Solution of Reliability of Multi-State *k*/*n* System with Arbitrary Distribution

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***Abstract*—The voting system is one of the important ways to improve system reliability by establishing redundancy. In order to break through the limitation of the assumption that components are independently and identically distributed or that components are exponentially distributed in the existing voting system, state-space method and recursive method were used to study *n* arbitrarily distributed components without considering maintenance. The reliability and MTTF of the 2-state *k*-out-of-*n* system were derived from the analysis of $k \geqslant (1+n)/2$ and $k<(1+n)/2$. When considering multi-state, the relationship between the reliability of the system and the state of each component was analyzed, and the corresponding analytical formula was obtained. The Monte-Carlo method was used to sample and simulate each example, and the simulation results are compared with the analytic results. The error is less than 3%, which proves the correctness of the relevant analytical formulas. This paper provides some theoretical support for the application and reliability design of k-out-of-n system in engineering.**

Keywords-voting system; state-space method; reliability; multi-state; arbitrary distribution

I. INTRODUCTION

With the increasing attention to reliability, redundant design is one of the effective methods to increase system reliability. Voting system is a common redundant design [1]. The voting system refers to the *k*-out-of-*n* system, which consists of *n* components in parallel and then a voter in series. According to the differences of voters, *k*-out-of-*n* systems can be divided into two types. When *k* of the *n* components are normal at least, the system can work normally. This kind of system is referred to as *k*/*n*(G) system or *k*/*n*:G system. Among the *n* components, as long as *k* components fail, the system will fail. And this type is referred to as *k*/*n*(F) system or *k*/*n*:F system. The *k*/*n*(G) system and *k*/*n*(F) system can be transformed into each other. That is, *k*/*n*(G) system is (*n*-*k*+1)/*n*(F) system. In the study of reliability mathematics theory, *k*/*n*(G) system is a kind of reliability model which has been studied more. It was first proposed by Birnbaum et.al [2]. The voting system studied in this paper refers to *k*/*n*(G) system without special explanation. And it is uniformly recorded as *k*/*n* system. In calculating the reliability of the voting system, it can be equivalent to a series-parallel system as shown in Fig. 1.

As the design and functional characteristics of engineering systems become more and more complex, the components of voting systems tend to be different and multi-state. When the failure probability density function of each component is independent and different, or when the components are multi-state, it is an essential problem to evaluate the reliability of the system correctly. In engineering practice, Monte-Carlo simulation method is used to solve this problem. Exponential distribution is the most widely used and discussed in engineering [3, 4]. The existing research on classical voting system mostly focuses on the assumption that each component is independent and has the same distribution [1,5,6]. Under this assumption, the reliability of the voting system is $R_{k/n}(t)=\sum_{r=k}^{n}\binom{n}{r}R^{r}*(1-R)^{n-r}$ [7]. This greatly simplifies the system, but its application to the prediction and evaluation of system reliability is limited. When the components are multi-state, the paper [8, 9] elaborated on the calculation principle of the reliability of multi-state voting system, but did not consider the changing rule of the states of the components over time.

For some products with high labor costs or too fast replacement, such as mobile phones and other electronic devices, it is now considered to be an unrepairable system [10].

Aiming at various distribution functions, the 2-state and multi-state voting systems were studied by the state-space method. The purpose is to get the analytical formula of the reliability of the unrepairable multi-state voting system.

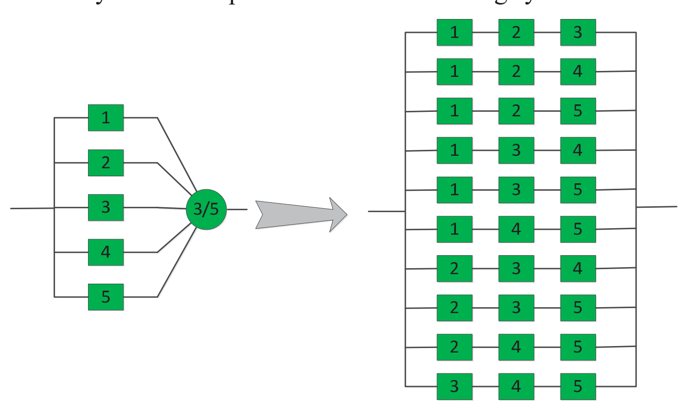

Figure 1. Equivalent Series-parallel System of 3-out-of-5 System.

