Notes on Analytical Mechanics

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2024 spring

教材: V. I. Arnold, *Mathematical methods of classical mechanics*

1. Newtonian mechanics
2. Lagrangian mechanics
   - variational principle 变分法
3. Hamiltonian mechanics
   - symplectic structure 辛结构
4. Integrable systems
5. Nonintegrability
6. Relativity

Kepler's 3 laws

1. Each planet moves on an elliptical orbit with the sun at one focus.
2. The area swept by the planet within unit time is a constant. (angular momentum conservation)
3. The ratio of the cube of the semimajor and the square of the period is a constant. ($\frac{a^3}{T^2}$ = const)

The principles of relativity and determinacy:

A. space($\mathbb{R}^3$) and time($\mathbb{R}$)

B. Galileo's principle of relativity

There exist coordinate systems (called inertial 惯性的) with the following properties:
- All the laws of nature at all moments of time are the same in all inertial coordinate systems
- All coordinate systems in uniform rectilinear motion w.r.t. an inertial one are themselves inertial 相对于惯性坐标系作匀速直线运动的坐标系是惯性的

C. Newton's principle of determinacy

The initial state of a mechanical system (the totality of positions and velocities of its points of some moment) uniquely determines all its motion.

Galileo structure

1. The universe - a four-dim affine space (仿射空间) A^4. The points of A^4 are called world points or events. The parallel displacements of A^4 constitute a vector space $\mathbb{R}^4$.
2. Time - a linear map $t : \mathbb{R}^4 \to \mathbb{R}$. If $t(a-b) = 0, a, b \in A^4$, we say a and b are simultaneous events.
3. The distance between simultaneous events a and b: $\rho(a, b) = \sqrt{\langle a-b, a-b \rangle}$

The *Galileo group* is the group of all transformations preserving the Galileo structure.

The elements of Galileo group are called *Galilean transformations*.
Thus, Galilean transformations are affine transformations of A^4 which preserve intervals of time and the distance between simultaneous events. (保持时间间隔和同时事件距离)

Three types of elements of Galileo group: (在伽利略坐标空间 $(t, \vec{x}) \in (\mathbb{R}, \mathbb{R}^3)$ 下)

1. uniform motion: $g_1(t, \vec{x}) = (t, \vec{x} + \vec{v}t)$, $\vec{v} \in \mathbb{R}^3$ velocity
2. translation: $g_2(t, \vec{x}) = (t + s, \vec{x} + \vec{s})$, $\forall t \in \mathbb{R}, \vec{x} \in \mathbb{R}^3$
3. rotation: $g_3(t, \vec{x}) = (t, G\vec{x})$, $G : \mathbb{R}^3 \to \mathbb{R}^3$ orthogonal transformation

Newton mechanics

$$m\ddot{x} = \vec{F}(x, \dot{x}, t)$$

1. By the time translation, we get that the equation $\frac{\mathrm{d}^2 x(t+s)}{\mathrm{d}(t+s)^2}$ does not depend on t explicitly
2. By the space translation, we get that the equation depends only on relative positions
3. By the rotation, we get $F(Gx, G\dot{x}) = G \cdot F(x, \dot{x})$, G orthogonal transformation of $\mathbb{R}^3$

$$m_1 \ddot{x}_1 = \frac{G m_1 m_2 (x_1 - x_2)}{\|x_1 - x_2\|^3}$$

$$m_2 \ddot{x}_2 = \frac{G m_1 m_2 (x_2 - x_1)}{\|x_1 - x_2\|^3}$$

Systems of 1 degree of freedom: discribe by

$$\ddot{x} = f(x), x \in \mathbb{R} \tag{1}$$

Kinetic energy: $T = \frac{1}{2}\dot{x}^2$

Potential energy: $U(x) = -\displaystyle\int_{x_0}^{x} f(t)\,\mathrm{d}t$

Total energy: $E = T + U$

Thm. For a system evolving according to Equation 1, its total energy is conserved.

Proof. $\dot{E} = \dot{T} + \dot{U} = \dot{x}\ddot{x} + (-f(x))\dot{x} = 0$

Setting $y = \dot{x}$, we convert Equation 1 into

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x) \end{cases} \tag{2}$$

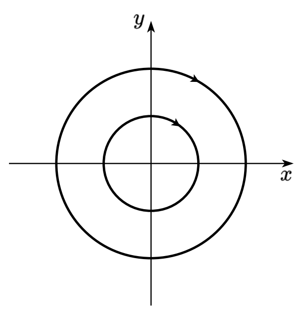
Phase space (plane): the space $\mathbb{R}^2$ of x and y
Phase point: a point in the phase plane
Phase curve: the image of a solution in the plane space

Example: Harmonic oscillator 谐振子

$$\ddot{x} = -x$$

$\Longrightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$K = \frac{1}{2}\dot{x}^2 = \frac{1}{2}y^2, U = -\int(-x)\,\mathrm{d}x = \frac{1}{2}x^2 \Longrightarrow E = \frac{1}{2}(x^2 + y^2)$



More generally, we consider

$$E = \frac{1}{2}y^2 + U(x)$$

$\Longrightarrow y = \pm\sqrt{2(E - U(x))}$

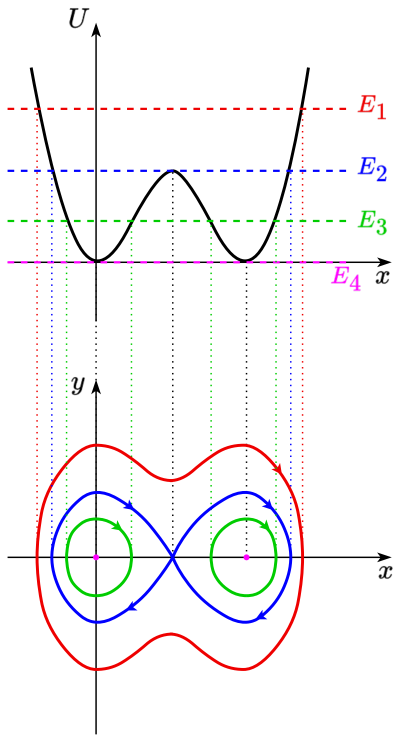
从 U 画相图的方法:

Rules: (1) $y=-\sqrt{2(E-U(x))}$ has the same monotonicity as U.

(2) In the upper half plane $y>0$, $y=\dot{x}$, this means, points move to the right.

(3) If the E-level set $\left\{\frac{1}{2}y^2+U(x)=E\right\}$ does not contains any critial point of E, then the level set is an entire periodic orbit.

(4) If the E-level set contains a critial point x_t of E and if the critial point is nondegenerate, then depending on the sign of $U''(x_t)$, it gives a *saddle* if $U''(x_t)<0$ and a *center* if $U''(x_t)>0$.



如图所示, E_2-level set 包含了一个 saddle, E_4-level set 包含了两个 center.

$\begin{cases}\dot{x}=y\\ \dot{y}=-U'(x)\end{cases}$, suppose x^* is a critical point.

The linearized equation $\begin{cases}\delta\dot{x}=\delta y\\ \delta\dot{y}=-U''(x^*)\end{cases}$ $\quad i.e. \begin{pmatrix}\delta\dot{x}\\ \delta\dot{y}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -U''(x^*) & 0\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$

Systems of 2 degree of freedom:

Nöther's law: *symmetry (invariance under a continuous group)* implies conservation law.

We say s system has *2 degrees of freedom* if it satisfies the equation

$$\ddot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x}),\boldsymbol{x}\in\mathbb{R}^2 \tag{3}$$

A system is called *conservative* (保守系统) if there exists a function $U:\mathbb{R}^2\to\mathbb{R}$ s.t.

$$\boldsymbol{F}=-\nabla U \tag{4}$$

The equation is

$$\ddot{\boldsymbol{x}} = -\nabla U(\boldsymbol{x}) \tag{5}$$

We define the total energy as $E = \frac{1}{2}\|\boldsymbol{y}\|^2 + U(\boldsymbol{x}), \boldsymbol{y} = \dot{\boldsymbol{x}}$.

Thm. A conservative system has the law of energy conservation.

Def. The motion of a point in a *central field* (中心力场) on a plane is defined by

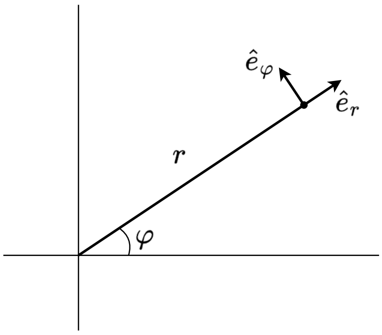
$$\ddot{\boldsymbol{r}} = \Phi(r)\hat{\boldsymbol{e}}_{\boldsymbol{r}} \tag{6}$$

Def. We define the *angular momentum* by

$$\boldsymbol{M} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{7}$$

Thm. For a point moving in a central field, the angular momentum is conserved.

$\boldsymbol{r} = r\hat{\boldsymbol{e}}_{\boldsymbol{r}}$



Lemma. $\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\dot{\varphi}\hat{\boldsymbol{e}}_{\varphi}, \quad \ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{\boldsymbol{e}}_{\boldsymbol{r}} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\hat{\boldsymbol{e}}_{\varphi}$.

Proof. $\boldsymbol{r} = r\hat{\boldsymbol{e}}_{\boldsymbol{r}}$,

$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \dot{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}\varphi}\dot{\varphi} = \dot{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\dot{\varphi}\hat{\boldsymbol{e}}_{\varphi}$

$\ddot{\boldsymbol{r}} = \ldots$

可以用复数证, $\left(re^{i\varphi}\right)' = \dot{r}e^{i\varphi} + r\dot{\varphi}ie^{i\varphi}, \quad \left(re^{i\varphi}\right)'' = (\ddot{r} - r\dot{\varphi}^2)e^{i\varphi} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})ie^{i\varphi}$ 即为该引理.

$$\begin{aligned}\boldsymbol{M} = \boldsymbol{r} \times \dot{\boldsymbol{r}} &= \boldsymbol{r} \times \left(\dot{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + r\dot{\varphi}\hat{\boldsymbol{e}}_{\varphi}\right) \\ &= r\hat{\boldsymbol{e}}_{\boldsymbol{r}} \times r\dot{\varphi}\hat{\boldsymbol{e}}_{\varphi} = r^2\dot{\varphi}\hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \hat{\boldsymbol{e}}_{\varphi} = r^2\dot{\varphi}\end{aligned}$$

Kepler's second law is exactly the angular momentum conservation.

Thm. Consider a point moving in a central field with the equation $\ddot{\boldsymbol{r}} = -\nabla U(\boldsymbol{r}), \boldsymbol{r} \in \mathbb{R}^2$ where $U = U(\|\boldsymbol{r}\|) = U(r)$. Then the radius r satisfies a system of 1 degree of freedom with effective potential

$$V(r) = U(r) + \frac{M^2}{2r^2} \tag{8}$$

(Reducing a motion in a central field to a system of 1 degree of freedom)

Proof. By the equation of motion, we have $\ddot{\boldsymbol{r}} = -\frac{\partial U}{\partial r}\hat{\boldsymbol{e}}_r$.

由引理 $\Longrightarrow \begin{cases} \ddot{r} - r\dot{\varphi}^2 = -\frac{\partial U}{\partial r} \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \quad \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) = 0 \end{cases}$

According to the angular momentum conservation (上面一行又再次给出了),

$r^2\dot{\varphi} = M \Longrightarrow \dot{\varphi} = \dfrac{M}{r^2}$

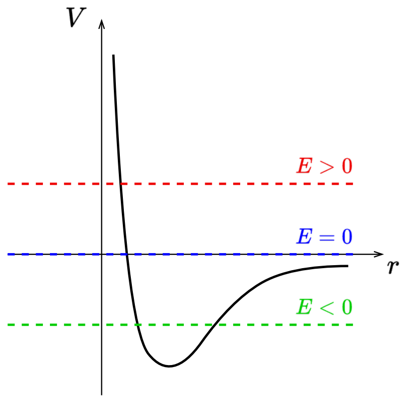
$\Longrightarrow \ddot{r} = -\dfrac{\partial U}{\partial r} + r\left(\dfrac{M}{r^2}\right)^2 = -\dfrac{\partial}{\partial r}\left(U + \dfrac{M^2}{2r^2}\right) = -\dfrac{\partial}{\partial r}V$, where $V = U + \dfrac{M^2}{2r^2}$.

注. The total energy in the derived 1-dim problem $E_1 = \dfrac{\dot{r}^2}{2} + V(r)$ is *the same* as the total energy in the original problem $E = \dfrac{\|\dot{\boldsymbol{r}}\|^2}{2} + U(\boldsymbol{r})$, since

$$\frac{\|\dot{\boldsymbol{r}}\|^2}{2} = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\varphi}^2}{2} = \frac{\dot{r}^2}{2} + \frac{M^2}{2r^2}$$

Kepler problem: $U = -\frac{k}{r} \Longrightarrow$

$$V(r) = U(r) + \frac{M^2}{2r^2} = -\frac{k}{r} + \frac{M^2}{2r^2} \tag{9}$$



$E = \dfrac{1}{2}\dot{r}^2 + V(r) \Longrightarrow \dfrac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2(E - V(r))}$

又由 angular momentum 守恒 $\Longrightarrow \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = \dfrac{M}{r^2}$

$\Longrightarrow \dfrac{\mathrm{d}\varphi}{\mathrm{d}r} = \dfrac{M/r^2}{\sqrt{2\left(E - \left(-\frac{k}{r} + \frac{M^2}{2r^2}\right)\right)}}$

$\Longrightarrow \varphi = \displaystyle\int \mathrm{d}\varphi = \int \dfrac{M/r^2}{\sqrt{2\left(E + \frac{k}{r} - \frac{M^2}{2r^2}\right)}}\,\mathrm{d}r = \cdots$

$$= \arccos\left(\frac{\frac{M}{r} - \frac{k}{M}}{\sqrt{2E + \frac{k^2}{M^2}}}\right) = \arccos\left(\frac{\frac{M^2}{kr} - 1}{\sqrt{1 + \frac{2EM^2}{k^2}}}\right)$$

记 $p := \frac{M^2}{k}$, $\quad e := \sqrt{1 + \frac{2EM^2}{k^2}}$ (eccentricity, 离心率)

$$\Longrightarrow \frac{\frac{p}{r}-1}{e} = \cos\varphi \Longrightarrow r = \frac{p}{1+e\cos\varphi}$$

$$r = \frac{p}{1+e\cos\varphi} \tag{10}$$

- when $0 < e < 1$ i.e. $E < 0$, the orbit is an ellipse
- when $e = 1$ i.e. $E = 0$, the orbit is a parabola
- when $e > 1$ i.e. $E > 0$, the orbit is a hyperbola

a: semi-major, b: semi-minor

$\Longrightarrow 2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2}$

$\Longrightarrow a = \frac{p}{1-e^2} = \frac{k}{2|E|}, b = a\sqrt{1-e^2} = \frac{M}{\sqrt{2|E|}}$

Kepler's third law: $\dfrac{a^3}{T^2} = \text{const}$

$\pi ab = \int_0^T \frac{M}{2}\,\mathrm{d}t = \frac{1}{2}MT \Longrightarrow T = \cdots = 2\pi a^{\frac{3}{2}} k^{-\frac{1}{2}}$.

Axially symmetric field 轴对称场

Def. A vector field in $\mathbb{R}^3$ is said to be *axially symmetric* if it is invariant under the group of rotations which fixes every point in the axis.

We choose the z-axis to be the axis fixed by the group of rotations.

Lemma. If a field is conservative and axially symmetric, then its potential energy U has the form $U(r, z)$, independent of φ.

