes the local angle of attack for each wing. In this case, the positive dihedral angle causes an increase in angle of attack on the left wing, but a decrease on the right wing

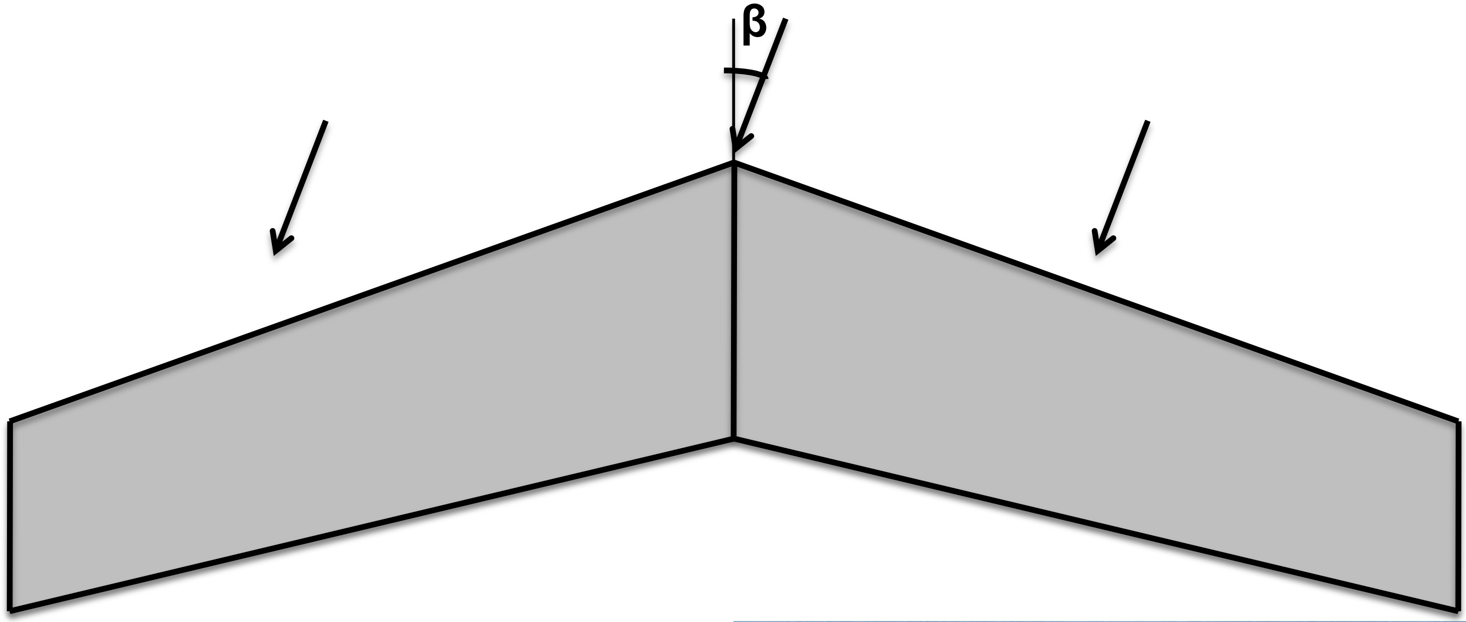




At a roll angle, the aircraft's lift vector is tilted, leading to a component of lift force in the lateral direction

The left wing produces a larger lift than the right, producing a restoring roll moment

Wing Sweep Dihedral Effect



Wing sweep also creates a restoring roll moment

High wing placement also increases roll stability

Too much roll stability can be uncomfortable, or can reduce maneuverability. Some aircraft employ negative dihedral (anhedral) to remove excess stability


Harrier - Image Courtesy of US Navy

Velocity Effects

- **Stability Derivatives can change depending on flight conditions**
  - **An interesting example is aileron reversal where wing flexibility can twist the wing at high speeds**
    - **When ailerons deflect, they increase the lift to provide a rolling moment**
    - **But if the wing twist is sufficient, it could reduce the overall AoA of the wing and therefore reduce the lift on the wing**
    - **This would decrease the effectiveness of the aileron or possibly overpower it to produce an opposite roll moment**

Important Stability Derivatives

- **Longitudinal Derivatives**

Z_{α} (-) $Z_{\dot{\alpha}}$ (-) X_{α} (-) M_{α} (-) $M_{\dot{\alpha}}$ (-)

Z_{q} (-) $Z_{\delta e}$ (-) $X_{\delta e}$ (-) M_{q} (-) $M_{\delta e}$ (-)

(Lift acts in negative Z-direction)

(Drag acts in negative X-direction)

- **Lateral-Directional Derivatives**

Y_{β} (-) L_{β} (-) $L_{\delta a}$ (-) N_{β} (+) $N_{\delta a}$ (+)

Y_{r} (+) L_{p} (-) N_{p} (-) $N_{\delta r}$ (-)

$Y_{\delta r}$ (+) L_{r} (+) N_{r} (-)

(+/-: typical signs for stable, conventional aircraft)

Summary of Stability Derivatives

- **Stability derivatives linearize the aircraft dynamics about a trim condition**
- **Stability derivatives that oppose the motion of the vehicle can be considered damping derivatives**
- **Lateral-directional derivatives are coupled. Roll induces yaw, yaw induces roll**
- **Since an aircraft is usually symmetric about the X-Z plane, the pitching terms are decoupled from the yaw and roll terms**
- **Derivatives are heavily dependant on aircraft geometry but can also change depending on flight conditions**

References

1. **B.L. Stevens, F.L. Lewis, *Aircraft Control and Simulation*, John Wiley & Sons, 1992**
2. **McCormick, B.W., *Aerodynamics, Aeronautics and Flight Mechanics*, 2nd edition, Wiley & Sons, 1995**
3. **E. Field, MAE 154S, Mechanical and Aerospace Engineering Department, UCLA, 2001**