II. NOTATION AND ASSUMPTIONS

A. Notation:

a) In The 2-State:

n: The total number of components in a system

k: The minimum number of normal components when the system is operating normally

R_i: Reliability of component i at a certain moment

p_i: the probability that component i is in a normal state

$\overline{p_i}$: the probability that component i is in a failure state

A_i: The sum of the reliability of all state events when there are i failed components

$\overline{A_i}$: The sum of the failure probability of all state events when there are i normal components

$R_{k/n}$: Reliability of k/n system at a certain moment

$MTTF_{k/n}$: Mean time to failure of the k/n system

b) In The Mult-State:

$\boldsymbol{S}=\{0,1,2,\cdots,\mathrm{M}\}$: A set of states of each component, M+1 states in total

x_i: The state of component i, $x_i \in \boldsymbol{S}$

$\mathrm{x}=(x_1,x_2,\cdots x_n)$: State vector of system

$P_{i,j}$: Probability that component i is superior to (or in) state j

$f_{i,(j,j-1)}$: Probability density function of component i from state j to state j-1. $f_{i,(j,j-1)}$ is related to the time interval in which component i degenerates from state j to state j-1.

$R_{(k_h/n),h}$: Reliability of voting system in state h

B. Assumptions

① All the components of the system are independent of each other and their probability density functions are arbitrary distribution.

② When each component has only two states, x_i=0 is the failure state, x_i=1 is the normal state. When the state of each component is multiple, x_i=0 is the state of complete fail, x_i=M is the completely normal state, x_i=1,2,⋯,M-1 are intermediate states.

③ Without considering maintenance

④ The voter of the system will not fail. That is, the reliability of the voter is 1.

III. RELIABILITY OF 2-STATE VOTING SYSTEM

The 2-state voting system means that each component of the system has only two states: normal (x_i=1) and failure (x_i=0). At the same time, the system has only two states. When at least k components are normal, the system is normal. Otherwise the system fails.

A. State-space Method

For complex systems, the state-space method is an effective way to list all logical state events [10]. Although the state explosion exists in the state-space method, it is easy to find the rules between state events for voting systems with only two states for each component. The state-space method can be used to solve the reliability of the voting system. The state space set of voting system in normal operation can be considered, and the state space set in a failure system can also be considered. Combining with the recursive method, the formula of system reliability can be obtained.

B. State Space Analysis of a Normal System

Considering the normal operation of the system (that is, the system has not failed), the state space set exists regularly. Each A_i can be obtained by recursive method. In this case, the state space is as follows:

Among them, the value of different A_i can be obtained by using the recursive method.

$$A_0=\prod_{i=1}^{n} p_i \tag{1}$$

$$A_1=\sum_{a_1=1}^{n}\frac{A_0*\overline{p_{a_1}}}{p_{a_1}}=A_0\sum_{a_1=1}^{n}\frac{\overline{p_{a_1}}}{p_{a_1}} \tag{2}$$

$$A_2=\sum_{a_1=1}^{n-1}\sum_{a_2=a_1+1}^{n}\frac{A_0*\overline{p_{a_1}}\,\overline{p_{a_2}}}{p_{a_1}p_{a_2}}=A_0\sum_{1\le a_1<a_2\le n}\frac{\overline{p_{a_1}}\,\overline{p_{a_2}}}{p_{a_1}p_{a_2}} \tag{3}$$

$$A_3=\sum_{a_1=1}^{n-2}\sum_{a_2=a_1+1}^{n-1}\sum_{a_3=a_2+1}^{n}\frac{A_0*\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}}}{p_{a_1}p_{a_2}p_{a_3}}=A_0\sum_{1\le a_1<a_2<a_3\le n}\frac{\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}}}{p_{a_1}p_{a_2}p_{a_3}} \tag{4}$$