证明. 见作业

Thm. For a particle moving in a conservative and axially symmetric field, then

$$M_z = \langle \hat{\boldsymbol{e}}_z, \boldsymbol{r} \times \dot{\boldsymbol{r}} \rangle \tag{11}$$

is conserved (角动量 $\boldsymbol{M}$ 在 z 方向的分量守恒).

Proof. $\dot{M}_z = \langle \hat{\boldsymbol{e}}_z, \boldsymbol{r} \times \ddot{\boldsymbol{r}} \rangle = \langle \hat{\boldsymbol{e}}_z, \boldsymbol{r} \times \boldsymbol{F} \rangle = 0$

(由引理, $\boldsymbol{F} = -\nabla U(r, z)$ *lies in the plane spanned by* $\hat{\boldsymbol{e}}_z$ *and* $\boldsymbol{r}$.)

The two-body problem

$$\begin{aligned} m_1 \ddot{\boldsymbol{r}}_{\boldsymbol{1}} &= -\frac{\partial U}{\partial \boldsymbol{r}_{\boldsymbol{1}}} \\ m_2 \ddot{\boldsymbol{r}}_{\boldsymbol{2}} &= -\frac{\partial U}{\partial \boldsymbol{r}_{\boldsymbol{2}}} \end{aligned} \tag{12}$$

where $U = U(|\boldsymbol{r}_{\boldsymbol{1}} - \boldsymbol{r}_{\boldsymbol{2}}|)$.

Thm. The time variation of the relative distance $\boldsymbol{r} = \boldsymbol{r}_{\boldsymbol{1}} - \boldsymbol{r}_{\boldsymbol{2}}$ in the two body problem is the same as the motion of a point mass $m = \frac{m_1 m_2}{m_1 + m_2}$ in a field with potential $U(|\boldsymbol{r}|)$.

Proof. $m_1 m_2 \ddot{\boldsymbol{r}} = m_1 m_2 \ddot{\boldsymbol{r}}_{\boldsymbol{1}} - m_1 m_2 \ddot{\boldsymbol{r}}_{\boldsymbol{2}} = -m_2 \frac{\partial U}{\partial \boldsymbol{r}_{\boldsymbol{1}}} + m_1 \frac{\partial U}{\partial \boldsymbol{r}_{\boldsymbol{2}}} = -(m_1 + m_2) \frac{\partial U}{\partial \boldsymbol{r}}$.

Lagrangian mechanics

Chapter 03 - Calculus of variations 变分法

Lagrangian mechanics: variational principle

$\updownarrow$ Legendre transformation

Hamiltonian mechanics: symplectic structure

Functional 泛函: A function on the space of curves.

$$\Phi(\gamma) = \int_a^b L(x(t), \dot{x}(t), t)\,\mathrm{d}t \tag{13}$$

where $\gamma : [a, b] \to \mathbb{R}^n$ is a C^1 curve.

Lagrangian $L : \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{\text{phase space}} \times \mathbb{R} \to \mathbb{R}$ is a C^2 function.

$\Phi : E \to \mathbb{R}$ functional, E is the space of C^1 curves in $\mathbb{R}^n$.

Def. A functional $\Phi : E \to \mathbb{R}$ is said to be *differentiable* at γ_0, if there exist a linear functional $F : E \to \mathbb{R}$ s.t.

$$\Phi(\gamma_0 + h) - \Phi(\gamma_0) = F \cdot h + o(h) \tag{14}$$

F is called the *differential* of Φ at γ_0.

Thm. Assume L is C^2, then $\Phi(\gamma) = \displaystyle\int_a^b L(x(t), \dot{x}(t), t)\,\mathrm{d}t$ is differentiable. Its differential is given by

$$F \cdot h = \int_a^b \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right)_{\gamma_0} h(t)\,\mathrm{d}t + \left.\frac{\partial L}{\partial \dot{x}} h\right|_a^b \tag{15}$$

Proof. 用分部积分公式.

$$\begin{aligned}
\Phi(\gamma + h) - \Phi(\gamma) &= \int_a^b L\big(x + h, \dot{x} + \dot{h}, t\big)\,\mathrm{d}t - \int_a^b L(x, \dot{x}, t)\,\mathrm{d}t \\
&= \int_a^b \left(\cancel{L(x, \dot{x}, t)} + \left.\frac{\partial L}{\partial x}\right|_\gamma h + \left.\frac{\partial L}{\partial \dot{x}}\right|_\gamma \dot{h} + o(h) - \cancel{L(x, \dot{x}, t)}\right)\mathrm{d}t \\
&= \int_a^b \left(\left.\frac{\partial L}{\partial x}\right|_\gamma h + \left.\frac{\partial L}{\partial \dot{x}}\right|_\gamma \dot{h}\right)\mathrm{d}t + o(h) \\
&= \int_a^b \left(\left.\frac{\partial L}{\partial x}\right|_\gamma h - \frac{\mathrm{d}}{\mathrm{d}t}\left.\frac{\partial L}{\partial \dot{x}}\right|_\gamma h\right)\mathrm{d}t + \left.\left.\frac{\partial L}{\partial \dot{x}}\right|_\gamma h\right|_a^b + o(h).
\end{aligned}$$

Def. A curve γ is called an extremal (极点) of Φ if $F|_\gamma \cdot h = 0, \forall h \in E$.

Thm. Let E_0 be the space of curves. $\gamma : [a, b] \to \mathbb{R}^n, \gamma(a) = x_0, \gamma(b) = x_1$ with fixed points. Then γ is an extremal of Φ if and only if

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0 \text{ along } \gamma \tag{16}$$

We call Equation 16 *the Euler-Lagrange equation.*

Proof. Since we consider only curves with fixed endpoints in E_0 i.e. $\gamma_1 = \gamma_0 + h$ has the same points as γ_0, this implies $h = 0$ at $t = a, b$.

Thus the differential of Φ reads

$$F_\gamma \cdot h = \int_a^b \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \right)\Bigg|_{\gamma(t)} h(t)\,\mathrm{d}t$$

Lemma. If a continuous function $f : [a, b] \to \mathbb{R}$ satisfies $\displaystyle\int_a^b f(t)h(t)\,\mathrm{d}t = 0$, $\forall h$ continuous with $h(a) = h(b) = 0$, then $f(t) \equiv 0$.

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which finishes the proof.

Thm. A curve is an extremal of Φ iff the Euler-Lagrangian equation is satisfied along γ.

Example. a free mass (不受力的质点静止或匀速直线运动)

$U = 0, \quad L = T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$

$\implies 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x_i} = m\ddot{x}_i \implies x_i = A_i t + B_i.$

Hamilton's principle of least action:

$$\underbrace{m\ddot{x} = -\nabla U(x), \quad x \in \mathbb{R}^3 \quad \text{Newton eqn}}_{\text{Mechanical systems}} \tag{17}$$

Thm. Motions of the mechanical system *coincide with* extremals of the functional

$$\Phi(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t), t)\,\mathrm{d}t \tag{18}$$

where $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$. (Kinetic energy - potential energy)

证明. 显然

Def. Given Lagrangian $L(x, \dot{x}, t), x \in \mathbb{R}^n$, we call

x the generalized coordinates/positions,

$\dot{x}$ the generalized velocities,

$\frac{\partial L}{\partial \dot{x}} = p$ the generalized momentum,

$\frac{\partial L}{\partial x}$ the generalized forces

Example: central field

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\varphi}\hat{\boldsymbol{e}}_\varphi$$

Kinetic energy $T = \frac{1}{2}m\|\dot{\boldsymbol{r}}\|^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$

Potential energy $U(r)$

Lagrangian $L(r, \varphi, \dot{r}, \dot{\varphi}) = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$

$$\overset{\text{E-L}}{\Longrightarrow} \begin{cases} m\ddot{r} = mr\dot{\varphi}^2 - \frac{\partial U}{\partial r} \Longrightarrow m\ddot{r} = \frac{M^2}{mr^3} - \frac{\partial U}{\partial r} \\ \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\varphi}) = 0 \qquad \Longrightarrow mr^2\dot{\varphi} = M \text{ angular momentum conservation} \end{cases}$$

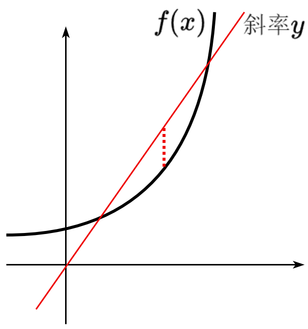
Def. If the Lagrangian does not depend on a generalized coordinate x_1 i.e. $\frac{\partial L}{\partial x_1} = 0$, we say x_1 is a *cyclic coordinate.*

Thm. The existence of a cyclic generalized coordinate x_1 implies the corresponding generalized momentum $\frac{\partial L}{\partial \dot{x}_1}$ is a conserved quantity.

Legendre transformation:

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function (额外假设 C^2, $f'' > 0$), the *Legendre transformation* of f is

$$f^*(y) = \sup_x (xy - f(x)) \tag{19}$$



Suppose the sup is attained at x_0, we have $\frac{\mathrm{d}}{\mathrm{d}x}(xy - f(x))\Big|_{x_0} = 0 \Longrightarrow y = f'(x_0)$

Prop. Let f be a convex function, then its Legendre transformation is still convex.

Proof.

$$\begin{aligned} f^*(\lambda y_1 + (1-\lambda)y_2) &= \sup_x ((\lambda y_1 + (1-\lambda)y_2)x - f(x)) \\ &= \sup_x [\lambda(y_1 x - f(x)) + (1-\lambda)(y_2 x - f(x))] \\ &\leq \lambda \sup_x (y_1 x - f(x)) + (1-\lambda)\sup_x (y_2 x - f(x)) \\ &= \lambda f^*(y_1) + (1-\lambda) f^*(y_2). \end{aligned}$$

Prop. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^2, strictly convex. Suppose the sup in the definition of $f^*(x)$ is attained at x_0 satisfying $y_0 = f'(x_0)$. Then we have

$$x_0 = (f^*)'(y_0), \quad f''(x_0) \cdot (f^*)''(y_0) = 1 \tag{20}$$

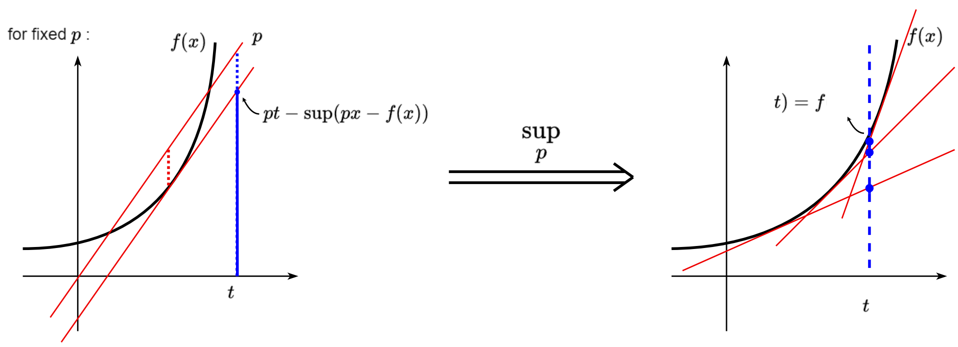
Proof. By assumption, we have $f^*(y_0) = x_0 y_0 - f(x_0)$, where $x_0 = (f')^{-1}(y_0)$
$\Longrightarrow (f^*)'(y_0) = \frac{\partial f^*}{\partial x_0}\frac{\partial x_0}{\partial y_0} + \frac{\partial f^*}{\partial y_0} = \frac{\partial x_0}{\partial y_0}(y_0 - f'(x_0)) + x_0 = x_0$

注. 说明 f' 与 $(f^*)'$ 互为反函数.

Prop. The Legendre transformation is involutive, i.e. $(f^*)^* = f$.

Proof. 若假设 $f \in C^2$, 则由上一命题容易证明 (略).

$h(t) = \sup\limits_p \left(pt - \sup\limits_x (px - f(x)) \right)$, 对每个 t,
对每个 p, $pt - \sup\limits_x (px - f(x))$ 的几何意义为 $f(x)$ 的斜率为 p 的切线在 t 处的函数值 (如下图),
再结合 $f(x)$ 是凸函数, 故这些切线都在 $f(x)$ 图像下方, 这些点的 sup 正是 $f(t)$.



For a convex function multi variables $f : \mathbb{R}^n \to \mathbb{R}$, we define its Legendre transformation

$$f^*(y) = \sup_x (\langle x, y \rangle - f(x)) \tag{21}$$

Hamiltonian mechanics:

Given a Lagrangian $L(x, \dot{x}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, suppose L is (strictly) convex in $\dot{x}$. Let the *Hamiltonian*

$$H(x, y) = \sup_{\dot{x}} (\langle y, \dot{x} \rangle - L(x, \dot{x})) \tag{22}$$

be the Legendre transformation of L w.r.t $\dot{x}$.

By the involutivity, we have

$$L(x, \dot{x}) = \sup_y (\langle \dot{x}, y \rangle - H(x, y))$$

When the sup is attained, we have

$$\begin{cases} y = \frac{\partial L}{\partial \dot{x}} & \text{generalized momentum} \\ \dot{x} = \frac{\partial H}{\partial y} \end{cases} \tag{23}$$

$$\overset{\text{E-L}}{\Longrightarrow} \begin{cases} \dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} = -\frac{\partial H}{\partial x} & \text{(why?)} \\ \dot{x} = \frac{\partial H}{\partial y} \end{cases}$$

We get *the Hamiltonian canonical equations*

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases} \tag{24}$$

Example. $L(x, \dot{x}) = \frac{1}{2}m\langle A\dot{x}, \dot{x}\rangle - U(x)$

$y = \frac{\partial L}{\partial \dot{x}} = mA\dot{x} \Longrightarrow \dot{x} = \frac{1}{m}A^{-1}y$

$H(x, y) = \langle y, \dot{x}\rangle - L(x, \dot{x}) = \dots = \underbrace{\frac{1}{2m}\langle y, A^{-1}y\rangle}_{\text{Kinetic energy}} + \underbrace{U(x)}_{\text{Potential energy}}$

(Lagrangian: 动能 $-$ 势能 $\Longrightarrow$ 有变分法
Hamiltonian: 动能 $+$ 势能 $\Longrightarrow$ 有辛结构)

Thm. For mechanical systems (where kinetic energy $K = \frac{1}{2}m\langle A\dot{x}, \dot{x}\rangle$ is a quadratic form w.r.t. $\dot{x}$), the Hamiltonian is the total energy.

Thm. (energy conservation) If H does not depend on t explicitly, then along any orbit $(x(t), y(t))$, we have $H(x(t), y(t)) \equiv$ const.

Proof. $\frac{\mathrm{d}}{\mathrm{d}t}H(x(t), y(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0.$

Cyclic coordinates:

Def. If a generalized coordinate x_1 does not enter H i.e. $\frac{\partial H}{\partial x_1} = 0$, we call x_1 a *cyclic coordinate.*
(We also have $\frac{\partial L}{\partial x_1} = 0$)

Prop. Suppose x_1 is a cyclic coordinate of H. Then the corresponding y_1 is a conserved quantity.

Proof. $\dot{y}_1 = -\frac{\partial H}{\partial x_1} = 0$

于是每一个 cyclic coordinate 都能将 n 个坐标减少到 $n-1$ 个 ($2n$ 个一阶方程 $\Longrightarrow 2(n-1)$ 个), 最后由 $\dot{x}_1 = \frac{\partial H}{\partial y_1}(y_1, \tilde{x}, \tilde{y}, t)$ 积分出 x_1 即可.

