# The Fractional Fourier Filtering without Edge Effect

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***Abstract*—The paper presents one kind of filter based on fractional Fourier transform for one dimensional signal. Compared with other one-dimensional filters, this filter can improve the edge effect before and after filtering. Not only has high signal-to-noise ratio, but also there is no leak at signal of two terminals. When calculating the signal-to-noise ratio, I take four examples with low noise to high noise. Through MATLAB simulation, it shows that has excellent filter characteristics and is very suitable for signal processing.**

**Keywords-*the fractional Fourier transform; one-dimensional signal; filter;* signal-noise ratio**

## I. INTRODUCTION

In the field of signal processing, the traditional Fourier transform is a mature study and widely used as mathematical tool. Fourier transform occupies an important status in the continuous and discrete time signal processing. It is also a powerful tool for stationary signal analysis and processing. The signal is decomposed into sinusoidal components with different frequencies by Fourier transform, and the overall spectrum of the signal is obtained. However, as the research object and research scope expanding, the Fourier transform exposed gradually in research limitations on certain issues. This limitation is mainly manifested in: The traditional Fourier transform is a global transform that gives you just the whole spectrum of the signal. It can neither describe the time-frequency local characteristics of the signal nor analyze the non-stationary signal. However, in practice, most signals are non-stationary. To process and analyze the non-stationary signal, people put forward the fractional Fourier transform (FrFt) based on the original traditional Fourier transform, which can be regarded as the extension of the traditional Fourier transform [1, 2].

As a new mathematical tool, FrFt is a widely used fractional order system. The application prospect of FrFt in signal processing is very broad. The FrFt domain filtering method is used to process non-stationary signal, which has no additional cost and smaller mean square error than traditional Fourier transform [3, 4].

The traditional Fourier transform is a linear operator. In the time-frequency plane, if the operator is regarded as rotating counterclockwise from the time axis to the frequency axis, then the fractional-order Fourier transform operator can be regarded as rotating at any Angle[5]. So, the FrFt is considered by us to be a generalized Fourier transform. The FrFt can be defined in several different ways. It can be derived from any one definition of other ways to define, and the latter can be regarded as the nature of the former, which are equivalent to each other[6]. Different properties have different physical interpretations and have different applications in practice. From various angles to know the definition of FrFt will make us a more complete understanding of the fractional Fourier transform.

The FrFt is essentially a unified time-frequency transform, which reflects the information of the signal in the time domain and frequency domain[7, 8]. It is with a single variable to represent the time and frequency information, and there are no cross terms. Compared with the traditional Fourier transform, it is suitable for the non-stationary case with one free parameter. Therefore, the fractal-order Fourier transform can get the effect that the traditional time-frequency distribution or Fourier transform cannot get under some conditions, Moreover, because it has a relatively mature and fast discrete algorithm, it does not need too much calculation while obtaining better results.

In this paper, the FrFt is applied to the one-dimensional signal filtering, and a narrow bandpass filter is designed in the frequency domain, which is equivalent to the multiplicative filter in the fractional Fourier domain, to achieve the filtering effect. The multiplication filter of the traditional frequency domain is extended to the FrFt domain, and the fractional Fourier multiplier filter is obtained. The multiplicative filter of the fractional Fourier domain is equivalent to a narrow band-pass filter in the frequency domain. Therefore, we design a band pass filter in frequency domain in this paper. The FrFt of the four sets of signals is firstly carried out, and then the effect of signal filtering is achieved through a bandpass filter[9]. Finally, the fractional-order Fourier reverse change of the filtered signal is carried out, to obtain the signal after noise suppression.

This paper is organized as follows. The FrFt is introduced in Section 2, where we also analyze the model characteristics and FrFt filtering. The Section 3 lists four simulation experiments. Section 4 concludes the paper.