TABLE I. STATE SPACE SET OF NORMAL SYSTEMS

N_n	N_f	State Space(Normal system)	Number of states	A_i
n	0	$p_1p_2p_3*\cdots*p_n$	1	A_0
n-1	1	$\overline{p_1}p_2p_3*\cdots*p_n$	n	A_1
		$p_1\overline{p_2}p_3*\cdots*p_n$		
		……		
		$p_1p_2p_3*\cdots*p_{n-1}\overline{p_n}$		
n-2	2	$\overline{p_1}\,\overline{p_2}p_3*\cdots*p_n$	$\frac{n^2-n}{2}$	A_2
		$\overline{p_1}p_2\overline{p_3}p_4*\cdots*p_n$		
		……		
		$p_1p_2*\cdots*p_{n-2}\overline{p_{n-1}}\,\overline{p_n}$		
n-3	3	$\overline{p_1}\,\overline{p_2}\,\overline{p_3}p_4*\cdots*p_n$	$\binom{n}{3}$	A_3
		$\overline{p_1}\,\overline{p_2}p_3\overline{p_4}p_5*\cdots*p_n$		
		……		
		$p_1p_2*\cdots*p_{n-3}\overline{p_{n-2}}\,\overline{p_{n-1}}\,\overline{p_n}$		
……	……	……	……	……
k	n-k	$\overline{p_1}\,\overline{p_2}\,\overline{p_3}*\cdots*\overline{p_{n-k-1}}\,\overline{p_{n-k}}p_{n-k+1}*\cdots*p_n$	$\binom{n}{k}$	A_{n-k}
		$\overline{p_1}\,\overline{p_2}*\cdots*\overline{p_{n-k-1}}p_{n-k}\overline{p_{n-k+1}}p_{n-k+2}*\cdots*p_n$		
		……		
		$p_1p_2p_3*\cdots*p_k\overline{p_{k+1}}\,\overline{p_{k+2}}*\cdots\overline{p_n}$		

a. N_n is the number of normal components

b. N_f is the number of failure components

$$A_{n-k} = \sum_{a_1=1}^{k+1} \sum_{a_2=a_1+1}^{k+2} \sum_{a_3=a_2+1}^{k+3} \cdots \sum_{a_{n-k}=a_{n-k-1}+1}^{n} \frac{A_0 * \overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}} *\ldots* \overline{p_{a_{n-k}}}}{p_{a_1} p_{a_2} p_{a_3} *\ldots* p_{a_{n-k}}}$$
$$= A_0 \sum_{1\le a_1<a_2<a_3<\ldots<a_{n-k}\le n}^{n} \frac{\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}} *\ldots* \overline{p_{a_{n-k}}}}{p_{a_1} p_{a_2} p_{a_3} *\ldots* p_{a_{n-k}}} \tag{5}$$

System reliability is the sum of probabilities of state events in a set of state spaces. So, the reliability of the voting system is:

$$R_{k/n} = A_0 + A_1 + A_2 + \ldots + A_{n-k} = \sum_{i=0}^{n-k} A_i \tag{6}$$

Let $\frac{\overline{p_i}}{p_i} = \frac{1-R_i}{R_i} = r_i$. Then, the reliability of the voting system is:

$$R_{k/n} = \sum_{i=0}^{n-k} A_i$$
$$= A_0\left(1+\sum_{a_1=1}^{n} r_{a_1} + \sum_{1\le a_1<a_2\le n}^{n} r_{a_1} r_{a_2} + \ldots + \sum_{1\le a_1<a_2<a_3<\cdots<a_{n-k}\le n}^{n} r_{a_1} r_{a_2} r_{a_3} *\ldots* r_{a_{n-k}}\right) \tag{7}$$
$$= A_0 * \left[1+\sum_{i=1}^{n-k}\left(\sum_{1\le a_1<a_2<\cdots<a_i\le n}^{n} r_{a_1} r_{a_2} *\ldots* r_{a_i}\right)\right]$$

It is applicable to $n \ge k \ge 1$.

C. State Space Analysis of a Failure System

For the *k/n*:G system, it can be equivalent to the (n-k+1)/n:F system. When more than or equal to (n-k+1) components fail, the voting system will fail. That is, when no more than (k-1) components work, the system will fail. Each $\overline{A_i}$ of can be obtained by recursive method. In this case, the state space is as follows:

TABLE II. STATE SPACE SET OF FAILURE SYSTEMS

N_n	N_f	State Space(Failure system)	Number of states	$\overline{A_i}$
0	n	$\overline{p_1}\,\overline{p_2}\,\overline{p_3}*\ldots*\overline{p_n}$	1	$\overline{A_0}$
1	n-1	$p_1\overline{p_2}\,\overline{p_3}*\ldots*\overline{p_n}$	n	$\overline{A_1}$
		$\overline{p_1}p_2\overline{p_3}*\ldots*\overline{p_n}$		
		……		
		$\overline{p_1}\,\overline{p_2}\,\overline{p_3}*\ldots*\overline{p_{n-1}}p_n$		
2	n-2	$p_1p_2\overline{p_3}*\ldots*\overline{p_n}$	$\frac{n^2-n}{2}$	$\overline{A_2}$
		$p_1\overline{p_2}p_3\overline{p_4}*\ldots*\overline{p_n}$		
		……		
		$\overline{p_1}\,\overline{p_2}*\ldots*\overline{p_{n-2}}p_{n-1}p_n$		
3	n-3	$p_1p_2p_3\overline{p_4}*\ldots*\overline{p_n}$	$\binom{n}{3}$	$\overline{A_3}$
		$p_1p_2\overline{p_3}p_4\overline{p_5}*\ldots*\overline{p_n}$		
		……		
		$\overline{p_1}\,\overline{p_2}*\ldots*\overline{p_{n-3}}p_{n-2}p_{n-1}p_n$		
……	……	……	……	……
k-1	n-k+1	$p_1p_2*\ldots*p_{k-1}\overline{p_k}\,\overline{p_{k+1}}*\ldots*\overline{p_n}$	$\binom{n}{k-1}$	$\overline{A_{k-1}}$
		$p_1p_2*\ldots*p_{k-2}\overline{p_{k-1}}p_k\overline{p_{k+1}}*\ldots*\overline{p_n}$		
		……		
		$\overline{p_1}\,\overline{p_2}\,\overline{p_3}*\ldots*\overline{p_{n-k+1}}p_{n-k+2}*\ldots*p_n$		