Liouville theorem

Thm. The Hamiltonian flow *preserves volume* of the phase space.

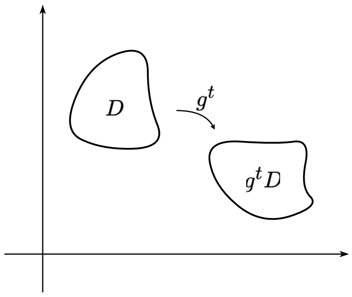
---

phase space: $\mathbb{R}^n \times \mathbb{R}^n$
phase flow: the one-parameter group of transformations of the phase space

$$\begin{aligned} g^t : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \times \mathbb{R}^n \\ (x(0), y(0)) &\mapsto (x(t), y(t)) \end{aligned} \tag{25}$$

where $(x(t), y(t))$ solves the Hamiltonian equation.

For any given t_0, g^{t_0} is a diffeomorphism (微分同胚) on $\mathbb{R}^n \times \mathbb{R}^n$



$\mathrm{Vol}(D) = \mathrm{Vol}(g^t(D))$

Proof of Liouville theorem. Let $D(t) := g^t(D(0))$, $v(t) := \mathrm{Vol}(D(t))$.

Suppose we are given a system of $\dot{x} = f(x)$. Then $g^t(x) = x + f(x)t + O(t^2) \quad (t \to 0)$

$v(t) = \int_{D(t)} \mathrm{d}x = \int_{D(0)} \det \frac{\partial g^t}{\partial x} \, \mathrm{d}x = \int_{D(0)} \det\left(I + \frac{\partial f}{\partial x} t + O(t^2)\right) \mathrm{d}x \quad (t \to 0)$

Lemma1. For any matrix A, we have

$$\det(I + tA) = 1 + t \operatorname{tr}(A) + O(t^2) \quad (t \to 0)$$

$\Longrightarrow v(t) = \int_{D(0)} \left(1 + t \operatorname{tr}\left(\frac{\partial f}{\partial x}\right) + O(t^2)\right) \mathrm{d}x = \int_{D(0)} (1 + t \nabla \cdot f + O(t^2)) \, \mathrm{d}x$

$\Longrightarrow \left.\dfrac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=0} = \displaystyle\int_{D(0)} \nabla \cdot f \, \mathrm{d}x$

Lemma2. If $\nabla \cdot f \equiv 0$ (divergence free), then g^t preserves volume.

Proof. Since $t = t_0$ is no worse than $t = 0$, we have

$$\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=t_0} = \int_{D(t_0)} \nabla \cdot f \, \mathrm{d}x$$

$\Longrightarrow \frac{\mathrm{d}v}{\mathrm{d}t} \equiv 0 \Longrightarrow v(t) \equiv \text{const.}$

For our Hamiltonian flow, we have $\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$

$\Longrightarrow \nabla \cdot f = \nabla \cdot (\dot{x}, \dot{y}) = \nabla \cdot \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right) = \sum \frac{\partial}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial}{\partial y_i} \frac{\partial H}{\partial x_i} = 0,$

which finishes the proof.

Thm. (Poincaré recurrence, 加强版本) Let $f : X \to X$ be a diffeomorphism <u>preserving the volume</u> where X is a domain with $\mathrm{Vol}(X) < \infty$. Then for any $A \subset X$ with $\mathrm{Vol}(A) > 0$, we have a.e. $x \in A$, $\exists \{n_k\}_{k=0}^{\infty}$ s.t. $f^{n_k}(x) \in A$.

Proof. We first prove that there is one point $x \in A$ and $n \in \mathbb{N}$ s.t. $f^n(x) \in A$.

Consider $A, f(A), f^2(A), \ldots$, since f preserves volume, we have $\mathrm{Vol}(A) = \mathrm{Vol}(f(A)) = \ldots$

Since $\mathrm{Vol}(A) > 0$, let $N = \left\lfloor \frac{\mathrm{Vol}(X)}{\mathrm{Vol}(A)} \right\rfloor + 1$, $\exists n_1 < n_2 \in \mathbb{N}$ s.t. $f^{n_1}(A) \cap f^{n_2}(A) \neq \emptyset$

$\Longrightarrow A \cap f^{n_2 - n_1}(A) \neq \emptyset$ (这一步需要单射).

修改以上证明 (考虑 $A, f^{-1}(A), \ldots$), 就不需要单射条件了.

余下证明见作业.

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holonomic constraints

Example. $L_N : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2), \quad U_N = \frac{1}{2} N \dot{q}_2^2 + U_0(q_1, q_2)$

$L_N = T - U_N$

$H_N = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + U_N(\boldsymbol{q}) = E \text{ (total energy) } \text{ fixed as } N \to \infty$

$\Longrightarrow q_2 \to 0$ as $N \to \infty$ (不然 $\frac{1}{2} N \dot{q}_2^2$ 项趋于无穷)

Consider initial condition $(*)$:

$$\begin{aligned} q_1(0) &= q_1^0, \quad \dot{q}_1(0) = \dot{q}_1^0, \\ q_2(0) &= 0, \quad \dot{q}_2(0) = 0 \end{aligned}$$

Thm. Let $q_1 = \varphi_N(t)$ be the evolution of q_1 under the initial condition $(*)$, then the limit

$$\lim_{N \to \infty} \varphi_N(t) \to \psi(t)$$

exists, where $q_1 = \psi(t)$ satisfies the E-L equation

$$\frac{\partial L_*}{\partial q_1} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_*}{\partial \dot{q}_1}, \tag{26}$$

where

$$L_*(q_1, \dot{q}_1) = T \mid_{q_2 = \dot{q}_2 = 0} - U_0 \mid_{q_2 = 0} \tag{27}$$

即: 被很大的势场限制住的运动, 其 Lagrangian 相当于不考虑这个势场的势能以及限制曲面法向的动能.

Def. (holonomic constraints) Let Γ be an m-dim surface in $\mathbb{R}^{3n}$ of points $\boldsymbol{r_1}, \ldots, \boldsymbol{r_n} \in \mathbb{R}^3$ with masses $m_1, \ldots, m_n$ respectively. Let $\boldsymbol{q} = (q_1, \ldots, q_m)$ be coordinates on Γ, $\boldsymbol{r_i} = \boldsymbol{r_i}(\boldsymbol{q})$. The system discribed by

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}}, \quad L = \frac{1}{2} \sum m_i \boldsymbol{r}_i^2 - U(\boldsymbol{q}) \tag{28}$$

is called *a system of n points with $3n - m$ ideal holonomic coordinates.*

Example. $\boldsymbol{r}_i \in \mathbb{S}^2 \subset \mathbb{R}^3, \quad q = (\theta, \varphi)$ is the spherical coordinates,
$\boldsymbol{r}_i = (\cos\theta_i \cos\varphi_i, \cos\theta_i \sin\varphi_i, \sin\theta_i)$

Differentiable manifold

Def. A set M is called a *manifold* if M is given a finite or countable collection of charts such that each point lies in at least one chart.

chart (坐标卡): (U, φ), where $U \subset \mathbb{R}^n$ is an open set, φ is an one-to-one map from U to some subset of M.

In the overlaping region of two charts, $\varphi_j^{-1} \circ \varphi_i \left[\varphi_i^{-1}\left(V_i \cap V_j\right)\right] \subset U_j$

$\Longrightarrow \varphi_j^{-1} \circ \varphi_i$ is a map from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$.

We say M is a *differentiable manifold* if $\varphi_j^{-1} \circ \varphi_i$ is differentiable for all i, j

---

注. **Projective space**

$$\mathbb{RP}^n = \left\{\text{all lines going through } O \text{ in } \mathbb{R}^{n+1}\right\} \tag{29}$$

作业题. $SO(3)$ homeomorphic to $\mathbb{RP}^3$.

Tangent space

We say two curves $\varphi(t), \psi(t)$ are equivalent at x if $\varphi(0) = \psi(0) = x$ and $\lim\limits_{t \to 0} \dfrac{\varphi(t) - \psi(t)}{t} = 0$ in some chart.

Def. A tangent vector to a manifold M at x is an equivalent class of curves $\varphi(t)$ with $\varphi(0) = x$.

tangent space (切空间) $T_x M$: the set of tangent vectors to M at x.

tangent bundle (切丛) $TM = \bigcup\limits_{x \in M} T_x M$

Riemannian manifold: a manifold endowed with a Riemannian metric (各点 *metric* 要是同一个).

A *Riemannian metric* on a manifold M is given by a positive definite quadratic form on $T_x M$ at each point $x \in M$:

$$\begin{aligned} T_x M \times T_x M &\to \mathbb{R} \\ (u, v) \qquad &\mapsto \langle u, v \rangle_x = \langle A_x u, v \rangle \end{aligned} \tag{30}$$

$\|v\|_x = \sqrt{\langle u, v \rangle_x}, \quad v \in T_x M$

For a curve $\gamma : [0, 1] \to M$, its length is given by $\ell(\gamma) = \displaystyle\int_0^1 \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle_{\gamma(t)}}\, \mathrm{d}t$

Derivative map:

Let $f : M \to N$ be a map between two manifolds. The derivative of f at point $x \in M$ is the linear map of the tangent spaces

$$Df_x = f_{*x} : T_xM \to T_{f(x)}N, \tag{31}$$

which is given in the following way:

$\forall v \in T_xM$, consider a curve $\varphi : \mathbb{R} \to M$ with $\varphi(0) = x, \dot{\varphi}(0) = v$, then $f_{*x}(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi(t))\right|_{t=0}$.

注. $f_{*x}(v)$ 的值与 φ 的选取无关.

注. f_{*x} 是线性的.

How to solve a system with holonomic constraints:

1. Find $\Gamma \subset \mathbb{R}^{3n}$, $\boldsymbol{r}_1, \dots, \boldsymbol{r}_n \in \mathbb{R}^3$, $\boldsymbol{r}_i = \boldsymbol{r}_i(\boldsymbol{q}) \in \Gamma$, $\boldsymbol{q} = (q_1, \dots, q_m) \in \mathbb{R}^m$
2. Kinetic energy
$$T = \frac{1}{2}\sum m_i \dot{\boldsymbol{r}}_i^2 = \frac{1}{2}\sum m_i \left|\frac{D\boldsymbol{r}_i}{D\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}\right|^2 = \frac{1}{2}\sum a_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j$$
3. Solve E-L equation $L = T - U(\boldsymbol{q})$

Example. motion of a particle of mass 1 on a surface of revolution in $\mathbb{R}^3$

Introduce polar coordinates (r, φ) on $\mathbb{R}^2$

Γ is given by $(r(z), \varphi, z), z \in [a, b], \varphi \in [0, 2\pi]$.
$x = r(z)\cos\varphi \Longrightarrow \dot{x} = r_z\dot{z}\cos\varphi - r\sin\varphi\ \dot{\varphi}$
$y = r(z)\sin\varphi \Longrightarrow \dot{y} = r_z\dot{z}\sin\varphi + r\cos\varphi\ \dot{\varphi}$

$\Longrightarrow T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \dots = \frac{1}{2}((1 + r_z^2)\dot{z}^2 + r^2(z)\dot{\varphi}^2)$

$L(= T) = L(\varphi, z, \dot{\varphi}, \dot{z}) : T\Gamma \to \mathbb{R}$ does not depend on φ explicitly (φ is a cyclic coordinate)

$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi} = 0 \Longrightarrow \frac{\partial L}{\partial \dot{\varphi}} = r^2\dot{\varphi} = \mathrm{const}$

Let α be the angle formed by $\boldsymbol{v}$ with z-axis, then the horizontal component of the velocity is
$r\dot{\varphi} = |\boldsymbol{v}|\sin\alpha$ (看下图理解)

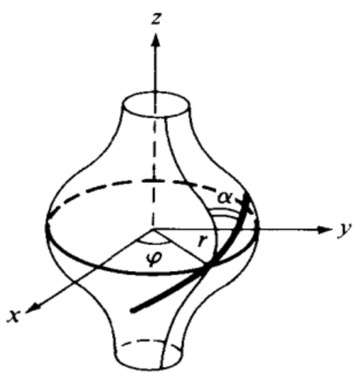

Figure 66 Surface of revolution

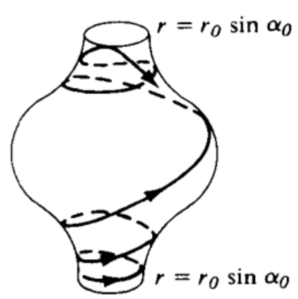

Figure 67 Geodesics on a surface of revolution

By energy conservation, $|\boldsymbol{v}|$ is a constant $\Longrightarrow \underbrace{r\sin\alpha = r^2\dot{\varphi}\ |\boldsymbol{v}|^{-1} = \mathrm{const}}_{\text{Clairaut's theorem}}$

Example. bead on a rotating circle

a bead with mass 1 moves along a vertical circle of radius 1 which rotates with angular velocity ω.

$\Gamma : \boldsymbol{r} = (\sin(\theta)\cos(\omega t), \sin(\theta)\sin(\omega t), \cos(\theta))$ ($\theta \in [-\pi, \pi]$ measured from the highest point)

$T = \frac{1}{2}\,|\dot{\boldsymbol{r}}|^2 = \frac{1}{2}\left(\omega^2 \sin^2\theta + \dot{\theta}^2\right), \quad U = g\cos\theta$

$\Longrightarrow L = T - U = \underbrace{\frac{1}{2}\dot{\theta}^2}\ + \underbrace{\frac{1}{2}\omega^2\sin^2\theta - g\cos\theta}$

与一个 1-dim system 相同: $T' = \frac{1}{2}\dot{q}^2, U' = A\cos q - B\sin^2 q$, where $A = g, B = \frac{1}{2}\omega^2$

- 当 $2B < A$ 即 $\omega^2 r < g$ 时 ($r = 1$), $q = \pm\pi$ (最低点) 是一个稳定点. (此时大圈转的很慢, 符合生活常识.)
- 当 $2B > A$ 时, $q = \pm\pi$ 将不再是稳定点, 但在 $(-\pi, \pi)$ 中新出现了两个稳定点, 分别对应 $\cos q = -\frac{A}{2B} = -\frac{g}{\omega^2 r}$, 如下图所示.

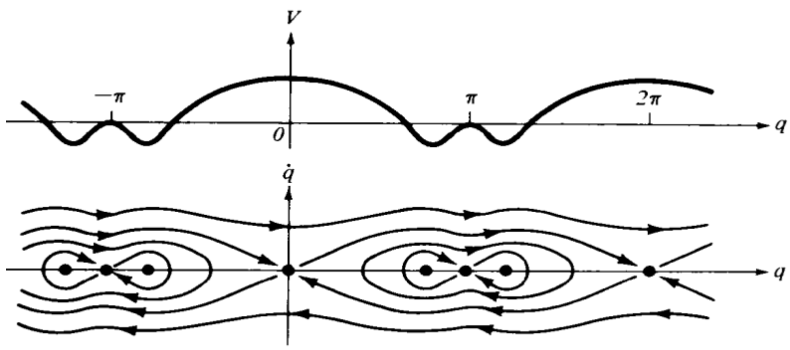

Figure 69 Effective potential energy and phase plane of the bead

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Nöther's law

以下考虑的是 $\underbrace{\text{自治系统}}_{L(q,\dot{q}):TM\to\mathbb{R}}$. $\underbrace{\text{非自治系统}}_{L(q,\dot{q},t):TM\times\mathbb{R}\to\mathbb{R}}$ 也存在类似定理, 见作业.