## II. THE FRACTIONAL FOURIER TRANSFORM

### A. A brief introduction of fractional Fourier transform

Many researchers pay attention to FrFt, because it has a lot of advantages that the traditional Fourier transform does not have. The FrFt retain the traditional Fourier transform nuclear properties and characteristics based on the original added new advantages, so you can think of FrFt is a kind of generalized Fourier transform. $\alpha$ order FrFt is described as [2]

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$$X_{\alpha}\left(t'\right)=F_{\alpha}\{x(t)\}=\int k_{\alpha}(t,t')x(t)dt \tag{1}$$

Corresponding to the transformation point of view, the first order FrFt is the ordinary Fourier transform, and the zero-order FrFt is its primary function [6].

When α is between 0 and 1, $x(t)$ can show $\alpha$ order FrFt. Fractional Fourier transform is also a kind of linear transformation. The arithmetic by rapid transformation algorithm, continuous FrFt can approximate as a collection of discrete Fourier transform.

The FrFt can be interpreted as the representation of the signal in the fractional Fourier domain, which is formed by the signal rotating at any angle around the origin in the time-frequency plane. Fractional Fourier transform operation corresponding to a rotational phase space, it is through form rotating of Wigner distribution (WD)[10]. Where the rotation angle has related with the order of FrFt. Rotation operator will generate two-dimensional operator $D(u,v)$, is described as

$$R_{\Phi}\{D(u,v)\}=D(u\cos\Phi+v\sin\Phi,-u\sin\Phi+v\cos\Phi) \tag{2}$$

This kind of the rotation of the time-frequency distribution properties are defined as a frequency distribution $D(t,f)$, when meet all the rotation of the $x(t)$ and $\alpha$ properties:

$$D_x(t,f)=R_{-\varphi}\{D_x(t,f)\}. \tag{3}$$

The WVD is by $W_{(t)}(t,f)$ signal representation:

$$W_x(t,f)=\int x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j2\pi f\tau}d\tau \tag{4}$$

An amplitude modulation frequency modulation signal, the time and frequency can be represented by the WD. In the Fourier transform of proper order, the clockwise linear frequency modulation signal is converted into an amplitude modulated sine signal. Therefore, FrFt can make chirp signal. The signal in the fractional Fourier domain refers to the frequency change as it passes through the smallest rectangle in a relatively short time bandwidth. According to the above ideas, FrFt filter signal is generated by narrow-band signal. Based on the same linear frequency modulation signal, FrFt domain transformation related to WD determines a single component, which can search FrFt for FrFt fractional order changes in different degrees.

### *B. The nature of fractional Fourier transform*

1) *·Linear:*

$$F^p\{\sum{}_n c_n f_n(u)\}=\sum{}_n c_n\{F^p f_n(u)\} \tag{6}$$

2) *·When the order is integer:*

$$F^n=(F)^n \text{ (n is a integer)} \tag{7}$$

3) *·The FrFt of inverse transformation:*

$$F^{-p}=(F^p)^{-1} \tag{8}$$

4) *·Unitarity:*

$$F^p=(F^p)^H \tag{9}$$

5) *·Order number additive:*

$$F^{p_1}F^{p_2}=F^{p_1+p_2} \tag{10}$$

6) *·Commutativity:*

$$F^{p_1}F^{p_2}=F^{p_2}F^{p_1} \tag{11}$$

7) *·Associativity:*

$$(F^{p_1}F^{p_2})F^{p_3}=F^{p_1}(F^{p_2}F^{p_3}) \tag{12}$$

### *C. The principle of fractional Fourier transform filtering*

#### *1) Fractional Fourier transform in the application of signal processing*

Signal analysis is the purpose of transforming some signal, extracted it from the signal information. The FrFt is a linear transformation that takes time and frequency as the horizontal axis and the vertical axis respectively, and rotates at a certain Angle. When signal and noise interference have strong coupling in time domain or frequency domain, FrFt has the property of removing time-frequency coupling and selecting the appropriate rotation Angle, good filtering and interference separation effect can be obtained in fractional-order domain.