a. N_n is the number of normal components

b. N_f is the number of failure components

Among them, the value of different $\overline{A_i}$ can be obtained by using the recursive method.

$$\overline{A_0} = \prod_{i=1}^{n} \overline{p_i} \tag{8}$$

$$\overline{A_1} = \sum_{a_1=1}^{n} \frac{\overline{A_0} * p_{a_1}}{\overline{p_{a_1}}} = \overline{A_0}\sum_{a_1=1}^{n} \frac{p_{a_1}}{\overline{p_{a_1}}} \tag{9}$$

$$\overline{A_2} = \sum_{a_1=1}^{n-1} \sum_{a_2=a_1+1}^{n} \frac{\overline{A_0} * p_{a_1} p_{a_2}}{\overline{p_{a_1}}\,\overline{p_{a_2}}} = \overline{A_0} \sum_{1\le a_1<a_2\le n}^{n} \frac{p_{a_1} p_{a_2}}{\overline{p_{a_1}}\,\overline{p_{a_2}}} \tag{10}$$

$$\overline{A_3} = \sum_{a_1=1}^{n-2} \sum_{a_2=a_1+1}^{n-1} \sum_{a_3=a_2+1}^{n} \frac{\overline{A_0} * p_{a_1} p_{a_2} p_{a_3}}{\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}}} = \overline{A_0} \sum_{1\le a_1<a_2<a_3\le n}^{n} \frac{p_{a_1} p_{a_2} p_{a_3}}{\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}}} \tag{11}$$

$$\overline{A_{k-1}} = \sum_{a_1=1}^{n-k+2} \sum_{a_2=a_1+1}^{n-k+3} \sum_{a_3=a_2+1}^{n-k+4} \cdots \sum_{a_{k-1}=a_{k-2}+1}^{n} \frac{\overline{A_0} * p_{a_1} p_{a_2} p_{a_3} *\ldots* p_{a_{k-1}}}{\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}} *\ldots* \overline{p_{a_{k-1}}}}$$
$$= \overline{A_0} \sum_{1\le a_1<a_2<a_3<\cdots<a_{k-1}\le n}^{n} \frac{p_{a_1} p_{a_2} p_{a_3} *\ldots* p_{a_{k-1}}}{\overline{p_{a_1}}\,\overline{p_{a_2}}\,\overline{p_{a_3}} *\ldots* \overline{p_{a_{k-1}}}} \tag{12}$$

The failure probability of the system is the sum of the probabilities of each state space in Table II, so the reliability of the voting system is:

$$R_{k/n} = 1-\left(\overline{A_0} + \overline{A_1} + \overline{A_2} + \ldots + \overline{A_{k-1}}\right) = 1-\sum_{i=0}^{k-1} \overline{A_i} \tag{13}$$

Let $\frac{p_i}{\overline{p_i}} = \frac{R_i}{1-R_i} = \overline{r_i}$. Then, the reliability of the voting system is:

$$R_{k/n} = 1-\sum_{i=0}^{k-1} \overline{A_i}$$
$$= 1-\overline{A_0}\left(1+\sum_{i=1}^{n} \overline{r_{a_1}} + \sum_{1\le a_1<a_2\le n}^{n} \overline{r_{a_1}}\,\overline{r_{a_2}} + \ldots + \sum_{1\le a_1<a_2<\cdots<a_{k-1}\le n}^{n} \overline{r_{a_1}}\,\overline{r_{a_2}} *\ldots* \overline{r_{a_{k-1}}}\right)$$
$$= 1-\overline{A_0} * \left[1+\sum_{i=1}^{k-1}\left(\sum_{1\le a_1<a_2<\cdots<a_i\le n}^{n} \overline{r_{a_1}}\,\overline{r_{a_2}} *\ldots* \overline{r_{a_i}}\right)\right]$$
(14)

It is applicable to $n \ge k \ge 1$.