M manifold, $L : TM \to \mathbb{R}$ a smooth function, $h : M \to M$ a smooth map

Def. A Lagrangian system $L : TM \to \mathbb{R}$ *admits* the map h if for any $v \in T_x M$ we have

$$L(x, v) = L(h(x), Dh(x)v) \tag{32}$$

Thm.(Nöther) If the system $L : TM \to \mathbb{R}$ admits the 1-parameter group of diffeomorphism $h^s : M \to M, s \in \mathbb{R}$, then the Lagrangian system *admits a first integral* $I : TM \to \mathbb{R}$

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}h^s(q)}{\mathrm{d}s}\bigg|_{s=0} \tag{33}$$

注. I 的表达式中两部分应该是行向量乘列向量, 得到个数量 (或写成向量点积).

Proof. Let $\varphi(t)$ be a solution to the E-L equation, $\Phi(t, s) := h^s(\varphi(t))$

By assumption, $L\left(\Phi(t, s), \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, s)\right) = L(\varphi(t), \dot{\varphi}(t))$

Take $\frac{\mathrm{d}}{\mathrm{d}s}$ on both sides, $0 = \frac{\partial L}{\partial q}\frac{\mathrm{d}\Phi}{\mathrm{d}s} + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Phi$

By the E-L equation $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\left(\Phi(t,s), \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s)\right) = \frac{\partial L}{\partial q}\left(\Phi(t,s), \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s)\right)$
$\Longrightarrow 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}\Phi}{\mathrm{d}s} + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\Phi = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}\Phi}{\mathrm{d}s}\right)$
$\Longrightarrow \dfrac{\partial L}{\partial \dot{q}}\dfrac{\mathrm{d}h^s}{\mathrm{d}s}$ is a conserved quantity.

(抄自教材, 其实没懂) In fact, the first integral $I = \frac{\partial L}{\partial \dot{q}}q'$ is *the rate of change of* $L(v)$ when the vector $v \in T_x M$ varies inside $T_x M$ with velocity $\frac{\mathrm{d}h^s(x)}{\mathrm{d}s}\big|_{s=0}$

Example. points with masses m_i, $L = \frac{1}{2}\sum m_i \dot{\boldsymbol{x}}_i^2 - U(\boldsymbol{x})$

1. Suppose the system admits the translation along $\boldsymbol{e}_1$: $h^s(\boldsymbol{x}_i) = \boldsymbol{x}_i + \boldsymbol{e}_1 s$

   By Nother's thm, $I = \sum \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i}\frac{\mathrm{d}h^s}{\mathrm{d}s} = \sum m_i \langle \dot{\boldsymbol{x}}_i, \boldsymbol{e}_1\rangle = \underbrace{\sum m_i \dot{x}_{i1}}_{\text{动量 } \boldsymbol{e}_1 \text{分量}}$ is conserved

   i.e. translation invariance $\Longrightarrow$ momentum conservation

2. Suppose L admits rotations around $\boldsymbol{e}_1$

   Let h^s be the rotation around $\boldsymbol{e}_1$-axis by angle s, then $\frac{\mathrm{d}h^s}{\mathrm{d}s}(\boldsymbol{x}_i) = \boldsymbol{e}_1 \times \boldsymbol{x}_i$ (思考)

   $\Longrightarrow I = \sum \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i}\frac{\mathrm{d}h^s}{\mathrm{d}s}(\boldsymbol{x}_i) = \sum m_i \langle \dot{\boldsymbol{x}}_i, \boldsymbol{e}_1 \times \boldsymbol{x}_i\rangle = \underbrace{\sum m_i \langle \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i, \boldsymbol{e}_1\rangle}_{\text{角动量在 } \boldsymbol{e}_1 \text{ 的投影}}$ conserved

   i.e. rotation invariance $\Longrightarrow$ angular momentum conservation

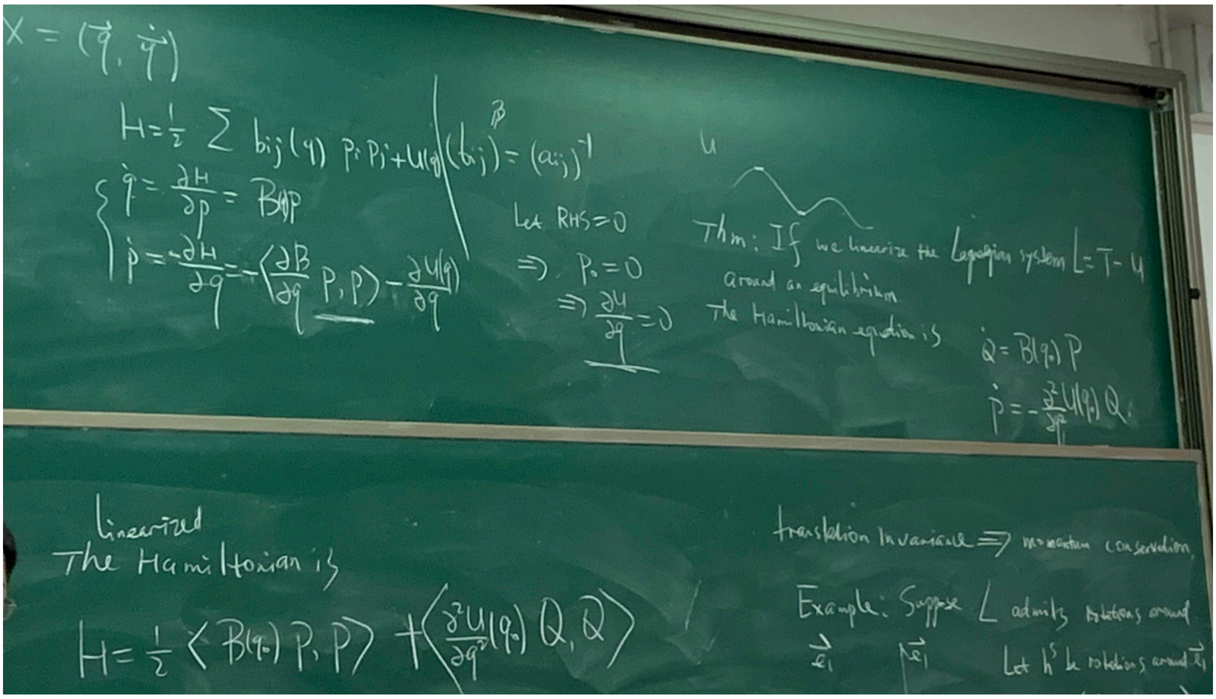
Linearization

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n \tag{34}$$

A point $\boldsymbol{x}_0$ is called an equilibrium (平衡点) or fixed point if $\boldsymbol{x}(t) \equiv \boldsymbol{x}_0$ is a solution. In other words, $\boldsymbol{f}(\boldsymbol{x}_0) = 0$.

Consider $L = T - U(\boldsymbol{q}), \quad T = \frac{1}{2} \sum a_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j$

The point $\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0$ is an equilibrium iff $\dot{\boldsymbol{q}}_0 = 0$ and $\boldsymbol{q}_0$ is a critical point of U i.e. $\left.\frac{\partial U}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}_0} = 0$



Linearization

fixed point, equilibrium.

$$L = \frac{1}{2} a_{ij}(q)\, \dot{q}_i \dot{q}_j - U(q)$$

$$H = \frac{1}{2} b_{ij}(q)\, p_i p_j + U(q) \quad , (b_{ij}) = (a_{ij})^{-1}.$$

At an equibrium, we have $p=0$, q is at a critical point of U.

(通过选取坐标系)

<u>Thm.</u> To linearize the above system around an equibrium point $(p=0, q=0)$, it is enough to replace the kinetic energy by $T = \frac{1}{2}\sum a_{ij}(0)\, \dot{q}_i \dot{q}_j = \frac{1}{2}\langle A\dot{q}, \dot{q}\rangle$ and the potential energy by its quadratic part $U_2 = \frac{1}{2}\sum C_{ij}\, q_i q_j$, where $C_{ij} = \frac{\partial^2 U(0)}{\partial q_i \partial q_j}$.

~~proof the~~ The linearized Hamiltonian is $\overline{H} = \frac{1}{2}\sum b_{ij}\, p_i p_j + \frac{1}{2}\sum C_{ij}\, q_i q_j$, $(b_{ij}) = (a_{ij})^{-1}$.

$$\begin{cases} \dot{q} = \dfrac{\partial \overline{H}}{\partial p} = Bp \\[2mm] \dot{p} = -\dfrac{\partial \overline{H}}{\partial q} = -Cq \end{cases}$$

E-L eqn reads $\dfrac{\partial L}{\partial q} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} \Rightarrow -Cq = \dfrac{d}{dt}(A\dot{q}) \Rightarrow \ddot{q} = -A^{-1}Cq$

We next find the eigenvalue of $-A^{-1}C$:

$$\det(-A^{-1}C - \lambda I) = 0 \Rightarrow \det(C + \lambda A) = 0.$$

(a_{ij}) is positive definite.

Suppose the point $q=0$ is a nondegenerate local minimum of U, then C is also positive definite.

$A = Q^T Q$, Q nonsingular $\qquad C = D^T D$, D nonsingular

$$\Rightarrow \det(Q^{-T} D^T D Q^{-1} + \lambda I) = 0 \Rightarrow \det\big(\underbrace{(DQ^{-1})^T (DQ^{-1})}_{\text{positive definite}} + \lambda I\big) = 0$$

$\Rightarrow$ the solutions are all negative

$\Rightarrow \exists$ matrix M s.t. $A^{-1}C = M^{-1}\Lambda M$, where $\Lambda = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$, $\lambda_i > 0$

$\ddot{q} = -M^{-1}\Lambda M q$, $M\ddot{q} = -\Lambda M q$, 令 $Mq = R$, $\ddot{R} + \Lambda R = 0$

$\Rightarrow \ddot{r}_i + \lambda_i r_i = 0 \Rightarrow r_i = \alpha_i \cos(\sqrt{\lambda_i}\, t) + \beta_i \sin(\sqrt{\lambda_i}\, t)$

另法:

$L = T - U = \frac{1}{2}\langle A\dot{q}, q\rangle - \frac{1}{2}\langle Cq, q\rangle$

$C = D^TD \Rightarrow \langle Cq, q\rangle = \langle D^TDq, q\rangle = \langle Dq, Dq\rangle \overset{Dq=Q}{=\!=} \langle Q, Q\rangle$

$q = D^{-1}Q,\ \dot{q} = D^{-1}\dot{Q}$

$T = \frac{1}{2}\langle A\dot{q}, \dot{q}\rangle = \frac{1}{2}\langle AD^{-1}\dot{Q}, D^{-1}\dot{Q}\rangle$

$A = M^TM$ (注意跟上面记号不同) $\Rightarrow T = \frac{1}{2}\langle M^TMD^{-1}\dot{Q}, D^{-1}\dot{Q}\rangle = \frac{1}{2}\langle \underbrace{(MD^{-1})^T(MD^{-1})}_{\text{positive definite}}\dot{Q}, \dot{Q}\rangle$

$(MD^{-1})^T(MD^{-1}) = O^T\Lambda O$, where O orthogonal. $\Rightarrow T = \frac{1}{2}\langle O^T\Lambda O\dot{Q}, \dot{Q}\rangle$

$\Lambda > 0$ diagonal

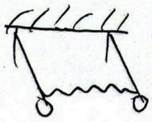
$= \frac{1}{2}\langle \Lambda O\dot{Q}, O\dot{Q}\rangle$

$OQ = \tilde{Q}$ $\quad = \frac{1}{2}\langle \Lambda\dot{\tilde{Q}}, \dot{\tilde{Q}}\rangle$

$= \frac{1}{2}\sum \lambda_i \dot{\tilde{Q}}_i^2$

<u>Thm.</u> Suppose $(p, q) = (0, 0)$ is an equilibrium of the above system where $q = 0$ is a local minimum of U. Then the linearized system can be decomposed into several 1-dim harmonic oscillators.

Example. p105-106.



跳过 Chap6, 先讲 Chap7 微分形式

We call the space of 1-forms on $\mathbb{R}^n$ the dual space of $\mathbb{R}^n$, denoted by $(\mathbb{R}^n)^*$.

$(\mathbb{R}^n)^*$ is a linear space of dimension n.

Let $x_1, \dots, x_n$ be a basis on $(\mathbb{R}^n)^*$. $\xi \in \mathbb{R}^n$ can be written as $(\xi_1, \dots, \xi_n)$.

We also take x_i as a basis of $(\mathbb{R}^n)^*$ as follows

$$x_i(\xi) = \xi_i.$$

A general $\omega \in (\mathbb{R}^n)^*$ can be written as $\omega = \sum \omega_i x_i$.

$$\omega(\xi) = \sum \omega_i x_i(\xi_i) = \sum \omega_i \xi_i \quad \text{Inner product.}$$

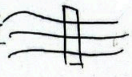
2-forms: A 2-form is a function on a pair of vectors $\omega^2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which is bilinear and skew symmetric (斜对称).

$$\omega^2(\lambda_1 \xi_1 + \lambda_2 \xi_2, \xi_3) = \lambda_1 \omega^2(\xi_1, \xi_3) + \lambda_2 \omega^2(\xi_2, \xi_3).$$

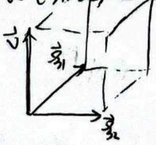
$$\omega^2(\xi_1, \xi_2) = -\omega^2(\xi_2, \xi_1).$$

~~Exa~~ example: ① outer product of vectors

② determinant on $\mathbb{R}^2$: $\xi_1, \xi_2 \in \mathbb{R}^2$, $\omega^2(\xi_1, \xi_2) := \det(\xi_1, \xi_2)$

③ flux 流量 

constant vector field $v \in \mathbb{R}^3$, $\omega^2(\xi_1, \xi_2) = \det(v, \xi_1, \xi_2)$.



平行六面体
flux = 体积
↳ 使之最大

The space of 2-forms is a linear space, denoted by $\Lambda^2 \mathbb{R}^n$.

---

k-forms: A k-form on $\mathbb{R}^n$ is a function of k vectors that is k-linear and anti-symmetric: $\omega^k : (\mathbb{R}^n)^k \to \mathbb{R}$

$$\omega^k(\lambda_1 \xi_1' + \lambda_2 \xi_1'', \xi_2, \dots, \xi_k) = \lambda_1 \omega^k(\xi_1', \xi_2, \dots, \xi_k) + \lambda_2 \omega^k(\xi_1'', \xi_2, \dots, \xi_k).$$

$$\omega^k(\xi_{i_1}, \dots, \xi_{i_k}) = (-1)^{\tau(i_1, \dots, i_k)} \omega^k(\xi_1, \dots, \xi_k)$$

外积

We next introduce the exterior product of forms

$\omega^k \in \Lambda^k \mathbb{R}^n$, $\omega^l \in \Lambda^l \mathbb{R}^n$, we want to define $\omega^k \wedge \omega^l \in \Lambda^{k+l} \mathbb{R}^n$.