The FrFt can be used as the basis of generalized spatial filtering, which is introduced into the filtering system by the Fourier spectrum, and is limited to the operation of linear space-invariant systems, while the FrFt itself is space-varying. So more can be done by introducing different filters on different fractional-order Fourier transform surfaces [11]. For some of the high frequency noise, for example, the chirp noise that the noise is empty. The bandwidth of the filter needs to be high resolution and large. Conventional Fourier transform can be difficult to achieve, at this moment if using FrFt filtering can filter out it easily.

#### *2) Waveform estimation*

Filtering is one of the waveforms to estimate the problem to be solved in many practical applications, the useful signals tend to be distorted by a known system, at the same time aliasing noise[12].

$$Y=\psi(x)+n \tag{13}$$

$\psi$ causes linear system model of signal distortion, $X$ is the useful signal, $n$ is the additive noise. This requires an estimation operator to estimate the waveform of the observed value $Y$, to reduce the influence of distortion and signal to noise. According to the actual needs, waveform estimation can be divided into three basic estimates: filtering, smoothing and prediction.

Eq. (13) by estimating operator $f$, the influence of signal noise can be reduced which is:

$$\hat{x} = x(t+a)$$

(14)

When $a=0$ is to use the current moment and previous observations, by estimate, after losing a moment in the useful Waveform estimation signal, this waveform estimation is called filtering.

When $a \neq 0$ is to use the current moment and previous observations, by estimating, after losing a moment the useful signal, the waveform estimation is called forecasting.

*3) Fractional Fourier filtering algorithm*

The FrFt has the property of releasing time-frequency coupling. By selecting the appropriate rotation angle $a_0$ and matching it with the processing object, a better signal and interference separation effect can be obtained in the FrFt domain. The correct estimation of parameters is the premise of implementing the filtering algorithm. The basic idea of the signal parameter estimation method based on the FrFt as follows: the rotation angle $a_0$ is taken as a variable to carry out the FrFt of the observation signal, and the parameter estimation value is obtained through two-dimensional search. To reduce the computation, this paper determines an appropriate search range by determining the transform range of signal parameters in advance. Related steps of filtering algorithm:

a) The parameters of the filter signal are estimated to obtained rotation angle $a_0$.

b) After the fraction-order Fourier transform of order $\alpha$, the signal of rotation angle $a_0$ can be expressed as:

$$X_\alpha(t') = S_\alpha(t') + N_\alpha(t') \qquad (15)$$

Where $S_\alpha(t')$ is the signal after the FrFt. If it is a finite length signal, most of its energy is concentrated in a narrow band of FrFt domain. $N_\alpha(t')$ is noise signal after the FrFt, generally in fractional Fourier aggregation behavior will not occur.

c)In fractional Fourier at rush make filtering processing

$$X_\alpha(t') = X_\alpha(t')M(t') = S_\alpha(t')M(u) + N_\alpha(t')M(t') \qquad (16)$$

If $M(t')$ is the ideal band-pass filter. The appropriate bandwidth is selected, and most of the noise energy is filtered, to achieve the effect of signal filtering.

d) The filtered signal is reversed rotated to the time domain through the $\alpha$ order fractional Fourier transform, and the noise suppression signal is obtained. The filtering process is shown in figure 1.

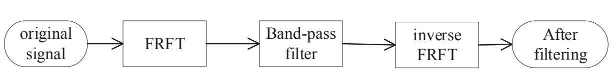


Figure 1. The fractional Fourier transform filtering process

If the signal and noise are not coupled in the time domain, the noise can be filtered out by appropriate filters. If the signal and noise are coupled in both the time and frequency domain, the noise will not be completely filtered through the time or frequency domain, but the coupling can be removed by rotating coordinates to an Angle. In this rotating coordinate system, the noise can be completely filtered out, that is to say, the fractional Fourier domain filtering at a certain Angle can get a better effect.

Time-frequency filtering that is commonly used in three kinds of filter, low-pass filter, high-pass filter, and band-pass filter. Only need to set up related parameters, can through the FrFt and generalized to fractional Fourier (it is important to note that not all angles can be promoted).