D. Reliability and MTTF of The System

Both (7) and (14) used the result of a ratio (r_i or $\overline{r_i}$). The r_i is the ratio of failure probability to reliability of component i. When r_i>1, it means that the component i is more likely to fail at time t. And r_i<1 means that the component i is less likely to fail at time t. However, $\overline{r_i}$ is the opposite of r_i. Here, r_i or $\overline{r_i}$ can be assigned a new concept, which shows that the system reliability can be expressed not only by the reliability of each component, but also by the ratio of each component. Compared with the analytical formula obtained by minimal path set method, they lack a coefficient and have a simpler structure.

From (7) and (14), It can be known that when n-k<k-1 (i.e. k>(n+1)/2), the number of normal components in the normal operation of the system must be more than the number of failure components. Moreover, when n-k>k-1 (i.e. k<(n+1)/2), the number of failure components will be more than that of normal components if the system fails. To clarify the scope of application of (7) and (14), system reliability was obtained:

$$R_{k/n}=\begin{cases} A_0\left[1+\sum_{i=1}^{n-k}\left(\sum_{1\le a_1<a_2<a_3<\cdots<a_i\le n}^{n} r_{a_1}r_{a_2}r_{a_3}\cdots r_{a_i}\right)\right], n\ge k\ge \dfrac{n+1}{2} \\ 1-\overline{A_0}\left[1+\sum_{i=1}^{k-1}\left(\sum_{1\le a_1<a_2<a_3<\cdots<a_i\le n}^{n} \overline{r_{a_1}r_{a_2}r_{a_3}}*\cdots*\overline{r_{a_i}}\right)\right], 1\le k<\dfrac{n+1}{2} \end{cases} \quad (15)$$

In addition to reliability, the MTTF of the system is also an important indicator for evaluating the system. It is the integral of system reliability. That is:

$$MTTF_{k/n}=\int_0^{\infty}R_{k/n}(t)dt$$
$$=\begin{cases} \int_0^{\infty}A_0\left[1+\sum_{i=1}^{n-k}\left(\sum_{1\le a_1<a_2<a_3<\cdots<a_i\le n}^{n} r_{a_1}r_{a_2}r_{a_3}\cdots r_{a_i}\right)\right]dt, n\ge k\ge \dfrac{n+1}{2} \\ \int_0^{\infty}\left[1-\overline{A_0}\left[1+\sum_{i=1}^{k-1}\left(\sum_{1\le a_1<a_2<a_3<\cdots<a_i\le n}^{n} \overline{r_{a_1}r_{a_2}r_{a_3}}*\cdots*\overline{r_{a_i}}\right)\right]\right]dt, 1\le k<\dfrac{n+1}{2} \end{cases}$$
(16)

The analytic formula of reliability obtained by the minimum path set method is as follows:

$$R_{k/n}=\begin{cases} \sum_{i=0}^{n-k}\left[(-1)^i\binom{k+i-1}{i}\sum_{1\le a_1<a_2<\cdots<a_{k+i}\le n}^{n} R_{a_1}R_{a_2}\cdots R_{a_{k+i}}\right], n\ge k\ge \dfrac{n+1}{2} \\ A_0*\sum_{i=0}^{n-k}\left[(-1)^i\binom{k+i-1}{i}\sum_{1\le a_1<a_2<\cdots<a_{n-k-i}\le n}^{n} \dfrac{1}{R_{a_1}R_{a_2}\cdots R_{a_{n-k-i}}}\right], 1\le k<\dfrac{n+1}{2} \end{cases} \quad (17)$$

Comparing (15) and (17), it can been seen that the form of (15) is simpler and requires less computation time.

IV. Reliability of Multi-State Voting System

A. Multi-state Voting System Base on Performance Degradation

When the number of states of each component is more than 2, a simple 2-state problem becomes a complex multi-state problem. Corresponding, system has multiple states. The multi-state voting system is called MS voting system. The MS voting system is much more complicated than the 2-state voting system [8]. In MS voting system, component i has M+1 states. And the performance of the component is a process of gradual degradation. The performance of state j is better than that of state j-1. The probability density function of degradation between two adjacent states of component i is $f_{i,(j,j-1)}$ (j=1,2,⋯,M). When state 0 is reached, the component fails completely. $f_{i,(j,j-1)}$ is related to the time interval in which component i degenerates from state j to state j-1.