$(k+l \leq n)$

We start by the exterior of 2 1-forms:

Let ω_1, ω_2 be 1-forms, we define $\omega_1 \wedge \omega_2(\xi_1, \xi_2) = \begin{vmatrix} \omega_1(\xi_1) & \omega_2(\xi_1) \\ \omega_1(\xi_2) & \omega_2(\xi_2) \end{vmatrix}$

We define a map from $\mathbb{R}^n$ to the ~~plane~~ plane $\mathbb{R} \times \mathbb{R}$ by associating each $\xi \in \mathbb{R}^n$ with the vector $(\omega_1(\xi), \omega_2(\xi))$

With the exterior product, we ~~start~~ start with the basis $x_1, \ldots, x_n$ of $(\mathbb{R}^n)^*$,

we get $x_i \wedge x_j$ $(i<j)$ totally $\binom{n}{2}$ of them. They form a basis of $\Lambda^2(\mathbb{R}^n)$

Suppose we has k 1-forms $\omega_1, \ldots, \omega_k \in (\mathbb{R}^n)^*$, we define $\omega_1 \wedge \cdots \wedge \omega_k$ as a k-form as follows

$$(\omega_1 \wedge \cdots \wedge \omega_k)(\xi_1, \ldots, \xi_k) = \begin{vmatrix} \omega_1(\xi_1) & \omega_2(\xi_1) & \cdots & \omega_k(\xi_1) \\ \omega_1(\xi_2) & \omega_2(\xi_2) & \cdots & \omega_k(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1(\xi_k) & \omega_2(\xi_k) & \cdots & \omega_k(\xi_k) \end{vmatrix}$$

$\Lambda^k \mathbb{R}^n$ has a basis of the form $x_{i_1} \wedge \cdots \wedge x_{i_k}$, $i_l \neq i_j$

$\dim(\Lambda^k \mathbb{R}^n) = \binom{n}{k}$.

We want define $\omega^k \wedge \omega^l \in \Lambda^{k+l}$.

We need $\omega^k \wedge \omega^k$ satisfy : 1. $\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k$

2. $(\lambda_1 \omega_1^k + \lambda_2 \omega_2^k) \wedge \omega^l = \cdots$

Forms on $\mathbb{R}^n$

- 1-form: linear functional on $\mathbb{R}^n$

  The space of 1-forms is denoted by $(\mathbb{R}^n)^* \cong \mathbb{R}^n$
- 2-form: $\omega^2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ bilinear, anti-symmetric

  $\omega^2 : (u, v) = \langle Au, v \rangle, \quad A$ anti-symmetric
- k-form: k-linear, skew-symmetric

Exterior product

1. The exterior product of 2 1-forms $\omega_1, \omega_2 \in \Lambda^1 \mathbb{R}^n$:

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \begin{vmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) \\ \omega_2(\xi_1) & \omega_2(\xi_2) \end{vmatrix} \tag{35}$$

2. The exterior product of k 1-forms $\omega_1, ..., \omega_k \in \Lambda^1 \mathbb{R}^n$

$$(\omega_1 \wedge ... \wedge \omega_k)(\xi_1, ..., \xi_k) = \begin{vmatrix} \omega_1(\xi_1) & ... & \omega_1(\xi_k) \\ \vdots & \ddots & \vdots \\ \omega_k(\xi_1) & ... & \omega_k(\xi_k) \end{vmatrix} = \det\left(\omega_i\left(\xi_j\right)\right) \tag{36}$$

Next, given k-form ω^k and l-form ω^l, we define their exterior product

$$\left(\omega^k \wedge \omega^l\right)(\xi_1, ..., \xi_{k+l}) = \sum_{\text{all permutations}} (-1)^\nu \omega^k\left(\xi_{i_1}, ..., \xi_{i_k}\right)\omega^l\left(\xi_{i_{k+1}}, ..., \xi_{i_{k+l}}\right) \tag{37}$$

properties of exterior product
- skew-symmetry: $\omega^k \wedge \omega^l = (-1)^{kl}\ \omega^l \wedge \omega^k$
- distributivity
- associativity

We shall prove the equivalence of the definition of $\omega^k \wedge \omega^l$ and the exterior product of 1-forms. (omitted)

Remark. $\omega^1 \wedge \omega^1 = 0$, but $\omega^2 \wedge \omega^2$ is in general not. Actually, $\omega^{2k+1} \wedge \omega^{2k+1} = 0$.

---

pull-back 拉回 (behavior under mappings)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, ω^k be a k-form on $\mathbb{R}^m$. Then

$$\left(f^* \omega^k\right)(\xi_1, ..., \xi_k) := \omega^k(f(\xi_1), ..., f(\xi_k)) \tag{38}$$

is a k-form on $\mathbb{R}^n$

Differential forms

We generalize the notion of forms from $\mathbb{R}^n$ to a manifold M

Let $f : M \to \mathbb{R}$ be a differentiable function, recall the differential $\mathrm{d}f(x)$ is a linear functional $T_x M \to \mathbb{R}$ given by:

Given $\xi \in T_x M$, let a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\gamma(0) = x, \gamma'(0) = \xi$, then

$$\mathrm{d}_x f(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma(t)) = \langle \mathrm{d}_x f, \gamma'(0) \rangle = \langle \mathrm{d}_x f, \xi \rangle$$

Def. A differential 1-form on M is a smooth map $\omega : TM \to \mathbb{R}$ linear on each $T_x M$.

Thm. Give $\mathbb{R}^n$ coordinates $x_1, ..., x_n$. Then *every* differential 1-form $\omega : T\mathbb{R}^n \to \mathbb{R}$ can be written as

$$\omega = a_1(x_1, ..., x_n)\,\mathrm{d}x_1 + ... + a_n(x_1, ..., x_n)\,\mathrm{d}x_n, \tag{39}$$

where $\mathrm{d}x_i(\xi) = \xi_i,\ \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$, and $a_i(x)$ are smooth functions.

Def. A differential k-form $\omega^k|_x$ at a point $x \in M$ is a k-form on the tangent space $T_x M$, i.e. a k-linear, skew-symmetric function of k vectors $\xi_1, ..., \xi_k \in T_x M$.

Thm. Give $\mathbb{R}^n$ coordinates $x_1, ..., x_n$. Then *every* differential k-form ω^k on $T\mathbb{R}^n$ has the form

$$\omega^k = \sum_{i_1 < ... < i_k} a_{i_1, ..., i_k}(x_1, ..., x_n)\,\mathrm{d}x_{i_1} \wedge ... \wedge \mathrm{d}x_{i_k} \tag{40}$$

Integration of differential forms

We shall prove a general Stokes formula

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega \tag{41}$$

or written as $\langle \omega, \partial\Omega \rangle = \langle\, \mathrm{d}\omega, \Omega \rangle$, i.e. ∂ and d are dual.

---

1. The integration of a differential 1-form along a curve

   Let $\gamma : [0, 1] \to M$ be a smooth curve and ω^1 a 1-form. We define

$$\int_\gamma \omega^1 = \lim_{\Delta \to 0} \sum_i \omega^1(\xi_i) \tag{42}$$

There, we partition $[0, 1]$ into intervals $\Delta_i = [t_i, t_{i+1}]$, $\xi_i = \mathrm{d}\gamma|_{t_i}(\Delta_i) \in T_{\gamma(t_i)} M$, 这里把 Δ_i 看成向量. 见下图.

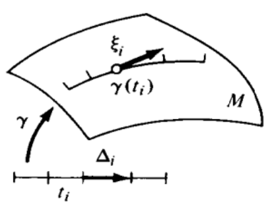


Figure 146 Integrating a 1-form along a path

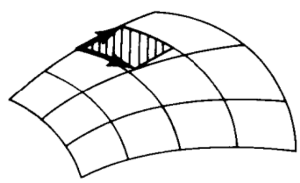


Figure 147 Integrating a 2-form over a surface

2. The integration of a differential k-form along a k-dim surface

   定义与 1-form 类似, 将曲面划分为小块, 每块用平行多面体代替. 见上图.

3. (*special case*) The integration of a differential k-form on oriented $\mathbb{R}^k$

Let $x_1, ..., x_k$ be a oriented coordinate system of $\mathbb{R}^k$.
Then every k-form has the form $\omega^k = a(x_1, ..., x_k)\, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_k$
Let D be a bounded convex polyhedron in $\mathbb{R}^k$. We define

$$\int_D \omega^k = \int_D a(x_1, ..., x_k)\, \mathrm{d}x_1 ... \mathrm{d}x_k \tag{43}$$

其中右式理解为通常的 Riemann 积分.

注. 这一定义与上面的一般定义相容, 因为此时 D 的 tangent space 仍是 D 本身.

上面只能定义 k-form 在 k-dim 曲面上的积分. 为定义一般的 k-form on n-dim space 的积分, 自然地, 通过拉回来定义.

The behavior of differential forms under mappings

Let $f : M \to N$ be a differentiable map between manifolds. Let ω be a differential k-form on N. Then there is a well-defined differential k-form on M:

$$(f^* \omega)(\xi_1, ..., \xi_k) \coloneqq \omega(f_*(\xi_1), ..., f_*(\xi_k)), \quad \forall \xi_1, ..., \xi_k \in T_x M \tag{44}$$

其中 f_* 是 f 的微分.

Integration of a k-form on n-dim manifolds

Def. A k-dim *cell* σ on M is a triple $\sigma = (D, f, Or)$, where
- D: a convex polyhedron in $\mathbb{R}^k$
- $f : D \to M$ a differentiable map
- Or: an orientation on $\mathbb{R}^k$

Def. The integration of a k-form ω^k on a cell σ is defined as

$$\int_\sigma \omega^k = \int_D f^* \omega^k \tag{45}$$

Def. A *chain* (链) of dim k on a manifold M consists of a finite collection of k-dim cells $\sigma_1, ..., \sigma_r$ and integers $m_1, ..., m_r$ called *multiplicities*. Denote it by $C_k = m_1 \sigma_1 + ... + m_r \sigma_r$.

$$\int_{C_k} = \sum m_i \int_{\sigma_i} \omega \tag{46}$$

Def. Let $\sigma = (D, f, Or)$ be a cell of dim k, we define its *boundary* $\partial\sigma$ to be a collection of cells of dim $k - 1$,

$$\partial\sigma = \sum \sigma_i, \quad \sigma_i = (D_i, f_i, Or_i), \tag{47}$$

where D_i are faces of D, $f_i = f|_{D_i}$.

Let $e_1, ..., e_k$ be a basis of $\mathbb{R}^k$. At each point of D_i, we choose an outer normal (外法向). An orientation on D_i is a choice of basis $f_1, ..., f_{k-1}$, we require $(n, f_1, ..., f_{k-1})$ to have the same orientation as $(e_1, ..., e_k)$.

Exterior differentiation

To see the divergence we assume the domain Ω is very small.

Suppose Ω is the parallelepiped (平行六面体) spanned by $\varepsilon\xi_1, \varepsilon\xi_2, \varepsilon\xi_3$

The *divergence* is obtained as the limit

$$\lim \frac{\int_{\partial\Omega_\varepsilon} \omega^2}{\varepsilon^3 V}, \quad V = \det(\xi_1, \xi_2, \xi_3) \tag{48}$$

Def. (exterior derivative) (*omitted, see page 189*)

Thm. Let ω^k be a k-form, then there exists a unique $(k+1)$-form Ω on TM, given as

$$F(\varepsilon\xi_1, ..., \varepsilon\xi_{k+1}) = \varepsilon^{k+1}\Omega(\xi_1, ..., \xi_{k+1}) + o(\varepsilon^{k+1}) \quad (\varepsilon \to 0) \tag{49}$$

where $F = \int_\sigma \omega^k$, σ be the boundary of the parallelepiped spanned by $(\varepsilon\xi_1, ..., \varepsilon\xi_{k+1})$.

Moreover, if $\omega^k = \sum a_{i_1,...,i_k} \,\mathrm{d}x_{i_1} \wedge ... \wedge \mathrm{d}x_{i_k}$, then

$$\Omega = \mathrm{d}\omega^k = \sum \mathrm{d}a_{i_1,...,i_k} \wedge \mathrm{d}x_{i_1} \wedge ... \wedge \mathrm{d}x_{i_k} \tag{50}$$

注. 这基本给出了 *stokes* 公式.

Example. $\omega^2 = A\,\mathrm{d}y \wedge \mathrm{d}z + B\,\mathrm{d}z \wedge \mathrm{d}x + C\,\mathrm{d}x \wedge \mathrm{d}y$, then $\Omega = \mathrm{d}\omega^2 = \operatorname{div}(A, B, C)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Proof. 只证最简单的 1-form $\omega^1 = a(x_1, x_2)\,\mathrm{d}x_1$.

Given ξ, η (small enough) shown as below, we calculate $F(\xi, \eta) = \int_\sigma \omega^1$, where σ is the boundary of the parallelogram spanned by η, ξ.

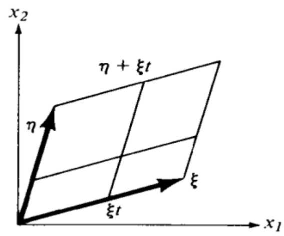


Figure 158 Theorem on exterior derivatives

$$\int_\sigma \omega^1 = \int_0^1 (a(\xi t) - a(\xi t + \eta))\xi_1 - (a(\eta t) - a(\eta t + \xi))\eta_1 \,\mathrm{d}t \qquad (\xi_i = \mathrm{d}x_i(\xi), \eta_i = \mathrm{d}x_i(\eta))$$

$$= \int_0^1 -\left(\cancel{\frac{\partial a}{\partial x_1}\eta_1} + \frac{\partial a}{\partial x_2}\eta_2\right)\xi_1 + \left(\cancel{\frac{\partial a}{\partial x_1}\xi_1} + \frac{\partial a}{\partial x_2}\xi_2\right)\eta_1 \,\mathrm{d}t + o(\|\xi\|^2 + \|\eta\|^2)$$

$$\text{(derivative taken at } x_1 = x_2 = 0)$$

$$= \frac{\partial a}{\partial x_2}(\eta_2\xi_1 - \xi_1\eta_2) + o(\|\xi\|^2 + \|\eta\|^2)$$

$\Longrightarrow \Omega(\xi, \eta) = \frac{\partial a}{\partial x_2}(\eta_2\xi_1 - \xi_1\eta_2) \Longrightarrow \Omega = \frac{\partial a}{\partial x_2}\,\mathrm{d}x_2 \wedge \mathrm{d}x_1 = \mathrm{d}\omega^1$.

Thm. (Stokes formula) Let ω be a k-form, Ω be a chain of dim $k+1$. Then

$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega \tag{51}$$

Proof. See page 192, using the thm above.

Def. We say a differential form ω is *closed* if $\mathrm{d}\omega = 0$.

A closed form, when integrated on a boundary, is always 0.

Def. A differential k-form ω is called *exact* if there exists a $(k-1)$-form α s.t. $\mathrm{d}\alpha = \omega$.

For any differential form, we have $\mathrm{d}^2\omega = 0 \implies$ an exact form is closed.

Dually, for any chain Ω, $\partial^2\Omega = 0$.

A *symplectic manifold* (M^{2n}, ω^2) is an even dimensional manifold M^{2n} endowed with a closed nondegenerate differential 2-form ω^2.

nondegenerate: $\forall \xi, \eta \in T_x M, \omega^2|_x \ (\xi, \eta) = \langle A\xi, \eta \rangle$, where A is nondegenerate and anti-symmetric.

Example. $\mathbb{R}^{2n}, \quad x_1, ..., x_n, y_1, ..., y_n, \quad \omega^2 = \mathrm{d}y_1 \wedge \mathrm{d}x_1 + ... + \mathrm{d}y_n \wedge \mathrm{d}x_n$

The most important symplectic manifold for us is $T^* M$ (*cotangent bundle*, 余切丛),

$$T^* M = \bigcup_{x \in M} T_x^* M \tag{52}$$

where $T_x^* M$ is the *cotangent space* of M at x, i.e. the space of 1-forms on $T_x M$.

The Lagrangian mechanics happens on TM,

$$L : TM \to \mathbb{R} \tag{53}$$

The Hamiltonian mechanics happens on $T^* M$,

$$H : T^* M \to \mathbb{R} \tag{54}$$

$H(p, q) = \sup\limits_{\dot{q}} (\langle p, \dot{q} \rangle - L(\dot{q}, q)), \quad p \in T_q M$

The generalized momentum p is a linear functional on $T_q M$.