In this paper, the input signal is firstly processed by FrFt with a rotation Angle of $\pi$, and then filtered by a multiplicity filter h(u) in the FrFt domain. Finally, the filtered time-domain signal is obtained by the inverse fractional Fourier transform with a rotation Angle of $-\pi$. Where Multiplier filter $h(u)$ is a narrow-band filter with $u_c$ as the center, which is equivalent to a narrow-band pass filter in the frequency domain. The center frequency is linearly scanned on the frequency axis as the instantaneous frequency of the signal changes.

It is similar with the traditional frequency domain filtering, by designing different $h(u)$ can get similar types of filter, divided into low pass filter, high pass filter, band pass filter. In this case, the similar is similar in the form of the transfer function.

Low pass filter: Only through $u \le u_1$ signal;

High pass filter: Only through $u \ge u_2$ signal;

Band pass filter: Only through $u_1 < u < u_2$ signal.

When the signal and noise are no longer coupled at the FrFt domain of a certain angle, the noise can be filtered out by a suitable filter.

## III. EXPERIMENT

Four groups of signals with different noise sizes were collected, and FrFt was used to filter the signals to analyze the signal-to-noise ratio changes before and after filtering, and observe whether there is edge effect. In this paper, Gaussian white noise of 40dB, 30dB, 20dB and 10dB are added as filtering signals respectively based on the pure signals. The horizontal axis is seen as the sampling time and the vertical axis is seen as the amplitude. Calculate the signal-to-noise ratio and observe the result.

Signal-to-Noise Ratio (SNR) refers to the ratio of signal to noise, the larger the value, the smaller the noise. The calculation method of SNR is shown as follows:

$$SNR = 20*\log_{10}(P_S / P_N) \qquad (17)$$

Where $P_S$ and $P_N$ represent the effective power of the signal and noise, respectively. The unit of measurement is the dB.

In this paper, four groups of filter signals are taken as input signals, and the FrFt filtering process is as follows:

a) There are four groups of signals, which are added 40 dB(fig.3), 30 dB(fug.5), 20 dB(fig.7) and 10 dB(fig.9) white Gaussian noise based on the pure signal(fig.2), respectively. Take the signal to be filtered as input.

b) Obtain the FrFt of the optimal order, namely the Fourier transform of order $\alpha$. In this paper, the optimal order is found by two-dimensional search. The optimal solution with higher parameter estimation accuracy is found by the traditional two-dimensional search with smaller search step, which leads to the increase of computation. Therefore, the optimal value is searched by a step-wise two-dimensional search algorithm, that is, the first step is big and then is small. First, this paper searches the maximum point $\alpha_0$ with step size of 0.01 between interval $[0\ 2]$, and then searches the optimal solution $\alpha$ with step size of 0.0001 in interval $\alpha_0 \pm 0.01$.

c) A narrow bandpass filter is designed in the frequency domain, which is equivalent to the multiplicative filter in the fractional Fourier domain. Its center frequency is scanned linearly along the frequency axis with the transformation of the instantaneous frequency of the signal, where the bandwidth is $u2 - u1 = 400$.

d) The signal is filtered through a bandpass filter.

e) The filtered results are rotated to the time domain by $\alpha$ inverse Fourier transform, and the filtered signals are obtained.

f) Calculate the output SNR after filtering:

$$SNR = 20*\log_{10}\left(P_S / P_N\right) \tag{18}$$

Where, $P_S = \left(pure\,signal\right)^2$ is power of signal, and is $P_N = (signal\ with\ noise - pure\ signal)^2$ power of noise.