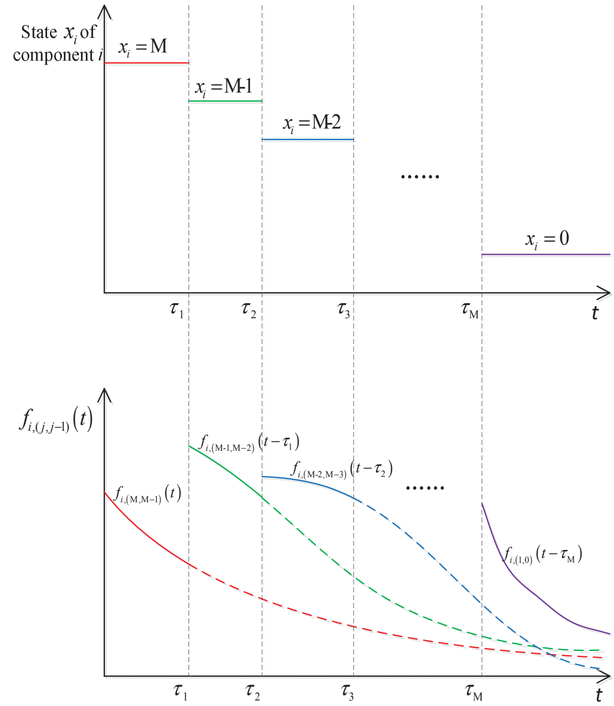

Figure 2. Change of component i between states.

In addition to this, a voting system also has multiple states. The state of component i is x_i, and the state vector of the voting system is $\mathrm{x}=(x_1,x_2,\cdots x_n)$, $x_i\in \boldsymbol{S}$, $i=1,2,\cdots,n$. $\boldsymbol{S}$ is a state set of components, $\boldsymbol{S}=\{0,1,2,\cdots,\mathrm{M}\}$. That is, each component has M+1 states (State M: perfect-functioning; state 0: complete failure). When the state of the component at time t is $x_i(t)$, $x_i(t')(t'>t)$ is only related to $x_i(t)$, regardless of the past history state of component i. Performance degradation of component i is shown in Fig. 2.

Multiple states of components will lead to state explosion of voting system. Assumptions of the MS voting system in this paper are as follows:

a) state h of the voting system: At least k_h components are in (or superior to) state j, but it does not satisfy the condition that at least k_{h+1} components are in (or superior to) state j'. ($0\le j\le j'\le \mathrm{M}$, $k_{h+1}\ge k_h$)

b) The performance of voting system in state h+1 is better than that of state h.

Assuming that the initial state of each component is x_i=M, the performance of each component and system will deteriorate continuously during the operation of the system. When the performance of the system degrades to a certain extent, it is considered that it can no longer meet the requirements of normal operation, even if the system does not have a complete failure [11]. In this case, it also needs to be treated as a failure. That is, when the system does not degenerate to state h-1, it can meet the operational requirements and need not to be treated as a failure, and the system is considered reliable.

B. System Reliability

The criterion for defining whether the system is reliable is the critical state h of the voting system. That is to say, the total number of components that are superior to (or in) state j is determined when the voting system is in state h.

It is assumed that the voting system can operate normally when the number of components that are not worse than state j is at least k_h. Then the probability that component i is superior to (or in) state j at time t is:

$$P_{i,j}(t)=P(x_i \geq j)=\sum_{l=j}^{M} P(x_i=l) \tag{18}$$

Where, $P(x_i=l)=p(x_i=l)\prod_{m=l+1}^{M}\left[1-p(x_i=m)\right]$. And $p(x_i=l)$ is a conditional probability, which indicates the probability that component i degenerates from state $l+1$ to state l under the condition of the state $l+1$.

In summary, the probability that component i is not worse than state j at time t can be obtained:

$$\begin{aligned}
P_{i,j}(t)&=P(x_i \geq j)=\sum_{l=j}^{\mathrm{M}} P(x_i=l)\\
&=P(x_i=\mathrm{M})+P(x_i=\mathrm{M}\text{-}1)+P(x_i=\mathrm{M}\text{-}2)+\cdots+P(x_i=j)\\
&=\left[1-F_{i,(\mathrm{M},\mathrm{M}-1)}(t)\right]+\int_0^t f_{i,(\mathrm{M},\mathrm{M}-1)}(\tau_1)*\left[1-F_{i,(\mathrm{M}-1,\mathrm{M}-2)}(t-\tau_1)\right]d\tau_1\\
&+\int_0^t f_{i,(\mathrm{M},\mathrm{M}-1)}(\tau_1)\int_{\tau_1}^t f_{i,(\mathrm{M}-1,\mathrm{M}-2)}(\tau_2-\tau_1)*\left[1-F_{i,(\mathrm{M}-2,\mathrm{M}-3)}(t-\tau_2)\right]d\tau_2 d\tau_1+\cdots\\
&+\int_0^t f_{i,(\mathrm{M},\mathrm{M}-1)}(\tau_1)\cdots\int_{\tau_{\mathrm{M}-j-1}}^t f_{i,(j,j-1)}(\tau_{\mathrm{M}-j}-\tau_{\mathrm{M}-j-1})*\left[1-F_{i,(j,j-1)}(t-\tau_{\mathrm{M}-j})\right]d\tau_{\mathrm{M}-j}\cdots d\tau_1\\
&=1-\int_0^t f_{i,(\mathrm{M},\mathrm{M}-1)}(\tau_1)\int_{\tau_1}^t f_{i,(\mathrm{M}-1,\mathrm{M}-2)}(\tau_2-\tau_1)\cdots\int_{\tau_{\mathrm{M}-j}}^t f_{i,(j,j-1)}(\tau_{\mathrm{M}-j+1}-\tau_{\mathrm{M}-j})d\tau_{\mathrm{M}-j+1}\cdots d\tau_1\\
&=1-\int_0^t\int_{\tau_1}^t\cdots\int_{\tau_{\mathrm{M}-j}}^t f_{i,(\mathrm{M},\mathrm{M}-1)}(\tau_1)f_{i,(\mathrm{M}-1,\mathrm{M}-2)}(\tau_2-\tau_1)\cdots f_{i,(j,j-1)}(\tau_{\mathrm{M}-j+1}-\tau_{\mathrm{M}-j})d\tau_{\mathrm{M}-j+1}\cdots d\tau_1
\end{aligned} \tag{19}$$

Equation (19) contains an M+1-j multiple integral. Generally, it is difficult to calculate multiple integrals by manual calculation. It can be solved by commercial software such as MATLAB. By (19), the probabilities of n components are obtained respectively. By substituting $P_{i,j}(t)$ for R_i in (15), the analytic formula of the reliability of the MS voting system in state h can be obtained as follows:

$$R_{(k_h/n),h}=\begin{cases} A_{0,j}\left[1+\sum_{i=1}^{n-k}\left(\sum_{1\leq a_1<a_2<\cdots<a_i\leq n}^{n} r_{a_1,j}r_{a_2,j}\cdots r_{a_i,j}\right)\right], k\geq (n+1)/2\\ 1-\overline{A_{0,j}}\left[1+\sum_{i=1}^{k-1}\left(\sum_{1\leq a_1<a_2<\cdots<a_i\leq n}^{n} \overline{r_{a_1,j}r_{a_2,j}}*\cdots*\overline{r_{a_i,j}}\right)\right], k<(n+1)/2 \end{cases} \tag{20}$$

Where,

$$r_{a_i,j}=\frac{1-P_{i,j}(t)}{P_{i,j}(t)} \tag{21}$$

$$\overline{r_{a_i,j}}=\frac{P_{i,j}(t)}{1-P_{i,j}(t)} \tag{22}$$

$$A_{0,j}=\prod_{i=1}^{n} P_{i,j}(t) \tag{23}$$

$$\overline{A_{0,j}}=\prod_{i=1}^{n}\left(1-P_{i,j}(t)\right) \tag{24}$$

Then, the probability that the MS voting system is in the state h is as follows:

$$P_h=R_{(k_h/n),h}-R_{(k_{h+1}/n),h+1} \tag{25}$$

C. Case Verification

A voting system is composed of three independent components, each of which has four states: 0, 1, 2, 3. State 0 is complete failure. State 3 is perfect-functioning. State 1 and state 2 are intermediate states, and the performance of state 1 is worse than that of the state 2. The probability density functions of the degradation between states of each component are shown in table III. When at least two components are located in state 2, the system can operate normally. Calculate the reliability of the voting system at t=125.

In this case, the curve of system reliability over time is shown in Fig. 3. The results of Monte Carlo simulation are very close to the analytical results, and the two curves almost coincide.

The result of Monte-Carlo simulation is 0.05237 at t=125, and the result of analytic method is 0.05112. The relative error between them is 2.387%. So the solution in this paper can be regarded as correct.