On $T^* M$, there is *a natual symplectic form*

$$\omega^2 = \mathrm{d}p \wedge \mathrm{d}q = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i \tag{55}$$

$\omega^2 = \mathrm{d}\omega^1$, where $\omega = p \, \mathrm{d}q$ is a 1-form on $T^* M$.

Let q be coordinates on M, p be coordinates on $T_q^* M$.

Let $N = T^* M$, $\omega = p \, \mathrm{d}q$ is a 1-form on N. At point $x = (p, q) \in N$,

$$\begin{aligned} \omega|_x : T_x N &\to \mathbb{R} \\ (p \, \mathrm{d}q)\xi &= p \cdot \xi_q \end{aligned} \tag{56}$$

We next introduce the Hamiltonian vector field.

Def. To every vector $\xi \in T_x M$, where (M, ω^2) is a symplectic manifold. We can associate a 1-form

$$\omega_\xi^1(\eta) = \omega^2(\eta, \xi), \forall \eta \in T_x M \tag{57}$$

This induces an isomorphism between $T_x M$ and $T_x^* M$, $\xi \mapsto \omega_\xi^1$

We denote this isomorphism as I:

$$\begin{aligned} I : T_x^* M &\to T_x M \\ \omega_\xi^1 = \omega^2(\cdot, \xi) &\mapsto \xi \end{aligned} \tag{58}$$

By the nondegeneracy, $\omega^2(\eta, \xi) = \langle A\eta, \xi \rangle = \langle \eta, A^T \xi \rangle$, thus ω_ξ^1 is exactly the vector $A^T \xi$.

Let $H : M \to \mathbb{R}$ be a function, then $\mathrm{d}H$ is a 1-form on T^*M. By the isomorphism, there exists a vector field on M denoted by $X_H = I\,\mathrm{d}H$, called the Hamiltonian vector field.

但反过来, 每个 M 上的 *vector field* 不一定是一个函数的微分. 如果是, 我们定义:

Def. We say a vector field X_H a *Hamiltonian vector field*, if there exists H s.t. $\omega(\cdot, X_H) = \mathrm{d}H$.

Example. $\mathbb{R}^2, \quad (y, x), \quad \omega = \mathrm{d}y \wedge \mathrm{d}x, \quad H : \mathbb{R}^2 \to \mathbb{R}$ a Hamiltonian function

设 H 对应的 Hamiltonian vector field 为 X, 则由定义有 $\omega(\xi, X) = \mathrm{d}H(\xi)$

$\Longrightarrow \begin{vmatrix} \xi_y & X_y \\ \xi_x & X_x \end{vmatrix} = \frac{\partial H}{\partial y}\xi_y + \frac{\partial H}{\partial x}\xi_x \Longrightarrow X = (X_y, X_x) = \left(-\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right)$

Given Hamiltonian vector field X_H, we define the *Hamiltonian flow* by solving the ODE

$$\dot{x} = X_H(x), \quad x \in N = T^*M \tag{59}$$

denoted by

$$\begin{aligned} g^t : N &\to N \\ x &\mapsto g^t x \end{aligned} \tag{60}$$

Example. 上面那个例题中, H 导出的 Hamiltonian flow 便由

$$(\dot{y}, \dot{x}) = \left(-\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right)$$

给出, 这即是 Hamiltonian canonical equation.

Thm. A Hamiltonian flow preserves the symplectic structure

$$\left(g^t\right)^* \omega = \omega \tag{61}$$

Proof. See page 204.

Law of energy conservation

Thm. Let $H : M \to \mathbb{R}$ be a Hamiltonian function, then it is constant under the corresponding Hamiltonian flow (with Hamiltonian function H).

Proof. $\frac{\mathrm{d}}{\mathrm{d}t} H(g^t x) = \mathrm{d}H(X_H) = \omega(X_H, X_H) = 0$.

The algebraic structure of Hamiltonian mechanics

The commutator of vector fields

Let $A(z)$ be a vector field on $\mathbb{R}^n$, we have the ODE $\dot{z} = A(z)$.

Let a^t be the flow of the ODE.

Def. Given a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the Lie derivative of φ along the vector field A is defined as

$$(\mathcal{L}_A \varphi)(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(a^t z) = \mathrm{d}\varphi \cdot A = \sum A_i \frac{\partial}{\partial z_i} \varphi \tag{62}$$

We denote

$$\mathcal{L}(A) \coloneqq A_1(z)\frac{\partial}{\partial z_1} + \ldots + A_n(z)\frac{\partial}{\partial z_n} \tag{63}$$

Lemma. The operator $\mathcal{L}_B\mathcal{L}_A - \mathcal{L}_A\mathcal{L}_B$ is a <u>first order</u> linear operator.

Proof.

$$\begin{cases} \mathcal{L}_B\mathcal{L}_A\varphi = \sum_i B_i\frac{\partial}{\partial z_i}\left(\sum_j A_j\frac{\partial}{\partial z_j}\varphi\right) = \sum_{i,j}\left(B_i\frac{\partial A_j}{\partial z_i}\frac{\partial \varphi}{\partial z_j} + \cancel{B_i A_j\frac{\partial^2\varphi}{\partial z_i\partial z_j}\varphi}\right) \\ \mathcal{L}_A\mathcal{L}_B\varphi = \sum_i A_i\frac{\partial}{\partial z_i}\left(\sum_j B_j\frac{\partial}{\partial z_j}\varphi\right) = \sum_{i,j}\left(A_i\frac{\partial B_j}{\partial z_i}\frac{\partial \varphi}{\partial z_j} + \cancel{A_i B_j\frac{\partial^2\varphi}{\partial z_i\partial z_j}\varphi}\right) \end{cases} \text{(指相减时消掉)}$$

$\Longrightarrow (\mathcal{L}_B\mathcal{L}_A - \mathcal{L}_A\mathcal{L}_B)\varphi = \sum_{i,j}\left(B_i\frac{\partial A_j}{\partial z_i} - A_i\frac{\partial B_j}{\partial z_i}\right)\frac{\partial}{\partial z_j}\varphi \eqqcolon \sum_{i,j}[A,B]_j\,\frac{\partial}{\partial z_j}\varphi = [A,B]\cdot\dot{\varphi}$

We denote

$$[A,B]_j \coloneqq \sum_i\left(B_i\frac{\partial A_j}{\partial z_i} - A_i\frac{\partial B_j}{\partial z_i}\right) \tag{64}$$

Def. The *commutator* of two vector field A and B is the vector field C for which

$$\mathcal{L}_C = \mathcal{L}_B\mathcal{L}_A - \mathcal{L}_A\mathcal{L}_B \tag{65}$$

denoted by

$$C = [A,B] \tag{66}$$

Prop. For all smooth vector fields A, B, C, we have

(1) linearity: $[aA + bB, C] = a[A,C] + b[B,C]$
(2) anti-symmetry: $[A,B] = -[B,A]$
(3) *Jacobi identity:* $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$

These implies the space of C^∞ vector fields forms a *Lie algebra.*

Thm. Let a^t, b^t be the flows generated by vector fields A, B respectively. Then the flows *commute* iff the commutator of the vector field vanish, i.e.

$$a^t \circ b^s = b^s \circ a^t \Longleftrightarrow [A,B] = 0 \tag{67}$$

Proof. We first have $\left.\frac{\partial^2}{\partial s\partial t}\right|_{t=s=0}(\varphi(a^t b^s z) - \varphi(b^s a^t z)) = (\mathcal{L}_B\mathcal{L}_A - \mathcal{L}_A\mathcal{L}_B)\varphi = [A,B]\cdot\dot{\varphi}$

This proves "$\Longrightarrow$".

For the "$\Longleftarrow$" part, if $[A,B] = 0$, then $\varphi(a^t b^s z) - \varphi(b^s a^t z) = o(s^2 + t^2) \quad (s, t \to 0)$

We partition the ractangle $[0,t) \times [0,s)$ into N^2 equal small ractangles.

We deform the path of $b^s a^t$ into $a^t b^s$ by N^2 steps. The total error is $o(1)$.

Letting $N \to \infty$, we get $\varphi(a^t b^s z) = \varphi(b^s a^t z), \forall\varphi$.

Poisson bracket

Def. Let F, H be two functions on a symplectic manifold (M, ω), the *Poisson bracket* $\{F, H\}$ is defined as the derivative of F along the Hamiltonian flow of H,

$$\{F,H\}(z)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}F(g_H^t z)=\mathrm{d}F(X_H)=\underbrace{\omega(X_H,X_F)}_{\text{注意}F\text{和}H\text{的位置是反的}} \tag{68}$$

Since ω is anti-symmetric, $\{F,H\}=-\{H,F\}$.

Example. $\mathbb{R}^2_{(x,y)}$, $\quad \omega=\mathrm{d}y\wedge\mathrm{d}x$

$\begin{cases}\dot{x}=\frac{\partial F}{\partial y}\\ \dot{y}=-\frac{\partial F}{\partial x}\end{cases}$ $\quad$ $\begin{cases}\dot{x}=\frac{\partial H}{\partial y}\\ \dot{y}=-\frac{\partial H}{\partial x}\end{cases}$

$\Longrightarrow \{F,H\}(z)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}F(g_H^t z)=\frac{\partial F}{\partial x}\dot{x}+\frac{\partial F}{\partial y}\dot{y}=\frac{\partial F}{\partial x}\frac{\partial H}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial H}{\partial x}$.

Prop. The Poisson bracket has the following properties: $\forall H_1,H_2,H_3\in C^\infty(M)$, we have

(1) linearity
(2) anti-symmetry
(3) *Jacobi identity:* $\{\{H_1,H_2\},H_3\}+\{\{H_2,H_3\},H_1\}+\{\{H_3,H_1\},H_2\}=0$

Thus, the space of $C^\infty(M)$ also forms a Lie algebra under the Poisson bracket.

Thm. Let B,Γ be two Hamiltonian vector field with Hamiltonians β,γ. Then $[\mathrm{B},\Gamma]$ is a Hamiltonian vector field whose Hamiltonian is exactly $\{\beta,\gamma\}$.

Proof. Let $\{\beta,\gamma\}=\delta$.

$\forall\alpha$, by Jacobi identity, we have $\{\alpha,\delta\}=\{\alpha,\{\beta,\gamma\}\}=\{\{\alpha,\beta\},\gamma\}-\{\{\alpha,\gamma\},\beta\}$.

$\{\{\alpha,\beta\},\gamma\}=\mathcal{L}_\Gamma\{\alpha,\beta\}=\mathcal{L}_\Gamma\mathcal{L}_\mathrm{B}\alpha,\ \ \{\{\alpha,\gamma\},\beta\}=\mathcal{L}_\mathrm{B}\{\alpha,\gamma\}=\mathcal{L}_\mathrm{B}\mathcal{L}_\Gamma\alpha$

Let Δ be the Hamiltonian vector field of δ, then $\{\alpha,\delta\}=\mathcal{L}_\Delta\alpha$

$\Longrightarrow \mathcal{L}_\Delta=\mathcal{L}_\Gamma\mathcal{L}_\mathrm{B}-\mathcal{L}_\mathrm{B}\mathcal{L}_\Gamma=\mathcal{L}_{[\mathrm{B},\Gamma]}$

Symplectic transformation

Example. $H:\mathbb{R}^2\to\mathbb{R}$, $H(y,x)=\frac{1}{2}(y^2+x^2)$, Hamiltonian equation $\begin{cases}\dot{y}=-x\\ \dot{x}=y\end{cases}$

Let $\begin{cases}x=r\sin\theta\\ y=r\cos\theta\end{cases}$ be the polar coordinates.

$\Longrightarrow \begin{cases}\dot{r}=\ldots=0\\ \dot{\theta}=\ldots=1\end{cases}$

$H(r,\theta)=\frac{1}{2}r^2$, if Hamiltonian equation holds, $\begin{cases}\dot{r}=-\frac{\partial H}{\partial\theta}=0\\ \dot{\theta}=\frac{\partial H}{\partial r}=r\end{cases}$, contradicts!

问题在于这一变换并不是辛变换!

Instead of (r,θ), we use (I,θ), where $I=\frac{r^2}{2}$. Then $H=I$, $\begin{cases}\dot{I}=-\frac{\partial H}{\partial\theta}=0\\ \dot{\theta}=\frac{\partial H}{\partial I}=1\end{cases}$ agrees.

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Let $H(Y,X)$ be a Hamiltonian on $\mathbb{R}^{2n}$

Suppose we have a coordinate change

$$(y,x) \longmapsto (Y,X) = (Y(y,x),\ X(y,x))$$

$K(y,x) := H(Y(y,x),\ X(y,x))$

We hope that the Hamiltonian structure is presened by the transformation.

$$\begin{cases} \dot{Y} = -\dfrac{\partial H}{\partial X} \\ \dot{X} = \dfrac{\partial H}{\partial Y} \end{cases} \qquad \begin{cases} \dot{y} = -\dfrac{\partial K}{\partial x} \\ \dot{x} = \dfrac{\partial K}{\partial y} \end{cases} \qquad \begin{pmatrix} \dot{Y} \\ \dot{X} \end{pmatrix} = \frac{D(Y,X)}{D(y,x)} \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{\partial Y}{\partial y} & \dfrac{\partial Y}{\partial x} \end{pmatrix} \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix}$$

$$\dot{Y} = \frac{\partial Y}{\partial y}\dot{y} + \frac{\partial Y}{\partial x}\dot{x}.$$

Let $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

$$\begin{pmatrix} -\dfrac{\partial H}{\partial X} \\ \dfrac{\partial H}{\partial Y} \end{pmatrix} = J\left(\left(\frac{\partial H}{\partial Y}, \frac{\partial H}{\partial X}\right)\right)^T = J\left(\left(\frac{\partial K}{\partial y}, \frac{\partial K}{\partial x}\right)\frac{D(y,x)}{D(Y,X)}\right)^T$$

$$\begin{pmatrix} \dot{Y} \\ \dot{X} \end{pmatrix} = \frac{D(Y,X)}{D(y,x)}\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} \qquad = J\left(\frac{D(y,x)}{D(Y,X)}\right)^T \begin{pmatrix} \dfrac{\partial K}{\partial y} \\ \dfrac{\partial K}{\partial x} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \frac{D(y,x)}{D(Y,X)} J \left(\frac{D(y,x)}{D(Y,X)}\right)^T \begin{pmatrix} \dfrac{\partial K}{\partial y} \\ \dfrac{\partial K}{\partial x} \end{pmatrix}$$

We need $= J$ to get $\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial K}{\partial x} \\ -\dfrac{\partial K}{\partial y} \end{pmatrix}$.

Def. Let $U \subset \mathbb{R}^{2n}$ be an open set. A map $\phi : U \to \mathbb{R}^{2n}$ is said symplectic (or canonical) if we have

$$(D\phi)\, J\, (D\phi)^T = J.$$

Let $\omega = \sum dy_i \wedge dx_i$ be the standard symplectic form on $\mathbb{R}^{2n}$. We consider the set of linear transformations on $\mathbb{R}^{2n}$ ~~pre~~ preserving ω. $((\phi^*\omega = \omega))$

M : linear transformation on $\mathbb{R}^{2n}$. $\quad M^*\omega = \omega$.

$$\omega(Mu, Mv) = \omega(u,v) \qquad M : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} Mu \\ Mv \end{pmatrix}$$

$$\left(\ \omega(u,v) = \langle Ju, v\rangle.\ \right)$$

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$\Rightarrow$ we ~~have~~ hope $\quad \langle JMu, Mv\rangle = \langle Ju, v\rangle \qquad \Rightarrow M^TJM = J$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad \langle M^TJMu, v\rangle$

We denote

$$Symp(2n) = \{ M \in \mathbb{R}^{2n\times 2n} \mid M^TJM = J \}$$

Called the linear symplectic group.