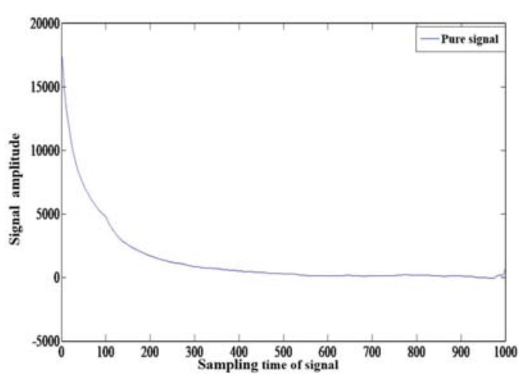


Figure 2.The pure signal

## A. *Example 1: An analog signal with a SNR of 40dB*

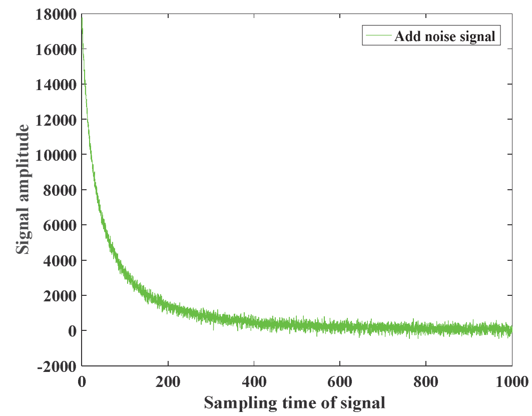


Figure 3. The SNR noise signal of 40dB

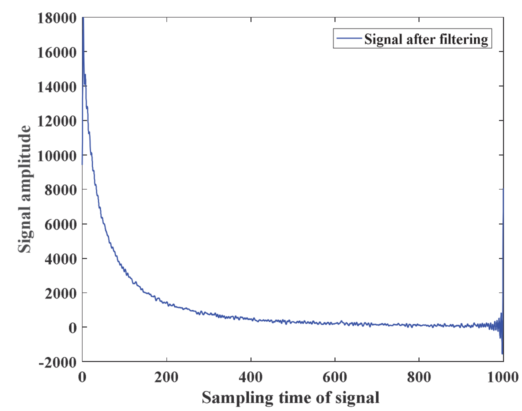


Figure 4.The signal after filtering

the process of transition didn't occur lag effect, and no significant loss of information before and after filtering,

SNR_IN = 40, SNR_OUT = 69.3563

## B. *Example 2: An analog signal with a SNR of 30dB*

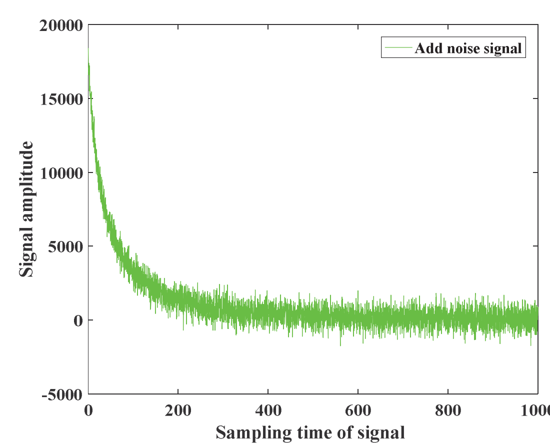

Figure 5. The SNR noise signal of 30dB

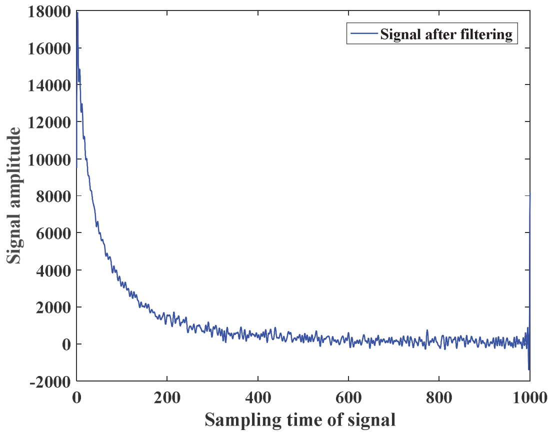


Figure 6.The signal after filtering

The process of transition didn't occur lag effect, and no significant loss of information before and after filtering,