TABLE III. INFORMATION OF EACH COMPONENT

	Component 1	**Component 2**	**Component 3**
state 3→2	$f_{1,(3,2)}=0.001\mathrm{e}^{-0.001*t}$	$f_{2,(3,2)}=\frac{1}{\sqrt{18\pi}}e^{-\frac{(t-50)^2}{18}}$	$f_{3,(3,2)}=0.1*\left(\frac{t}{50}\right)^4 e^{-\left(\frac{t}{50}\right)^5}$
state 2→1	$f_{1,(2,1)}=0.002\mathrm{e}^{-0.002*t}$	$f_{2,(2,1)}=\frac{1}{\sqrt{32\pi}}e^{-\frac{(t-50)^2}{32}}$	$f_{3,(2,1)}=0.1*\left(\frac{t}{60}\right)^5 e^{-\left(\frac{t}{60}\right)^6}$
state 1→0	$f_{1,(1,0)}=0.003\mathrm{e}^{-0.003*t}$	$f_{2,(1,0)}=\frac{1}{\sqrt{32\pi}}e^{-\frac{(t-50)^2}{32}}$	$f_{3,(1,0)}=0.1*\left(\frac{t}{70}\right)^6 e^{-\left(\frac{t}{70}\right)^7}$

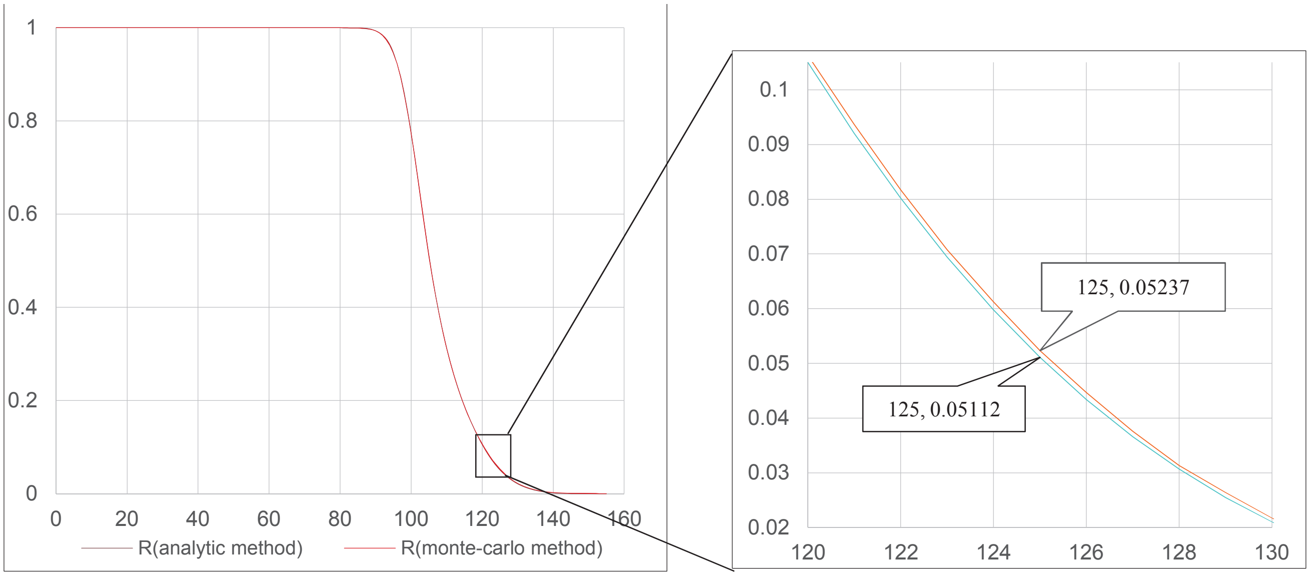

Figure 3. System reliability of case.

Conclusion

(1) By using state-space method and recursive method, the analytic formulas of reliability and MTTF of the system were obtained when the components were arbitrarily distributed and x_i=0 and x_i=1 were considered only. For the cases of $k \geqslant (1+n)/2$ and $k<(1+n)/2$, the analytic formula was divided into applicable ranges, which enhances its applicability. The analytic formula obtained by this method is simpler in structure than that obtained by the minimum path set method.

(2) When considering that the component had multiple states and the components were degraded continuously, the analytic formula for the reliability of the MS voting system was obtained.

(3) When the information of each component and the maximum working time t are known, the reliability of the system can be quickly evaluated by using the analytic formula, which is more accurate and stable than the Monte-Carlo method. This is of great significance to the reliability design and analysis of voting system.

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