(By anology, in the Euclidean space $\mathbb{R}^n$, with the inner product $\langle \cdot, \cdot \rangle$,

$$O(n) = \{ M \in \mathbb{R}^{n\times n} \mid M^TM = I \}. \qquad \langle Mu, Mv\rangle = \langle u, v\rangle \;)$$

Thm. Let M be a symplectic matrix and λ an eigenvalue of M.

Then $\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}}$ are all eigenvalue of M.

$\det J = 1$

proof. We need to use the fact $\det M = 1$

$J^{-1} = -J$

$M = -J (M^T)^{-1} J$

$$\det(M - \lambda I) = \det(-J(M^T)^{-1}J - \lambda I) \overset{}{=} \det\big(J(-(M^T)^{-1} + \lambda I)J\big) = \det(-(M^T)^{-1} + \lambda I)$$

$$= \underbrace{\det (M^T)^{-1}}_{=1} \cdot \det(\lambda M^T - I) = \det(\lambda M^T - I) \overset{\text{取转置}}{=} \det(\lambda M - I)$$

$$= \lambda^{2n} \det(M - \tfrac{1}{\lambda} I)$$

$\Rightarrow$ if λ is an eigenvalue of M, then $\frac{1}{\lambda}$ is also an eigenvalue

$$\big(p(\lambda) = \lambda^{2n} p(\tfrac{1}{\lambda}) \big).$$

p is reflective

On the other hand, $p(\lambda)$ is real $\Rightarrow \bar{\lambda}$ also eigenvalue

$\Rightarrow$ eigenvalues come in 4-tuples $(\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}})$

2

$\omega(u,v) = \langle Ju, v\rangle$

<u>Def</u>.. We say $u \perp v$ (u skew orthogonal to v) iff $\omega(u,v) = 0$

- We say a linear subspace $P \perp P$ iff $\omega(u,v) = 0$, $\forall u, v \in P$.
- A linear subspace $P \perp P$ and $\dim P = n$ in $\mathbb{R}^{2n}$ is called <u>Lagrangian</u>

<u>example</u>. In $\mathbb{R}^{2n}$, we have $2n$ basis vectors labeled by e_{y_i}, e_{x_i}

$n=2$, in $\mathbb{R}^4$. $\quad e_{y_1} \ e_{y_2}$, $\quad e_{x_1} \ e_{x_2}$. $\qquad \omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$

$\cancel{\begin{vmatrix} y_1 & \tilde{y}_1 \\ x_1 & \tilde{x}_1 \end{vmatrix} = y_1 \tilde{x}_1 - \tilde{y}_1 x_1}$

The following are Lagrangian subspaces: $\operatorname{span}\{e_{y_1}, e_{y_2}\}$ $\quad \operatorname{span}\{e_{y_1}, e_{x_2}\}$

$\operatorname{span}\{e_{x_1}, e_{x_2}\}$ $\quad \operatorname{span}\{e_{y_2}, e_{x_1}\}$.

However, $\operatorname{span}\{e_{x_1}, e_{y_1}\}$; $\operatorname{span}\{e_{x_2}, e_{y_2}\}$ are not Lagrangian. ✓

Lagrangian $L : TM \to \mathbb{R}$

$\updownarrow$ Legendre transformation

Hamiltonian $H : T^*M \to \mathbb{R}$

$\circlearrowright$ symplectic transformation.

Hamilton – Jacobi PDE.

Chap 9.

<u>Lemma</u>. Let ω be a 2-form on an odd dimensional vector space $\mathbb{R}^{2n+1}$.

Then $\exists \xi \in \mathbb{R}^{2n+1}$, $\xi \neq 0$ s.t. $\omega(\xi, \eta) = 0$, $\forall \eta \in \mathbb{R}^{2n+1}$.

<u>proof</u>. $\omega(\xi, \eta) = \langle A\xi, \eta\rangle$. $\quad A$ skew-symmetric with order $2n+1$

$\det A = \det A^T = \det(-A) = (-1)^{2n+1}\det A = -\det A \quad \Rightarrow \det A = 0$

$\Rightarrow A$ has a zero eigenvalue.

Let ξ be the eigenvector with eigenvalue 0, $(\neq 0)$ ✓

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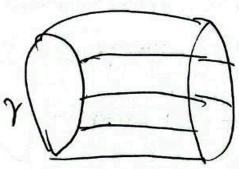
Let ω^1 be a 1-form, $\omega^2 = d\omega^1$ is a 2-form on a manifold M^{2n+1}.

We further assume ω^2 has a unique 0 eigenvalue everywhere

This determines uniquely a direction ξ_x at every x, as the ~~eigenvalue of~~ eigenvector of $d\omega^1|_x$ of eigenvalue 0.

The integral ~~lines~~ curves of this direction field is called vortex lines (涡线).

Let γ be a closed curve.



The vortex lines issued from points on γ form a vortex tube.

Thm. For any two closed curves γ_1, γ_2 encircling the same vortex tube, we have

$$\int_{\gamma_1} \omega^1 = \int_{\gamma_2} \omega^1 .$$

proof. Let σ be the part on the vortex tube with boundary $\partial\sigma = \gamma_2 - \gamma_1$

$$\Rightarrow \int_{\gamma_2} \omega^1 - \int_{\gamma_1} \omega^1 = \int_{\partial\sigma} \omega^1 = \int_\sigma d\omega^1 = 0$$

Let $\omega^1 = y\,dx - H\,dt$ be a 1-form defined on $\underset{(y,x)}{T^*M^n} \times \mathbb{R}$,

called Poincaré – Cartan 1-form

$$\left(\begin{array}{ll} H: T^*M^n \times \mathbb{R} \to \mathbb{R} & x \in M \\ H(y, x, t) & y \in T_x^* M \end{array} \right).$$

Thm. The vortex lines of the Poincaré – Cartan 1-form are given by solutions to the Hamiltonian equations
$$\begin{cases} \dot{y} = -\dfrac{\partial H}{\partial x} \\ \dot{x} = \dfrac{\partial H}{\partial y} \end{cases}$$

Proof. $d\omega' = dy\wedge dx - \underline{\frac{\partial H}{\partial y}\, dy\wedge dt - \frac{\partial H}{\partial x}\, dx\wedge dt}$

The 2-form $d\omega'$ has a matrix representation

$$\begin{array}{cccc} y & x & t & \\ \left(\begin{array}{cc|c} 0 & -I & \frac{\partial H}{\partial y} \\ I & 0 & \frac{\partial H}{\partial x} \\ \hline -\frac{\partial H}{\partial y} & -\frac{\partial H}{\partial x} & 0 \end{array}\right) & & & \begin{array}{c} y \\ x \\ t \end{array} \end{array}$$

The eigenvector associated to the zero eigenvalue is

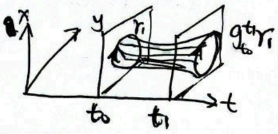
$$\begin{array}{ccc} \dot y & \dot x & \dot t \\ \Big(-\frac{\partial H}{\partial x}, & \frac{\partial H}{\partial y}, & 1\Big)^T \end{array} \rightarrow \text{vortex lines}$$

which is also exactly the Velocity vector!

Thm. Let γ_1, γ_2 be any two closed curves encircling the same vortex tube of $\omega' = ydx - Hdt$. Then $\int_{\gamma_1} \omega' = \int_{\gamma_2} \omega'$

Cor. If we choose γ_1 on the hyperplane $t = t_0$, $\gamma_2 = g^t \gamma_1$. → lies in $t=t_1$.

Then we have $\int_{\gamma_1} ydx = \int_{\gamma_2} ydx$.



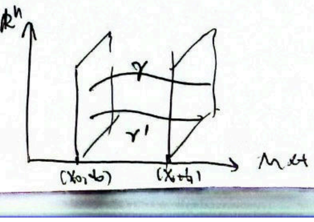
Let σ be a 2-chain with boundary $\partial\sigma = \gamma$, then we got

$$\int_\gamma ydx = \int_{\partial\sigma} ydx = \iint_\sigma dy\wedge dx$$

Cor. $\iint_\sigma dy\wedge dx = \iint_{g^t\sigma} dy\wedge dx$ (i.e. Hamiltonian flow preserves the symplectic form. 这可是记的)

The least action principle in phase space (Hamiltonian version)

Thm. Let γ be a vortex line of the Poincáre-Cartan 1-form. Then the integral $\int_\gamma ydx - Hdt$ has γ as an extremal under variations with endpoints remaining in the subspaces $(t=t_0, x=x_0)$ and $(t=t_1, x=x_1)$.



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proof. $\delta\int_\gamma y\,dx - H\,dt = \delta\int_\gamma (y\cdot\dot{x} - H)\,dt = \int_\gamma \Big((\delta y)\dot{x} + \underbrace{y(\delta\dot{x})}_{\text{分部积分}} - \frac{\partial H}{\partial y}\delta y - \frac{\partial H}{\partial x}\delta x \Big)\,dt$

$$= \underbrace{\cancel{y\delta x\Big|_{t_0}^{t_1}}}_{=0} + \int_\gamma \cancel{\Big((\dot{x} - \frac{\partial H}{\partial y})\delta y - (\dot{y} + \frac{\partial H}{\partial x})\delta x \Big)}\,dt = 0.$$

Thm. If the Hamiltonian H does not depend on time explicitly, then a vortex line γ lying on a level of constant $\{H \equiv E\}$ are extremals of the integral in the class of curves lying on $\{H(y,x) \equiv E\}$ connecting endpoints x_0, x_1.

$$\int_\gamma y\,dx$$

Maupertuis principle

$L = T - U$ $\qquad T = \sum \frac{1}{2} g_{ij}\dot{x}_i\dot{x}_j = \frac{1}{2}\frac{ds^2}{dt^2}$

Riemann matric $ds^2 = \sum g_{ij}\,dx_i\,dx_j$.

For the Hamiltonian system $H = T + U(x)$

Thm In the region of configuration space $\{U(x) < E\}$, we define a Riemannian matric $d\rho^2 = (E - U(x))ds^2$. Then the Hamiltonian orbit of H are geodesics of $d\rho^2$.

proof. $\int_\gamma y\,dx = \int_\gamma \frac{\partial L}{\partial \dot{x}}\dot{x}\,dt = \int 2T\,d\tau = \int \frac{ds^2}{d\tau^2}\,d\tau$

we rescale time s.t. $\left(\frac{ds}{d\tau}\right)^2 = 2(E-U)$ $\qquad = \int 2(E-U)\frac{1}{\sqrt{2(E-U)}}\,ds = \sqrt{2}\int d\rho$.

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Lagragian $L: TM \to \mathbb{R}$ $\quad S(\gamma) = \int_\gamma L\,dt$

$\updownarrow$ Legendre transformation

Hamiltonian $H: T^*M \to \mathbb{R}$ $\quad$ Symplectic ω

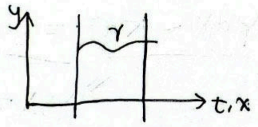
$S(\gamma) = \int_\gamma y\,dx - H\,dt$

Symplectic transformation.

Hamiltono-Jacobi equation $\quad \partial_t S + H(\boxed{\partial_x S}, x) = 0$, $\quad S$: Action.

The least action principle in phase space.

Thm. The Integral $\int_\gamma y\,dx - H\,dt$ has γ as an extremal under variations of γ for which the ends of the curves remains in the n-dim subspace $(t=t_0,\ x=x_0)$ $(t=t_1,\ x=x_1)$



Thm. Suppose that the Hamiltonian does not depend on time explicitly.

Then the trajectories of the system lying on the energy level $\{H(y,x) = h\}$ are extremals of the integral $\int y\,dx$ in the class of curves lying in $\{H(y,x)=h\}$ and connecting $x = x_0$ and $x = x_1$ $\quad$ the integral of "reduced action". $\quad$ H fixed

How to convert between autonomous & nonautonomous systems:

- Let $H(y,x,t)$ be a nonautonomous system.

Let $K(y,x,E,t) = H(y,x,t) + E$, we treat t as ~~a spacial~~ spatial variable and pair it with the dual momentum E.

We treat K as the new Hamiltonian

$$\dot{y} = -\frac{\partial H}{\partial x} = -\frac{\partial K}{\partial x} \qquad \dot{t} = \frac{dt}{dt} = 1 = \frac{\partial K}{\partial E}$$

$$\dot{x} = \frac{\partial H}{\partial y} = \frac{\partial K}{\partial y} \qquad \dot{E} = -\frac{\partial K}{\partial t}$$

1

• 反过来. Let $H(y,x)$ be an autonomous system.

$$\begin{cases} x=(x_1, X), & x_1=-T \\ y=(y_1, Y), & \end{cases}$$

Suppose on the energy level $H(y,x)=h$, we can solve $y_1 = K(Y, X, T)$

$$\left(H(K, Y, -T, X) = h\right)$$

$$\frac{dX}{dT} = -\frac{dX}{dx_1} = -\frac{\frac{dX}{dt}}{\frac{dx_1}{dt}} = -\frac{\frac{\partial H}{\partial y}}{\frac{\partial H}{\partial y_1}} = +\frac{\partial K}{\partial Y} \quad \text{(隐函数定理)}$$

$$\frac{dk}{dY} = \frac{-\frac{\partial H}{\partial Y}}{\frac{\partial H}{\partial K}}$$

$$\frac{dY}{dT} = -\frac{dY}{dx_1} = -\frac{\frac{dY}{dt}}{\frac{dx_1}{dt}} = \frac{\frac{\partial H}{\partial X}}{\frac{\partial H}{\partial y_1}} = -\frac{\partial K}{\partial X}.$$

$$\frac{\partial H}{\partial K}\frac{\partial K}{\partial Y} + \frac{\partial H}{\partial Y} = 0.$$

K is non autonomous

The Poincaré-Cartan form was originally $y\,dx - H\,dt$

For the system K it becomes $Y\,dX - K\,dT = y\,dx$.

$$= Y\,dX + y_1\,dx_1$$

We next consider a mechanical system

$$L(\dot{x}, x) = T - U(x), \quad T = \frac{1}{2}\sum g_{ij}(x)\,\dot{x}_i\,\dot{x}_j$$

The Hamiltonian $H = T + U(x)$

~~We wan~~ On the energy level E, we want to write the orbit as geodesics of certain metric. (Maupertuis metric)

Thm. Let $ds^2 = \sum g_{ij}(x)\,dx_i\,dx_j$. be a Riemanian metric on M.

The trajectories of Hamiltonian system $H(y,x) = T + U(x)$ on energy level E are geodesics of the metric

$$d\rho^2 = (E - U(x))\,ds^2$$

proof. $\int y\,dx = \int \frac{\partial L}{\partial \dot{x}}\dot{x}\,dt = 2\int \sum g_{ij}(x)\,\dot{x}_i\,\dot{x}_j\,dt = 2\int (E-U)\,dt = 2\int (E-U)\frac{1}{\sqrt{2(E-U)}}\,ds$

$ds^2 = 2T\,dt^2 = \left(\sum g_{ij}(x)\,\dot{x}_i\,\dot{x}_j\right)dt^2 \Rightarrow ds = \sqrt{2T}\,dt = \sqrt{2(E-U)}\,dt$

$$= \int \sqrt{2(E-U)}\,ds = \sqrt{2}\int d\rho$$

2

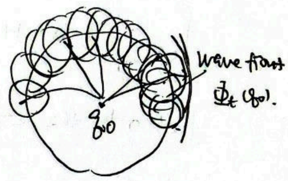
Huygen's principle

Wave front 波前

We consider the set of points q reached by light starting from q_0 within time t.