SNR_IN = 30, SNR_OUT = 67.0462

### *C. **Example 3**: An analog signal with a SNR of 20dB*

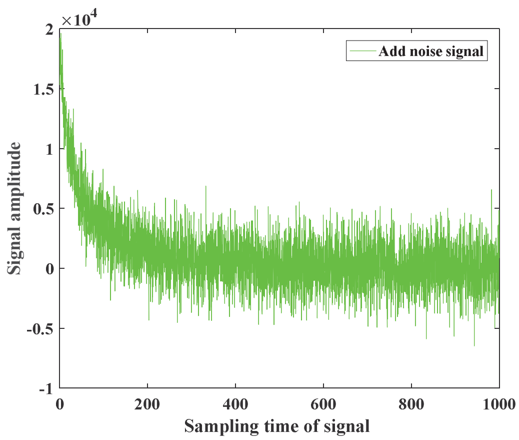


Figure 7. The SNR noise signal of 20dB

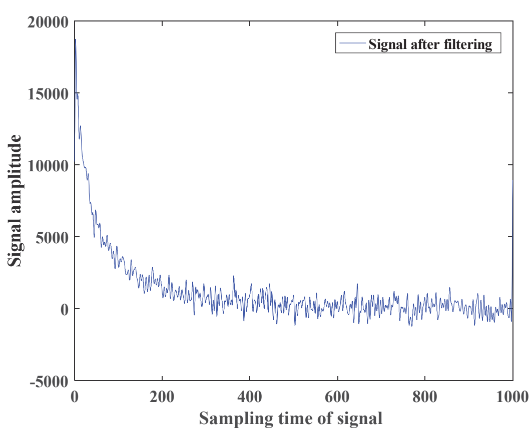


Figure 8. The signal after filtering

The process of transition didn't occur lag effect, and no significant loss of information before and after filtering,

SNR_IN = 20, SNR_OUT = 66.9212

### *D. **Example 4:** An analog signal with a SNR of 10dB*

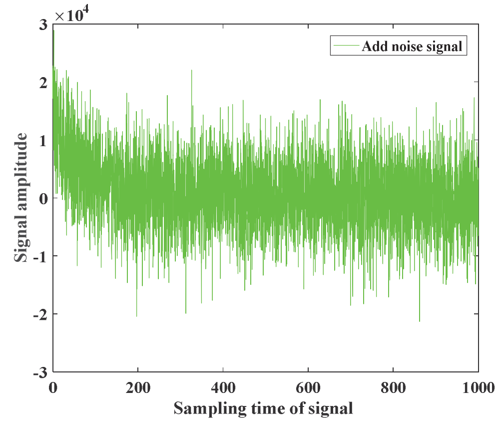


Figure 9. The SNR noise signal of 10dB

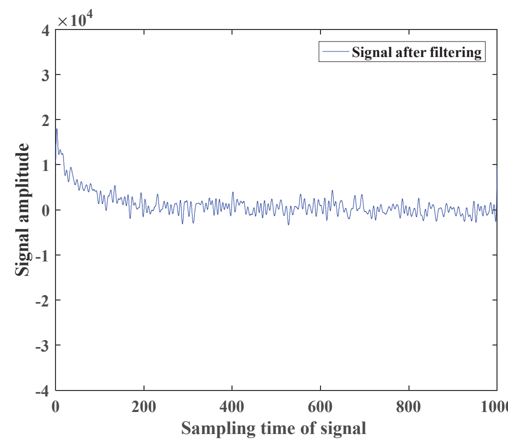


Figure 10.The signal after filtering

The process of transition didn't occur lag effect, and no significant loss of information before and after filtering,

SNR_IN = 10, SNR_OUT = 65.0576

## IV. CONCLUSION

In this paper, we have introduced a new adaptive filtering method that based on the fractional Fourier transform and its characteristics. As an ordinary Fourier transformation promotion, fractional Fourier transform is used to analyze the linear frequency modulation signal constitute as a powerful tool. That fractional Fourier adaptive filtering of linear frequency modulation signal adaptive filtering error is small and fast convergence. It is often applied to adaptive with various frequency modulation signal and multi-component signal in noise control system. The result of simulation shows that fractional Fourier adaptive filtering performance is better than that of time domain adaptive filtering, and the main effect is a high signal-to-noise ratio before and after the effect of filter. Both ends of the signal doesn't have any distortion and leak, so it has a great value in engineering application.