The boundary of the set is called the wave front, denoted by $\Phi_t(q_0)$

<u>Thm.</u> Let $\Phi_t(q_0)$ be as above. For every point q on $\Phi_t(q_0)$, consider the wave front $\Phi_s(q)$ of time s. Then the wave front $\Phi_{t+s}(q_0)$ is the envelop of $\Phi_s(q)$, $\forall q \in \Phi_t(q_0)$



↳ The envelop is tangent to each wave front $\Phi_s(q)$, $\forall q \in \Phi_t(q_0)$ whenever they intersect.

<u>Def.</u> We define the action as $S_{x_0,t_0}(x,t) = \int_\gamma L\,dt = \int_\gamma y\,dx - H\,dt$.

where the integral is taken along an extremal γ connecting (x_0,t_0) to (x,t)

<u>Thm.</u> The differential of S (with (x_0,t_0) fixed) is

$$dS = y\,dx - H\,dt$$

where $y = \frac{\partial L(\dot{x},x)}{\partial \dot{x}}$, $H = y\cdot\dot{x} - L(\dot{x},x)$ are evaluated at the terminal point of γ

<u>Thm</u> (Hamilton-Jacobi) The action S satisfies

$$\partial_t S + H(\partial_x S, x) = 0.$$

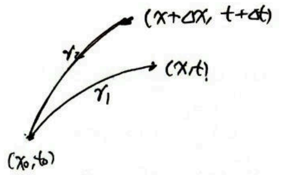
要去找 γ (extremal)

⇒ 得到 S (积分算出) ⇒ S满足HJ

<u>proof.</u> $\partial_t S = -H(y,x) = -H(\partial_x S, x)$.

证明见后

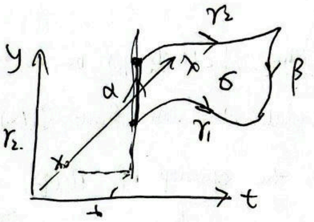
proof.



Let γ_1 be the extremal connecting (x_0, t_0) to (x,t)
γ_2 $(x+\Delta x, t+\Delta t)$.

$$S_{x_0,t_0}(x,t) = \int_{\gamma_1} y\,dx - H\,dt \qquad S_{x_0,t_0}(x+\Delta x, t+\Delta t) = \int_{\gamma_2} y\,dx - H\,dt.$$

$$\int_{\gamma_2-\gamma_1} y\,dx - H\,dt$$



Let σ be the 2-dim cell whose boundary is given by $\alpha, \beta, \gamma_1, \gamma_2$.

$$0 = \int_\sigma d(y\,dx - H\,dt) = \int_{\partial\sigma} y\,dx - H\,dt$$
$$= \int_{\alpha+\beta+\gamma_2-\gamma_1} y\,dx - H\,dt$$

- Along α, we have x_0, t_0 fixed $\Rightarrow dx = 0,\ dt = 0 \Rightarrow \int_\alpha y\,dx - H\,dt = 0$.

$$\Rightarrow \overbrace{\int_{\gamma_2-\gamma_1} y\,dx - H\,dt}^{S_{x_0,t_0}(x+\Delta x, t+\Delta t) - S_{x_0,t_0}(x,t) =} = \int_{-\beta} y\,dx - H\,dt = y\Delta x - H\Delta t + o(\Delta x, \Delta t) \qquad \square$$

Generating function for performing symplectic transformations on $\mathbb{R}^{2n}$.

Suppose $\exists$ symplectic transformation sending (y,x) to (Y,X) $\quad \phi(x,y) = (X,Y)$.

$\phi^*\omega = \omega, \quad (D\phi) J (D\phi)^T = J.$

$\omega = dy \wedge dx = dY \wedge dX. \qquad \Rightarrow dy \wedge dx - dY \wedge dX = 0 \Rightarrow d(y\,dx - Y\,dX) = 0$

$\Rightarrow \exists S \quad$ s.t. $\ y\,dx - Y\,dX = dS$

We can treat S as a function of x and X.

$$\begin{cases} \dfrac{\partial S(x,X)}{\partial x} = y \\[2mm] \dfrac{\partial S(x,X)}{\partial X} = -Y. \end{cases} \qquad (*)$$

Thm. If $\det \dfrac{\partial^2 S}{\partial x \partial X} \neq 0$ at some point (X_0, x_0), then by solving the above equations, we get a map $\phi(x,y) = (X,Y)$ that is symplectic.

proof. Using the assumption, near X_0, x_0 we apply the implicit function theorem 隐函数定理 to write $X(x,y)$, then we substitute it into the second equation, we get $Y(x,y)$.

$\Rightarrow$ We get a map sending (x,y) to (X,Y), denoted by ϕ.

In general, using a function $S(x,X)$, we can introduce a symplectic transformation by $(*)$.

W16

Integral Systems.

Thm. (Liouville - Arnold theorem) Let $H_1 = H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian and suppose that there are $H_1, H_2, \dots, H_n : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying

(1) $\{H_i, H_j\} = 0, \quad \forall i, j \in \{1, \dots, n\}$

(2) The level set $M_a = \{(y, x) \in \mathbb{R}^{2n} \mid H_i(y, x) = a_i, \ i = 1, 2, \dots, n\}$ is compact, $a = (a_1, \dots, a_n)$

(3) At each point of M_a, the n vectors dH_i are linearly independent.

Then:

(1) M_a is diffeomorphic to $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ and is invariant under the Hamiltonian flow of each H_i.
$\{(\varphi_1, \dots, \varphi_n) \bmod 2\pi\}$.

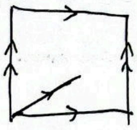
(2) M_a is a Lagrangian submanifold, i.e. $\forall u, v \in T_x M_a$, we have $\omega_x(u, v) = 0$.

(3) In a neighborhood U of M_a, there is a symplectic transformation $\Phi(y, x) = (I, \theta)$ such that $\Phi(U) = (-\delta, \delta)^n \times \mathbb{T}^n$ for some $\delta > 0$.

(4) In the new coordinates, each $K_i = H_i \circ \Phi^{-1}$ is a function of I only, (independent of θ) so that its Hamiltonian equation is

$$\begin{cases} \dot{I} = 0 & \Rightarrow I \text{ const} \\ \dot{\theta} = \dfrac{\partial K_i}{\partial I}(I) \triangleq \omega_i(I). & \Rightarrow \dot{\theta} \text{ const} \Rightarrow \theta \text{ linear} \end{cases}$$

$\mathbb{T}^2$:



The Hamiltonian flow is a linear flow on each torus $\{I\} \times \mathbb{T}^n$:

$$\theta \mapsto \theta + \omega_i(I) t.$$

$I : T_x^* M \to T_x M$.
(I non-singular).

Proof: By assumption, dH_i linearly independent $\Rightarrow$ $I dH_i$ linearly independent.

$[I dH_i, I dH_j]$ is a Hamiltonian vector field with Hamiltonian $\{H_i, H_j\} = 0$

$\Rightarrow [I dH_i, I dH_j] = 0. \Rightarrow$ The Hamiltonian flow $\phi_{H_i}^{t_i}$, $i = 1, \dots, n$ commute.

The derivative of H_i along the flow of H_j is $\{H_i, H_j\} = 0 \Rightarrow$ the flow $\phi_{H_i}^{t_i}$

1

lies entirely in M_a. This implies that IdH_i are tangent to M_a.

We next show M_a is a ~~Lag~~ Lagrangian Submanifold.

· First, we have $\omega(IdH_i, IdH_j) = \{H_j, H_i\} = 0$.

· Since IdH_i are linearly independent, they span $T_x M_a$, $\forall x$.

$\forall u, v \in T_x M_a$ can be written as linear combinations as IdH_i, $i=1,\dots,n$.

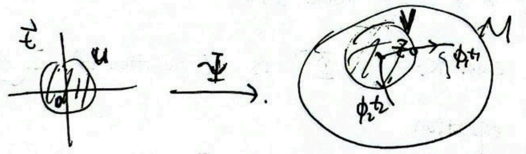
$\rightarrow$ $\omega(u,v)=0$.

Prop. Let M^n be a compact connected n-dim manifold, on which we have a ~~connect~~ commuting vector fields linearly independent at each point. Then M^n is diffeomorphic to $\mathbb{T}^n$.

proof. Denote by $\phi_i^{t_i}$ the flow of the vector fields. $\phi_i^{t_i} \circ \phi_j^{t_j} = \phi_j^{t_j} \circ \phi_i^{t_i}$.

We introduce $\Phi^{\vec{t}} : M \to M$, $\Phi^{\vec{t}} = \prod_{i=1}^{n} \phi_i^{t_i}$, $\vec{t} = (t_1, \dots, t_n)$.

We fix a point $z_0 \in M$ and introduce $\Psi : \mathbb{R}^n \to M : \vec{t} \mapsto \Phi^{\vec{t}}(z_0)$.



Lemma1. Ψ maps a small neighborhood of 0 in $\mathbb{R}^n$ to a small neighborhood V of z_0 in M diffeomorphically.

proof of Lemma1. By implicit function theorem, it's enough to verify $D\Psi$ is nondegenerate. $D\Psi$ is exactly the matrix formed by the vectors given the flow, which are linear independent.

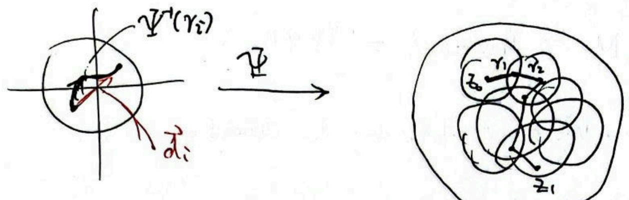
Lemma2. Ψ is onto. (满)

proof of Lemma2. We let z_0 run over M. For each $z_0 \in M$, by lemma1, we get a neighborhood $V(z_0)$ on which Ψ^{-1} maps $V(z_0)$ to $U(z_0) \in \mathbb{R}^n$ diffeomorphically.

Using the compactness of M, we cover M with finitely many of such neighborhoods $V(z_i)$, $i=1,\dots,m$. For any two points $z_0, z_1 \in M$, we connect them by a path $\gamma = \bigcup \gamma_j$, where each γ_i lies entirely in some neighborhood $V(z_j)$.

2

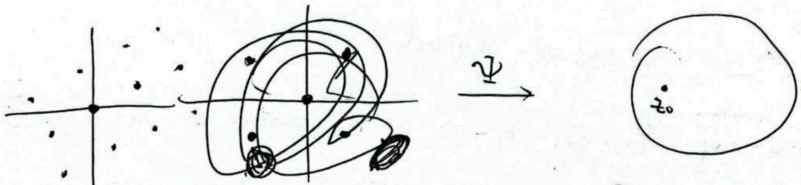
Let $\vec{d_i}$ be the difference of endpoints of γ_i in Ψ^{-1}.



Let $\vec{t} = \sum \vec{d_i}$, then $\Phi^{\vec{t}}(z_0) = z_1$. □.

注. Ψ 不可能单, since $\mathbb{R}^n$ 不紧但 M 紧.

Let $\mathrm{Stab}(z_0) = \{\vec{t} \in \mathbb{R}^n \mid \Phi^{\vec{t}}(z_0) = z_0\}$ be the stabilizer (group) of z_0.



The local diffeomorphism property implies that $\mathrm{Stab}(z_0)$ is discrete. (无聚点) no accumulation point

由$\mathbb{R}^n$中离散子群之刻画 $\Rightarrow$ $\mathrm{Stab}(z_0) = \mathrm{span}\{e_1, \dots, e_k\}$.

We consider the map $\tilde{\Psi} : \mathbb{R}^n / \mathrm{Stab}(z_0) \to M$ (induced from Ψ) $\Rightarrow$ 既满又单.

$\tilde{\Psi}$ is a diffeomorphism from $\mathbb{R}^n/\mathrm{Stab}(z_0) \to M$, since M compact $\Rightarrow$ $k = n$.

$\mathrm{Stab}(z_0) \cong \mathbb{Z}^n \quad \Rightarrow \quad \mathbb{R}^n/\mathrm{Stab}(z_0) = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. !!!

We finally work on the action - angle coordinates

(I, θ).

We need to find a symplectic transformation $(y, x) \mapsto (I, \theta)$, $\omega = dy \wedge dx = dI \wedge d\theta$

We need to find $S(I, x)$ such that $y\,dx + \theta\,dI = dS(I, x)$

The symplectic transformation is given by $\begin{cases} y = \frac{\partial S}{\partial x} \\ \theta = \frac{\partial S}{\partial I} \end{cases}$

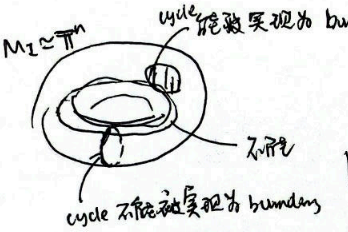
3

Moreover, we expect $H_i\left(\frac{\partial S}{\partial x}(I,x), x\right) = K_i(I)$.

This implies $M_a = \{H_i = a_i\} = \{K_i(I) = a_i\}$

So for each M_a, we can label it using I, denoted by M_I.

Let $\lambda = y\,dx$ be Liouville 1-form. $\omega = d\lambda$.



$\left.\begin{matrix}\mathbb{T}^2\text{中:}\\ 2\text{个.}\end{matrix}\right\}$

Let $\gamma_1, \dots, \gamma_n$ be n loops representing the basis elem

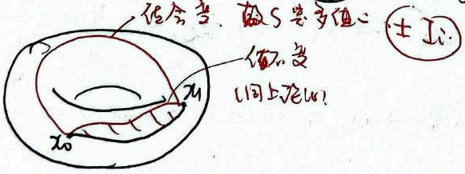
$$H_1(\mathbb{T}^n) = \text{cycle}/\text{boundary}.$$

We define $I_i = \int_{\gamma_i} \lambda$. ← well-defined.

(Suppose we choose $\overset{\text{another}}{\text{loops}}$ $\tilde{\gamma}_i$ s.t. $\gamma_i - \tilde{\gamma}_i$ is the boundary of some surface $\Sigma \subset M_a$.

$\Rightarrow \int_{\gamma_i - \tilde{\gamma}_i} \lambda = \int_{\partial\Sigma} \lambda = \int_\Sigma d\lambda = \int_\Sigma \omega = 0$ ← $\Sigma \subset \underline{M_a}$ 是 Lagrangian 子流形!)

Fix $x_0 \in M_a$, we define $S(I,x) = \int_{x_0}^{x} \lambda \Big|_{M_a}$.



On each M_a, the variable I is constant by the formula $y\,dx + Q\,dI = dS$,

we get $y = \frac{dS}{dx}\Big|_{M_a}$.

We simply define $\theta = \frac{\partial S}{\partial I}$. We next show θ is defined on $\mathbb{T}^n$.

We know S is a multi-valued function with increment I_i when we turn around γ_i once.

$\Rightarrow \frac{\partial S|_{\gamma_i}}{\partial I_j} = \frac{\partial I_i}{\partial I_j} = \delta_{ij}$, θ_i is a multi-valued function with increment 1 when we turn around γ_i once.

$\Downarrow$ $\boxed{\theta_i \text{ gives coordinate on the } i\text{-th component of } \mathbb{T}^n}$ $\square$.