## References


[1] A. Singh and P. K. Banerji, "Fractional Integrals of Fractional Fourier Transform for Integrable Boehmians," *Proceedings of the National Academy of Sciences India,* vol. 88, no. 1, pp. 49-53, 2018.

[2] P. K. Anh, L. P. Castro, P. T. Thao, and N. M. Tuan, "New sampling theorem and multiplicative filtering in the FRFT domain," *Signal, Image and Video Processing,* no. 1, pp. 1-8, 2019.Durak L and Aldirmaz S. “Adaptive fractional Fourier domain filtering Signal Processing”, 2010,90(3):1188–1196.

[3] L. Durak and S. Aldirmaz, "Adaptive fractional Fourier domain filtering," *Signal Processing,* vol. 90, no. 4, pp. 1188-1196, 2010.

[4] S. Kumar, R. Saxena, and K. Singh, "Fractional Fourier Transform and Fractional-Order Calculus-Based Image Edge Detection," *Circuits Systems & Signal Processing,* vol. 36, no. 4, pp. 1-21, 2016.

[5] S. C. Pei and J. J. Ding, "Simplified fractional Fourier transforms," *Journal of the Optical Society of America A Optics Image Science & Vision,* vol. 17, no. 12, pp. 2355-67, 2000.Rojas H E, Cortés, Camilo A. Denoising of measured lightning electric field signals using adaptive filters in the fractional Fourier domain[J]. Measurement, 2014, 55(9):616-626.

[6] Zayed and Ahmed, "A New Perspective on the Two-Dimensional Fractional Fourier Transform and Its Relationship with the Wigner Distribution," *Journal of Fourier Analysis & Applications,* vol. 25, no. 2, pp. 460-487, 2019.

[7] H. E. Rojas and C. A. Cort├Ｅs, "Denoising of measured lightning electric field signals using adaptive filters in the fractional Fourier domain," *Measurement,* vol. 55, no. 9, pp. 616-626, 2014.Zhang and Zhi-Chao. “Algebraic representation for fractional Fourier transform on one-dimensional discrete signal models”. IET Signal Processing, 2018, 12(2):143-148.

[8] Almeida and L.B., "The fractional Fourier transform and time-frequency representations," *IEEE Transactions on Signal Processing,* vol. 42, no. 11,pp.3084-3091, 1994.Rongtao Sun and Nikita Zaveri,“Electrochemical noise signal processing using R/S analysis and Fractional Fourier transform ”.IFAC Proceedings Volumes, 2010, Vol.39 (11)

[9] Zhang and Zhi-Chao, "Algebraic representation for fractional Fourier transform on one-dimensional discrete signal models," *Iet Signal Processing,* vol. 12, no. 2, pp. 143-148, 2018.

[10] J. J. Ding and S. C. Pel, "Fractional Fourier Transforms and Wigner Distribution Functions for Stationary and Non-Stationary Random Process," in *IEEE International Conference on Acoustics*, 2006.

[11] R. Sun, N. Zaveri, Y. Chen, A. Zhou, and N. Zufelt, "ELECTROCHEMICAL NOIS E SIGNAL PROCESSING USING R/S ANALYSIS AND FRACTIONAL FOURIER TRANSFORM," *Ifac Proceedings Volumes,* vol. 39, no. 11, pp. 182-187, 2010.

[12] S. E. Azoug and S. Bouguezel, "A non-linear preprocessing for opto-digital image encryption using multiple-parameter discrete fractional Fourier transform," *Optics Communications,* vol. 359, no. 1, pp. 85-94, 2016.